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Application of Director Theory to Models of Flow in Straight and Curved Pipes

Application of Director Theory to Models of Flow in Straight and Curved Pipes

By

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Abstract

This thesis is about the application of director theory to modelling fluid flow in pipes and whether it could be beneficial to the field of cardiovascular modelling. Director theory was first developed in the field of solid mechanics, primarily by the Cosserat brothers, but was later applied in fluid dynamics by Green and Naghdi. However there has been limited literature since applying director theory in this field.

Director theory simplifies the solution of the full 3D Navier-Stokes equations in thin pipe-like geometries with arbitrary variation of thickness and orientation. Instead of solving the equations pointwise, the theory solves integrated versions of the equations over the cross-section of the pipe. Some information about the flow properties in the cross-section are retained by the weighting functions of the velocity directors (vectors) that depend on the cross-sectional coordinates. It is this property of director theory that is thought could provide more accuracy than the classical 1D models for cardiovascular modelling.

This thesis considers the development of director theory in regards to modelling fluid flow through various pipe geometries of increasing complexity. The accuracy of the model solutions are then assessed against full 3D computational simulations. The models considered include straight pipes of constant and varying radius, pipes with constant curvature of constant and varying radius and pipes of varying curvature.

With the director theory approach, the velocity of the fluid is approximated by an expansion of directors (vectors that can depend on the the coaxial coordinate and time) multiplied by weighting functions (which depend on the cross-sectional coordinates and are which take polynomial form in the models presented in this thesis). In Chapter 3, which focuses on the modelling of fluid flow through straight pipes, the order of this expansion is specified by K which corresponds to the highest order polynomials in the weighting functions. In Chapter 4, which focuses on the modelling of fluid flow though pipes of constant curvature, the order of the expansion in later stages of the derivation is specified by H, this also corresponds to the highest order polynomials in the weighting functions, but in this case the weighting functions will take a more complicated form in the end, after satisfying the relevant conditions to the case being considered. The difference in notation stems from this, which will become more apparent where the derivation in presented in the Chapter 4.

It is found that in a straight pipe of constant radius, the director theory approach with a velocity expansion order of $K = 3$ is sufficient to recover the exact Poiseuille flow solution. Director theory solutions were obtained for straight tapering and wavy

(sinusoidal) walled pipes at two orders, $K = 3$ and $K = 5$. For the tapered pipe, the maximum relative error between the director theory solution and the computational simulations, at Reynolds number $Re = 4$ was approximately 5% at order $K = 3$ and 3% at order $K = 5$. For the wavy walled pipe, the maximum relative error between the director theory solution and the computational simulations, at $Re = 4$ was approximately 4.5% at order $K = 3$ and 2.5% at order $K = 5$.

For the pipe with constant curvature and constant radius, director theory solutions were obtained at a range of orders from $H = 7$ to $H = 15$. The average relative error between the director theory solution and the computational simulation (with mesh base size $0.025m$), for $Re = 1000$ and curvature ratio $\delta = 0.01$, was approximately 2% for orders above $H = 8$. The average relative error between the director theory solution and the computational simulation (with mesh base size $0.025m$), for $Re = 1000$ and $\delta = 0.1$, was under 4% for orders above $H = 11$.

The central aim of this thesis is to modernise the presentation of director theory in the application of fluid mechanics to improve accessibility and help promote future work in the topic. In addition, a novel approach to deriving the system of equations for fluid flow in curved pipes is presented. This novel approach is more intuitive within the field of fluid dynamics as it is derived directly from the Navier-Stokes equations, while achieving the same end results as the method put forward by Green and Naghdi.

Dedication and acknowledgements

I would like to thank my supervisors Dr Alberto Gambaruto and Prof Alan Champneys for their support, advice and patience throughout this project. I would also like to thank the Engineering and Physical Sciences Research Council for funding this project. Additional thanks to my reviewer Dr Andrew Laurie for their helpful advice and to my mentor Mark Williams for their support.

I also appreciate the support and companionship from my friends, family and particularly my partner Miguel Lagunes Fortiz. I would like to dedicate this thesis to my late father Ronald Webster, who always remains an inspiration.

Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

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TABLE OF CONTENTS

Page

List of Tables

Table **Page**

List of Figures

Figure **Page**

Chapter 1

Introduction

Pipe flow is ubiquitous throughout the world, and is often associated with a transport process. Examples from man-made structures include the water networks that provide homes with running water, surgical tools and intravenous bags, as well as many machines in industry [15]. Examples from the natural world include plants where water is absorbed through the roots and transported throughout the plant as well as the cardiovascular system in humans where blood is continually pumped round the body, collecting oxygen from the lungs to be distributed throughout the body and expelling carbon dioxide [37]. In these examples and indeed generally, the conduits are rarely of simple design, but rather they exhibit changes in shape, diameter, curvature and torsion, and may include branching to form a network.

The motivation for this thesis is the mathematical modelling and numerical analysis of blood flow and associated transport phenomena of oxygen and drug solutes. The modelling is commonly carried out at a range of scales to find the balance between the accuracy of 3D models at a higher computational cost, and the computationally effective 1D models which provide simplified solutions. Non-invasive medical techniques such as magnetic resonance imaging (MRI) make it possible to reconstruct a 3D model of part of a patient's cardiovascular system [1]. This virtual model can be used to reliably simulate blood flow and provide an indication of the health state of the patient for a number of conditions and diseases.

The problem with 3D simulations is that they are too computationally expensive to run for large portions of the cardiovascular system. Simulations that take too long cannot

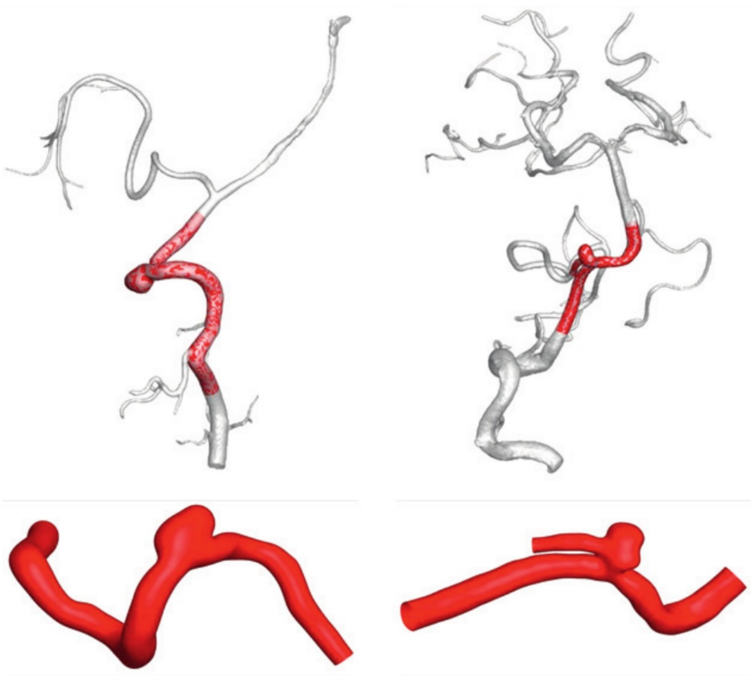


Figure 1.1: Geometries of cerebral arteries reconstructed from medical image data. The sections investigated in 3D CFD simulations are small portions of the larger cardiovascular network. [11]

be used effectively as a clinical tool. A way to overcome this is to couple 3D modelling techniques with 1D modelling techniques. This is done by modelling a small section of the arterial system in 3D and the remaining vasculature in 1D [1]. Fig 1.1 shows examples of this coupling for geometries of cerebral arteries. The sections in red represent what is typically used in 3D simulations. Fig 1.2 shows 3D and 1D modelling techniques of the branching of an artery, in which the 1D models provide boundary conditions at artificially truncated sections in the 3D models. Here, reduced order modelling is meaningful since it is expected that the flow in the coaxial direction will behave at a larger scale than the in-plane flow. However, flow in curved pipes is known to contain secondary flows which are not negligible.

In some previous works, a velocity profile is assumed at inflow for the 3D simulation, which is unlikely to be accurate. Using 1D models for the extensive arterial geometry is useful in providing the 3D simulations with accurate boundary conditions. Whilst 3D-1D-0D coupled models currently exist for numerical simulations of the cardiovascular system [1, 35], the intent is to look particularly at making use of director theory for the

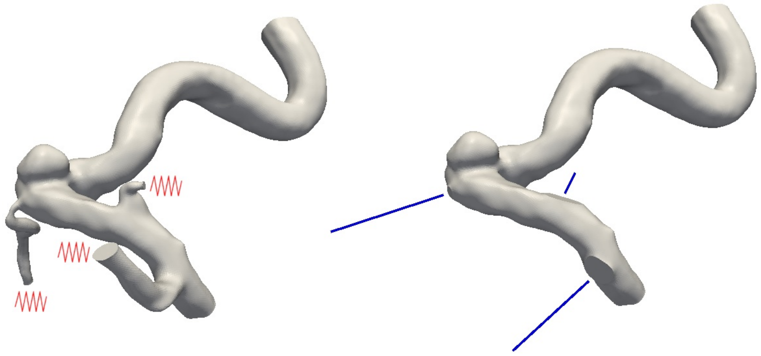


Figure 1.2: Modelling techniques of the branching of an artery. Left: branches are modelled as 3D geometries with 0D boundary conditions. Right: branches are modelled as 1D model boundary conditions to retain accuracy at a reduced computational cost. [44]

1D models. This has some benefits over more traditional 1D models. By making use of director theory (see Chapter 2), the model is not limited to straight line segments, and a more geometrically realistic representation of the arteries can be made. This should allow the 3D simulation regions to be provided with more accurate boundary conditions.

The biological tissues that make up blood vessels are alive and respond to flow conditions, amongst other factors. It has been found that vessels which remodel inwards, and can lead to stenosis, tend to have low wall shear stress and vessels that remodel outwards, which can lead to an aneurysm, tend to have high wall shear stress [12]. In diseases, this remodelling gets out of control, due to either a continued response that gets exaggerated, like an unstable feedback loop, or due to a fault with the biochemical reactions or the signaling pathways. The conditions that the tissue responds to can be biochemistry related (nutrient or waste transport for example) or mechanics related. Fluid mechanics plays an important role through mechanotransduction, in which mechanical stimulus is sensed by the cells and initiates a signaling pathway. Mechanical stimulus is often related to wall shear stress, or their derivatives, which is sensed by endothelial cells. While highly debated, it is thought that the remodelling occurs in part as an adapted response to alleviate undesirable haemodynamical conditions, such as a lowered or elevated wall shear stress and derived measures. Such discussions can be found in papers relating to cardiovascular modelling [10, 12, 11, 13, 27, 44].

The intent of this thesis is to provide a 1D modelling approach which employs director theory, resulting in a computationally efficient tool which retains good accuracy.

1.1 Related Work

1.1.1 Director theory

In this section a literature review on director theory and applications is presented, as well as relevant topics in cardiovascular modelling.

Director theory, also known as Cosserat theory due to the work on the subject by brothers Eugéne and François Cosserat, has been developed for many applications over the past century. The Cosserat brothers [22] lay out the foundations and development of director theory in detail in their book, although with quite different notation to that of more modern work.

In some of their earlier work on the subject, Green et al [8] present what they consider to be a complete general theory of a Cosserat surface that builds on work by previous authors by taking constitutive equations into account. They then go on to consider some special cases of the general theory, an elastic membrane being one case.

In later papers, they apply director theory to the thermodynamics of slender rods [3], with a novelty that a temperature change is allowed across the cross-section of the rod, and develop thermodynamic theories which account for electromagnetic effects [5]. Special cases considered in this paper include a magnetic thermoelastic rod and a non-conducting rod in free space.

In 1981, Naghdi [41] presented an account of recent developments in the direct formulation of theories of rods and shells.

A more recent application of director theory was in the field of computer graphics. Spillmann and Teschner [32] worked on modelling and dynamically simulating Cosserat nets, which are networks of elastic rods linked by elastic joints. These structures can be used to represent many objects such as elastic rings and coarse nets.

Over the past four or five decades, director theory has been applied in many areas within the field of fluid mechanics, such as Bogey at al's work on Cosserat jets [19]. Green and Naghdi have been particularly prolific authors in this area. They have published work on directed fluid sheets [2, 4], and used director theory to construct a general approach to viscous fluid flow in straight and curved pipes [6], for which the pipes are not required to be slender. They go on to use this direct approach to find an analytical solution for incompressible viscous fluid flow in curved pipes [7]. They find their results to compare well with experimental data and numerical solutions of the same problem, based on the Navier-Stokes equations. A similar model to this will be presented in Chapter 4, the novelty being in the way the system of equations are derived. They will be derived directly from the Navier-Stokes equations, presenting a more intuitive approach within the field of fluid mechanics.

Naghdi [40] applies the theory of directed fluid jets to Newtonian and non-Newtonian flows. He begins with the theory of a general case for any finite number of directors and later records results for the special cases of 2 and 5 directors. He considers the applicability to straight Newtonian jets and how it relates to problems of instability or breakup. He then considers Poiseuille flow through a circular pipe. He then focuses on Poiseuille flow in the theory of directed jets. The final section focuses on the determination on the unknown coefficients and constitutive response functions in the previous section. This is done by making use of results from the three-dimensional theory.

An approach similar to director theory was used by Caulk and Naghdi [18] to study slender axisymmetric surfaces of revolution. They make an assumption that the fluid velocity can be approximated by a Taylor expansion. They then simplify the form of the velocity by making use of the axisymmetry of the flow. They impose integral conditions over the cross-section rather than satisfying the momentum equation pointwise. After some manipulation and carrying out the integration, they end up with a system of six scalar partial differential equations describing the fluid flow. They then consider how different forms of the lateral surface simplify these equations. They find some solutions to specific cases which compare well with the corresponding exact analytical solution and experimental results.

The approach in the above paper is applied specifically to blood flow in small vessels by Carapau and Sequeira [23]. They state that using directed curves can be of use as an independent theory to predict some of the main properties of the 3D problems. Some advantages of the director theory are listed, including that the theory incorporates all components of the linear momentum and that invariance under superimposed rigid body motions is satisfied at each order. They specifically apply the theory to modelling blood flow in a straight, radially circular, rigid and impermeable walled vessel with constant radius. From this, they derive the unsteady relationship between mean pressure gradient and volume flow rate, along with the corresponding equation for wall shear stress. They found their results to compare well with others. They later used the same approach to model visco-elastic non-Newtonian generalised Oldroyd-B flows in an axisymmetric pipe with circular cross-section [24].

Another application of director theory to blood flow was by Robertson and Sequeira [13], who found the director theory approach to provide better results than the classical 1D models in geometries relevant to blood flow. Specifically considered is steady axisymmetric flow of incompressible linearly viscous fluids in slender bodies of revolution with rigid walls, where it is appropriate for the fluid to be modelled as quasi-steady.

Most of the works mentioned in the preceding paragraphs focus on flow through straight pipes and using what is often termed a nine director theory, which will correspond to order $K = 3$ in Chapter 3. Novelties of the work presented in this thesis will include presenting solutions for a wavy walled pipe, and presenting solutions at order $K = 5$ in addition to $K = 3$ to assess whether increasing the order increases the accuracy of the solution.

It would be fair to say that the full power of Naghdi and Green's method has not yet been brought to bear on the problem of flow in highly curved pipes of varying cross-section. One might have expected that such ideas would lie at the heart of 1D numerical approximation algorithms. At first sight, this might seem surprising because it was their work which laid the foundation of this theory. As shall be seen in this thesis though, their formulation requires a lot of additional work before it can be transformed into a practical tool. Addressing this challenge provides the primary aim for this thesis.

1.1.2 Cardiovascular flow modelling

The more mainstream approach to cardiovascular modelling [16] is to use classical 1D modelling or to model a small section of the arterial system in 3D and couple it with 1D or lumped parameter models. The classical 1D models have some in-built fluid-structure interaction modelling, and hence are widely used to investigate pressure pulses travelling in the cardiovascular system. However, they are known to provide unrealistic boundary conditions for velocity profiles when using a coupled 3D-1D model.

For 3D techniques, there is the pressure splitting approach (also known as the projection method) which was first developed by Chorin [17]. The algorithm involves decomposing the velocity into a divergence-free part and an irrotational part. Firstly, an intermediate velocity that does not satisfy the incompressibility constraint is computed at each time step. Then the pressure is used to project the intermediate velocity into a space of divergence-free velocity field to get the next update of velocity and pressure.

Papers of coupled modelling of the cardiovascular system include those by Formaggia and Veneziani [35], who derive the classical 1D models for blood flow in arteries and then focus on how to use multiscale models for the cardiovascular system, making use of lumped (0D) parameter models; Bui et al [1], who use fractal tree models of all regions of small cerebral vasculature; and Kashefi et al [10], who also develop a new extensive lumped parameter model of the entire arterial network which they find to have similar outcomes to those of the coupled 3D healthy model.

Further 1D cardiovascular modelling works include that of Low et al [34], who present an improved and robust 1D human arterial network model by adopting parts of the physical models from different authors to establish an accurate baseline model; and Flores et al [29], who developed a novel linear 1D dynamical theory of blood flow in networks of flexible vessels based on a generalised Darcy's model.

A particular area of interest is the relationship between blood flow properties and cardiovascular disease. Gambaruto and João [11] obtained a set of measures that have been related to diseased states, from analysis of the simulations in cerebral saccular aneurysm cases. These measures include the impingement region, separation lines, convective transport near the wall and vortex core lines or structures. Gambaruto et al [12] use post-operative MRI scans of patients with bypass grafts in the peripheral vasculature to assess how local haemodynamic parameters relate to vascular remodelling where the graft rejoins the host artery. It is found that regions of both low wall shear stress and convective transport towards the wall tend to correlate with regions of inward modelling.

Other interesting works in the area include that of Alastruey et al [28] look at the coupling of lumped parameter and 1D models. An algorithm is proposed and verified to accurately estimate peripheral resistance and compliances from in vivo data. In another paper, Alastruey et al [26] assess 1D numerical simulations against in vitro measurements. It was found that, compared to the purely elastic model, viscoelasticity significantly reduced the average relative root mean square errors between numerical and experimental waveforms over the 70 locations measured in the in vitro model.

Lee et al [33] aim to better the understanding of steady flow in 3D non-planar double bend geometries. These are chosen to loosely model a right coronary or femoral artery, neglecting branches. The 3D computations are performed using a high accuracy spectral/hp element Navier-Stokes solver. It is found that non-planarity has the biggest effects on mixing and asymmetric secondary flow streamlines.

Boileau et al [21] test the accuracy of some of the most common numerical methods applied to 1D modelling of blood vessels, including Galerkin methods, finite volume method, a finite difference method and a simplified trapezium rule method. They apply the methods to six benchmark test cases with an increasing degree of difficulty, from blood flow in a tube to blood flow in the ADAN56 model which contains the largest 56 systemic arteries of the cardiovascular system. They are compared with theoretical results, 3D numerical data and experimental data to assess their accuracy. Good agreement was found among all the numerical methods tested and they were able to capture the main features of pressure, flow and area waveforms. These findings support the use of 1D modelling as a computationally cost-effective tool.

Ramalho et al [44] study how geometric description and the prescription of outflow boundary conditions influence the computed flow field. Results of the effects of outflow boundary modelling choice on computed haemodynamic parameters are used to identify appropriateness of the models based on the physical interpretation.

Aguado-Sierra et al [25] propose separating the pressure and velocity into reservoir and wave parts. The separation algorithm is applied to in-vivo human and canine data as well as numerical data from a validated model of pulse wave propagation in the larger conduit arteries. The algorithm is found to be reasonably robust, indicating potential clinical usefulness.

There has also been previous works in the area that include curvature in the pipes. There were two papers by Siggers and Waters [30, 31] focusing on steady and unsteady flows, respectively, in curved pipes. The novelty in these papers is that finite curvature is considered rather than asymptotically small curvature and the Coriolis force is considered in addition to the centrifugal force. Flow is considered to be driven by either a steady or oscillatory axial pressure gradient and a local coordinate system following the curvature is used, similar to the director theory approach. For small Dean number and ratio of curvature, an asymptotic solution in these parameters is considered. For large Dean number and finite curvature, an integrated form of the boundary layer equation in considered.

Alastruey et al [27] presented a new method to investigate the mechanisms by which vascular curvature and torsion affect blood flow. It is applied to steady flow in single bends, helices, double bends and a rabbit thoraic aorta based on image data. The roles of each of the forces on the patterns of primary and secondary velocities, vortical structures and wall stresses in each cross-section are analysed.

The majority of the current work, with a few exceptions, on cardiovascular modelling use either lumped parameter models or classical 1D models, sometimes coupled with a section of a full 3D model. Classical 1D models can be useful for modelling a large network that would be infeasible to model in 3D due to computation costs. However, oversimplifications of the geometry can lead to inaccuracies and loss of information. The director theory method offers an appealing alternative to classical 1D models, with the ability to allow for curvature and capture secondary flows. However, for this to be an implementable tool, a clearer methodology for the theory and research into how to model more complex geometries than have been considered in the past is required. The intent of this thesis is to work toward overcoming these barriers.

1.2 Overview of the Director Theory Methodology

Firstly the equations of motion are derived from the conservation of mass and linear momentum, and nondimensionalised, then solutions are obtained by considering specific pipe geometries which allows the system of equations to be simplified and solved.

The method for deriving the equations of motion involves approximating the velocity field as a summation of vectors (directors) that may depend on the coaxial coordinate along the pipe (hence a 1D model) and time, multiplied by weighting functions that depend on the cross-sectional coordinates. The form of the weighting functions is chosen, which will generally be polynomials of the cross-sectional coordinates. Then an order for the velocity expansion is chosen, which will generally relate to the polynomial order of the weighting functions. The directors are the terms that will be solved for, for a specified Reynolds number or pressure drop. The number of unknown directors is first reduced any considering boundary conditions and fluid properties such as incompressibility, along with any symmetry conditions. The reduced form of the velocity expansion is substituted into the Navier-Stokes equations. The conservation of momentum of each director is considered by forming a system of equations of the Navier-Stokes equations multiplied by each respective weighting function and integrated over the cross-section. This system of equations is then solved for using an appropriate method, such as least squares or an iterative solver, to find the value of the directors, which can then be substituted back into the velocity expansion to give the velocity profile. The particular constraints on the velocity form and how the pressure terms are dealt with will vary depending on the geometry being considered.

The method can be summarised in algorithmic form as follows:

1. Set up a coordinate system that follows the centreline of the pipe.

2. Derive the Navier-Stokes equations in this coordinate system.

3. Choose a form for the weighting functions (e.g polynomials) and the order of the velocity expansion.

4. Reduce the number of unknown directors by requiring the velocity expansion to satisfy any conditions being considered (such as steady state, fully developed flow, no-slip, incompressibility and symmetry conditions).

5. Substitute the reduced form of the velocity expansion into the derived Navier-Stokes equations.

6. Form the system of equations, that will depend at most of the coaxial coordinate and time, by multiplying the Navier-Stokes equations by each weighting function and integrating over the cross-section of the pipe. Prescribe the coaxial pressure gradient, while how to deal with the cross-sectional pressure gradients will depend the geometry being considered.

7. Nondimensionalise the system of equations.

8. Solve the equations using an appropriate method for the system, such as least squares or an iterative solver, to obtain the values of the directors.

9. Substitute into the reduced velocity expansion to recover the velocity profile.

10. Comparisons can then be made with known exact solutions or results of 3D simulations to assess the accuracy.

1.3 Objectives

This chapter has provided a review of literature of director theory and cardiovascular modelling. Director theory was originally developed within the field of solid mechanics and later adapted as of 1D modelling technique in fluid mechanics. Director theory has been applied in fluid mechanics to model fluid flow in channels, straight Newtonian jets and in straight and curved pipes. Classical approaches to cardiovascular modelling involve coupling 3D models with typical 1D or lumped parameter models. The intent of this thesis is to build on the existing body of literature, by introducing curved pipes into the director theory approach to modelling the cardiovascular system. By having a more realistic geometry in the model, the hope is to obtain improved results to feed into the 3D model. An accurate model that is time efficient to run could be of use in a clinical setting. The director theory approach offers improved accuracy over traditional 1D models due to the allowance of curvature and secondary flows, while being computationally cheaper than full 3D models.

There has been a limited amount of work involving director theory in recent years. A secondary aim of this thesis is to modernise the presentation of director theory for fluid flow in pipes, which may improve accessibility. In addition, a novel approach for deriving the system of equations for the director theory model will be introduced, namely they will be derived directly from the Navier-Stokes equations, which is more intuitive than the solid mechanics based approach presented by Green and Naghdi [6] and which will be outlined in Chapter 2.

To allow for useful applications going forward, future researchers will need to be able to turn director theory into a practical computational tool. Hence the objective of this thesis is to turn the somewhat abstract methods of Naghdi and Green into something that can be applied algorithmically, while at the same time, providing a robust comparison of the accuracy of the method with the modern 3D computational simulations.

1.4 Outline

The rest of this thesis is outlined as follows.

Chapter 2, which follows, introduces the concept of director theory in relation to fluid mechanics and contains a mathematical derivation of the conservation laws in regards to director theory. It also provides a summary of the approach that will be taken, to model fluid flow in pipes using director theory, in later chapters. Finally, a background is given on the computational fluid dynamics (CFD) simulations that will be used for comparison with the director theory models.

Chapter 3 contains a director theory approach to modelling fluid flow in straight pipes. Pipe geometries considered in Chapter 3 are a pipe of constant radius, a tapered pipe and a wavy walled (sinusoidal) pipe. The results are then compared with those of equivalent CFD simulations. The tapered and wavy walled pipes are novel applications of the director theory method.

Chapter 4 contains a director theory approach to modelling fluid flow in a toroidally curved (constant curvature) pipe of constant radius. A similar method to Chapter 3 is followed, in which the equations of motion are derived and non-dimensionalised, although in this case, a new coordinate system is firstly set up which follows the centreline of the curved pipe. A larger number of directors are required to model the more complicated flow in a curved pipe, which leads to a larger system of equations to solve for. The system is solved using an in-built solver (*fsolve*) in Maple [38] at a range of orders and compared with corresponding CFD simulations.

Chapter 5 contains an insight into how to approach fluid flow modelling in more generally curved pipes using director theory. The equations of motion are derived for a toroidally curved pipe with varying radius and for a pipe with varying curvature.

Chapter 6 contains conclusions and suggests avenues for future work.

CHAPTER 2

Development of Director Theory for Fluid Mechanics

2.1 Introduction

The first goal is to focus on gaining a solid understanding of director theory and how to apply it within the field of fluid mechanics, specifically looking at fluid flow in pipes as this will be relevant for modelling blood flow in arteries.

Director theory (also known as Cosserat theory) is a powerful technique that uses vectors called directors to describe a geometry. There are hierarchical theories for 3D continua, 2D shells, 1D rods and 0D points, which are detailed in the book by Rubin [43]. Cosserat theory was first applied to solid mechanics and later also to fluid mechanics. In applications to fluid mechanics, it is akin to approximating the velocity field by expressing it as a summation of vectors that depend only on the coaxial coordinate and time, multiplied by weighting (or shape) functions that depend on the in-plane coordinates, resulting in a higher-order 1D model.

Relevant to this thesis, is the potential to use these one-dimensional Cosserat rods to help accurately model the complicated geometry of the cardiovascular system. The advantage of Cosserat rods is that the directors ($\mathbf{d}_i$) depend only on the tangent of the space curve and time (as shown in Fig 2.1), allowing a simpler description of curves that bend in space and time than traditional coordinate systems.

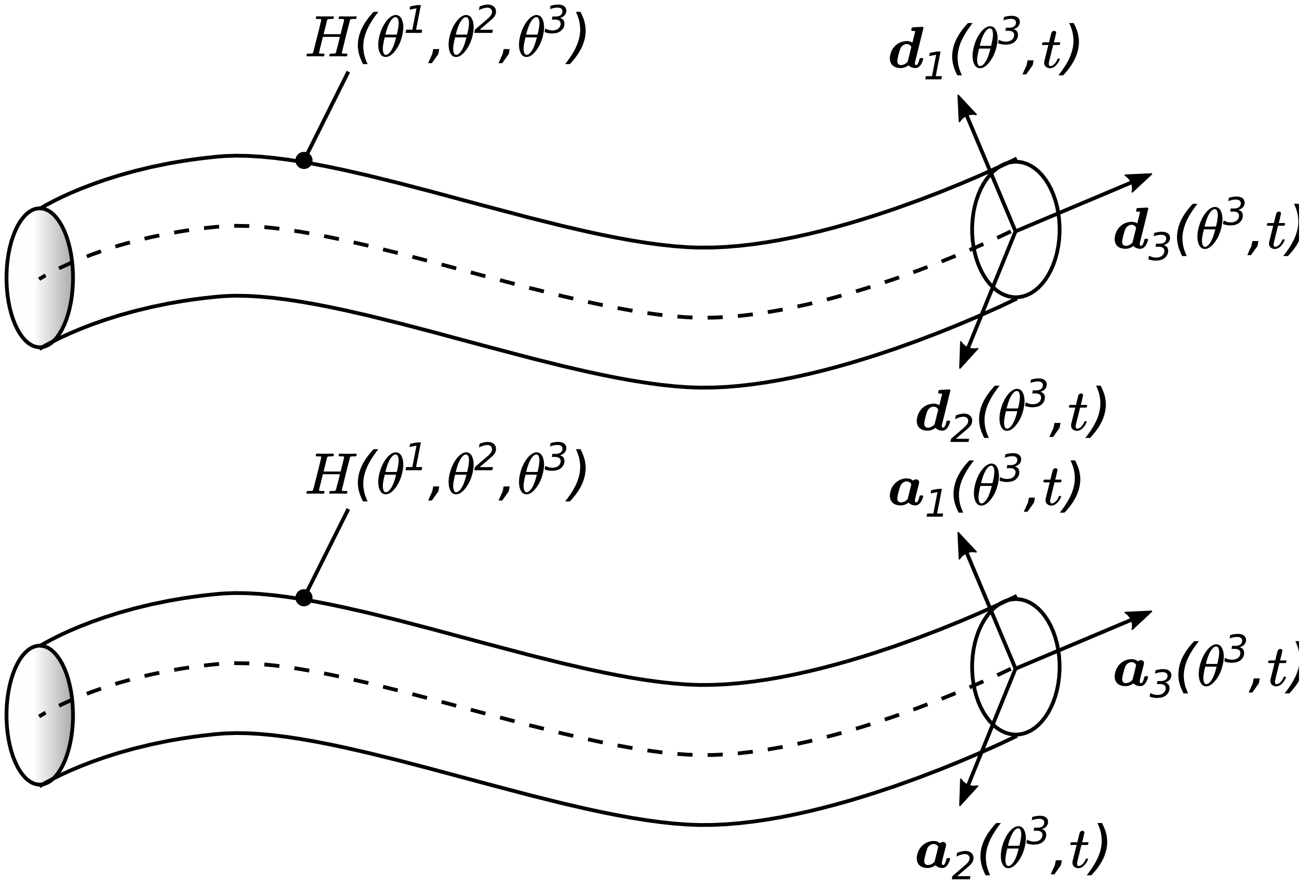


Figure 2.1: Cosserat rod along with directors $\mathbf{d}_1, \mathbf{d}_2, \mathbf{d}_3$ (top) and curvilinear base vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ (bottom), with H being the lateral surface [43].

2.2 Development of the Theory

A brief outline of how the theory is developed is provided in Appendix A of the paper by Green and Naghdi [6]. Owing to the complexity of the theory, and the brevity of the treatment in that Appendix, a detailed derivation and explanation is given below, in so doing, also providing a clearer presentation of the methodology. This lays out how to form the new coordinate system, the expansions for position and velocity and how to rewrite various balance laws in these terms.

This will be done to explore and understand the original approach by Green and Naghdi of applying director theory to fluid mechanics. However, a different approach to deriving the system of equations will be presented, that comes directly from the Navier-Stokes equations. This presents a more intuitive method than the approach presented in this chapter, which was first developed within the field of solid mechanics.

2.2.1 Coordinate systems and covariant and contravariant basis vectors

Presently, a Lagrangian coordinate system will be set up and the mapping between the Eulerian (fixed) and Lagrangian (convective) system will be shown. A convective frame is often appealing in setting up the problem and equations in fluid mechanics. Covariant and contravariant basis vectors for the Lagrangian system will be defined. Expressions for the position vector and velocity vector of a particle in the Lagrangian system will be stated. The Lagrangian system will then be related to a curvilinear system at a fixed time t, and consequently can be readily related to a fixed reference frame.

The motivation for this is to to be able to go from the Lagrangian form of the balance equations, which the director curve is defined in, to a Eulerian form which is the standard system to work in within fluid dynamics.

