Uncertainty propagation in subspace methods for operational modal analysis under misspecified model orders

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Abstract
The quantification of statistical uncertainty in modal parameter estimates has become a standard tool, used in applications to, e.g., damage diagnosis, reliability analysis, modal tracking and model calibration. Although efficient multi-order algorithms to obtain the (co)variance of the modal parameter estimates with subspace methods have been proposed in the past, the effect of a misspecified model order on the uncertainty estimates has not been investigated. In fact, the covariance estimates may be inaccurate due to the presence of small singular values in the supposed signal space. In this paper we go back to the roots of the uncertainty propagation in subspace methods and revise it to account for the case when a part of the noise space is erroneously added to the signal space. What is more, the proposed scheme adapts a different approach for the sensitivity analysis of the signal space, which improves the numerical efficiency. The performance is illustrated on an extensive Monte Carlo simulation of a simple mechanical system and applied to real data from a bridge.

1 Introduction

The identification of dynamic system characteristics from vibration measurements is a fundamental task in engineering. Amongst others, subspace-based system identification methods are well-suited for this purpose. They identify the system matrices of a linear time-invariant state-space model that describes the dynamic system behavior [1], from which the modal parameters are retrieved. The estimates are inherently afflicted with statistical uncertainties due to the unknown ambient excitation, measurement noise and limited data length. The quantification of these uncertainties is important for engineering applications related to Operational Modal Analysis (OMA) and Structural Health Monitoring, e.g., as in [2–5]. However, the subspace identification methods only produce point estimates but not their related confidence bounds.

The objective for uncertainty quantification in OMA is to obtain the modal parameter estimates *and* their confidence bounds from the same dataset. While the statistical properties of estimates from subspace methods have been analyzed in great detail in the automatic control literature in the past, e.g. in [6–8], the expressions therein cannot be directly used for an actual covariance estimation in practical applications, since they require in addition e.g. the estimation of the unknown states and their covariances, which are not computed in the modal parameter estimation. A different approach was proposed in [9], where the covariance of estimated parameters is computed easily from the sample covariances of the underlying output data covariances and their related sensitivities. A memory efficient and fast computation scheme for this method has been developed in [10] based on a mathematical reformulation of the algorithm.

The sensitivity-based covariance propagation is a simple and powerful tool for uncertainty quantification, and is theoretically justified by the statistical delta method [11]. It turns out that the sensitivity computation in one of the first steps of this propagation through the subspace algorithm, where the SVD of the output covariance Hankel matrix is truncated at an assumed model order, is sensitive to the actual model order choice. This is particularly relevant for the analysis of stabilization diagrams, where the system is identified at many successive model orders. In fact, the developments in [9, 10] that are based on the SVD sensitivities from [12] are only valid when the model order is chosen lower than the theoretical true order of the system, i.e., none of the considered singular values in the truncation of the SVD converge to zero when the number of data samples increases. If part of the noise space is erroneously added to the signal space, the sensitivity estimation from [9, 10] is perturbed, leading to inaccurate covariance estimates of the modal parameters. In this paper, the uncertainty quantification for the modal parameter estimates is adapted considering both signal and noise space contributions of the underlying SVD based on [13], assuming that the model order for the separation between signal and noise spaces is known. What is more, the sensitivity analysis of the signal space from [13] in the proposed scheme uses a different derivation than in the previous approaches [9, 10, 12] and yields a simpler computation, improving the numerical efficiency of the existing efficient approaches. The performance of the proposed approach is illustrated on an extensive Monte Carlo simulation of an simple mechanical system and applied to real data from a bridge.

2 Background

2.1 Subspace identification

Assume that the vibration behavior of a structure can be modelled by a linear time-invariant system, and that only outputs of the structure (accelerations, velocities, displacements or strains) are measured while inputs (acting forces) are unknown. Then the system dynamics can be described by the discrete-time state space model [14]

$$\begin{cases} x_{k+1} = Ax_k + w_k \\ y_k = Cx_k + v_k \end{cases} \tag{1}$$

where $x_k \in \mathbb{R}^n$ are the states, $y_k \in \mathbb{R}^r$ are the outputs, vectors w_k and v_k denote the white process and output noise respectively, $A \in \mathbb{R}^{n\times n}$, $C \in \mathbb{R}^{r\times n}$ are the state and observation matrices, and n is the model order.

Let $R_i = \mathrm{E}(y_k y_{k-i}^T)$ be the theoretical output covariances of the measurements, which yield, following from (1), $R_i = CA^{i-1}G$, where $G = \mathrm{E}(x_{k+1} y_k^T)$ denotes the cross-covariance between the states and the outputs. The collection of R_i can be stacked to form a block Hankel matrix

$$\mathcal{H} = \begin{bmatrix} R_1 & R_2 & \dots & R_q \\ R_2 & R_3 & \dots & R_{q+1} \\ \vdots & \vdots & \ddots & \vdots \\ R_{p+1} & R_{p+2} & \dots & R_{p+q} \end{bmatrix} \in \mathbb{R}^{(p+1)r\times qr}, \tag{2}$$

where p and q are chosen such that $\min(pr, qr) \geq n$ with often $p+1 = q$. Matrix $\mathcal{H}$ enjoys the factorization property

$$\mathcal{H} = \mathcal{O}\,\mathcal{C}, \tag{3}$$

where the observability and stochastic controllability matrices $\mathcal{O}$ and $\mathcal{C}$ are

$$\mathcal{O} = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^p \end{bmatrix}, \quad \mathcal{C} = \begin{bmatrix} G & AG & \dots & A^{q-1}G \end{bmatrix}. \tag{4}$$

