An exact robust hyperexponential differentiator

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***Abstract*—A simple differentiator is proposed, which is modeled by a second order time-varying linear differential equation. It is shown that for any signal of interest, whose second derivative is an essentially bounded function of time, the differentiation error converges to zero with a hyperexponential rate (faster than any exponential). An implicit discretization scheme of the differentiator is given, which preserves all main properties of the continuous-time counterpart. In addition, the differentiation error is robustly stable with respect to the measurement noise with a linear gain. The efficiency of the suggested differentiator is illustrated through comparison in numeric experiments with popular alternatives.**

I. Introduction

The estimation problem of the derivative of a signal through its noisy measurements in real time is a mature and well-known challenge, which has numerous solutions (*e.g.*, the popular differentiators can be found in [1], [2], [3], [4], [5]), but still attracting a lot of attention [6] (see also the related special issue).

The main performance characteristics demanded from a differentiator include: the time of convergence of the estimate to the true value and asymptotic precision in the perturbation-free case, noise sensitivity and the implementation complexity, to mention the most important ones. The existence of multiple solutions is explained by the fact that it is difficult to design a differentiation algorithm outperforming others by all existing criteria.

In this note we are going to propose a new differentiator having an accelerated (faster than any exponential) rate of convergence, providing asymptotically exact estimates in the noise-free scenario for the signals with bounded second derivatives (no knowledge of the upper bound is required for the tuning), robust, and admitting a simple digital implementation. The gain selection rules are formulated using linear matrix inequalities, and it is demonstrated in simulations that the new differentiator has very advantageous performance qualities even in the case of a slow sampling.

The paper is organized as follows. The brief preliminaries are given in Section II. The problem statement is introduced in Section III. The properties of the proposed differentiator in continuous time are investigated in Section IV. The properties of its implicit Euler discretization are considered in Section V. The results of numeric comparison of the presented differentiator with a linear high-gain observer and super-twisting exact differentiator are shown in Section VI.

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Notation

- $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$, where $\mathbb{R}$ is the set of real numbers, $\mathbb{Z}$ is the set of integer numbers, $\mathbb{Z}_+ = \mathbb{Z} \cap \mathbb{R}_+$.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ is used for the Euclidean norm on $\mathbb{R}^n$.
- For a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ and $[t_0, t_1) \subset \mathbb{R}_+$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess}\sup_{t\in[t_0,t_1)} \|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions).
- A continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and it is strictly increasing. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s \in \mathbb{R}_+$.
- Denote the identity matrix of dimension $n \times n$ by I_n.
- $\text{diag}\{g\}$ represents a diagonal matrix of dimension $n\times n$ with a vector $g \in \mathbb{R}^n$ on the main diagonal.
- The relation $P \prec 0$ ($P \succ 0$) means that a symmetric matrix $P \in \mathbb{R}^{n\times n}$ is negative (positive) definite, $\lambda_{\min}(P)$ denotes the minimal eigenvalue of such a matrix P.
- $\exp(1) = \mathbf{e}$.

II. Preliminaries

The used standard stability notions and their definitions can be found in [7].

A. Uniform hyperexponential stability

Consider a non-autonomous differential equation:

$$dx(t)/dt = f(t, x(t), d(t)),\ t \geq t_0,\ t_0 \in \mathbb{R}_+, \qquad (1)$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $d(t) \in \mathbb{R}^m$ is the vector of external disturbances and $d \in \mathcal{L}_\infty^m$; $f : \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous function with respect to x, d and piecewise continuous with respect to t, $f(t, 0, 0) = 0$ for all $t \in \mathbb{R}_+$. A solution of the system (1) for an initial condition $x_0 \in \mathbb{R}^n$ at time instant $t_0 \in \mathbb{R}_+$ and some $d \in \mathcal{L}_\infty^m$

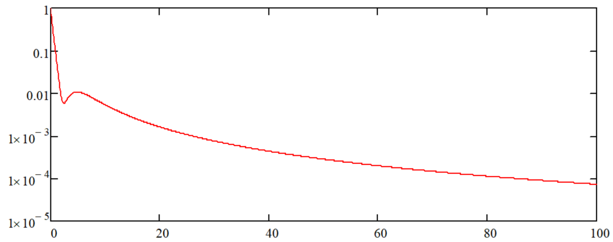

Figure 1. $\delta(\tau)$ versus $\tau \in [0, 100]$

is denoted as $X(t, t_0, x_0, d)$, and we assume that f ensures existence and uniqueness of solutions $X(t, t_0, x_0, d)$ at least locally in forward time.