The system of Cartesian coordinates shall be denoted by (x^1, x^2, x^3), with corresponding base vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$. As such, the position vector $\mathbf{x}$ of an arbitrary point in this space, from the origin O, is given by

$$\mathbf{x} = x^i\ \mathbf{e}_i = x^1\ \mathbf{e}_1 + x^2\ \mathbf{e}_2 + x^3\ \mathbf{e}_3 \tag{2.1}$$

Now consider a system of convective coordinates denoted by $(\theta^1, \theta^2, \theta^3)$. At each time t, there will be a bijective mapping from the Cartesian coordinate system to the convective coordinate system. There therefore exist functions σ^i and ψ^i, $(i = 1, 2, 3)$ such that

$$\theta^i = \sigma^i(x^1, x^2, x^3, t)$$

and

$$x^i = \psi^i(\theta^1, \theta^2, \theta^3, t).$$

So the position vector from (2.1) can also be expressed as follows

$$\begin{aligned}\mathbf{x} &= \psi^i(\theta^j, t)\ \mathbf{e}_i \\ &= \psi^1(\theta^1, \theta^2, \theta^3, t)\ \mathbf{e}_1 + \psi^2(\theta^1, \theta^2, \theta^3, t)\ \mathbf{e}_2 + \psi^3(\theta^1, \theta^2, \theta^3, t)\ \mathbf{e}_3.\end{aligned}$$

There are two useful properties that the basis vectors, $\mathbf{e}_i$, of the Cartesian coordinate system have. Firstly, each $\mathbf{e}_i$ is the direction of increase of a single coordinate, with the other coordinates being held constant. Secondly, each $\mathbf{e}_i$ is orthogonal to a plane formed by holding one coordinate constant and allowing the others to vary. [20]

If the basis vectors for our convective coordinate system are formed using the first property, covariant base vectors are obtained. These are obtained from the Cartesian basis vectors $\mathbf{u}_i$ by observing how a change of a single convective coordinate, while holding the others constant, affects the Cartesian coordinates. That is, the covariant basis vectors $\mathbf{g}_i$ are defined by

$$\begin{aligned}\mathbf{g}_i &= \frac{\partial x^j}{\partial \theta^i}\mathbf{e}_j \\ &= \frac{\partial x^1}{\partial \theta^i}\mathbf{e}_1 + \frac{\partial x^2}{\partial \theta^i}\mathbf{e}_2 + \frac{\partial x^3}{\partial \theta^i}\mathbf{e}_3.\end{aligned}$$

If the basis vectors for our convective coordinate system are formed using the second property, the contravariant base vectors are obtained. The contravariant basis vectors $\mathbf{g}^i$ are defined by

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j,$$

where δ^i_j is the Kronecker delta defined by

$$\delta^i_j = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{otherwise.} \end{cases}$$

Now let g_{ij} be defined as

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j,$$

and g^{ij} be defined as

$$g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j.$$

Here, g_{ij} and g^{ij} are the covariant and contravariant metric tensors, respectively. See [9, 14] for more information. Let $g^{(1/2)}$ be defined to be the determinant of the matrix whose columns are the covariant base vectors $\mathbf{g}_i$. That is

$$\begin{aligned}g^{(1/2)} &= \det(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3) \\ &= \begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix}.\end{aligned}$$

The velocity of a typical particle is given by

$$\begin{aligned}\mathbf{v} &= \dot{\mathbf{x}} \\ &= \begin{pmatrix} \frac{\partial x^1}{\partial t} \\ \frac{\partial x^2}{\partial t} \\ \frac{\partial x^3}{\partial t} \end{pmatrix}.\end{aligned} \tag{2.2}$$

The convective coordinates, θ^i, are held constant in the above as they always represent the same particle. Now assume the body is bounded by the surface

$$H(\theta^1,\theta^2,\theta^3) = 0. \tag{2.3}$$

Suppose that this surface represents the lateral surface of a tube-like body along the θ^3-direction. For this tube-like body under consideration, let the position vector $\mathbf{x}$ admit the representation

$$\mathbf{x} = \sum_{N=0}^{K} \lambda_N(\theta^1,\theta^2)\,\mathbf{d}_N(\theta^3,t), \tag{2.4}$$

with $\mathbf{d}_0 = \mathbf{r}(\theta^3,t)$ and $\lambda_0 = 1$. The $\lambda_N(\theta^1,\theta^2)$ can be thought of as weighting functions while the $\mathbf{d}_N(\theta^3,t)$ represent the directors; $\mathbf{r}$ can be identified as the position vector representing points on a one-dimensional curve in the tube-like body. Its tangent vector is defined by

$$\mathbf{a}_3 = \frac{\partial \mathbf{r}}{\partial \theta^3} = \begin{pmatrix} \frac{\partial r^1}{\partial \theta^3} \\ \frac{\partial r^2}{\partial \theta^3} \\ \frac{\partial r^3}{\partial \theta^3} \end{pmatrix}.$$

Using Eq. (2.4), the velocity vector in Eq. (2.2) takes the form

$$\mathbf{v} = \sum_{N=0}^{K} \lambda_N(\theta^1,\theta^2)\,\mathbf{w}_N(\theta^3,t),$$

where $\mathbf{w}_0 = \mathbf{v}^* = \frac{\partial \mathbf{r}}{\partial t}$ and $\mathbf{w}_N = \frac{\partial \mathbf{d_N}}{\partial t}$. So the velocity is considered a summation of vectors (directors) that depend on a single spatial coordinate, that is the coordinate following the centreline of the pipe, multiplied by scalar function of the cross-sectional coordinates, which could be thought of as weighting functions.

Now let ζ^i with $i = 1,2,3$ be a system of fixed curvilinear coordinates in the same Euclidean 3-space and let points in this space be specified by the position vector

$$\bar{\mathbf{x}} = \bar{\mathbf{r}}(\zeta^i).$$

The corresponding covariant base vectors, $\bar{\mathbf{g}}_i$, are given by

$$\bar{\mathbf{g}}_i = \frac{\partial x^j}{\partial \zeta^i}\mathbf{e}_j = \frac{\partial x^1}{\partial \zeta^i}\mathbf{e}_1 + \frac{\partial x^2}{\partial \zeta^i}\mathbf{e}_2 + \frac{\partial x^3}{\partial \zeta^i}\mathbf{e}_3. \tag{2.5}$$

The contravariant base vectors, $\bar{\mathbf{g}}^i$, are defined by

$$\bar{\mathbf{g}}^i \cdot \bar{\mathbf{g}}_j = \delta^i_j. \tag{2.6}$$

Now let $\bar{g}_{ij}$ be defined as

$$\bar{g}_{ij} = \bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j,$$

and $\bar{g}^{ij}$ be defined as

$$\bar{g}^{ij} = \bar{\mathbf{g}}^i \cdot \bar{\mathbf{g}}^j.$$

Here, $\bar{g}_{ij}$ and $\bar{g}^{ij}$ are the covariant and contravariant metric tensors, respectively. Then, $\bar{g}^{(1/2)}$ is defined to be the determinant of the matrix whose columns are the covariant base vectors $\bar{\mathbf{g}}_i$. That is

$$\begin{aligned} \bar{g}^{(1/2)} &= \det(\bar{\mathbf{g}}_1, \bar{\mathbf{g}}_2, \bar{\mathbf{g}}_3) \\ &= \begin{vmatrix} \frac{\partial x^1}{\partial \zeta^1} & \frac{\partial x^1}{\partial \zeta^2} & \frac{\partial x^1}{\partial \zeta^3} \\ \frac{\partial x^2}{\partial \zeta^1} & \frac{\partial x^2}{\partial \zeta^2} & \frac{\partial x^2}{\partial \zeta^3} \\ \frac{\partial x^3}{\partial \zeta^1} & \frac{\partial x^3}{\partial \zeta^2} & \frac{\partial x^3}{\partial \zeta^3} \end{vmatrix}. \end{aligned} \tag{2.7}$$

A fixed reference curve may be selected in this space, represented by the position vector

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}(\zeta^3),$$

and its tangent vector is defined by

$$\begin{aligned} \bar{\mathbf{a}}_3 &= \frac{\partial \bar{\mathbf{r}}}{\partial \zeta^3} \\ &= \begin{pmatrix} \frac{\partial \bar{r}^1}{\partial \zeta^3} \\ \frac{\partial \bar{r}^2}{\partial \zeta^3} \\ \frac{\partial \bar{r}^3}{\partial \zeta^3} \end{pmatrix}. \end{aligned}$$

In terms of the fixed coordinates, ζ^i, the velocity of the tube-like body at time t may be represented by

$$\begin{aligned} \mathbf{v} &= \bar{\mathbf{v}}(\zeta^i, t) \\ &= \bar{v}^i \bar{\mathbf{g}}_i \\ &= \sum_{N=0}^{K} \bar{\lambda}_N(\zeta^1, \zeta^2) \bar{\mathbf{w}}_N(\zeta^3, t), \end{aligned} \tag{2.8}$$

where $\bar{\mathbf{w}}_0 = \bar{\mathbf{v}}^*$.

The surface described in Eq. (2.3), which bounds the body, can now be specified by

$$\bar{H}(\zeta^1,\zeta^2,\zeta^3,t) = 0.$$

This is a material surface and moves with the body. Therefore its material derivative will be equal to zero:

$$\begin{aligned}\frac{D\bar{H}}{Dt} &= \frac{\partial\bar{H}}{\partial t}\frac{dt}{dt} + \frac{\partial\bar{H}}{\partial x}\frac{dx}{dt}\\ &= \frac{\partial\bar{H}}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\bar{H}\\ &= \frac{\partial\bar{H}}{\partial t} + \bar{v}^i\bar{\mathbf{g}}_i\cdot\frac{\partial\bar{H}}{\partial\zeta^j}\bar{\mathbf{g}}^j\\ &= \frac{\partial\bar{H}}{\partial t} + \bar{v}^i\frac{\partial\bar{H}}{\partial\zeta^i}\\ &= 0.\end{aligned}$$

2.2.2 Representation of the physics

Here, expressions for important physical laws, such as conservation of mass density and conservation of linear momentum will be derived in the curvilinear coordinate system.

Any function associated with the body may be expressed either in terms of (θ^i,t) or in terms of (ζ^i,t). Thus

$$F(\theta^i,t) = \bar{F}(\zeta^i,t). \tag{2.9}$$

In particular, the mass density has representations

$$\rho(\theta^i,t) = \bar{\rho}(\zeta^i,t),$$

and the conservation of mass requires that

$$\begin{aligned}0 &= \dot{\rho} + \rho(\nabla\cdot\mathbf{v})\\ &= \frac{\partial\bar{\rho}}{\partial t} + \bar{\mathbf{v}}\cdot\nabla\bar{\rho} + \bar{\rho}(\nabla\cdot\bar{\mathbf{v}}) \qquad \left(\text{here the derivatives } \nabla \text{ are } \frac{\partial}{\partial\theta}\right)\\ &= \frac{\partial\bar{\rho}}{\partial t} + \nabla\cdot(\bar{\rho}\bar{\mathbf{v}})\\ &= \frac{\partial\bar{\rho}}{\partial t} + \frac{1}{\bar{g}^{(1/2)}}\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{v}^i). \qquad \left(\text{here } \bar{g}^{-1/2} \text{ is used as } \bar{g}^{(1/2)}\partial\zeta = \partial\theta\right)\end{aligned} \tag{2.10}$$

More generally, the conservation laws may be written for ρF as

$$0 = \dot{\overline{\rho F}} + \rho F(\nabla\cdot\mathbf{v}) = \frac{\partial}{\partial t}(\bar{\rho}\bar{F}) + \frac{1}{\bar{g}^{(1/2)}}\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{F}\bar{v}^i). \qquad \left(\text{here } \dot{\overline{\rho F}} = \frac{\partial(\rho F)}{\partial t}\right) \tag{2.11}$$

Now the convected coordinates θ^i are chosen such that at time t the θ^i curves coincide with the fixed ζ^i curves. Also, the moving curve represented by the position vector $\mathbf{r}$ coincides with the fixed reference curve represented by the position vector $\bar{\mathbf{r}}$ at this time. With this choice of θ^i,

$$\mathbf{g}_i = \bar{\mathbf{g}}_i$$
$$g_{ij} = \bar{g}_{ij}$$
$$g^{ij} = \bar{g}^{ij}$$
$$g^{(1/2)} = \bar{g}^{(1/2)}$$
$$\mathbf{a}_3 = \bar{\mathbf{a}}_3$$
$$\lambda(\theta^1,\theta^2) = \bar{\lambda}(\zeta^1,\zeta^2).$$

2.2.3 Averaging across the cross-section

The key step in the theory of Green and Naghdi [6] is creating averaged versions of the equations of motion Eq. (2.10) and Eq. (2.11) by first projecting onto each of the directors and then averaging by integration across the cross section. The details are presented below. The cases of Eq. (2.10) and Eq. (2.11) are treated separately, to obtain the integrations equations for the conservation of mass density and linear momentum respectively. Beginning with mass density, first multiply both sides of Eq. (2.10) by $\lambda_N\lambda_M = \bar{\lambda}_N\bar{\lambda}_M$:

$$\begin{aligned}
0 &= \left(\dot{\rho} + \rho(\nabla\cdot\mathbf{v})\right)\lambda_N\lambda_M \\
&= \frac{\partial}{\partial t}(\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M) + \frac{1}{\bar{g}^{(1/2)}}\bar{\lambda}_N\bar{\lambda}_M\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{v}^i) \\
&= \frac{\partial}{\partial t}\left(\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\right) + \frac{1}{\bar{g}^{(1/2)}}\frac{\partial}{\partial\zeta^i}\left(\bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\bar{v}^i\right) - \bar{\rho}\bar{v}^\alpha\left(\frac{\partial\bar{\lambda}_M}{\partial\zeta^\alpha}\bar{\lambda}_N + \frac{\partial\bar{\lambda}_N}{\partial\zeta^\alpha}\bar{\lambda}_M\right),
\end{aligned}$$

where $\alpha = 1,2$ and repeated indices are summed. This is because the second term on the right hand side is obtained from

$$\begin{aligned}
\frac{1}{\bar{g}^{(1/2)}}\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\rho\bar{\lambda}_N\bar{\lambda}_M\bar{v}^i) &= \frac{1}{\bar{g}^{(1/2)}}\bar{\lambda}_N\bar{\lambda}_M\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{v}^i) + \frac{1}{\bar{g}^{(1/2)}}\bar{g}^{(1/2)}\rho\bar{v}^i\frac{\partial}{\partial\zeta^i}(\bar{\lambda}_N\bar{\lambda}_M) \\
&= \frac{1}{\bar{g}^{(1/2)}}\bar{\lambda}_N\bar{\lambda}_M\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{v}^i) + \bar{\rho}\bar{v}^i\left(\bar{\lambda}_N\frac{\partial\bar{\lambda}_M}{\partial\zeta^i} + \bar{\lambda}_M\frac{\partial\bar{\lambda}_N}{\partial\zeta^i}\right)
\end{aligned}$$

Now, multiplying both sides by $\bar{g}^{(1/2)}$ (note that $g^{(1/2)} = \bar{g}^{(1/2)}$ as under consideration is when the curves represented in the curvilinear and Lagrangian coordinate systems coincide, so the left hand side is multiplied by $g^{(1/2)}$ and the right hand side by $\bar{g}^{(1/2)}$), to obtain

$$0 = \left(\dot{\rho} + \rho(\nabla \cdot \mathbf{v})\right) \lambda_N \lambda_M g^{(1/2)} = \frac{\partial}{\partial t}\left(\bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \bar{\lambda}_M\right) + \frac{\partial}{\partial \zeta^i}\left(\bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \bar{\lambda}_M \bar{v}^i\right) - \bar{g}^{(1/2)} \bar{\rho} \bar{v}^\alpha \left(\frac{\partial \bar{\lambda}_M}{\partial \zeta^\alpha} \bar{\lambda}_N + \frac{\partial \bar{\lambda}_N}{\partial \zeta^\alpha} \bar{\lambda}_M\right).$$

As shown in [14],

$$\overline{\dot{g^{(1/2)}}} = \frac{d}{dt} \begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{d}{dt}\left(\frac{\partial x^1}{\partial \theta^1}\right) & \frac{d}{dt}\left(\frac{\partial x^1}{\partial \theta^2}\right) & \frac{d}{dt}\left(\frac{\partial x^1}{\partial \theta^3}\right) \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{d}{dt}\left(\frac{\partial x^2}{\partial \theta^1}\right) & \frac{d}{dt}\left(\frac{\partial x^2}{\partial \theta^2}\right) & \frac{d}{dt}\left(\frac{\partial x^2}{\partial \theta^3}\right) \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{d}{dt}\left(\frac{\partial x^3}{\partial \theta^1}\right) & \frac{d}{dt}\left(\frac{\partial x^3}{\partial \theta^2}\right) & \frac{d}{dt}\left(\frac{\partial x^3}{\partial \theta^3}\right) \end{vmatrix},$$

and

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial x^i}{\partial \theta^j}\right) &= \frac{\partial}{\partial \theta^j}\left(\frac{dx^i}{dt}\right) \\ &= \frac{\partial v^i}{\partial \theta^j} \\ &= \frac{\partial v^i}{\partial x^k} \frac{\partial x^k}{\partial \theta^j}. \end{aligned}$$

So

$$\overline{\dot{g^{(1/2)}}} = \begin{vmatrix} \frac{\partial v^1}{\partial x^k} \frac{\partial x^k}{\partial \theta^1} & \frac{\partial v^1}{\partial x^k} \frac{\partial x^k}{\partial \theta^2} & \frac{\partial v^1}{\partial x^k} \frac{\partial x^k}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial v^2}{\partial x^k} \frac{\partial x^k}{\partial \theta^1} & \frac{\partial v^2}{\partial x^k} \frac{\partial x^k}{\partial \theta^2} & \frac{\partial v^2}{\partial x^k} \frac{\partial x^k}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix} + \begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial v^3}{\partial x^k} \frac{\partial x^k}{\partial \theta^1} & \frac{\partial v^3}{\partial x^k} \frac{\partial x^k}{\partial \theta^2} & \frac{\partial v^3}{\partial x^k} \frac{\partial x^k}{\partial \theta^3} \end{vmatrix}$$

and

$$\begin{aligned}
\begin{vmatrix} \frac{\partial v^1}{\partial x^k}\frac{\partial x^k}{\partial \theta^1} & \frac{\partial v^1}{\partial x^k}\frac{\partial x^k}{\partial \theta^2} & \frac{\partial v^1}{\partial x^k}\frac{\partial x^k}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix} &= \left(\frac{\partial v^1}{\partial x^1}\frac{\partial x^1}{\partial \theta^1} + \frac{\partial v^1}{\partial x^2}\frac{\partial x^2}{\partial \theta^1} + \frac{\partial v^1}{\partial x^3}\frac{\partial x^3}{\partial \theta^1}\right)\left(\frac{\partial x^2}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^3} - \frac{\partial x^3}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^3}\right) \\
&- \left(\frac{\partial v^1}{\partial x^1}\frac{\partial x^1}{\partial \theta^2} + \frac{\partial v^1}{\partial x^2}\frac{\partial x^2}{\partial \theta^2} + \frac{\partial v^1}{\partial x^3}\frac{\partial x^3}{\partial \theta^2}\right)\left(\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^3} - \frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^3}\right) \\
&+ \left(\frac{\partial v^1}{\partial x^1}\frac{\partial x^1}{\partial \theta^3} + \frac{\partial v^1}{\partial x^2}\frac{\partial x^2}{\partial \theta^3} + \frac{\partial v^1}{\partial x^3}\frac{\partial x^3}{\partial \theta^3}\right)\left(\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2} - \frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\right) \\
&= \frac{\partial v^1}{\partial x^1}\left(\frac{\partial x^1}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^3} - \frac{\partial x^1}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^3} - \frac{\partial x^1}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^3}\right. \\
&\left. + \frac{\partial x^1}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^3} + \frac{\partial x^1}{\partial \theta^3}\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2} - \frac{\partial x^1}{\partial \theta^3}\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\right) \\
&+ \frac{\partial v^1}{\partial x^2}\left(\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^3} - \frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^3} - \frac{\partial x^2}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^3}\right. \\
&\left. + \frac{\partial x^2}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^3} + \frac{\partial x^2}{\partial \theta^3}\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2} - \frac{\partial x^2}{\partial \theta^3}\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\right) \\
&+ \frac{\partial v^1}{\partial x^3}\left(\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^3} - \frac{\partial x^3}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^3} - \frac{\partial x^3}{\partial \theta^2}\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^3}\right. \\
&\left. + \frac{\partial x^3}{\partial \theta^2}\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^3} + \frac{\partial x^3}{\partial \theta^3}\frac{\partial x^2}{\partial \theta^1}\frac{\partial x^3}{\partial \theta^2} - \frac{\partial x^3}{\partial \theta^3}\frac{\partial x^3}{\partial \theta^1}\frac{\partial x^2}{\partial \theta^2}\right) \\
&= \frac{\partial v^1}{\partial x^1} g^{(1/2)}.
\end{aligned}$$

Similarly,

$$\begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial v^2}{\partial x^k}\frac{\partial x^k}{\partial \theta^1} & \frac{\partial v^2}{\partial x^k}\frac{\partial x^k}{\partial \theta^2} & \frac{\partial v^2}{\partial x^k}\frac{\partial x^k}{\partial \theta^3} \\ \frac{\partial x^3}{\partial \theta^1} & \frac{\partial x^3}{\partial \theta^2} & \frac{\partial x^3}{\partial \theta^3} \end{vmatrix} = \frac{\partial v^2}{\partial x^2} g^{(1/2)}$$

and

$$\begin{vmatrix} \frac{\partial x^1}{\partial \theta^1} & \frac{\partial x^1}{\partial \theta^2} & \frac{\partial x^1}{\partial \theta^3} \\ \frac{\partial x^2}{\partial \theta^1} & \frac{\partial x^2}{\partial \theta^2} & \frac{\partial x^2}{\partial \theta^3} \\ \frac{\partial v^3}{\partial x^k}\frac{\partial x^k}{\partial \theta^1} & \frac{\partial v^3}{\partial x^k}\frac{\partial x^k}{\partial \theta^2} & \frac{\partial v^3}{\partial x^k}\frac{\partial x^k}{\partial \theta^3} \end{vmatrix} = \frac{\partial v^3}{\partial x^3} g^{(1/2)}.$$

So

$$\begin{aligned}
\overline{g^{(1/2)}}^{\,\cdot} &= \left(\frac{\partial v^1}{\partial x^1} + \frac{\partial v^2}{\partial x^2} + \frac{\partial v^3}{\partial x^3}\right) g^{(1/2)} \\
&= (\nabla \cdot \mathbf{v})\, g^{(1/2)}.
\end{aligned}$$

Alternatively, this can be derived in tensor notation (see e.g. pp61-63, [42]). Therefore

$$
\begin{aligned}
0 &= (\dot{\rho} + \rho(\nabla\cdot\mathbf{v}))\lambda_N\lambda_M g^{(1/2)} \\
&= \left(\dot{\rho} g^{(1/2)} + \rho\dot{\overline{g^{(1/2)}}}\right)\lambda_N\lambda_M \\
&= \dot{\overline{\rho g^{(1/2)}\lambda_N\lambda_M}},
\end{aligned}
$$

leading to

$$
\dot{\overline{\rho g^{(1/2)}\lambda_N\lambda_M}} = \frac{\partial}{\partial t}\left(\bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\right) + \frac{\partial}{\partial\zeta^i}\left(\bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\bar{v}^i\right) - \bar{g}^{(1/2)}\bar{\rho}\bar{v}^\alpha\left(\frac{\partial\bar{\lambda}_M}{\partial\zeta^\alpha}\bar{\lambda}_N + \frac{\partial\bar{\lambda}_N}{\partial\zeta^\alpha}\bar{\lambda}_M\right) = 0.
$$

Integrals are taken over the cross-section rather than satisfying the conservation equations pointwise in the fluid. Integrating over an area of constant θ^3 bounded by $H(\theta^1,\theta^2,\theta^3) = 0$, which at time t coincides with an area of constant ζ^3 bounded by $\bar{H}(\zeta^1,\zeta^2\zeta^3,t) = 0$, the conservation of mass equation, gives

$$
\begin{aligned}
\int\int_A \dot{\overline{\rho g^{(1/2)}\lambda_N\lambda_M}}\, d\theta^1 d\theta^2 = \int\int_{\bar{A}} \bigg(&\frac{\partial}{\partial t}(\bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M) + \frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\bar{v}^i) \\
&- \bar{g}^{(1/2)}\bar{\rho}\bar{v}^\alpha\left(\frac{\partial\bar{\lambda}_M}{\partial\zeta^\alpha}\bar{\lambda}_N + \frac{\partial\bar{\lambda}_N}{\partial\zeta^\alpha}\bar{\lambda}_M\right)\bigg) d\zeta^1 d\zeta^2 = 0.
\end{aligned}
$$

Assuming zero flux in the ζ^1 and ζ^2 directions, this becomes

(2.12)

$$
\begin{aligned}
\dot{\overline{\int\int_A \rho g^{(1/2)}\lambda_N\lambda_M\, d\theta^1 d\theta^2}} &= \frac{\partial}{\partial t}\int\int_{\bar{A}} \bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\, d\zeta^1 d\zeta^2 + \frac{\partial}{\partial\zeta^3}\int\int_{\bar{A}}(\bar{g}^{(1/2)}\bar{\rho}\bar{\lambda}_N\bar{\lambda}_M\bar{v}^3)\, d\zeta^1 d\zeta^2 \\
&- \int\int_{\bar{A}} \bar{g}^{(1/2)}\bar{\rho}\bar{v}^\alpha\left(\frac{\partial\bar{\lambda}_M}{\partial\zeta^\alpha}\bar{\lambda}_N + \frac{\partial\bar{\lambda}_N}{\partial\zeta^\alpha}\bar{\lambda}_M\right) d\zeta^1 d\zeta^2 = 0.
\end{aligned}
$$

So Eq. (2.12) is the integral form of the equation for conservation of mass density.

By following a similar methodology to above for Eq. (2.11), first will be obtained a general conservation law in integral form, and then by appropriate substitution, the conservation of linear momentum.

Multiplying both sides of Eq. (2.11) by $\lambda_N(\theta^1,\theta^2) = \bar{\lambda}_N(\zeta^1,\zeta^2)$ gives

$$
\lambda_N\left(\dot{\overline{\rho F}} + \rho F(\nabla\cdot\mathbf{v})\right) = \bar{\lambda}_N\left(\frac{\partial}{\partial t}(\bar{\rho}\bar{F}) + \frac{1}{\bar{g}^{(1/2)}}\frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{F}\bar{v}^i)\right) = 0.
$$

After multiplying both sides by $g^{(1/2)} = \bar{g}^{(1/2)}$, this can be written as

$$
\dot{\overline{\rho F g^{(1/2)}\lambda_N}} = \frac{\partial}{\partial t}(\bar{g}^{(1/2)}\bar{\rho}\bar{F}\bar{\lambda}_N) + \frac{\partial}{\partial\zeta^i}(\bar{g}^{(1/2)}\bar{\rho}\bar{F}\bar{\lambda}_N\bar{v}^i) - \bar{g}^{(1/2)}\bar{\rho}\bar{F}\frac{\partial\lambda_N}{\partial\zeta^\alpha}\bar{v}^\alpha = 0. \tag{2.13}
$$

Similar to Eq. (2.4), F is considered as an expansion of components multiplied by the weighting functions λ_M, and making the substitutions

$$F = \sum_{M=0}^{K} f_M \lambda_M \tag{2.14}$$

and

$$\bar{F} = \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M \tag{2.15}$$

gives

$$\begin{aligned} 0 &= \overline{\rho g^{(1/2)} \lambda_N \sum_{M=0}^{K} f_M \lambda_M}^{\cdot} \\ &= \frac{\partial}{\partial t}\left(\bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M\right) + \frac{\partial}{\partial \zeta^i}\left(\bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \bar{v}^i \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M\right) - \bar{g}^{(1/2)} \bar{\rho} \frac{\partial \bar{\lambda}_N}{\partial \zeta^\alpha} \bar{v}^\alpha \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M. \end{aligned}$$

Similarly to before, integrating over an area of constant θ^3 bounded by $H(\theta^1, \theta^2, \theta^3) = 0$, which at time t coincides with an area of constant ζ^3 bounded by $\bar{H}(\zeta^1, \zeta^2, \zeta^3, t) = 0$, and again assuming zero flux in the ζ^1 and ζ^2 directions gives

$$\begin{aligned} 0 &= \overline{\int\int_A \rho g^{(1/2)} \lambda_N \sum_{M=0}^{K} f_M \lambda_M d\theta^1 d\theta^2}^{\cdot} \\ &= \frac{\partial}{\partial t} \int\int_{\bar{A}} \left(\bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M\right) d\zeta^1 d\zeta^2 + \frac{\partial}{\partial \zeta^3} \int\int_{\bar{A}} \left(\bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \bar{v}^3 \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M\right) d\zeta^1 d\zeta^2 \\ &- \int\int_{\bar{A}} \left(\bar{g}^{(1/2)} \bar{\rho} \frac{\partial \bar{\lambda}_N}{\partial \zeta^\alpha} \bar{v}^\alpha \sum_{M=0}^{K} \bar{f}_M \bar{\lambda}_M\right) d\zeta^1 d\zeta^2. \end{aligned}$$

If $f_M = f_M(\theta^3, t)$ and $\bar{f}_M = \bar{f}_M(\zeta^3, t)$, then this can be written as

$$\begin{aligned} 0 &= \overline{\sum_{M=0}^{K} \int\int_A g^{(1/2)} \rho \lambda_M \lambda_N d\theta^1 d\theta^2 f_M}^{\cdot} \\ &= \frac{\partial}{\partial t} \sum_{M=0}^{K} \int\int_{\bar{A}} \bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \bar{\lambda}_M d\zeta^1 d\zeta^2 \bar{f}_M + \frac{\partial}{\partial \zeta^3} \sum_{M=0}^{K} \bar{f}_M \int\int_{\bar{A}} \bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_M \bar{\lambda}_N \bar{v}^3 d\zeta^1 d\zeta^2 \\ &- \sum_{M=0}^{K} \bar{f}_M \int\int_{\bar{A}} \bar{g}^{(1/2)} \bar{\rho} \frac{\partial \bar{\lambda}_N}{\partial \zeta^\alpha} \bar{\lambda}_M \bar{v}^\alpha d\zeta^1 d\zeta^2. \end{aligned} \tag{2.16}$$

Making the substitutions $f_M = w_M$ and $\bar{f}_M = \bar{w}_M$ gives

(2.17)

$$
\begin{aligned}
0 &= \dot{\overline{\sum_{M=0}^{K} \int\int_{A} g^{(1/2)} \rho \lambda_M \lambda_N d\theta^1 d\theta^2 w_M}} \\
&= \frac{\partial}{\partial t} \sum_{M=0}^{K} \int\int_{\bar{A}} \bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_N \bar{\lambda}_M d\zeta^1 d\zeta^2 \bar{w}_M + \frac{\partial}{\partial \zeta^3} \sum_{M=0}^{K} \bar{w}_M \int\int_{\bar{A}} \bar{g}^{(1/2)} \bar{\rho} \bar{\lambda}_M \bar{\lambda}_N \bar{v}^3 d\zeta^1 d\zeta^2 \\
&\quad - \sum_{M=0}^{K} \bar{w}_M \int\int_{\bar{A}} \bar{g}^{(1/2)} \bar{\rho} \frac{\partial \bar{\lambda}_N}{\partial \zeta^\alpha} \bar{\lambda}_M \bar{v}^\alpha d\zeta^1 d\zeta^2.
\end{aligned}
$$

So Eq. (2.17) is the integral form of the equation for conservation of linear momentum. The above analysis gives a connection between integral balances in Lagrangian and Eulerian forms and hence direct connections between Lagrangian and Eulerian forms of field equations.

This section has derived a relation between the balance laws in convective and curvilinear coordinate systems, based on that presented by Green and Naghdi [6], although displaying more details of the derivation in an effort to aid accessibility. Green and Naghdi also provide an alternative method of derivation, in addition to other relations such as those for energy and entropy balance, however, since these will not be considered in the models presented in this thesis, they have not been presented in this section.

2.3 Solving steady state Newtonian and incompressible Navier-Stokes equations using director theory

This chapter has given an overview of the development of director theory. However in the following chapters where director theory models will be presented for solving fluid flow in pipes, it will not be necessary to consider many of these details. The main point is to consider the velocity as an expansion of *directors* (vectors) that follow depend on the centreline of the pipe multiplied by weighting (or shape) functions that depend on the cross-sectional pipe.

The relevant equations of motion will then be derived in the curvilinear coordinate system that follows the centreline of the pipe. Then instead of solving the momentum conservation equations pointwise, *director momentum* will be considered. That is, the equations that will be solved for will be the momentum equations multiplied by the weighting function λ_N of each director w_N in turn, integrated over the cross-section of the pipe. The form of the weighing functions will be chosen, such as polynomials of the cross-sectional coordinates, then the weighting functions will be determined more specifically and the number of unknown directors reduced by required the velocity expansion to satisfy relevant boundary and fluid property conditions. The size of the system of equations will depend on the number of directors considered for the velocity expansion. For large systems in-built solvers just as *fsolve* in Maple [38] or *lsqr* in MATLAB [39] will be used to solve for the weighting functions, after which the velocity profile can be reconstructed.

