



# International Journal of Mathematical Education in Science and Technology

ISSN: (Print) (Online) Journal homepage: <https://www.tandfonline.com/loi/tmes20>

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To cite this article: Mark McCartney (2022): Chaos on the hypercube and other places, International Journal of Mathematical Education in Science and Technology, DOI: [10.1080/0020739X.2022.2113466](https://doi.org/10.1080/0020739X.2022.2113466)

To link to this article: <https://doi.org/10.1080/0020739X.2022.2113466>



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Published online: 09 Sep 2022.



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## Chaos on the hypercube and other places

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### ABSTRACT

Using the sawtooth map as the basis of a coupled map lattice enables simple analytic results to be obtained for the global Lyapunov spectra of a number of standard lattice networks. The results presented can be used to enrich a course on chaos or dynamical systems by providing tractable examples of higher dimensional maps and links to a number of standard results in matrices. A number of suggestions for classroom use are given.

### ARTICLE HISTORY

Received 4 February 2022

### KEYWORDS

Coupled map lattice;  
Lyapunov exponents;  
hypercube; discrete chaos

## 1. Introduction

Chaos is most frequently introduced to students via the logistic map

$$x_{n+1} = \alpha x_n(1 - x_n). \quad (1)$$

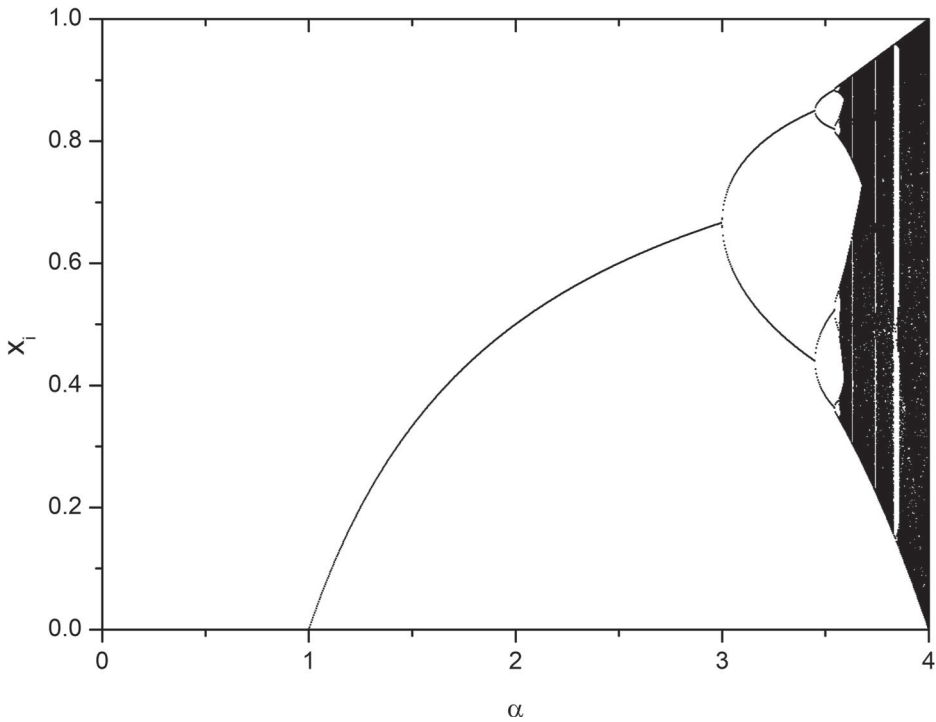
As the control parameter  $\alpha$  is increased the logistic map illustrates post transient behaviour which decays to fixed point  $x^* = 0$  ( $0 < \alpha \leq 1$ ); fixed point  $x^* = 1 - \frac{1}{\alpha}$  ( $1 \leq \alpha < 3$ ); period two orbit  $x_{1,2}^* = \frac{\alpha+1 \pm \sqrt{(\alpha-3)(\alpha+1)}}{2\alpha}$  ( $3 \leq \alpha < 1 + \sqrt{6}$ ); and for  $\alpha \geq 1 + \sqrt{6}$  the post transient behaviour repeatedly period doubles to period 4, 8, 16, 32 orbits, reaching aperiodic behaviour at the so-called accumulation point  $\alpha_\infty \simeq 3.5699456718$ . At this point the behaviour of the map is chaotic and beyond the accumulation point the map exhibits rich behaviour with chaos abruptly ceasing to give periodic windows which then period double back into chaos. All of this can be visually summarised in a Feigenbaum diagram where post-transient behaviour is plotted as a function of control parameter  $\alpha$  (Figure 1). The logistic map is a particularly powerful pedagogic example, illustrating as it does how a seemingly innocuous quadratic iterator leads to enormously rich behaviour.

