An Analysis of Convergence Properties of Finite-Time Homogeneous Controllers Through Its Implementation in a Flexible-Joint Robot

Jesús Mendoza-Avila, Denis Efimov, Leonid Fridman, Jaime Alberto Moreno Pérez

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***Abstract*— A study of the stability of interconnected homogeneous systems, affected by singular perturbations, is presented by means of the implementation of finite-time composite control in a single-link flexible-joint robot. Previous results suggest that the implementation of finite-time convergent controllers leads to the arising of chattering. Now, throughout a practical example we show that the design of a controller making the whole system homogeneous avoids the undesired chattering and recovers the ideal finite-time convergence properties. Regrettably, information of the states of the fast dynamics is not commonly available, then the proposed strategy is not applicable at all. Nevertheless, the main interest of our study lays in the expansion of the panorama for a better understanding of the causes of chattering, which contributes to the development of chattering reduction techniques, and has attracted a lot of attention, nowadays.**

I. Introduction

Homogeneous systems constitute a subclass of nonlinear dynamics admitting special and very useful properties like scalability of solutions, global expansion of local behaviors, robustness, and different rates of convergence: rational, exponential and finite-time [1]–[3]. Particularly, homogeneous controllers of negative degree possesses an infinite gain near to the origin, so that, they are able to provide finite-time convergence [4]–[7]. However, it is a well-known fact that the interconnection of a finite-time convergent dynamics and a linear Parasitic Dynamics (PD) produces the so-called chattering phenomenon: a high-frequency oscillatory behavior of a system in steady-state. Consequently, the problems of chattering reduction and its analysis have attracted a lot of attention (see for example [8]–[15]).

Recently, [16] has presented an analysis of the stability properties of singularly perturbed systems in the framework of homogeneity notions. Three types of stability were discovered by depending on the relation between the homogeneity degrees (HD) of the fast dynamics (FD) and slow dynamics (SD): global asymptotic stability (GAS) when both dynamics have the same HD and the singular perturbation parameter (SPP) is sufficiently small, practical GAS when the SD has a smaller HD, and local asymptotic stability when the SD has a greater HD. In the last two cases, both the final bound of the trajectories and the domain of attraction depends on the SPP. Loosely speaking, the first case coincides with the concept of motion separation predicted by classical results on smooth (at least Lipschitz continuous) singularly perturbed systems (see [17] for details about it), hence GAS at the origin can be expected. Nevertheless, the hypothesis of such an analysis is quite restrictive and it limits their application to a wider class of system.

A flexible-joint robot is a n-link manipulator with revolute joints actuated by DC motors, where the links are assume perfectly rigid but the actuator are elastically coupled to the link [18]. Commonly, the joint is model by a linear torsional spring with certain stiffness. Moreover, flexible-joint robots are represented by singularly perturbed system, where the motions of the link are the slow dynamics and those in the elastic joint constitute the fast ones. In [19], the idea of composite control for flexible-joint robots was developed by designing a control law as the sum of two terms: a linear feedback for the slow variables and a correction term of viscous friction for the fast ones.

The present paper is devoted to the study of stability of interconnected homogeneous systems, affected by singular perturbations, through the implementation of finite-time composite control in a single -link flexible-joint robot. To the best of our knowledge, there is not available literature regarding stability analysis of the consider class of systems and controllers. Thus, inspired by the results of [16], we show that the design of a controller making the whole system homogeneous avoids the undesired chattering and recovers the ideal finite-time convergence properties for the case under study. Regrettably, in most cases, information on the states of the fast dynamics is not available, then the presented strategy is not applicable at all. But it is still interesting because it allows the expansion of the panorama for a better understanding of the causes of chattering.

The outline of the paper is as follows. Section II present some useful definitions and concepts for the development of our study. Section III introduces a family of homogeneous finite-time convergent controllers. In Section IV, the model of a single-link flexible-joint robot is provided. The stability a analysis of the homogeneous finite-time convergent controllers implemented in a single-link flexible-joint robot is developed in Section V. Finally, the conclusions are given in Section VI.

