On Relaxed Conditions of Integral ISS for Multistable Periodic Systems

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Abstract— **A novel characterization of the integral Input-to-State Stability (iISS) property is introduced for multistable systems whose dynamics are periodic with respect to a part of the state. First, the concepts of *iISS-Leonov functions* and *output smooth dissipativity* are introduced, then their equivalence to the properties of *bounded-energy-bounded-state* and *global attractiveness* of solutions in the absence of disturbances are proven. The proposed approach permits to relax the usual requirements of positive definiteness and periodicity of the iISS-Lyapunov functions. Moreover, the usefulness of the theoretical results is illustrated by a robustness analysis of a nonlinear pendulum with a constant bias input and an unbounded state-dependent input coefficient.**

I. Introduction

The study of stability properties of dynamical systems under uncertainty conditions is one of the main topics in control theory, and related subjects like mechanics, power systems, biological systems, etc. Among others, the Lyapunov function method has gained popularity in the analysis of stability of unperturbed dynamical systems [1]–[4]. This approach offers an innovative characterization of boundedness and convergence properties of the system trajectories with respect to an equilibrium (or an invariant set) by means of the existence of a continuously differentiable (or at least Lipschitz continuous) Lyapunov function, which is positive definite with respect to such an equilibrium (or invariant set) and its time-derivative is negative definite along the trajectories of the system under study. Similarly, instability of an equilibrium can be studied by using the Chetaev function approach [3], [5]. But in this case, a Chetaev function may be sign-indefinite with a negative or positive definite derivative.

By extending the classical notion of stability, the concept of Input-to-State Stability (ISS) allows the analysis of stability and robustness of nonlinear systems affected by bounded external inputs, e.g., exogenous disturbances or measurement noises [6], [7]. A weaker notion of ISS has been introduced with the concept of integral ISS (iISS) [8], which is suitable for investigation of robustness against disturbances with bounded energy. iISS is a weaker property than ISS since the latter implies the former. Despite that, the results of [8] have shown that it is the most natural property of well-behaved systems, so that, it is indeed a very useful notion.

The classical theory of Lyapunov stability is mainly applicable to systems with a single isolated equilibrium or invariant set [9]. However, many engineering applications concern systems with multiple equilibria or invariant sets (multistable systems), such as power or biological systems, etc. Hence, extensions of the Lyapunov stability approach and related notions have been developed for systems with several disjoint invariant solutions [4], [10]–[14]. These extensions are far from being trivial and require significant modifications and relaxations of the employed stability concepts, see for example the results of [12], [15]–[19]. Particularly, the works in [15] and [16] introduce extensions of the ISS and iISS properties, respectively, providing necessary and sufficient conditions for global analysis of robustness for systems with multiple invariant solutions constituting a decomposable set.[1] Furthermore, in both of them, the ISS and iISS properties are characterized by the existence of a nonnegative (taking zero values only at some elements of the decomposable invariant a set) and continuously differentiable (in the manifold where the system evolves) Lyapunov function with a negative definite (with respect to such set) time-derivative along the solutions of the system under investigation.

A particular class of systems with multiple invariant solutions is formed by those with periodic dynamics with respect to a part of the state. Some examples of such a class of systems include the forced nonlinear pendulum [20], [21], power systems [22], [23], microgrids [24], [25], and phase-locked loops [26]. Unfortunately, the study of stability and robustness of this class of systems requires a more sophisticated solution and the aforementioned Lyapunov-based approaches cannot be applied in a straightforward manner. For instance, the implementation of the results in [15] or [16] leads to the construction of periodic ISS- (iISS-respectively) Lyapunov functions with respect to the part of the state in which the system is also periodic. This fact represents a strong limitation, and just local results can be obtained in the most cases, see for example [24], [25], [27]. To overcome these difficulties, in [28] the results of [15] have been extended for periodic systems with respect to a part of the state by defining the concept of an *ISS-Leonov function*, which significantly relaxes the conditions of periodicity, differentiability and positive definiteness of ISS-Lyapunov functions in the multistable sense. These results are based on the ideas of [29] (see also [4], [30], [31]), who have proposed the so-called *cell structure* approach,

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[1] In the sense of [14].

which allows to conclude global boundedness of solutions for systems whose dynamics are periodic with respect to a scalar state variable. Furthermore, an extension of the cell structure approach for systems which are periodic with respect to multiple state variables is presented in [13] by introducing the concept of a *Leonov function*, a sign-indefinite function with respect to the periodic states and radially unbounded with respect to the non-periodic ones whose derivative is sign-definite.

