# Mean-Variance efficient strategies in proportional reinsurance under group correlation in a gaussian framework

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**Abstract**

The paper concerns optimal mean-variance proportional reinsurance under group correlation. In order to solve the corresponding constrained quadratic optimization problem, we make large recourse both to the smart friendly technique originally proposed by B. de Finetti in his pioneering paper [8] and to the well known Karush-Kuhn-Tucker conditions for constrained optimization. In our paper we offer closed form results and insightful considerations about our problem. In detail, we give closed form formulae to express the efficient mean-variance retention set both in the retention space and in the mean-variance one.

**Keywords.** Mean-variance efficiency; constrained quadratic optimization; proportional reinsurance; group correlation.

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## 1 Introduction

Proportional reinsurance played a relevant role in the history of modern finance. Keeping account of the high practical relevance of this type of reinsurance arrangements and in order to throw light on the characteristics of optimal proportional reinsurance strategies in a gaussian framework, B. de Finetti introduced in a pioneering paper (see de Finetti [8]) the mean-variance approach in financial decisions under uncertainty, which gained universal fame in the subsequent decade through the papers by H. Markowitz (see Markowitz [20], [21], [22]) concerning efficient diversification of investments.

The actuarial world undoubtedly recognized the importance of de Finetti's approach, but restricting the application areas to the case of no correlated risks.

The issue has been treated e.g. by Barone (see Barone [3]) who analyzed a specific point of the controversial issue raised by H. Markowitz about the applicability of de Finetti's approach to the case of correlated risks; Bühlmann (see Bühlmann [5], and more recently, Bühlmann [6]) who was well aware of de Finetti's treatment in the no correlation case; Dickson and Waters (see Dickson and Waters [9]) who, always with reference to the no correlation case, treated the issue of minimizing the ruin probability; Glineur and Walhin (see Glineur and Walhin [13]) who gave rigorous proof in terms of Kuhn-Tucker conditions of de Finetti's results under no correlation; Lampaert and Walhin (see Lampaert and Walhin [17]), who compared different type of reinsurance strategies including proportional reinsurance; Borch (see Borch [4]), who in his milestone book, paid a lot of tributes to de Finetti's ideas about insurance issues, including reinsurance arrangements and Gerber, who also in many papers (see Gerber [10], Gerber et al. [11], and jointly with Bühlmann [7]) recalled de Finetti's anticipatory thoughts in reinsurance arrangements.

The reason of this attitude is probably due to the widespread opinion that closed form formulae of the whole set of proportional efficient retentions may be obtained only under no correlation, while in the correlation case there is the need to make recourse to a sequential procedure (the reinsurance counterpart of the critical line algorithm of the portfolio problem). Surely, practical problems could be solved this way even if at a high computational burden, but losing the elegance and insightfullness coming from closed form formulae. While this statement is true in general, there is at least one specific structure of correlation which allows to recover nice results with closed form formulae. This special structure was named group correlation by de Finetti, who treated the point in his paper and gave a quick hint to some key properties of the efficient set, but without giving closed form formulae.

At our knowledge, the only subsequent treatment of this issue may be found in a paper by Gigante (see Gigante [12]). Recognizing the strict connection between the proportional reinsurance problem and the portfolio selection problem, early signalled by Pressacco (see Pressacco [26]), she was able to apply the Karush-Kuhn-Tucker conditions to the constrained quadratic optimization problem arising from the mean-variance approach to proportional reinsurance. This way she showed that, under group correlation, a fast version of the critical line algorithm could be applied to obtain rather quickly the whole mean-variance set of efficient retentions. Anyway, she also did not get closed form formulae.

In this paper, we fill this gap and we offer a complete closed form solution of the proportional reinsurance problem under group correlation, jointly with an insightful interpretation of the economic meaning of the solution set. To explain our results, we make large recourse to de Finetti's original approach, based as explained in Pressacco and Serafini (see Pressacco and Serafini [27]), on the so called key functions (or single advantage functions) technique.

In detail, we show that, in the $n-$dimensional space of retentions, the mean-variance efficient set is piecewise linear and continuous at corner points. That there is a one to one correspondence between the above efficient set and the closed interval of values of a proper global advantage parameter; it reveals to be nothing but the counterpart of the set of optimal values of the Lagrange multiplier of the expectation constraint in the quadratic optimization approach. We provide closed form formulae of the efficient retention vectors as a function of the global advantage parameter, and specify the values of the retention vector and of the advantage parameter corresponding to corner points of the efficient path, explaining the economic meaning of such corners. Finally, we show that the geometric picture of the efficient retention set in a mean-variance space is a piecewise, continuous and without kinks union of parabolas, and offer their explicit equations.

We are well aware of the fact that, for non gaussian worlds, alternative risk measures are used looking for mean-risk efficient sets in financial models and in proportional reinsurance too. Indeed, starting from the pioneer paper by Artzner et al. (see Artzner et al. [1]) on coherent risk measures, recent literature is focusing on some new approaches about risk measurement and is addressing to the topic of finding the proper corresponding measure of risk. See for instance and among others, Landsman and Sherris (see Landsman and Sherris [18]), who discuss the properties of a risk measure including risk aversion, diversification, additivity and consistency; Rockafellar et al. (see Rockafellar et al. [28]), who introduce the general deviations and the expectation bounded risk measures; Goovaerts et al. (see Goovaerts et al. [14]), who propose to use the so called consistent risk measures; even more, Miller and Ruszczynski (see Miller and Ruszczynski [24]), who provide a model for the portfolio optimization problem, using alternative risk functionals, such as semideviation, deviation from quantile and spectral risk measures; Balbás et al. (see Balbás et al. [2]) study the optimal reinsurance problem when risk is measured by general risk measures, such as deviation measures, expectation bounded measure of risk or coherent measures of risk; in particular the authors propose a unified approach which results seem to not depend on the concrete risk measure to be used. And many others investigating on the shortcomings of the well known VaR measure and proposing other alike risk measures, such as CVaR or Expected Shortfall, Compatible Conditional Value at Risk (CCVaR) and other tail risk measures.

We hope anyway that our results about group correlation in a gaussian world may be useful for the sake of comparison between results coming from the traditional mean-variance approach and those coming from modern mean-risk alternative models.

The plan of the paper is as follows: in section 2 there is a short recall of de Finetti's mean-variance approach to proportional reinsurance. In section 3 we describe the basic properties of proportional reinsurance under group correlation and the structure of the mean-variance efficient retentions. Section 4 is devoted to a geometric interpretation of the efficient path in the single group retention space through application of de Finetti's approach. In section 5 we provide closed form formulae for the efficient set in the global retention space as a function of the global advantage parameter. In section 6 we derive explicit formulae of the parabolic efficient set in mean-variance space. Conclusions follow in section 7. Examples, figures, comments and some proofs are presented in several appendixes.

## 2 A short recall of de Finetti's approach

Let us briefly recall the essentials of de Finetti's mean-variance approach to the proportional reinsurance problem. An insurance company is faced with $n$ risks (policies). The net profit of these risks is represented by a vector of random variables with expected value $E := \{m_i > 0; i = 1, \dots, n\}$ and non singular covariance matrix $\mathbf{V} := \{\sigma_{ij}; i, j = 1, \dots, n\}$.

The company has to choose a proportional reinsurance or retention strategy specified by a retention vector $\mathbf{x}$. The retention strategy is feasible if $0 \leq x_i \leq 1$ for all $i$. Applying reinsurance on original terms, a retention $\mathbf{x}$ induces a random profit with expected value $E = \mathbf{x}^\top \mathbf{m}$ and variance $V = \mathbf{x}^\top \mathbf{V}\,\mathbf{x}$.

A retention $\mathbf{x}$ is by definition Pareto efficient if, for no feasible retention $\mathbf{y}$, we have both $E(\mathbf{y}) \geq E(\mathbf{x})$ and $V(\mathbf{y}) \leq V(\mathbf{x})$, with at least one of the two inequalities expressed in a strict sense.

Let $X^*$ be the set of optimal retentions. The core of de Finetti's approach is represented by the following simple and clever ideas, where the geometric intuition plays an important role.

The set of feasible retentions is represented by points of the $n-$dimensional unit cube. The set $X^*$ is a path in this cube. It connects the natural starting point, the vertex $\mathbf{1}$ of full retention (with the largest expectation $E = \sum_i m_i$), to the opposite vertex $\mathbf{0}$ of full reinsurance (zero retention and hence minimum null variance).

De Finetti argues that the optimum path is characterized by the property that, at any point $\mathbf{x}^*$ of $X^*$, it moves in such a way to get locally the largest benefit measured by the ratio decrease of variance over decrease of expectation. To translate this idea in an operational setting de Finetti introduces the so called individual advantage functions:

$$F_i(\mathbf{x}) = \frac{1}{2} \frac{\frac{\partial V}{\partial x_i}}{\frac{\partial E}{\partial x_i}} := \sum_{j=1}^{n} \frac{\sigma_{ij}}{m_i} x_j \qquad i = 1, \dots, n \tag{1}$$

Intuitively these functions capture the individual advantage coming at $\mathbf{x}$ from a small (additional or initial) reinsurance of the $i$-th risk.

The connection between the efficient path and the set of individual advantage functions is then straightforward: at any point of $X^*$, one should move in such a way to provide additional or initial reinsurance only to the set of those risks giving the largest benefit (that is with the largest value of their advantage functions). If this set is a singleton the direction of the optimum path is obvious; otherwise, $x$ should be left in the direction preserving the equality of the advantage functions among all the best performers.

Given the form of the advantage functions, it is easily seen that this implies a movement on a segment of the cube characterized by the set of equations $F_i(\mathbf{x}) = \lambda(\mathbf{x})$ for all the current best performers. Hence $\lambda(\mathbf{x})$ plays the role of a global advantage parameter. Precisely, global advantage here means the largest advantage which may be obtained moving from $\mathbf{x}$ toward the best direction. And we continue on this segment until the individual advantage function of another risk matches the current decreasing value of the global advantage parameter, thus becoming a member of the best performers set. Accordingly, at this point, the direction of the efficient path is changed as it is defined by a new set of equations $F_i(\mathbf{x}) = \lambda(\mathbf{x})$, with the addition of the equation for the newcomer.

A repeated sequential application of this matching logic defines the whole efficient set. De Finetti (see de Finetti [8]) offered closed form formulae in case of no correlation, and a largely informal sketch of the sequential procedure in case of correlated risks[1]. Moreover, he gave some insights about the handling of a particular type of correlation structure, named group correlation.

## 3 Proportional reinsurance under group correlation: basics and the structure of mean-variance efficient retentions

According to de Finetti's paper, group correlation means that the set of insured risks, may be partitioned into a number $g$ of groups characterized by the following properties.

Let $q$ be the label of a given group, $q = 1, \ldots, g$. The correlation between each couple of policies belonging to the same group (shortly the *correlation within*) is a fixed group specific positive constant $\rho_q > 0$, whereas the correlation between couples of risks of different groups (shortly the *correlation between*) is identically zero. Furthermore, the ratio between standard deviation and expected return of the gain of any policy is a group specific constant ratio $\frac{\sigma_{i,q}}{m_{i,q}} = a_q$ for any risk $i$ of the group $q$. In technical terms, this amounts to say that the insurance premiums are charged by a safety loading based on a standard deviation principle, with a loading coefficient which is the same for all risks within the same group.

To avoid misunderstanding let us introduce and explain some notations.

- $x_{i,q} \quad \forall i = 1, \ldots, n_q$ is the individual retention of the $i-$th risk in group $q$
- $\mathbf{x}_q \quad \forall q = 1, \ldots, g$ is the vector of retentions of the group $q$. In particular $\mathbf{1}_q$ (respectively $\mathbf{0}_q$) is the vector of full (zero) retentions for all risks of the group $q$
- $\mathbf{x}$ is the vector of retention of the risks of all groups
- $F_{i,q}(\mathbf{x})$ is the value at $\mathbf{x}$ of the individual advantage function of risk $i$ in group $q$

It is straightforward to check that, under group correlation, the individual advantage function of the $i$-th risk of the group $q$ becomes

$$F_{i,q}(\mathbf{x}) = a_q \cdot \left( x_{i,q}\sigma_{i,q} + \rho_q \cdot \sum_{j \neq i} x_{j,q}\sigma_{j,q} \right) \tag{2}$$

[1] As pointed out by Markowitz (see Markowitz [23]) in his recent critical review, de Finetti overlooked the (say non-regular) case in which at some step it is not possible to find a matching point along an optimum segment before one of the currently active variables reaches a boundary value (0 or 1). The proper adjustment of de Finetti's procedure to the non regular case has been discussed by Pressacco-Serafini (see Pressacco and Serafini [27], par. 4 and 5.)

which may rewritten as follows:

$$F_{i,q}(\mathbf{x}) = a_q \cdot \left[x_{i,q}\sigma_{i,q} \cdot (1-\rho_q) + \rho_q \cdot \sum_{j=1}^{n_q} x_{j,q}\sigma_{j,q}\right] \tag{3}$$

Note that, owing to zero correlation between groups, $F_{i,q}(\mathbf{x}) = F_{i,q}(\mathbf{x}_q)$.