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A. Notation

- $\mathbb{N}$ is the set of natural numbers, $\mathbb{Q}$ is the set of rational ones, and $\mathbb{R}$ is the set the real ones. Moreover, $\mathbb{R}_+$ represents the set of non-negative real numbers, i.e., $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$.
- $\mathcal{C}^0$ and $\mathcal{C}^1$ represent the families of continuous and continuously differentiable functions, respectively.
- $|\cdot|$ denotes the absolute value in $\mathbb{R}$, $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^n$.
- Expressions like $|\cdot|^{\gamma}\text{sign}(\cdot), \gamma \in \mathbb{R}$ are written as $\lceil\cdot\rfloor^{\gamma}$.
- A function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if it is continuous, strictly increasing and $\alpha(0) = 0$. The function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is unbounded. A continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if, for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$ and, for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and it tends to zero for $t \to \infty$.
- The space $\mathcal{L}_\infty^m$ is the set of piecewise continuous, bounded functions $u : [0, \infty) \to \mathbb{R}^m$ such that

$$\|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} |u(t)| < \infty.$$

II. Preliminaries

Consider the system

$$\dot{x} \;=\; f(x, u), \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector and $u \in \mathbb{R}^m$ is an input. In addition, $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ ensures forward existence and uniqueness of solutions at least locally in time, and it also satisfies $f(0, 0) = 0$. For any initial condition $x_0 \in \mathbb{R}^n$ we denote such a solution by $\chi(t, x_0)$ for $t \geq 0$ where it is defined.

A. Weighted homogeneity

Following [1], [2], we have that for real numbers $r_i > 0$ $(i = 1, \ldots, n)$ called weights and $\lambda > 0$, one can define

- the vector of weights $r = (r_1, \ldots, r_n)^T$, $r_{\max} = \max_{1 \leq j \leq n} r_j$ and $r_{\min} = \min_{1 \leq j \leq n} r_j$;
- the dilation matrix function $\Lambda_r(\lambda) = \text{diag}(\lambda^{r_i})_{i=1}^n$, such that, for all $x \in \mathbb{R}^n$ and for all $\lambda > 0$, $\Lambda_r(\lambda)x = (\lambda^{r_1}x_1, \ldots, \lambda^{r_i}x_i, \ldots, \lambda^{r_n}x_n)^T$ (along the paper the dilation matrix is represented by Λ_r wherever λ can be omitted);
- the r-homogeneous norm of $x \in \mathbb{R}^n$ is given by $||x||_r = \left(\sum_{i=1}^n |x_i|^{\frac{\rho}{r_i}}\right)^{\frac{1}{\rho}}$ for $\rho \geq r_{\max}$ (it is not a norm in the usual sense, since it does not satisfy the triangle inequality);
- for $s > 0$, the sphere and the closed ball in the homogeneous norm are defined as $S_r(s) = \{x \in \mathbb{R}^n : ||x||_r = s\}$ and $B_r(s) = \{x \in \mathbb{R}^n : ||x||_r \leq s\}$, respectively.

Definition 1 ([1], [2]). A function $g : \mathbb{R}^n \to \mathbb{R}$ is r-homogeneous with a degree $\mu \in \mathbb{R}$ if for all $\lambda > 0$ and all $x \in \mathbb{R}^n$:

$$\lambda^{-\mu} g(\Lambda_r(\lambda)x) = g(x).$$

A vector field $f : \mathbb{R}^n \to \mathbb{R}^n$ is r-homogeneous with a degree $\nu \geq -r_{\min}$ if for all $x \in \mathbb{R}^n$ and all $\lambda > 0$,

$$f(\Lambda_r(\lambda)x) = \lambda^{\nu}\Lambda_r(\lambda)f(x).$$

The system (1) (with $u = 0$) is r-homogeneous of degree ν if the vector filed f is r-homogeneous of degree ν.

B. Stability

Now, following [2] we say that the system (1) is:

- *Lyapunov stable at the origin* if the solution $\chi(t, x_0)$ is defined for all $t \geq 0$ and for each $\epsilon > 0$ there exists $\delta > 0$ such that for any $\|x_0\| < \delta$, the solution satisfies $\|\chi(t, x_0)\| < \epsilon$ for all $t \geq 0$.
- *(Globally) Asymptotically stable at the origin* if it is Lyapunov stable at the origin and, in addition, there exists $\delta_0 > 0$ such that $\lim_{t \to +\infty} \|\chi(t, x_0)\| = 0$ for each $\|x_0\| < \delta_0$ (δ_0 arbitrarily large).
- *(Globally) Finite-time stable at the origin* if it is (globally) asymptotically stable and there exists $\delta_0 > 0$ such that $\|\chi(t, x_0)\| = 0$ for all $t \geq T_s(x_0)$ and all $\|x_0\| < \delta_0$ (with δ_0 arbitrary large), where $T_s : \mathbb{R}^n \to \mathbb{R}_+$ is a settling time function.