A system is defined as chaotic if it fulfils three criteria:

- (1) The orbit is bounded in a finite region of space.
- (2) The orbit is aperiodic.
- (3) The orbit is sensitive to initial conditions.

Further, chaos can be quantified via the global Lyapunov exponent, which gives a measure of the rate of separation of neighbouring points as they are iterated – and thus gives a

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**Figure 1.** Feigenbaum diagram for the logistic map showing period doubling into chaos.

numerical measure of criterion 3 above. For a general one-dimensional map of the form

$$x_{n+1} = f(x_n) \quad (2)$$

the global Lyapunov exponent is defined as

$$\lambda_g = \lim_{M \rightarrow \infty} \frac{1}{M} \sum_{i=1}^M \ln |f'(x_i)| \quad (3)$$

with evaluation being carried out numerically over a large number (say  $10^7$ ) of post-transient iterations. If  $\lambda_g > 0$ , and the first two criteria hold, the system is chaotic.

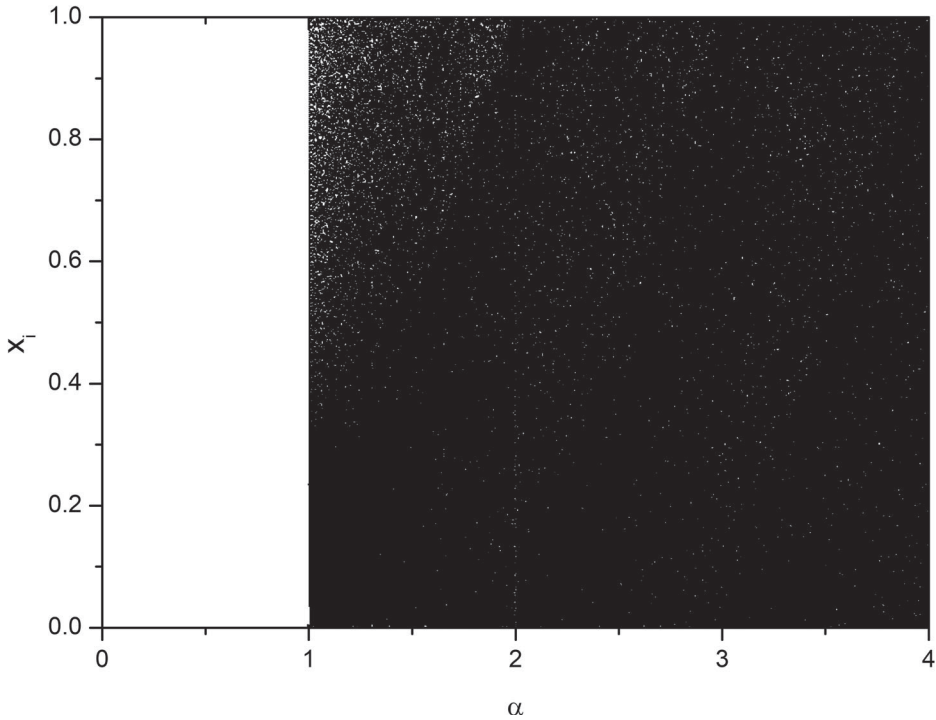
An even simpler map than the logistic map, but one not so frequently introduced to students, is the sawtooth, or ‘chopper’ map,

$$x_{n+1} = \alpha x_n \bmod 1. \quad (4)$$

This map has the advantage that, apart from at a finite set of discontinuities (which we will ignore), its first derivative is a constant,  $\alpha$ , and so the global Lyapunov exponent can be immediately evaluated as

$$\lambda_g = \ln \alpha.$$

Clearly if  $\alpha > 1$  we expect (4) to be chaotic, though formally a certain amount of care is required (see question 2 of Classroom Exercises). The Feigenbaum diagram of the sawtooth map is shown in Figure 2.



**Figure 2.** Feigenbaum diagram for the sawtooth map. Note the absence of period doubling behaviour and immediate coverage of the entire unit interval in comparison to the logistic map.

## 2. Coupled maps

A simple and powerful way to generalise a one-dimensional map is to use the idea of a coupled map lattice (CML). Visually this can be thought of as a network with a one-dimensional map of functional form  $f(x)$  at each node whose value is updated via a linear combination of the outputs of the other maps to which it is connected.