Definition 1. For a given set $\mathbb{D} \subset \mathcal{L}_\infty^m$ the system (1) is called *uniformly hyperexponentially stable* if it is uniformly globally asymptotically stable and for any $\alpha > 0$ there exists $\rho_\alpha \in \mathcal{K}$ and $\beta_\alpha \in \mathcal{KL}$ such that for all $x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}_+$ and $d \in \mathbb{D}$:

$$\|X(t, t_0, x_0, d)\| \leq \mathbf{e}^{-\alpha(t-t_0)}\rho_\alpha(\|x_0\|) + \beta_\alpha(\|d\|_\infty, t - t_0)$$

for all $t \geq t_0$.

A simple example of a uniformly hyperexponentially stable system (1) is

$$\dot{x}(t) = -(1 + t)x(t) + d(t), \ t \geq 0$$

with $x(t), d(t) \in \mathbb{R}$, whose solutions admit an estimate:

$$|x(t)| \leq e^{-\frac{t^2}{2}-t}|x(0)| + \frac{2\|d\|_\infty}{1 + t}, \ t \geq 0$$

for any $x(0) \in \mathbb{R}$ and $d \in \mathcal{L}_\infty^1$ (hence, for $t_0 = 0$, $\rho_\alpha(s) = e^{\frac{(\alpha-1)^2}{2}} s$ and $\beta_\alpha(s, t) = \frac{2s}{1+t}$ for any $\alpha > 0$).

In this definition the hyperexponential rate of convergence is demanded only in initial conditions, while uniformity is understood in double meaning: as independence in the initial time t_0 and in the input $d \in \mathbb{D}$. Despite it is assumed that $\mathbb{D} \subset \mathcal{L}_\infty^m$, any other suitable class of inputs can be considered.

B. Auxiliary properties

We will use the following estimate:

Lemma 1. *For all $\tau \geq 0$ it holds:*

$$\mathbf{e}^{-\tau}\int_0^\tau \frac{\mathbf{e}^s}{(2s + 1)^2}ds \leq \frac{1}{(\tau + 1)^2}.$$

Proof. The integral above can be calculated using the exponential integral special function, but there is no analytical expression for it. Since this inequality depends only on a scalar variable $\tau \geq 0$, its verification can be performed numerically: the discrepancy $\delta(\tau) = \frac{1}{(\tau+1)^2} - \mathbf{e}^{-\tau}\int_0^\tau \frac{\mathbf{e}^s}{(2s+1)^2}ds$ is shown in Fig. 1 in logarithmic scale. As we can conclude from the above figure, $\delta(\tau) \geq 0$ for all $\tau \geq 0$ (only initial domain of values for τ is presented), which implies the desired property. □

The following blockwise matrix inversion formula is used in the sequel:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}^{-1} = \begin{bmatrix} A^{-1} + A^{-1}BSCA^{-1} & -A^{-1}BS \\ -SCA^{-1} & S \end{bmatrix},$$
$$S = (D - CA^{-1}B)^{-1},$$

where A, B, C and D are matrices of appropriate dimensions, A and S should be nonsingular.

III. Problem statement

Assume that a continuously differentiable signal $\phi(t) \in \mathbb{R}$ is measured with a noise $v(t) \in \mathbb{R}$:

$$y(t) = \phi(t) + v(t),$$

where $y(t) \in \mathbb{R}$, $v \in \mathcal{L}_\infty^1$, and $\phi : \mathbb{R}_+ \to \mathbb{R}$ has the second derivative $\ddot{\phi}(t) = \frac{d^2\phi(t)}{dt^2}$ with $\ddot{\phi} \in \mathcal{L}_\infty^1$ (without a known constant upper bound).

It is required to estimate the first derivative $\dot{\phi}(t) = \frac{d\phi(t)}{dt}$ of the signal ϕ with accelerated time of convergence and robustly with respect to the perturbation v.

The problem can be equivalently stated as the state estimation for the system

$$\dot{x}(t) = Ax(t) + b\ddot{\phi}(t), \ y(t) = Cx(t) + v(t), \quad (2)$$

where $x(t) \in \mathbb{R}^2$ is the state, $x(0) = [\phi(0) \ \dot{\phi}(0)]^\top$,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \ b = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ C = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

and $\ddot{\phi} \in \mathcal{L}_\infty^1$ corresponds to an unknown external input. And further in this work an observer for this system will be designed estimating $x(t)$ with a hyperexponential rate of convergence uniformly in $\ddot{\phi} \in \mathbb{D} = \mathcal{L}_\infty^1$ while $\|v\|_\infty = 0$, and having a bounded estimation error for $v \in \mathcal{L}_\infty^1$.

IV. Differentiator in continuous time

Let the state observer for (2) be chosen in the form:

$$\dot{\hat{x}}(t) = A\hat{x}(t) + D(t)L(y(t) - C\hat{x}(t)), \quad (3)$$
$$D(t) = \mathrm{diag}\{[\varrho(t) \quad \varrho^2(t)]^\top\},$$

where $\hat{x}(t) \in \mathbb{R}^2$ is the estimate of the state $x(t)$, $L \in \mathbb{R}^2$ is the observer gain that will be selected later, $\varrho(t) = 1 + t$ is a strictly growing function of time.