One important feature of r-homogeneous systems is that their rate of convergence is related to their HD.

Theorem 1 ([20]). *Let the system* (1) *be r-homogeneous of degree ν, then its origin is a globally finite-time stable equilibrium point if and only if it is GAS and the homogeneity degree ν is negative.*

C. Stability of singularly perturbed homogeneous systems

Now, let us recall the results of [16] about the stability of a particular class of singularly perturbed homogeneous systems with different HD. For this, consider the interconnected system

$$\dot{x} \;=\; f(x, y), \tag{2}$$

$$\epsilon\dot{y} \;=\; g(x, y) = A(x)y + R(x), \tag{3}$$

where $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$ are the state variables, and $\epsilon > 0$ is a small parameter, f is of class $\mathcal{C}^0$, A is of class $\mathcal{C}^1$ with $\det(A(x)) \neq 0$ for all $x \in \mathbb{R}^n$ and $R \in C^1$ with $R(0) = 0$ and $R(x) \neq 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Moreover, the system (2) is $(r, \tilde{r})$-homogeneous with a degree ν, while the system (3) is $(r, \tilde{r})$-homogeneous with a degree μ for some vectors of weights r and $\tilde{r}$. Under the introduced restrictions on A and R, the equation $g(x, h(x)) = 0$ admits a solution $h(x) = -A^{-1}(x)R(x)$, which is locally Lipschitz continuous.

Assumption 1. *For the system* (2)-(3)*:*

- *The reduced-order dynamics $\dot{x} = f(x, h(x))$, is GAS at the origin.*
- *The system $\dot{y} = A(x)y$ is GAS at the origin, uniformly w.r.t. x, and there exists $P = P^\top > 0$ and $Q = Q^\top > 0$ such that*

$$A^\top(x)P + PA(x) \leq -Q$$

for all $x \in \mathbb{R}^n$.

Under the aforementioned hypothesis, the main result of [16] establishes a trade-off between the kind of stability of the interconnected system (2)-(3) and the relation of the homogeneity degrees of each subsystems.

Theorem 2 ([16]). *Let the subsystems* (2) *and* (3) *be* r*-homogeneous with a degree* ν *and* $\tilde{r}$*-homogeneous with a degree* μ*, respectively. Moreover, consider functions* $f \in C^0$ *and* $A, R \in C^1$*. If Assumption 1 is satisfied then there is* $\epsilon_0 > 0$ *such that for all* $\epsilon \in (0, \epsilon_0]$ *the interconnected system* (2)-(3) *is*

- *globally asymptotically stable for* $\mu = \nu$*,*
- *locally asymptotically stable, for* $\mu < \nu$*,*
- *globally asymptotically practically stable, for* $\mu > \nu$*.*

Roughly speaking, the first case validates the concept of motion separation for homogeneous singularly perturbed systems (which are not necessarily Lipschitz continuous), such that GAS can be expected for the system (2)-(3). On the other hand, such a concept of motion separation is only valid outside of a neighborhood of the origin for the second case, and near to the origin for the third one, hence, just local or practical stability can be concluded, respectively.

D. Generalized forms approach for Lyapunov functions design

The concept of generalized forms was introduced in [21] as r-homogeneous generalized polynomials of degree $m \in \mathbb{R}$, consisting of a linear combination of a finite number of homogeneous monomials, which are sums, products and sums of products of terms like: $a\lceil x \rfloor^p$ or $b|x|^q$, where $a, b \in \mathbb{R}$, and $p, q > 0$. This class of functions has some important properties like closure with respect to the addition, product and partial derivatives operators.