Thus, if we have a set of  $N$  maps, with the value of the  $j$ th map at the  $n$ th timestep being defined as  $x_n^j$  then a CML can be defined by

$$\underline{x}_{n+1} = A\underline{F}(\underline{x}_n) \quad (5)$$

where

$$\underline{x}_n = \begin{pmatrix} x_n^1 \\ x_n^2 \\ \vdots \\ x_n^N \end{pmatrix} \quad \underline{F}(\underline{x}_n) = \begin{pmatrix} f(x_n^1) \\ f(x_n^2) \\ \vdots \\ f(x_n^N) \end{pmatrix} \quad (6)$$

and  $A$  is a constant coupling matrix, with the constraints that none of the elements of  $A$  are negative and that the sum of each row adds to unity. For  $N$  coupled maps, instead of one Lyapunov exponent we have a spectrum of  $N$  of them and the evaluation of this spectrum is a generalisation of the one dimensional procedure defined by (3). Instead of

the log of a single derivative averaged over a large number of iterations we have the log of the eigenvalues of the product of the Jacobians of the system averaged over a large number of iterations.

Given the Jacobian of the CML,

$$J(\underline{x}_n) = \begin{bmatrix} a_{11}f'(x_n^1) & a_{12}f'(x_n^2) & \dots & \dots & a_{1N}f'(x_n^N) \\ a_{21}f'(x_n^1) & a_{22}f'(x_n^2) & \dots & \dots & a_{2N}f'(x_n^N) \\ \vdots & \vdots & & & \vdots \\ \vdots & \vdots & & & \vdots \\ a_{N1}f'(x_n^1) & a_{N2}f'(x_n^2) & \dots & \dots & a_{NN}f'(x_n^N) \end{bmatrix} \quad (7)$$

we evaluate the matrix product,

$$\tilde{J}_M = \prod_{n=1}^M J(\underline{x}_n). \quad (8)$$

If the  $N$  eigenvalues of  $\tilde{J}_M$  are  $\{\Lambda_M^1, \Lambda_M^2, \dots, \Lambda_M^N\}$  then the corresponding global Lyapunov exponents of the system are given by  $\{\lambda_g^1, \lambda_g^2, \dots, \lambda_g^N\}$  where

$$\lambda_g^j = \lim_{M \rightarrow \infty} \frac{1}{M} \ln |\Lambda_M^j| \quad (9)$$

If any of the global Lyapunov exponents is greater than zero then the CML is chaotic, and if more than one of the global Lyapunov exponents is greater than zero then the CML is said to be hyper-chaotic.

As in the one-dimensional case, the use of the sawtooth map simplifies matters dramatically with the Jacobian matrix immediately reducing to the constant matrix

$$J = \alpha A \quad (10)$$

meaning that the global Lyapunov spectrum  $\{\lambda_g^1, \lambda_g^2, \dots, \lambda_g^N\}$  is given by

$$\lambda_g^j = \ln |\alpha e_j| \quad (11)$$

where  $e_j$  is the  $j$ th eigenvalue of  $A$ . Thus, we have reduced the significant numerical problem of evaluating the Lyapunov spectrum to the problem of evaluating the eigenvalues of a single constant matrix.

## 2.1. The eigenvalues of some matrices

Before going further, we will state without proof the eigenvalues of the following  $N \times N$  matrices.

(a)

$$B = \begin{bmatrix} b & 1 & 1 & \dots & 1 \\ 1 & b & 1 & \dots & 1 \\ 1 & 1 & b & \dots & 1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & b \end{bmatrix} \quad (12)$$

has eigenvalues  $b + N - 1$  and  $b - 1$ , with the second of these being of multiplicity  $N - 1$ .

(b) The circulant matrix

$$C = \begin{bmatrix} a & b & 0 & \dots & b \\ b & a & b & \dots & 0 \\ 0 & b & a & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ b & 0 & 0 & \dots & a \end{bmatrix} \quad (13)$$

has eigenvalues,

$$e_j = a + 2b \cos\left(\frac{2\pi j}{N}\right) \quad j = 1 \dots N. \quad (14)$$

(c) Finally, the  $2N+1$  distinct eigenvalues for the adjacency matrix  $H_N$  for the  $2^N$  vertices of an  $N$  dimensional hypercube are

$$e_i = 2i - N \quad i = 0 \dots N \quad (15)$$

with the  $i$ th eigenvalue having multiplicity  $\binom{N}{i}$ .

Clearly the size of the adjacency matrix grows exponentially with the hypercube dimension, with  $H_N$  being  $2^N \times 2^N$ , but an illustrative example is the adjacency matrix  $H_2$  for the square:

$$H_2 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}. \quad (16)$$

See question 2 of classroom exercises for the evaluation of  $H_3$  and  $H_4$ .