In what follows, we will find convenient to order the risks within each group coherently with their standard deviation in decreasing order, that is use the labeling induced by $\sigma_{1,q} > \sigma_{2,q} > \ldots > \sigma_{n_q,q}$. Note that multiplying by $a_q$ for any risk does not alter the ordering that is then the same induced by the ratio variance over expectation, which played a key role in the no correlation case (see Pressacco-Serafini [27], par. 3).

In the same line of reasoning, an ordering among groups will be introduced according to the values at $\mathbf{x} = \mathbf{1}$ (or $\mathbf{x}_q = \mathbf{1}_q$) of the individual advantage functions of the first risk of each group. Coherently, we will label groups so as:

$$F_{1,1}(\mathbf{1}_1) > F_{1,2}(\mathbf{1}_2) > \ldots > F_{1,g}(\mathbf{1}_g)$$

After that, let us resume here some fundamental results about the structure of mean-variance efficient retentions (under group correlation) which will be proved in Appendix II.1 and discussed in the next paragraphs.

**Theorem 1.** *A necessary condition for $\mathbf{x}_q$ being the group $q$ subset of retention of a global efficient retention $\mathbf{x}$ is that $\mathbf{x}_q$ satisfies:*

$$0 \leq x_{1,q} \leq 1 \tag{4}$$

$$x_{i,q} = \min\left[\frac{x_{1,q} \cdot \sigma_{1,q}}{\sigma_{i,q}}; 1\right] \quad \forall i = 2, \ldots, n_q \tag{5}$$

Note that once $x_{1,q}$ has been chosen, the candidate $\mathbf{x}_q$, member of an efficient global retention $\mathbf{x}$, is univocally defined; we may look at $x_{1,q}$ as the "driver" of corresponding retention vector $\mathbf{x}_q$.

Let us now take $\mathbf{x}_1$ as the group 1 retentions generated by a given driver $x_{1,1}$ and $\mathbf{x}_q$ as the group $q$ retentions generated by a driver $x_{1,q}$, and consider the following equation in the unknown $x_{1,q}$:

$$F_{1,1}(\mathbf{x}_1) = F_{1,q}(\mathbf{x}_q) \tag{6}$$

that is in the extended form:

$$a_1 \cdot \left[x_{1,1}\sigma_{1,1} + \rho_1 \cdot \sum_{i=2}^{n_1} \sigma_{i,1} \cdot \min\left(\frac{x_{1,1} \cdot \sigma_{1,1}}{\sigma_{i,1}}; 1\right)\right] = a_q \cdot \left[x_{1,q}\sigma_{1,q} + \rho_q \cdot \sum_{i=2}^{n_q} \sigma_{i,q} \cdot \min\left(\frac{x_{1,q} \cdot \sigma_{1,q}}{\sigma_{i,q}}; 1\right)\right] \tag{7}$$

Now the following Lemma holds:

**Lemma 1.** *Given a feasible $x_{1,1}$, equation (7) has exactly one feasible solution $\widehat{x}_{1,q}$ iff $F_{1,q}(\mathbf{1}_q) \geq F_{1,1}(\mathbf{x}_1)$; otherwise ($F_{1,q}(\mathbf{1}_q) < F_{1,1}(\mathbf{x}_1)$) there are no feasible solutions.*

Now, we use the Lemma to obtain a theorem which gives the necessary and sufficient conditions to be satisfied by a set of drivers $x_{1,1}, x_{1,2}, \ldots, x_{1,g}$ such that the retention $\mathbf{x}$ generated by them is efficient.

**Theorem 2.** *A set $x_{1,q}$, $\forall q = 1, \ldots, g$, of drivers generates an efficient retention $\mathbf{x}$ (coherently with Theorem 1 rules) iff:*

$$0 \leq x_{1,1} \leq 1 \tag{8}$$

$$x_{1,q} = \begin{cases} \widehat{x}_{1,q} \quad if \quad F_{1,q}(\mathbf{1}_q) \geq F_{1,1}(\mathbf{x}_1) \\ 1 \quad if \quad F_{1,q}(\mathbf{1}_q) < F_{1,1}(\mathbf{x}_1) \end{cases} \tag{9}$$

A comment to this result is now in order: the whole efficient set is in biunivocal correspondence with the closed interval [0,1] of feasible retentions of the first risk of the first group. Given the choice of $x_{1,1}$, all other components of the corresponding efficient retention vector $\mathbf{x}$ are obliged; precisely, they come from the application of the rule given by Theorem 2, as for the set of drivers $x_{1,q}$ and, given this way such a set, by the application of Theorem 1 for all the other components.

# 4 A geometric look at the efficient path in the single group space of retentions

To understand the geometric behavior of the efficient path it is convenient to introduce the following new notation:

$$\mathbf{x}_q^h = \left[x_{1,q}^h, x_{2,q}^h, \ldots, x_{n_q,q}^h\right] \tag{10}$$

for any $h = 1, \ldots, n_q$. It denotes a set of efficient retentions of the group $q$ generated (according to the Theorem 1) by the driver $x_{1,q}^h = \frac{\sigma_{h,q}}{\sigma_{1,q}}$. Then, it is:

$$\mathbf{x}_q^h = \left[\frac{\sigma_{h,q}}{\sigma_{1,q}}, \frac{\sigma_{h,q}}{\sigma_{2,q}}, \frac{\sigma_{h,q}}{\sigma_{h-1,q}}, 1, \ldots, 1\right] \tag{11}$$

In the $n_q-$dimension subspace of retentions of the group $q$, this implies that the set of efficient retentions is a piecewise linear path with corner points given by the formula (11).

For the sake of completeness, it is convenient to denote by $\mathbf{x}_q^{n_q+1} = \mathbf{0}$ the last corner (the end point) of the efficient path.

Let us now explain the economic meaning of the corner points. At the $h-$th corner point the policy labeled $h$ begins to be reinsured. Why? According to the enlightening de Finetti's logic, because exactly at that point its individual advantage function matches from below the value of the global advantage parameter, which in turn is the common value of the individual advantage functions of all the $(h-1)$ risks previously sharing the largest value among all the individual advantage functions. All the other policies, those labeled from $(h+1)$ to $n_q$, have smaller values of their individual advantage functions and are fully retained.

What happens in each segment joining two adjacent corner points? There, the set of the partially reinsured policies does not change. Simply their retentions regularly decrease in a way preserving the equality of the product $x_{i,q} \cdot \sigma_{i,q}$, which in turn implies preserving the equality of the individual advantage functions of this set of risks.

With only one group of policies this would be enough, for a complete understanding of the geometric behavior of the set of efficient retentions. In the next paragraph, we will manage the case with more than one group at hand. In order to do that, it is convenient to introduce the following notation:

$$\lambda_{h,q} = F_{h,q}(\mathbf{x}_q^h)$$

Indeed $\mathbf{x}_q^h$ is, by definition, the corner point at which the $h-$th risk of the group $q$ enters in reinsurance. There, its individual advantage function $F_{h,q}(\mathbf{x}_q^h)$ matches the common value of individual advantage functions of all other risks (of the group) already reinsured. Hence, $\lambda_{h,q} = F_{h,q}(\mathbf{x}_q^h)$ denotes the value of the global advantage function at that corner.

# 5 The efficient set in the global retention space

The starting point of this section is the set of $g$ sequences of the $\lambda_{h,q}$ values, which, we repeat, are the values of the global advantage parameter at corner point $h$ of the group $q$. By $(\lambda_{h,q}, \lambda_{h-1,q}]$ we denote the interval of values of the global advantage parameter between two adjacent corner points of the group $q$, where exactly $(h-1)$ policies of the group are currently reinsured. Obviously for $h = 1$, $\lambda_{h-1,q} = \lambda_{0,q}$ does not have a precise meaning, but it would be convenient to put $\lambda_0, q = +\infty$ for any $q$. Let then put

$$\lambda_q = \lambda_{1,q} = \max_{i \in q} F_{i,q}(\mathbf{1}_q) = F_{1,q}(\mathbf{1}_q) \tag{12}$$

and

$$\overline{\lambda} = \max_q \lambda_q = F_{1,1}(\mathbf{1}_1) = F_{1,1}(\mathbf{1}) \tag{13}$$

which is, keeping account of the ordering of risks and groups, the largest value of the global advantage parameter at the starting point of the efficient path.

For any choice of $0 \leq \lambda \leq \overline{\lambda}$, an efficient global retention $\mathbf{x}(\lambda)$ is derived according to the following rule.

Choose a group, say $q$. Localize the interval $(\lambda_{h,q}, \lambda_{h-1,q}]$ to which $\lambda$ belongs; this defines indirectly $h(q, \lambda)$, which means that there, for that $\lambda$, exactly $(h-1)$ policies of the group are reinsured. If $h(q, \lambda) = 1$, then $x_{i,q} = 1$ for any $i$. Otherwise, if $h \geq 2$, that is $h - 1 \geq 1$, and keeping account of the general rules provided in the previous paragraphs, the following equation must hold:

$$\begin{aligned}\lambda &= a_q \cdot \Big[x_{1,q}\sigma_{1,q} \cdot (1-\rho_q) + \rho_q \cdot (h-1) \cdot x_{1,q}\sigma_{1,q} + \rho_q \cdot \sum_{j=h}^{n_q} \sigma_{j,q}\Big] \\ &= a_q \cdot \Big[x_{1,q}\sigma_{1,q} \cdot (1 + \rho_q \cdot (h-2)) + \rho_q \cdot \sum_{j=h}^{n_q} \sigma_{j,q}\Big]\end{aligned} \tag{14}$$

with $h = h(q, \lambda)$.

Solving for $x_{1,q}$, it turns out[2]:

$$x_{1,q}(\lambda) = \frac{\lambda}{a_q \cdot \sigma_{1,q} \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{\sigma_{1,q} \cdot [1 + \rho_q \cdot (h-2)]} \tag{15}$$

as for the other active policies, due to the R7 in Appendix II.1, we have:

$$x_{i,q}(\lambda) = \frac{\lambda}{a_q \cdot \sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{\sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]} \tag{16}$$

Finally, repeat the procedure for any $q$.

We underline that the above expressions hold, for the group $q$, for any value of $\lambda \in (\lambda_{h,q}, \lambda_{h-1,q}]$.

It is worth noting that such expressions are linear equations: more precisely, in any interval between two critical values (of course of the same group), each retention quota $x_{i,q}$ shows a linear behavior whose slope depends on the interval of $\lambda$ (that is as the number of active policies in the interval changes). In other terms, the function $x(\lambda)$ may be written in the form:

$$x_{i,q}(\lambda) = A_{i,q} \cdot \lambda + B_{i,q} \qquad i = 1, \dots, (h-1) \tag{17}$$

with obviously:

$$A_{i,q} = \frac{1}{a_q \cdot \sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]}$$

$$B_{i,q} = -\rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{\sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]}$$

While, for $i = h, \dots, n_q$, $x_{i,q}(\lambda) = 1$. Thus, for any risk of the group $q$, the picture of $x_{i,q}(\lambda)$ is piecewise linear, starting from the origin (zero retention point) and showing more and more increasing slopes up to the level $x_{i,q} = 1$ (full retention point). It would be straightforward to prove (see Appendix II.2) that it is continuous also at the critical point $\lambda_h$. As an example, look at the picture 1 in Appendix I.

Let $\{I_q\}$ denote the set of $\lambda$ intervals of the group $q$. The intersection $\bigcap_q \{I_q\}$ of such a family of intervals gives rise to a new set of subintervals. For each one of these subintervals all the retentions

[2]Note that, this way, the set of drivers are explicitly obtained as a function of $\lambda$, whereas in par. 3 they were obtained solving an equation as a function of $x_{1,1}$. In our opinion, the approach of this paragraph is convenient from a computational point of view, while the one used in par. 3 is more insightful from the theoretical side.

are, at the same time, fully linear in $\lambda$, that is for any value of a subinterval, the retentions (of the reinsured risks) of all groups may be described by formulas (16). This property would be decisive to describe the behavior of the efficient set in the mean-variance space.

# 6 The efficient set in the mean-variance space

## 6.1 Global expected return of the portfolio

In the modern portfolio theory, when expected return is plotted against variance the efficient frontier is an union of parabolas. Points along this line represent optimal portfolios for which there is lowest risk for any given level of expected return.