Further, for $r \in \mathbb{Q}^n$ any generalized form can be represented in the state space by a set of associated forms (classical homogeneous polynomials). For this, a suitable change of coordinates $x = d(z)$ has to be applied in each hyper-octant of the state space, where every element of the variable $z \in \mathbb{R}^n$ is nonnegative. One option is

$$x_i = \begin{cases} z_i^{2r_i} & \text{for} \quad x_i \geq 0 \\ -z_i^{2r_i} & \text{for} \quad x_i < 0 \end{cases} \qquad i = 1, ..., n. \qquad (4)$$

where $z_i > 0$ and r_i is the homogeneity weight of the corresponding variable x_i. With this transformation we obtain associated forms whose domains are restricted to the positive hyper-octant, i.e., $z_i > 0$, $i = 1, ..., n$, and all their exponents are even. So, the problem of analysis of the positive definiteness of a generalized form can be reduced to determine the positive definiteness of its associated forms, for which there are classical tools like Polya's theorem [22] or sum of squares methods [23].

In [21], [24], [25], a methodology for construction of Lyapunov functions based on generalized forms is presented for a system $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, whose vector field f is described by generalized forms and is r-homogeneous of degree k. This procedure is summarized in the following steps:

1) Select a generalized form $V : \mathbb{R}^n \to \mathbb{R}_+$, with a homogeneity degree m and a vector of weights r:
$$V(x) = \sum_{i=1}^{n} \alpha_i |x_i|^{\frac{m}{r_i}} + P(\alpha_j, x); \quad j > n \qquad (5)$$
where α_i are coefficients and $P(\alpha_j, x)$ contains cross terms between the variables x_i for $i = 1, ..., n$.
2) Compute the function $W(x) = -\frac{\partial V}{\partial x} f(x)$, i.e., the negative of the derivative of V along the trajectories of the system under study.
3) Apply an adequate change of coordinates like (4) to get the sets of associated forms $\{V_i(z)\}$ and $\{W_i(z)\}$ of $V(x)$ and $W(x)$, respectively.
4) Find the parameters to guarantee that every form in the sets $\{V_i(z)\}$ and $\{W_i(z)\}$ is positive definite. This can be efficiently done by using software like SOSTOOLS with SEDUMI solver, where the parameters are calculated by means of the solution of some matrix inequalities.

III. A CLASS OF NONLINEAR HOMOGENEOUS FINITE-TIME CONVERGENT CONTROLLERS

Consider the following chain of integrators of order n:

$$\begin{array}{rcl} \dot{x}_1 & = & x_2 \\ \vdots & \vdots & \vdots \\ \dot{x}_n & = & u, \end{array} \qquad (6)$$

where $x_1, \ldots, x_n \in \mathbb{R}$ are the state variables and $u \in \mathbb{R}$ is a control input. Moreover, let a nonlinear homogeneous finite-time convergent controller be given by

$$u = -k_1\lceil x_1 \rfloor^{\frac{1}{n+1}} - k_2\lceil x_2 \rfloor^{\frac{1}{n}} - \ldots - k_n\lceil x_n \rfloor^{\frac{1}{2}}, \qquad (7)$$

where $k_1, \ldots, k_n > 0$ are parameters to be designed. So, the closed-loop system (6)-(7) is r-homogeneous of degree $\nu = -\frac{1}{2}$ with a vector of weights $r = [(n+1)/2, n/2, (n-1)/2, \ldots, 1]$. Thus, if the closed-loop system (6)-(7) is GAS at the origin, then by Theorem 1, its trajectories exhibit finite-time convergence.

An idea to select Lyapunov functions candidates for homogeneous systems like (6)-(7) is presented in [26]. This procedure exploits the structure of the system to construct suitable Lyapunov function candidates in the family of generalized forms. So, starting with a second-order system, Lyapuvno function candidates for higher-order system are obtained in a recursive way (see [26] for more details). Then, a r-homogeneous of degree μ Lyapunov function candidate for the closed-loop system (6)-(7) has the form

$$V_n(x) = V_{n-1}(x) + P_n(\alpha_j, x) + \alpha_1 |x_1|^{\frac{\mu}{n+1}}, \qquad (8)$$

where $V_{n-1}(x)$ is a Lyapunov function for a system with the same structure of (6)-(7) but of order $n-1$, and $P_n(\alpha_j, x)$ are some additional cross-terms between the variables x_i for $i = 1, ..., n$. Clearly, both $V_n(x)$ and $V_{n-1}(x)$ must be of the same homogeneity degree.