Motivated by [13], [16] and [28], this paper introduces a novel characterization of the iISS property for multistable systems whose dynamics are periodic with respect to a part of the state. For this purpose, we introduce the concepts of *iISS-Leonov function* and *output smooth dissipativity*, and we prove that they are equivalent to the properties of *bounded-energy-bounded-state* and global attractiveness of solutions in the absence of disturbances. The last two notions were introduced by [16] as a weaker definition of iISS (comparing with the original one given in [8]), but more appropriate for systems with multiple equilibria (including periodic dynamics). Moreover, the usefulness of our results is illustrated by studying the robustness of a nonlinear pendulum with bias force and unbounded state-dependent input coefficient.

The remainder of the paper is structured as follows. Section II presents the problem statement and some useful definitions. Section III contains the main results of this work and the corresponding proofs. Section IV provides an example to illustrate the advantage of our proposal. Finally, the conclusions are given in Section V.

II. Preliminaries

A. Notation

$\mathbb{N}$ and $\mathbb{R}$ stand for the sets of natural and real numbers, respectively. Moreover, $\mathbb{R}_+$ represents the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.

A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}_\infty$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.

The notation $DV(\tilde{x})v$ stands for the directional (or Dini) derivative of a continuously differentiable (or locally Lipschitz continuous) function $V : \mathbb{R}^n \to \mathbb{R}_+$ in the direction of the vector $v \in \mathbb{R}^n$ evaluated at the point $\tilde{x}$.

The distance from a point $\tilde{x} \in \mathbb{R}^n$ to the set $S \subset \mathbb{R}^n$ is defined as $|\tilde{x}|_S = \inf_{a\in S} |\tilde{x}-a|$, where $|\tilde{x}| = |\tilde{x}|_{\{0\}}$ for $\tilde{x} \in \mathbb{R}^n$ is a usual Euclidean norm for a vector $\tilde{x} \in \mathbb{R}^n$. Furthermore, we introduce the vector norm $|\tilde{x}|_\infty = \max_{1\leq i\leq n} |\tilde{x}_i|$ then $|\tilde{x}|_\infty \leq |\tilde{x}| \leq \sqrt{n}|\tilde{x}|_\infty$. Besides, for a locally essentially bounded and measurable signal $d : \mathbb{R}_+ \to \mathbb{R}^m$ the essential supremum norm is defined as $\|d\|_\infty = \text{ess sup}_{t\geq 0} |d(t)|$, and the set of such inputs with a finite norm is further denoted by $\mathcal{L}^m_\infty$.

B. Multistable periodic system

Let a map $f(\tilde{x}, d) : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^n$ be continuously differentiable with respect to its arguments, and consider the system

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t), d(t)), \tag{1}$$

where $\tilde{x}(t) \in \mathbb{R}^n$ is the state and $d(t) \in \mathcal{L}^m_\infty$ is an input signal. Moreover, for any $\tilde{x} \in \mathbb{R}^n$ and $d \in \mathcal{L}^m_\infty$ we denote by $\tilde{X}(t, \tilde{x}; d)$ the uniquely defined solution of (1) at time t fulfilling $\tilde{X}(0, \tilde{x}; d) = \tilde{x}$. Together with (1), consider its unperturbed version

$$\dot{\tilde{x}}(t) = f(\tilde{x}(t), 0). \tag{2}$$

Hence, a set $S \subset \mathbb{R}^n$ is invariant for the unperturbed system (2) if $\tilde{X}(t, \tilde{x}; 0) \in S$ for all $t \in \mathbb{R}$ and for all $\tilde{x} \in S$. For $\tilde{x} \in \mathbb{R}^n$, the point $y \in \mathbb{R}^n$ belongs to the ω-limit (α-limit) set of (2) if there is a sequence t_i, $\lim_{i\to+\infty} t_i = +\infty$, such that $\lim_{i\to+\infty} \tilde{X}(t_i, \tilde{x}; 0) = y$ ($\lim_{i\to+\infty} \tilde{X}(-t_i, \tilde{x}; 0) = y$). For any $\tilde{x} \in \mathbb{R}^n$, the $\alpha-$ and $\omega-$limit sets of the unperturbed system (2) are invariant [32].

Now, let us consider a special class of systems which are periodic with respect to a part of the state [4], [31].

Assumption 1: Let $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$, where $\tilde{z} \in \mathbb{R}^k$ and $\tilde{\theta} \in \mathbb{R}^q$ are two subsets of the state vector, $n = k + q$, $k > 0$ and $q > 0$. The vector field f in (2) is $2\pi-$periodic with respect to $\tilde{\theta}$.

In other words, we suppose that the system (2) can be embedded in a manifold $M = \mathbb{R}^k \times \mathbb{S}^q$, where $\mathbb{S}$ is the unit sphere, by a simple projection of the variables $\tilde{\theta}$ on the sphere $\mathbb{S}^q$. Denote $x = (z, \theta) \in M$ with $z \in \mathbb{R}^k$ and $\theta \in \mathbb{S}^q$.