In our context, how is the efficient frontier made? Intuitively, it will be still a sequence of pieces of parabolas connecting each other with continuity and derivability, as we will show hereinafter. Let us demonstrate this claim.

Let us consider the overall expected return of the generic group $q$, in each interval where there are $(h-1)$ active policies of the group; it is a function of $\lambda$:

$$E_q(\lambda) = \sum_{i=1}^{h-1} x_{i,q}(\lambda) \cdot m_{i,q} + \sum_{i=h}^{n_q} m_{i,q} \tag{18}$$

with $x_{i,q}(\lambda)$ given by the (16). Taking into account that $\frac{m_{i,q}}{\sigma_{i,q}} = \frac{1}{a_q}$, we obtain:

$$\begin{aligned} E_q(\lambda) &= \sum_{i=1}^{h-1} \left[ \frac{\lambda}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-2)]} \right] + \sum_{i=h}^{n_q} m_{i,q} \\ &= \lambda \cdot \left[ \frac{(h-1)}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} \right] - \rho_q \cdot \frac{(h-1) \cdot \sum_{j=h}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-2)]} + \sum_{i=h}^{n_q} m_{i,q} \end{aligned} \tag{19}$$

which again could be well-rewritten in a more expressive way:

$$E_q(\lambda) = \lambda \cdot \alpha_q + \beta_q \tag{20}$$

as a linear function of the advantage parameter $\lambda$. Note that both:

$$\alpha_q = \frac{(h-1)}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} \tag{21}$$

$$\beta_q = -\frac{\rho_q \cdot (h-1) \cdot \sum_{j=h}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-2)]} + \sum_{i=h}^{n_q} m_{i,q} \tag{22}$$

are group and interval specific coefficients, through the dependence on the $(a_q, \rho_q)$ parameters couple and, of course, in the specific interval, also on $h_q$. Not surprisingly, the slope $\alpha_q$ is positive for any $h = 2, \ldots, n_q + 1$.

Now, in order to consider the global expected return of the entire portfolio, $E(\lambda)$, coming from all groups $g$, we need the remark at the end of the previous section.

Precisely, on a specific subinterval of $\lambda$, where each group has a group specific number $(h-1)_q$ of active policies, the global expected return is given simply by the sum of the expectations of each of them.

In other terms, keeping in consideration all groups, there is still a subinterval specific linear relation between the global expectation and the value of the advantage parameter $\lambda$:

$$E(\lambda) = \lambda \cdot \alpha + \beta \tag{23}$$

where, of course:

$$\alpha = \sum_{q=1}^{g} \alpha_q$$

$$\beta = \sum_{q=1}^{g} \beta_q$$

We stress again that the coefficients $\alpha$ and $\beta$ are interval specific. Precisely, it is possibile to show that the slope is increasing with $h$ (see Appendix II.3), while the role of $\beta$ is to grant continuity at the connection points (see Appendix II.4). See also graph 4 in Appendix I.

From (23) we get also the inverse, still interval specific, relation:

$$\lambda = \frac{E-\beta}{\alpha} \tag{24}$$

which will be helpful hereinafter.

## 6.2 Global variance of the portfolio

With regard to the variance of the generic group $q$, we should have to deal with a potentially huge covariance matrix, but exploiting the relation $x_{i,q}\sigma_{i,q} = c(\lambda) = c$, $\forall i = 1, \ldots, (h-1)$ and $x_{i,q} = 1$, $\forall i = h, \ldots, n_q$, it is tedious but straightforward to see that:

$$V_q = \sum_{i=1}^{h-1} c^2 + c^2 \cdot \rho_q \cdot \sum_{i=1}^{h-1} \sum_{j \neq i} 1 + \sum_{i=h}^{n_q} \sigma_{i,q}^2 + 2 \cdot \rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{j=1}^{h-1} c \cdot \rho_q \sum_{i=h}^{n_q} \sigma_{i,q} + \sum_{i=1}^{h-1} c \cdot \rho_q \sum_{j=h}^{n_q} \sigma_{j,q} \tag{25}$$

Then, manipulating the above expression and writing the expression as a second order polynomial in the variable $c(\lambda)$, we obtain:

$$V_q = c^2 \cdot [(h-1) + \rho_q \cdot (h-1) \cdot (h-2)] + c \cdot 2(h-1) \cdot \rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{i=h}^{n_q} \sigma_{i,q}^2 \tag{26}$$

which gives the overall variance, of the $q$−th group, in the subinterval where $(h-1)$ policies are active. Now, by recalling that there (see also (16)):

$$x_{i,q}(\lambda)\sigma_{i,q} = c(\lambda) = \frac{\lambda}{a_q \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{[1 + \rho_q \cdot (h-2)]} \tag{27}$$

we are able to rewrite the formulation of the variance emphasizing the dependence from the advantage parameter $\lambda$. In particular, we may write:

$$V_q(\lambda) = \alpha_q^{'} \cdot \lambda^2 + \beta_q^{'} \cdot \lambda + \gamma_q \tag{28}$$

where, specifically (see Appendix II.7):

$$\alpha_q^{'} = \frac{(h-1)}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} \tag{29}$$

$$\beta_q^{'} = 0 \tag{30}$$

$$\gamma_q = -\frac{\rho_q^2 \cdot (h-1) \cdot \left(\sum_{i=h}^{n_q} \sigma_{i,q}\right)^2}{1 + \rho_q \cdot (h-2)} + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{i=h}^{n_q} \sigma_{i,q}^2 \tag{31}$$

noting also that $\alpha_q^{'} = \alpha_q$ given in the (21).

Now, let us evaluate the global variance of the entire portfolio, $V(\lambda)$, coming from all groups $g$. As before, in any subinterval of $\lambda$, where each group has a group specific number $(h-1)_q$ of active policies, the global variance is given simply by the sum of the variances of each of them, owing to the null correlation between groups.

$$V(\lambda) = \lambda^2 \cdot \sum_{q=1}^{g} \alpha_q + \sum_{q=1}^{g} \gamma_q = \lambda^2 \cdot \alpha + \gamma \tag{32}$$

We underline that the graph of the variance is an union of parabolas, whose symmetry axis is, for each of them, the vertical axis ($\lambda = 0$) with concavity coefficient positive and increasing with $h$ (recall that such a coefficient is also the slope of the straight line of the corresponding expectation), while the role of $\gamma$ is to grant continuity at the connection points (see Appendix II.5). See also graphs from 5 to 7 in Appendix I.

Finally, according to the (24), it is possible to write the relation which links, in any subinterval, the global expected return and the global variance, through a function $V(E)$. It is:

$$V(E) = \frac{(E-\beta)^2}{\alpha^2} \cdot \alpha + \gamma = \frac{(E-\beta)^2}{\alpha} + \gamma \tag{33}$$

that is, in every single subinterval, we are able to write $V$ as a quadratic function of $E$. Overall the intervals of the efficient set the graph of $V(E)$ is therefore an union of parabolas. It would be tedious but straightforward to prove (see Appendix II.6) that also in the connection points $V(E)$ is a continuous function. It turns out also to be differentiable (without kinks) as intuitive considering $\frac{1}{2}\frac{\partial V}{\partial E} = \lambda$ and recalling the continuity of $\lambda$ along the optimum path. For a formal proof see Appendix II.6. See also graphs 8 and 9 in Appendix I.

## 7 Conclusions

We analyzed in this paper the problem of finding an elegant mean-variance efficient solution to the so called group correlation problem in proportional reinsurance. This problem was originally raised by de Finetti in his path-breaking paper (see de Finetti [8]); there, he gave a quick hint to some key properties of the solution but without giving closed form formulae.

At our knowledge, the only subsequent treatment of this issue may be found in a paper by Gigante (see Gigante [12]); she was able to apply the Karush-Kuhn-Tucker conditions to the constrained quadratic optimization problem arising from the mean-variance approach to proportional reinsurance. This way she showed that, under group correlation, a fast version of the critical line algorithm could be applied to obtain rather quickly the whole mean-variance set of efficient retentions. Anyway, she did not get closed form formulae. In this paper we fill this gap and offer a complete closed form solution of the proportional reinsurance problem under group correlation, jointly with an insightful interpretation of the economic meaning of the solution set. To explain our results we make large recourse to de Finetti's original approach, based on the so called advantage functions technique.

More precisely, we show that there is a one-to-one correspondence between the set of efficient mean-variance retentions and the closed interval of values of a proper "advantage" parameter; we offer closed form formulae of these vectors as a function of the advantage parameter; we show that the geometric picture of the efficient retention set in a mean-variance space is a, piecewise continuous and without kinks, union of parabolas, whose equations are explicitly derived.

We hope anyway that our results about group correlation in a gaussian world may be useful for the sake of comparison between results coming from the traditional mean-variance approach and those coming from modern mean-risk alternative models.

## Appendix I: examples and comments

In this Appendix we will present some examples and results concerning a sample of hypothetical insurance policies. We highlight here that our sample is just a toy model, without any attempt to be realistic, but functional to the purpose of checking the working of the model.

In detail, we consider 5 groups characterized by the couple of constants $(a_q, \rho_q)$, each one with 10 policies, ordered by the ranking of corresponding expected returns, $m_i$, which is the same of the ranking of the standard deviations. Let us summarize then the entire set in the following table:

Table 1: The sample.

Each group has a constant positive group specific correlation coefficient, $\rho_q$ and a constant group specific ratio $a_q$. Note also that the first risk of each group is, by definition, the more risky one being associated with the highest standard deviation.

| $q$ | **1** | **2** | **3** | **4** | **5** |
|---|---|---|---|---|---|
| $\rho_q$ | 0.05 | 0.10 | 0.15 | 0.20 | 0.25 |
| $a_q$ | 4.013 | 3.156 | 4.891 | 2.955 | 2.650 |
| $i$ | *Expected Return* $m_{i,q}$ | | | | |
| **1** | 55 | 86 | 34 | 205 | 101 |
| **2** | 49 | 75 | 29 | 180 | 91 |
| **3** | 35 | 69 | 24 | 155 | 79 |
| **4** | 31 | 58 | 20 | 147 | 62 |
| **5** | 30 | 55 | 19 | 137 | 59 |
| **6** | 28 | 46 | 17 | 121 | 50 |
| **7** | 23 | 39 | 14 | 101 | 42 |
| **8** | 20 | 32 | 11 | 93 | 39 |
| **9** | 15 | 27 | 9 | 80 | 26 |
| **10** | 13 | 20 | 7 | 67 | 17 |

The first step of the procedure[3] is to calculate the set of $\lambda_{h,q}$ values for each $q-$th group.

In the following table we show the corresponding results. As might be expected, such values are within any group decreasingly ordered because of the lowering of the advantage arising from subsequent entries in reinsurance.

[3]The procedure has been implemented in VBA code.

Table 2: Advantage function values of the whole set.

| | $\lambda_{h,1}$ | $\lambda_{h,2}$ | $\lambda_{h,3}$ | $\lambda_{h,4}$ | $\lambda_{h,5}$ |
|---|---|---|---|---|---|
| $\lambda_{1,q}$ | 1082.404 | 1275.772 | 1351.754 | 3677.677 | 1525.842 |
| $\lambda_{2,q}$ | 985.761 | 1166.221 | 1232.130 | 3459.392 | 1455.607 |
| $\lambda_{3,q}$ | 748.985 | 1100.491 | 1094.562 | 3197.449 | 1350.256 |
| $\lambda_{4,q}$ | 678.113 | 969.029 | 970.153 | 3099.657 | 1171.158 |
| $\lambda_{5,q}$ | 659.590 | 930.188 | 935.462 | 2959.954 | 1134.285 |
| $\lambda_{6,q}$ | 620.933 | 804.703 | 858.902 | 2708.489 | 1007.863 |
| $\lambda_{7,q}$ | 520.265 | 700.131 | 733.297 | 2359.232 | 881.441 |
| $\lambda_{8,q}$ | 457.445 | 588.588 | 596.925 | 2205.559 | 828.765 |
| $\lambda_{9,q}$ | 348.721 | 503.935 | 498.833 | 1933.138 | 577.678 |
| $\lambda_{10,q}$ | 303.621 | 378.449 | 393.564 | 1638.016 | 388.045 |

For instance, $\lambda_{2,1} = 985.761$ gives the value of the advantage parameter at the point where the 2−nd risk, of the first group, begins to be reinsured. At that point, we know that the risk 1 is already entered in reinsurance, whereas risks from 3 to 10 are still fully retained.

To each $\lambda_{h,q}$ we are able to associate the corresponding vector $\mathbf{x}_q^h(\lambda_{h,q})$ of efficient retentions of the group. As an example, table 3 shows the collection of efficient retentions at corner points for the group number one.