Estimates $\hat{R}_i$ and consequently $\hat{\mathcal{H}}$ can be computed from the output covariance estimates of the measurements $\{y_k\}_{k=1,\dots,N+p+q}$, e.g. by

$$\hat{\mathcal{H}} = \mathcal{Y}^{+}\mathcal{Y}^{-T}, \tag{5}$$

where the data Hankel matrices $\mathcal{Y}^+$ and $\mathcal{Y}^-$ contain future and past time horizons

$$\mathcal{Y}^+ = \frac{1}{\sqrt{N}} \begin{bmatrix} y_{q+1} & y_{q+2} & \cdots & y_{N+q} \\ y_{q+2} & y_{q+3} & \cdots & y_{N+q+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{p+q+1} & y_{p+q+2} & \cdots & y_{p+q+N} \end{bmatrix}, \ \mathcal{Y}^- = \frac{1}{\sqrt{N}} \begin{bmatrix} y_q & y_{q+1} & \cdots & y_{N+q-1} \\ y_{q-1} & y_q & \cdots & y_{N+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ y_1 & y_2 & \cdots & y_N \end{bmatrix}. \tag{6}$$

Thanks to decomposition (3), an estimate of $\mathcal{O}$ can be retrieved from $\hat{\mathcal{H}}$ by a Singular Value Decomposition truncated at model order n,

$$\hat{\mathcal{H}} = \begin{bmatrix} \hat{U}_{\text{sig}} & \hat{U}_{\text{null}} \end{bmatrix} \begin{bmatrix} \hat{D}_{\text{sig}} & 0 \\ 0 & \hat{D}_{\text{null}} \end{bmatrix} \hat{V}^T, \ \hat{\mathcal{O}} = \hat{U}_{\text{sig}} \hat{D}_{\text{sig}}^{1/2}. \tag{7}$$

An estimate of the observation matrix C is then obtained from the first block-row of $\hat{\mathcal{O}}$. The state transition matrix A is estimated from the shift invariance property of $\hat{\mathcal{O}}$, namely $\hat{A} = \underline{\hat{\mathcal{O}}}^{\dagger} \overline{\hat{\mathcal{O}}}$, where $\overline{\hat{\mathcal{O}}}, \underline{\hat{\mathcal{O}}} \in \mathbb{R}^{pr \times n}$ are respectively the matrix $\hat{\mathcal{O}}$ without the first and the last block rows. Finally, the modal parameter estimates are obtained from the eigenvalues and eigenvectors of $\hat{A}$, and from $\hat{C}$.

2.2 Uncertainty quantification

The delta method is a statistical tool that helps to estimate the covariance of a function of an asymptotically Gaussian variable [11]. It is used to propagate the sample covariance of the output covariances that are computed in the first step of the subspace algorithm through all steps of the algorithm down to the modal parameters. The output covariances are asymptotically Gaussian, i.e.,

$$\sqrt{N}\text{vec}(\hat{\mathcal{H}} - \mathcal{H}) \to \mathcal{N}(0, \Sigma_{\mathcal{H}}), \tag{8}$$

where $\text{vec}(\cdot)$ denotes the column stacking vectorization. An estimate $\hat{\Sigma}_{\mathcal{H}}$ of the covariance can be easily evaluated by the sample covariance based on partitions of the available data. The propagation of this covariance to the modal parameter estimates is then based on the delta method, stating that a matrix function $\hat{Y} = f(\hat{\mathcal{H}})$ has also asymptotically Gaussian entries with

$$\sqrt{N}\text{vec}(\hat{Y} - Y) \to \mathcal{N}(0, \mathcal{J}_{Y,\mathcal{H}} \Sigma_{\mathcal{H}} \mathcal{J}_{Y,\mathcal{H}}^T). \tag{9}$$

The derivative $\mathcal{J}_{Y,\mathcal{H}}$ of the function with respect to $\mathcal{H}$ is obtained from perturbation theory. For a first-order perturbation it holds $\Delta Y \approx \mathcal{J}_{Y,\mathcal{H}} \Delta\mathcal{H}$. Hence, perturbing the functional relationship between $\mathcal{H}$ and Y analytically and neglecting higher-order terms yields the desired derivative, in particular for cases where the functional relationship is not explicit like for the SVD or eigenvalue decomposition. Subsequently, covariance expressions for the estimates satisfy

$$\hat{\Sigma}_Y \approx \hat{\mathcal{J}}_{Y,\mathcal{H}} \hat{\Sigma}_{\mathcal{H}} \hat{\mathcal{J}}_{Y,\mathcal{H}}^T. \tag{10}$$

With this principle, the uncertainties of the output covariances from the first step of the subspace method can be propagated step by step through the algorithm down to the modal parameters. The analytical sensitivities are derived in detail in [9], with a computationally efficient algorithm proposed in [10]. The modal parameter covariance follows from (10) based on the Hankel matrix sample covariance and estimates of the respective sensitivities.