So, by Assumption 1 for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$

$$f(\tilde{x}, 0) = f(\tilde{x} + \xi_j, 0), \tag{3}$$

where $\xi_j = [0_k, 2\pi j] \in \mathbb{R}^n$, where 0_k is a zero vector of dimension k and $j \in \mathbb{Z}^q$. Denote the projection from $\mathbb{R}^n$ to M by $\mathcal{P} : \mathbb{R}^n \to M$, which is just a modulus of the last q coordinates over 2π. Hence, for any $\tilde{x}_0 \in \mathbb{R}^n$ the solution $\tilde{X}(t, \tilde{x}_0; d) \in \mathbb{R}^n$ of the system (1) can be projected to the solution $X(t, x_0; d) \in M$ with $x_0 = \mathcal{P}(\tilde{x}_0) \in M$, then both solutions are defined on the same time interval and $X(t, x_0; d) = \mathcal{P}(\tilde{X}(t, \tilde{x}_0; d))$ for all such instants of time. Similarly, the set $\tilde{\mathcal{W}} \subset \mathbb{R}^n$, containing all $\alpha-$ and $\omega-$limit sets of the system (2), can be projected to M by using the periodicity of the last q variables, which will be denoted by $\mathcal{W}$, and $|x|_\mathcal{W} = \inf_{y\in\mathcal{W}} |x - y|$ is a distance to the set $\mathcal{W}$ for $x \in M$ (then $|\cdot|$ represents the distance on M). Note that due to the periodicity of (2), even if $\mathcal{W}$ is compact, the set $\tilde{\mathcal{W}}$ becomes unbounded in $\mathbb{R}^n$, in a common case.

C. Decomposable sets

Let $\Lambda \subset M$ be a compact invariant set for the system (2).

Definition 1 ([14]): A decomposition of Λ is a finite and disjoint family of compact invariant sets $\Lambda_1, \ldots, \Lambda_k$ such that

$$\Lambda = \cup_{i=1}^k \Lambda_i.$$

Each Λ_i will be called an *atom* of the decomposition.

For an invariant set Λ, its attracting and repulsing subsets are defined as follows:

$$\mathfrak{A}(\Lambda) = \{x \in M : |X(t,x;0)|_\Lambda \to 0 \,\text{as}\, t \to +\infty\},$$
$$\mathfrak{R}(\Lambda) = \{x \in M : |X(t,x;0)|_\Lambda \to 0 \,\text{as}\, t \to -\infty\}.$$

Define a relation on $\mathcal{B} \subset M$ and $\mathcal{D} \subset M$ by $\mathcal{B} \prec \mathcal{D}$ if $\mathfrak{A}(\mathcal{B}) \cap \mathfrak{R}(\mathcal{D}) \neq \emptyset$.

Definition 2 ([14]): Let $\Lambda_1, \ldots, \Lambda_k$ be a decomposition of Λ, then

1) An r-cycle ($r \geq 2$) is an ordered r-tuple of distinct indices $i_1, \ldots, i_r$ such that $\Lambda_{i_1} \prec \ldots \prec \Lambda_{i_r} \prec \Lambda_{i_1}$.
2) A 1-cycle is an index i such that $\mathfrak{A}(\Lambda_i) \cap \mathfrak{R}(\Lambda_i) \setminus \Lambda_i \neq \emptyset$.
3) A filtration ordering is a numbering of the Λ_i so that $\Lambda_i \prec \Lambda_j \implies i \leq j$.

So, the existence of an r-cycle with $r \geq 2$ is equivalent to the existence of a heteroclinic orbit for the system (2), while the existence of a 1-cycle implies the existence of a homoclinic orbit for the system (2) [32].

Definition 3: The set Λ is called decomposable if it admits a finite decomposition without cycles, $\Lambda = \cup_{i=1}^k \Lambda_i$, for some non-empty disjoint compact sets Λ_i, which form a filtration ordering of Λ, as detailed in Definitions 1 and 2.

Assumption 2: Let a set $\tilde{\mathcal{W}} \subset \mathbb{R}^n$ contain all $\alpha-$ and $\omega-$limit sets of (2) and $\mathcal{W}$, its projection on $M = \mathbb{R}^k \times \mathbb{S}^q$, be compact and decomposable (in the sense of Definition 3).

D. Integral ISS properties of multistable systems

The following notions of robustness for systems modeled by (1), with $\mathcal{W}$ as in Assumption 2, has been introduced by [16].