Remark 1. Any strictly growing function of time $\varrho(t)$ can be used in (3), *e.g.*, $\varrho(t) = a\mathbf{e}^{\alpha t}$ or $\varrho(t) = (b + at)^\alpha$ for $a > 0$, $b > 0$ and $\alpha > 0$.

Define the estimation error as $e(t) = x(t) - \hat{x}(t)$, whose dynamics can be written as follows:

$$\dot{e}(t) = (A - D(t)LC)\, e(t) + b\ddot{\phi}(t) - D(t)Lv(t).$$

We are in position to formulate the main result of this section:

Theorem 1. *Let there exist $P=P^\top\in\mathbb{R}^{2\times 2}$, $U\in\mathbb{R}^2$, $\gamma_1>0$ and $\gamma_2>0$ such that the linear matrix inequalities are verified:*

$$P\succ 0,\ Q\preceq 0,\ Q=\begin{bmatrix} Q_{11} & Pb & -U \\ b^\top P & -\gamma_1 & 0 \\ -U^\top & 0 & -\gamma_2 \end{bmatrix},$$
$$Q_{11}=A^\top P+PA-UC-C^\top U^\top+\frac{C^\top CPC^\top C}{2}+P.$$

Then for $L=P^{-1}U$ and any $e(0)\in\mathbb{R}^2$ in (2), (3):

$$\sqrt{\lambda_{\min}(P)}\begin{bmatrix} |e_1(t)| \\ |e_2(t)| \end{bmatrix}\leq\begin{bmatrix} 1 \\ \varrho(t) \end{bmatrix}(\mathbf{e}^{-\frac{t+2}{4}t}\sqrt{e(0)^\top Pe(0)}$$
$$+2\sqrt{\gamma_1}\frac{\|\ddot{\phi}\|_\infty}{\varrho^2(t)+1}+\sqrt{\gamma_2}\|v\|_\infty)$$

for all $t\geq 0$.

For $\|v\|_\infty=0$ the result of the theorem implies uniform hyperexponential stability of the estimation error $e(t)$ in (2), (3) with $\ddot{\phi}\in\mathcal{L}_\infty^m$. This property means that (3) is an asymptotically exact differentiator (as the super-twisting algorithm [2] in finite time), and the class of second derivatives $\ddot{\phi}\in\mathcal{L}_\infty^m$, for which the uniformity of the estimates is kept, can be enlarged by ones growing *not faster than a linear* in time.

Note also that the given linear matrix inequalities are always feasible under the conditions of the theorem.

Proof. Define new auxiliary variable $\epsilon(t)=\Gamma(t)e(t)$, where

$$\Gamma(t)=\mathrm{diag}\{[1\quad \varrho^{-1}(t)]^\top\},$$

whose dynamics takes the form:

$$\begin{aligned}\dot{\epsilon}(t)&=-\mathrm{diag}\{[0\quad \varrho^{-2}(t)]^\top\}e(t)+\Gamma(t)\dot{e}(t)\\ &=\varrho(t)\Big((A-LC-\Delta(t))\,\epsilon(t)+\varrho^{-2}(t)b\ddot{\phi}(t)-Lv(t)\Big),\\ \Delta(t)&=\mathrm{diag}\{[0\quad \varrho^{-2}(t)]^\top\}.\end{aligned}$$

Note that $\epsilon(0)=e(0)$. Finally, let us define a new time variable

$$\tau=\varphi(t)=\frac{t(t+2)}{2},\ t=\varphi^{-1}(\tau)=\sqrt{2\tau+1}-1,$$
$$d\tau=(t+1)dt,$$

then after the change of the time the auxiliary error $\epsilon(t)$ dynamics is reduced to an equivalent useful representation (note that $\tau\geq 0$):

$$\frac{d\epsilon(\tau)}{d\tau}=\Big(A-LC-\tilde{\Delta}(\tau)\Big)\epsilon(\tau)+\frac{1}{2\tau+1}b\ddot{\phi}(\tau)-Lv(\tau),\tag{4}$$

where $\ddot{\phi}(\tau)=\ddot{\phi}(\varphi^{-1}(\tau))$ and $v(\tau)=v(\varphi^{-1}(\tau))$ are the external signals in the new time,

$$\tilde{\Delta}(\tau)=\Delta(\varphi^{-1}(\tau))=\mathrm{diag}\{[0\quad (2\tau+1)^{-1}]^\top\}.$$