For instance, following the idea of [26] and applying the generalized forms approach of [21], [24], [25], we can find

Lyapunov functions and parameters $k_1, \ldots, k_n > 0$ for the closed-loop system (6)-(7) up to order $n = 4$, as it is shown in the following. Obviously, we omit the case $n = 1$ because it is trivial.

- For $n = 2$, we have $k_1 = 6$ and $k_2 = 8.5$, with

$$V_2(x) = \alpha_1 |x_1|^3 + \alpha_2 |x_2|^{\frac{9}{2}} + \alpha_3 \lceil x_1 \rfloor^{\frac{7}{3}} x_2,$$

  where $\alpha_1 = 33.61$, $\alpha_2 = 4.084$, $\alpha_3 = 7.3544$.
- For $n = 3$, we have $k_1 = 2.7$, $k_2 = 6$ and $k_3 = 8.5$, with

$$\begin{aligned} V_3(x) &= \alpha_1 |x_1|^{\frac{9}{4}} + \alpha_2 |x_2|^3 + \alpha_3 |x_3|^{\frac{9}{2}} \\ &+ \alpha_4 \lceil x_1 \rfloor^{\frac{3}{2}} x_2 + \alpha_5 \lceil x_1 \rfloor^{\frac{7}{4}} x_3 + \alpha_6 \lceil x_2 \rfloor^{\frac{7}{3}} x_3, \end{aligned}$$

  where $\alpha_1 = 130.1$, $\alpha_2 = 539.1$, $\alpha_3 = 36.07$, $\alpha_4 = 199.4$, $\alpha_5 = 13.63$, $\alpha_6 = 149.6$.
- For $n = 4$, we have $k_1 = 1$, $k_2 = 2.7$, $k_3 = 6$ and $k_4 = 8.5$, with

$$\begin{aligned} V_4(x) =& \alpha_1 |x_1|^{\frac{9}{5}} + \alpha_2 |x_2|^{\frac{9}{4}} + \alpha_3 |x_3|^3 + \alpha_4 |x_4|^{\frac{9}{2}} \\ &+ \alpha_5 x_1 x_2 + \alpha_6 \lceil x_1 \rfloor^{\frac{6}{5}} x_3 + \alpha_7 \lceil x_1 \rfloor^{\frac{7}{5}} x_4 \\ &+ \alpha_8 \lceil x_2 \rfloor^{\frac{3}{2}} x_3 + \alpha_9 \lceil x_2 \rfloor^{\frac{7}{4}} x_4 + \alpha_{10} \lceil x_3 \rfloor^{\frac{7}{3}} x_4 \\ &+ \alpha_{11} \lceil x_3 \rfloor^2 \lceil x_4 \rfloor^{\frac{3}{2}}. \end{aligned}$$

  where $\alpha_1 = 234.7$, $\alpha_2 = 51255$, $\alpha_3 = 1137000$, $\alpha_4 = 49488$, $\alpha_5 = 713.5$, $\alpha_6 = 232.6$, $\alpha_7 = 3.288$, $\alpha_8 = 77522$, $\alpha_9 = 2818$, $\alpha_{10} = 127700$, $\alpha_{11} = 125800$.

Unfortunately, for higher orders the problem to select suitable Lyapunov function candidates and apply the generalized form approach becomes more and more complicated.

IV. Flexible joint robot

Consider a single-link robot with a revolute joint actuated by a DC motor, where the elasticity of the joint is modeled by a linear torsional spring (see Figure 1).

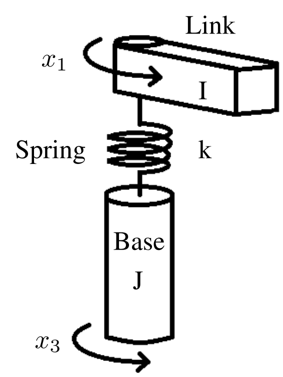

Fig. 1: Flexible joint robot.

Let a state-space model of a flexible joint robot in horizontal position be given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{k}{I}(x_1 - x_3) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= \tfrac{k}{J}(x_1 - x_3) + \tfrac{1}{J} u \end{aligned} \tag{9}$$

where x_1 and x_2 are position and velocity of the link, while x_3 and x_4 are position and velocity of the base. Moreover, k is the stiffness of the spring, I and J represents the inertia of the link and the base, respectively.