Definition 4 ([16]): The system (1) is said to have the *uniform bounded-energy bounded-state* (UBEBS) property if, for some class $\mathcal{K}_\infty$ functions χ, ς and η, and some positive constant c, the following estimate holds for all $t \geq 0$, all $x \in M$ and all locally essentially bounded and measurable input signals $d(\cdot)$:

$$\chi(|X(t,x;d)|_\mathcal{W}) \leq \varsigma(|x|_\mathcal{W}) + \int_0^t \eta(|d(s)|)ds + c. \qquad (4)$$

Definition 5 ([16]): The system (1) is said to have the *zero-global attraction* (0-GATT) property with respect to a compact invariant set $\mathcal{W}$, if every trajectory $X(t,x;0)$ of the unperturbed system (2) satisfies

$$\lim_{t\to+\infty} |X(t,x;0)|_\mathcal{W} = 0. \qquad (5)$$

Definition 6: ([16]) A $\mathcal{C}^1$ function $V : M \to \mathbb{R}_+$ is said to be an *iISS-Lyapunov function* for the system (1) if there exist functions $\alpha_1, \alpha_2, \gamma \in \mathcal{K}_\infty$, a continuous positive definite function $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$, and a scalar $g \geq 0$, such that, for all $x \in M$:

$$\alpha_1(|x|_\mathcal{W}) \leq V(x) \leq \alpha_2(|x|_\mathcal{W} + g) \qquad (6)$$

and the following dissipation inequality is satisfied for all $(x,d) \in M \times \mathbb{R}^m$:

$$DV(x)f(x,d) \leq -\alpha_3(|x|_\mathcal{W}) + \gamma(|d|). \qquad (7)$$

In [16] the iISS properties of systems with multiple invariant sets are characterized by the equivalence between several robustness notions and the existences of an iISS-Lyapunov function as in Definition 6. The following theorem presents an extract of the main result of [16].

Theorem 1 ([16]): Consider a nonlinear system as in (1) and let $\mathcal{W}$ be as in Assumption 2. Then the following properties are equivalent:

1) 0-GATT and UBEBS;
2) existence of a smooth iISS-Lyapunov function V such that $DV(x) = 0$ for all $x \in \mathcal{W}$;
3) existence of a $\mathcal{C}^1$ iISS-Lyapunov function V;

Definition 7 ([16]): The system (1) is said to be iISS *in the multistable sense* with respect to the set $\mathcal{W}$ and the input $d(\cdot)$ if and only if it satisfies Assumption 2, and the UBEBS and 0-GATT properties.

Due to the typical lack of global Lyapunov stability in multistable systems, the definition of iISS given above is weaker than the classical one introduced by [8]. Clearly, classical iISS implies iISS in the multistable sense but the contrary is not true in the general case when $\mathcal{W}$ admits a decomposition with multiple atoms. However, [16, Lemma 5] shows that if $\mathcal{W}$ consists of one single atom, then iISS in the multistable case is equivalent to classical iISS.

E. Boundedness of solution of periodic systems

A sufficient criterion to establish boundedness of solutions for periodic systems, which satisfy Assumption 1 for $q = 1$, has been introduced in [4], [29], [31]. This criterion can be seen as a fine handling of instability and periodicity, which leads to boundedness of trajectories. That is, the existence of periodically repeated invariant solutions separating the domain of the periodic variables allows to establish certain cell structure, which bounds the admissible behavior of the trajectories.

In [13], a generalization of this cell structure approach is developed for systems whose dynamics are periodic with respect to multiple state variables, i.e., for the case when the system (2) satisfies Assumption 1 for $q > 1$. To this end, the concept of Leonov functions was introduced.

Following [13], consider the sets: $\mathcal{V} = (\mathbb{R}^k \times \mathcal{S})$, where $\mathcal{S}$ is a qth sphere (a set topologically equivalent to $\mathbb{S}^q$), such that $\pi \leq |\theta|_\infty < 2\pi$ for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathcal{V}$, and $\mathcal{U} = \cup_{r\in\mathbb{Z}_+} \mathcal{U}_r$, where

$$\mathcal{U}_r = \{\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n : \tilde{z} = 0,\, |\tilde{\theta}|_\infty = 2r\pi,\, f(\tilde{x}) = 0\}. \qquad (8)$$

Accordingly, the $\mathcal{U}$ includes all equilibria obtained by shifting the one at the origin using the property that, by assumption, f is $2\pi-$periodic in $\tilde{\theta}$. However, in general, the system (2) may possess other equilibria that do not belong to $\mathcal{U}$.