For analysis of stability properties in (4) let us choose a candidate Lyapunov function $V(\epsilon)=\epsilon^\top P\epsilon$ (that is positive definite since $P\succ 0$), whose derivative in the new time τ for the dynamics (4) can be written as follows:

$$\frac{dV(\epsilon(\tau))}{d\tau}=\epsilon(\tau)^\top\left((A-LC)^\top P+P(A-LC)\right)\epsilon(\tau)$$
$$-2\epsilon(\tau)^\top P\left(\tilde{\Delta}(\tau)\epsilon(\tau)+\frac{1}{2\tau+1}b\ddot{\phi}(\tau)-Lv(\tau)\right).$$

Due to the form of $\tilde{\Delta}(\tau)$ we have:

$$\frac{1}{2}C^\top CPC^\top C\geq -P\tilde{\Delta}(\tau)-\tilde{\Delta}(\tau)P$$
$$-\frac{1}{2}\frac{C^\top CPC^\top C-C^\top CPC^\top C}{2\tau+1}$$

for all $\tau\geq 0$. Therefore,

$$\frac{dV(\epsilon(\tau))}{d\tau}=\begin{bmatrix}\epsilon(\tau)\\ \frac{\ddot{\phi}(\tau)}{2\tau+1}\\ v(\tau)\end{bmatrix}^\top\tilde{Q}\begin{bmatrix}\epsilon(\tau)\\ \frac{\ddot{\phi}(\tau)}{2\tau+1}\\ v(\tau)\end{bmatrix}-V(\epsilon(\tau))$$
$$+\gamma_1\left(\frac{\ddot{\phi}(\tau)}{2\tau+1}\right)^2+\gamma_2v^2(\tau),$$
$$\tilde{Q}=\begin{bmatrix}\tilde{Q}_{11} & Pb & -PL\\ b^\top P & -\gamma_1 & 0\\ -L^\top P & 0 & -\gamma_2\end{bmatrix},\ \tilde{Q}_{11}=(A-LC)^\top P$$
$$+P(A-LC)+\frac{C^\top CPC^\top C}{2}+P.$$

It is easy to see that $\tilde{Q}$ equals to Q under substitution $U=PL$, then $\tilde{Q}\preceq 0$ and

$$\frac{dV(\epsilon(\tau))}{d\tau}\leq -V(\epsilon(\tau))+\gamma_1\left(\frac{\ddot{\phi}(\tau)}{2\tau+1}\right)^2+\gamma_2v^2(\tau)$$
$$\leq -V(\epsilon(\tau))+\gamma_1\left(\frac{\|\ddot{\phi}\|_\infty}{2\tau+1}\right)^2+\gamma_2\|v\|_\infty^2.$$

Passing to the time domain we get an estimate:

$$V(\epsilon(\tau))\leq\mathbf{e}^{-\tau}V(\epsilon(0))+\gamma_1\mathbf{e}^{-\tau}\int_0^\tau\mathbf{e}^s\left(\frac{\|\ddot{\phi}\|_\infty}{2s+1}\right)^2ds$$
$$+\gamma_2\mathbf{e}^{-\tau}\int_0^\tau\mathbf{e}^s\|v\|_\infty^2ds$$

for all $\tau\geq 0$. According to Lemma 1:

$$\gamma_1\mathbf{e}^{-\tau}\int_0^\tau\mathbf{e}^s\left(\frac{\|\ddot{\phi}\|_\infty}{2s+1}\right)^2ds\leq\gamma_1\left(\frac{\|\ddot{\phi}\|_\infty}{\tau+1}\right)^2,$$

and obviously $\gamma_2\mathbf{e}^{-\tau}\int_0^\tau\mathbf{e}^s\|v\|_\infty^2ds\leq\gamma_2\|v\|_\infty^2$ for all $\tau\geq 0$, then

$$V(\epsilon(\tau))\leq\mathbf{e}^{-\tau}V(\epsilon(0))+\gamma_1\left(\frac{\|\ddot{\phi}\|_\infty}{\tau+1}\right)^2+\gamma_2\|v\|_\infty^2,\ \tau\geq 0,$$

and in the original time

$$V(\epsilon(t))\leq\mathbf{e}^{-\frac{t(t+2)}{2}}V(\epsilon(0))+4\gamma_1\left(\frac{\|\ddot{\phi}\|_\infty}{t(t+2)+2}\right)^2+\gamma_2\|v\|_\infty^2$$

for all $t \geq 0$. Since $\epsilon(t) = \Gamma(t)e(t)$, the desired estimate on the behavior of $e_1(t)$ and $e_2(t)$ can be derived for all $t \geq 0$ from this inequality. $\square$

The upper bound on the estimation error of (2), (3) calculated in Theorem 1 implies that the gain of $|e_2(t)|$ in v is linear in time. For a small $\|v\|_\infty$ and having the computations on a bounded interval of time it can be not a problem in application, but let us check that happens after discretization of (3).