Table 3: Set of efficient retentions of group 1.

| $h$ | $x(1,1)$ | $x(2,1)$ | $x(3,1)$ | $x(4,1)$ | $x(5,1)$ | $x(6,1)$ | $x(7,1)$ | $x(8,1)$ | $x(9,1)$ | $x(10,1)$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | 0.891 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 3 | 0.636 | 0.714 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 4 | 0.564 | 0.633 | 0.886 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 5 | 0.545 | 0.612 | 0.857 | 0.968 | 1 | 1 | 1 | 1 | 1 | 1 |
| 6 | 0.509 | 0.571 | 0.800 | 0.903 | 0.933 | 1 | 1 | 1 | 1 | 1 |
| 7 | 0.418 | 0.469 | 0.657 | 0.742 | 0.767 | 0.821 | 1 | 1 | 1 | 1 |
| 8 | 0.364 | 0.408 | 0.571 | 0.645 | 0.667 | 0.714 | 0.870 | 1 | 1 | 1 |
| 9 | 0.273 | 0.306 | 0.429 | 0.484 | 0.500 | 0.536 | 0.652 | 0.750 | 1 | 1 |
| 10 | 0.236 | 0.265 | 0.371 | 0.419 | 0.433 | 0.464 | 0.565 | 0.650 | 0.867 | 1 |

More generally, through the application of rule expressed in (16), we are also able to give the vector of efficient retentions of all risks of the group for any interval $(\lambda_{h,q}, \lambda_{h-1,q}]$. Note that this describes $x_{i,q}(\lambda_q)$ as a piecewise linear relation for any given $i$. It may be easily seen, as shown in Appendix II.2, that the relation is continuous also at the points $\lambda_{h,q}$ where there is a change in the slope. As $h$ grows, that is going leftwards in the horizontal axis, the slope is decreasing coherently with what may be deduced from formula (16).

For the sake of comparison, we present the pictures of the efficient retentions of groups 1 (low correlation) and 5 (highest correlation).

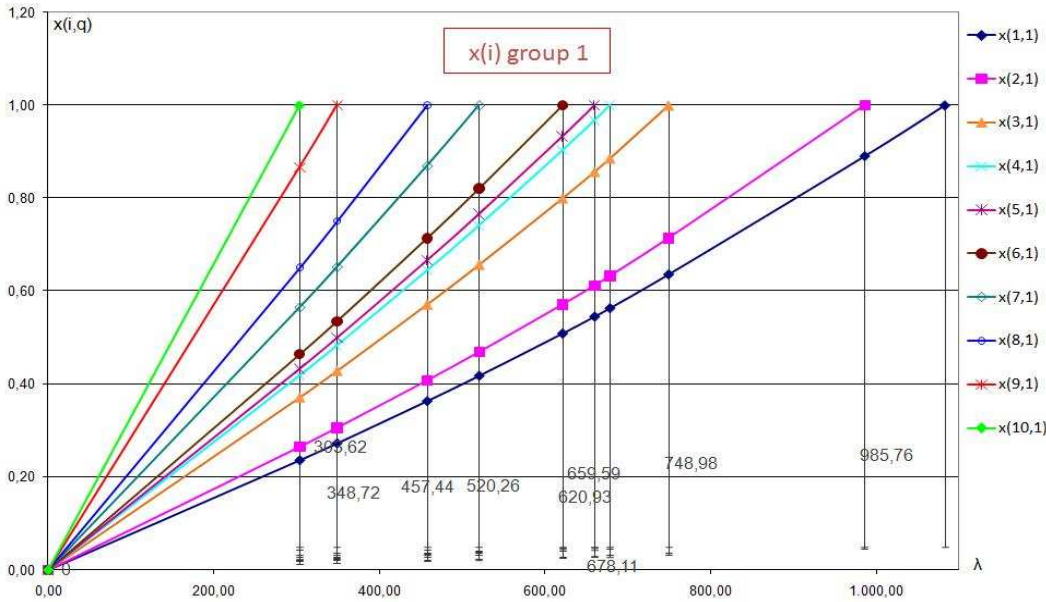


Figure 1: Plot of the relation between $\lambda$ (horizontal axis) and $x$ (vertical axis), group 1.

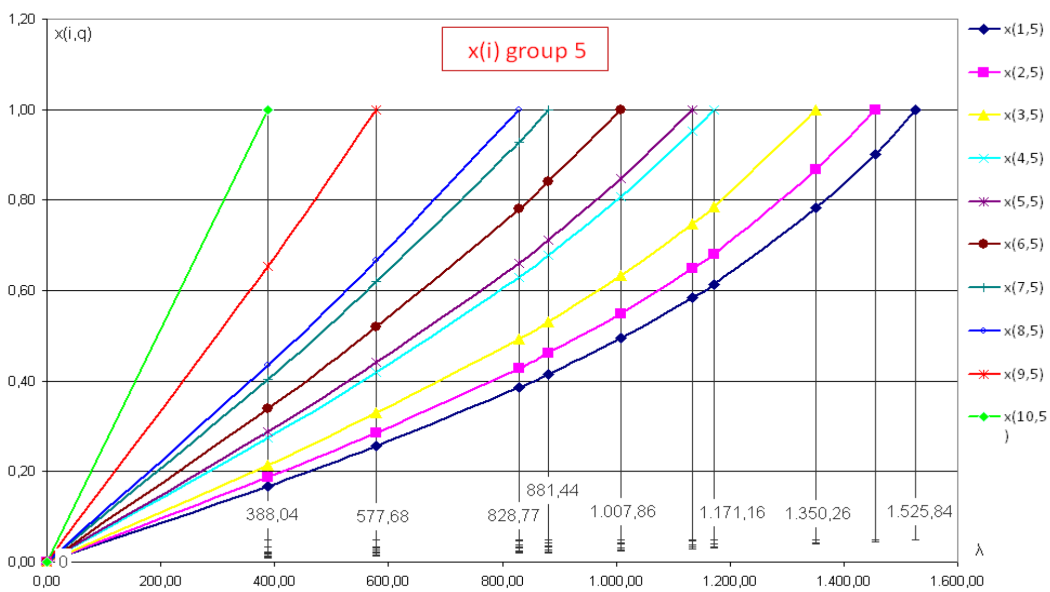


Figure 2: Plot of the relation between $\lambda$ (horizontal axis) and $x$ (vertical axis), group 5.

It is interesting to note that, for low correlation, the graph of each risk is almost perfectly linear, approximating the well-known result of no correlation case. On the contrary, for high enough value of correlation, the slope of the different segments exhibits a clear variation. Note also that, for each couple of policies actively reinsured, the ratio between their retentions is constant at any point of the efficient set, as the relation between the two retentions is described by an homogeneous linear relation.

In a geometric interpretation, the path of the efficient set, in any facet of the $n-$ dimensional group subspace of feasible retentions, points toward the null vertex of the corresponding facet.

Surprisingly, the behavior of the path of retentions, within any specific group, is exactly the same that the group would exhibit under no correlation. In other words, the union of the two fundamental conditions (of constant correlation and constant loading) generates, within each group, the same path of mean-variance efficient retentions that comes under no correlation at all. Of course, what is changed in going from no correlation to constant correlation plus constant loading is the value of the advantage function at each point of the path (this property was already noted by de Finetti in his paper).

Let us now take account of overall set of 50 policies. Consider the set of subintervals generated as the intersection of the family of the intervals for any group $q$. Under the hypothesis that all $\lambda_{h,q}$ are different, the total number of subintervals obtained in this way is 50.

As an example, in the following graph (figure 3) we put together pictures 1 and 2 of groups 1 and 5. More generally, if we put together pictures of all groups, we will observe an image of the efficient retentions of all policies as a function of the advantage parameter.

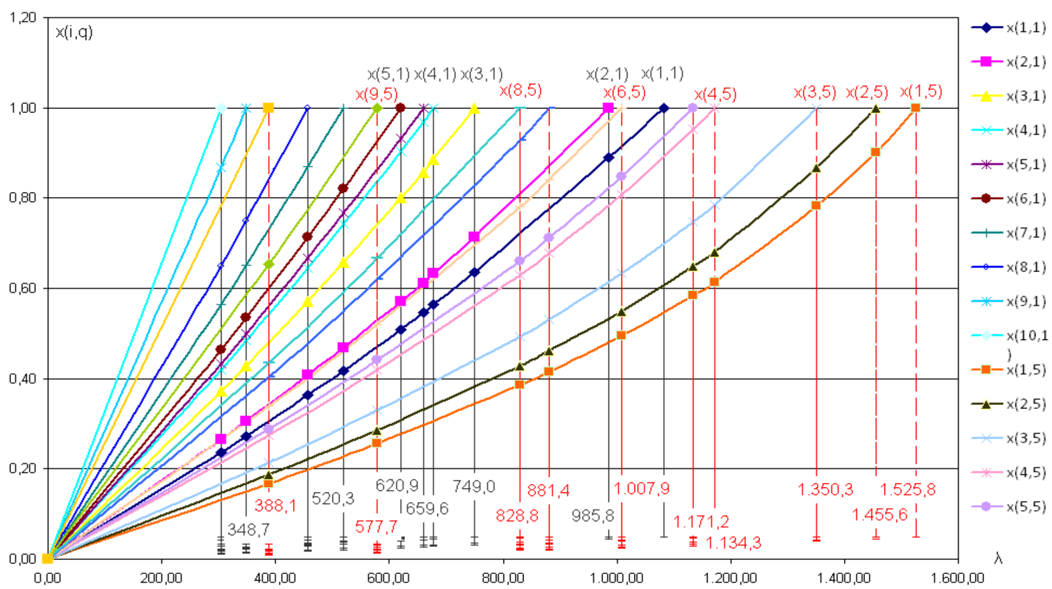


Figure 3: Plot of the relation between $\lambda$ (horizontal axis) and $x$ (vertical axis), groups 1 and 5.

The following graph describes the linear relation between expectation $E$ and the advantage parameter $\lambda$, restricted to the first three intervals of the latter.

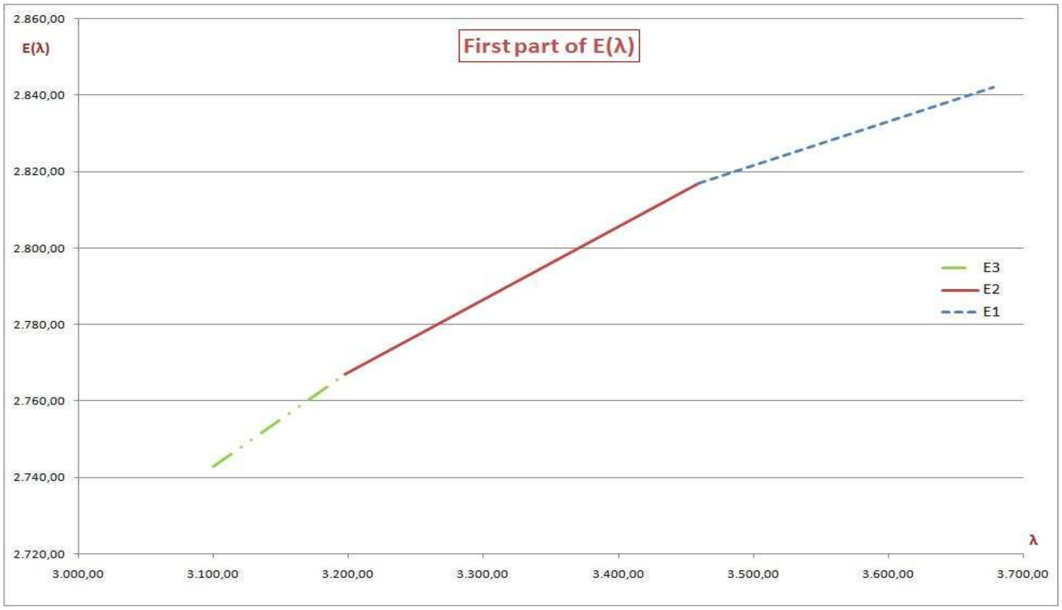


Figure 4: Behavior of $E(\lambda)$ in the first three intervals of $\lambda$.

As regard the relation between $V$ and advantage parameter $\lambda$ in the same intervals, let us have a look at the picture 5.