Definition 8 ([13]): A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a Leonov function for (2) if there exist a constant $g \geq 0$, functions $\alpha \in \mathcal{K}_\infty$, $\psi \in \mathcal{K}$ and a continuous function $\lambda : \mathbb{R} \to \mathbb{R}$ satisfying $\lambda(0) = 0$, and $\lambda(s)s > 0$ for all $s \neq 0$, such that

$$\alpha(|\tilde{z}|) - \psi(|\tilde{\theta}|) - g \leq V(\tilde{x}) \quad \forall \tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n, \quad \inf_{\tilde{x}\in\mathcal{V}} V(\tilde{x}) > 0, \quad \sup_{\tilde{x}\in\mathcal{U}} V(\tilde{x}) \leq 0, \tag{9}$$

and the following dissipation inequality holds:

$$\frac{\partial V(\tilde{x})}{\partial \tilde{x}} f(\tilde{x}, 0) \leq -\lambda(V(\tilde{x})) \quad \forall \tilde{x} \in \mathbb{R}^n. \tag{10}$$

Loosely speaking, a Leonov function V is sign-indefinite with respect to the periodic variables $\tilde{\theta}$ and radially unbounded with respect to the non-periodic ones z. Furthermore, a Leonov function V is negative definite with respect to the distance to the boundary of the cell containing an equilibrium of (2). In other words, it takes positive values on the set $\mathcal{V}$ and negative values in a vicinity of the set $\mathcal{U}$.

Theorem 2 ([13]): Let Assumption 1 be satisfy. If there exists a Leonov function for the system (2), then for all $\tilde{x}_0 \in \mathbb{R}^n$ the trajectories $\tilde{X}(t, \tilde{x}_0; 0)$ are bounded for all $t \geq 0$.

By exploiting the periodicity of the system (2), the proof of Theorem 2 in [13] consists of two steps: first, it is shown that the sets $\Omega_j = \{\tilde{x} \in \mathbb{R}^n : V_j(\tilde{x}) \leq 0\}$, where $j = [j_1, \ldots, j_q] \in \mathbb{Z}^q$ and $V_j(\tilde{x}) = V(\tilde{x} - \left[\begin{smallmatrix} 0_k \\ 2\pi j \end{smallmatrix}\right])$, are globally attractive and forward invariant for (2). Note that $\mathcal{U} \in \Omega_j$ for any j. Then, it is proven that the intersection of the sets Ω_j is composed by compact and isolated "cells". Take any $\tilde{x}_0 \in \mathbb{R}^n$, then the solution $\tilde{X}(t, \tilde{x}_0; 0)$ asymptotically enters and remains in such a cell. Therefore, for any $\tilde{x}_0 \in \mathbb{R}^n$, the corresponding solution $\tilde{X}(t, \tilde{x}_0; 0)$ is bounded for all $t \geq 0$.

III. Main results

Now, inspired by [13], [28] and [16], we introduce a new characterization of the iISS property with respect to the set $\tilde{\mathcal{W}}$ for periodic systems evolving in $\mathbb{R}^n$.

Recall that by assumption, the solutions $\tilde{X}(t, \tilde{x}_0; 0)$ of (2) are defined in $\mathbb{R}^n$ for all $\tilde{x}_0 \in \mathbb{R}^n$ at least locally in time for $t \in [0, T)$ for some $T > 0$.

Definition 9: A $\mathcal{C}^1$ function $V_1 : \mathbb{R}^n \to \mathbb{R}$ is said to be an iISS-Leonov function for the system (1) if there exist functions $\vartheta_1, \vartheta_2, \sigma_1, \gamma_1 \in \mathcal{K}_\infty$, a continuous non-negative function $\lambda_1 : \mathbb{R}_+ \to \mathbb{R}_+$, and scalars $g_1, \ell_1 \geq 0$, such that for all $\tilde{x} = (\tilde{z}, \tilde{\theta}) \in \mathbb{R}^n$:

$$\vartheta_1(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \sigma_1(|\tilde{\theta}|) - g_1 \leq V_1(\tilde{x}) \leq \vartheta_2(|\tilde{x}|_{\tilde{\mathcal{W}}} + \ell_1), \tag{11}$$

and the following dissipation inequality

$$DV_1(\tilde{x}) f(\tilde{x}, d) \leq -\lambda_1(V(\tilde{x})) + \gamma_1(|d|). \tag{12}$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V_1(\tilde{x}) \geq 0\}$ and $d \in \mathbb{R}^m$.

Definition 10: The system (1) is said to be *output smoothly dissipative* (OSD) if there exists a $\mathcal{C}^1$ function $V_2 : \mathbb{R}^n \to \mathbb{R}$, functions $\vartheta_3, \vartheta_4, \sigma_2, \gamma_2 \in \mathcal{K}_\infty$, a continuous non-negative function $\lambda_2 : \mathbb{R}_+ \to \mathbb{R}_+$, constants $g_2, \ell_2 \geq 0$, and a continuous output map $h : \mathbb{R}^n \to \mathbb{R}^p$, such that for all $(\tilde{x}, d) \in \mathbb{R}^n \times \mathbb{R}^m$:

$$\vartheta_3(|\tilde{x}|_{\tilde{\mathcal{W}}}) - \sigma_2(|\tilde{\theta}|) - g_2 \leq V_2(\tilde{x}) \leq \vartheta_4(|\tilde{x}|_{\tilde{\mathcal{W}}} + \ell_2), \tag{13}$$

and the following dissipation inequality

$$DV_2(\tilde{x}) f(\tilde{x}, d) \leq -\lambda_2(|h(\tilde{x})|) + \gamma_2(|d|). \tag{14}$$

holds for all $\tilde{x} \in \{\tilde{x} \in \mathbb{R}^n : V_2(\tilde{x}) \geq 0\}$ and $d \in \mathbb{R}^m$.