2.3 Uncertainty propagation for $\mathcal{O}$

In this paper, the influence of the model order selection on the uncertainty quantification of the modal parameters is considered, which takes place in the truncation of the SVD in (7) for the estimation of the observability matrix. The related part of the uncertainty quantification is recalled from [9, 10] in this section, i.e., the respective sensitivity $\mathcal{J}_{\mathcal{O},\mathcal{H}}$ satisfying

$$\text{vec}(\Delta\mathcal{O}) = \mathcal{J}_{\mathcal{O},\mathcal{H}} \, \text{vec}(\Delta\mathcal{H}). \tag{11}$$

This sensitivity is directly related to the sensitivity of the SVD of $\mathcal{H}$ by $\Delta\mathcal{O} = U_s \Delta D_s^{1/2} + \Delta U_s D_s^{1/2}$. The sensitivities of the SVD as shown in [12] and used in [9, 10] are based on perturbing the relationships $\mathcal{H} v_i = \sigma_i u_i$, $u_i = \mathcal{H} v_i/\sigma_i$ and $v_i = \mathcal{H}^T u_i/\sigma_i$, respectively, where σ_i, u_i and v_i are the singular values in D_s, the left singular vectors in U_s and the right singular vectors in V_s for $i = 1, \ldots, n$. This yields

$$\Delta\sigma_i = u_i^T \, \Delta\mathcal{H} \, v_i, \tag{12}$$

$$\begin{bmatrix} I & -\mathcal{H}/\sigma_i \\ -\mathcal{H}^T/\sigma_i & I \end{bmatrix} \begin{bmatrix} \Delta u_i \\ \Delta v_i \end{bmatrix} = \frac{1}{\sigma_i} \begin{bmatrix} (I - u_i u_i^T)\Delta\mathcal{H} v_i \\ (I - v_i v_i^T)\Delta\mathcal{H}^T u_i \end{bmatrix}. \tag{13}$$

To obtain the singular vector sensitivities from the second equation, the matrix on the left side needs to be inverted. However, it is a rank-deficient matrix. The solution in [9] is obtained by using the pseudoinverse. In [10], the condition $u_i^T \Delta u_i + v_i^T \Delta v_i = 0$ is added to (13), which results from the norm of the singular vectors being constantly one. This results in an invertible matrix on the left side, for which the matrix inversion lemma is used to efficiently obtain the solution for Δu_i only in [10]. With these sensitivities, matrix $\mathcal{J}_{\mathcal{O},\mathcal{H}}$ is assembled.

An important requirement in these developments is that $\sigma_i \neq 0$. A consistent estimate of the sensitivities can thus only be achieved if $\hat{\sigma}_i$ is an estimate of a non-zero singular value, i.e., when the data length $N \to \infty$ then $\hat{\sigma}_i \to \sigma_i \neq 0$. To ensure that the sensitivity computations are correct, it is thus not sufficient to ensure that $\hat{\sigma}_i$ is non-zero (this will always be the case for noisy data), but that the SVD in (7) is truncated at a model order that is not higher than the "true" theoretical order of the considered system.

In contrast, when the SVD in (7) is truncated at a model order $\hat{n}$ that is higher than the theoretical order n of the considered system, some of the considered singular values and the associated singular vectors are estimates corresponding to singular values that are theoretically zero. While this does not affect the consistency of the estimates of the true system modes, the singular value and vector sensitivities as presented in the previous section are only consistent until model order n, but not for $\hat{n} > n$. Since the sensitivities of the true system modes identified at order $\hat{n}$ depend on matrix A and thus also on the singular vector sensitivities for $\hat{n} > n$, this can lead to an erroneous covariance computation of the modal parameter estimates, as illustrated in the next section.

2.4 Numerical application for illustration and validation

Consider a 6 DOF chain-like system as illustrated in Figure 1 that, for any consistent set of units, is modeled with spring stiffnesses $k_1 = k_3 = k_5 = 100$ and $k_2 = k_4 = k_6 = 200$, mass $m_i = 1/20$ and a proportional damping matrix such that each mode has a damping ratio of 2%. The exact modal parameters of the system are depicted in Table 1. The system is excited by a white noise signal in all DOFs and sampled with a frequency of 50 Hz for a duration of 2000 seconds. The responses are measured only at DOFs 1, 3 and 5. To emulate measurement noise, Gaussian white noise with 5% of the standard deviation of the output is added to the response for each channel.

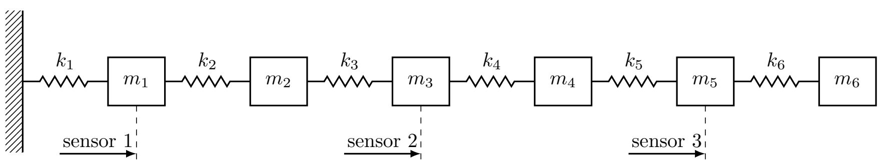

Figure 1: Illustration of 6 DOF chain system used for Monte Carlo simulation.

A Monte Carlo experiment is performed, where 1000 realizations of the described signal are computed.

Table 1: Exact modal parameters of the chain system.

Natural frequency (Hz)						Damping ratio (%)					
f_1	f_2	f_3	f_4	f_5	f_6	ζ_1	ζ_2	ζ_3	ζ_4	ζ_5	ζ_6
1.936	5.618	8.682	14.494	15.798	17.007	2	2	2	2	2	2

In the following, the behavior of the existing modal parameter uncertainty quantification approach as outlined in Section 2.3 is examined at different model orders. The modes of the chain are identified with the covariance-driven subspace method with $p + 1 = q = 26$ at the (true) model order $n = 12$ as well as model orders 24, 48 and 60, where the observability matrix is obtained in (7). In Fig. 2 and 3, the means of the estimated standard deviations with the presented uncertainty quantification strategy are shown for different model orders and compared to the empirical standard deviation from the Monte Carlo simulation for frequency and damping ratio estimates. First, it can be observed that the computed standard deviations are identical either using the original derivation from [9] or the efficient reformulation from [10] at all model orders. Second, it can be seen that the estimated standard deviations always match the empirical ones very well at the true model order $n = 12$ for all cases. However, the situation is different at higher model orders: the estimated standard deviations are in general inaccurate for higher model orders, which is particularly