Now, consider a small parameter $\epsilon > 0$, such that k is $\mathbb{O}(\frac{1}{\epsilon^2})$, and the variables $z_1 = k(x_1 - x_3)$ and $z_2 = \epsilon \dot{z}_1$, then the single-link flexible-joint robot can be rewritten as a singularly perturbed model given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= -\tfrac{1}{I} z_1, \\ \epsilon \dot{z}_1 &= z_2, \\ \epsilon \dot{z}_2 &= -\tfrac{k_1(I+J)}{IJ} z_1 - \tfrac{k_1}{J} u, \end{aligned} \tag{10}$$

where $k_1 = k\epsilon^2$. We call the x-subsystem as the slow dynamics, and the z-subsystem as the fast dynamics. Note that if $\epsilon = 0$ (i.e., the stiffness k tends to infinite) then $z_1 = -\frac{I}{I+J} u$, and the standard model of a single-link rigid robot

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \tfrac{1}{I+J} u, \end{aligned} \tag{11}$$

is recovered. So, we call the system (11) as the reduced-order dynamics.

V. Finite-time composite control

The concept of composite control for flexible-joint robots modeled by (10) was introduced in [19], where the control law is the sum of two terms: a linear feedback for the slow variables and a correction term of viscous friction for the fast ones. Following such an idea, in this section we present an analysis about what happen when the linear feedback is replaced by a finite-time convergent controller.

For this purpose, consider the controller

$$u_s = (I + J)(-c_1 \lceil x_1 \rfloor^{\frac{1}{3}} - c_2 \lceil x_2 \rfloor^{\frac{1}{2}}) \tag{12}$$

where $c_1, c_2 > 0$ are parameters, such that if they are properly set then the closed-loop system (11)-(12) is GAS at the origin. Besides, (11)-(12) is r-homogeneous of degree $\nu = -\frac{1}{2}$ with a vector of weights $r = [\frac{3}{2}, 1]$, hence it exhibits finite-time stability at the origin, according to Theorem 1. On the other hand, the fast dynamics in (10) can be stabilized by adding a viscous friction term, that is,

$$u_f = k_v z_2, \tag{13}$$

where k_v is a positive constant. Then, a finite-time composite control law is given by

$$u_1 = u_s + u_f. \tag{14}$$

So, the closed-loop system (10)-(14) is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{1}{I} z_1 \\ \epsilon \dot{z}_1 &= z_2 \\ \epsilon \dot{z}_2 &= -\tfrac{k_1(I+J)}{IJ} z_1 - \tfrac{k_2}{J} z_2 \\ &\quad - \tfrac{k_1(I+J)}{J}(-c_1 \lceil x_1 \rfloor^{\frac{1}{3}} - c_2 \lceil x_2 \rfloor^{\frac{1}{2}}), \end{aligned} \tag{15}$$

where $k_2 = k_1 k_v$. Note that for the system (15) the reduced order dynamics is r-homogeneous of degree $\nu = -\frac{1}{2}$ with $r = [\frac{3}{2}, 1]$, while the fast dynamics is $\tilde{r}$-homogeneous of degree $\mu = 1$ with $\tilde{r} = [1, 1]$. Therefore, we have that $\nu < \mu$. However, the system (15) does not satisfy the hypothesis of Theorem 2 (the right-hand side of the fast dynamics part fails to be $\mathcal{C}^1$), hence, we cannot conclude anything about its stability properties. To the best of our knowledge, there is not literature regarding the problem of stability analysis for system like (15). But it is a well-known fact that the implementation of finite-time convergent controllers in presence of parasitic dynamics leads to the undesired phenomena of chattering (see for example [9]–[15]). Let us check the following example.

Motivational example. Select the parameters of the system (9) as $k = 80$, $I = 1$, $J = 1.5$ and $k_1 = 1$, and the gains of the controllers as $c_1 = 7$, $c_2 = 10$ and $k_v = 3.5$. By using the Euler's integration method and a sampling step of 1[ms], the simulations results of the closed-loop system (9)-(14) with initial condition $x_0 = [0, 0, 0.2\pi, 0]$ are provided in Figure 2.

Note that the states of the system (15) converge to a permanent oscillatory regime as it is expected. The system parameters and controller's gains have been selected to make easily visible the oscillatory behavior of the trajectories in steady state. Obviously, a better setting of such constants leads to a smaller amplitude and higher frequency of the oscillations, but there is no way them because it is an intrinsic property of the closed-loop system (9)-(14).