## 2.2. Evaluating the Lyapunov Spectra for some standard CMLs

These three results allow us to evaluate global Lyapunov spectra for corresponding standard CMLs.

(a) A fully connected CML

The first is a fully connected lattice of  $N$  nodes where each node is connected to each of the  $N-1$  other nodes with a coupling constant  $\frac{\varepsilon}{N-1}$  and connected to itself with coupling constant  $1 - \varepsilon$ , giving rise to  $N \times N$  coupling matrix,

$$A_1 = \begin{bmatrix} 1 - \varepsilon & \frac{\varepsilon}{N-1} & \frac{\varepsilon}{N-1} & \dots & \frac{\varepsilon}{N-1} \\ \frac{\varepsilon}{N-1} & 1 - \varepsilon & \frac{\varepsilon}{N-1} & \dots & \frac{\varepsilon}{N-1} \\ \frac{\varepsilon}{N-1} & \frac{\varepsilon}{N-1} & 1 - \varepsilon & \dots & \frac{\varepsilon}{N-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\varepsilon}{N-1} & \frac{\varepsilon}{N-1} & \frac{\varepsilon}{N-1} & \dots & 1 - \varepsilon \end{bmatrix}. \quad (17)$$

Comparing with (12) we note that for the sawtooth map the Jacobian can be written as

$$J_1 = \alpha A_1 = \frac{\alpha \varepsilon}{N-1} B \quad (18)$$

where

$$b = (N-1) \frac{1-\varepsilon}{\varepsilon}. \quad (19)$$

This leads immediately to the eigenvalues of  $J_1$  being given by  $\alpha$  and  $\alpha \left(1 - \frac{\varepsilon N}{N-1}\right)$  with the second of these being of multiplicity  $N-1$  and hence the distinct global Lyapunov exponents are

$$\lambda_g^1 = \ln |\alpha|, \quad \lambda_g^2 = \ln \left| \alpha \left(1 - \frac{\varepsilon N}{N-1}\right) \right|. \quad (20)$$

### (b) Nearest Neighbour Coupling

The most common form of CML is where each node is symmetrically coupled to its nearest neighbours via a coupling matrix of the form:

$$A_2 = \begin{bmatrix} 1-\varepsilon & \frac{\varepsilon}{2} & 0 & \dots & \frac{\varepsilon}{2} \\ \frac{\varepsilon}{2} & 1-\varepsilon & \frac{\varepsilon}{2} & \dots & 0 \\ 0 & \frac{\varepsilon}{2} & 1-\varepsilon & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{\varepsilon}{2} & 0 & 0 & \dots & 1-\varepsilon \end{bmatrix} \quad (21)$$

where we note that the lattice forms a closed loop with the first node being connected to the last.

Comparing with (13) we note that for the sawtooth map the Jacobian of this system can be written as

$$J_2 = \alpha C \quad (22)$$

where

$$\begin{aligned} a &= \alpha(1-\varepsilon) \\ b &= \frac{\alpha \varepsilon}{2}. \end{aligned} \quad (23)$$

Using (14) this immediately leads to the global Lyapunov spectrum being given by

$$\lambda_g^i = \ln \left| \alpha \left(1 - 2\varepsilon \sin^2 \left(\frac{i\pi}{N}\right)\right) \right| \quad i = 1 \dots N. \quad (24)$$

### (c) A CML on the hypercube

For a hypercube in  $N$  dimensional space each of the  $2^N$  vertices will be attached to  $N$  others. We assume the coupling strength between the connected vertices is  $\frac{\varepsilon}{N}$  and the coupling of a vertex to itself is  $1 - \varepsilon$ . We next note that the diagonal elements of the adjacency matrix (16) are all zero, with all other elements either zero or one. Thus, the coupling matrix for the  $N$  dimensional hypercube can be written as:

$$A^N = \frac{\varepsilon}{N}H_N + (1 - \varepsilon)I_{2^N} \quad (25)$$

where  $I_n$  is the  $n \times n$  unit matrix. Finally, we note that for the sawtooth map the Jacobian of this system can be written as,

$$J^N = \alpha A^N \quad (26)$$

and using (15) give the eigenvalues to be

$$e_i = \alpha \left( 1 - \frac{2\varepsilon i}{N} \right) \quad i = 0 \dots N \quad (27)$$

with the  $i$ th eigenvalue being of multiplicity  $\binom{N}{i}$  and thus the spectrum of  $N+1$  distinct global Lyapunov exponents is

$$\lambda_g^i = \ln \left| \alpha \left( 1 - \frac{2\varepsilon i}{N} \right) \right| \quad i = 0 \dots N. \quad (28)$$

It should be noted that in each of the three cases above, (20), (24), (28), the largest Lyapunov exponent is  $\ln |\alpha|$  and thus, irrespective of the lattice structure the system can only be chaotic if  $\alpha > 1$ . Further, as  $\varepsilon \rightarrow 0$  (ie the lattice becomes unconnected) each system becomes maximally hyperchaotic as all the Lyapunov exponents approach  $\ln |\alpha|$ , each corresponding to an independent map at each lattice node.