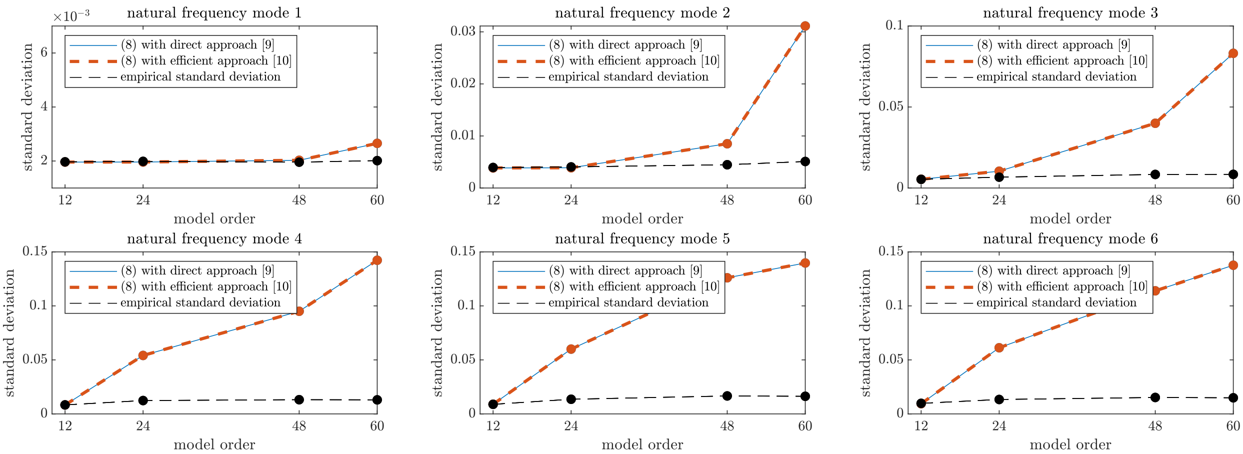

Figure 2: Mean of computed standard deviations and empirical standard deviations of frequencies (in Hz) from Monte Carlo simulation at different model orders.

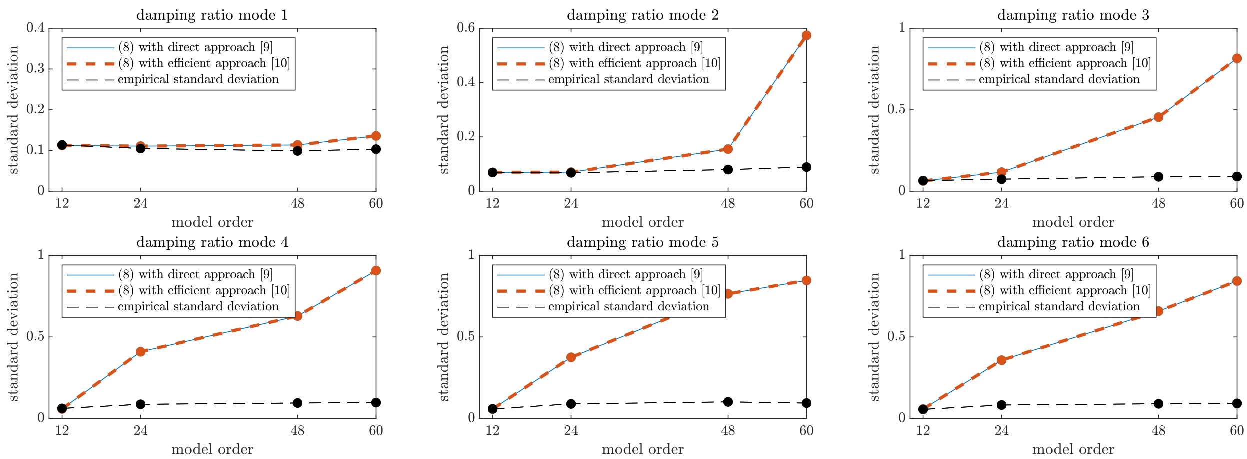

Figure 3: Mean of computed standard deviations and empirical standard deviations of damping ratios (in %) from Monte Carlo simulation at different model orders.

pronounced for modes 4–6. Only for the first modes 1 and 2, and there only for moderately over-estimated model orders the previous uncertainty quantification is still accurate. The empirical standard deviations seem to be quite stable for the different model orders.

3 Uncertainty propagation for overmodelled systems

In the following it is assumed that the theoretical model order n is known, and that the system is identified at some order $\hat{n} > n$. The sensitivities of the observability matrix through the SVD are revisited for this case, being a crucial part in the uncertainty quantification of the modal parameters.

The classical computation of the observability matrix in (7) depends on both the singular vectors and singular values. When $\hat{n} > n$, this would mean that the sensitivities of singular values are required, for which some of the singular values are estimates of zero. The sensitivities of zero singular values are complex and first-order perturbations insufficient in their approximation [15, 16]. However, we do not have to deal with them if the observability matrix estimate is defined at the chosen model order $\hat{n}$ directly by the singular vectors as

$$\hat{\mathcal{O}} = \begin{bmatrix} \hat{U}_{\text{sig}} & \hat{U}_{\text{null},1} \end{bmatrix}, \tag{14}$$

where $\hat{U}_{\text{null},1}$ consists of the first $\hat{n} - n$ columns of $\hat{U}_{\text{null}}$ in (7).