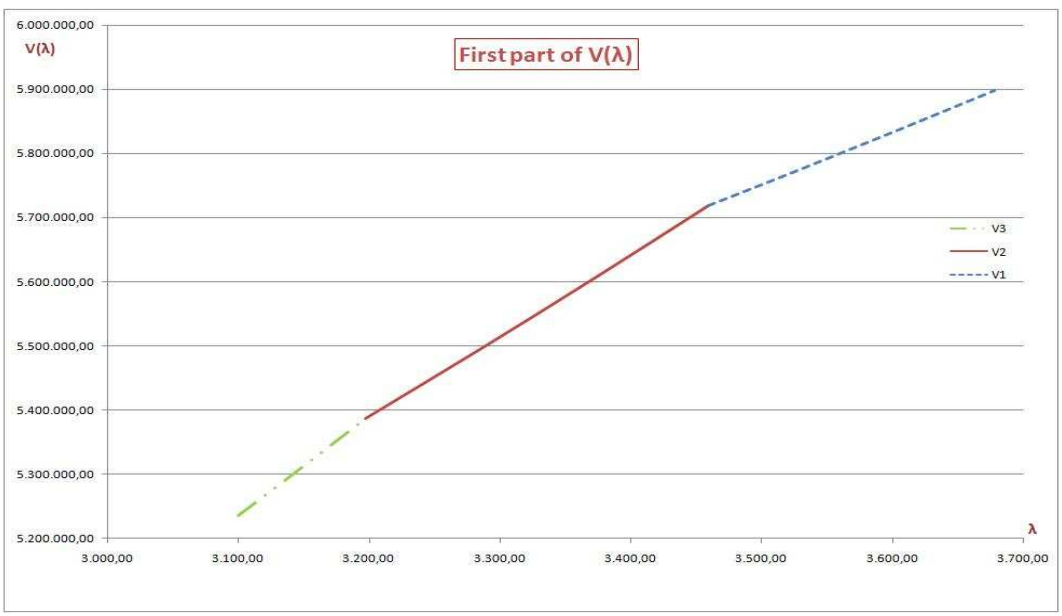


Figure 5: Behavior of $V(\lambda)$ in the first three intervals of $\lambda$.

At first glance, the parabolic behavior of $V(\lambda)$ cannot be appreciated. To this aim, see next graphs. In picture 6 we extended to the entire interval of $\lambda$ the graph of the three parabolas, describing the variance $V(\lambda)$ of the efficient set, in the first three intervals of $\lambda$.

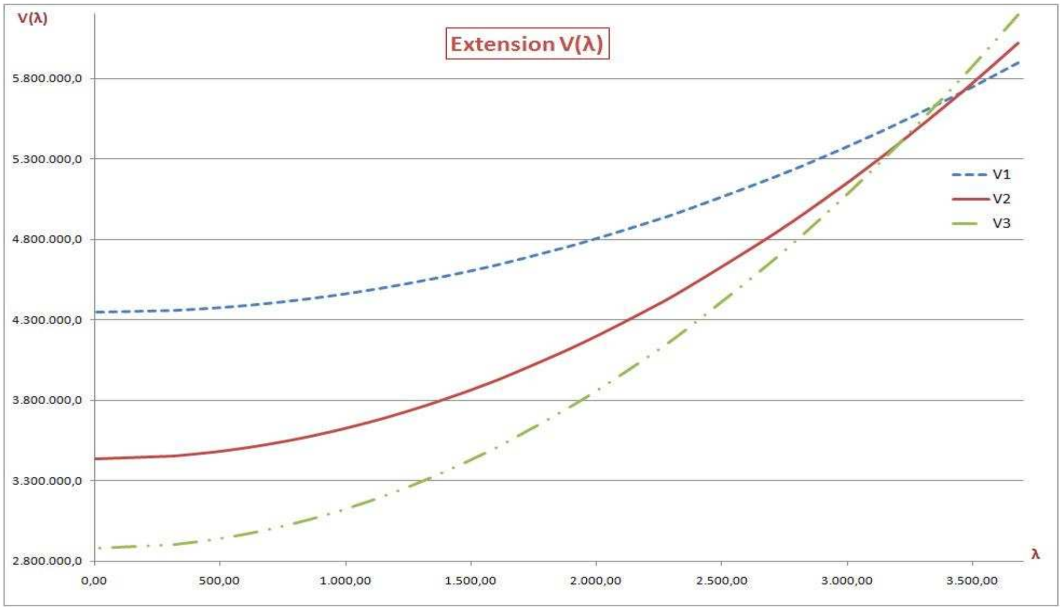


Figure 6: Behavior of the three parabolas $V(\lambda)$ in the whole interval of $\lambda$.

Actually, this graph allows to appreciate the parabolic behavior of $V(\lambda)$. In picture 7 we zoomed in on the extension to a restricted interval of $\lambda$ in order to appreciate the variation of the pieces of the parabolas. In each subinterval the lowest curve.

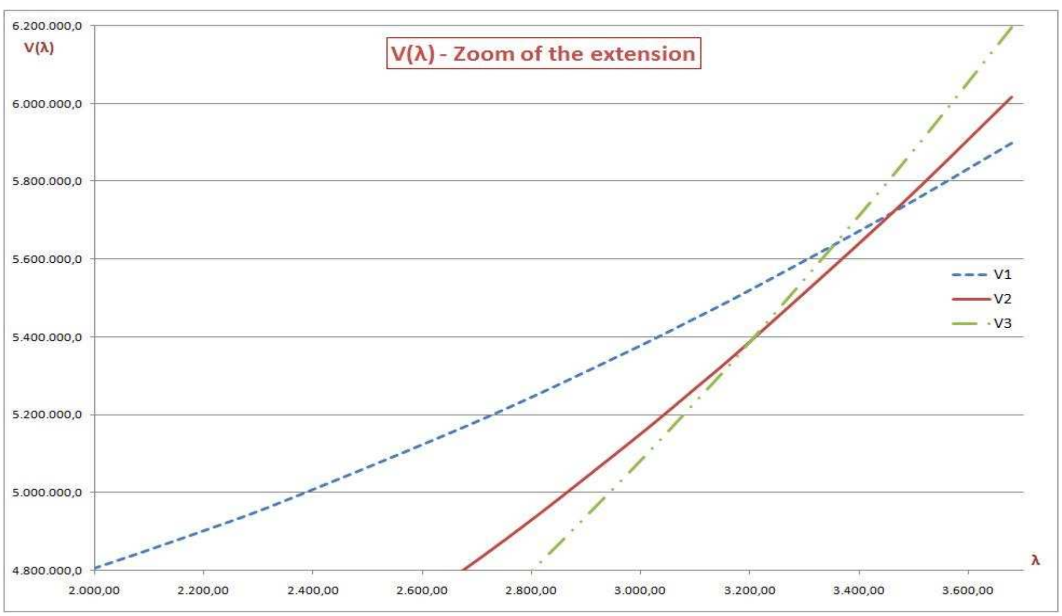


Figure 7: Zoom in on the three parabolas $V(\lambda)$ in the whole interval of $\lambda$.

The picture of the efficient frontier in the plane $(E, V)$ is the following one.

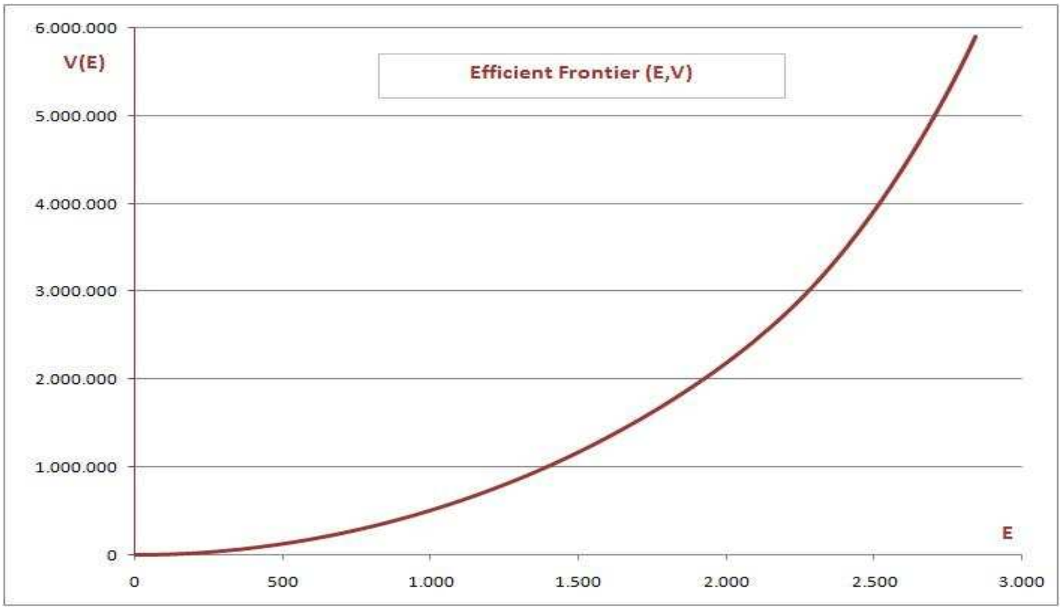


Figure 8: Variance as function of expected return plot.

At first glance, it should be clear that $V(E)$ is represented by a continuous and without kinks graph.

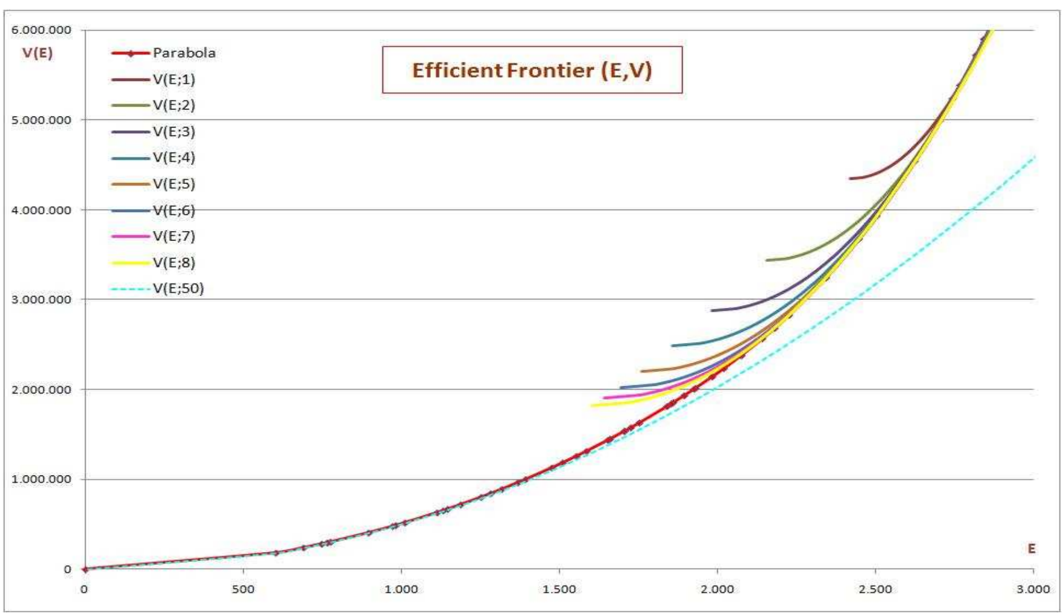


Figure 9: Variance as function of expected return plot as an union of different parabolas.

In the previous graph we plotted variances as a function of expectation $E$ for portfolios with efficient retentions only for a subset of risks. More precisely, such retentions regard, respecting the optimal proportions, only those risks already reinsured up to the $h$ interval, the retention of all the other being fixed at level 1. Obviously, in the $h-$th interval exactly $h$ risks are actively reinsured. This gives an idea of the loss of efficiency arising by the failure to begin properly reinsuring the next risks. Each of the internal parabola, starting from the top ($V(E,i)$ with $i = 1, 2, \ldots, h$), shows the behavior of $V(E)$ under optimal reinsurance of only the first $h$ policies. The dot line is the picture of $V(E)$ if we could make a change of scale in such a way that the upper bound 1 of retention would not be anymore a binding constraint.

## Appendix II.1

In this Appendix we give proofs of the fundamental theorems introduced in section 3.