V. Differentiator in discrete time

Note that (3) is modeled by a linear time-varying system with an external known input $y(t)$. Since the time-varying gain $D(t)$ is strictly growing, the explicit Euler discretization cannot be used for all $t \geq 0$, however, the implicit one can be effectively applied [8]. Let $h > 0$ be constant discretization step, denote by $t_k = hk$ for $k \in \mathbb{Z}_+$ the discretization time instants, then application of the implicit Euler discretization method to (3) gives for $k \in \mathbb{Z}_+$:

$$\hat{x}_{k+1} = Z(t_{k+1})\left(\hat{x}_k + hD(t_{k+1})Ly_{k+1}\right), \qquad (5)$$

$$Z(t) = (I_2 - h(A - D(t)LC))^{-1} = \frac{K(t)}{O(t)},$$

$$K(t) = \begin{bmatrix} 1 & h \\ -L_2 h\varrho^2(t) & L_1 h\varrho(t) + 1 \end{bmatrix},$$

$$O(t) = L_2 h^2 \varrho^2(t) + L_1 h\varrho(t) + 1,$$

where $\hat{x}_k$ is an approximation of $\hat{x}(t_k)$ and $y_k = y(t_k)$. As we can conclude from these expressions, the discrete state transition matrix $Z(t)$ is nonsingular (its determinant equals 1) and elementwise bounded for all $t \geq 0$. Moreover, the input gain matrix

$$F(t) = Z(t)D(t)L = \frac{\varrho(t)}{O(t)} \begin{bmatrix} L_2 h\varrho(t) + L_1 \\ L_2\varrho(t) \end{bmatrix}$$

is also elementwise bounded with

$$h \lim_{t \to +\infty} F(t) = \begin{bmatrix} 1 \\ h^{-1} \end{bmatrix},$$

which evaluates the asymptotic noise sensitivity.

Consider a discrete-time model (the solutions of (2) can be approximated using the same discretization method):

$$x_{k+1} = (I_2 - hA)^{-1}\left(x_k + hb\ddot{\phi}_{k+1}\right), \; y_k = Cx_k + v_k,$$

for $k \in \mathbb{Z}_+$, where x_k should approach to $x(t_k)$ as h converges to zero, $\ddot{\phi}_{k+1} = \ddot{\phi}(t_{k+1})$ and $v_k = v(t_k)$ (formally this perturbation v_k is different from the one used in (2) since it should also include the discretization error, but with a light ambiguity in notation we will continue to use the same symbol). In this work we will assume that v_k and $\ddot{\phi}_k$ take finite values with bounded norms as before, *i.e.*, $|v|_\infty = \sup_{k\in\mathbb{Z}_+} |v_k|$ and $|\ddot{\phi}|_\infty = \sup_{k\in\mathbb{Z}_+} |\ddot{\phi}_k|$ are well defined.

To analyze the properties of (5) we will consider the discretization error $e_k = x_k - \hat{x}_k$, whose stability and hyperexponential convergence rate have been analyzed in Theorem 1 for the continuous-time scenario. The direct computations show that

$$\begin{aligned} e_{k+1} &= (I_2 - hA)^{-1}\left(x_k + hb\ddot{\phi}_{k+1}\right) \\ &\quad - Z(t_{k+1})\left(\hat{x}_k + hD(t_{k+1})LCx_{k+1} + hD(t_{k+1})Lv_{k+1}\right) \\ &= (I_2 - hZ(t_{k+1})D(t_{k+1})LC)(I_2 - hA)^{-1}\left(x_k + hb\ddot{\phi}_{k+1}\right) \\ &\quad - Z(t_{k+1})\left(\hat{x}_k + hD(t_{k+1})Lv_{k+1}\right) \end{aligned}$$

for all $k \in \mathbb{Z}_+$ and that

$$(I_2 - hZ(t_{k+1})D(t_{k+1})LC)(I_2 - hA)^{-1} = Z(t_{k+1}),$$

therefore, we finally get:

$$e_{k+1} = Z(t_{k+1})\left(e_k + hb\ddot{\phi}_{k+1} - hD(t_{k+1})Lv_{k+1}\right). \qquad (6)$$

To investigate stability and the rate of convergence in (6), let us define a time-varying Lyapunov function candidate (the same was used before):

$$V_k = e_k^\top \Pi_k e_k, \; \Pi_k = \Gamma(t_{k+1})P\Gamma(t_{k+1}),$$

where $P = P^\top \succ 0$ has been introduced in Theorem 1 (more precise requirements will be defined below). For a time-varying parameters $\alpha_k > 0$, $\gamma_k > 0$ and $\sigma_k > 0$ determined later for $k \in \mathbb{Z}_+$ we obtain