To analyze the perturbations of the signal and the null space singular vectors, there is a considerable body of literature over the last decades. Based on the works by Stewart [17], expressions for the first-order perturbations of the singular vector spaces have been developed in [13], yielding

$$\Delta U_{\text{sig}} = U_{\text{null}} U_{\text{null}}^T \Delta\mathcal{H} V_{\text{sig}} D_{\text{sig}}^{-1}, \tag{15}$$

$$\Delta U_{\text{null}} = -U_{\text{sig}} D_{\text{sig}}^{-1} V_{\text{sig}}^T \Delta\mathcal{H}^T U_{\text{null}}. \tag{16}$$

In comparison to (13), the perturbation of the left singular vectors in the signal space in (15) is decoupled from the perturbation of the right singular vectors, and no pseudoinverse is required, leading to a more direct and simpler computation. This expression is further analyzed and developed in [18]. The expression for the left null space perturbation in (16) has been used in recent subspace-based identification, damage detection and localization approaches, e.g. [19, 20].

Finally, combine (14)–(16) to obtain the perturbation on the observability matrix

$$\begin{aligned} \operatorname{vec}(\Delta\mathcal{O}) &= \begin{bmatrix} \operatorname{vec}(U_{\text{null}} U_{\text{null}}^T \Delta\mathcal{H} V_{\text{sig}} D_{\text{sig}}^{-1}) \\ \operatorname{vec}(-U_{\text{sig}} D_{\text{sig}}^{-1} V_{\text{sig}}^T \Delta\mathcal{H}^T U_{\text{null},1}) \end{bmatrix} \\ &= \begin{bmatrix} D_{\text{sig}}^{-1} V_{\text{sig}}^T \otimes U_{\text{null}} U_{\text{null}}^T \\ -(U_{\text{null},1}^T \otimes U_{\text{sig}} D_{\text{sig}}^{-1} V_{\text{sig}}^T) \mathcal{P}_{(p+1)r,qr} \end{bmatrix} \operatorname{vec}(\Delta\mathcal{H}), \end{aligned} \tag{17}$$

where $\otimes$ denotes the Kronecker product and $\mathcal{P}_{a,b}$ is a permutation matrix with $\mathcal{P}_{a,b} \operatorname{vec}(X) = \operatorname{vec}(X^T)$ for $X \in \mathbb{R}^{a,b}$. A consistent estimate of the sensitivity in (17) is obtained by replacing $U_{\text{sig}}, D_{\text{sig}}, V_{\text{sig}}, U_{\text{null}}$ by their estimated counterparts from the SVD of $\hat{\mathcal{H}}$ in (7).

The remainder of the uncertainty propagation to the modal parameters, i.e., from the observability matrix to the state-space matrix estimates of A and C, then to the eigenvalues and eigenvectors of A and finally to the natural frequencies, damping ratios and mode shapes, is carried out as described in [9, 10]. The modal parameter covariances are then obtained based on (10).

3.1 Numerical validation

The behavior of the proposed modifications for the uncertainty quantification from the previous section are examined in the following, where the observability matrix is computed directly from the singular vectors in (14) for the different model orders. The following approaches for the singular vector sensitivities are considered:

- Based on (13) as in [9, 10], assuming that all considered singular vectors from SVD of Hankel matrix belong to signal space. Since the results based on the original computation in [9] and on the efficient computation in [10] are identical as shown in Fig. 2 and 3, no difference between both computations is made in the following;
- Based on (15), assuming that all considered singular vectors from SVD of Hankel matrix belong to signal space;
- Based on (15)–(16) as shown in (17), assuming the separation of signal and noise spaces at the theoretical model order $n = 12$.

In Fig. 4 and 5 the proposed approaches are compared. First of all, it can be seen that the empirical standard deviations from the Monte Carlo simulation as well as the estimated standard deviations based on the existing method (Eq. (13)) are identical to the results in Fig. 2 and 3, either using the observability matrix estimate on the singular vectors only (Eq. (14)) or both the singular vectors and values (Eq. (7)). This indicates that there is no change in precision of the modal parameters when estimating the observability matrix now in (14). Second, changing the singular vector sensitivity computation to the formula in (15), while again assuming that all the considered singular vectors (including the ones above the theoretical system order) belong to the signal space, leads already to smaller errors on the standard deviation estimates than the previous computa-

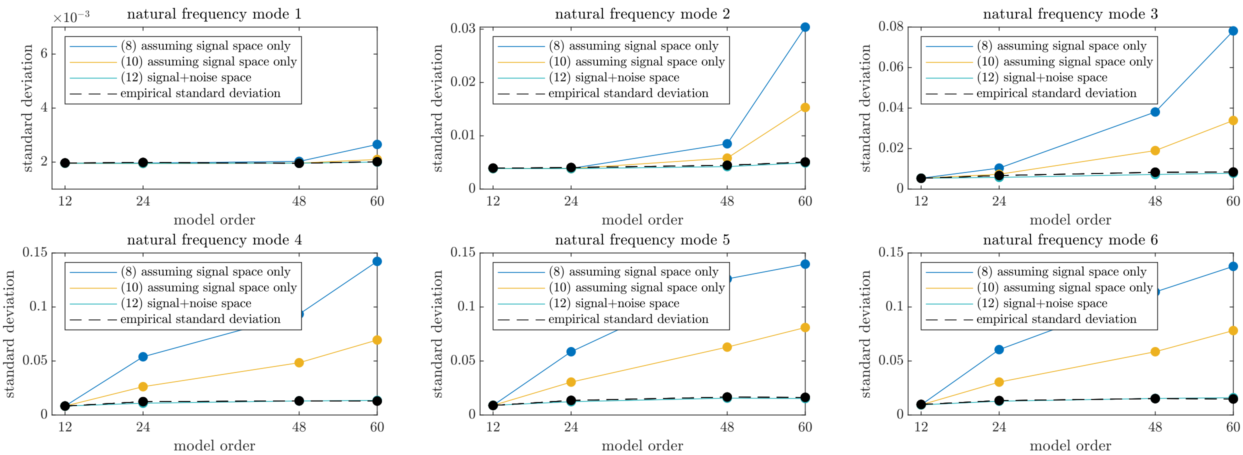

Figure 4: Mean of computed standard deviations and empirical standard deviations of frequencies (in Hz) from Monte Carlo simulation at different model orders, based on the computation of $\hat{\mathcal{O}}$ in (14).