Our proof exploits Karush-Kuhn-Tucker conditions in a form which uses the individual advantage function. According to this approach, given the following constrained quadratic optimization version of the proportional reinsurance problem:

$$\begin{aligned} \min \quad & \frac{1}{2}\mathbf{x}^T\mathbf{V}\mathbf{x} \\ & \mathbf{m}^T\mathbf{x} = E \\ & \mathbf{0} \le \mathbf{x} \le \mathbf{1} \\ & 0 \le E \le \sum_i m_i \end{aligned} \tag{II.1.1}$$

the Karush-Kuhn-Tucker conditions are resumed by the following sentence (see Pressacco and Serafini [27], pg. 32): a retention $\mathbf{x}$ is mean-variance efficient iff it exists a $\lambda$ (Lagrange multiplier of the expectation constraint) such that:

**a.** $0 < x_i < 1$ implies $F_i(\mathbf{x}) = \lambda$;

**b.** $x_i = 0$ implies $F_i(\mathbf{x}) \ge \lambda$;

**c.** $x_i = 1$ implies $F_i(\mathbf{x}) \le \lambda$.

In the case of group correlation, such conditions become:

**I)** $0 < x_{i,q} < 1$ implies $a_q \cdot \left( x_{i,q}\sigma_{i,q} + \rho_q \cdot \sum_{j \ne i} x_{j,q}\sigma_{j,q} \right) = \lambda$

**II)** $x_{i,q} = 0$ implies $a_q \cdot \rho_q \cdot \sum_{j \ne i} x_{j,q}\sigma_{j,q} \ge \lambda$

**III)** $x_{i,q} = 1$ implies $a_q \cdot \left( \sigma_{i,q} + \rho_q \cdot \sum_{j \ne i} x_{j,q}\sigma_{j,q} \right) \le \lambda$

The following results are straightforward:

**R 1.** *Let* $\mathbf{x}_q = \mathbf{0}_q$ *be the group* $q$ *retentions of an efficient retention* $\mathbf{x}$*. It is, recalling the equation* (2)*,* $F_{i,q}(\mathbf{x}) = F_{i,q}(\mathbf{0}_q) = 0$*. On the other side, the b. Karush-Kuhn-Tucker condition implies* $F_{i,q}(\mathbf{x}) \geq \lambda$*. Then* $\lambda \leq 0$*.*

**R 2.** *Let* $\mathbf{x}$ *be an efficient retention with at least one strictly positive retention* $x_{i,q} > 0$*. Then, still recalling* (2)*, it is* $F_{i,q}(\mathbf{x}) > 0$*. On the other side, a. or c. Karush-Kuhn-Tucker conditions imply* $F_{i,q}(\mathbf{x}) \leq \lambda$*. Then* $\lambda > 0$*.*

R1 and R2 jointly imply:

**R 3.** $\mathbf{x}_q = \mathbf{0}_q$ *for one group* $q$ *implies* $\mathbf{x} = \mathbf{0}$*.*

Note that this is equivalent to say that, if in a given group at least a retention is strictly positive, then there is a strictly positive retention in any group.

**R 4.** *Let* $\mathbf{x}_{i,q} > 0$ *then* $x_{h,q} > 0$ *for any* $h$*.*

Ad absurdum, let $x_{h,q} = 0$, then $F_{i,q} - F_{h,q} = a_q \cdot (1 - \rho_q) \cdot (x_{i,q}\sigma_{i,q}) > 0$. On the other side, a. and c. Karush-Kuhn-Tucker conditions imply that $F_{i,q} \leq \lambda$, while b. Karush-Kuhn-Tucker conditions would imply $F_{h,q} \geq \lambda$, that is $F_{h,q} - F_{i,q} \geq 0$, contradictory with the previous inequation.

On the basis of R3 and R4 we have now:

**R 5.** *In any efficient retention, either* $\mathbf{x} = \mathbf{0}$ *or* $\mathbf{x} > \mathbf{0}$*. Verbally, either all null retentions or all strictly retentions.*

Let us now concentrate on some properties of $\mathbf{x}_q$ vector belonging to an efficient $\mathbf{x}$ vector.

**R 6.** $x_{i,q} = 1$ *implies* $x_{h,q} = 1$ *for any* $h > i$*.*

Indeed, $F_{i,q} - F_{h,q} = a_q \cdot (1 - \rho_q) \cdot (\sigma_{i,q} - x_{h,q}\sigma_{h,q}) > 0$ owing to the positivity of the third factor induced by $\sigma_{i,q} > \sigma_{h,q}$.

Suppose ad absurdum, it would be $0 < x_{h,q} < 1$, then a. Karush-Kuhn-Tucker condition would imply $F_{h,q} = \lambda$, while c. Karush-Kuhn-Tucker condition would imply $F_{h,q} \leq \lambda$, that is jointly $F_{i,q} - F_{h,q} < 0$ contradictory with the previous inequation.

**R 7.** *Let* $0 < x_{i,q}, x_{h,q} < 1$ *then* $x_{i,q}\sigma_{i,q} = x_{h,q}\sigma_{h,q}$*.*

Indeed, $F_{i,q} - F_{h,q} = a_q \cdot (1 - \rho_q) \cdot (x_{i,q}\sigma_{i,q} - x_{h,q}\sigma_{h,q})$; on the other side, a. Karush-Kuhn-Tucker condition would imply $F_{i,q} = \lambda$ as well $F_{h,q} = \lambda$, that is $F_{i,q} - F_{h,q} = 0$.

We signal that R5, R6 and R7 have been discovered by de Finetti as an informal empirical consequence of his approach and later rigorously proved by Gigante (see Gigante [12]) making recourse to the Karush-Kuhn-Tucker conditions. The following results are, at our knowledge, original.

Previous Results give the following structure of any group retention $\mathbf{x}_q$ which is part of an efficient retention $\mathbf{x}$. Given a proper and of course feasible choice of $x_{1,q}$, it is $x_{i,q} = \min\left[\frac{x_{1,q}\cdot\sigma_{1,q}}{\sigma_{i,q}};1\right]$, $\forall i = 2,\ldots,n_q$. This way Theorem 1 has been proved.

Now, let us consider $F_{i,q}(\mathbf{x}) = F_{i,q}(\mathbf{x}_q)$ as a function of the driver $x_{1,q}$ of the retention $\mathbf{x}_q$. Recall

$$F_{1,q}(x_{1,q}) = a_q \cdot \left[x_{1,q}\sigma_{1,q} + \rho_q \cdot \sum_{i=2}^{n_q} \sigma_{i,q} \cdot \min\left(\frac{x_{1,q}\cdot\sigma_{1,q}}{\sigma_{i,q}};1\right)\right]$$

It is straightforward to check that $F_{1,q}$ is an increasing continuous function of the driver $x_{1,q}$ with

$$F_{1,q}(0) = 0$$

and

$$F_{1,q}(1) = a_q \cdot \left[\sigma_{1,q} + \rho_q \cdot \sum_{i=2}^{n_q} \sigma_{i,q}\right]$$

In order to prove Theorem 2, let us choose freely a feasible value of $x_{1,1}$.

For $x_{1,1} = 1$, c. Karush-Kuhn-Tucker condition would imply $F_{1,1}(\mathbf{1}_1) \leq \lambda$. On the other side, the largest value of $F_{1,q}$ is $F_{1,q}(\mathbf{1}_q) < F_{1,1}(\mathbf{1}_1)$ (due to the ordering of groups). Then, $F_{1,q}(\mathbf{1}_q) < \lambda$ which implies for Karush-Kuhn-Tucker condition $x_{1,q} = 1$. This is equivalent to say that the equation (6) has no feasible solution in $x_{1,q}$. In this case Theorem 2 says that $x_{1,q} = 1$.

For $0 < x_{1,1} < 1$, a. Karush-Kuhn-Tucker condition would imply $F_{1,1}(\mathbf{x}_1) = \lambda$. If the equation (6) has feasible solutions in $x_{1,q}$, then the only value coherent with the Karush-Kuhn-Tucker conditions is exactly the one $\widehat{x}_{1,q}$ which give the solution; in case there is no solution, it is $F_{1,q}(x_{1,q}) < \lambda$ for any feasible value of the driver $x_{1,q}$ which, according to the Karush-Kuhn-Tucker conditions, implies $x_{1,q} = 1$.

Finally, for $x_{1,1} = 0$ it is straightforward to check that $\widehat{x}_{1,q}$ is the trivial solution of the equation.

## Appendix II.2

Proof that $x_{i,q}(\lambda)$ is continuous at the connection points $\lambda_h$.

At the connection point, using (16) corresponding to two adjacent intervals of $\lambda$ around $\lambda(h)$ we have:

$$\frac{\lambda_h}{a_q \cdot \sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{\sigma_{i,q} \cdot [1 + \rho_q \cdot (h-2)]} =$$
$$= \frac{\lambda_h}{a_q \cdot \sigma_{i,q} \cdot [1 + \rho_q \cdot (h-1)]} - \rho_q \cdot \frac{\sum_{j=h+1}^{n_q} \sigma_{j,q}}{\sigma_{i,q} \cdot [1 + \rho_q \cdot (h-1)]}$$

$$\frac{\lambda_h \cdot \big[[1+\rho_q \cdot (h-1)] - [1+\rho_q \cdot (h-2)]\big]}{a_q \cdot \sigma_{i,q} \cdot [1+\rho_q \cdot (h-2)] \cdot [1+\rho_q \cdot (h-1)]} =$$

$$= \frac{a_q \cdot \rho_q \cdot \Big[\sum_{j=h}^{n_q} \sigma_{j,q} \cdot [1+\rho_q \cdot (h-1)] - \sum_{j=h+1}^{n_q} \sigma_{j,q} \cdot [1+\rho_q \cdot (h-2)]\Big]}{a_q \cdot \sigma_{i,q} \cdot [1+\rho_q \cdot (h-2)] \cdot [1+\rho_q \cdot (h-1)]}$$

The numerator of the left hand side is

$$\lambda \cdot \rho_q$$

and the one of the right hand side is

$$a_q \rho_q \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q} \cdot \rho_q + \sigma_{h,q} \cdot [1+\rho_q \cdot (h-1)]$$

Dividing both members by $\rho_q$ we found for $\lambda_h$ exactly the equation

$$\lambda_h = F_h(\mathbf{x}_h) = a_q \cdot \Big[\sigma_{h,q} \cdot [1+\rho_q \cdot (h-1)] + \rho_q \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q}\Big] \tag{II.2.1}$$

which is the value of the advantage parameter corresponding to the new entrance in reinsurance of the policy $h$.

## Appendix II.3

Proof that $\alpha_h$, as function of $h$, is increasing with $h$.

$$\alpha_h = \sum_{q=1}^{g} \alpha_{q,h}$$

First of all note that, moving from $(h-1)$ to $h$ determines a change only in one of the $\alpha_q$ coefficient, corresponding to the group whose the new risk entering in reinsurance is member.

With reference to this group, consider $\alpha_h$ and $\alpha_{(h-1)}$ in order to show that $\alpha_h > \alpha_{(h-1)}$, or, keeping account of formula (21), that:

$$\frac{h}{a^2 \cdot [1+\rho_q \cdot (h-1)]} > \frac{(h-1)}{a^2 \cdot [1+\rho_q \cdot (h-2)]} \tag{II.3.1}$$

immediately verified for $h = 1$ and easily verified, by elementary algebra, for $h \geq 2$.

## Appendix II.4

Proof of continuity of the expectation $E(\lambda)$ at the connection points $\lambda_h$.

First of all, note that $E(\lambda)$ is a sum of $g$ functions $E_q(\lambda)$ and it would be straightforward provided that the single functions are continuous. At a connection point only one of these functions could raise a problem of continuity, the one of the group to which the risk (generating the corner) belongs.

It is then enough to show the continuity of this single function.