2.4 Computational fluid dynamics simulations

To evaluate the accuracy of the models developed using director theory, the solutions obtained will be compared with computational fluid dynamics (CFD) simulations, which use the finite volume method [36]. The CFD software that will be used to create the simulations is Simcenter STAR-CCM+ [45]. An outline of how the software works will be presented here, further details of which can be found in the Simcenter STAR-CCM+ index. A system of equations is formed from the fundamental conservation laws, with additional equations from constitutive laws, which will vary depending on the material being considered, added to form a closed set. When integrated over a finite control volume, the continuity, momentum and energy equations can be expressed respectively as

$$\frac{\partial}{\partial t}\int_V \rho dV + \oint_A \rho \mathbf{v}\cdot d\mathbf{a} = \int_V S dV$$

$$\frac{\partial}{\partial t}\int_V \rho \mathbf{v} dV + \oint_A \rho \mathbf{v}\otimes\mathbf{v}\cdot d\mathbf{a} = -\oint_A p\mathbf{I}\cdot d\mathbf{a} + \oint_A \mathbf{T}\cdot d\mathbf{a} + \int_V \mathbf{f}_b dV + \int_V \mathbf{s} dV$$

$$\frac{\partial}{\partial t}\int_V \rho E dV + \oint_A \rho H\mathbf{v}\cdot d\mathbf{a} = -\oint_A \mathbf{q}\cdot d\mathbf{a} + \oint_A \mathbf{T}\cdot \mathbf{v} d\mathbf{a} + \int_V \mathbf{f}_b\cdot\mathbf{v} dV + \int_V S dV$$

where

- t is time
- V is the control volume
- $\mathbf{a}$ is the area vector
- ρ is the density of the fluid
- $\mathbf{v}$ is the velocity of the fluid

- S and $\mathbf{s}$ are source terms
- p is pressure
- $\mathbf{T}$ is the viscous stress tensor
- $\mathbf{f}_b$ is the resultant of body forces
- E is the total energy
- H is the total enthalpy
- $\mathbf{q}$ is the heat flux

Later chapters will will focus on incompressible flows without source terms, so the energy equation can be disregarded and the continuity and momentum equations can be simplified to

$$\oint_A \mathbf{v} \cdot d\mathbf{a} = 0$$

$$\rho \frac{\partial}{\partial t} \int_V \mathbf{v} dV + \rho \oint_A \mathbf{v} \otimes \mathbf{v} \cdot d\mathbf{a} = -\oint_A p\mathbf{I} \cdot d\mathbf{a} + \oint_A \mathbf{T} \cdot d\mathbf{a}$$

A mesh is created dividing the continuous domain being considered into a finite number of cells and the continuous equations are discretised using either the finite volume or finite element method, depending on the mathematical model. Models are selected with the physics and mesh continua that are appropriate to the flow and geometry being solved for. For the simulations presented in later chapters, models for the physics continuum have been chosen to represent a 3D laminar, steady state flow for a liquid of constant density and viscosity.

For the mesh continuum, the Surface Remesher and Polyhedral Mesher tools were selected. The Surface Remesher improves the overall quality of an existing surface and optimises it for the volume mesh models. The Polyhedral Mesher uses an arbitrary polyhedral cell shape in order to build the core mesh. The Prism Layer Mesher adds prismatic cell layers next to wall boundaries, they are important for allowing the solver to resolve near wall flow accurately. The Generalized Cylinder is used with the Polyhedral Mesher to generate an extruded mesh along the length of cylindrical type parts, this typically reduces the overall cell count and in some cases can improve the rate of convergence. Examples of meshes with these models are shown in Fig. 2.2.

For steady state problems the SIMPLE algorithm is used, and the solution process can be summarised as follows

1. Set the boundary conditions;

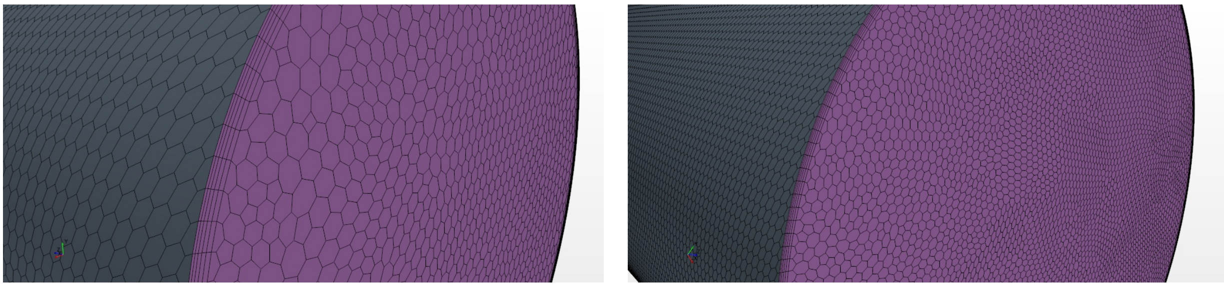


Figure 2.2: Example of finite volume mesh used in Chapter 4 (a similar mesh is used in Chapter 3. The left hand image depicts a mesh with base size $0.05m$, while the right hand image depicts a finer mesh with base size $0.025m$, with pipe diameter $2m$. The grey area is the wall of the pipe and the purple area is the fluid inlet. The prism layers can be seen around the circumference of the inlet, close to the wall.

2. Compute the gradients of velocity and pressure;
3. Solve the discretised momentum equation to compute the intermediate velocity field $\mathbf{v}^*$;
4. Compute the uncorrected mass fluxes at faces $\dot{m}_f^*$;
5. Solve the Poisson equation for the pressure correction to produce cell values of the pressure correction p';
6. Update the pressure field as the previous pressure plus the pressure correction multiplied by the under-relaxation factor: $p^{n+1} = p^n + \omega p'$;
7. Update the boundary pressure corrections p'_b;
8. Correct the face mass fluxes: $\dot{m}_f^{n+1} = \dot{m}_f^* + \dot{m}'_f$;
9. Correct the cell velocities
$$\mathbf{v}_p^{n+1} = \mathbf{v}_p^* - \frac{V\nabla p'}{\mathbf{a}'^v_p}$$
where $\nabla p'$ is the gradient of the pressure corrections, $\mathbf{a}'^v_p$ is the vector of central coefficients for the discretised velocity equation, and V is the cell volume.

2.5 Comparison of solutions

In subsequent chapters comparisons between the 1D director theory solutions and those observed from the 3D finite volume solver are presented and discussed. The relative error between these solutions is obtained by

1. Taking a cross-section from the 3D finite volume solution at the specified location;

2. Extracting coordinates and solution at the nodes in this section;
3. Compute solutions at these points using the 1D director theory;
4. Compute the relative error, given by

$$\text{Error} = \frac{1}{N}\sum_{i=1}^{N}\frac{\left|\phi^{i}_{FVM} - \phi^{i}_{\text{director}}\right|}{\phi^{Av}_{FVM}} \tag{2.18}$$

   where

$$\phi^{Av}_{FVM} = \frac{1}{N}\sum_{i=1}^{N}\left|\phi^{i}_{FVM}\right|.$$

2.6 Summary

This chapter has introduced the concept of director theory in relation to fluid flow in pipes. A Lagrangian coordinate system was introduced and the relation for various conservation laws between the curvilinear and Lagrangian coordinate system was derived. This provides some context on the historical development of the application of director theory to fluid mechanics with the approach presented by Green and Naghdi [6].

It has also given a brief introduction to the computational fluid dynamics simulations that will later be used to compare with the solutions found using the director theory model. In the following chapters, the director theory model will be applied to specific pipe flow geometries and the accuracy assessed against the 3D simulations. A alternative method to that laid out above will be used, that presents a more intuitive approach.

Chapter 3

Straight Axisymmetric Pipe

3.1 Introduction

The first case to which the theory was applied was fluid flow through a straight axisymmetric pipe. This work is a study of a paper by Caulk and Naghdi [18]. Their paper investigates axisymmetric flow of an incompressible fluid through a slender surface of revolution (e.g. pipe or jet flow), where the radius can vary along the axial coordinate and with time, as shown in Fig. 3.1. Starting from the Navier-Stokes equations and then assuming that the fluid velocity can be approximated by a Taylor expansion, Caulk and Naghdi derive a system of partial differential equations that describe the fluid flow. They then go on to consider these equations for special cases.

The next section follows the derivation of Caulk and Naghdi [18], filling in some of the algebraic steps omitted in their paper. Maple (2019) was used as part of the present derivations. In the following section, the system of equations are simplified for special forms of the geometry, namely for a constant radius and then for a radius depending only on the axial coordinate (independent of time). In the latter case, the specific forms of a tapered pipe and wavy (sinusoidal) walled pipe are then considered. For the case of a constant radius, solutions are found for a steady axial flow and for a steady swirling flow.

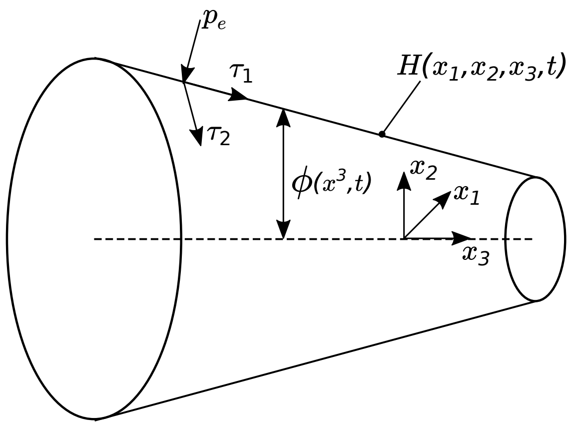


Figure 3.1: Straight axisymmetric pipe where the radius ϕ can vary along the coaxial direction and with time. H denotes the lateral surface and p_e, τ_a, τ_h are the normal and tangential components of the stress vector [18].

3.2 Derivation of Equations

Let the Cartesian coordinates be denoted by x_i and their associated unit base vectors be denoted by $\mathbf{e}_i$ for $i = 1,2,3$. In a fixed reference frame the equations governing the fluid motion are then stated to be

$$\frac{\partial \mathbf{T}_i}{\partial x_i} = \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i} \right), \tag{3.1}$$

representing conservation of linear momentum, and

$$\frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} + \frac{\partial v_3}{\partial x_3} = 0, \tag{3.2}$$

representing conservation of mass for an incompressible fluid. The Cauchy (or *full*) stress tensor is given by

$$\mathbf{T}_i = -p\mathbf{e}_i + \sigma_{ij}\mathbf{e}_j, \tag{3.3}$$

and the stress vector on a surface is given by

$$\mathbf{t} = \nu_i \mathbf{T}_i. \tag{3.4}$$

Here the variables are defined as follows

- $\mathbf{t}$ denotes the stress vector on a surface whose outward unit normal is $\boldsymbol{\nu} = \nu_i \mathbf{e}_i$;
- ρ is the constant fluid density;
- $\mathbf{v} = v_i \mathbf{e}_i$ is the velocity vector;

- p is an arbitrary pressure determined by the solutions to Eqs. (3.1) and (3.2);
- σ_{ij} is the deviatoric stress response, which is determined by a constitutive equation and is responsible for the viscous forces.

The boundary of the fluid is defined by

$$H(x_1, x_2, x_3, t) = \phi^{(2)} - x_1^2 - x_2^{(2)} = 0, \tag{3.5}$$

where $\phi = \phi(x_3, t)$ denotes the instantaneous radius of the surface of revolution. This boundary moves with the fluid, therefore its material derivative is zero. Therefore

$$\begin{aligned}
\frac{DH}{Dt} &= \frac{\partial H}{\partial t} + \mathbf{v} \cdot \nabla H \\
&= \frac{\partial \phi^{(2)}}{\partial t} + v_1(-2x_1) + v_2(-2x_2) + v_3 \frac{\partial \phi^{(2)}}{\partial t} \\
&= 2\phi \frac{\partial \phi}{\partial t} - 2x_1 v_1 - 2x_2 v_2 + 2\phi \frac{\partial \phi}{\partial x_3} v_3 \\
&= 0.
\end{aligned}$$

This gives the boundary condition

$$\phi \frac{\partial \phi}{\partial t} + \phi \frac{\partial \phi}{\partial x_3} v_3 - x_1 v_1 - x_2 v_2 = 0. \tag{3.6}$$

The components of the outward unit normal to the surface are given by

$$\nu_i = -\frac{\frac{\partial H}{\partial x_i}}{[(\frac{\partial H}{\partial x_1})^{(2)} + (\frac{\partial H}{\partial x_2})^{(2)} + (\frac{\partial H}{\partial x_3})^{(2)}]^{(1/2)}}.$$

Considering each component explicitly:

$$\begin{aligned}
\nu_1 &= \frac{2x_1}{[4x_1^{(2)} + 4x_2^{(2)} + 4\phi(\frac{\partial \phi}{\partial x_3})^{(2)}]^{(1/2)}} \\
&= \frac{2x_1}{2[(x_1^{(2)} + x_2^{(2)}) + \phi^{(2)}(\frac{\partial \phi}{\partial x_3})^{(2)}]^{(1/2)}} \\
&= \frac{x_1}{[\phi^{(2)} + \phi^{(2)}(\frac{\partial \phi}{\partial x_3})^{(2)}]^{(1/2)}} \\
&= \frac{x_1}{\phi[1 + (\frac{\partial \phi}{\partial x_3})^{(2)}]^{(1/2)}};
\end{aligned}$$

similarly

$$\nu_2 = \frac{x_2}{\phi[1 + (\frac{\partial \phi}{\partial x_3})^{(2)}]^{(1/2)}};$$

and finally

$$v_3 = -\frac{2\phi\frac{\partial\phi}{\partial x_3}}{2\phi[1+\frac{\partial\phi}{\partial x_3}]}$$
$$= -\frac{\frac{\partial\phi}{\partial x_3}}{[1+(\frac{\partial\phi}{\partial x_3})^{(2)}]^{(1/2)}}.$$

The velocity is then assumed to be approximated by the finite polynomial series of order K given by

$$\mathbf{v} = \mathbf{w}_{00} + \sum_{N=1}^{K}\sum_{j=0}^{N} x_1^{(j)} x_2^{(N-j)} \mathbf{w}_{j,N-j}, \tag{3.7}$$

where

- $\mathbf{w}_{00} = w^i_{0,0}(x_3,t)\mathbf{e}_i$ $(i = 1,2,3)$ is the velocity along the centre line (x_3-axis) of the flow;
- $\mathbf{w}_{j,N-j} = w^i_{j,N-j}(x_3,t)\mathbf{e}_i$ $(i = 1,2,3)$.

In the context of a director theory approach, the polynomials $x_1^{(j)}x_2^{(N-j)}$ could be considered shape functions and the vectors $\mathbf{w}_{j,N-j}$ considered directors. As an example, if the order is set to $K = 3$, then the velocity expansion takes the form

$$\mathbf{v} = (w^i_{00}+x_1w^i_{1,0}+x_2w^i_{0,1}+x_1^{(2)}w^i_{2,0}+x_1x_2w^i_{1,1}+x_2^{(2)}w^i_{0,2}+x_1^{(3)}w^i_{3,0}+x_1^{(2)}x_2w^i_{2,1}+x_1x_2^{(2)}w^i_{1,2}+x_2^{(3)}w^i_{0,3})\mathbf{e}_i,$$

where $i = 1,2,3$ (where the directors have been split up into the components of the coordinate basis).

Assuming the flow is axisymmetric, some of the directors will have to be set to zero due to symmetry conditions. This can be determined by considering that rotating the body through an angle θ about its centreline, the velocity profile should look the same. This is to say that if the velocity is expressed in cylindrical polar coordinates, it should be independent of θ i.e. $\mathbf{v}(r,\theta,z,t) = \mathbf{v}(r,z,t)$. The unit base vectors of the cylindrical polar coordinate system are given in terms of the Cartesian base vectors by

$$\hat{\mathbf{r}} = \cos(\theta)\mathbf{e}_1 + \sin(\theta)\mathbf{e}_2; \quad \hat{\boldsymbol{\theta}} = -\sin(\theta)\mathbf{e}_1 + \cos(\theta)\mathbf{e}_2; \quad \hat{\mathbf{z}} = \mathbf{e}_3.$$

Therefore, the velocity components in cylindrical polar coordinates are

$$v_r = \cos(\theta)v_1 + \sin(\theta)v_2; \quad v_\theta = -\sin(\theta)v_1 + \cos(\theta)v_2; \quad v_z = v_3.$$

Also, noting the transformation from Cartesian coordinates to cylindrical polar coordinates

$$x_1 = r\cos(\theta); \quad x_2 = r\sin(\theta); \quad x_3 = z;$$

such that for order $K = 3$:

$$\begin{aligned} v_r = \cos(\theta)\Big(& w^1_{0,0} + r\cos(\theta)w^1_{1,0} + r\sin(\theta)w^1_{0,1} + r^{(2)}\cos^{(2)}(\theta)w^1_{2,0} + r^{(2)}\cos(\theta)\sin(\theta)w^1_{1,1} + r^{(2)}\sin^{(2)}(\theta)w^1_{0,2} \\ & + r^{(3)}\cos^{(3)}(\theta)w^1_{3,0} + r^{(3)}\cos^{(2)}(\theta)\sin(\theta)w^1_{2,1} + r^{(3)}\cos(\theta)\sin^{(2)}(\theta)w^1_{1,2} + r^{(3)}\sin^{(3)}(\theta)w^1_{0,3}\Big) \\ & + \sin(\theta)\Big(w^2_{0,0}(r\cos(\theta)w^2_{1,0} + r\sin(\theta)w^2_{0,1} + r^{(2)}\cos^{(2)}(\theta)w^2_{2,0} + r^{(2)}\cos(\theta)\sin(\theta)w^2_{1,1} + r^{(2)}\sin^{(2)}(\theta)w^2_{0,2} \\ & + r^{(3)}\cos^{(3)}(\theta)w^2_{3,0} + r^{(3)}\cos^{(2)}(\theta)\sin(\theta)w^2_{2,1} + r^{(3)}\cos(\theta)\sin^{(2)}(\theta)w^2_{1,2} + r^{(3)}\sin^{(3)}(\theta)w^2_{0,3}\Big); \end{aligned}$$

$$\begin{aligned} v_\theta = -\sin(\theta)\Big(& w^1_{0,0} + r\cos(\theta)w^1_{1,0} + r\sin(\theta)w^1_{0,1} + r^{(2)}\cos^{(2)}(\theta)w^1_{2,0} + r^{(2)}\cos(\theta)\sin(\theta)w^1_{1,1} + r^{(2)}\sin^{(2)}(\theta)w^1_{0,2} \\ & + r^{(3)}\cos^{(3)}(\theta)w^1_{3,0} + r^{(3)}\cos^{(2)}(\theta)\sin(\theta)w^1_{2,1} + r^{(3)}\cos(\theta)\sin^{(2)}(\theta)w^1_{1,2} + r^{(3)}\sin^{(3)}(\theta)w^1_{0,3}\Big) \\ & + \cos(\theta)\Big((w^2_{0,0} + r\cos(\theta)w^2_{1,0} + r\sin(\theta)w^2_{0,1} + r^{(2)}\cos^{(2)}(\theta)w^2_{2,0} + r^{(2)}\cos(\theta)\sin(\theta)w^2_{1,1} + r^{(2)}\sin^{(2)}(\theta)w^2_{0,2} \\ & + r^{(3)}\cos^{(3)}(\theta)w^2_{3,0} + r^{(3)}\cos^{(2)}(\theta)\sin(\theta)w^2_{2,1} + r^{(3)}\cos(\theta)\sin^{(2)}(\theta)w^2_{1,2} + r^{(3)}\sin^{(3)}(\theta)w^2_{0,3}\Big); \end{aligned}$$

$$\begin{aligned} v_z = & w^3_{0,0} + r\cos(\theta)w^3_{1,0} + r\sin(\theta)w^3_{0,1} + r^{(2)}\cos^{(2)}(\theta)w^3_{2,0} + r^{(2)}\cos(\theta)\sin(\theta)w^3_{1,1} + r^{(2)}\sin^{(2)}(\theta)w^3_{0,2} \\ & + r^{(3)}\cos^{(3)}(\theta)w^3_{3,0} + r^{(3)}\cos^{(2)}(\theta)\sin(\theta)w^3_{2,1} + r^{(3)}\cos(\theta)\sin^{(2)}(\theta)w^3_{1,2} + r^{(3)}\sin^{(3)}(\theta)w^3_{0,3}. \end{aligned}$$

For these expressions to be independent of θ, we can consider the values of the directors at each order, as these terms will have matching powers of r. For v_r at order 0 the expression is

$$\cos(\theta)\ w^1_{0,0} + \sin(\theta)\ w^2_{0,0}.$$

For this expression to be independent of θ, it must be the case that $w^1_{0,0} = w^2_{0,0} = 0$. For v_r at order 1 the expression is

$$r\left(\cos^{(2)}(\theta)w^1_{1,0} + \cos(\theta)\sin(\theta)(w^1_{0,1} + w^2_{1,0}) + \sin^{(2)}(\theta)w^2_{0,1}\right). \tag{3.8}$$

Given that

$$\cos^{(2)}(\theta) + \sin^{(2)}(\theta) = 1, \tag{3.9}$$

for the expression given by Eq. (3.8) to be independent of θ it is required that $w^1_{1,0} = w^2_{0,1}$ and $w^1_{0,1} = -w^2_{1,0}$. For v_r at order 2 the expression is

$$r^{(2)}\left(\cos^{(3)}(\theta)w^1_{2,0} + \cos^{(2)}(\theta)\sin(\theta)(w^1_{1,1} + w^2_{2,0}) + \cos(\theta)\sin^{(2)}(\theta)(w^1_{0,2} + w^2_{1,1}) + \sin^{(3)}(\theta)w^2_{0,2}\right).$$

For this expression to be independent of θ it is required that $w^1_{2,0} = w^2_{0,2} = 0$, $w^1_{1,1} = -w^2_{2,0}$ and $w^1_{0,2} = -w^2_{1,1}$. For v_r at order 3 the expression is

$$
\begin{aligned}
&r^{(3)}\Big(\cos^{(4)}(\theta)w^1_{3,0} + \cos^{(3)}(\theta)\sin(\theta)(w^1_{2,1}+w^2_{3,0}) + \cos^{(2)}(\theta)\sin^{(2)}(\theta)(w^1_{1,2}+w^2_{2,1}) \\
&+ \cos(\theta)\sin^{(3)}(\theta)(w^1_{0,3}+w^2_{1,2}) + \sin^{(4)}(\theta)w^2_{0,3}\Big).
\end{aligned}
\tag{3.10}
$$

Given that

$$
\cos^{(4)}(\theta) + 2\cos^{(2)}(\theta)\sin^{(2)}(\theta) + \sin^{(4)}(\theta) = \left(\sin^{(2)}(\theta)+\cos^{(2)}(\theta)\right)^{(2)} = 1,
\tag{3.11}
$$

for the expression given by Eq. (3.10) to be independent of θ it is required that $w^1_{2,1} = -w^2_{3,0}$, $w^1_{0,3} = -w^2_{1,2}$, $w^1_{3,0} = w^2_{0,3}$ and $w^1_{1,2} = 2w^2_{0,3} - w^2_{2,1}$.

The relationships between the directors that have been found thus far can be used when examining the terms for each order of v_θ. For v_θ at order 0 the expression is 0. For v_θ at order 1 the expression is

$$
r\left(\sin^{(2)}(\theta)+\cos^{(2)}(\theta)\right)w^2_{1,0} = rw^2_{1,0}.
$$

So this expression is already independent of θ.

For v_θ at order 2 the expression is

$$
r^{(2)}\left(\cos(\theta)(\sin^{(2)}(\theta)+\cos^{(2)}(\theta))w^2_{2,0} + \sin(\theta)(\sin^{(2)}(\theta)+\cos^{(2)}(\theta))w^2_{1,1}\right) = r^{(2)}\left(\cos(\theta)w^2_{2,0}+\sin(\theta)w^2_{1,1}\right).
$$

For this expression to be independent of θ it is required that $w^2_{1,1} = w^2_{2,0} = 0$.

For v_θ at order 3 the expression is

$$
\begin{aligned}
&r^{(3)}\Big(\cos^{(4)}(\theta)w^2_{3,0} + \cos^{(3)}(\theta)\sin(\theta)(w^2_{2,1}-w^2_{0,3}) + \cos^{(2)}(\theta)\sin^{(2)}(\theta)(w^2_{1,2}+w^2_{3,0}) \\
&+ \cos(\theta)\sin^{(3)}(\theta)((w^2_{2,1}-w^2_{0,3}) + \sin^{(4)}(\theta)w^2_{1,2}\Big).
\end{aligned}
$$

Given Eq. (3.11), for this expression to be independent of θ, it is required that $w^2_{1,2} = w^2_{3,0}$ and $w^2_{2,1} = w^2_{0,3}$.

Lastly, consider the values of the directors at each order for v_z. For v_z at order 0, the expression is w^3_{00} which is already independent of θ. For v_z at order 1 the expression is

$$
r\left(\cos(\theta)w^3_{1,0} + \sin(\theta)w^3_{0,1}\right).
$$

For this expression to be independent of θ it is required that $w^3_{1,0} = w^3_{0,1} = 0$.

For v_z at order 2 the expression is

$$
r^{(2)}\left(\cos^{(2)}(\theta)w^3_{2,0} + \cos(\theta)\sin(\theta)w^3_{1,1} + \sin^{(2)}(\theta)w^3_{0,2}\right).
$$

Given Eq. (3.9), for this expression to be independent of θ it is required that $w^3_{1,1} = 0$ and $w^3_{0,2} = w^3_{2,0}$.

For v_z at order 3, the only way for the expression to be independent of θ is for all the directors at this order to equal zero, i.e. $w^3_{3,0} = w^3_{2,1} = w^3_{1,2} = w^3_{0,3} = 0$.

Setting $w^2_{1,0} = w^1_{0,1}$, $w^2_{0,1} = w^1_{1,0}$, $w^2_{3,0} = w^1_{0,3}$, $w^2_{0,3} = w^1_{3,0}$, $w^3_{0,0} = v$ and $w^3_{2,0} = w^3_{2,0}$, the velocity now takes the form

(3.12)

$$\begin{aligned}\mathbf{v} &= [(w^1_{1,0} + (x_1^{(2)} + x_2^{(2)})w^1_{3,0})x_1 - (w^1_{0,1} + (x_1^{(2)} + x_2^{(2)})w^1_{0,3})x_2]\mathbf{e}_1 \\ &+ [(w^1_{1,0} + (x_1^{(2)} + x_2^{(2)})w^1_{3,0})x_2 + (w^1_{0,1} + (x_1^{(2)} + x_2^{(2)})w^1_{0,3})x_1]\mathbf{e}_2 + [w^3_{0,0} + (x_1^{(2)} + x_2^{(2)})w^3_{2,0}]\mathbf{e}_3,\end{aligned}$$

where $w^3_{0,0}$, $w^1_{1,0}$, $w^1_{0,1}$, $w^3_{2,0}$, $w^1_{3,0}$, $w^1_{0,3}$ are scalar functions of x_3 and t.

- $w^3_{0,0}$ represents the velocity along the axis of symmetry;
- $w^3_{2,0}$ is related to transverse shearing motion;
- $w^1_{0,1}$ and $w^1_{0,3}$ are related to rotational motion about $\mathbf{e}_3$;
- $w^1_{1,0}$ and $w^1_{3,0}$ are related to transverse elongation.

For simplicity and to avoid confusion of superscripts and powers in later equations, the following notational changes will be made and used henceforth. The directors pointing in the cross-section of the form $w^1_{j,k}$ will now be written as $u_{j,k}$ and the directors pointing along the pipe of the form $w^3_{j,k}$ will now be written as $w_{j,k}$. So the velocity for order $K = 3$ given by Eq. (3.12) is now written as

(3.13)

$$\begin{aligned}\mathbf{v} &= [(u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0})x_1 - (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3})x_2]\mathbf{e}_1 \\ &+ [(u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0})x_2 + (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3})x_1]\mathbf{e}_2 + [w_{0,0} + (x_1^{(2)} + x_2^{(2)})w_{2,0}]\mathbf{e}_3.\end{aligned}$$

The above argument for $K = 3$ can be generalised to any K by considering axisymmetry and it is found that there has to be a certain symmetry in the form of the directors and their weighting functions.

If the order $K = 5$ was chosen then the velocity would take the form

(3.14)

$$\begin{aligned}\mathbf{v} &= [(u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0} + (x_1^{(2)} + x_2^{(2)})^{(2)}u_{5,0})x_1 - (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3} + (x_1^{(2)} + x_2^{(2)})^{(2)}u_{0,5})x_2]\mathbf{e}_1 \\ &+ [(u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0} + (x_1^{(2)} + x_2^{(2)})^{(2)}u_{5,0})x_2 + (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3} + (x_1^{(2)} + x_2^{(2)})^{(2)}u_{0,5})x_1]\mathbf{e}_2 \\ &+ [w_{0,0} + (x_1^{(2)} + x_2^{(2)})w_{2,0} + (x_1^{(2)} + x_2^{(2)})^{(2)}w_{4,0}]\mathbf{e}_3.\end{aligned}$$

Substituting the expression given by Eq. (3.13) for **v** into the boundary condition, given by Eq. (3.6), gives

(3.15)
$$\phi\frac{\partial\phi}{\partial t}+\phi\frac{\partial\phi}{\partial x_3}(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})-x_1[(u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2]$$
$$-x_2[(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1+(u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2]$$
$$=\phi\frac{\partial\phi}{\partial t}+\phi\frac{\partial\phi}{\partial x_3}(w_{0,0}+\phi^{(2)}w_{2,0})-u_{1,0}(x_1^{(2)}+x_2^{(2)})-\phi^{(2)}u_{3,0}(x_1^{(2)}+x_2^{(2)})$$
$$=\frac{\partial\phi}{\partial t}+\frac{\partial\phi}{\partial x_3}(w_{0,0}+\phi^{(2)}w_{2,0})-u_{1,0}\phi-u_{3,0}\phi^{(3)}$$
$$=\frac{\partial\phi}{\partial t}+(w_{0,0}+\phi^{(2)}w_{2,0})\frac{\partial\phi}{\partial x_3}-(u_{1,0}+\phi^{(2)}u_{3,0})\phi$$
$$=0.$$

The corresponding expression for the boundary condition at order $K=5$, found by substituting Eq. (3.14) into Eq. (3.6) is given by

(3.16)
$$\frac{\partial\phi}{\partial t}+(w_{0,0}+\phi^{(2)}w_{2,0}+\phi^{(4)}w_{4,0})\frac{\partial\phi}{\partial z}-(u_{1,0}+\phi^{(2)}u_{3,0}+\phi^{(4)}u_{5,0})\phi=0.$$

Substituting the expression for **v**, given by Eq. (3.13), into the incompressibility condition, given by Eq. (3.2), gives

$$\frac{\partial}{\partial x_1}[(u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1+(u_{0,1}-(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2]$$
$$+\frac{\partial}{\partial x_2}[(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1+(u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2]+\frac{\partial}{\partial x_3}[w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0}]$$
$$=u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3}+2x_1x_2u_{0,3}+u_{1,0}+(x_1^{(2)}+3x_2^{(2)})u_{3,0}+\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}$$
$$=2u_{1,0}+4(x_1^{(2)}+x_2^{(2)})u_{3,0}+\frac{\partial v}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}$$
$$=\left(\frac{\partial w_{0,0}}{\partial x_3}+2u_{1,0}\right)+x_\alpha^{(2)}\left(\frac{\partial w_{2,0}}{\partial x_3}+4u_{3,0}\right)$$
$$=0.$$

For the above equation to hold at every point in the fluid, the velocity coefficients must satisfy the separate conditions

(3.17)
$$\frac{\partial w_{0,0}}{\partial x_3}+2u_{1,0}=0$$

and

(3.18)
$$\frac{\partial w_{2,0}}{\partial x_3}+4u_{3,0}=0.$$

For order $K = 5$, these two equations and the following one are required to satisfy incompressibility

$$\frac{\partial w_{4,0}}{\partial z} + 6u_{5,0} = 0. \tag{3.19}$$

Instead of satisfying the momentum equation Eq. (3.1) pointwise in the fluid, recalling the method detailed in Chapter 2, the following integral conditions are imposed:

$$\int_A \left[\frac{\partial \mathbf{T}_i}{\partial x_i} - \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i} \right) \right] da = 0; \tag{3.20}$$

$$\int_A \left[\frac{\partial \mathbf{T}_i}{\partial x_i} - \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i} \right) \right] x_{\alpha_1} \cdots x_{\alpha_N}\, da = 0; \tag{3.21}$$

for $N = 1,2,3$. Here, the notation $x_{\alpha_1} \cdots x_{\alpha_N}$ is used to represent the weighting functions $x_1^{(j)} x_2^{(N-j)}$ in the velocity expansion given by Eq. (3.7), with α_i taking the values of 1 or 2. This notation is temporarily adopted to assist with the following analysis. A is an arbitrary cross-section of the slender surface of revolution defined by x_3 = constant and bounded by the circle given by Eq. (3.5).

By separating the cross-sectional and coaxial component of the stress tensor, Eq. (3.20) can be rewritten as

$$\int_A \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} da + \int_A \frac{\partial \mathbf{T}_3}{\partial x_3} da = \int_A \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i} \right) da.$$

As x_3 is constant over the surface of integration, the derivative in the second term can be taken outside the integral. Applying the divergence theorem to the first term gives

$$\int_A \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} = \int_{\partial A} \mathbf{t}_\alpha \cdot \boldsymbol{\nu} ds = \int_{\partial A} \mathbf{t} \cdot \hat{\boldsymbol{\nu}} ds, \tag{3.22}$$

where $\hat{\boldsymbol{\nu}}$ is the unit vector comprised of only the $\mathbf{e}_1$ and $\mathbf{e}_2$ components of $\boldsymbol{\nu}$. To find the scale factor to ensure $\hat{\boldsymbol{\nu}}$ is a unit vector, the norm of $\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2$ is calculated:

$$\begin{aligned} |\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2| &= ({\nu_1}^2 + {\nu_2}^2)^{(1/2)} \\ &= \left(\frac{x_1^{(2)} + x_2^{(2)}}{\phi^{(2)} \left(1 + \frac{\partial \phi}{\partial x_3}^{(2)} \right)} \right)^{(1/2)} \\ &= \frac{1}{\left(1 + \frac{\partial \phi}{\partial x_3}^{(2)} \right)^{(1/2)}}. \end{aligned}$$

So

$$\hat{\boldsymbol{\nu}} = \left(1 + \frac{\partial \phi}{\partial x_3}^{(2)} \right)^{(1/2)} (\nu_1 \mathbf{e}_1 + \nu_2 \mathbf{e}_2).$$

Substituting this expression for $\hat{\boldsymbol{v}}$ into Eq. (3.22) gives

$$\int_A \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} = \int_{\partial A} \left(1 + \frac{\partial \phi}{\partial x_3}^{(2)}\right)^{(1/2)} \mathbf{t} \cdot (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) ds.$$

However, from Eq. (3.4)

$$\int_{\partial A} \left(1 + \frac{\partial \phi}{\partial x_3}^{(2)}\right)^{(1/2)} \mathbf{t} \cdot (v_1 \mathbf{e}_1 + v_2 \mathbf{e}_2) ds = \int_{\partial A} \left(1 + \left(\frac{\partial \phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)} \mathbf{t} ds.$$

With this simplification, Eq. (3.20) becomes

$$\begin{aligned} &\frac{\partial}{\partial x_3} \int_A \mathbf{T}_3 da + \int_{\partial A} \left(1 + \left(\frac{\partial \phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)} \mathbf{t} ds \\ &= \int_A \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i}\right) da. \end{aligned} \tag{3.23}$$

This gives the equation for conversation of momentum considered over the cross-section of the pipe. Next, the conservation of momentum of the individual directors over the cross-section will be considered, starting from Eqs. (3.21). A similar method of manipulation to that detailed above for conservation of momentum will be carried for the conservation of director momentum to obtain the form of the equations that will solved for. This process now follows.