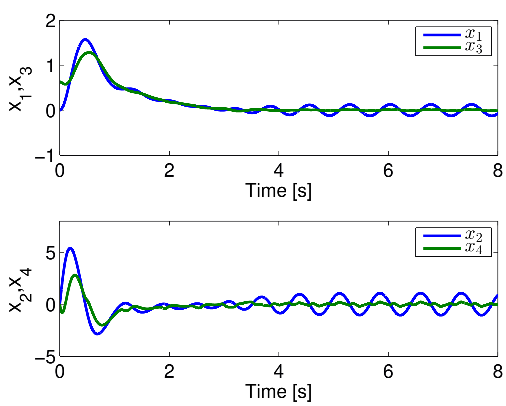

Fig. 2: Behavior of the closed-loop system (15) with initial condition $x_0 = [0, 0, 0.2\pi, 0]$.

A. Ideal finite-time composite control

Inspired in the results of [16] (despite they cannot be applied for the consider case), the objective of this work is to verify if a "homogenization" of the closed-loop system (9)-(14) can avoid the oscillatory steady-state response of its trajectories, i.e., if the first point of Theorem 2 can be realize for this particular case.

For this purpose, let us propose the following control schemes for the slow and fast dynamics, respectively:

$$u_s = \varrho_1 I^{\frac{1}{3}} \left(-c_1 \lceil x_1 \rfloor^{\frac{1}{5}} - c_2 \lceil x_2 \rfloor^{\frac{1}{4}} \right) \tag{16}$$

and

$$u_f = -\frac{I + J}{I} z_1 + \varrho_1 \lceil z_1 \rfloor^{\frac{1}{3}} + \varrho_2 \lceil z_2 \rfloor^{\frac{1}{2}} \tag{17}$$

where c_1, c_2, ϱ_1 and ϱ_2 are some positive constants. Then, the system (10) in closed-loop with our ideal finite-time composite control law $u_2 = u_s + u_f$ is given by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\tfrac{1}{I} z_1 \\ \epsilon \dot{z}_1 &= z_2 \\ \epsilon \dot{z}_2 &= -\tfrac{k_1 \varrho_1}{J} \lceil z_1 \rfloor^{\frac{1}{3}} - \tfrac{k_1 \varrho_2}{J} \lceil z_2 \rfloor^{\frac{1}{2}} \\ &\quad - \tfrac{k_1 \varrho_1 I^{\frac{1}{3}}}{J} \left(-c_1 \lceil x_1 \rfloor^{\frac{1}{5}} - c_2 \lceil x_2 \rfloor^{\frac{1}{4}} \right). \end{aligned} \tag{18}$$

Note that the system (18) is r-homogeneous of degree $\nu = -\frac{1}{2}$ with a vector of weights $r = [\frac{5}{2}, 2, \frac{3}{2}, 1]$. Hence, it is expected to be GAS at the origin for a proper selection of the gains c_1, c_2, ϱ_1 and ϱ_2, and a sufficiently small parameter $\epsilon > 0$. Moreover, according to Theorem 1, if a r-homogeneous system has negative degree then GAS implies finite-time convergence. So, we can establish the following result:

Theorem 3. *Consider the controllers u_s given by* (16) *and u_f given by* (17)*. For some positive parameters c_1, c_2, ϱ_1 and ϱ_2, and a sufficiently small parameter $\epsilon > 0$, the closed-loop system* (18) *is finite-time stable at the origin.*

Proof. Consider

$$\begin{aligned} V(x, z) = \; & \alpha_1 |x_1|^{\frac{9}{5}} + \alpha_2 |x_2|^{\frac{9}{4}} + \alpha_3 |z_1|^3 + \alpha_4 |z_2|^{\frac{9}{2}} \\ & + \alpha_5 x_1 x_2 + \alpha_6 \lceil x_1 \rfloor^{\frac{6}{5}} z_1 + \alpha_7 \lceil x_1 \rfloor^{\frac{7}{5}} z_2 \\ & + \alpha_8 \lceil x_2 \rfloor^{\frac{3}{2}} z_1 + \alpha_9 \lceil x_2 \rfloor^{\frac{7}{4}} z_2 + \alpha_{10} \lceil z_1 \rfloor^{\frac{7}{3}} z_2 \\ & + \alpha_{11} |z_1| |z_2|^3 + \alpha_{12} x_2 \lceil z_1 \rfloor^{\frac{5}{3}} + \alpha_{13} |x_1|^{\frac{6}{5}} |z_2|^{\frac{3}{2}}. \end{aligned} \tag{19}$$

as a Lyapunov function candidate for the system (18), where α_i for $i = 1, \ldots, 12$ are constant parameter. Note that it is r-homogeneous of degree $\kappa = -\frac{9}{2}$ with a vector of weights $r = [\frac{5}{2}, 2, \frac{3}{2}, 1]$ and it belongs to the family of generalized forms.