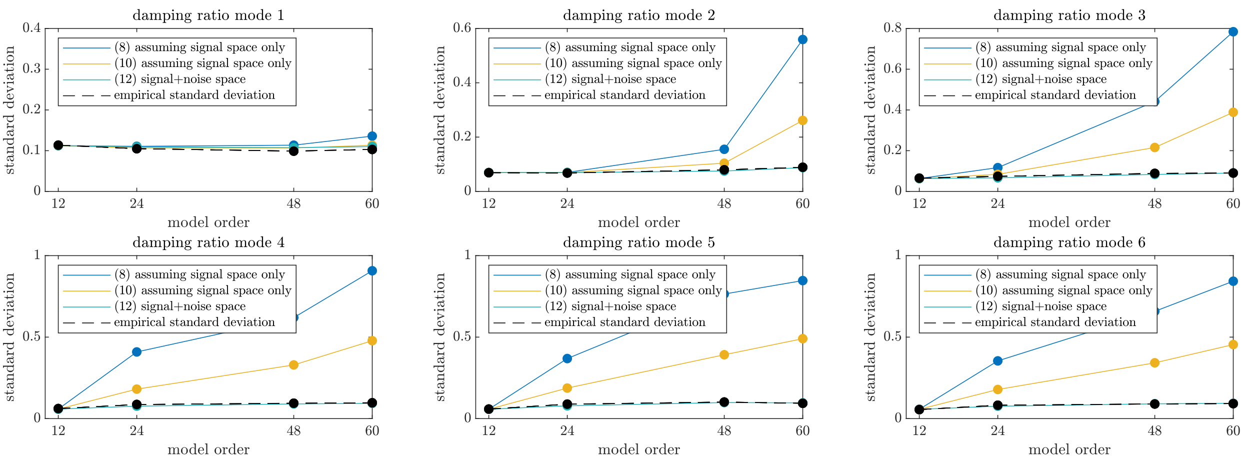

Figure 5: Mean of computed standard deviations and empirical standard deviations of damping ratios (in %) from Monte Carlo simulation at different model orders, based on the computation of $\hat{\mathcal{O}}$ in (14).

tion. However, the errors are still non-negligible, especially for the higher modes and higher model orders. Finally, the proposed uncertainty computation in (17) considering the sensitivities related to the signal and noise spaces separately leads to accurate standard deviation estimates for all considered model orders for both frequencies and damping ratios.

In the final part of the numerical validation the precision of the standard deviation estimates is evaluated. In Fig. 6 and 7 the coefficients of variation (CV, empirical standard deviation of the delta method-based standard deviation estimates divided by their mean) are shown for the different considered approaches for the frequencies and damping ratios. It can be seen that the CV for all approaches is small at the true model order $n = 12$, being around 5% of the standard deviation estimate. With the approaches that do not consider a separation between signal and noise spaces, the CV increases strongly up to 200% and more for the higher model orders, where the sensitivity computation based on (15) shows better precision than the previous one based on (13). Finally, the proposed uncertainty computation in (17) considering the sensitivities related to the signal and noise spaces separately leads to the best precision. The CV still increases for the higher model orders in particular for the higher modes, but remains reasonable and below 50% in all cases.

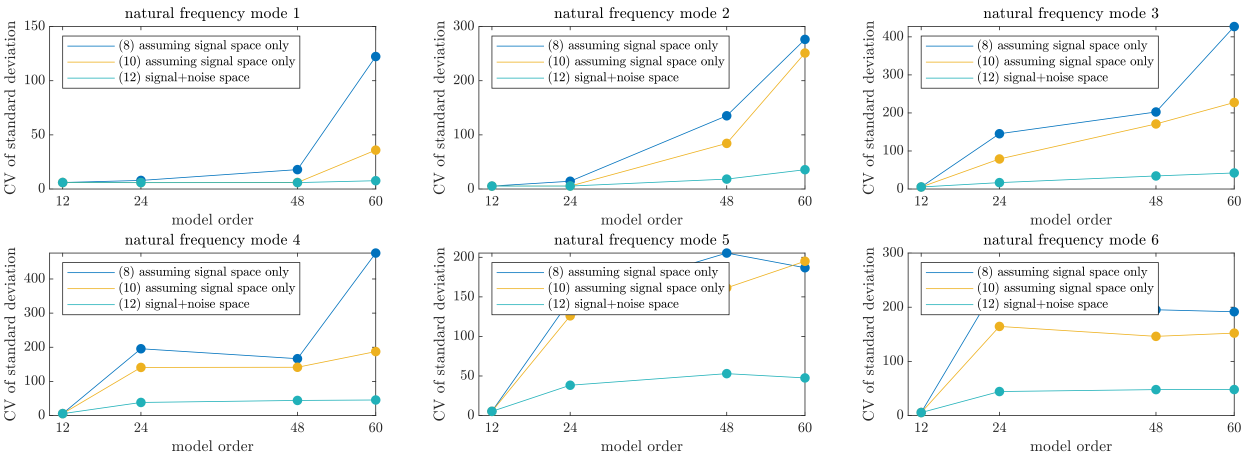

Figure 6: Coefficients of variation (in %) of the frequency standard deviation estimates.