Eqs. (3.21) can be rewritten as

$$\int_A \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} x_{\alpha_1} \cdots x_{\alpha_N} da + \int_A \frac{\partial \mathbf{T}_3}{\partial x_3} x_{\alpha_1} \cdots x_{\alpha_N} da = \int_A \rho \left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i}\right) x_{\alpha_1} \cdots x_{\alpha_N} da, \tag{3.24}$$

for $N = 1,2,3$. Looking at the equation for $N = 1$ for Eq. (3.24), by the chain rule

$$\frac{\partial}{\partial x_\alpha}(x_{\alpha_1} \mathbf{T}_\alpha) = \mathbf{T}_{\alpha_1} + x_{\alpha_1} \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha},$$

where α is summed over 1 and 2, and α_1 takes the value of 1 or 2 depending on the weighting function. Rearranging gives

$$x_{\alpha_1} \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} = \frac{\partial}{\partial x_\alpha}(x_{\alpha_1} \mathbf{T}_\alpha) - \mathbf{T}_{\alpha_1}.$$

Taking the integral over the cross-section gives

$$\int_A \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} x_{\alpha_1} da = \int_A \frac{\partial}{\partial x_\alpha}(\mathbf{T}_\alpha x_{\alpha_1}) da - \int_A \mathbf{T}_\alpha da.$$

Applying the divergence theorem, similarly to before, gives

$$\int_A \frac{\partial \mathbf{T}_\alpha}{\partial x_\alpha} x_{\alpha_1} da = \int_{\partial A} \left(1 + \left(\frac{\partial \phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)} \mathbf{t} x_{\alpha_1} ds - \int_A \mathbf{T}_\alpha da.$$

So, for $N = 1$, Eq. (3.21) becomes

$$\begin{aligned} &\frac{\partial}{\partial x_3}\int_A \mathbf{T}_3 x_{\alpha_1} da + \int_{\partial A}\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)} \mathbf{t} x_{\alpha_1} ds \\ &= \int_A \mathbf{T}_{\alpha_1} da + \int_A \rho\left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i}\right) x_{\alpha_1} da. \end{aligned} \tag{3.25}$$

Similar derivation for Eqs. (3.21) when $N = 2$ and $N = 3$ gives

$$\begin{aligned} &\frac{\partial}{\partial x_3}\int_A \mathbf{T}_3 x_{\alpha_1} x_{\alpha_2} da + \int_{\partial A}\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)} \mathbf{t} x_{\alpha_1} x_{\alpha_2} ds \\ &= \int_A (\mathbf{T}_{\alpha_1} x_{\alpha_2} + \mathbf{T}_{\alpha_2} x_{\alpha_1}) da + \int_A \rho\left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i}\right) x_{\alpha_1} x_{\alpha_2} da; \end{aligned} \tag{3.26}$$

$$\begin{aligned} &\frac{\partial}{\partial x_3}\int_A \mathbf{T}_3 x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} da + \int_{\partial A}\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)} \mathbf{t} x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} ds \\ &= \int_A (\mathbf{T}_{\alpha_1} x_{\alpha_2} x_{\alpha_3} + \mathbf{T}_{\alpha_2} x_{\alpha_1} x_{\alpha_3} + \mathbf{T}_{\alpha_3} x_{\alpha_1} x_{\alpha_2}) da + \int_A \rho\left(\frac{\partial \mathbf{v}}{\partial t} + v_i \frac{\partial \mathbf{v}}{\partial x_i}\right) x_{\alpha_1} x_{\alpha_2} x_{\alpha_3} da. \end{aligned} \tag{3.27}$$

The stress vector on the lateral surface is resolved in terms of its normal and tangential components, as shown in Fig. 3.1 in the form

$$\mathbf{t} = \tau_a \mathbf{e}_a - p_e \mathbf{v} + \tau_h \mathbf{e}_\theta, \tag{3.28}$$

where

- p_e is the external pressure acting normal to the surface;
- τ_a and τ_h are stress acting tangential to the surface;
- $\mathbf{e}_a$ and $\mathbf{e}_\theta$ are unit tangent vectors.

The tangent vectors $\mathbf{e}_\theta$ and $\mathbf{e}_a$ are defined as

$$\mathbf{e}_\theta = -\frac{x_2}{\phi}\mathbf{e}_1 + \frac{x_1}{\phi}\mathbf{e}_2$$

and

$$\begin{aligned}
\mathbf{e}_a &= \nu \times \mathbf{e}_\theta \\
&= \left[\frac{x_1}{\theta\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_1 + \frac{x_2}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_2 - \frac{\frac{\partial\phi}{\partial x_3}}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}} \right] \times \left[-\frac{x_2}{\phi}\mathbf{e}_1 + \frac{x_1}{\phi}\mathbf{e}_2 \right] \\
&= \frac{x_1\frac{\partial\phi}{\partial x_3}}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_1 + \frac{x_2\frac{\partial\phi}{\partial x_3}}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_2 + \frac{x_1^{(2)}+x_2^{(2)}}{\phi^{(2)}\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_3 \\
&= \frac{x_1\frac{\partial\phi}{\partial x_3}}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_1 + \frac{x_2\frac{\partial\phi}{\partial x_3}}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_2 + \frac{1}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_3.
\end{aligned}$$

From this, the stress vector on the lateral surface can be rewritten in terms of Cartesian base vectors as

(3.29)

$$\begin{aligned}
\mathbf{t} = \tau_a &\left[\frac{x_1\frac{\partial\phi}{\partial x_3}}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_1 + \frac{x_2\frac{\partial\phi}{\partial x_3}}{\phi\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_2 + \frac{1}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\mathbf{e}_3 \right] \\
- p_e &\left[\frac{x_1}{\phi\left(1+(\frac{\partial\phi}{\partial x_3})^{(2)}\right)^{(1/2)}}\mathbf{e}_1 + \frac{x_2}{\phi\left(1+(\frac{\partial\phi}{\partial x_3})^{(2)}\right)^{(1/2)}}\mathbf{e}_2 - \frac{\frac{\partial\phi}{\partial x_3}}{\left(1+(\frac{\partial\phi}{\partial x_3})^{(2)}\right)^{(1/2)}}\mathbf{e}_3 \right] + \tau_h \left[-\frac{x_2}{\phi}\mathbf{e}_1 + \frac{x_1}{\phi}\mathbf{e}_2 \right] \\
= &\left[\phi^{-1}\left(\frac{x_1\left(\tau_a\frac{\partial\phi}{\partial x_3} - p_e\right)}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}} - x_2\tau_h \right) \right]\mathbf{e}_1 + \left[\phi^{-1}\left(\frac{x_2\left(\tau_a\frac{\partial\phi}{\partial x_3} - p_e\right)}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}} + x_1\tau_h \right) \right]\mathbf{e}_2 \\
+ &\left[\frac{\tau_a + p_e\frac{\partial\phi}{\partial x_3}}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}} \right]\mathbf{e}_3.
\end{aligned}$$

The deviatoric stress response in a linear viscous fluid is given by

$$\sigma_{ij} = \mu\left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i}\right).$$

Substituting this into Eq. (3.3) gives the following expressions for $\mathbf{T}_1$, $\mathbf{T}_2$ and $\mathbf{T}_3$:

$$\mathbf{T}_1 = \left(-p + 2\mu\frac{\partial v_1}{\partial x_1}\right)\mathbf{e}_1 + \mu\left(\frac{\partial v_1}{\partial x_2} + \frac{\partial v_2}{\partial x_1}\right)\mathbf{e}_2 + \mu\left(\frac{\partial v_1}{\partial x_3} + \frac{\partial v_3}{\partial x_1}\right)\mathbf{e}_3;$$

$$\mathbf{T}_2 = \mu\left(\frac{\partial v_2}{\partial x_1} + \frac{\partial v_1}{\partial x_2}\right)\mathbf{e}_1 + \left(-p + 2\mu\frac{\partial v_2}{\partial x_2}\right)\mathbf{e}_2 + \mu\left(\frac{\partial v_2}{\partial x_3} + \frac{\partial v_3}{\partial x_2}\right)\mathbf{e}_3;$$

$$\mathbf{T}_3 = \mu\left(\frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3}\right)\mathbf{e}_1 + \mu\left(\frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3}\right)\mathbf{e}_2 + \left(-p + 2\mu\frac{\partial v_3}{\partial x_3}\right)\mathbf{e}_3.$$

Substituting in the expression for the velocity, given by Eq. (3.13), gives

$$\begin{aligned}\mathbf{T}_1 &= [-p + 2\mu(u_{1,0} + (3x_1^{(2)} + x_2^{(2)})u_{3,0} - 2x_1x_2u_{0,3})]\mathbf{e}_1 \\ &+ \mu[4x_1x_2u_{3,0} + (2x_1^{(2)} - 2x_2^{(2)})u_{0,3}]\mathbf{e}_2 \\ &+ \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2 + 2x_1w_{2,0}\right]\mathbf{e}_3;\end{aligned} \tag{3.30}$$

$$\begin{aligned}\mathbf{T}_2 &= \mu[4x_1x_2u_{3,0} + (2x_1^{(2)} - 2x_2^{(2)})u_{0,3}]\mathbf{e}_1 \\ &+ [-p + 2\mu(u_{1,0} + (x_1^{(2)} + 3x_2^{(2)})u_{3,0} + 2x_1x_2u_{0,3})]\mathbf{e}_2 \\ &+ \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_2 + \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_1 + 2x_2w_{2,0}\right]\mathbf{e}_3;\end{aligned} \tag{3.31}$$

$$\begin{aligned}\mathbf{T}_3 &= \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2 + 2x_1w_{2,0}\right]\mathbf{e}_1 \\ &+ \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_2 + \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_1 + 2x_2w_{2,0}\right]\mathbf{e}_2 \\ &+ \left[-p + 2\mu\left(\frac{\partial v}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]\mathbf{e}_3.\end{aligned} \tag{3.32}$$

Correspondingly, in the case of order $K = 5$, the expressions are given by

(3.33)

$$\begin{aligned}\mathbf{T}_1 &= \left[-p + 2\mu(u_{1,0} + (3x_1^{(2)} + x_2^{(2)})u_{3,0} + (5x_1^{(2)} + x_2^{(2)})(x_1^{(2)} + x_2^{(2)})u_{5,0} - 2x_1x_2u_{0,3} - 4x_1x_2(x_1^{(2)} + x_2^{(2)})u_{0,5})\right]\mathbf{e}_1 \\ &+ \left[2\mu(2x_1x_2u_{3,0} + 4x_1x_2(x_1^{(2)} + x_2^{(2)})u_{5,0} + (x_1^{(2)} - x_2^{(2)})u_{0,3} + 2(x_1^{(4)} - x_2^{(4)})u_{0,5})\right]\mathbf{e}_2 \\ &+ \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)}\frac{\partial u_{5,0}}{\partial x_3}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)}\frac{\partial u_{0,5}}{\partial x_3}\right)x_2\right. \\ &\left.+2x_1w_{2,0} + 4x_1(x_1^{(2)} + x_2^{(2)})w_{4,0}\right]\mathbf{e}_3;\end{aligned}$$

(3.34)

$$\begin{aligned}\mathbf{T}_2 &= \left[2\mu(2x_1x_2u_{3,0} + 4x_1x_2(x_1^{(2)} + x_2^{(2)})u_{5,0} + (x_1^{(2)} - x_2^{(2)})u_{0,3} + 2(x_1^{(4)} - x_2^{(4)})u_{0,5})\right]\mathbf{e}_1 \\ &\left[-p + 2\mu\left(u_{1,0} + (x_1^{(2)} + 3x_2^{(2)})u_{3,0} + (x_1^{(2)} + 5x_2^{(2)})(x_1^{(2)} + x_2^{(2)})u_{5,0} + 2x_1x_2u_{0,3} + 4x_1x_2(x_1^{(2)} + x_2^{(2)})u_{0,5}\right)\right] \\ &+ \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)}\frac{\partial u_{5,0}}{\partial x_3}\right)x_2 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)}\frac{\partial u_{0,5}}{\partial x_3}\right)x_1\right. \\ &\left.+2x_2w_{2,0} + 4x_2(x_1^{(2)} + x_2^{(2)})w_{4,0}\right]\mathbf{e}_3;\end{aligned}$$

(3.35)

$$
\begin{aligned}
\mathbf{T}_3 = \mu \Bigg[& \left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)}) \frac{\partial u_{3,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)} \frac{\partial u_{5,0}}{\partial x_3} \right) x_1 \\
& - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)}) \frac{\partial u_{0,3}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)} \frac{\partial u_{0,5}}{\partial x_3} \right) x_2 + 2x_1 w_{2,0} + 4x_1 (x_1^{(2)} + x_2^{(2)}) w_{4,0} \Bigg] \mathbf{e}_1 \\
& + \mu \Bigg[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)}) \frac{\partial u_{3,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)} \frac{\partial u_{5,0}}{\partial x_3} \right) x_2 \\
& - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)}) \frac{\partial u_{0,3}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)} \frac{\partial u_{0,5}}{\partial z} \right) x_1 + 2x_2 w_{2,0} + 4x_2 (x_1^{(2)} + x_2^{(2)}) w_{4,0} \Bigg] \mathbf{e}_2 \\
& + \left[-p + 2\mu \left(\frac{\partial w_{0,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)}) \frac{\partial w_{2,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})^{(2)} \frac{\partial w_{0,5}}{\partial x_3} \right) \right] \mathbf{e}_3 .
\end{aligned}
$$

By substituting the stress and velocity terms into Eqs. (3.23) and (3.24), after converting to cylindrical polar coordinates, integrating and some algebraic manipulation, which is detailed in Appendix A (Appendix A), the following six resulting partial differential equations with unknown functions dependent on the coaxial coordinate and time are obtained for $K = 3$:

(3.36)

$$
\begin{aligned}
& 2\pi\mu \frac{\partial}{\partial z} \left[\phi^{(2)} \left(\frac{\partial w_{0,0}}{\partial z} + \frac{\phi^{(2)}}{2} \frac{\partial w_{2,0}}{\partial z} \right) \right] + 2\pi\phi \left(\tau_a + p_e \frac{\partial \phi}{\partial z} \right) \\
& = \frac{\partial p}{\partial z} + \pi\rho\phi^{(2)} \left[\frac{\partial w_{0,0}}{\partial t} + \frac{\partial w_{0,0}}{\partial z} \left(w_{0,0} + \frac{\phi^{(2)}}{2} w_{2,0} \right) + \frac{\phi^{(2)}}{2} \left(\frac{\partial w_{2,0}}{\partial t} + \frac{\partial w_{2,0}}{\partial z} \left(w_{0,0} + \frac{2}{3}\phi^{(2)} w_{2,0} \right) \right. \right. \\
& \left. \left. + 2w_{2,0} \left(u_{1,0} + \frac{2}{3}\phi^{(2)} u_{3,0} \right) \right) \right] ;
\end{aligned}
$$

(3.37)

$$
\begin{aligned}
& \frac{\pi\mu}{4} \frac{\partial}{\partial z} \left[\phi^{(4)} \left(\frac{\partial u_{1,0}}{\partial z} + \frac{2}{3}\phi^{(2)} \frac{\partial u_{3,0}}{\partial z} \right) \right] + \frac{\pi\mu}{2} \frac{\partial}{\partial z} (w_{2,0}\phi^{(4)}) - 2\pi\mu\phi^{(2)} (u_{1,0} + \phi^{(2)} u_{3,0}) + \pi\phi^{(2)} \left(\tau_a \frac{\partial \phi}{\partial z} - p_e \right) \\
& = -p_I + \frac{\rho\pi\phi^{(4)}}{4} \left[\frac{\partial u_{1,0}}{\partial t} + \frac{\partial u_{1,0}}{\partial z} \left(w_{0,0} + \frac{2}{3}\phi^{(2)} w_{2,0} \right) + u_{1,0} \left(u_{1,0} + \frac{2}{3}\phi^{(2)} u_{3,0} \right) - u_{0,1} \left(u_{0,1} + \frac{2}{3} u_{0,3}\phi^{(2)} \right) \right. \\
& \left. + \frac{2}{3}\phi^{(2)} \left[\frac{\partial u_{3,0}}{\partial t} + \frac{\partial u_{3,0}}{\partial z} \left(w_{0,0} + \frac{3}{4}\phi^{(2)} w_{2,0} \right) + 3u_{3,0} \left(u_{1,0} + \frac{3}{4}\phi^{(2)} u_{3,0} \right) - u_{0,3} \left(u_{0,1} + \frac{3}{4}\phi^{(2)} u_{0,3} \right) \right] \right] ;
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\mu}{4} \frac{\partial}{\partial z} \left[\phi^{(4)} \left(\frac{\partial u_{0,1}}{\partial z} + \frac{2\phi^{(2)}}{3} \frac{\partial u_{0,3}}{\partial z} \right) \right] + \phi^{(2)} \tau_h \left(1 + \left(\frac{\partial \phi}{\partial z} \right)^{(2)} \right)^{(1/2)} \\
& = \frac{\rho\phi^{(4)}}{4} \left[\frac{\partial u_{0,1}}{\partial t} + \frac{\partial u_{0,1}}{\partial z} \left(w_{0,0} + \frac{2\phi^{(2)} w_{2,0}}{3} \right) + 2u_{0,1} \left(u_{1,0} + \frac{2\phi^{(2)} u_{3,0}}{3} \right) \right. \\
& \left. \frac{2\phi^{(2)}}{3} \left(\frac{\partial u_{0,3}}{\partial t} + \frac{\partial u_{0,3}}{\partial z} \left(w_{0,0} + \frac{3\phi^{(2)} w_{2,0}}{4} \right) + 4u_{0,3} \left(u_{1,0} + \frac{3\phi^{(2)} u_{3,0}}{4} \right) \right) \right] ;
\end{aligned}
\tag{3.38}
$$

$$\begin{aligned}
&\frac{\pi\mu}{2}\frac{\partial}{\partial z}\left[\phi^{(4)}\left(\frac{\partial w_{0,0}}{\partial z}+\frac{2\phi^{(2)}}{3}\frac{\partial w_{2,0}}{\partial z}\right)\right]-\pi\mu\phi^{(4)}w_{2,0}\\
&-\frac{\pi\mu\phi^{(4)}}{2}\left(\frac{\partial u_{1,0}}{\partial z}+\frac{2\phi^{(2)}}{3}\frac{\partial u_{3,0}}{\partial z}\right)+\pi\phi^{(3)}\left(\tau_a+p_e\frac{\partial\phi}{\partial z}\right)\\
&=\frac{\partial q_I}{\partial z}+\frac{\pi\rho^*\phi^{(4)}}{4}\left[\frac{\partial w_{0,0}}{\partial t}+\frac{\partial w_{0,0}}{\partial z}\left(w_{0,0}+\frac{2\phi^{(2)}w_{2,0}}{3}\right)\right.\\
&\left.\frac{2\phi^{(2)}}{3}\left(\frac{\partial w_{2,0}}{\partial t}+\frac{\partial w_{2,0}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+2w_{2,0}\left(u_{1,0}+\frac{2\phi^{(2)}u_{3,0}}{4}\right)\right)\right];
\end{aligned}\tag{3.39}$$

$$\begin{aligned}
&\frac{\pi\mu}{24}\frac{\partial}{\partial z}\left[\phi^{(6)}\left(\frac{\partial u_{1,0}}{\partial z}+\frac{3\phi^{(2)}}{4}\frac{\partial u_{3,0}}{\partial z}\right)\right]+\frac{\pi\mu}{12}\frac{\partial}{\partial z}(\phi^{(6)}w_{2,0})\\
&-\frac{\pi\mu\phi^{(4)}}{2}\left(u_{1,0}+\frac{5\phi^{(2)}u_{3,0}}{3}\right)+\frac{\pi\phi^{(4)}}{4}\left(\tau_a\frac{\partial\phi}{\partial z}-p_e\right)\\
&=-q+\frac{\pi\rho\phi^{(6)}}{24}\left[\frac{\partial u_{1,0}}{\partial t}+\frac{\partial u_{1,0}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+u_{1,0}\left(u_{1,0}+\frac{3\phi^{(2)}u_{3,0}}{4}\right)\right.\\
&-u_{0,1}\left(u_{0,1}+\frac{3\phi^{(2)}u_{0,3}}{4}\right)+\frac{3\phi^{(2)}}{4}\left(\frac{\partial u_{3,0}}{\partial t}+\frac{\partial u_{3,0}}{\partial z}\left(w_{0,0}+\frac{4\phi^{(2)}w_{2,0}}{5}\right)\right.\\
&\left.\left.+3u_{3,0}\left(u_{1,0}+\frac{4\phi^{(2)}u_{3,0}}{5}\right)-u_{0,3}\left(u_{0,1}+\frac{4\phi^{(2)}u_{0,3}}{5}\right)\right)\right];
\end{aligned}\tag{3.40}$$

$$\begin{aligned}
&\frac{\mu}{24}\frac{\partial}{\partial z}\left[\phi^{(6)}\left(\frac{\partial u_{0,1}}{\partial z}+\frac{3\phi^{(2)}}{4}\frac{\partial u_{0,3}}{\partial z}\right)\right]-\frac{\mu\phi^{(6)}u_{0,3}}{6}+\frac{\phi^{(4)}\tau_h}{4}\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\\
&=\frac{\rho\phi^{(6)}}{24}\left[\frac{\partial u_{0,1}}{\partial t}+\frac{\partial u_{0,1}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+2u_{0,1}\left(u_{1,0}+\frac{3\phi^{(2)}u_{3,0}}{4}\right)\right.\\
&\left.+\frac{3\phi^{(2)}}{4}\left(\frac{\partial u_{0,3}}{\partial t}+\frac{\partial u_{0,3}}{\partial z}\left(w_{0,0}+\frac{4\phi^{(2)}w_{2,0}}{5}\right)+4u_{0,3}\left(u_{1,0}+\frac{4\phi^{(2)}u_{3,0}}{5}\right)\right)\right].
\end{aligned}\tag{3.41}$$

The corresponding set of partial differential equations for order $K=5$ are:

$$\begin{aligned}
&\frac{2}{3}\pi\mu\phi^{(6)}\frac{\partial^{(2)}w_{4,0}}{\partial z^{(2)}}+\pi\mu\phi^{(4)}\frac{\partial^{(2)}w_{2,0}}{\partial z^{(2)}}+2\pi\mu\phi^{(2)}\frac{\partial^{(2)}w_{0,0}}{\partial z^{(2)}}-\frac{1}{3}\pi\rho\phi^{(6)}\frac{\partial w_{4,0}}{\partial t}-\frac{1}{2}\pi\rho\phi^{(4)}\frac{\partial w_{2,0}}{\partial t}-\pi\rho\phi^{(2)}\frac{\partial w_{0,0}}{\partial t}\\
&-\frac{1}{5}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial w_{4,0}}{\partial z}-\frac{4}{5}\pi\rho\phi^{(10)}w_{4,0}u_{5,0}-\frac{1}{4}\pi\rho\phi^{(8)}w_{4,0}\frac{\partial w_{2,0}}{\partial z}-\frac{1}{4}\pi\rho\phi^{(8)}w_{2,0}\frac{\partial w_{4,0}}{\partial z}-\pi\phi^{(8)}w_{4,0}u_{3,0}\\
&-\frac{1}{2}\pi\rho\phi^{(8)}w_{2,0}u_{5,0}-\frac{1}{3}\pi\rho\phi^{(6)}w_{4,0}\frac{\partial w_{0,0}}{\partial z}-\frac{1}{3}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial w_{4,0}}{\partial z}-\frac{4}{3}\pi\rho\phi^{(6)}u_{1,0}w_{4,0}-\frac{2}{3}\pi\rho\phi^{(6)}w_{2,0}u_{3,0}\\
&-\frac{1}{2}\pi\rho\phi^{(4)}w_{2,0}\frac{\partial w_{0,0}}{\partial z}-\frac{1}{2}\pi\rho\phi^{(4)}w_{0,0}\frac{\partial w_{2,0}}{\partial z}-\pi\rho\phi^{(4)}u_{1,0}w_{2,0}-\pi\rho\phi^{(2)}w_{0,0}\frac{\partial w_{0,0}}{\partial z}+4\pi\mu\phi^{(3)}\frac{\partial\phi}{\partial z}\frac{\partial w_{2,0}}{\partial z}\\
&+4\pi\mu\phi\frac{\partial\phi}{\partial z}\frac{\partial w_{0,0}}{\partial z}+4\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial w_{4,0}}{\partial z}-\frac{\partial p}{\partial z}+2\pi\phi\left(\tau_a+p_e\frac{\partial\phi}{\partial z}\right)=0;
\end{aligned}$$

$$\begin{aligned}
&\frac{2}{3}\pi\mu\phi^{(6)}\frac{\partial w_{4,0}}{\partial z}+\frac{1}{2}\pi\mu\phi^{(4)}\frac{\partial w_{2,0}}{\partial z}+p_I+\frac{1}{5}\pi\rho\phi^{(10)}u_{0,3}u_{0,5}-2\pi\mu\phi^{(6)}u_{5,0}-2\pi\mu\phi^{(4)}u_{3,0}-2\pi\mu\phi u_{1,0}\\
&-\frac{1}{8}\pi\rho\phi^{(8)}\frac{\partial u_{5,0}}{\partial t}-\frac{3}{8}\pi\rho\phi^{(8)}u_{3,0}^{(2)}-\frac{1}{6}\pi\rho\phi^{(6)}\frac{\partial u_{3,0}}{\partial t}+\frac{1}{12}\pi\rho\phi^{(12)}u_{0,5}^{(2)}-\frac{5}{12}\pi\rho\phi^{(12)}u_{5,0}^{(2)}\\
&-\frac{1}{4}\pi\rho\phi^{(4)}u_{1,0}^{(2)}-\frac{1}{4}\pi\rho\phi^{(4)}\frac{\partial u_{1,0}}{\partial t}-\pi\phi^{(2)}p_e+4\pi\mu\phi^{(5)}w_{4,0}\frac{\partial\phi}{\partial z}+\pi\mu\phi^{(3)}\frac{\partial\phi}{\partial z}\frac{\partial u_{1,0}}{\partial z}+2\pi\mu\phi^{(3)}w_{2,0}\frac{\partial\phi}{\partial z}\\
&+\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial u_{5,0}}{\partial z}+\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial u_{3,0}}{\partial z}+\frac{1}{4}\pi\rho\phi^{(4)}w_{0,0}\frac{\partial u_{1,0}}{\partial z}-\frac{3}{4}\pi\rho\phi^{(8)}u_{1,0}u_{5,0}-\frac{1}{8}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial u_{5,0}}{\partial z}\\
&-\frac{1}{8}\pi\rho\phi^{(8)}w_{4,0}\frac{\partial u_{1,0}}{\partial z}-\frac{1}{8}\pi\rho\phi^{(8)}w_{2,0}\frac{\partial u_{3,0}}{\partial z}+\frac{1}{8}\pi\rho\phi^{(8)}u_{0,3}^{(2)}-\frac{1}{10}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial u_{3,0}}{\partial z}-\frac{4}{5}\pi\rho\phi^{(10)}u_{3,0}u_{5,0}\\
&-\frac{1}{10}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial u_{5,0}}{\partial z}-\frac{2}{3}\pi\rho\phi^{(6)}u_{1,0}u_{3,0}-\frac{1}{6}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}-\frac{1}{6}\pi\rho\phi^{(6)}w_{2,0}\frac{\partial u_{1,0}}{\partial z}\\
&-\frac{1}{12}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial u_{5,0}}{\partial z}+\pi\phi^{(2)}\tau_a\frac{\partial\phi}{\partial z}+\frac{1}{8}\pi\mu\phi^{(8)}\frac{\partial^{(2)}u_{5,0}}{\partial z^{(2)}}+\frac{1}{6}\pi\mu\phi^{(6)}\frac{\partial^{(2)}u_{3,0}}{\partial z^{(2)}}+\frac{1}{4}\pi\mu\phi^{(4)}\frac{\partial^{(2)}u_{1,0}}{\partial z^{(2)}}=0;
\end{aligned}$$

$$\begin{aligned}
&\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial u_{0,3}}{\partial z}+\pi\mu\phi^{(3)}\frac{\partial\phi}{\partial z}\frac{\partial u_{0,1}}{\partial z}+\frac{1}{8}\pi\mu\phi^{(8)}\frac{\partial^{(2)}u_{0,5}}{\partial z^{(2)}}+\frac{1}{6}\pi\mu\phi^{(6)}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}+\frac{1}{4}\pi\mu\phi^{(4)}\frac{\partial^{(2)}u_{0,1}}{\partial z^{(2)}}\\
&+\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial u_{0,5}}{\partial z}+\pi\phi^{(2)}\tau_h\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}-\frac{1}{4}\pi\rho\phi^{(4)}w_{0,0}\frac{\partial u_{0,1}}{\partial z}\\
&-\frac{1}{6}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial u_{0,3}}{\partial z}-\frac{1}{2}\pi\rho\phi^{(8)}u_{0,3}u_{3,0}-\frac{1}{6}\pi\rho\phi^{(6)}w_{2,0}\frac{\partial u_{0,1}}{\partial z}\\
&-\frac{1}{8}\pi\rho\phi^{(8)}w_{2,0}\frac{\partial u_{0,3}}{\partial z}-\frac{2}{5}\pi\rho\phi^{(10)}u_{0,3}u_{5,0}-\frac{1}{8}\pi\rho\phi^{8}w_{4,0}\frac{\partial u_{0,1}}{\partial z}-\frac{1}{10}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial u_{0,3}}{\partial z}-\frac{2}{3}\pi\rho\phi^{(6)}u_{0,3}u_{1,0}\\
&-\frac{1}{8}\pi\rho\phi^{(8)}\frac{\partial u_{0,5}}{\partial t}-\frac{1}{12}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial u_{0,5}}{\partial z}-\frac{1}{2}\pi\rho\phi^{(12)}u_{5,0}u_{0,5}-\frac{3}{4}\pi\rho\phi^{(8)}u_{1,0}u_{0,5}-\frac{1}{8}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial u_{0,5}}{\partial z}\\
&-\frac{1}{6}\pi\rho\phi^{(6)}u_{0,3}\frac{\partial u_{0,1}}{\partial t}-\frac{1}{6}\pi\rho\phi^{(6)}u_{0,1}\frac{\partial u_{0,3}}{\partial t}-\frac{1}{10}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial u_{0,5}}{\partial z}-\frac{3}{5}\pi\rho\phi^{(10)}u_{3,0}u_{0,5}=0;
\end{aligned}$$

$$\begin{aligned}
&\pi\phi^{(3)}\tau_a-\frac{1}{12}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial w_{4,0}}{\partial z}-\frac{1}{3}\pi\rho\phi^{(12)}u_{5,0}w_{4,0}-\frac{1}{6}\pi\rho\phi^{(6)}w_{2,0}\frac{\partial w_{0,0}}{\partial z}-\frac{1}{6}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial w_{2,0}}{\partial z}\\
&-\frac{1}{3}\pi\rho\phi^{(6)}u_{1,0}w_{2,0}-\frac{1}{10}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial w_{2,0}}{\partial z}-\frac{1}{10}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial w_{4,0}}{\partial z}\\
&-\frac{2}{5}\pi\rho\phi^{(10)}u_{3,0}w_{4,0}-\frac{1}{5}\pi\rho\phi^{(10)}u_{5,0}w_{2,0}-\frac{1}{8}\pi\rho\phi^{(8)}w_{4,0}\frac{\partial w_{0,0}}{\partial z}-\frac{1}{8}\pi\rho\phi^{(8)}w_{2,0}\frac{\partial w_{2,0}}{\partial z}\\
&-\frac{1}{8}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial w_{4,0}}{\partial z}-\frac{1}{2}\pi\rho\phi^{(8)}u_{1,0}w_{4,0}-\frac{1}{4}\pi\rho\phi^{(8)}u_{3,0}w_{2,0}\\
&-\frac{1}{4}\pi\rho\phi^{(4)}w_{0,0}\frac{\partial w_{0,0}}{\partial z}-\frac{1}{4}\pi\mu\phi^{(8)}\frac{\partial u_{5,0}}{\partial z}+2\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial w_{2,0}}{\partial z}+2\pi\mu\phi^{(3)}\frac{\partial\phi}{\partial z}\frac{\partial w_{0,0}}{\partial z}-\frac{\partial q_I}{\partial z}+\frac{1}{4}\pi\mu\phi^{(8)}\frac{\partial^{(2)}w_{4,0}}{\partial z^{(2)}}\\
&+\frac{1}{3}\pi\mu\phi^{(6)}\frac{\partial^{(2)}w_{2,0}}{\partial z^{(2)}}+\frac{1}{2}\pi\mu\phi^{(4)}\frac{\partial^{(2)}w_{0,0}}{\partial z^{(2)}}+\pi\phi^{(3)}p_e\frac{\partial\phi}{\partial z}-\frac{1}{3}\pi\mu\phi^{(6)}\frac{\partial u_{3,0}}{\partial z}-\frac{4}{3}\pi\mu\phi^{(6)}w_{4,0}\\
&-\frac{1}{2}\pi\mu\phi^{(4)}\frac{\partial u_{1,0}}{\partial z}-\pi\mu\phi^{(4)}w_{2,0}+2\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial w_{4,0}}{\partial z}-\frac{1}{4}\pi\rho\phi^{(4)}\frac{\partial w_{0,0}}{\partial t}-\frac{1}{8}\pi\rho\phi^{(8)}\frac{\partial w_{4,0}}{\partial t}-\frac{1}{6}\pi\rho\phi^{(6)}\frac{\partial w_{2,0}}{\partial t}=0;
\end{aligned}$$