### 3. Conclusions

Undergraduate teaching of discrete chaos frequently does not move beyond one dimension, covering systems such as the logistic, tent and Ricker maps. While two dimensional coupled maps, such as the Hénon map, kicked rotor, or ball bouncing on a vibrating plate, can be found in textbooks, (see for example Elaydi, 2008, pp. 228–237; Hilborn, 2000, p. 197; Tél & Márton, 2006, p. 174, 234) the evaluation of Lyapunov exponents for two dimensional systems is rarely discussed (with a significant, and very helpful exception being Sprott, 2003, p. 110). Further, the investigation of coupled maps of dimension greater than two is usually ignored altogether in undergraduate texts due to the difficulties in finding analytic results (e.g. fixed points and their stability) in larger systems and the need for more advanced numerical methods to evaluate global Lyapunov spectra.

However, as this paper has sought to show the use of the sawtooth map allows for the analytic evaluation of Lyapunov spectra for a range of coupled maps via the use of some standard results in matrix algebra, thus bringing the topic within reach of the undergraduate curriculum. The introduction of these ideas could of course be made at a more elementary level than described here, with the coupling matrix  $A$  being restricted to a two-



or three-dimensional system with numerical elements. But the generalisation of the system to higher dimensions is both elegant and relatively straightforward and provides the student with an introduction to some standard eigenvalue results and to the research area of coupled map lattices.

### 3.1. Classroom exercises

- (1) Plot the sawtooth map (4) for a range of values of  $\alpha$ .
- (2) It is noted in the paper that ‘a certain amount of care’ is required in using the sawtooth map. As an example, consider  $\alpha = 10$ . In such a case, starting with  $x_0$  as any  $n$  digit decimal the map would reach a fixed point of zero after  $n$  iterations, and formally the map is only chaotic if the initial condition is irrational. What would the behaviour of the map be if  $x_0 = \frac{1}{7}$ ? How would you expect the map to behave if  $\alpha = 2$ ? (Hint: think in terms of binary).
- (3) In the paper the adjacency matrix of  $H_2$  is given. What is the adjacency matrix for  $H_3$  and  $H_4$ ? For the cube  $H_3$  can be constructed visually, but  $H_4$  is less obvious. Use the fact that in general if the vertices of an  $N$  dimensional hypercube are labelled in *binary* from 0 to  $2^N - 1$ , then a pair of vertices are connected if their binary representation has a Hamming distance of one, i.e. differs by only one digit. For both  $H_3$  and  $H_4$  use a computer algebra package to verify the form of the eigenvalues.
- (4) By using repeated column subtractions, show that the eigenvalues of (12) are as stated in the paper.
- (5) The Kolmogorov-Sinai entropy density,  $K$ , of a CML is defined as the sum of the positive global Lyapunov exponents divided by the number of lattice nodes:

$$K = \frac{1}{N} \sum_{\lambda_g^i > 0} \lambda_g^i. \quad (29)$$

Show that for the fully connected lattice (17) with the sawtooth map (4),

$$K = \ln |\alpha| + \frac{N-1}{N} \ln \left| \left( 1 - \frac{\varepsilon N}{N-1} \right) \right| \quad (30)$$

provided  $\alpha > 1$  and  $\varepsilon < \frac{(\alpha-1)(N-1)}{\alpha N}$  or  $\varepsilon > \frac{(\alpha+1)(N-1)}{\alpha N}$ .

- (6) Construct a coupling matrix for a CML based on the sawtooth map corresponding to an octahedron where each vertex is connected to its four neighbours by a coupling strength  $\frac{\varepsilon}{4}$  and to itself with strength  $1 - \varepsilon$ . Given that the adjacency matrix for the octahedron has eigenvalues 0,  $-2$ , 4 (with multiplicities 3, 2, 1 respectively), what is the global Lyapunov spectrum for the system?
- (7) Write a computer program to calculate the time series for a four node CML with nearest neighbour coupling as defined by the coupling matrix (21) and the sawtooth map on each node. Investigate the behaviour of the system by plotting Feigenbaum diagrams as a function of  $\alpha$  for a range of coupling constants  $\varepsilon$ .

## Disclosure statement

No potential conflict of interest was reported by the author.

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