The following lemma is the key-point of this work because it establishes the relation of iISS-Leonov functions with the 0-GATT and UBEBS properties.

Lemma 1: Let Assumptions 1 and 2 be satisfied. Then the existence of an iISS-Leonov function for (1) implies 0-GATT and UBEBS properties with respect to $\tilde{\mathcal{W}}$.

Based on Lemma 1, the main result of this paper shows that the existence of an iISS-Leonov function or the OSD property implies the iISS property for systems, whose dynamics is periodic with respect to a part of the state.

Theorem 3: Let Assumptions 1 and 2 be satisfied. Then, for the system (1) the following properties are equivalent:

(a) iISS property with respect to the set $\tilde{\mathcal{W}}$;
(b) existence of an iISS-Leonov function;
(c) existence of an output function that makes the system OSD.

Note that an iISS-Lyapunov function $V : M \to \mathbb{R}_+$ is only useful to investigate the iISS property of the system (1) in the manifold $M = \mathbb{R}^k \times \mathbb{S}^q$. In order to perform analysis in $\mathbb{R}^n$, we need to extend such an iISS-Lyapunov function V to $\mathbb{R}^n$, where it will be continuously differentiable, and positive definite with respect to the set $\tilde{\mathcal{W}}$ (whose elements form a periodic arrangement in $\mathbb{R}^n$). However, all this properties are only preserved if such a function V is periodic with respect to $\tilde{\theta}$, which is quite restrictive.

Clearly, an iISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}_+$ satisfies the restrictions (13) and (14), hence it can be considered as an iISS-Leonov function for (1). On the other hand, iISS-Leonov functions are not required to be positive definite nor periodic, which ease the study of robustness in periodic systems. Thus, we can say that the existence of an iISS-Leonov function is a relaxed characterization of the iISS property for periodic systems.

The proofs of Lemma 1 and Theorem 3 are omitted due to space limitation. The usefulness of our proposal is illustrated in the next section with an academical example.

IV. An Illustrative Example

Consider the following mathematical model:

$$\begin{aligned} \dot{x}_1 &= x_2 + d_1, \\ \dot{x}_2 &= -\omega^2 \sin(x_1) - kx_2 + c + \tfrac{x_2}{1+|x_1|^2} d_2, \end{aligned} \tag{15}$$

where $x = [x_1, x_2]^\top \in \mathbb{R}^2$ is the state, $\omega, k > 0$ are parameters, $c \in \mathbb{R}$ is a constant bias, and $d_1, d_2 \in \mathcal{L}_\infty$ are perturbations.

For $|c| < \omega^2$ and $d_1 = d_2 = 0$, the system (15) has equilibria at $x_0 = [x_{10}, x_{20}]^\top \in \{[\operatorname{asin}(c\omega^{-2}) + 2j\pi, 0]^\top \cup [(1+2j)\pi - \operatorname{asin}(c\omega^{-2}), 0]^\top\} = \tilde{\mathcal{W}}$, where $j \in \mathbb{Z}$ (note that this set $\tilde{\mathcal{W}}$ contain all α- and ω-limit sets of (15)). By linearizing the system (15) around the point $[\operatorname{asin}(c\omega^{-2}), 0]$, we can check that it is a locally asymptotically stable equilibrium point, and the same is true for every point $[\operatorname{asin}(c\omega^{-2}) + 2j\pi, 0]$ for all $j \in \mathbb{Z}$. Therefore, we can say that the set $\tilde{\mathcal{W}}$ is decomposable. Additionally, consider the coordinates $\zeta = [\zeta_1, \zeta_2]^\top \in \mathbb{R}^2$, where $\zeta_1 = x_1 - x_{10}$ and

$\zeta_2 = x_2$, that is, a translation of the point $[x_{10}, 0]$ to the origin.

Note that for $d_1 = d_2 = 0$, the system (15) describes a simple pendulum, where the parameter ω is a function of the gravity and the pendulum's length, k is a function of the viscous fiction and pendulum's mass, and c can be seen as a constant input. So, the system (15) is a good academical example to illustrate the theoretical results given in previous sections.