$$3q_I+\frac{1}{8}\pi\rho\phi^{(12)}u_{0,3}u_{0,5}-\frac{3}{4}\pi\phi^{(4)}p_e-3\pi\mu\phi^{(8)}u_{5,0}-\frac{5}{2}\pi\mu\phi^{(6)}u_{3,0}-\frac{3}{2}\pi\mu\phi^{(4)}u_{1,0}-\frac{1}{16}\pi\rho\phi^{(12)}w_{2,0}\frac{\partial u_{5,0}}{\partial z}$$
$$-\frac{1}{16}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial u_{3,0}}{\partial z}-\frac{1}{2}\pi\rho\phi^{(12)}u_{3,0}u_{5,0}-\frac{1}{8}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial u_{1,0}}{\partial z}-\frac{3}{56}\pi\rho\phi^{(14)}w_{4,0}\frac{\partial u_{5,0}}{\partial z}-\frac{3}{32}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}$$
$$-\frac{3}{32}\pi\rho\phi^{(8)}w_{2,0}\frac{\partial u_{1,0}}{\partial z}-\frac{3}{8}\pi\rho\phi^{(8)}u_{1,0}u_{3,0}-\frac{3}{40}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial u_{1,0}}{\partial z}-\frac{3}{40}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial u_{3,0}}{\partial z}+\frac{3}{40}\pi\rho\phi^{(10)}u_{0,3}^{(2)}$$
$$-\frac{9}{20}\pi\rho\phi^{(10)}u_{1,0}u_{5,0}-\frac{3}{40}\pi\rho\phi^{(10)}w_{0,0}\frac{\partial u_{5,0}}{\partial z}+\frac{1}{8}\pi\mu\phi^{(6)}\frac{\partial^{(2)}u_{1,0}}{\partial z^{(2)}}+\frac{3}{8}\pi\mu\phi^{(8)}\frac{\partial w_{4,0}}{\partial z}+\frac{1}{4}\pi\mu\phi^{(6)}\frac{\partial w_{2,0}}{\partial z}$$
$$-\frac{9}{40}\pi\rho\phi^{(10)}u_{3,0}^{(2)}-\frac{3}{40}\pi\rho\phi^{(10)}\frac{\partial u_{5,0}}{\partial t}-\frac{3}{32}\pi\rho\phi^{(8)}\frac{\partial u_{3,0}}{\partial t}-\frac{15}{56}\pi\rho\phi^{(14)}u_{5,0}^{(2)}+\frac{3}{56}\pi\rho\phi^{(14)}u_{0,5}^{(2)}-\frac{1}{8}\pi\rho\phi^{(6)}u_{1,0}^{(2)}$$
$$-\frac{1}{8}\pi\rho\phi^{(6)}\frac{\partial u_{1,0}}{\partial t}+\frac{3}{4}\pi\mu\phi^{(9)}\frac{\partial\phi}{\partial z}\frac{\partial u_{5,0}}{\partial z}+\frac{3}{4}\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial u_{3,0}}{\partial z}+3\pi\mu\phi^{(7)}w_{4,0}\frac{\partial\phi}{\partial z}+\frac{3}{4}\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial u_{1,0}}{\partial z}$$
$$+\frac{3}{2}\pi\mu\phi^{(5)}w_{2,0}\frac{\partial\phi}{\partial z}+\frac{3}{4}\pi\phi^{(4)}\tau_a\frac{\partial\phi}{\partial z}+\frac{3}{40}\pi\mu\phi^{(10)}\frac{\partial^{(2)}u_{5,0}}{\partial z^{(2)}}+\frac{3}{32}\pi\mu\phi^{(8)}\frac{\partial^{(2)}u_{3,0}}{\partial z^{(2)}}=0;$$

$$\frac{3}{4}\phi^{(4)}\tau_h\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}+\frac{3}{4}\pi\mu\phi^{(9)}\frac{\partial\phi}{\partial z}\frac{\partial u_{0,5}}{\partial z}+\frac{3}{40}\pi\mu\phi^{(10)}\frac{\partial^{(2)}u_{0,5}}{\partial z^{(2)}}+\frac{3}{32}\pi\mu\phi^{(8)}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}+\frac{1}{8}\pi\mu\phi^{(6)}\frac{\partial^{(2)}u_{0,1}}{\partial z^{(2)}}$$
$$+\frac{3}{4}\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial u_{0,3}}{\partial z}+\frac{3}{4}\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial u_{0,1}}{\partial z}-\frac{3}{32}\pi\rho\phi^{(8)}u_{0,3}\frac{\partial u_{0,1}}{\partial t}-\frac{3}{32}\pi\rho\phi^{(8)}u_{0,1}\frac{\partial u_{0,3}}{\partial t}-\frac{3}{40}\pi\rho\phi^{(10)}w_{0,0}\frac{\partial u_{0,5}}{\partial z}$$
$$-\frac{9}{20}\pi\rho\phi^{(10)}u_{1,0}u_{0,5}-\frac{9}{28}\pi\rho\phi^{(14)}u_{5,0}u_{0,5}-\frac{3}{56}\pi\rho\phi^{(14)}w_{4,0}\frac{\partial u_{0,5}}{\partial z}-\frac{3}{8}\pi\rho\phi^{(12)}u_{3,0}u_{0,5}-\frac{1}{16}\pi\rho\phi^{(12)}w_{2,0}\frac{\partial u_{0,5}}{\partial z}$$
$$-\frac{3}{40}\pi\rho\phi^{(10)}\frac{\partial u_{0,5}}{\partial t}-\frac{3}{4}\pi\mu\phi^{(8)}u_{0,5}-\frac{1}{2}\pi\mu\phi^{(6)}u_{0,3}-\frac{3}{40}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial u_{0,1}}{\partial z}-\frac{1}{16}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial u_{0,3}}{\partial z}$$
$$\frac{1}{4}\pi\rho\phi^{(12)}u_{0,3}u_{5,0}-\frac{3}{8}\pi\rho\phi^{(8)}u_{0,3}u_{1,0}-\frac{1}{8}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial u_{0,1}}{\partial z}-\frac{3}{32}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial u_{0,3}}{\partial z}-\frac{3}{10}\pi\rho\phi^{(10)}u_{0,3}u_{3,0}$$
$$-\frac{3}{32}\pi\rho\phi^{8}w_{2,0}\frac{\partial u_{0,1}}{\partial z}-\frac{3}{40}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial u_{0,3}}{\partial z}=0;$$

$$-\pi\mu\phi^{(6)}w_{2,0}+\frac{3}{4}\pi\phi^{(5)}\tau_a-\frac{3}{10}\pi\mu\phi^{(10)}\frac{\partial u_{5,0}}{\partial z}-\frac{3}{8}\pi\mu\phi^{(8)}\frac{\partial u_{3,0}}{\partial z}-\frac{3}{2}\pi\mu\phi^{(8)}w_{4,0}-\frac{1}{2}\pi\mu\phi^{(6)}\frac{\partial u_{1,0}}{\partial z}$$
$$-\frac{3}{40}\pi\rho\phi^{(10)}\frac{\partial w_{4,0}}{\partial t}-\frac{3}{32}\pi\rho\phi^{(8)}\frac{\partial w_{2,0}}{\partial t}-\frac{1}{8}\pi\rho\phi^{(6)}\frac{\partial w_{0,0}}{\partial t}-\frac{3}{56}\pi\rho\phi^{(14)}w_{4,0}\frac{\partial w_{4,0}}{\partial z}-\frac{3}{14}\pi\rho\phi^{(14)}u_{5,0}w_{4,0}$$
$$-\frac{1}{16}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial w_{2,0}}{\partial z}-\frac{1}{16}\pi\rho\phi^{(12)}w_{2,0}\frac{\partial w_{4,0}}{\partial z}-\frac{1}{4}\pi\rho\phi^{(12)}u_{3,0}w_{4,0}-\frac{1}{8}\pi\rho\phi^{(12)}u_{5,0}w_{2,0}$$
$$-\frac{3}{40}\pi\rho\phi^{(10)}w_{4,0}\frac{\partial w_{0,0}}{\partial z}-\frac{3}{40}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial w_{2,0}}{\partial z}-\frac{3}{40}\pi\rho\phi^{(10)}w_{0,0}\frac{\partial w_{4,0}}{\partial z}-\frac{3}{10}\pi\rho\phi^{(10)}u_{1,0}w_{4,0}$$
$$-\frac{3}{20}\pi\rho\phi^{(10)}u_{3,0}w_{2,0}-\frac{3}{32}\pi\rho\phi^{(8)}w_{2,0}\frac{\partial w_{0,0}}{\partial z}-\frac{3}{32}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial w_{2,0}}{\partial z}-\frac{3}{16}\pi\rho\phi^{(8)}u_{1,0}w_{2,0}$$
$$-\frac{1}{8}\pi\rho\phi^{(6)}w_{0,0}\frac{\partial w_{0,0}}{\partial z}+\frac{3}{4}\pi\phi^{(5)}p_e\frac{\partial\phi}{\partial z}+\frac{3}{20}\pi\mu\phi^{(10)}\frac{\partial^{(2)}w_{4,0}}{\partial z^{(2)}}+\frac{3}{16}\pi\mu\phi^{(8)}\frac{\partial^{(2)}w_{2,0}}{\partial z^{(2)}}+\frac{1}{4}\pi\mu\phi^{(6)}\frac{\partial^{(2)}w_{0,0}}{\partial z^{(2)}}$$
$$+\frac{3}{2}\pi\mu\phi^{(9)}\frac{\partial\phi}{\partial z}\frac{\partial w_{4,0}}{\partial z}-\frac{\partial h}{\partial z}+\frac{3}{2}\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial w_{2,0}}{\partial z}+\frac{3}{2}\pi\mu\phi^{(5)}\frac{\partial\phi}{\partial z}\frac{\partial w_{0,0}}{\partial z}=0;$$

$$\frac{5}{8}\pi\mu\phi^{(11)}\frac{\partial\phi}{\partial z}\frac{\partial u_{5,0}}{\partial z}+\frac{5}{8}\pi\mu\phi^{(9)}\frac{\partial\phi}{\partial z}\frac{\partial u_{3,0}}{\partial z}+\frac{5}{2}\pi\mu\phi^{(9)}w_{4,0}\frac{\partial\phi}{\partial z}+\frac{5}{8}\pi\mu\phi^{(7)}\frac{\partial\phi}{\partial z}\frac{\partial u_{1,0}}{\partial z}+\frac{5}{4}\pi\mu\phi^{(7)}w_{2,0}\frac{\partial\phi}{\partial z}$$
$$-\frac{13}{4}\pi\mu\phi^{(10)}u_{5,0}-\frac{5}{2}\pi\mu\phi^{(8)}u_{3,0}-\frac{5}{4}\pi\mu\phi^{(6)}u_{1,0}+5h+\frac{5}{56}\pi\rho\phi^{(14)}u_{0,3}u_{0,5}$$
$$+\frac{5}{8}\pi\phi^{(6)}\tau_a\frac{\partial\phi}{\partial z}-\frac{5}{64}\pi\rho\phi^{(8)}u_{1,0}^{(2)}-\frac{5}{64}\pi\rho\phi^{(8)}\frac{\partial u_{1,0}}{\partial t}+\frac{5}{128}\pi\rho\phi^{(16)}u_{0,5}^{(2)}$$
$$-\frac{25}{128}\pi\rho\phi^{(16)}u_{5,0}^{(2)}-\frac{1}{16}\pi\rho\phi^{(10)}\frac{\partial u_{3,0}}{\partial t}-\frac{5}{32}\pi\rho\phi^{(12)}u_{3,0}^{(2)}-\frac{5}{96}\pi\rho\phi^{(12)}\frac{\partial u_{5,0}}{\partial t}$$
$$-\frac{5}{8}\pi\phi^{(6)}p_e+\frac{1}{4}\pi\mu\phi^{(10)}\frac{\partial w_{4,0}}{\partial z}+\frac{5}{32}\pi\mu\phi^{(8)}\frac{\partial w_{2,0}}{\partial z}+\frac{5}{96}\pi\mu\phi^{(12)}\frac{\partial^{(2)}u_{5,0}}{\partial z^{(2)}}+\frac{1}{16}\pi\mu\phi^{(10)}\frac{\partial^{(2)}u_{3,0}}{\partial z^{(2)}}$$
$$+\frac{5}{64}\pi\mu\phi^{(8)}\frac{\partial^{(2)}u_{1,0}}{\partial z^{(2)}}-\frac{1}{4}\pi\rho\phi^{(10)}u_{1,0}u_{3,0}-\frac{1}{16}\pi\rho\phi^{(10)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}-\frac{1}{16}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial u_{1,0}}{\partial z}$$
$$-\frac{5}{14}\pi\rho\phi^{(14)}u_{3,0}u_{5,0}-\frac{5}{112}\pi\rho\phi^{112}w_{4,0}\frac{\partial u_{3,0}}{\partial z}-\frac{5}{112}\pi\rho\phi^{(14)}w_{2,0}\frac{\partial u_{5,0}}{\partial z}-\frac{5}{128}\pi\rho\phi^{(16)}w_{4,0}\frac{\partial u_{5,0}}{\partial z}$$
$$-\frac{5}{96}\pi\rho\phi^{(12)}w_{0,0}\frac{\partial u_{5,0}}{\partial z}-\frac{5}{96}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial u_{1,0}}{\partial z}-\frac{5}{96}\pi\rho\phi^{(12)}w_{2,0}\frac{\partial u_{3,0}}{\partial z}+\frac{5}{96}\pi\rho\phi^{(12)}u_{0,3}^{(2)}$$
$$-\frac{5}{16}\pi\rho\phi^{(12)}u_{1,0}u_{5,0}-\frac{5}{64}\pi\rho\phi^{(8)}w_{0,0}\frac{\partial u_{1,0}}{\partial z}=0;$$

$$-\pi\mu\phi^{(10)}u_{0,5}-\frac{5}{8}\pi\mu\phi^{(8)}u_{0,3}-\frac{15}{56}\pi\rho\phi^{(14)}u_{3,0}u_{0,5}-\frac{5}{112}\pi\rho\phi^{(14)}w_{2,0}\frac{\partial u_{0,5}}{\partial z}+\frac{5}{8}\pi\phi^{(6)}\tau_h\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}$$
$$-\frac{15}{64}\pi\rho\phi^{(16)}u_{5,0}u_{0,5}-\frac{5}{128}\pi\rho\phi^{(16)}w_{4,0}\frac{\partial u_{0,5}}{\partial z}-\frac{5}{16}\pi\rho\phi^{(12)}u_{1,0}u_{0,5}-\frac{5}{96}\pi\rho\phi^{(12)}w_{0,0}\frac{\partial u_{0,5}}{\partial z}$$
$$-\frac{1}{16}\pi\rho\phi^{(10)}u_{0,3}\frac{\partial u_{0,1}}{\partial t}-\frac{1}{16}\pi\rho\phi^{(10)}u_{0,1}\frac{\partial u_{0,3}}{\partial t}+\frac{5}{96}\pi\mu\phi^{(12)}\frac{\partial^{(2)}u_{0,5}}{\partial z^{(2)}}+\frac{1}{16}\pi\mu\phi^{(10)}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}$$
$$+\frac{5}{64}\pi\mu\phi^{(8)}\frac{\partial^{(2)}u_{0,1}}{\partial z^{(2)}}-\frac{5}{28}\pi\rho\phi^{(14)}u_{0,3}u_{5,0}-\frac{5}{96}\pi\rho\phi^{(12)}w_{4,0}\frac{\partial u_{0,1}}{\partial z}-\frac{1}{4}\pi\rho\phi^{(10)}u_{0,3}u_{1,0}$$
$$-\frac{5}{64}\pi\rho\phi^{8}w_{0,0}\frac{\partial u_{0,1}}{\partial z}-\frac{1}{16}\pi\rho\phi^{(10)}w_{0,0}\frac{\partial u_{0,3}}{\partial z}-\frac{5}{112}\pi\rho\phi^{(14)}w_{4,0}\frac{\partial u_{0,3}}{\partial z}-\frac{1}{16}\pi\rho\phi^{(10)}w_{2,0}\frac{\partial u_{0,1}}{\partial z}$$
$$+\frac{5}{8}\pi\mu\phi^{(9)}\frac{\partial u_{0,3}}{\partial z}\frac{\partial\phi}{\partial z}+\frac{5}{8}\pi\mu\phi^{(7)}\frac{\partial u_{0,1}}{\partial z}\frac{\partial\phi}{\partial z}+\frac{5}{8}\pi\mu\phi^{(11)}\frac{\partial u_{0,5}}{\partial z}\frac{\partial\phi}{\partial z}-\frac{5}{96}\pi\rho\phi^{(12)}\frac{\partial u_{0,5}}{\partial t}$$
$$-\frac{5}{96}\pi\rho\phi^{(12)}w_{2,0}\frac{\partial u_{0,3}}{\partial z}-\frac{5}{24}\pi\rho\phi^{(12)}u_{0,3}u_{3,0}=0.$$

3.3 Nondimensionalisation

In this section the system of partial differential equations will be nondimensionalised, before seeking solutions to them for special cases in the next section. To nondimensionalise, standardised velocity and length parameters are introduced, these are U and L

respectively. Dimensionless variables are introduced with the following definitions

$$\bar{x}_1 = \frac{x_1}{L}; \qquad \bar{x}_2 = \frac{x_2}{L}; \qquad \bar{x}_3 = \frac{x_3}{L};$$
$$\bar{w}_{0,0} = \frac{w_{0,0}}{U}; \qquad \bar{w}_{2,0} = \frac{L^{(2)}w_{2,0}}{U}; \qquad \bar{w}_{4,0} = \frac{L^{(4)}w_{4,0}}{U};$$
$$\bar{u}_{1,0} = \frac{ReLu_{1,0}}{U}; \qquad \bar{u}_{3,0} = \frac{ReL^{(3)}u_{3,0}}{U}; \qquad \bar{u}_{5,0} = \frac{ReL^{(5)}u_{5,0}}{U};$$
$$\bar{u}_{0,1} = \frac{ReLu_{0,1}}{U}; \qquad \bar{u}_{0,3} = \frac{ReL^{(3)}u_{0,3}}{U}; \qquad \bar{u}_{0,5} = \frac{ReL^{(5)}u_{0,5}}{U};$$
$$\bar{p}_e = \frac{Lp_e}{\mu U}; \qquad \bar{\tau}_a = \frac{L\tau_a}{\mu U}; \qquad \bar{\tau}_h = \frac{L\tau_h}{\mu U};$$
$$\bar{p}_I = \frac{p_I}{\mu L U}; \qquad \bar{q}_I = \frac{q_I}{\mu L^{(3)} U}; \qquad \bar{h}_I = \frac{h_I}{\mu L^{(5)} U};$$
$$\bar{p} = \frac{Lp}{\mu U}; \qquad \bar{t} = \frac{Ut}{L}; \qquad \bar{\phi} = \frac{\phi}{L};$$
$$\bar{\mathbf{T}}_1 = \frac{L\bar{T}_1}{\mu U}; \qquad \bar{\mathbf{T}}_2 = \frac{L\bar{T}_2}{\mu U}; \qquad \bar{\mathbf{T}}_1 = \frac{L\bar{T}_1}{\mu U};$$
$$\bar{\mathbf{t}} = \frac{L^{(2)}\mathbf{t}}{\mu U}; \qquad \bar{\mathbf{v}} = \frac{\mathbf{v}}{U};$$

where Re is the Reynolds number defined by $Re = \frac{\rho L U}{\mu}$. Here, the flow regime of interest is that of thin pipes which are not so thin that asymptotic reduction of the Navier-Stokes equations such as lubrication theory applies, however the fundamental flow is such that is dominated by the flow in the coaxial direction so a quasi-1D numerical method might be more efficient than a full-scale solution of the Navier-Stokes equations. The pressure integral terms p_I, q_I and h_I are defined respectively by

$$p_I = \int\int_A p\,da; \tag{3.42}$$

$$q_I = \int\int_A p x_1^{(2)} da = \int\int_A p x_2^{(2)} da \tag{3.43}$$

$$h_I = \int\int_A p x_1^{(4)} da = \int\int_A p x_2^{(4)} da; \tag{3.44}$$

The nondimensionalised velocity at order $K = 3$, corresponding to Eq.(3.13) is

(3.45)
$$\bar{\mathbf{v}} = \frac{1}{Re}[(\bar{u}_{1,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{3,0})\bar{x}_1 - (\bar{u}_{0,1} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{0,3})\bar{x}_2]\mathbf{e}_1$$
$$+ \frac{1}{Re}[(\bar{u}_{1,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{3,0})\bar{x}_2 + (\bar{u}_{0,1} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{0,3})\bar{x}_1]\mathbf{e}_2 + [\bar{w}_{0,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{w}_{2,0}]\mathbf{e}_3.$$

The nondimensionalised velocity at order $K = 5$, corresponding to Eq.(3.14) is

(3.46)

$$\begin{aligned}\bar{\mathbf{v}} = &\frac{1}{Re}[(\bar{u}_{1,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{3,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})^{(2)}\bar{u}_{5,0})\bar{x}_1 - (\bar{u}_{0,1} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{0,3} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})^{(2)}\bar{u}_{0,5})\bar{x}_2]\mathbf{e}_1\\ &+ \frac{1}{Re}[(\bar{u}_{1,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{3,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})^{(2)}\bar{u}_{5,0})\bar{x}_2 + (\bar{u}_{0,1} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{u}_{0,3} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})^{(2)}\bar{u}_{0,5})\bar{x}_1]\mathbf{e}_2\\ &+ [\bar{w}_{0,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})\bar{w}_{2,0} + (\bar{x}_1^{(2)} + \bar{x}_2^{(2)})^{(2)}\bar{w}_{4,0}]\mathbf{e}_3.\end{aligned}$$

The stress vector given by Eq. (3.29) can be written in nondimensional form as

$$\begin{aligned}\bar{\mathbf{t}} = &\left[\bar{\phi}^{-1}\left(\frac{\bar{x}_1\left(\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{x}_3} - \bar{p}_e\right)}{\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{x}_3}\right)^{(2)}\right)^{(1/2)}} - \bar{x}_2\bar{\tau}_h\right)\right]\mathbf{e}_1 + \left[\bar{\phi}^{-1}\left(\frac{\bar{x}_2\left(\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{x}_3} - \bar{p}_e\right)}{\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{x}_3}\right)^{(2)}\right)^{(1/2)}} + \bar{x}_1\bar{\tau}_h\right)\right]\mathbf{e}_2\\ &+ \left[\frac{\bar{\tau}_a + \bar{p}_e\frac{\partial\bar{\phi}}{\partial\bar{x}_3}}{\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{x}_3}\right)^{(2)}\right)^{(1/2)}}\right]\mathbf{e}_3.\end{aligned}$$

The conditions that must be satisfied for incompressibility to hold at order $K = 3$, given by Eqs. (3.17) and (3.18) , are given in nondimensional form as

$$\frac{\partial\bar{w}_{0,0}}{\partial\bar{x}_3} + \frac{2}{Re}\bar{u}_{1,0} = 0 \tag{3.47}$$

and

$$\frac{\partial\bar{w}_{2,0}}{\partial\bar{x}_3} + \frac{4}{Re}\bar{u}_{3,0} = 0. \tag{3.48}$$

The extra equation that must be satisfied for incompressibility to hold at order $K = 5$, given by Eq. (3.19), is given in nondimensional form as

$$\frac{\partial\bar{w}_{4,0}}{\partial\bar{x}_3} + \frac{6\bar{u}_{5,0}}{Re} = 0. \tag{3.49}$$

The boundary condition for order $K = 3$, given by Eq. (3.15), is given in nondimensional form by

$$\frac{\partial\bar{\phi}}{\partial\bar{t}} + (\bar{w}_{0,0} + \bar{\phi}^{(2)}\bar{w}_{2,0})\frac{\partial\bar{\phi}}{\partial\bar{x}_3} - (\bar{u}_{1,0} + \bar{\phi}^{(2)}\bar{u}_{3,0})\frac{\bar{\phi}}{Re} = 0.$$

The corresponding boundary condition for order $K = 5$, given by Eq (3.16) , is given in nondimensional form by

$$\frac{\partial\bar{\phi}}{\partial\bar{t}} + (\bar{w}_{0,0} + \bar{\phi}^{(2)}\bar{w}_{2,0} + \bar{\phi}^{(4)}\bar{w}_{4,0})\frac{\partial\bar{\phi}}{\partial\bar{x}_3} - (\bar{u}_{1,0} + \bar{\phi}^{(2)}\bar{u}_{3,0} + \bar{\phi}^{(4)}\bar{u}_{5,0})\frac{\bar{\phi}}{Re} = 0.$$

By substituting these nondimensional variables into Eqs. (3.36) - (3.41) and multiplying through by a suitable term, the dimensionless system of partial differential equations for $K = 3$ are obtained as the following:

(3.50)
$$2\pi\frac{\partial}{\partial\bar{z}}\left[\bar{\phi}^{(2)}\left(\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}+\frac{\bar{\phi}^{(2)}}{2}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}\right)\right]+2\pi\bar{\phi}\left(\bar{\tau}_a+\bar{p}_e\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)=\frac{\partial p_I}{\partial\bar{z}}+\pi Re\bar{\phi}^{(2)}\left[\frac{\partial\bar{w}_{0,0}}{\partial\bar{t}}\right.$$
$$\left.+\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{\bar{\phi}^{(2)}}{2}\bar{w}_{2,0}\right)+\frac{\bar{\phi}^{(2)}}{2}\left(\frac{\partial\bar{w}_{2,0}}{\partial\bar{t}}+\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{2}{3}\bar{\phi}^{(2)}\bar{w}_{2,0}\right)+\frac{2}{Re}\bar{w}_{2,0}\left(\bar{u}_{1,0}+\frac{2}{3}\bar{\phi}^{(2)}\bar{u}_{3,0}\right)\right)\right];$$

(3.51)
$$\frac{\pi}{4}\frac{\partial}{\partial\bar{z}}\left[\bar{\phi}^{(4)}\left(\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}+\frac{2}{3}\bar{\phi}^{(2)}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}\right)\right]+\frac{\pi}{2}\frac{\partial}{\partial\bar{z}}(\bar{w}_{2,0}\bar{\phi}^{(4)})-\frac{2}{Re}\pi\bar{\phi}^{(2)}(\bar{u}_{1,0}+\bar{\phi}^{(2)}\bar{u}_{3,0})+\pi\bar{\phi}^{(2)}\left(\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{z}}-\bar{p}_e\right)$$
$$-p+\frac{\pi\bar{\phi}^{(4)}}{4}\left[\frac{\partial\bar{u}_{1,0}}{\partial\bar{t}}+\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{2}{3}\bar{\phi}^{(2)}\bar{w}_{2,0}\right)+\bar{u}_{1,0}\left(\bar{u}_{1,0}+\frac{2}{3}\bar{\phi}^{(2)}\bar{u}_{3,0}\right)-\bar{u}_{0,1}\left(\bar{u}_{0,1}+\frac{2}{3}\bar{u}_{0,3}\bar{\phi}^{(2)}\right)\right.$$
$$\left.+\frac{2}{3}\bar{\phi}^{(2)}\left[\frac{\partial\bar{u}_{3,0}}{\partial\bar{t}}+\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{3}{4}\bar{\phi}^{(2)}\bar{w}_{2,0}\right)+3\bar{u}_{3,0}\left(\bar{u}_{1,0}+\frac{3}{4}\bar{\phi}^{(2)}\bar{u}_{3,0}\right)-\bar{u}_{0,3}\left(\bar{u}_{0,1}+\frac{3}{4}\bar{\phi}^{(2)}\bar{u}_{0,3}\right)\right]\right];$$

(3.52)
$$\frac{1}{4Re}\frac{\partial}{\partial\bar{z}}\left[\bar{\phi}^{(4)}\left(\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}+\frac{2\bar{\phi}^{(2)}}{3}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}\right)\right]+\bar{\phi}^{(2)}\bar{\tau}_h\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)^{(2)}\right)^{(1/2)}$$
$$=\frac{\bar{\phi}^{(4)}}{4}\left[\frac{\partial\bar{u}_{0,1}}{\partial\bar{t}}+\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{2\bar{\phi}^{(2)}\bar{w}_{2,0}}{3}\right)+\frac{2}{Re}\bar{u}_{0,1}\left(\bar{u}_{1,0}+\frac{2\bar{\phi}^{(2)}\bar{u}_{3,0}}{3}\right)\right.$$
$$\left.\frac{2\bar{\phi}^{(2)}}{3}\left(\frac{\partial\bar{u}_{0,3}}{\partial\bar{t}}+\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{3\bar{\phi}^{(2)}\bar{w}_{2,0}}{4}\right)+\frac{4}{Re}\bar{u}_{0,3}\left(\bar{u}_{1,0}+\frac{3\bar{\phi}^{(2)}\bar{u}_{3,0}}{4}\right)\right)\right];$$

(3.53)
$$\frac{\pi}{2}\frac{\partial}{\partial\bar{z}}\left[\bar{\phi}^{(4)}\left(\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}+\frac{2\bar{\phi}^{(2)}}{3}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}\right)\right]-\pi\bar{\phi}^{(4)}\bar{w}_{2,0}$$
$$-\frac{\pi\bar{\phi}^{(4)}}{2Re}\left(\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}+\frac{2\bar{\phi}^{(2)}}{3}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}\right)+\pi\bar{\phi}^{(3)}\left(\bar{\tau}_a+\bar{p}_e\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)$$
$$=\frac{\partial\bar{q}_I}{\partial\bar{z}}+\frac{\pi Re\bar{\phi}^{(4)}}{4}\left[\frac{\partial\bar{w}_{0,0}}{\partial\bar{t}}+\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{2\bar{\phi}^{(2)}\bar{w}_{2,0}}{3}\right)\right.$$
$$\left.\frac{2\bar{\phi}^{(2)}}{3}\left(\frac{\partial\bar{w}_{2,0}}{\partial\bar{t}}+\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{3\bar{\phi}^{(2)}\bar{w}_{2,0}}{4}\right)+\frac{2}{Re}\bar{w}_{2,0}\left(\bar{u}_{1,0}+\frac{2\bar{\phi}^{(2)}\bar{u}_{3,0}}{4}\right)\right)\right];$$

$$
\begin{aligned}
&\frac{\pi}{Re}\frac{\partial}{\partial\bar{z}}\left[\bar{\phi}^{(6)}\left(\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}+\frac{3\bar{\phi}^{(2)}}{4}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}\right)\right]+\frac{\pi}{12}\frac{\partial}{\partial\bar{z}}(\bar{\phi}^{(6)}\bar{w}_{2,0})\\
&-\frac{\pi\bar{\phi}^{(4)}}{2Re}\left(\bar{u}_{1,0}+\frac{5\bar{\phi}^{(2)}\bar{u}_{3,0}}{3}\right)+\frac{\pi\bar{\phi}^{(4)}}{4}\left(\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{z}}-\bar{p}_e\right)\\
&=-\bar{q}_I+\frac{\pi\bar{\phi}^{(6)}}{24}\left[\frac{\partial\bar{u}_{1,0}}{\partial\bar{t}}+\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{3\bar{\phi}^{(2)}\bar{w}_{2,0}}{4}\right)+\frac{1}{Re}\bar{u}_{1,0}\left(\bar{u}_{1,0}+\frac{3\bar{\phi}^{(2)}\bar{u}_{3,0}}{4}\right)\right.\\
&-\frac{1}{Re}\bar{u}_{0,1}\left(\bar{u}_{0,1}+\frac{3\bar{\phi}^{(2)}\bar{u}_{0,3}}{4}\right)+\frac{3\bar{\phi}^{(2)}}{4}\left(\frac{\partial\bar{u}_{3,0}}{\partial\bar{t}}+\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{4\bar{\phi}^{(2)}\bar{w}_{2,0}}{5}\right)\right.\\
&\left.\left.+\frac{3}{Re}\bar{u}_{3,0}\left(\bar{u}_{1,0}+\frac{4\bar{\phi}^{(2)}\bar{u}_{3,0}}{5}\right)-\frac{1}{Re}\bar{u}_{0,3}\left(\bar{u}_{0,1}+\frac{4\bar{\phi}^{(2)}\bar{u}_{0,3}}{5}\right)\right)\right];
\end{aligned}
\tag{3.54}
$$

$$
\begin{aligned}
&\frac{1}{24Re}\frac{\partial}{\partial\bar{z}}\left[\bar{\phi}^{(6)}\left(\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}+\frac{3\bar{\phi}^{(2)}}{4}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}\right)\right]-\frac{\bar{\phi}^{(6)}\bar{u}_{0,3}}{6Re}+\frac{\bar{\phi}^{(4)}\bar{\tau}_h}{4}\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)^{(2)}\right)^{(1/2)}\\
&=\frac{\bar{\phi}^{(6)}}{24}\left[\frac{\partial\bar{u}_{0,1}}{\partial\bar{t}}+\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{3\bar{\phi}^{(2)}\bar{w}_{2,0}}{4}\right)+\frac{2}{Re}\bar{u}_{0,1}\left(\bar{u}_{1,0}+\frac{3\bar{\phi}^{(2)}\bar{u}_{3,0}}{4}\right)\right.\\
&\left.+\frac{3\bar{\phi}^{(2)}}{4}\left(\frac{\partial\bar{u}_{0,3}}{\partial\bar{t}}+\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}\left(\bar{w}_{0,0}+\frac{4\bar{\phi}^{(2)}\bar{w}_{2,0}}{5}\right)+\frac{4}{Re}\bar{u}_{0,3}\left(\bar{u}_{1,0}+\frac{4\bar{\phi}^{(2)}\bar{u}_{3,0}}{5}\right)\right)\right].
\end{aligned}
\tag{3.55}
$$