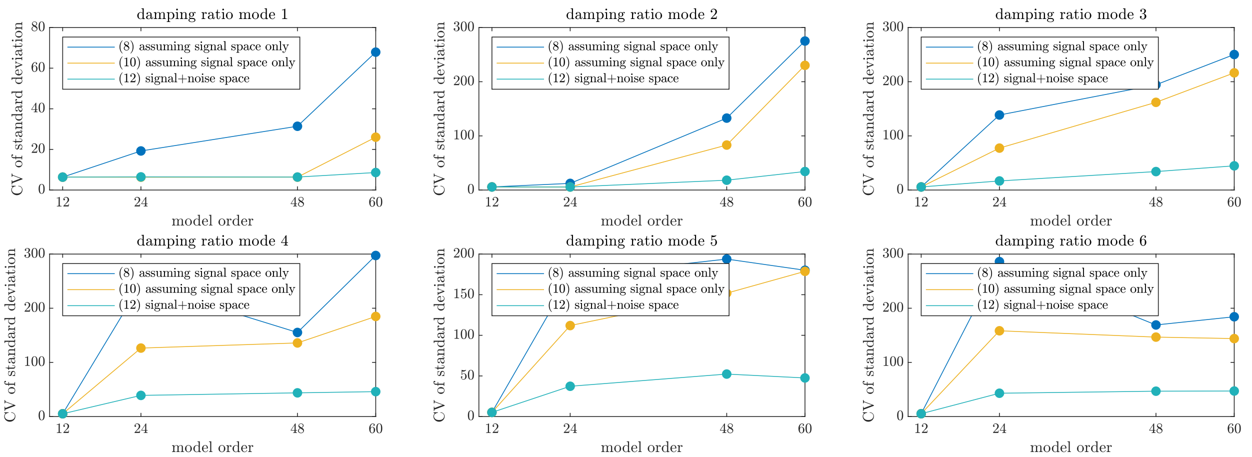

Figure 7: Coefficients of variation (in %) of the damping ratio standard deviation estimates.

4 Application

This section presents an application of the proposed uncertainty quantification approach for deriving confidence intervals of modal parameters obtained from real data. The application is carried out on measurements from the monitoring campaign of the S101 Bridge in Austria [21], shown in Fig. 8.

Figure 8: S101 Bridge before demolition.

In this study, the response of the bridge to ambient loads in vertical direction is used from 15 acceleration sensors mounted on its deck. Measurements were sampled with a frequency of 500 Hz and were decimated to 27.78 Hz prior to modal parameter estimation and uncertainty quantification. The modal parameters and the related uncertainties are estimated with $p = 20$ and model orders ranging from $n_{\min} = 12$ to $n_{\max} = 60$. A heuristic division of the signal space and the null space is conducted at model order 12 and used for the uncertainty propagation to the observability matrix in (17). In Fig. 9 the resulting stabilization diagram of natural frequencies with the corresponding 95% confidence intervals obtained with the proposed uncertainty propagation approach is shown. In the stabilization diagram the natural frequency estimates and their confidence intervals are plotted on top of the singular values of the cross power spectral density (CPSD) matrix evaluated for each frequency line [22]. A threshold on the Coefficient of Variation (CV) for the natural frequency equal to 20% and for the damping ratio equal to 50% is enforced to filter the uncertain modal parameter estimates from the stabilization diagram, i.e., any mode whose CV of the natural frequency exceeds 20% or 50% for the damping ratio, is discarded from the diagram in Figure 9.

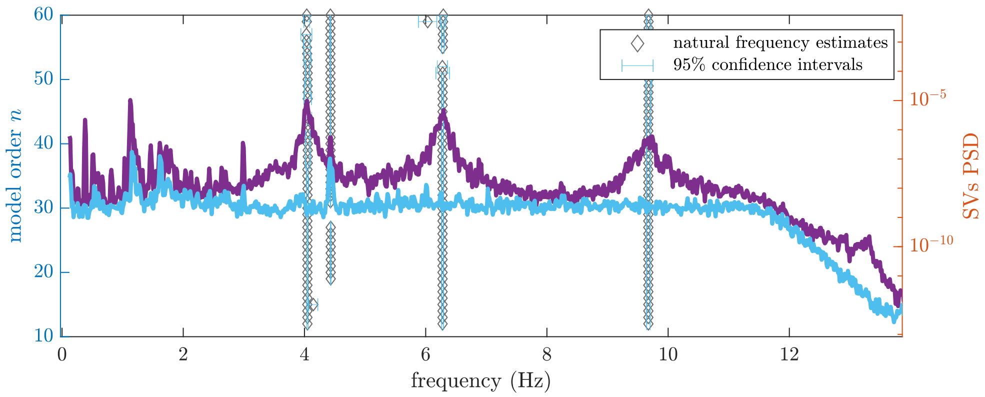

Figure 9: Stabilization diagram of the natural frequency estimates with the corresponding 95% confidence intervals.

Three modal alignments, i.e., groups of modal parameters obtained at different model orders that correspond to the same mode, are extracted manually at 4.05 Hz, 6.28 Hz and 9.67 Hz. To assess the real-life performance of the proposed method and to compare it to the previous approaches, a comparison similar to Section 3.1 is carried out, including:

- The previous approach from [9, 10] based on (13), assuming that all considered singular vectors from SVD of Hankel matrix belong to signal space;
- Based on (15), assuming that all considered singular vectors from SVD of Hankel matrix belong to signal space;
- The proposed approach based on (15)–(16) as shown in (17), assuming the separation of signal and noise spaces at the model order $n = 12$.