$$V_{k+1} - \alpha_k V_k = \begin{bmatrix} e_k \\ \ddot{\phi}_{k+1} \\ v_{k+1} \end{bmatrix}^\top Q_k \begin{bmatrix} e_k \\ \ddot{\phi}_{k+1} \\ v_{k+1} \end{bmatrix} + \gamma_k \ddot{\phi}_{k+1}^2 + \sigma_k v_{k+1}^2$$

for

$$\begin{aligned} Q_k &= \begin{bmatrix} I_2 \\ hb^\top \\ -hL^\top D(t_{k+1}) \end{bmatrix} Z^\top(t_{k+1})\Pi_{k+1}Z(t_{k+1}) \\ &\times \begin{bmatrix} I_2 \\ hb^\top \\ -hL^\top D(t_{k+1}) \end{bmatrix}^\top - \begin{bmatrix} \alpha_k\Pi_k & 0 & 0 \\ 0 & \gamma_k & 0 \\ 0 & 0 & \sigma_k \end{bmatrix}. \end{aligned}$$

We need to find the restrictions on P and the gains $\alpha_k, \gamma_k, \sigma_k$ such that $Q_k \preceq 0$. To this end, using the Schur complement and multiplying the obtained matrix from both sides on $\mathrm{diag}\{I_2, Z^{-1}(t_{k+1})\Gamma^{-1}(t_{k+1}), 1, 1\}$ and its transpose the latter property is equivalent to

$$\tilde{Q}_k \succeq 0, \; \tilde{Q}_k = \begin{bmatrix} \tilde{Q}_{11}^k & \tilde{Q}_{12}^k \\ (\tilde{Q}_{12}^k)^\top & \tilde{Q}_{22}^k \end{bmatrix},$$

$$\tilde{Q}_{11}^k = \begin{bmatrix} \Pi_{k+1}^{-1} & \Gamma^{-1}(t_{k+1}) \\ \Gamma^{-1}(t_{k+1}) & \alpha_k R(t_{k+1}) \end{bmatrix},$$

$$\tilde{Q}_{12}^k = h \begin{bmatrix} Z(t_{k+1})b & -F(t_{k+1}) \\ 0 & 0 \end{bmatrix}, \; \tilde{Q}_{22}^k = \begin{bmatrix} \gamma_k & 0 \\ 0 & \sigma_k \end{bmatrix},$$

where

$$R(t_{k+1}) = \Gamma^{-1}(t_{k+1})Z^{-\top}(t_{k+1})\Pi_k Z^{-1}(t_{k+1})\Gamma^{-1}(t_{k+1}).$$

Noting that

$$R(t) = \Gamma^{-1}(t)\left(I_2 - h(A - D(t)LC)\right)^\top \Gamma(t)P \times\Gamma(t)\left(I_2 - h(A - D(t)LC)\right)\Gamma^{-1}(t)$$

and recalling that

$$\Gamma(t)(A - D(t)LC)\Gamma^{-1}(t) = \varrho(t)(A - LC),$$

we obtain

$$R(t) = P - h\varrho(t)\left((A - LC)^\top P + P(A - LC)\right) +h^2\varrho^2(t)(A - LC)^\top P(A - LC),$$

which implies that this matrix is positive definite under the restrictions of Theorem 1 for any $t \geq 0$.

To formulate the conditions implying $\tilde{Q}_k \succeq 0$, first, let us investigate the restrictions for $\tilde{Q}_{11}^k \succ 0$. Calculating the Schur complement of $\tilde{Q}_{11}^k$ we get an inequality:

$$\alpha_k R(t_{k+1}) - \Gamma^{-1}(t_{k+1})\Gamma(t_{k+2})P\Gamma(t_{k+2})\Gamma^{-1}(t_{k+1}) \succ 0,$$

and since

$$P \succ \Gamma^{-1}(t_{k+1})\Gamma(t_{k+2})P\Gamma(t_{k+2})\Gamma^{-1}(t_{k+1}),$$

the property $\tilde{Q}_{11}^k \succ 0$ follows the conditions of Theorem 1 for $\alpha_k = a$ or $\alpha_k = \frac{a}{h\varrho(t_{k+1})}$ for some $a \geq 1$. If $\alpha_k = \frac{a}{h^2\varrho^2(t_{k+1})}$ with $a > 0$, then an auxiliary linear matrix inequality should be verified:

$$a(A - LC)^\top P(A - LC) - P \succ 0,$$

i.e., the matrix P should be a solution of Lyapunov equation for the matrix $A-LC$ in both, continuous and discrete, times. Reformulating this inequality through the Schur complement we obtain:

$$\begin{bmatrix} a^{-1}P & PA - UC \\ A^\top P - C^\top U^\top & P \end{bmatrix} \prec 0,$$

where as before $U = PL$ and, obviously, always there is $a > 0$ such that it is verified. Returning back to verification of $\tilde{Q}_k \succeq 0$, and having $\tilde{Q}_{11}^k \succ 0$ we can also use the Schur complement of $\tilde{Q}_k$ to check the desired property:

$$\tilde{Q}_{22}^k - (\tilde{Q}_{12}^k)^\top(\tilde{Q}_{11}^k)^{-1}\tilde{Q}_{12}^k \succeq 0,$$

then denote $T_k = (\tilde{Q}_{11}^k)^{-1}$, which can be calculated using the blockwise inversion formula given in the preliminaries with the first block element

$$T_k^{11} = \Pi_{k+1} + \Pi_{k+1}\Gamma^{-1}(t_{k+1})(\alpha_k R(t_{k+1}) -\Gamma^{-1}(t_{k+1})\Pi_{k+1}\Gamma^{-1}(t_{k+1}))^{-1}\Gamma^{-1}(t_{k+1})\Pi_{k+1},$$

leading to

$$(\tilde{Q}_{12}^k)^\top(\tilde{Q}_{11}^k)^{-1}\tilde{Q}_{12}^k = h^2 \times\begin{bmatrix} b^\top Z^\top(t_{k+1})T_k^{11}Z(t_{k+1})b & -F^\top(t_{k+1})T_k^{11}Z(t_{k+1})b \\ -b^\top Z^\top(t_{k+1})T_k^{11}F(t_{k+1}) & F^\top(t_{k+1})T_k^{11}F(t_{k+1}) \end{bmatrix}.$$

It is straightforward to check that $Z(t)b = \frac{1}{O(t)}\begin{bmatrix} h \\ L_1 h\varrho(t) + 1 \end{bmatrix}$ is of order $\varrho^{-1}(t)$ for any $t \geq 0$, while it has been discussed that $F(t)$ is globally bounded for all $t \geq 0$. For $\alpha_k = \frac{a}{h^2\varrho^2(t_{k+1})}$ with $a > 0$, T_k^{11} is of the order $\Gamma^2(t_{k+1})$. Therefore, there exists $g > 0$ and $s > 0$ such that for $\gamma_k = \frac{g}{h^2\varrho^4(t_{k+1})}$ and $\sigma_k = \frac{s}{h^2}$ the property $\tilde{Q}_k \succeq 0$ is verified for all $k \in \mathbb{Z}_+$, and the following result can be formulated:

Theorem 2. *Let there exist $P = P^\top \in \mathbb{R}^{2\times 2}$, $U \in \mathbb{R}^2$ such that the linear matrix inequalities are verified:*

$$P \succ 0,\ A^\top P + PA - C^\top U^\top - UC \prec 0.$$

Then for $L = UP^{-1}$ there exist $a > 0$, $g > 0$ and $s > 0$ such that for any $e_0 \in \mathbb{R}^2$ in (6)*:*

$$\lambda_{\min}(P)\|e_k\|^2 \leq \varrho^2(t_{k+1})(\prod_{i=0}^{k-1}\frac{a}{h^2\varrho^2(t_{i+1})}e_0^\top Pe_0 +\frac{g|\ddot{\phi}|_\infty^2}{h^2}\sum_{i=0}^{k-1}\frac{1}{\varrho^4(t_{i+1})}\prod_{\ell=0}^{k-i-2}\frac{a}{h^2\varrho^2(t_{\ell+1})} +\frac{s|v|_\infty^2}{h^2}\sum_{i=0}^{k-1}\prod_{\ell=0}^{k-i-2}\frac{a}{h^2\varrho^2(t_{\ell+1})})$$

for all $k \in \mathbb{Z}_+$.

Proof. Under the introduced restrictions, it has been shown above that

$$V_{k+1} \leq \frac{a}{h^2\varrho^2(t_{k+1})}V_k + \frac{g|\ddot{\phi}|_\infty^2}{h^2\varrho^4(t_{k+1})} + \frac{s}{h^2}|v|_\infty^2$$

for all $k \in \mathbb{Z}_+$. Hence,

$$V_k \leq \prod_{i=0}^{k-1}\frac{a}{h^2\varrho^2(t_{i+1})}V_0 +\frac{g|\ddot{\phi}|_\infty^2}{h^2}\sum_{i=0}^{k-1}\frac{1}{\varrho^4(t_{i+1})}\prod_{\ell=0}^{k-i-2}\frac{a}{h^2\varrho^2(t_{\ell+1})} +\frac{s|v|_\infty^2}{h^2}\sum_{i=0}^{k-1}\prod_{\ell=0}^{k-i-2}\frac{a}{h^2\varrho^2(t_{\ell+1})}$$

for all $k \in \mathbb{Z}_+$, which gives the required estimate. □

As we can see from the obtained results, the convergence in the initial error is faster than any exponential and there is no asymptotic dependence in $\ddot{\phi}$, while the noise gain admits a static linear bound.