Similarly, the dimensionless system of partial differential equations for $K=5$ is given by:

(3.56)

$$
\begin{aligned}
&\frac{2}{3}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{w}_{4,0}}{\partial\bar{z}^{(2)}}+\pi\bar{\phi}^{(4)}\frac{\partial^{(2)}\bar{w}_{2,0}}{\partial\bar{z}^{(2)}}+2\pi\bar{\phi}^{(2)}\frac{\partial^{(2)}\bar{w}_{0,0}}{\partial\bar{z}^{(2)}}-\frac{1}{3}\pi Re\bar{\phi}^{(6)}\frac{\partial\bar{w}_{4,0}}{\partial\bar{t}}-\frac{1}{2}\pi Re\bar{\phi}^{(4)}\frac{\partial\bar{w}_{2,0}}{\partial\bar{t}}-\pi Re\bar{\phi}^{(2)}\frac{\partial\bar{w}_{0,0}}{\partial\bar{t}}\\
&-\frac{1}{5}\pi Re\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{4}{5}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\bar{u}_{5,0}-\frac{1}{4}\pi Re\bar{\phi}^{(8)}\bar{w}_{4,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}-\frac{1}{4}\pi Re\bar{\phi}^{(8)}\bar{w}_{2,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\pi\bar{\phi}^{(8)}\bar{w}_{4,0}\bar{u}_{3,0}\\
&-\frac{1}{2}\pi\bar{\phi}^{(8)}\bar{w}_{2,0}\bar{u}_{5,0}-\frac{1}{3}\pi Re\bar{\phi}^{(6)}\bar{w}_{4,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}-\frac{1}{3}\pi Re\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{4}{3}\pi\bar{\phi}^{(6)}\bar{u}_{1,0}\bar{w}_{4,0}-\frac{2}{3}\pi\bar{\phi}^{(6)}\bar{w}_{2,0}\bar{u}_{3,0}\\
&-\frac{1}{2}\pi Re\bar{\phi}^{(4)}\bar{w}_{2,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}-\frac{1}{2}\pi Re\bar{\phi}^{(4)}\bar{w}_{0,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}-\pi\bar{\phi}^{(4)}\bar{u}_{1,0}\bar{w}_{2,0}-\pi Re\bar{\phi}^{(2)}\bar{w}_{0,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}+4\pi\bar{\phi}^{(3)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}\\
&+4\pi\bar{\phi}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}+4\pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{\partial\bar{p}_I}{\partial\bar{z}}+2\pi\bar{\phi}\left(\bar{\tau}_a+\bar{p}_e\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)=0;
\end{aligned}
$$

(3.57)

$$\frac{2}{3}\pi\bar{\phi}^{(6)}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}+\frac{1}{2}\pi\bar{\phi}^{(4)}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}+\bar{p}_I+\frac{1}{5Re}\pi\bar{\phi}^{(10)}\bar{u}_{0,3}\bar{u}_{0,5}-\frac{2}{Re}\pi\bar{\phi}^{(6)}\bar{u}_{5,0}-\frac{2}{Re}\pi\bar{\phi}^{(4)}\bar{u}_{3,0}-\frac{2}{Re}\pi\bar{\phi}\bar{u}_{1,0}$$
$$-\frac{1}{8}\pi\bar{\phi}^{(8)}\frac{\partial\bar{u}_{5,0}}{\partial\bar{t}}-\frac{3}{8Re}\pi\bar{\phi}^{(8)}\bar{u}^{(2)}_{3,0}-\frac{1}{6}\pi\bar{\phi}^{(6)}\frac{\partial\bar{u}_{3,0}}{\partial\bar{t}}+\frac{1}{12Re}\pi\bar{\phi}^{(12)}\bar{u}^{(2)}_{0,5}-\frac{5}{12Re}\pi\bar{\phi}^{(12)}\bar{u}^{(2)}_{5,0}$$
$$-\frac{1}{4Re}\pi\bar{\phi}^{(4)}\bar{u}^{(2)}_{1,0}-\frac{1}{4}\pi\bar{\phi}^{(4)}\frac{\partial\bar{u}_{1,0}}{\partial\bar{t}}-\pi\bar{\phi}^{(2)}\bar{p}_e+4\pi\bar{\phi}^{(5)}\bar{w}_{4,0}\frac{\partial\bar{\phi}}{\partial\bar{z}}+\frac{1}{Re}\pi\bar{\phi}^{(3)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}+2\pi\bar{\phi}^{(3)}\bar{w}_{2,0}\frac{\partial\bar{\phi}}{\partial\bar{z}}$$
$$+\frac{1}{Re}\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}+\frac{1}{Re}pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}+\frac{1}{4}\pi\bar{\phi}^{(4)}\bar{w}_{0,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}-\frac{3}{4Re}\pi\bar{\phi}^{(8)}\bar{u}_{1,0}\bar{u}_{5,0}-\frac{1}{8}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}$$
$$-\frac{1}{8}\pi\bar{\phi}^{(8)}\bar{w}_{4,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}-\frac{1}{8}\pi\bar{\phi}^{(8)}\bar{w}_{2,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}+\frac{1}{8Re}\pi\bar{\phi}^{(8)}\bar{u}^{(2)}_{0,3}-\frac{1}{10}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}-\frac{4}{5Re}\pi\bar{\phi}^{(10)}\bar{u}_{3,0}\bar{u}_{5,0}$$
$$-\frac{1}{10}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}-\frac{2}{3Re}\pi\bar{\phi}^{(6)}\bar{u}_{1,0}\bar{u}_{3,0}-\frac{1}{6}\pi\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}-\frac{1}{6}\pi\bar{\phi}^{(6)}\bar{w}_{2,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}-\frac{1}{12}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}$$
$$+\pi\bar{\phi}^{(2)}\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{z}}+\frac{1}{8Re}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{u}_{5,0}}{\partial\bar{z}^{(2)}}+\frac{1}{6Re}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{u}_{3,0}}{\partial\bar{z}^{(2)}}+\frac{1}{4Re}\pi\bar{\phi}^{(4)}\frac{\partial^{(2)}\bar{u}_{1,0}}{\partial\bar{z}^{(2)}}=0;$$

(3.58)

$$\frac{1}{Re}\pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}+\frac{1}{Re}\pi\bar{\phi}^{(3)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}+\frac{1}{8Re}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{u}_{0,5}}{\partial\bar{z}^{(2)}}+\frac{1}{6Re}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{u}_{0,3}}{\partial\bar{z}^{(2)}}+\frac{1}{4Re}\pi\bar{\phi}^{(4)}\frac{\partial^{(2)}\bar{u}_{0,1}}{\partial\bar{z}^{(2)}}$$
$$+\frac{1}{Re}\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}$$
$$+\pi\bar{\phi}^{(2)}\bar{\tau}_h\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)^{(2)}\right)^{(1/2)}-\frac{1}{4}\pi\bar{\phi}^{(4)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{1}{6}\pi\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{1}{2Re}\pi\bar{\phi}^{(8)}\bar{u}_{0,3}\bar{u}_{3,0}-\frac{1}{6}\pi\bar{\phi}^{(6)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}$$
$$-\frac{1}{8}\pi\bar{\phi}^{(8)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{2}{5Re}\pi\bar{\phi}^{(10)}\bar{u}_{0,3}\bar{u}_{5,0}-\frac{1}{8}\pi\bar{\phi}^{8}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{1}{10}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{2}{3Re}\pi\bar{\phi}^{(6)}\bar{u}_{0,3}\bar{u}_{1,0}$$
$$-\frac{1}{8}\pi\bar{\phi}^{(8)}\frac{\partial\bar{u}_{0,5}}{\partial\bar{t}}-\frac{1}{12}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}-\frac{1}{2Re}\pi\bar{\phi}^{(12)}\bar{u}_{5,0}\bar{u}_{0,5}-\frac{3}{4Re}\pi\bar{\phi}^{(8)}\bar{u}_{1,0}\bar{u}_{0,5}-\frac{1}{8}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}$$
$$-\frac{1}{6Re}\pi\bar{\phi}^{(6)}\bar{u}_{0,3}\frac{\partial\bar{u}_{0,1}}{\partial\bar{t}}-\frac{1}{6Re}\pi\bar{\phi}^{(6)}\bar{u}_{0,1}\frac{\partial\bar{u}_{0,3}}{\partial\bar{t}}-\frac{1}{10}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}-\frac{3}{5Re}\pi\bar{\phi}^{(10)}\bar{u}_{3,0}\bar{u}_{0,5}=0;$$

(3.59)

$$\pi\bar{\phi}^{(3)}\bar{\tau}_a - \frac{Re}{12}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}} - \frac{1}{3}\pi\bar{\phi}^{(12)}\bar{u}_{5,0}\bar{w}_{4,0} - \frac{Re}{6}\pi\bar{\phi}^{(6)}\bar{w}_{2,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}$$
$$- \frac{Re}{6}\pi\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}} - \frac{1}{3}\pi\bar{\phi}^{(6)}\bar{u}_{1,0}\bar{w}_{2,0} - \frac{Re}{10}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}} - \frac{Re}{10}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}$$
$$- \frac{2}{5}\pi\bar{\phi}^{(10)}\bar{u}_{3,0}\bar{w}_{4,0} - \frac{1}{5}\pi\bar{\phi}^{(10)}\bar{u}_{5,0}\bar{w}_{2,0} - \frac{Re}{8}\pi\bar{\phi}^{(8)}\bar{w}_{4,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}$$
$$- \frac{Re}{8}\pi\bar{\phi}^{(8)}\bar{w}_{2,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}} - \frac{Re}{8}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}} - \frac{1}{2}\pi\bar{\phi}^{(8)}\bar{u}_{1,0}\bar{w}_{4,0} - \frac{1}{4}\pi\bar{\phi}^{(8)}\bar{u}_{3,0}\bar{w}_{2,0} - \frac{Re}{4}\pi\bar{\phi}^{(4)}\bar{w}_{0,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}$$
$$- \frac{1}{4Re}\pi\bar{\phi}^{(8)}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}} + 2\pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}} + 2\pi\bar{\phi}^{(3)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}} - \frac{\partial\bar{q}_I}{\partial\bar{z}} + \frac{1}{4}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{w}_{4,0}}{\partial\bar{z}^{(2)}}$$
$$+ \frac{1}{3}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{w}_{2,0}}{\partial\bar{z}^{(2)}} + \frac{1}{2}\pi\bar{\phi}^{(4)}\frac{\partial^{(2)}\bar{w}_{0,0}}{\partial\bar{z}^{(2)}} + \pi\bar{\phi}^{(3)}\bar{p}_e\frac{\partial\bar{\phi}}{\partial\bar{z}} - \frac{1}{3Re}\pi\bar{\phi}^{(6)}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}} - \frac{4}{3}\pi\bar{\phi}^{(6)}\bar{w}_{4,0}$$
$$- \frac{1}{2Re}\pi\bar{\phi}^{(4)}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}} - \pi\bar{\phi}^{(4)}\bar{w}_{2,0} + 2\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}} - \frac{Re}{4}\pi\bar{\phi}^{(4)}\frac{\partial\bar{w}_{0,0}}{\partial\bar{t}} - \frac{Re}{8}\pi\bar{\phi}^{(8)}\frac{\partial\bar{w}_{4,0}}{\partial\bar{t}} - \frac{Re}{6}\pi\bar{\phi}^{(6)}\frac{\partial\bar{w}_{2,0}}{\partial\bar{t}} = 0;$$

(3.60)

$$3\bar{q}_I + \frac{1}{8Re}\pi\bar{\phi}^{(12)}\bar{u}_{0,3}\bar{u}_{0,5} - \frac{3}{4}\pi\bar{\phi}^{(4)}\bar{p}_e - \frac{3}{Re}\pi\bar{\phi}^{(8)}\bar{u}_{5,0} - \frac{5}{2Re}\pi\bar{\phi}^{(6)}\bar{u}_{3,0}$$
$$- \frac{3}{2Re}\pi\bar{\phi}^{(4)}\bar{u}_{1,0} - \frac{1}{16}\pi\bar{\phi}^{(12)}\bar{w}_{2,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}} - \frac{1}{16}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}} - \frac{1}{2Re}\pi\bar{\phi}^{(12)}\bar{u}_{3,0}\bar{u}_{5,0}$$
$$- \frac{1}{8}\pi\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}} - \frac{3}{56}\pi\bar{\phi}^{(14)}\bar{w}_{4,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}} - \frac{3}{32}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}$$
$$- \frac{3}{32}\pi\bar{\phi}^{(8)}\bar{w}_{2,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}} - \frac{3}{8Re}\pi\bar{\phi}^{(8)}\bar{u}_{1,0}\bar{u}_{3,0} - \frac{3}{40}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}} - \frac{3}{40}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}} + \frac{3}{40Re}\pi\bar{\phi}^{(10)}\bar{u}_{0,3}^{(2)}$$
$$- \frac{9}{20Re}\pi\bar{\phi}^{(10)}\bar{u}_{1,0}\bar{u}_{5,0} - \frac{3}{40}\pi\bar{\phi}^{(10)}\bar{w}_{0,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}} + \frac{1}{8Re}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{u}_{1,0}}{\partial\bar{z}^{(2)}} + \frac{3}{8}\pi\bar{\phi}^{(8)}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}} + \frac{1}{4}\pi\bar{\phi}^{(6)}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}$$
$$- \frac{9}{40Re}\pi\bar{\phi}^{(10)}\bar{u}_{3,0}^{(2)} - \frac{3}{40}\pi\bar{\phi}^{(10)}\frac{\partial\bar{u}_{5,0}}{\partial\bar{t}} - \frac{3}{32}\pi\bar{\phi}^{(8)}\frac{\partial\bar{u}_{3,0}}{\partial\bar{t}}$$
$$- \frac{15}{56Re}\pi\bar{\phi}^{(14)}\bar{u}_{5,0}^{(2)} + \frac{3}{56Re}\pi\bar{\phi}^{(14)}\bar{u}_{0,5}^{(2)} - \frac{1}{8Re}\pi\bar{\phi}^{(6)}\bar{u}_{1,0}^{(2)} - \frac{1}{8}\pi\bar{\phi}^{(6)}\frac{\partial\bar{u}_{1,0}}{\partial\bar{t}} + \frac{3}{4Re}\pi\bar{\phi}^{(9)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}$$
$$+ \frac{3}{4Re}\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}} + 3\pi\bar{\phi}^{(7)}\bar{w}_{4,0}\frac{\partial\bar{\phi}}{\partial\bar{z}} + \frac{3}{4Re}\pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}$$
$$+ \frac{3}{2}\pi\bar{\phi}^{(5)}\bar{w}_{2,0}\frac{\partial\bar{\phi}}{\partial\bar{z}} + \frac{3}{4}\pi\bar{\phi}^{(4)}\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{z}} + \frac{3}{40Re}\pi\bar{\phi}^{(10)}\frac{\partial^{(2)}\bar{u}_{5,0}}{\partial\bar{z}^{(2)}} + \frac{3}{32Re}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{u}_{3,0}}{\partial\bar{z}^{(2)}} = 0;$$

(3.61)

$$\frac{3}{4}\bar{\phi}^{(4)}\bar{\tau}_h\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)^{(2)}\right)^{(1/2)}+\frac{3}{4Re}\pi\bar{\phi}^{(9)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}+\frac{3}{40Re}\pi\bar{\phi}^{(10)}\frac{\partial^{(2)}\bar{u}_{0,5}}{\partial\bar{z}^{(2)}}+\frac{3}{32Re}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{u}_{0,3}}{\partial\bar{z}^{(2)}}$$
$$+\frac{1}{8Re}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{u}_{0,1}}{\partial\bar{z}^{(2)}}+\frac{3}{4Re}\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}+\frac{3}{4Re}\pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{3}{32Re}\pi\bar{\phi}^{(8)}\bar{u}_{0,3}\frac{\partial\bar{u}_{0,1}}{\partial\bar{t}}-\frac{3}{32Re}\pi\bar{\phi}^{(8)}\bar{u}_{0,1}\frac{\partial\bar{u}_{0,3}}{\partial\bar{t}}$$
$$-\frac{3}{40}\pi\bar{\phi}^{(10)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}-\frac{9}{20Re}\pi\bar{\phi}^{(10)}\bar{u}_{1,0}\bar{u}_{0,5}-\frac{9}{28Re}\pi\bar{\phi}^{(14)}\bar{u}_{5,0}\bar{u}_{0,5}-\frac{3}{56}\pi\bar{\phi}^{(14)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}$$
$$-\frac{3}{8Re}\pi\bar{\phi}^{(12)}\bar{u}_{3,0}\bar{u}_{0,5}-\frac{1}{16}\pi\bar{\phi}^{(12)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}-\frac{3}{40}\pi\bar{\phi}^{(10)}\frac{\partial\bar{u}_{0,5}}{\partial\bar{t}}-\frac{3}{4Re}\pi\bar{\phi}^{(8)}\bar{u}_{0,5}-\frac{1}{2Re}\pi\bar{\phi}^{(6)}\bar{u}_{0,3}$$
$$-\frac{3}{40}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{1}{16}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{1}{4Re}\pi\bar{\phi}^{(12)}\bar{u}_{0,3}\bar{u}_{5,0}-\frac{3}{8Re}\pi\bar{\phi}^{(8)}\bar{u}_{0,3}\bar{u}_{1,0}-\frac{1}{8}\pi\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}$$
$$-\frac{3}{32}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{3}{10Re}\pi\bar{\phi}^{(10)}\bar{u}_{0,3}\bar{u}_{3,0}-\frac{3}{32}\pi\bar{\phi}^{8}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{3}{40}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}=0;$$

(3.62)

$$-\pi\bar{\phi}^{(6)}\bar{w}_{2,0}+\frac{3}{4}\pi\bar{\phi}^{(5)}\bar{\tau}_a-\frac{3}{10Re}\pi\bar{\phi}^{(10)}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}-\frac{3}{8Re}\pi\bar{\phi}^{(8)}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}-\frac{3}{2}\pi\bar{\phi}^{(8)}\bar{w}_{4,0}-\frac{1}{2Re}\pi\bar{\phi}^{(6)}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}$$
$$-\frac{3Re}{40}\pi\bar{\phi}^{(10)}\frac{\partial\bar{w}_{4,0}}{\partial\bar{t}}-\frac{3Re}{32}\pi\bar{\phi}^{(8)}\frac{\partial\bar{w}_{2,0}}{\partial\bar{t}}-\frac{Re}{8}\pi\bar{\phi}^{(6)}\frac{\partial\bar{w}_{0,0}}{\partial\bar{t}}-\frac{3Re}{56}\pi\bar{\phi}^{(14)}\bar{w}_{4,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{3}{14}\pi\bar{\phi}^{(14)}\bar{u}_{5,0}\bar{w}_{4,0}$$
$$-\frac{Re}{16}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}-\frac{Re}{16}\pi\rho\bar{\phi}^{(12)}\bar{w}_{2,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{1}{4}\pi\bar{\phi}^{(12)}\bar{u}_{3,0}\bar{w}_{4,0}-\frac{1}{8}\pi\bar{\phi}^{(12)}\bar{u}_{5,0}\bar{w}_{2,0}$$
$$-\frac{3Re}{40}\pi\bar{\phi}^{(10)}\bar{w}_{4,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}-\frac{3Re}{40}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}-\frac{3Re}{40}\pi\bar{\phi}^{(10)}\bar{w}_{0,0}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{3}{10}\pi\bar{\phi}^{(10)}\bar{u}_{1,0}\bar{w}_{4,0}$$
$$-\frac{3}{20}\pi\bar{\phi}^{(10)}\bar{u}_{3,0}\bar{w}_{2,0}-\frac{3Re}{32}\pi\bar{\phi}^{(8)}\bar{w}_{2,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}-\frac{3Re}{32}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}-\frac{3}{16}\pi\bar{\phi}^{(8)}\bar{u}_{1,0}\bar{w}_{2,0}$$
$$-\frac{Re}{8}\pi\bar{\phi}^{(6)}\bar{w}_{0,0}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}+\frac{3}{4}\pi\bar{\phi}^{(5)}\bar{p}_e\frac{\partial\bar{\phi}}{\partial\bar{z}}+\frac{3}{20}\pi\bar{\phi}^{(10)}\frac{\partial^{(2)}\bar{w}_{4,0}}{\partial\bar{z}^{(2)}}+\frac{3}{16}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{w}_{2,0}}{\partial\bar{z}^{(2)}}+\frac{1}{4}\pi\bar{\phi}^{(6)}\frac{\partial^{(2)}\bar{w}_{0,0}}{\partial\bar{z}^{(2)}}$$
$$+\frac{3}{2}\pi\bar{\phi}^{(9)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}-\frac{\partial\bar{h}_I}{\partial\bar{z}}+\frac{3}{2}\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}+\frac{3}{2}\pi\bar{\phi}^{(5)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{w}_{0,0}}{\partial\bar{z}}=0;$$

(3.63)

$$
\begin{aligned}
&\frac{5}{8Re}\pi\bar{\phi}^{(11)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}+\frac{5}{8Re}\pi\bar{\phi}^{(9)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}+\frac{5}{2}\pi\bar{\phi}^{(9)}\bar{w}_{4,0}\frac{\partial\bar{\phi}}{\partial\bar{z}}+\frac{5}{8Re}\pi\bar{\phi}^{(7)}\frac{\partial\bar{\phi}}{\partial\bar{z}}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}+\frac{5}{4}\pi\bar{\phi}^{(7)}\bar{w}_{2,0}\frac{\partial\bar{\phi}}{\partial\bar{z}}\\
&-\frac{13}{4Re}\pi\bar{\phi}^{(10)}\bar{u}_{5,0}-\frac{5}{2Re}\pi\bar{\phi}^{(8)}\bar{u}_{3,0}-\frac{5}{4Re}\pi\bar{\phi}^{(6)}\bar{u}_{1,0}+5\bar{h}_I+\frac{5}{56Re}\pi\bar{\phi}^{(14)}\bar{u}_{0,3}\bar{u}_{0,5}+\frac{5}{8}\pi\bar{\phi}^{(6)}\bar{\tau}_a\frac{\partial\bar{\phi}}{\partial\bar{z}}\\
&-\frac{5}{64Re}\pi\bar{\phi}^{(8)}\bar{u}_{1,0}^{(2)}-\frac{5}{64}\pi\bar{\phi}^{(8)}\frac{\partial\bar{u}_{1,0}}{\partial\bar{t}}+\frac{5}{128Re}\pi\bar{\phi}^{(16)}\bar{u}_{0,5}^{(2)}-\frac{25}{128Re}\pi\bar{\phi}^{(16)}\bar{u}_{5,0}^{(2)}\\
&-\frac{1}{16}\pi\bar{\phi}^{(10)}\frac{\partial\bar{u}_{3,0}}{\partial\bar{t}}-\frac{5}{32Re}\pi\bar{\phi}^{(12)}\bar{u}_{3,0}^{(2)}-\frac{5}{96}\pi\bar{\phi}^{(12)}\frac{\partial\bar{u}_{5,0}}{\partial\bar{t}}-\frac{5}{8}\pi\bar{\phi}^{(6)}\bar{p}_e\\
&+\frac{1}{4}\pi\bar{\phi}^{(10)}\frac{\partial\bar{w}_{4,0}}{\partial\bar{z}}+\frac{5}{32}\pi\bar{\phi}^{(8)}\frac{\partial\bar{w}_{2,0}}{\partial\bar{z}}+\frac{5}{96Re}\pi\bar{\phi}^{(12)}\frac{\partial^{(2)}\bar{u}_{5,0}}{\partial\bar{z}^{(2)}}+\frac{1}{16Re}\pi\bar{\phi}^{(10)}\frac{\partial^{(2)}\bar{u}_{3,0}}{\partial\bar{z}^{(2)}}\\
&+\frac{5}{64Re}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{u}_{1,0}}{\partial\bar{z}^{(2)}}-\frac{1}{4Re}\pi\bar{\phi}^{(10)}\bar{u}_{1,0}\bar{u}_{3,0}-\frac{1}{16}\pi\bar{\phi}^{(10)}\bar{w}_{0,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}-\frac{1}{16}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}\\
&-\frac{5}{14Re}\pi\bar{\phi}^{(14)}\bar{u}_{3,0}\bar{u}_{5,0}-\frac{5}{112}\pi\bar{\phi}^{112}\bar{w}_{4,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}-\frac{5}{112}\pi\bar{\phi}^{(14)}\bar{w}_{2,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}-\frac{5}{128}\pi\bar{\phi}^{(16)}\bar{w}_{4,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}\\
&-\frac{5}{96}\pi\bar{\phi}^{(12)}\bar{w}_{0,0}\frac{\partial\bar{u}_{5,0}}{\partial\bar{z}}-\frac{5}{96}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}-\frac{5}{96}\pi\bar{\phi}^{(12)}\bar{w}_{2,0}\frac{\partial\bar{u}_{3,0}}{\partial\bar{z}}+\frac{5}{96Re}\pi\bar{\phi}^{(12)}\bar{u}_{0,3}^{(2)}\\
&-\frac{5}{16Re}\pi\bar{\phi}^{(12)}\bar{u}_{1,0}\bar{u}_{5,0}-\frac{5}{64}\pi\bar{\phi}^{(8)}\bar{w}_{0,0}\frac{\partial\bar{u}_{1,0}}{\partial\bar{z}}=0;
\end{aligned}
$$

(3.64)

$$
\begin{aligned}
&-\frac{1}{Re}\pi\bar{\phi}^{(10)}\bar{u}_{0,5}-\frac{5}{8Re}\pi\bar{\phi}^{(8)}\bar{u}_{0,3}-\frac{15}{56Re}\pi\bar{\phi}^{(14)}\bar{u}_{3,0}\bar{u}_{0,5}\\
&-\frac{5}{112}\pi\bar{\phi}^{(14)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}+\frac{5}{8}\pi\bar{\phi}^{(6)}\bar{\tau}_h\left(1+\left(\frac{\partial\bar{\phi}}{\partial\bar{z}}\right)^{(2)}\right)^{(1/2)}-\frac{15}{64Re}\pi\bar{\phi}^{(16)}\bar{u}_{5,0}\bar{u}_{0,5}\\
&-\frac{5}{128}\pi\bar{\phi}^{(16)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}-\frac{5}{16Re}\pi\bar{\phi}^{(12)}\bar{u}_{1,0}\bar{u}_{0,5}-\frac{5}{96}\pi\bar{\phi}^{(12)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}\\
&-\frac{1}{16}\pi\bar{\phi}^{(10)}\frac{\partial\bar{u}_{0,1}}{\partial\bar{t}}-\frac{1}{16}\pi\bar{\phi}^{(10)}\frac{\partial\bar{u}_{0,3}}{\partial\bar{t}}+\frac{5}{96Re}\pi\bar{\phi}^{(12)}\frac{\partial^{(2)}\bar{u}_{0,5}}{\partial\bar{z}^{(2)}}+\frac{1}{16Re}\pi\bar{\phi}^{(10)}\frac{\partial^{(2)}\bar{u}_{0,3}}{\partial\bar{z}^{(2)}}\\
&+\frac{5}{64Re}\pi\bar{\phi}^{(8)}\frac{\partial^{(2)}\bar{u}_{0,1}}{\partial\bar{z}^{(2)}}-\frac{5}{28Re}\pi\bar{\phi}^{(14)}\bar{u}_{0,3}\bar{u}_{5,0}-\frac{5}{96}\pi\bar{\phi}^{(12)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{1}{4Re}\pi\bar{\phi}^{(10)}\bar{u}_{0,3}\bar{u}_{1,0}\\
&-\frac{5}{64}\pi\bar{\phi}^{8}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}-\frac{1}{16}\pi\bar{\phi}^{(10)}\bar{w}_{0,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{5}{112}\pi\bar{\phi}^{(14)}\bar{w}_{4,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{1}{16}\pi\bar{\phi}^{(10)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}\\
&+\frac{5}{8Re}\pi\bar{\phi}^{(9)}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}\frac{\partial\bar{\phi}}{\partial\bar{z}}+\frac{5}{8Re}\pi\bar{\phi}^{(7)}\frac{\partial\bar{u}_{0,1}}{\partial\bar{z}}\frac{\partial\bar{\phi}}{\partial\bar{z}}+\frac{5}{8Re}\pi\bar{\phi}^{(11)}\frac{\partial\bar{u}_{0,5}}{\partial\bar{z}}\frac{\partial\bar{\phi}}{\partial\bar{z}}-\frac{5}{96}\pi\bar{\phi}^{(12)}\frac{\partial\bar{u}_{0,5}}{\partial\bar{t}}\\
&-\frac{5}{96}\pi\bar{\phi}^{(12)}\bar{w}_{2,0}\frac{\partial\bar{u}_{0,3}}{\partial\bar{z}}-\frac{5}{24Re}\pi\bar{\phi}^{(12)}\bar{u}_{0,3}\bar{u}_{3,0}=0.
\end{aligned}
$$

In the rest of this chapter, the overbars will be omitted for convenience, where it is understood that all variables hence forth are in dimensionless form.

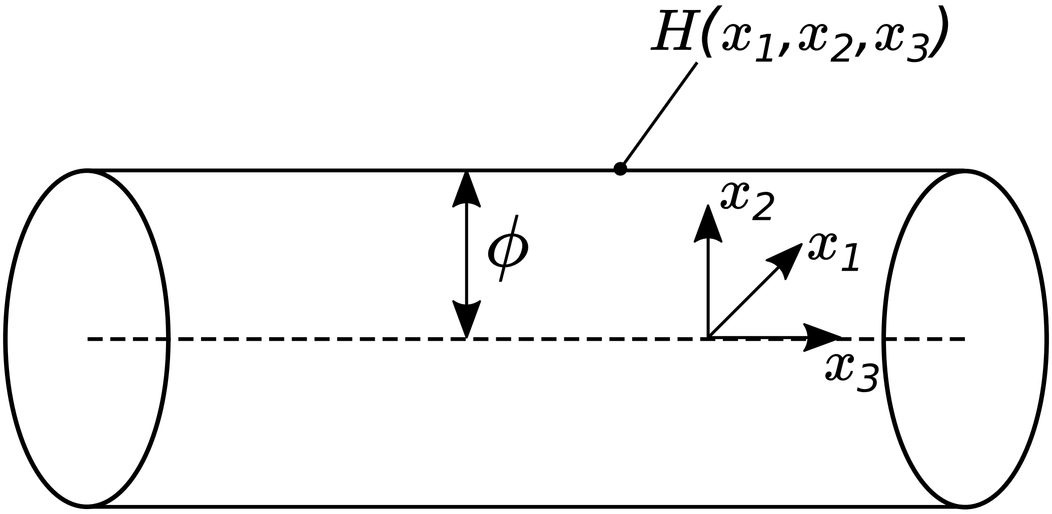


Figure 3.2: A straight axisymmetric pipe constant radius ϕ with lateral surface H.