A. iISS-Lyapunov function approach

First, let us show that the iISS-Lyapunov function approach fails to prove the iISS property for the system (15). For this purpose, consider the following function

$$W_1(\zeta) = \ln(1 + E(\zeta)), \tag{16}$$

where $V_1(\zeta) = \frac{1}{2}\zeta_2^2 + \omega^2(\cos(x_{10}) - \cos(\zeta_1 + x_{10}))$. Clearly, the function (16) is positive definite with respect to the set $\tilde{\mathcal{W}}$, and radially unbounded with respect to ζ_2. Furthermore, it is periodic in ζ_1 keeping its continuous differentiability in the whole $\mathbb{R}^2$. So, the function (16) qualifies as an iISS-Lyapunov function candidate according to Definition 9.

Now, it remains to check the derivative of (16) along the trajectories of the system (15). For this, consider the simplest case $d_1 = d_2 = 0$, such that $\dot{W}_1(\zeta) = \frac{\dot{V}_1(\zeta)}{1+V_1(\zeta)}$, where

$$\dot{V}_1(\zeta) = -k\zeta_2^2 + c\zeta_2. \tag{17}$$

Note that for $c = 0$, we can recall existing results to conclude the iISS property for the system (15). However, any $c \neq 0$ makes impossible to continue with the analysis. Note also that polynomial-like functions of ζ_2, periodic functions of ζ_1, or a combination are useless to deal with the undesired term in (17). So, there is not an evident way to add extra terms to the function V_1 to deal with the undesired one in (17), and at the same time, preserve the positive definiteness, radially unboundedness, and periodicity of the function (16).

B. iISS-Leonov function approach

Recall that positive definiteness and periodicity are not essential properties of iISS-Leonov function, hence a suitable candidate is given by

$$V_2(\zeta) = \tfrac{1}{2}\zeta_2^2 - \tfrac{1}{2}a\zeta_1^2 + \omega^2(\cos(x_{10}) - \cos(\zeta_1 + x_{10})) - c\zeta_1,$$

where a is a constant parameter to be determined. Let us define

$$\varrho(\zeta_1) = \omega^2(\cos(x_{10}) - \cos(\zeta_1 + x_{10})) - c\zeta_1 - \tfrac{1}{2}\upsilon\omega^2\zeta_1^2,$$

such that,

$$\varrho'(\zeta_1) = \omega^2 \sin(\zeta_1 + x_{10}) - c - \upsilon\omega^2\zeta_1,$$
$$\varrho''(\zeta_1) = \omega^2 \cos(\zeta_1 + x_{10}) - \upsilon\omega^2.$$

Then, for any $\upsilon > 1$ the equation $\varrho'(\zeta_1) = 0$ has a unique solution at $\zeta_1 = 0$ and the inequality $\varrho''(0) < 0$ holds. Moreover, the equation $\varrho''(x_1) = 0$ has no solution for $\upsilon > 1$. Hence $\varrho'(\zeta_1)$ is strictly decreasing in such a case. Thus, for $\upsilon > 1$ we have that $\varrho(\zeta_1)$ is negative definite, which implies that $\omega^2(\cos(x_{10}) - \cos(\zeta_1 + x_{10})) - c\zeta_1 \leq \frac{1}{2}\upsilon\omega^2\zeta_1^2$ under such a condition. By using similar arguments, we can deduce that $\omega^2(\cos(x_{10}) - \cos(\zeta_1 + x_{10})) - c\zeta_1 \geq -\frac{1}{2}\upsilon\omega^2\zeta_1^2$ for $\upsilon > 1$ as well. Therefore, for all $\zeta \in \mathbb{R}^2$:

$$\tfrac{1}{2}\zeta_2^2 - \tfrac{1}{2}(a + \upsilon\omega^2)\zeta_1^2 \leq V_2(\zeta) \leq \tfrac{1}{2}\zeta_2^2 + \tfrac{1}{2}(\upsilon\omega^2 - a)\zeta_1^2, \tag{18}$$

where $\upsilon > 1$. Now, the derivative of the function V_2 along the trajectories of the system (15), under the change of coordinates $x \mapsto \zeta$, is given by

$$\dot{V}_2(\zeta) = -k\zeta_2^2 - a\zeta_1\zeta_2 + (\omega^2 \sin(\zeta_1 + x_{10}) - c - a\zeta_1)d_1 + \tfrac{\zeta_2^2}{1+|\zeta_1+x_{10}|^2}d_2.$$

Then,

$$\dot{V}_2(\zeta) + \lambda V_2(\zeta) \leq -\frac{1}{2}\Phi^\top \Xi\, \Phi + \varpi_1 d_1^2 + |\omega^2 - c||d_1| + \tfrac{\zeta_2^2}{1+|\zeta_1+x_{10}|^2}d_2,$$

where $\lambda, \varpi_1 > 0$ are parameters, $\Phi = [\zeta_1, \zeta_2, d_1]^\top$, and

$$\Xi = \begin{bmatrix} \lambda(a - \upsilon\omega^2) & a & a \\ a & 2k - \lambda & 0 \\ a & 0 & 2\varpi_1 \end{bmatrix}.$$