The resulting zoomed stabilization diagrams with the 95% confidence intervals obtained with the different approaches are illustrated in Fig. 10 for the three modes. It can be seen that with the previous approach (left), the confidence bounds at high model orders sometimes become very large, up to the point that they exceed the predefined threshold of the CV so that some of the modes disappear from the diagram. The over-estimation of the standard deviations at high model orders with this approach was already observed on the numerical

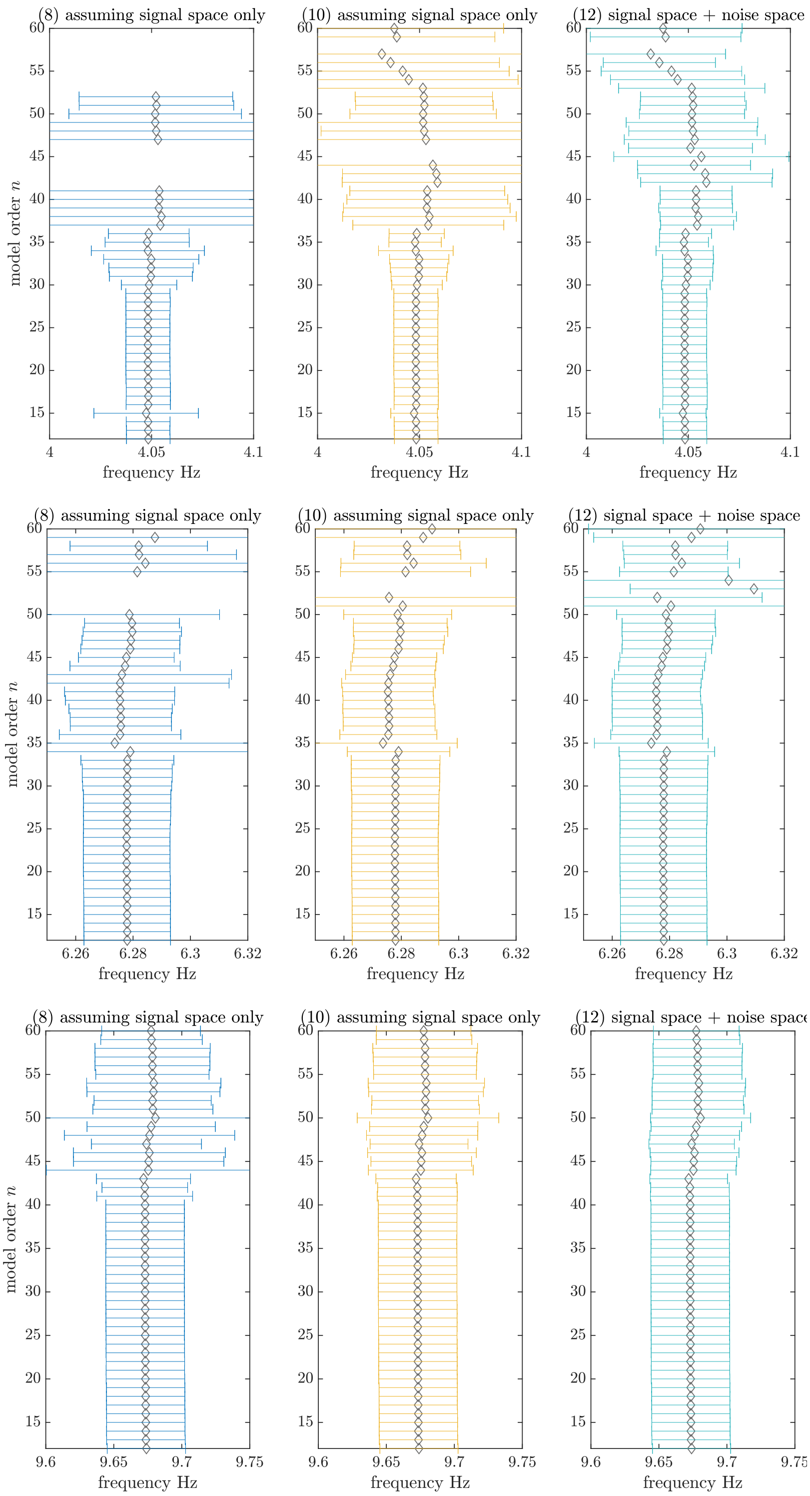

Figure 10: Confidence intervals of the natural frequency estimates of modes 1 (top), 2 (middle) and 3 (bottom) obtained with the different uncertain propagation methods.

example in the previous section. Changing the singular vector sensitivity computation to the formula in (15), while again assuming that all the considered singular vectors belong to the signal space (Fig. 10, middle), leads already to a more stable computation of the confidence intervals. Finally, the proposed uncertainty computation in (17) considering the sensitivities related to the signal and noise spaces separately (Fig. 10, right) seems to yield the most stable computation of the confidence intervals at high model orders compared to the other approaches.

5 Conclusions

In this paper, the problem of misspecified model order selection has been addressed in the uncertainty quantification of modal parameter estimates from covariance-driven stochastic subspace identification. While the selection of a model order that is higher than the true system order should not affect the consistency of the modal parameter estimates, it has been shown that the associated estimates of the modal parameter standard deviations are affected in the previous uncertainty quantification approaches in [9, 10]. The separate consideration of the signal and noise spaces related to the true model order amends this problem in the uncertainty computation, leading to accurate and more precise estimates of the modal parameter standard deviations. It can also be applied in the uncertainty quantification of other subspace methods, such as [23, 24].

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