3.4 Results: Constant Radius

Taking the special case where ϕ is a constant, as shown in Fig. 3.2, a no-slip condition can be imposed requiring the velocity to vanish identically on the boundary, given by Eq. (3.5). From Eq. (3.13), this implies that

$$u_{1,0} + \phi^{(2)} u_{3,0} = 0; \tag{3.65}$$

$$u_{0,1} + \phi^{(2)} u_{0,3} = 0; \tag{3.66}$$

$$w_{0,0} + \phi^{(2)} w_{2,0} = 0. \tag{3.67}$$

Then the kinematic condition Eq. (3.15) is satisfied exactly. Using Eqs. (3.65) - (3.67), then the incompressibility conditions Eq. (3.17) and Eq. (3.18) can be reduced as follows:

$$\begin{aligned}
0 &= \frac{\partial w_{2,0}}{\partial z} + 4u_{3,0} \\
&= \frac{\partial}{\partial z}\left(-\frac{w_{0,0}}{\phi^{(2)}}\right) + 4\left(-\frac{u_{1,0}}{\phi^{(2)}}\right) \\
&= -\frac{1}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial z} + \frac{2}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial z} \\
&= \frac{1}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial z} \\
&= \frac{\partial w_{0,0}}{\partial z}.
\end{aligned}$$

This in turn implies from Eqs. (3.17) and (3.65) that $u_{1,0} = u_{3,0} = 0$. Simplifying the six partial differential equations derived in the previous section and making the substitutions $u_{0,1} = -\phi^{(2)} u_{0,3}$ and $w_{2,0} = -\frac{v}{\phi^{(2)}}$, from Eqs. (3.66) and (3.67), leads to the following simplified equations

(3.68) $$2\pi\phi\tau_a = \frac{\partial p_I}{\partial z} + \frac{\pi Re \phi^{(2)}}{2}\frac{dw_{0,0}}{dt};$$

(3.69) $$p_I = \pi\phi^{(2)} p_e - \frac{\pi\phi^{(8)} u_{0,3}^{(2)}}{24Re};$$

(3.70) $$\frac{\phi^{(4)}}{24}\left(2\frac{\partial u_{0,3}}{\partial t} + w_{0,0}\frac{\partial u_{0,3}}{\partial z}\right) - \frac{\phi^{(4)}}{12Re}\frac{\partial^{(2)} u_{0,3}}{\partial z^{(2)}} + \tau_h = 0;$$

(3.71) $$\pi\phi^{(2)} w_{0,0} + \pi\phi^{(3)}\tau_a = \frac{\partial q_I}{\partial z} + \frac{\pi\phi^{(4)} Re}{12}\frac{dw_{0,0}}{dt};$$

(3.72) $$q_I = \frac{\pi\phi^{(4)} p_e}{4} - \frac{\pi\phi^{(10)} u_{0,3}^{(2)}}{240Re};$$

(3.73) $$\frac{\phi^{(4)}}{24}\left(\frac{1}{4}\frac{\partial u_{0,3}}{\partial t} + \frac{w_{0,0}}{10}\frac{\partial u_{0,3}}{\partial z}\right) + \frac{\tau_h}{4} = \frac{\phi^{(4)}}{96Re}\frac{\partial^{(2)} u_{0,3}}{\partial z^{(2)}} + \frac{\phi^{(2)} u_{0,3}}{6Re}.$$

3.4.1 Poiseuille flow

Making the simplifications consistent with Poiseuille flow, taking the velocity given by Eq. (3.13), set $u_{1,0}, u_{3,0}, u_{0,1}, u_{0,3} = 0$, the velocity simply takes the form

$$\mathbf{v} = (w_{0,0} + (x_1^{(2)} + x_2^{(2)}) w_{2,0})\mathbf{e}_3,$$

or in cylindrical polar coordinates

(3.74) $$\mathbf{v} = (w_{0,0} + r^{(2)} w_{2,0})\mathbf{e}_z.$$

Also, looking for a steady solution, Eqs. (3.68) - (3.73) reduce to

(3.75) $$2\pi\phi\tau_a = \frac{dp_I}{dz};$$

(3.76) $$p_I = \pi\phi^{(2)} p_e;$$

$$\tau_h = 0; \tag{3.77}$$

$$\pi\phi^{(2)} w_{0,0} + \pi\phi^{(3)} \tau_a = \frac{dq_I}{dz}; \tag{3.78}$$

$$q_I = \frac{\pi\phi^{(4)} p_e}{4}; \tag{3.79}$$

$$\frac{\tau_h}{4} = 0. \tag{3.80}$$

Rearranging Eq. (3.76) and substituting into Eq. (3.79) gives

$$q_I = \frac{\phi^{(2)} p_I}{4}.$$

Differentiating with respect to z gives

$$\frac{dq_I}{dz} = \frac{\phi^{(2)}}{4} \frac{dp_I}{dz}. \tag{3.81}$$

Rearranging and substituting Eqs. (3.75) and (3.81) into Eq. (3.78) and rearranging gives

$$w_{0,0} = -\frac{1}{4\pi} \frac{dp_I}{dz}. \tag{3.82}$$

Then from Eq. (3.67),

$$w_{2,0} = -\frac{w_{0,0}}{\phi^{(2)}} = \frac{1}{4\pi\phi^{(2)}} \frac{dp}{dz}.$$

Then, substituting into Eq (3.74), the velocity is given by

$$\mathbf{v} = -\frac{1}{4\pi} \frac{dp_I}{dz} \left(1 - \frac{r^{(2)}}{\phi^{(2)}}\right) \mathbf{e}_z.$$

Simplifying Eqs. (3.30), (3.31) and (3.32) based on this example gives

$$\mathbf{T}_1 = -p\ \mathbf{e}_1 + 2\ x_1\ w_{2,0}\ \mathbf{e}_3;$$

$$\mathbf{T}_2 = -p\ \mathbf{e}_2 + 2\ x_2\ w_{2,0}\ \mathbf{e}_3;$$

$$\mathbf{T}_3 = 2\ x_1\ w_{2,0}\ \mathbf{e}_1 + 2\ x_2\ w_{2,0}\ \mathbf{e}_2 - p\ \mathbf{e}_3.$$

Due to the constant radius, the components of the outward unit normal to the surface are simply

$$\nu_1 = \frac{x_1}{\phi};$$

$$v_2 = \frac{x_2}{\phi};$$
$$v_3 = 0.$$

Then, from Eq. (3.4),

$$\begin{aligned}\mathbf{t} &= \frac{x_1}{\phi}(-p\mathbf{e}_1 + 2x_1 w_{2,0}\mathbf{e}_3) + \frac{x_2}{\phi}(-p\mathbf{e}_2 + 2x_2 w_{2,0}\mathbf{e}_3)\\ &= -\frac{px_1}{\phi}\mathbf{e}_1 - \frac{px_2}{\phi}\mathbf{e}_2 + 2\phi w_{2,0}\mathbf{e}_3.\end{aligned}$$

Simplifying Eq. (3.29) for this example gives

$$\mathbf{t} = -\frac{p_e x_1}{\phi}\mathbf{e}_1 - \frac{p_e x_2}{\phi}\mathbf{e}_2 + \tau_a \mathbf{e}_3.$$

Equating these two expressions for the stress vector gives

$$p = p_e.$$

Then from Eq. (3.76),

$$p = \frac{p_I}{\pi\phi^{(2)}}.$$

Given that $w_{0,0}$ does not depend on z, Eq. (3.82) implies

$$\frac{dp_I}{dz} = A,$$

where A is a constant. Then, integrating with respect to z gives

$$p_I = Az + B,$$

where B is a constant. Therefore

$$p = \frac{8}{\phi^{(2)}}(B - z) \tag{3.83}$$

and

$$\mathbf{v} = 2\left(1 - \frac{r^{(2)}}{\phi^{(2)}}\right)\mathbf{e}_z. \tag{3.84}$$

where the value of A has been suitably chosen considering the non-dimensional scale and B takes on a different value to immediately above to simplify the notation. Eqs. (3.83) and (3.84) represent the classical Poiseuille flow. If a higher order of velocity expansion is taken, such as $K = 5$, the additional terms come out to be zero as Poiseuille flow is an exact solution. A plot of the solution using Maple is shown in Fig 3.3.

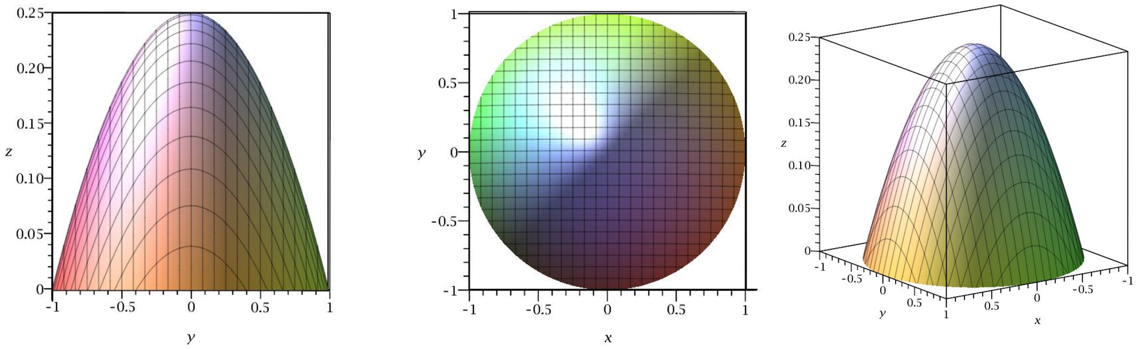


Figure 3.3: Different views of the 3D plot of the Poiseuille flow, given by Eq. (3.84) (multiplied by Re), with coefficients $\phi = 1$, $Re = 1/8$.

3.4.2 Steady swirling flow

Now looking for a steady swirling flow within a pipe of constant radius, dropping the time dependence, for $K = 3$, Eqs (3.68) - (3.73) reduce to

$$2\pi\phi\tau_a = \frac{dp_I}{dz}; \tag{3.85}$$

$$p_I = \pi\phi^{(2)}p_e - \frac{\pi\phi^{(8)}u^{(2)}_{0,3}}{24Re}; \tag{3.86}$$

$$\frac{\phi^{(4)}w_{0,0}}{24}\frac{du_{0,3}}{dz} - \frac{\phi^{(4)}}{12Re}\frac{d^{(2)}u_{0,3}}{dz^{(2)}} + \tau_h = 0; \tag{3.87}$$

$$\pi\phi^{(2)}w_{0,0} + \pi\phi^{(3)}\tau_a = \frac{dq_I}{dz}; \tag{3.88}$$

$$q_I = \frac{\pi\phi^{(4)}p_e}{4} - \frac{\pi\phi^{(10)}u^{(2)}_{0,3}}{240}; \tag{3.89}$$

$$\frac{\phi^{(4)}w_{0,0}}{240Re}\frac{du_{0,3}}{dz} + \frac{\tau_h}{4} = \frac{\phi^{(4)}}{96}\frac{d^{(2)}u_{0,3}}{dz^{(2)}} + \frac{\phi^{(2)}u_{0,3}}{6}. \tag{3.90}$$

Rearranging Eq. (3.87) and substituting into Eq. (3.90) multiplied by 4 and collecting terms gives

$$\frac{\phi^{(4)}}{24Re}\frac{d^{(2)}u_{0,3}}{dz^{(2)}} - \frac{\phi^{(4)}w_{0,0}}{40}\frac{du_{0,3}}{dz} - \frac{2\phi^{(2)}u_{0,3}}{3Re} = 0. \tag{3.91}$$

Noting that from above $w_{0,0}$ does not depend on z and now a steady solution is being sought so $w_{0,0}$ does not depend on t, therefore $w_{0,0} = v_0$, a constant. Considering the non-dimensional scaling, set $v_0 = 2$ Hence Eq. (3.91) is an ODE with constant coefficients. So a solution is sought of the form

$$u_{0,3} = Ae^{kz}, \tag{3.92}$$

where A and k are constants. Substituting this into Eq. (3.91) gives

$$Ae^{kz}\left(\frac{\phi^{(4)}}{24}k^{(2)} - \frac{2Re\phi^{(4)}}{40}k - \frac{2\phi^{(2)}}{3}\right) = 0.$$

So

$$k = \frac{\frac{Re\phi^{(4)}}{20} \pm \left(\frac{Re^{(2)}\phi^{(8)}}{400} + \frac{\phi^{(6)}}{9}\right)^{(1/2)}}{\frac{\phi^{(4)}}{12}}.$$

For a physically realistic solution, $u_{0,3}$ should decay rather than grow with z, so the solution with the + sign is to be disregarded. After some tidying up this gives

$$k = \frac{3\phi - (9Re^{(2)}\phi^{(2)} + 400)^{(1/2)}}{5\phi}.$$

Substituting this expression into Eq. (3.92) gives

$$u_{0,3} = A\exp\left(\frac{3\phi - (9Re^{(2)}\phi^{(2)} + 400)^{(1/2)}}{5\phi}z\right).$$

Then from Eq. (3.66),

$$u_{0,1} = -\phi^{(2)}A\exp\left(\frac{3\phi - (9Re^{(2)}\phi^{(2)} + 400)^{(1/2)}}{5\phi}z\right)$$

and from Eq. (3.67),

$$w_{2,0} = -\frac{2}{\phi^{(2)}}.$$

Then, substituting into Eq. (3.13), the velocity is given by

$$\begin{aligned}\mathbf{v} = {} & (\phi^{(2)} - (x_1^{(2)} + x_2^{(2)}))x_2A\exp\left(\frac{3\phi - (9Re^{(2)}\phi^{(2)} + 400)^{(1/2)}}{5\phi}x_3\right)\mathbf{e}_1 \\ & - (\phi^{(2)} - (x_1^{(2)} + x_2^{(2)}))x_1A\exp\left(\frac{3\phi - (9Re^{(2)}\phi^{(2)} + 400)^{(1/2)}}{5\phi}x_3\right)\mathbf{e}_2 \\ & + 2\left(1 - \frac{x_1^{(2)} + x_2^{(2)}}{\phi^{(2)}}\right)\mathbf{e}_3,\end{aligned}$$

or in cylindrical polar coordinates by

$$\mathbf{v} = -(\phi^{(2)} - r^{(2)})rA\exp\left(\frac{3\phi - (9Re^{(2)}\phi^{(2)} + 400)^{(1/2)}}{5\phi}z\right)\mathbf{e}_\theta + 2\left(1 - \frac{r^{(2)}}{\phi^{(2)}}\right)\mathbf{e}_z. \tag{3.93}$$

Plots of this solution, using Maple, are shown in Fig. 3.4 where the affect of varying the Reynold's on the swirling motion can be seen.

Now, the pressure solution is sought. Simplifying Eqs. (3.30), (3.31) and (3.32) based on this case, the equations become

$$\mathbf{T}_1 = (-p - \frac{4}{Re}x_1x_2u_{0,3})\mathbf{e}_1 + \frac{2}{Re}(x_1^{(2)} - x_2^{(2)})u_{0,3}\mathbf{e}_2 + \left(2x_1w_{2,0} - \left(\frac{du_{0,1}}{dx_3}\right)\frac{x_2}{Re}\right)\mathbf{e}_3;$$
$$\mathbf{T}_2 = \frac{2}{Re}(x_1^{(2)} - x_2^{(2)})u_{0,3}\mathbf{e}_1 + (-p + \frac{4}{Re}x_1x_2u_{0,3})\mathbf{e}_2 + \left(2x_2w_{2,0} + \left(\frac{du_{0,1}}{dx_3} + (x_1^{(2)} + x_2^{(2)})\frac{du_{0,3}}{dx_3}\right)\frac{x_1}{Re}\right)\mathbf{e}_3;$$
$$\mathbf{T}_3 = \left(2x_1w_{2,0} - \left(\frac{du_{0,1}}{dx_3} + (x_1^{(2)} + x_2^{(2)})\frac{du_{0,3}}{dx_3}\right)\frac{x_2}{Re}\right)\mathbf{e}_1 + \mu\left(2x_2w_{2,0} + \left(\frac{du_{0,1}}{dx_3} + (x_1^{(2)} + x_2^{(2)})\frac{du_{0,3}}{dx_3}\right)\frac{x_1}{Re}\right)\mathbf{e}_2 - p\mathbf{e}_3.$$

Then, from Eq. (3.4),

$$\begin{aligned}\mathbf{t} &= \frac{x_1}{\phi}\mathbf{T}_1 + \frac{x_2}{\phi}\mathbf{T}_2 \\ &= \frac{-px_1 - \frac{4}{Re}x1^{(2)}x_2u_{0,3} + \frac{2}{Re}x_2(x_1^{(2)} - x_2^{(2)})u_{0,3}}{\phi}\mathbf{e}_1 + \frac{\frac{2}{Re}x_1(x_1^{(2)} - x_2^{(2)}) - px_2 + \frac{4}{Re}x_1x_2^{(2)}u_{0,3}}{\phi}\mathbf{e}_2 \\ &+ \frac{\left(2x_1^{(2)}w_{2,0} - \left(\frac{du_{0,1}}{dx_3} + (x_1^{(2)} + x_2^{(2)})\frac{du_{0,3}}{dx_3}\right)\frac{x_1x_2}{Re}\right) + \left(2x_2^{(2)}w_{2,0} + \left(\frac{du_{0,1}}{dx_3} + (x_1^{(2)} + x_2^{(2)})\frac{du_{0,3}}{dx_3}\right)\frac{x_1x_2}{Re}\right)}{\phi}\mathbf{e}_3 \\ &= \frac{-px_1 - \frac{2}{Re}\phi^{(2)}x_2u_{0,3}}{\phi}\mathbf{e}_1 + \frac{-px_2 + \frac{2}{Re}\phi^{(2)}x_1u_{0,3}}{\phi}\mathbf{e}_2 + 2\phi w_{2,0}\mathbf{e}_3.\end{aligned}$$

Simplifying Eq. (3.29) based on this case gives

$$\mathbf{t} = \frac{-p_ex_1 - \tau_hx_2}{\phi}\mathbf{e}_1 + \frac{-p_ex_2 + \tau_hx_1}{\phi}\mathbf{e}_2 + \tau_a\mathbf{e}_3.$$

Equating the two expressions for the stress vector above gives $p_e = p^*$. Then substituting into Eq. (3.86) gives

$$p_I = \pi\phi^{(2)}p - \frac{\pi\phi^{(8)}u_{0,3}^{(2)}}{24Re}.$$

Differentiating with respect to z gives

$$\frac{dp_I}{dz} = \pi\phi^{(2)}\frac{dp}{dz} - \frac{\pi\phi^{(8)}u_{0,3}}{12Re}\frac{du_{0,3}}{dz}.$$

Substituting this into Eq. (3.85) gives

$$\tau_a = \frac{\phi}{2}\frac{dp}{dz} - \frac{\phi^{(7)}u_{0,3}}{24Re}\frac{du_{0,3}}{dz}. \tag{3.94}$$

From Eq. (3.89),

$$q_I = \frac{\pi\phi^{(4)}p}{4} - \frac{\pi\phi^{(10)}u_{0,3}^{(2)}}{240Re}.$$

Differentiating with respect to z gives

$$\frac{dq_I}{dz} = \frac{\pi\phi^{(4)}}{4}\frac{dp}{dz} - \frac{\pi\phi^{(10)}u_{0,3}}{120Re}\frac{du_{0,3}}{dz}. \tag{3.95}$$

Substituting Eqs. (3.94) and (3.95) into Eq. (3.88) gives

$$\pi\phi^{(2)}w_{0,0} + \pi\phi^{(3)}\left(\frac{\phi}{2}\frac{dp}{dz} - \frac{\phi^{(7)}u_{0,3}}{24Re}\frac{du_{0,3}}{dz}\right) = \frac{\pi\phi^{(4)}}{4}\frac{dp}{dz} - \frac{\pi\phi^{(10)}u_{0,3}}{120Re}\frac{du_{0,3}}{dz}.$$

Rearranging gives

$$\frac{dp}{dz} = \frac{2\phi^{(6)}u_{0,3}}{15Re}\frac{du_{0,3}}{dz} - \frac{4w_{0,0}}{\phi^{(2)}}.$$

Substituting in the expressions for $u_{0,3}$ and $w_{0,0}$ gives

$$\frac{dp}{dz} = 2\frac{\phi^{(6)}A^{(2)}}{15}\left(\frac{3Re\phi-(9Re^{(2)}\phi^{(2)}+400)^{(1/2)}}{5\phi}\right)\exp 2\left(\frac{3Re\phi-(9Re^{(2)}\phi^{(2)}+400)^{(1/2)}}{5\phi}z\right) - \frac{8}{\phi^{(2)}}.$$

Integrating with respect to z gives

$$p = \frac{2\phi^{(6)}A^{(2)}}{15}\exp\left(\frac{3Re\phi-(9Re^{(2)}\phi^{(2)}+400)^{(1/2)}}{5\phi}z\right) - \frac{8z}{\phi^{(2)}} + B, \tag{3.96}$$

where B is a constant. The pressure drop along the pipe, at varying Reynold's numbers, is shown in Fig. 3.5.

3.5 Results: Radius Depends Only on z

This section will consider the case where the radius can vary along the pipe. First we will be considered the case where the order of the velocity expansion is $K = 3$. As for the case of a constant radius, in the case where the radius depends only on z, the no-slip condition leads to Eqs. (3.65) - (3.67). Then Eq. (3.15) is satisfied exactly. Then using

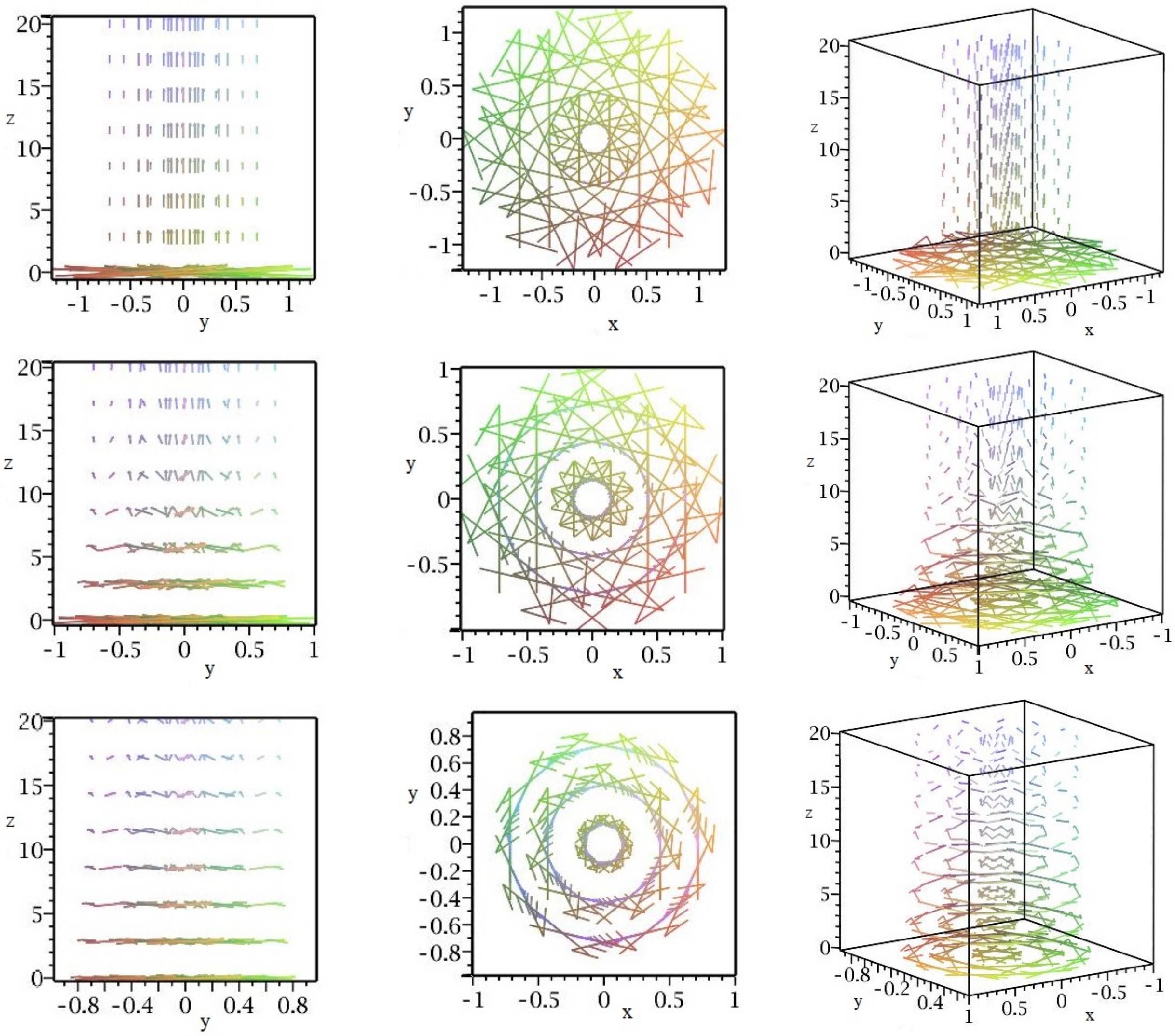


Figure 3.4: Different views of the steady swirling flow, given by Eq. (3.93), with coefficients $\phi = 1$ and $A = 5$ for $Re = 1$ (top), $Re = 50$ (middle), $Re = 100$ (bottom).

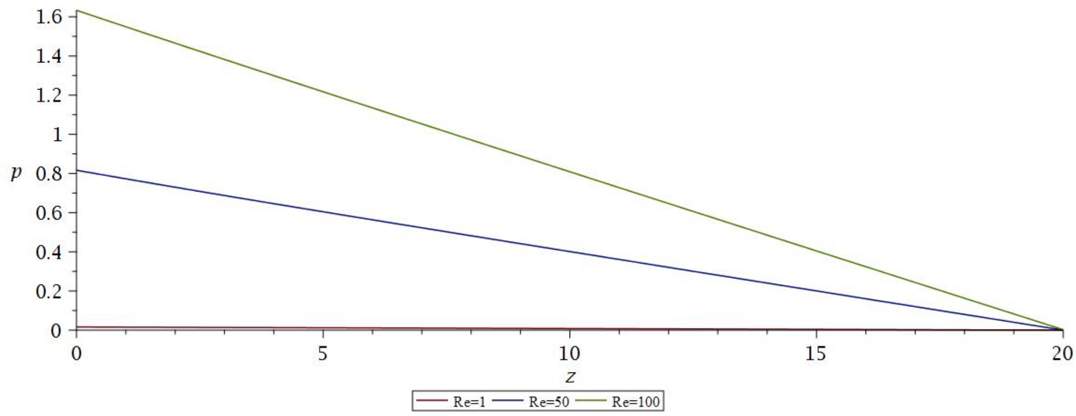


Figure 3.5: The pressure drop along the pipe, given by Eq. (3.96) with coefficients $\phi = 1$, $A = 5$ and B set so the outlet pressure is zero for $Re = 1$ (red), $Re = 50$ (blue) and $Re = 100$ (green).

Eqs. (3.17), (3.65) and (3.67) to manipulate Eq. (3.18) gives

$$\begin{aligned}
\frac{\partial}{\partial x_3}\left(-\frac{w_{0,0}}{\phi^{(2)}}\right) - \frac{4u_{1,0}}{\phi^{(2)}} &= 0;\\
\frac{\partial}{\partial x_3}\left(-\frac{w_{0,0}}{\phi^{(2)}}\right) + \frac{2}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial x_3} &= 0;\\
-\frac{1}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial x_3} + \frac{2w_{0,0}}{\phi^{(3)}}\frac{d\phi}{dx_3} + \frac{2}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial x_3} &= 0;\\
\frac{1}{\phi^{(2)}}\frac{\partial w_{0,0}}{\partial x_3} + \frac{2w_{0,0}}{\phi^{(3)}}\frac{d\phi}{dx_3} &= 0;\\
\frac{1}{\phi^{(4)}}\left(\phi^{(2)}\frac{\partial w_{0,0}}{\partial x_3} + 2w_{0,0}\phi\frac{d\phi}{dx_3}\right) &= 0;\\
\frac{1}{\phi^{(4)}}\frac{\partial}{\partial x_3}(\phi^{(2)}w_{0,0}) &= 0.
\end{aligned}$$

As the radius is finite, this implies that

$$\frac{\partial}{\partial x_3}(\phi^{(2)}w_{0,0}) = 0 \tag{3.97}$$

and hence $w_{0,0}$ takes the form

$$w_{0,0} = \frac{A}{\phi^{(2)}}, \tag{3.98}$$

where A depends only on time and is related to the magnitude of the bulk flow.

Then from the no-slip condition, given by Eqs. (3.65) - (3.67), and the incompressibility condition, given by Eqs. (3.47),(3.48), the forms of the following variables can be found:

$$u_{1,0} = -\frac{ReA}{\phi^{(3)}}\frac{d\phi}{dz}; \tag{3.99}$$

(3.100)
$$u_{3,0} = \frac{ReA}{\phi^{(5)}}\frac{d\phi}{dz};$$

(3.101)
$$w_{2,0} = -\frac{A}{\phi^{(4)}};$$

(3.102)
$$u_{0,1} = -\phi^{(2)}u_{0,3}.$$

Then simplifying Eqs. (3.36) - (3.41) for this case gives

(3.103)
$$2\pi\phi\left(\tau_a + p_e\frac{d\phi}{dz}\right) = \frac{\partial p_I}{\partial z} + \pi Re\left(\frac{1}{2}\frac{dA}{dt} - \frac{A^{(2)}}{\phi^{(3)}} + \frac{4A^{(2)}}{3\phi^{(5)}}\right);$$

(3.104)
$$-\frac{\pi A}{4}\left[\frac{d\phi}{dz}\frac{d^{(2)}\phi}{dz^{(2)}} + \phi\frac{d^{(3)}\phi}{dz^{(3)}}\right] + \pi\phi^{(2)}\left(\tau_a\frac{d\phi}{dz} - p_e\right) = -p_I - \frac{Re\pi}{12}\left(\phi\frac{dA}{dt}\frac{d\phi}{dz} + \frac{5A^{(2)}}{2\phi^{(2)}}\left(\frac{d\phi}{dz}\right)^{(2)} + \frac{A^{(2)}}{2\phi}\frac{d^{(2)}\phi}{dz^{(2)}}\right);$$

(3.105)
$$-\phi^{(4)}\left[\frac{5}{2Re}\left(\frac{d\phi}{dz}\right)^{(2)}u_{0,3} + \frac{\phi}{Re}\frac{d\phi}{dz}\frac{\partial u_{0,3}}{\partial z} + \frac{\phi^{(2)}}{12Re}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}\right] + \phi^{(2)}\tau_h\left(1 + \left(\frac{d\phi}{dz}\right)^{(2)}\right)^{(1/2)}$$
$$= -\frac{Re\phi^{(3)}}{6}\left(\frac{\phi^{(3)}}{2Re}\frac{\partial u_{0,3}}{\partial z} - \frac{A}{Re}u_{0,3}\frac{d\phi}{dz} - \frac{A}{4Re}\frac{\partial u_{0,3}}{\partial z}\right);$$

(3.106)
$$\frac{\pi A}{2}\left[\left(\frac{d\phi}{dz}\right)^{(2)} + \phi\frac{d^{(2)}\phi}{dz^{(2)}}\right] + \pi A + \pi\phi^{(3)}\left(\tau_a + p_e\frac{d\phi}{dz}\right) = \frac{\partial q_I}{\partial z} + \frac{\pi Re}{4}\left(\phi^{(2)}\frac{dA}{dt} - \frac{7A^{(2)}}{3\phi}\frac{d\phi}{dz}\right);$$

(3.107)
$$\frac{-\pi A\phi}{24}\left[\frac{3}{2}\left(\frac{d\phi}{dz}\right)^{(3)} + \frac{9}{2}\phi\frac{d\phi}{dz}\frac{d^{(2)}\phi}{dz^{(2)}} + \phi^{(2)}\frac{d^{(3)}\phi}{dz^{(3)}}\right] - \frac{\pi A\phi}{2}\frac{d\phi}{dz} + \frac{\pi\phi^{(4)}}{4}\left(\tau_a\frac{d\phi}{dz} - p_3\right) = -q_I$$
$$-\frac{\pi Re}{24}\left(\frac{\phi^{(3)}}{4}\frac{dA}{dt}\frac{d\phi}{dz} + \frac{A^{(2)}}{5}\left(\frac{d\phi}{dz}\right)^{(2)} + A^{(2)}\phi\frac{d^{(2)}\phi}{dz^{(2)}} + \frac{\phi^{(10)}}{10Re}u_{0,3}^{(2)}\right);$$

(3.108)
$$-\frac{\phi^{(6)}}{12}\left[\frac{7}{Re}\left(\frac{d\phi}{dz}\right)^{(2)}u_{0,3} + \frac{\phi}{Re}\frac{d^{(2)}\phi}{dz^{(2)}}u_{0,3} + \frac{2\phi}{Re}\frac{d\phi}{dz}\frac{\partial u_{0,3}}{\partial z} + \frac{\phi^{(2)}}{8Re}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}\right] - \frac{\phi^{(6)}}{6Re}u_{0,3} + \frac{\phi\tau_h}{4}\left(1 + \left(\frac{d\phi}{dz}\right)^{(2)}\right)^{(1/2)}$$
$$= -\frac{Re\phi^{(6)}}{8}\left(\frac{\phi^{(2)}}{12Re}\frac{\partial u_{0,3}}{\partial t} + \frac{A}{30Re}\frac{\partial u_{0,3}}{\partial z} + \frac{A}{\phi Re}\frac{d\phi}{dz}u_{0,3}\right).$$

For $K = 5$, the corresponding no-slip conditions are given by

(3.109)
$$w_{1,0} + \phi^{(2)}w_{3,0} + \phi^{(4)}w_{5,0} = 0;$$

(3.110)
$$w_{0,1} + \phi^{(2)}w_{0,3} + \phi^{(4)}w_{0,5};$$

(3.111)
$$w_{0,0} + \phi^{(2)}w_{2,0} + \phi^{(4)}w_{4,0}.$$

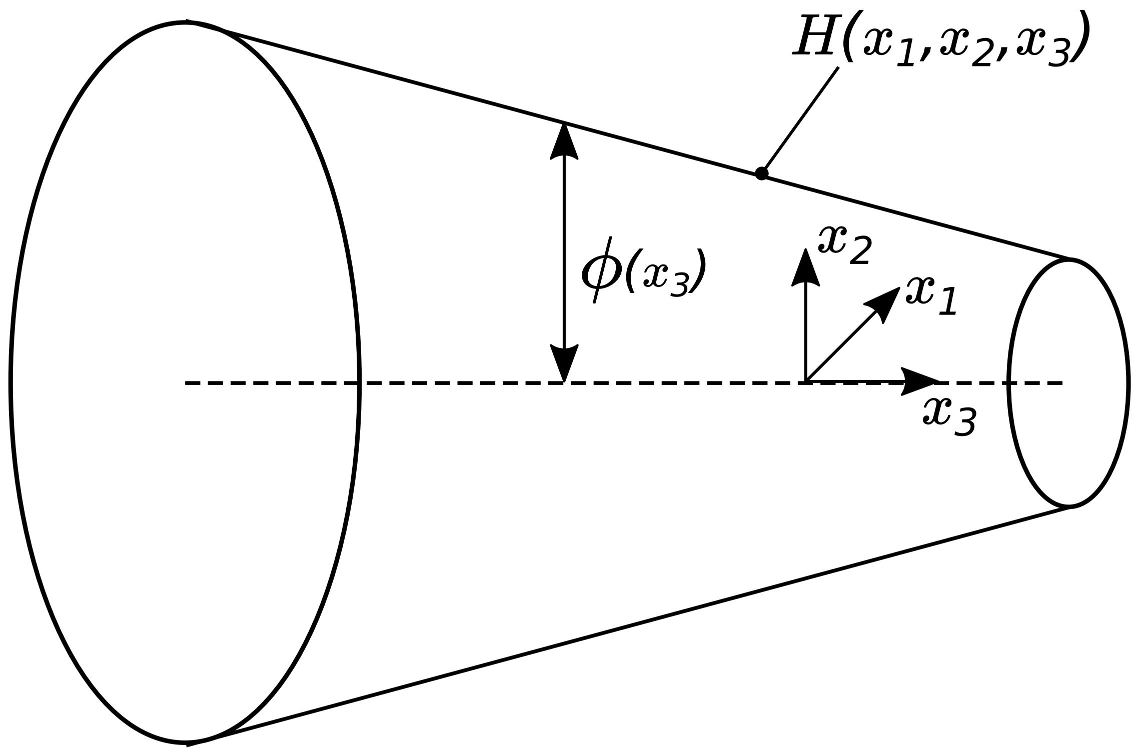


Figure 3.6: A straight axisymmetric uniformly tapering pipe with radius $\phi(z)$ given by Eq. (3.112), with lateral surface H.