By applying Sylvester's criterion we obtain that $\Xi \geq 0$ hold true for any $\lambda > 0$, $a > \upsilon\omega^2 > 0$, $\varpi_1 > \frac{a^2}{2\lambda(a-\upsilon\omega^2)}$, and

$$k > \tfrac{1}{2}\lambda + \frac{\varpi_1 a^2}{2\varpi_1\lambda(a - \upsilon\omega^2) - a^2}. \tag{19}$$

Now, consider the set $\Gamma = \{\zeta \in \mathbb{R}^2 : V_2(\zeta) > 0\}$ and the auxiliary function

$$W_2(\zeta) = \begin{cases} \ln(1 + V_2(\zeta)) & \text{for} \quad \zeta \in \Gamma, \\ V_2(\zeta) & \text{for} \quad \zeta \in \mathbb{R}^2 \setminus \Gamma, \end{cases}$$

which is continuous since $W_2(\zeta) = V_2(\zeta) = \ln(1+V_2(\zeta)) = 0$ for all $\zeta \in \partial\Gamma$, where $\partial\Gamma$ denotes the boundary of Γ. Note that the term $\ln(1 + V_2(\zeta))$ is well-defined for all $\zeta \in \Gamma$, besides, by properties of the natural logarithm, we have $0 \leq \ln(1 + V_2(\zeta)) \leq V_2(\zeta)$ for all $\zeta \in \Gamma$. Then, from (18) it is clear that for all $\zeta \in \mathbb{R}^2$:

$$\ln(1+\tfrac{1}{2}\zeta_2^2) - \tfrac{1}{2}(a+\upsilon\omega^2)\zeta_1^2 \leq W_2(\zeta) \leq \tfrac{1}{2}\zeta_2^2 - \tfrac{1}{2}(a-\upsilon\omega^2)\zeta_1^2,$$

where $\upsilon > 1$. Moreover, due to periodicity of the system (15) with respect to ζ_1 we have that $\frac{1}{2}\zeta_2^2 \leq \vartheta(|\zeta|_{\tilde{\mathcal{W}}}) \leq \frac{1}{2}\zeta_2^2 + \tilde{g}$ for some function ϑ of class $\mathcal{K}_\infty$ and some $\tilde{g} \geq 0$ sufficiently large. So, we can readily see that W satisfies inequality (11). Hence, we can consider it as an iISS-Leonov function candidate.

Now, let us check the derivative of W_2 along the trajectories of the system (15) for all $\zeta \in \Gamma$, which is given by

$$\dot{W}_2(\zeta) + \tfrac{\lambda V_2(\zeta)}{1+V_2(\zeta)} = \tfrac{\dot{V}_2(\zeta)+\lambda V_2(\zeta)}{1+V_2(\zeta)} \leq \tfrac{1}{1+V_2(\zeta)}\left(\varpi_1 d_1^2 + \varpi_2|d_1| + \tfrac{\zeta_2^2}{1+|\zeta_1+x_{10}|^2}|d_2|\right),$$

where $\varpi_2 \geq \omega^2 + |c|$. Note that for all $\zeta \in \Gamma$ the term $\frac{\lambda V_2(\zeta)}{1+V_2(\zeta)}$ is positive definite, and the terms $\frac{1}{1+V_2(\zeta)}$ and $\frac{\zeta_2^2}{(1+V_2(\zeta))(1+|\zeta_1+x_{10}|^2)}$ are bounded. Hence, we obtain

$$\dot{W}_2(\zeta) \leq \delta_1(|d_1|) + \delta_2(|d_2|)$$

for all $\zeta \in \Gamma$, where δ_1 and δ_2 are class $\mathcal{K}_\infty$ functions majorizing those terms of d_1 and d_2, respectively.

Thus, W_2 satisfies all conditions of Definition 9. Hence, we can say that it is an iISS-Leonov function for the system (15). Moreover, we have shown above that the set $\tilde{\mathcal{W}}$ is decomposable in the sense of Definition 3, then Assumptions 1 and 2 are satisfied by the system (15). Therefore, by Theorem 3 we can say that the system (15) has the iISS property with respect to the set $\tilde{\mathcal{W}}$.

V. Conclusions

A novel characterization of the iISS property has been developed for multistable systems whose dynamics is periodic with respect to a part of the state. The introduced theory consists in an analysis of boundedness of the system's trajectories in presence of bounded-energy perturbations by means of sign-indefinite functions with a sign-definite derivative called iISS-Leonov functions. The proposed approach permits to relax the usual requirements of positive definiteness and periodicity of the iISS-Lyapunov function approach. The theoretical result is illustrated by analyzing the iISS properties of a nonlinear pendulum with a constant bias input and unbounded state-dependent input coefficient.

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