VI. Comparison in simulations

Let us illustrate performance of the proposed hyperexponential differentiator (5) in numeric experiments, and in comparison with the high-gain differentiator from [1] and the super-twisting sliding mode exact differentiator from [2]. The former can be discretized also using the implicit Euler method, then its realization will take the same form as (5) by choosing a constant time value $t_{k+1} = T$, where $T > 0$ is a design parameter; the latter differentiator, due to its nonlinearity and discontinuity, it is difficult to discretize

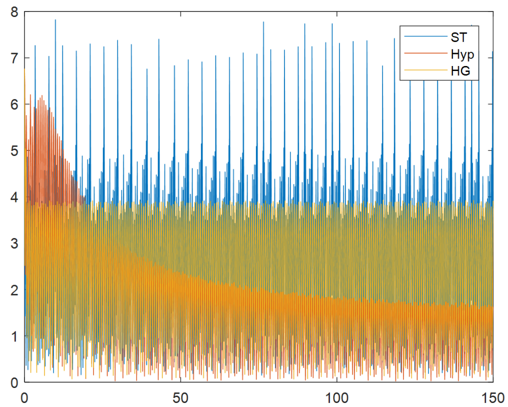

Figure 2. The results of estimation, $\|e_k\|$ versus time t_k, $h = 0.1$

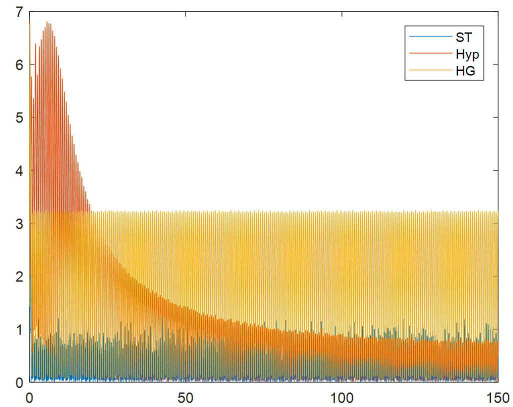

Figure 3. The results of estimation, $\|e_k\|$ versus time t_k, $h = 0.01$

using an implicit method [9], then usually the explicit Euler discretization is implemented despite its drawbacks [10], [11]:

$$\hat{x}_{k+1} = \hat{x}_k + h \left[\begin{array}{c} L_1^{ST}\sqrt{|y_k - \hat{x}_{k,1}|}\text{sign}(y_k - \hat{x}_{k,1}) + \hat{x}_{k,2} \\ L_2^{ST}\text{sign}(y_k - \hat{x}_{k,1}) \end{array} \right],$$

where $\hat{x}_k = [\hat{x}_{k,1}\ \ \hat{x}_{k,2}]^\top \in \mathbb{R}^2$ is the state, whose components have to converge to the signals $\phi_k = \phi(t_k)$ and $\dot{\phi}_k = \dot{\phi}(t_k)$, respectively, $L^{ST} = \left[\begin{array}{c} L_1^{ST} \\ L_2^{ST} \end{array} \right]$ is the observer gain to be chosen.

For simulations, let

$$\phi(t) = \sqrt{t+2} + \sin(5t)$$

and the noise

$$v(t) = 0.01\left(1 + \text{rnd}(1) + \sin(15t)\right),$$

where $\text{rnd}(1)$ denotes a uniformly distributed in the interval $[0, 1]$ random number. A solution of linear matrix inequalities from Theorem 2 gives:

$$L = \left[\begin{array}{c} 0.8144 \\ 0.3292 \end{array} \right].$$

The results of simulations are presented in figures 2 and 3 for $h = 0.1$ and $h = 0.01$, respectively. On the plots the error norm, $\|e_k\|$, is shown versus the time, t_k ("ST" corresponds to the curve generated by super-twisting differentiator for $L^{ST} = D(10)L$, "Hyp" stands for the proposed here, and "HG" is for high-gain one for $T = 20$). As we can conclude from these results, asymptotically the hyperexponential differentiator outperforms its alternatives in both cases.

For the hyperexponential differentiator, the error stays asymptotically bounded and not growing with time, which implies boundedness of the gain with respect to the noise $v(t)$. In addition, due to the differentiator gain $D(t)L$ grows gradually, there is no peaking in the transients, which is a usual drawback in the high-gain solution. The convergence decay and the shape of transients (*e.g.*, the amplitude of initial discrepancy) in (5) can be optimized by the choice and tuning of $\varrho(t)$.

VII. Conclusion

A new differentiator is proposed, which has a simple discrete-time implementation, guaranteeing a hyperexponential rate of convergence of the estimation error being asymptotically exact in the noise-free case. It has also certain robustness with respect to the measurement noise. The tuning rules are formulated using feasible linear matrix inequalities. The results of simulation demonstrate a good performance of the proposed differentiator, especially in the case of slow sampling. Extension to a higher order differentiator can be considered as a direction of future research.

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