It is to be proved that:

$$\lim_{\lambda \to \lambda_h^+} \alpha_{(h-1)} \cdot \lambda + \beta_{(h-1)} = \alpha_h \cdot \lambda_h + \beta_h \tag{II.4.1}$$

namely:

$$\alpha_{(h-1)} \cdot \lambda_h + \beta_{(h-1)} = \alpha_h \cdot \lambda_h + \beta_h \tag{II.4.2}$$

Now, keeping account of formula (21) and after some elementary algebra, we have:

$$\begin{aligned}[\alpha_{(h-1)} - \alpha_h] \cdot \lambda_h &= \left[ \frac{h-1}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]} - \frac{h}{a_q^2 \cdot [1 + \rho_q \cdot (h-1)]} \right] \cdot \lambda_h \\ &= \frac{-(1-\rho_q)}{a_q^2 \cdot [1 + \rho_q \cdot (h-1)] \cdot [1 + \rho_q \cdot (h-2)]} \cdot \lambda_h \end{aligned} \tag{II.4.3}$$

Keeping account of formula (22), it is:

$$[\beta_{(h-1)} - \beta_h] = \frac{\rho_q \cdot h \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-1)]} - \sum_{i=h+1}^{n_q} m_{i,q} - \frac{\rho_q \cdot (h-1) \cdot \sum_{j=h}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-2)]} + \sum_{i=h}^{n_q} m_{i,q} \tag{II.4.4}$$

noting also that it is:

$$\sum_{j=h}^{n_q} \sigma_{j,q} = \sum_{j=h+1}^{n_q} \sigma_{j,q} + \sigma_{h,q}$$

$$-\sum_{i=h+1}^{n_q} m_{i,q} + \sum_{i=h}^{n_q} m_{i,q} = m_{h,q}$$

then we obtain:

$$[\beta_{(h-1)} - \beta_h] =$$

$$= \frac{\rho_q \cdot h \cdot [1+\rho_q(h-2)] \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q}}{a_q \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} +$$

$$- \frac{\rho_q \cdot (h-1) \cdot [1+\rho_q(h-1)] \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q} - \rho_q \cdot (h-1) \cdot [1+\rho_q(h-1)] \cdot \sigma_{h,q}}{a_q \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} + m_{h,q} =$$

$$= \frac{\rho_q \cdot (1-\rho_q) \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q} - \rho_q(h-1) \cdot [1+\rho_q(h-1)] \cdot \sigma_{h,q}}{a_q \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q \cdot (h-2)]} + m_{h,q} =$$

$$= \frac{a_q \cdot \left[\rho_q(1-\rho_q) \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q} - \rho_q \cdot (h-1) \cdot [1+\rho_q(h-1)] \cdot \sigma_{h,q}\right]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} + m_{h,q} \tag{II.4.5}$$

Now, recalling that:

$$a_q = \frac{\sigma_{h,q}}{m_{h,q}} \Rightarrow a_q \cdot m_{h,q} = \sigma_{h,q}$$

and the formula (II.2.1):

$$\lambda_h = a_q \cdot \left[\sigma_{h,q} \cdot [1+\rho_q \cdot (h-1)] + \rho_q \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q}\right]$$

and putting together (II.4.3) and (II.4.5), let us verify the identity:

$$[\alpha_{(h-1)} - \alpha_h] \cdot \lambda_h + [\beta_{(h-1)} - \beta_h] = 0$$

$$[\alpha_{(h-1)} - \alpha_h] \cdot \lambda_h + [\beta_{(h-1)} - \beta_h] =$$

$$= \frac{-a_q(1-\rho_q)\left[\sigma_{h,q}[1+\rho_q(h-1)] + \rho_q \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q}\right]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} +$$

$$+ \frac{a_q\left[\rho_q(1-\rho_q) \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q} - \rho_q(h-1)[1+\rho_q(h-1)] \cdot \sigma_{h,q}\right]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} +$$

$$+ \frac{a_q \cdot \sigma_{h,q} \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} =$$

$$= \frac{-a_q \cdot \sigma_{h,q}(1-\rho_q)[1+\rho_q(h-1)] - a_q \cdot \sigma_{h,q} \cdot \rho_q \cdot (h-1)[1+\rho_q(h-1)]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} +$$

$$+ \frac{a_q \cdot \sigma_{h,q}[1+\rho_q(h-1)][1+\rho_q(h-2)]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} =$$

$$= \frac{-(1-\rho_q) - \rho_q \cdot (h-1) + [1+\rho_q(h-2)]}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} = 0 \quad \text{Q.E.D.} \tag{II.4.6}$$

## Appendix II.5

Proof of continuity of the variance $V(\lambda)$ at the connection points $\lambda_h$.

First of all let us recall also the $V(\lambda)$ is a sum of $g$ functions $V_q(\lambda)$ and, just as in the case of the continuity of the expectation, it is enough to prove the continuity of the function of the group to which the risk (generating the corner) belongs.

It is to be proved that:

$$\lim_{\lambda \to \lambda_h^+} \alpha_{(h-1)} \cdot \lambda^2 + \gamma_{(h-1)} = \alpha_h \cdot \lambda_h^2 + \gamma_h \tag{II.5.1}$$

namely:

$$\alpha_{(h-1)} \cdot \lambda_h^2 + \gamma_{(h-1)} = \alpha_h \cdot \lambda_h^2 + \gamma_h \tag{II.5.2}$$

Now, exploiting results from Appendix II.4 (see formula (II.4.3)), we already know that:

$$[\alpha_{(h-1)} - \alpha_h] \cdot \lambda_h^2 = \frac{-(1-\rho_q)}{a_q^2 \cdot [1+\rho_q \cdot (h-1)] \cdot [1+\rho_q \cdot (h-2)]} \cdot \lambda_h^2 \tag{II.5.3}$$

where, according also to formula (II.2.1):

$$\lambda_h^2 = a_q^2 \cdot \left[\sigma_{h,q} \cdot [1+\rho_q \cdot (h-1)] + \rho_q \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q}\right]^2 \tag{II.5.4}$$

Now, keeping account of formula (31) and after some elementary algebra, we have:

$$
\begin{aligned}
[\gamma_{(h-1)} - \gamma_h] = & \frac{\rho_q^2 \cdot h \cdot \Big(\sum\limits_{i=h+1}^{n_q} \sigma_{i,q}\Big)^2}{1+\rho_q \cdot (h-1)} - 2\rho_q \cdot \sum_{i=h+1}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} - \sum_{i=h+1}^{n_q} \sigma_{i,q}^2 - \\
& - \frac{\rho_q^2 \cdot (h-1) \cdot \Big(\sum\limits_{i=h}^{n_q} \sigma_{i,q}\Big)^2}{1+\rho_q \cdot (h-2)} + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{i=h}^{n_q} \sigma_{i,q}^2
\end{aligned}
\tag{II.5.5}
$$

Let us note that:

$$-2\rho_q \cdot \sum_{i=h+1}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} = 2\rho_q \cdot \sigma_{h,q} \sum_{i=h+1}^{n_q} \sigma_{i,q}$$

and that:

$$-\sum_{i=h+1}^{n_q} \sigma_{i,q}^2 + \sum_{i=h}^{n_q} \sigma_{i,q}^2 = \sigma_{h,q}^2$$

and moreover:

$$\left(\sum_{j=h}^{n_q} \sigma_{j,q}\right)^2 = \left(\sum_{j=h+1}^{n_q} \sigma_{j,q} + \sigma_{h,q}\right)^2 = \left(\sum_{j=h+1}^{n_q} \sigma_{j,q}\right)^2 + \sigma_{h,q}^2 + 2\sigma_{h,q} \cdot \sum_{j=h+1}^{n_q} \sigma_{j,q} = A$$

Therefore, by defining with:

$$B = [1+\rho_q \cdot (h-1)] \cdot [1+\rho_q \cdot (h-2)]$$

we have:

$$
\begin{aligned}
[\gamma_{(h-1)} - \gamma_h] = & \frac{\rho_q^2 \cdot h \cdot [1+\rho_q \cdot (h-2)] \cdot \Big(\sum\limits_{i=h+1}^{n_q} \sigma_{i,q}\Big)^2 - \rho_q^2 \cdot (h-1) \cdot [1+\rho_q \cdot (h-1)] \cdot A}{B} + \\
& + \frac{B \cdot 2\rho_q \cdot \sigma_{h,q} \sum\limits_{i=h+1}^{n_q} \sigma_{i,q} + B \cdot \sigma_{h,q}^2}{B}
\end{aligned}
\tag{II.5.6}
$$

Finally, putting together (II.5.3) and (II.5.6) and after some elementary algebra, we verify the following identity:

$$[\alpha_{(h-1)} - \alpha_h] \cdot \lambda_h^2 + [\gamma_{(h-1)} - \gamma_h] = 0$$

$$[\alpha_{(h-1)} - \alpha_h] \cdot \lambda_h^2 + [\gamma_{(h-1)} - \gamma_h] =$$

$$= \frac{-a_q^2(1-\rho_q)\Big[\sigma_{h,q}^2 \cdot [1+\rho_q(h-1)]^2 + \rho_q^2\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)^2 + 2\rho_q\sigma_{h,q}[1+\rho_q(h-1)] \cdot \sum_{j=h+1}^{n_q}\sigma_{j,q}\Big]}{a_q^2[1+\rho_q \cdot (h-1)] \cdot [1+\rho_q \cdot (h-2)]} +$$

$$+ \frac{a_q^2 \cdot \rho_q^2 \cdot h \cdot [1+\rho_q \cdot (h-2)] \cdot \Big(\sum_{i=h+1}^{n_q}\sigma_{i,q}\Big)^2 - a_q^2\rho_q^2 \cdot (h-1) \cdot [1+\rho_q \cdot (h-1)] \cdot \Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)^2}{a_q^2 \cdot [1+\rho_q \cdot (h-1)] \cdot [1+\rho_q \cdot (h-2)]} +$$

$$+ \frac{-a_q^2\rho_q^2 \cdot (h-1) \cdot [1+\rho_q \cdot (h-1)] \cdot \sigma_{h,q}^2 - a^2\rho_q^2 \cdot (h-1) \cdot [1+\rho_q \cdot (h-1)] \cdot 2\sigma_{h,q} \cdot \sum_{j=h+1}^{n_q}\sigma_{j,q}}{a_q^2 \cdot [1+\rho_q(h-1)] \cdot [1+\rho_q(h-2)]} +$$

$$+ \frac{a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)] \cdot 2\rho_q\sigma_{h,q}\sum_{i=h+1}^{n_q}\sigma_{i,q} + a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)] \cdot \sigma_{h,q}^2}{a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]} \tag{II.5.7}$$

Now, gathering the quadratic sum's terms:

$$\frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)^2[a_q^2\rho_q^2 \cdot h \cdot [1+\rho_q(h-2)] - a_q^2\rho_q^2 \cdot (1-\rho_q) - a_q^2\rho_q^2 \cdot (h-1)[1+\rho_q \cdot (h-1)]]}{a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]} =$$

$$= \frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)^2[h \cdot [1+\rho_q \cdot (h-2)] - (1-\rho_q) - (h-1) \cdot [1+\rho_q \cdot (h-1)]]}{a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]} =$$

$$= \frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)^2 \cdot 0}{a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]} = 0 \tag{II.5.8}$$

then the other sum's terms:

$$\frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)\Big[-2a_q^2(1-\rho_q)\rho_q\sigma_h[1+\rho_q(h-1)]+a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]2\rho_q\sigma_{h,q}\Big]}{a_q^2[1+\rho_q\cdot(h-1)][1+\rho_q\cdot(h-2)]}+$$

$$+\frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)\Big[-a_q^2\rho_q^2(h-1)[1+\rho_q(h-1)]2\sigma_{h,q}\Big]}{a_q^2[1+\rho_q\cdot(h-1)][1+\rho_q\cdot(h-2)]}=$$

$$=\frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)\Big[-(1-\rho_q)+[1+\rho_q(h-2)]-\rho_q(h-1)\Big]}{a_q^2[1+\rho_q\cdot(h-1)][1+\rho_q\cdot(h-2)]}=$$

$$=\frac{\Big(\sum_{j=h+1}^{n_q}\sigma_{j,q}\Big)\cdot 0}{a_q^2[1+\rho_q\cdot(h-1)]\cdot[1+\rho_q\cdot(h-2)]}=0 \tag{II.5.9}$$

and, finally all the others:

$$\frac{-a_q^2(1-\rho_q)\sigma_{h,q}^2[1+\rho_q(h-1)]^2+a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]\sigma_{h,q}^2}{a_q^2[1+\rho_q\cdot(h-1)][1+\rho_q\cdot(h-2)]}+$$

$$-\frac{a_q^2\rho_q^2(h-1)[1+\rho_q(h-1)]\sigma_{h,q}^2}{a_q^2[1+\rho_q\cdot(h-1)][1+\rho_q\cdot(h-2)]}=$$

$$\frac{-(1-\rho_q)[1+\rho_q(h-1)]+[1+\rho_q(h-2)]-\rho_q^2(h-1)}{a_q^2[1+\rho_q\cdot(h-1)][1+\rho_q\cdot(h-2)]}=$$

$$=\frac{0}{a_q^2[1+\rho_q(h-1)][1+\rho_q(h-2)]}=0 \quad \text{Q.E.D.} \tag{II.5.10}$$

# Appendix II.6

Proof of continuity and derivability of $V(E)$.

As for the continuity of $V(E)$, we have just proved the continuity of $E$ and $V$ as functions of $\lambda$, at the connection values $\lambda_h$.

Now, recall that $E$ and $V$ are monotone increasing functions of $\lambda$. Hence, for any couple of neighborhoods of $E_h$ and $V_h$ we may find a neighborhood $J(\lambda)$ such that, for any $\lambda \in J$, $E(\lambda)$ and $V(\lambda)$ are in the respective neighborhoods.