Hence, fixing $\epsilon = 0.02$ and the system parameter as $k = 2500$, $I = 1$, $J = 1.5$ and $k_1 = 1$, and by applying the generalized form approach, we obtain that (19) is positive definite and its derivative along the trajectories of (18) is negative definite, provided that the controller gains are set as $c_1 = 0.5$, $c_2 = 1$, $\varrho_1 = 1.965$ and $\varrho_2 = 2.682$ and the coefficients of the Lyapunov function are given by

$$\begin{array}{lll} \alpha_1 = 28.87, & \alpha_2 = 99.15, & \alpha_3 = 1.583, \\ \alpha_4 = 0.3285, & \alpha_5 = 21.6, & \alpha_6 = -0.7182, \\ \alpha_7 = -0.1044, & \alpha_8 = -5.111, & \alpha_9 = -2.492, \\ \alpha_{10} = 1.035, & \alpha_{11} = 0.9904, & \alpha_{12} = 3.702, \\ \alpha_{13} = 0.1553. & & \end{array}$$

Thus, we have that (19) is actually a Lyapunov function for the system (18), and we can say that it is GAS at the origin. Moreover, since it is r-homogeneous of negative degree, according to Theorem 1, we can conclude that the system (18) is finite-time stable. □

Thus, we have shown that using a nonlinear controller capable to make both, slow and fast dynamics, homogeneous of the same degree $\nu = \mu$, i.e., make the whole system homogeneous, we achieve finite-time convergence for the trajectories of the system (10) to the origin, avoiding the undesired chattering. Let us comeback to the motivational example.

Motivational example: revisited. Select the parameters of the system (9) as $k = 80$, $I = 1$, $J = 1.5$ and $k_1 = 1$, and the gains of the controllers as $c_1 = 1$, $c_2 = 2$, $\varrho_1 = 5.5$ and $\varrho_2 = 6$. By using the Euler's integration method and an integration step of 1[ms], the simulations results of the closed-loop system (18) with initial condition $x_0 = [0, 0, 0.2\pi, 0]$ are provided in Figure 3.

So, we can see in Figure 3 that the states of the system (18) converge to the origin in finite-time, which confirms the theoretical results.

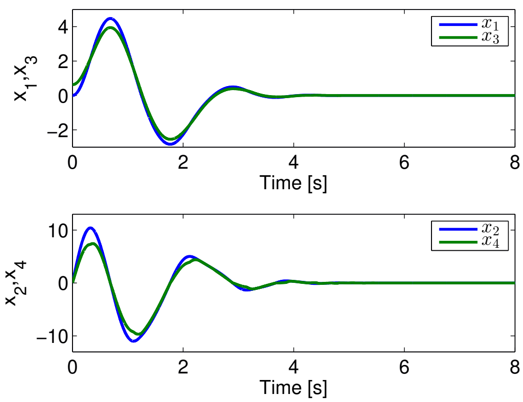

Fig. 3: Behavior of the closed-loop system (18) with initial condition $x_0 = [0, 0, 0.2\pi, 0]$.

VI. Conclusions

The stability of interconnected homogeneous systems affected by singular perturbations was studied through the implementation of finite-time composite control in a single-link flexible-joint robot. As previous results has suggested, the implementation of finite-time convergent controllers in presence of parasitic dynamics produces the undesired chattering effect. Pleasantly, in the case of study, our results show that the design of a controller making the whole system homogeneous avoids such chattering and recovers the ideal finite-time convergence. Unfortunately, in the most of the cases information of the states of the fast dynamics is not available, then the presented strategy is not applicable at all. But the design of similar strategies allowing the reduction of the amplitude of chattering is still possible and will be subject of future work.

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