As for the derivability of $V(E)$, let us recall that $V(E)$ is an union of parabolas whose formal expression holds on interval closed at right and open at left. Then, the left derivative of the variance is immediately obtained just by computing the first derivative of the expression (33):

$$V^{'}(E_h) = 2 \cdot \frac{(E_h - \beta_h)}{\alpha_h} \tag{II.6.1}$$

and inserting:

$$E_h = \alpha_h \cdot \lambda_h + \beta_h$$

we obtain:

$$V^{'}(E_h) = 2 \cdot \lambda_h \tag{II.6.2}$$

As for the right derivative, we must compute the limit:

$$\lim_{E \to E_h^+} V^{'}(E) \tag{II.6.3}$$

In a right neighborhood of $E_h$:

$$V^{'}(E) = 2 \cdot \frac{(E - \beta_h)}{\alpha_h} \tag{II.6.4}$$

and substituting:

$$E = \alpha_{h-1}\lambda + \beta_{h-1}$$

$$V^{'}(E) = 2 \cdot \lambda \tag{II.6.5}$$

Now, when $E \to E_h^+$, $\lambda \to \lambda_h^+$ and so

$$\lim_{E \to E_h^+} V^{'}(E) = lim_{\lambda \to \lambda_h^+} = 2 \cdot \lambda_h \quad \text{Q.E.D.} \tag{II.6.6}$$

## Appendix II.7

In order to calculate the variance of a generic group, we have to deal with a covariance matrix:

$$V = \mathbf{x}^\top \mathbf{V}\, \mathbf{x} = \sum_i \sum_j x_{i,q} x_{j,q} \rho_q(ij) \sigma_{i,q} \sigma_{j,q} \tag{II.7.1}$$

bearing in mind the useful position $x_{i,q}\sigma_{i,q} = c(\lambda) = c, \forall i = 1, \ldots, (h-1)$ and the fact that $x_{i,q} = 1, \forall i = h, \ldots, n_q$. Solving the matrix, we have

$$V_q = \sum_{i=1}^{h-1} c^2 + c^2 \rho_q \cdot \sum_{i=1}^{h-1} \sum_{j \neq i} 1 + \sum_{i=h}^{n_q} \sigma_{i,q}^2 + 2 \cdot \rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{j=1}^{h-1} c\rho_q \sum_{i=h}^{n_q} \sigma_{i,q} + \sum_{i=1}^{h-1} c\rho_q \sum_{j=h}^{n_q} \sigma_{j,q} \tag{II.7.2}$$

Then, manipulating the above expression and writing it as a second order polynomial in the variable $c(\lambda)$, we obtain:

$$V_q = c^2 \cdot [(h-1) + \rho_q \cdot (h-1) \cdot (h-2)] + c \cdot 2(h-1) \cdot \rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \sum_{j>i} \sigma_{j,q} + \sum_{i=h}^{n_q} \sigma_{i,q}^2 \tag{II.7.3}$$

Now, we exploit the following positions:

$$x_{i,q}(\lambda)\sigma_{i,q} = c(\lambda) = \frac{\lambda}{a_q \cdot [1 + \rho_q \cdot (h-2)]} - \rho_q \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{[1 + \rho_q \cdot (h-2)]} \tag{II.7.4}$$

and

$$c(\lambda)^2 = \frac{\lambda^2}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]^2} - 2\rho_q \cdot \lambda \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-2)]^2} + \rho_q^2 \cdot \frac{\left[\sum_{j=h}^{n_q} \sigma_{j,q}\right]^2}{[1 + \rho_q \cdot (h-2)]^2} \tag{II.7.5}$$

in order to rewrite the formulation of the variance as a function of the key parameter $\lambda$. In particular, we may opportunely write:

$$\begin{aligned}
V_q(\lambda) = {} & (h-1) \cdot [1 + \rho_q(h-2)] \cdot \left[\frac{\lambda^2}{a_q^2 \cdot [1 + \rho_q \cdot (h-2)]^2} - 2\rho_q \cdot \lambda \cdot \frac{\sum_{j=h}^{n_q} \sigma_{j,q}}{a_q \cdot [1 + \rho_q \cdot (h-2)]^2}\right] + \\
& + (h-1) \cdot [1 + \rho_q(h-2)] \cdot \rho_q^2 \cdot \frac{\left(\sum_{j=h}^{n_q} \sigma_{j,q}\right)^2}{[1 + \rho_q \cdot (h-2)]^2} + \\
& + 2(h-1) \cdot \rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \cdot \left[\frac{\lambda}{a_q \cdot [1 + \rho_q(h-2)]} - \rho_q \cdot \frac{\sum_{i=h}^{n_q} \sigma_{i,q}}{[1 + \rho_q(h-2)]}\right] + \\
& + 2\rho_q \cdot \sum_{i=h}^{n_q} \sigma_{i,q} \cdot \sum_{j>i} \sigma_{j,q} + \sum_{i=h}^{n_q} \sigma_{i,q}^2
\end{aligned} \tag{II.7.6}$$

Now, a convenient reformulation as a function of $\lambda$:

$$
\begin{aligned}
V_q(\lambda) = & \left[\frac{(h-1)\cdot[1+\rho_q(h-2)]}{a_q^2\cdot[1+\rho_q\cdot(h-2)]^2}\right]\cdot\lambda^2- \\
& -2\rho_q\cdot(h-1)\cdot\sum_{j=h}^{n_q}\sigma_{j,q}\cdot\left[\frac{[1+\rho_q(h-2)]}{a_q\cdot[1+\rho_q\cdot(h-2)]^2}\right]\cdot\lambda+ \\
& +2\rho_q\cdot(h-1)\cdot\sum_{j=h}^{n_q}\sigma_{j,q}\cdot\left[\frac{1}{a_q\cdot[1+\rho_q\cdot(h-2)]}\right]\cdot\lambda+ \\
& +(h-1)\cdot\rho_q^2\cdot\frac{\left(\sum_{j=h}^{n_q}\sigma_{j,q}\right)^2}{[1+\rho_q\cdot(h-2)]}+ \\
& -2(h-1)\cdot\rho_q^2\cdot\frac{\sum_{i=h}^{n_q}\sigma_{i,q}\cdot\sum_{i=h}^{n_q}\sigma_{i,q}}{[1+\rho_q(h-2)]}+ \\
& +2\rho_q\cdot\sum_{i=h}^{n_q}\sigma_{i,q}\cdot\sum_{i+1}^{n_q}\sigma_{i,q}+\sum_{i=h}^{n_q}\sigma_{i,q}^2
\end{aligned}
\tag{II.7.7}
$$

and finally:

$$
V_q(\lambda) = \frac{(h-1)}{a_q^2\cdot[1+\rho_q\cdot(h-2)]}\cdot\lambda^2-(h-1)\cdot\rho_q^2\cdot\frac{\sum_{i=h}^{n_q}\sigma_{i,q}\sum_{i=h}^{n_q}\sigma_{i,q}}{[1+\rho_q(h-2)]}+2\rho_q\cdot\sum_{i=h}^{n_q}\sigma_{i,q}\sum_{i+1}^{n_q}\sigma_{i,q}+\sum_{i=h}^{n_q}\sigma_{i,q}^2 \tag{II.7.8}
$$

so that:

$$
V_q(\lambda) = \alpha_q^{'}\cdot\lambda^2+\beta_q^{'}\cdot\lambda+\gamma_q^{'} \tag{II.7.9}
$$

where, specifically:

$$
\begin{aligned}
\alpha_q^{'} &= \frac{(h-1)}{a_q^2\cdot[1+\rho_q\cdot(h-2)]} \\
\beta_q^{'} &= 0 \\
\gamma_q^{'} &= -\frac{\rho_q^2\cdot(h-1)\cdot\left(\sum_{i=h}^{n_q}\sigma_{i,q}\right)^2}{1+\rho_q\cdot(h-2)}+2\rho_q\cdot\sum_{i=h}^{n_q}\sigma_{i,q}\sum_{j=h+1}^{n_q}\sigma_{j,q}+\sum_{i=h}^{n_q}\sigma_{i,q}^2
\end{aligned}
\tag{II.7.10}
$$

## References


[1] Artzner, P., F. Delbaen, J.M. Eber and D. Heath (1999), Coherent measures of risk, *Mathematical Finance*, **9**, 203-228

[2] Balbás, A., B. Balbás and A. Heras (2009), Optimal reinsurance with general risk measures, *Insurance: Mathematics and Economics*, **44**, 374-384

[3] Barone, L. (2008), Bruno de Finetti and the case of the critical line's last segment, *Insurance: Mathematics and Economics*, **42-1**, 359-377

[4] Borch, K. (1974), *The mathematical theory of insurance*, Lexington Books

[5] Bühlmann, H. (1970), *Mathematical methods in risk theory.* Springer

[6] Bühlmann, H. (2006), Encounters with ideas of Bruno De Finetti, presented at the Sixth Scientific Conference on Insurance and Finance of the German Association for Actuarial and Financial Mathematics, Berlin

[7] Bühlmann, H. and H. Gerber (1978), Risk bearing and the reinsurance market, *The ASTIN Bullettin*, **10**, 12-24.

[8] de Finetti, B. (1940), Il problema dei pieni, *Giornale Istituto Italiano Attuari*, **9**, 1-88; English translation by L. Barone available as "The problem of Full-risk insurances", Ch. 1 'The problem in a single accounting period', *Journal of Investment Management*, **4**, 19-43, 2006

[9] Dickson, D.C.M. and H.R. Waters (1997), Relative Reinsurance retention levels, *The ASTIN Bullettin*, **27-2**, 207-227

[10] Gerber, H.U. (1984), Equilibria in a proportional reinsurance market, *Insurance, Mathematics and Economics* **3**, 97-100

[11] Gerber, H. U. and E.S W. Shiu (2003), Economic Ideas of Bruno De Finetti in the Wiener Process Model, *Metodi Statistici per la Finanza e le Assicurazioni*, 75-95

[12] Gigante, P. (1990), L'approccio di B. de Finetti per la riassicurazione individuale proporzionale. Modelli semplificati e considerazioni operative, *Rivista di Matematica per le Scienze Economiche e Sociali*, **13**, 1°-2°, 147-161

[13] Glineur, F. and J.F. Walhin (2006), de Finetti's retention problem for proportional reinsurance revisited, *German Actuarial Bulletin*, **28-1**, 451-462

[14] Goovaerts, M., R. Kaas, J. Dhaene, and Q. Tang (2004), Some new classes of consistent risk measures, *Insurance: Mathematics and Economics*, **34**, 505-516

[15] Karush, W. (1939), *Minima of functions of several variables with inequalities as side constraints*, M.Sc. dissertation, Department of Mathematics, University of Chicago, Chicago, IL, USA

[16] Kuhn, H.W. and A.W. Tucker (1951), Nonlinear programming, *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability*, J. Neyman ed., University of California Press, Berkeley, CA, USA

[17] Lampaert, I. and J.F. Walhin (2005), On the optimality of proportional reinsurance, *Scandinavian Actuarial Journal*, **3**, 225-239

[18] Landsman, Z. and M. Sherris (2001), Risk measures and insurance premium principles, *Insurance: Mathematics and Economics*, **29**, 103-115

[19] Lintner, J. (1965), The valuation of risky assets and the selection of risky investments in stock portfolios and capital budgets, *Review of Economic and Statistics*, **47**, 13-37

[20] Markowitz, H. (1952), Portfolio selection, *Journal of Finance*, **6**, 77-91

[21] Markowitz, H. (1956), The optimization of quadratic functions subject to linear constraints, *Naval Research Logistics Quarterly*, **3**, 111-133

[22] Markowitz, H. (1959), Portfolio selection: efficient diversification of investments. New York: John Wiley & Sons. (reprinted by Yale University Press, 1970; 2nd ed. Basil Blackwell, 1991)

[23] Markowitz, H. (2006), de Finetti scoops Markowitz, *Journal of Investment Management*, **4**, 5-18

[24] Miller, N. and A. Ruszczynski (2008), Risk-adjusted probability measures in portfolio optimization with coherent measures of risk, *European Journal of Operational Research*, **191**, 193-206

[25] Mossin, J. (1966), Equilibrium in a capital asset market, *Econometrica*, **34**, 768-783

[26] Pressacco, F. (1986), Separation theorems in proportional reinsurance, *Insurance and Risk Theory*, M. Goovaerts et al. eds., Reidel Publishing, 209-215

[27] Pressacco, F. and P. Serafini (2007), The origins of the mean-variance approach in finance: revisiting de Finetti 65 years later, *Decisions in Economics and Finance*, **10-1**, 19-49

[28] Rockafellar, R.T., S. Uryasev and M. Zabarankin (2006), Generalized deviations in risk analysis, *Finance & Stochastics*, **10**, 51-74

[29] Rubinstein, M. (2006), Bruno de Finetti and mean-variance portfolio selection, *Journal of Investment Management*, **4**, 3-4

[30] Sharpe, W. (1964), Capital asset prices: a theory of capital market equilibrium under conditions of risk, *Journal of Finance*, **19**, 425-442