3.5.1 Tapered pipe

This subsection will consider the case where the radius is decreasing uniformly along the pipe, as shown in Fig. 3.6. Specific orders of the velocity expansion of $K = 3$ and $K = 5$ will be considered.

3.5.1.1 Order K=3

For a tapered pipe, the radius is given by

$$\phi(z) = 1 - \epsilon z, \tag{3.112}$$

where ϵ on a constant term representing the rate of change of the radius along the pipe. Substituting Eq. (3.112) into Eqs. (3.98)-(3.102) gives

$$w_{0,0} = \frac{A}{(1-\epsilon z)^{(2)}}; \tag{3.113}$$

$$u_{1,0} = \frac{-Re\epsilon A}{(1-\epsilon z)^{(3)}}; \tag{3.114}$$

(3.115)
$$u_{3,0} = \frac{Re\epsilon A}{(1-\epsilon z)^{(5)}};$$

(3.116)
$$u_{0,1} = -(1-\epsilon z)^{(2)} u_{0,3};$$

(3.117)
$$w_{2,0} = \frac{-A}{(c_2 - c_2 z)^{(4)}}.$$

With these substitutions, Eqs. (3.103) - (3.108) become:

$$2\pi(1-\epsilon z)(\tau_a - \epsilon p_e) = \pi Re\left[\frac{1}{2}\frac{dA}{dt} + \frac{2\epsilon A^{(2)}}{3(1-\epsilon z)^{(3)}}\right] + \frac{\partial p_I}{\partial z};$$

$$\pi(1-\epsilon z)^{(2)}(\epsilon\tau_a + p_e) = \pi Re\left[\frac{u_{0,3}^{(2)}}{24Re^{(2)}}(1-\epsilon z)^{(8)} - \frac{\epsilon(1-\epsilon z)}{12}\frac{dA}{dt} + \frac{\epsilon^{(2)}A}{12}\right] + p_I;$$

$$\frac{1}{12Re}(1-\epsilon z)^{(2)}\left[(1-\epsilon z)^{(2)}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}} - 12\epsilon(1-\epsilon z)\frac{\partial u_{0,3}}{\partial z} + 30\epsilon^{(2)}u_{0,3}\right] - \tau_h(\epsilon^{(2)}+1)^{(1/2)}$$
$$= \frac{1}{24}(1-\epsilon z)\left[A(1-\epsilon z)\frac{\partial u_{0,3}}{\partial z} + 2(1-\epsilon z)^{(3)}\frac{\partial u_{0,3}}{\partial t} - 4\epsilon A u_{0,3}\right];$$

$$\frac{\pi}{6}\left[A(6+\epsilon^{(2)}) - 6\epsilon p_e(1-\epsilon z)^{(3)} + 6\tau_a(1-\epsilon z)^{(3)}\right] = \frac{\pi Re}{12}(\epsilon z-1)^{(2)}\left[-\frac{dA}{dt} - \epsilon A^{(2)}\right] + \frac{\partial q_I}{\partial z};$$

$$\frac{\pi}{48}(1-\epsilon z)\left[3\epsilon^{(3)}A + 8\epsilon A + 12\epsilon\tau_a(1-\epsilon z)^{(3)} + 12p_e(1-\epsilon z)^{(3)}\right]$$
$$= \frac{\pi Re}{960}(1-\epsilon z)^{(4)}\left[8\epsilon^{(2)}A^{(2)} + 10\epsilon(1-\epsilon z)^{(3)}\frac{dA}{dt} + 4(1-\epsilon z)^{(10)}\right] + q_I;$$

$$\frac{1}{96Re}(1-\epsilon z)^{(2)}\left[\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}(1-\epsilon z)^{(2)} - 16\epsilon\frac{\partial u_{0,3}}{\partial z}(1-\epsilon z) + 56\epsilon^{(2)}u_{0,3}\right] - \frac{\tau_h}{4}(\epsilon^{(2)}+1)^{(1/2)}$$
$$= \frac{1}{480}(1-\epsilon z)\left[2A(1-\epsilon z)\frac{\partial u_{0,3}}{\partial z} - 8\epsilon A u_{0,3} + 5\frac{\partial u_{0,3}}{\partial t}(1-\epsilon z)^{(3)}\right].$$

If a steady solution is sought, these equations reduce to:

(3.118)
$$2\pi(1-\epsilon z)(\tau_a - \epsilon p_e) = \frac{2\pi Re\epsilon A^{(2)}}{3(1-\epsilon z)^{(3)}} + \frac{\partial p_I}{\partial z};$$

$$\pi(1-\epsilon z)^{(2)}(\epsilon\tau_a+p_e)=\pi Re\left[\frac{u_{0,3}^{(2)}}{24Re^{(2)}}(1-\epsilon z)^{(8)}+\frac{\epsilon^{(2)}A}{12}\right]+p_I; \tag{3.119}$$

$$\begin{aligned}&\frac{1}{12Re}(1-\epsilon z)^{(2)}\left[(1-\epsilon z)^{(2)}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}-12\epsilon(1-\epsilon z)\frac{\partial u_{0,3}}{\partial z}+30\epsilon^{(2)}u_{0,3}\right]-\tau_h(\epsilon^{(2)}+1)^{(1/2)}\\&=\frac{1}{24}(1-\epsilon z)\left[A(1-\epsilon z)\frac{\partial u_{0,3}}{\partial z}-4\epsilon A u_{0,3}\right];\end{aligned} \tag{3.120}$$

$$\frac{\pi}{6}\left[A(6+\epsilon^{(2)})-6\epsilon p_e(1-\epsilon z)^{(3)}+6\tau_a(1-\epsilon z)^{(3)}\right]=-\frac{\pi Re\epsilon A^{(2)}}{12(1-\epsilon z)}+\frac{\partial q_I}{\partial z}; \tag{3.121}$$

$$\begin{aligned}&\frac{\pi}{48}(1-\epsilon z)\left[3\epsilon^{(3)}A+8\epsilon A+12\epsilon\tau_a(1-\epsilon z)^{(3)}+12p_e(1-\epsilon z)^{(3)}\right]\\&=\frac{\pi Re}{960}\left[8\epsilon^{(2)}A^{(2)}+4(1-\epsilon z)^{(10)}\right]+q_I;\end{aligned} \tag{3.122}$$

$$\begin{aligned}&\frac{1}{96Re}(1-\epsilon z)^{(2)}\left[\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}(1-\epsilon z)^{(2)}-16\epsilon\frac{\partial u_{0,3}}{\partial z}(1-\epsilon z)+56\epsilon^{(2)}u_{0,3}\right]-\frac{\tau_h}{4}(\epsilon^{(2)}+1)^{(1/2)}\\&=\frac{1}{480}(1-\epsilon z)\left[2A(1-\epsilon z)\frac{\partial u_{0,3}}{\partial z}-8\epsilon A u_{0,3}\right].\end{aligned} \tag{3.123}$$

Substituting Eq. (3.112) into Eq. (3.29) gives

$$\mathbf{t}=\frac{-1}{1-\epsilon z}\left(\frac{x_1(\epsilon\tau_a+p_e)}{(1+\epsilon^{(2)})^{(1/2)}}+x_2\tau_h\right)\mathbf{e}_1+\frac{1}{1-\epsilon z}\left(x_1\tau_h-\frac{x_2(\epsilon\tau_a+p_e)}{(1+\epsilon^{(2)})^{(1/2)}}\right)\mathbf{e}_2+\left(\frac{\tau_a-\epsilon p_e}{(1+\epsilon^{(2)})^{(1/2)}}\right)\mathbf{e}_3.$$

For the tapered pipe, the components of the outward unit normal to the surface are given by

$$\nu_1=\frac{x_1}{(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}};$$

$$\nu_2=\frac{x_2}{(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}};$$

$$\nu_3=\frac{-\epsilon}{(1+\epsilon^{(2)})^{(1/2)}}.$$

Then substituting Eqs .(3.30)- (3.32) into Eq. (3.4) gives

$$
\begin{aligned}
\mathbf{t} = &\frac{x_1}{(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}}\left(-p+\frac{2}{Re}(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})\right)\mathbf{e}_1\\
&+\frac{x_1}{Re(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}}\left(4x_1x_2u_{3,0}+2(x_1^{(2)}-x_2^{(2)})u_{0,3}\right)\mathbf{e}_2\\
&+\frac{x_1}{Re(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}}\left(\left(\frac{du_{1,0}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{3,0}}{dz}\right)x_1-\left(\frac{du_{0,1}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{0,3}}{dz}\right)x_2+2Rex_1w_{2,0}\right)\mathbf{e}_3\\
&+\frac{x_2}{Re(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}}\left(4x_1x_2u_{3,0}+2(x_1^{(2)}-x_2^{(2)})u_{0,3}\right)\mathbf{e}_1\\
&+\frac{x_2}{(1-\epsilon z)(1+\epsilon^{(2)})^{(1/2)}}\left(-p+2Re(u_{1,0}+(x_1^{(2)}+3x_2^{(2)})u_{3,0}+2x_1x_2u_{0,3})\right)\mathbf{e}_2\\
&+\frac{x_2}{Re(1-\epsilon z)(1+\epsilon^{(2)})^(1/2)}\left(\left(\frac{du_{1,0}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{3,0}}{dz}\right)x_2+\left(\frac{du_{0,1}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{0,3}}{dz}\right)x_1+2Rex_2w_{2,0}\right)\mathbf{e}_3\\
&-\frac{\epsilon}{Re(1+\epsilon^{(2)})^{(1/2)}}\left(\left(\frac{du_{1,0}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{3,0}}{dz}\right)x_1-\left(\frac{du_{0,1}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{0,3}}{dz}\right)x_2+2Rex_1w_{2,0}\right)\mathbf{e}_1\\
&-\frac{\epsilon}{Re(1+\epsilon^{(2)})^{(1/2)}}\left(\left(\frac{du_{1,0}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{3,0}}{dz}\right)x_2+\left(\frac{du_{0,1}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{du_{0,3}}{dz}\right)x_1+2Rex_2w_{2,0}\right)\mathbf{e}_2\\
&-\frac{\epsilon}{(1+\epsilon^{(2)})^{(1/2)}}\left(-p+2\mu\left(\frac{dw_{0,0}}{dz}+(x_1^{(2)}+x_2^{(2)})\frac{dw_{2,0}}{dz}\right)\right)\mathbf{e}_3.
\end{aligned}
$$

Collecting together the unit vectors and using the boundary definition given by Eq. (3.5) leads to

$$
\begin{aligned}
\mathbf{t} = &\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\frac{-px_1}{1-\epsilon z}+\frac{2}{Re(1-\epsilon z)}(u_{1,0}x_1+3u_{3,0}x_1(1-\epsilon z)^{(2)}-u_{0,3}x_2^{(3)})\right.\\
&\left.+\frac{\epsilon}{Re}\left(x_2\frac{du_{0,1}}{dz}-x_1\frac{du_{1,0}}{dz}-2Rew_{2,0}x_1+(1-\epsilon z)^{(2)}\left(x_2\frac{du_{0,3}}{dz}-x_1\frac{du_{3,0}}{dz}\right)\right)\right)\mathbf{e}_1\\
&+\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\frac{2}{Re(1-\epsilon z)}(3u_{3,0}x_2(1-\epsilon z)^{(2)}+u_{0,3}x_1^{(3)}+x_2u_{1,0})-\frac{px_2}{1-\epsilon z}\right.\\
&\left.-\frac{\epsilon}{Re}\left(x_2\frac{du_{1,0}}{dz}+x_1\frac{du_{0,1}}{dz}+2Rex_2w_{2,0}+(1-\epsilon z)^{(2)}\left(x_1\frac{du_{0,3}}{dz}+x_2\frac{du_{3,0}}{dz}\right)\right)\right)\mathbf{e}_2\\
&+\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\epsilon p+\frac{1}{Re}(1-\epsilon z)\left(\frac{du_{1,0}}{dz}+\frac{du_{3,0}}{dz}(1-\epsilon z)+2Rew_{2,0}\right)-2\epsilon\left(\frac{dw_{0,0}}{dz}+\frac{dw_{2,0}}{dz}(1-\epsilon z)^{(2)}\right)\right)\mathbf{e}_3.
\end{aligned}
$$

Then substituting in Eqs.(3.113)-(3.117) leads to

$$
\begin{aligned}
\mathbf{t} = &\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\frac{-px_1}{1-\epsilon z}+2\left(\frac{3A\epsilon x_1}{(1-\epsilon z)^{(4)}}-\frac{\epsilon^{(3)}Ax_1}{(1-\epsilon z)^{(4)}}-\frac{u_{0,3}x_2^{(3)}}{Re(1-\epsilon z)}+\frac{\epsilon^{(2)}}{Re}x_2u_{0,3}(1-\epsilon z)\right)\right)\mathbf{e}_1\\
&+\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\frac{-px_2}{1-\epsilon z}+2\left(\frac{3A\epsilon x_2}{(1-\epsilon z)^{(4)}}-\frac{\epsilon^{(3)}Ax_2}{(1-\epsilon z)^{(4)}}+\frac{u_{0,3}x_1^{(3)}}{Re(1-\epsilon z)}-\frac{\epsilon^{(2)}}{Re}u_{0,3}x_1(1-\epsilon z)\right)\right)\mathbf{e}_2\\
&+\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\epsilon p+2\left(\frac{3\epsilon^{(2)}A}{(1-\epsilon z)^{(3)}}-\frac{A}{(1-\epsilon z)^{(4)}}\right)\right)\mathbf{e}_3.
\end{aligned}
$$

Equating the two expressions for the stress vector gives

(3.124)
$$\frac{-1}{1-\epsilon z}\left(\frac{x_1(\epsilon\tau_a+p_e)}{(1+\epsilon^{(2)})^{(1/2)}}+x_2\tau_h\right)=\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\frac{-px_1}{1-\epsilon z}\right.$$
$$\left.+2\left(\frac{3A\epsilon x_1}{(1-\epsilon z)^{(4)}}-\frac{\epsilon^{(3)}Ax_1}{(1-\epsilon z)^{(4)}}-\frac{u_{0,3}x_2^{(3)}}{Re(1-\epsilon z)}+\frac{\epsilon^{(2)}}{Re}x_2u_{0,3}(1-\epsilon z)\right)\right);$$

(3.125)
$$\frac{1}{1-\epsilon z}\left(x_1\tau_h-\frac{x_2(\epsilon\tau_a+p_e)}{(1+\epsilon^{(2)})^{(1/2)}}\right)=\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\frac{-px_2}{1-\epsilon z}\right.$$
$$\left.+2\left(\frac{3A\epsilon x_2}{(1-\epsilon z)^{(4)}}-\frac{\epsilon^{(3)}Ax_2}{(1-\epsilon z)^{(4)}}+\frac{u_{0,3}x_1^{(3)}}{Re(1-\epsilon z)}-\frac{\epsilon^{(2)}}{Re}u_{0,3}x_1(1-\epsilon z)\right)\right);$$

(3.126)
$$\left(\frac{\tau_a-\epsilon p_e}{(1+\epsilon^{(2)})^{(1/2)}}\right)=\frac{1}{(1+\epsilon^{(2)})^{(1/2)}}\left(\epsilon p+2\left(\frac{3\epsilon^{(2)}A}{(1-\epsilon z)^{(3)}}-\frac{A}{(1-\epsilon z)^{(3)}}\right)\right).$$

Then Eq. (3.126) can be rearranged to give

$$p=\frac{-(1-\epsilon z)^{(3)}\tau_a+\epsilon(1-\epsilon z)^{(3)}p_e-2A(\epsilon^{(2)}+1)}{\epsilon(1-\epsilon z)^{(3)}}.$$

This expression for p can be substituted back into Eqs. (3.124) and (3.125), then the following expressions can be found for τ_a and τ_h:

$$\tau_a=\frac{-2A(\epsilon^{(2)}+1)}{(1-\epsilon z)^{(3)}};$$

$$\tau_h=\frac{2}{Re}u_{0,3}(\epsilon+1)^{(1/2)}(1-\epsilon z)^{(2)}.$$

Substituting the expression for τ_a back into the expression for p gives $p=p_e$. Substituting the derived expressions back into the partial differential equations (PDEs), Eqs. (3.120) and (3.123) simplify to ordinary differential equations (ODEs) for $u_{0,3}$. Rearranging Eq. (3.120) gives an expression for the second order differential of $u_{0,3}$. Substituting this into Eq. (3.123) gives a first order ODE for $u_{0,3}$ which can be solved to give

$$u_{0,3}=\frac{B(-40\epsilon(1-\epsilon z)+ARe)^{\frac{7\epsilon^{(2)}+4}{2\epsilon^{(2)}}}}{(1-\epsilon z)^{(4)}},$$

where B is a constant of integration. Then $u_{0,1}$ is given by

$$u_{0,1}=\frac{B(-40\epsilon(1-\epsilon z)+ARe)^{\frac{7\epsilon^{(2)}+4}{2\epsilon^{(2)}}}}{(1-\epsilon z)^{(2)}}.$$

Substituting the expression for $u_{0,3}$ into the expression for τ_h gives

$$\tau_h = \frac{2B(\epsilon^{(2)}+1)^{(1/2)}(-40\epsilon(1-\epsilon z)+ARe)^{\frac{7\epsilon^{(2)}+4}{2\epsilon^{(2)}}}}{(1-\epsilon z)^{(2)}}.$$

Substituting everything back into Eq. (3.120), it is found that for the equation to balance it is required that $B=0$, which gives $u_{0,3}=u_{0,1}=\tau_h=0$. With the current expressions for the velocity, pressure and stress terms, Eq. (3.118) takes the form

$$\frac{-2\pi(\epsilon(1-\epsilon z)^{(3)}p_e+2A\mu(\epsilon^{(2)}+1))}{(1-\epsilon z)^{(2)}} = \frac{dp_I}{dz} + \frac{2A^{(2)}Re\epsilon\pi\rho}{3(1-\epsilon z)^{(3)}}.$$

This can be rearranged to give the follow expression for p_e:

$$p_e = -\frac{3\frac{dp_I}{dz} - 2A^{(2)}\epsilon\pi R + 12A\pi(\epsilon^{(2)}+1)}{6\epsilon\pi(1-\epsilon z)^{(3)}}.$$

Using the substitutions $p_e = p$ and Eq. (3.42) (in Appendix A) in this expression and rearranging gives

$$\frac{\partial p}{\partial z} = \frac{-4A(6(\epsilon^{(2)}+1)(1-\epsilon z)+AR}{6(1-\epsilon z)^{(5)}}.$$

Integrating gives

$$p = -4A\left(\frac{(\epsilon^{(2)}+1)}{3\epsilon(1-\epsilon z)^{(3)}} + \frac{ARe}{24(1-\epsilon z)^{(4)}}\right) + k,$$

where k is a constant of integration. Then from Eq. (3.42),

(3.127)
$$p_I = -A\pi\left(\frac{4(\epsilon^{(2)}+1)}{3\epsilon(1-\epsilon z)} + \frac{ARe}{6(1-\epsilon z)^{(2)}}\right) + k\pi(1-\epsilon z)^{(2)}$$

and from Eq. (3.43),

(3.128)
$$q_I = \frac{kz^{(4)}\epsilon^{(4)}\pi}{4} - kz^{(3)}\epsilon^{(3)}\pi + \frac{3kz^{(2)}\epsilon^{(2)}\pi}{2} - kz\epsilon\pi + \frac{Az\epsilon^{(2)}\pi}{3} + \frac{k\pi}{4} - \frac{A\epsilon\pi}{3} - \frac{ReA^{(2)}}{24} + \frac{Az}{3} - \frac{3A}{\epsilon}.$$

Substituting the expressions for the velocity form (Eqs. (3.113)-(3.117)) into Eq. (3.13) gives

(3.129)
$$\mathbf{v} = \left[\left(\frac{-\epsilon A}{(1-\epsilon z)^{(3)}} + (x_1^{(2)}+x_2^{(2)})\frac{\epsilon A}{(1-\epsilon z)^{(5)}}\right)x_1\right]\mathbf{e}_1 + \left[\left(\frac{-\epsilon A}{(1-\epsilon z)^{(3)}} + (x_1^{(2)}+x_2^{(2)})\frac{\epsilon A}{(1-\epsilon z)^{(5)})}\right)x_2+\right]\mathbf{e}_2$$
$$+\left[\frac{A}{(1-\epsilon z)^{(2)}} - (x_1^{(2)}+x_2^{(2)})\frac{A}{(1-\epsilon z)^{(4)}}\right]\mathbf{e}_3.$$

A plot of this is shown in Fig. 3.7.

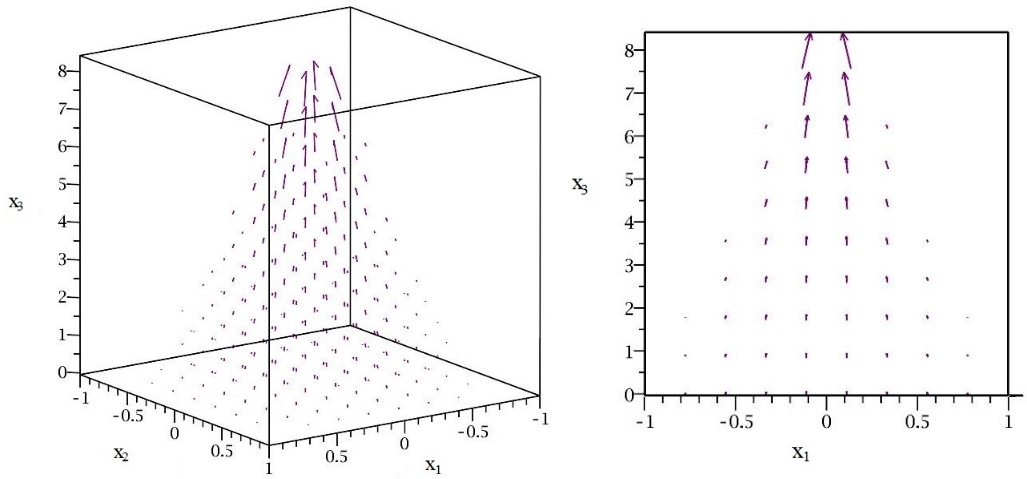


Figure 3.7: Plot of flow through a tapered pipe, given by Eq. (3.129) with $Re = 2$, $\epsilon = 0.1$.

Note that as this is the non-dimensionalised velocity, it must be the case that $A = 1$ for unity. Then the true velocity will be Eq. (3.129) multiplied by Re. The solution is plotted in Fig. 3.7 for $Re = 2$ and $\epsilon = 0.1$. This 1D solution is compared with the 3D finite volume solution, as shown in Figs. 3.8-3.9, and the solution errors are presented in Fig. 3.10. This shows a good agreement between the simulation and the director theory model, with a maximum relative error of approximately 3% for the $Re = 2$ case and 5% for the $Re = 4$ case. In the finite volume simulation, μ and ρ are set to 1 and the inlet condition (at $z = 0$) is a flow rate of π, and the tapered pipe geometry has length $8m$. The Reynolds number for the model is calculated by setting the flow rate at $z = 0$ to π and finding Re to satisfy this condition. While the solutions for the 1D model represent a fully developed flow, those obtained from the 3D finite volume simulations require the solution to develop from a uniform velocity (Dirichlet) boundary condition prescribed on the inflow section. It is expected therefore that some comparison errors which appear will be due to this. Finally the finite volume mesh had a base size of $0.02m$ with 5 prism layers around the boundary.

Figs. 3.8-3.10 show good agreement between the director theory solution and the CFD simulation, considering the low order of expansion used in the director theory approach.

3.5.1.2 Order K=5

Increasing the order to K=5 and seeking a steady non-swirling solution involves solving the PDE system given by Eqs (3.56)-(3.64) with $u_{0,1}, u_{0,3}, u_{0,5}$ set to 0. Motivated by

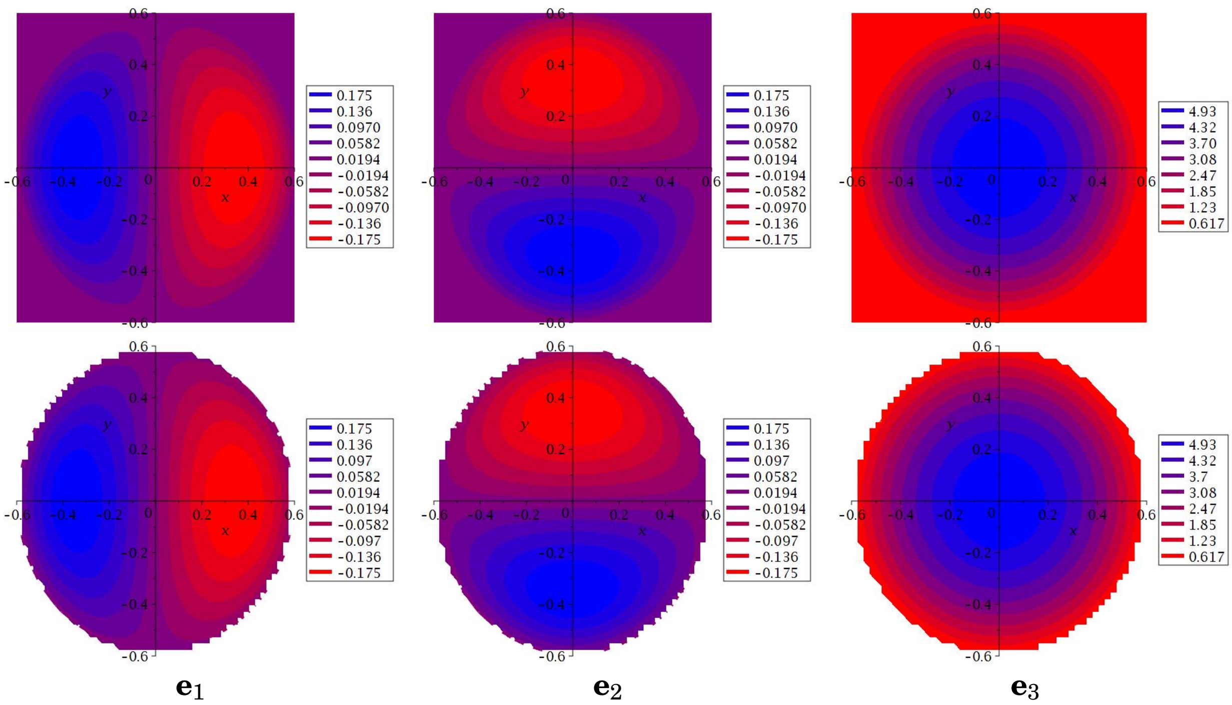


Figure 3.8: Contour plots of flow through a tapered pipe cross-section, given by Eq. (3.129) for $K = 3$ (top) and from a corresponding simulation using STAR-CCM+ (bottom), at location $z = 4$, with $Re = 2$ at the inflow section, $\epsilon = 0.1$, in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

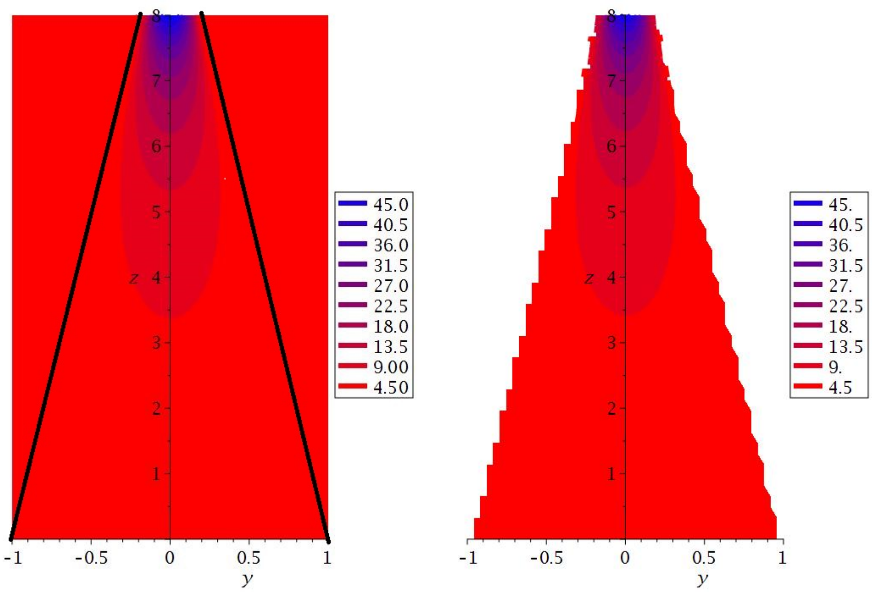


Figure 3.9: Contour plots of coaxial flow ($\mathbf{e}_3$ direction) along a tapered pipe cross-section, given by Eq. (3.129) for $K = 3$ (left) and from a corresponding simulation using STAR-CCM+ (right), with $Re = 2$ at the inflow section.

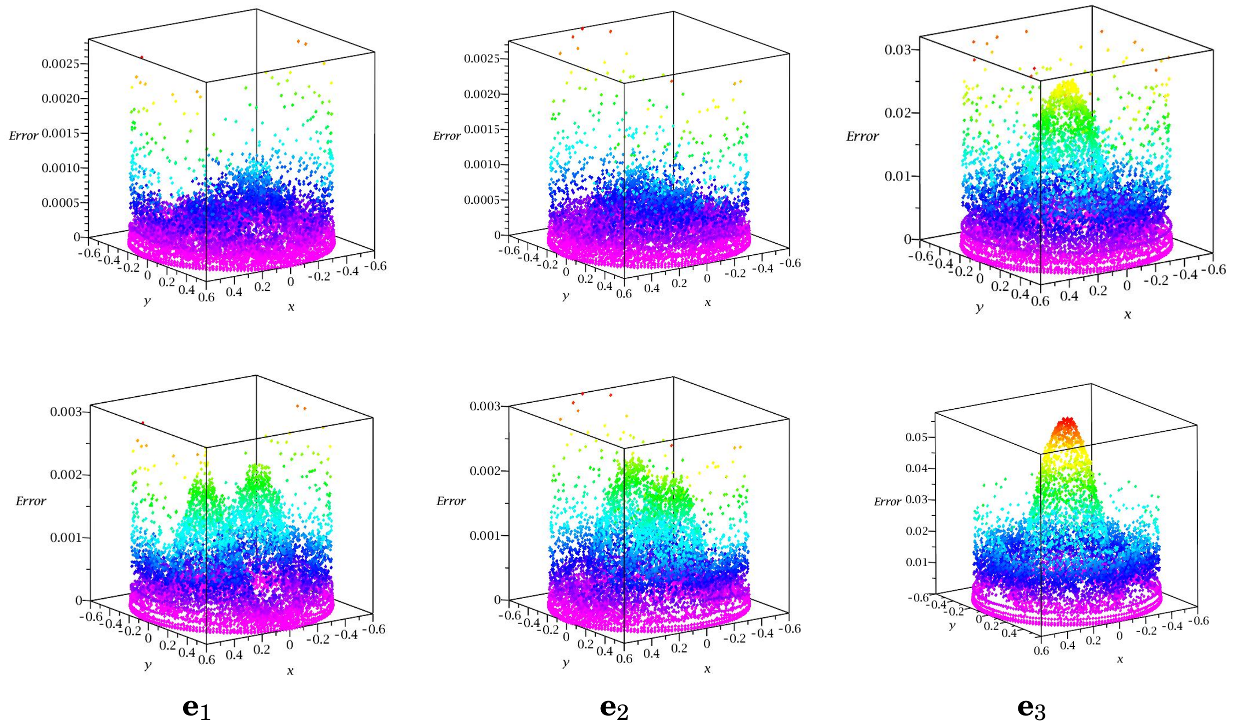


Figure 3.10: Plots of the relative error, calculated by Eq. (2.18), in a cross-section of the tapered pipe, between the 3D finite volume simulation and the 1D solution given by Eq. (3.145) for $K = 3$. Solution are obtained for $Re = 2$ (top) and $Re = 4$ (bottom), $\epsilon = 0.1$, in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

Poiseuille flow, it is assumed that the zeroth order term takes the following form

$$w_{0,0} = \frac{A}{(1-\epsilon z)^{(2)}}, \tag{3.130}$$

where A is a constant related to the bulk flow rate. Then from Eqs (3.47)-(3.49),(3.65) and (3.67) the form of the following velocity terms are found:

$$w_{2,0} = -\frac{A+4B}{(1-\epsilon z)^{(4)}}; \tag{3.131}$$

$$u_{5,0} = -\frac{\epsilon Re(3A+B)}{(1-\epsilon z)^{(7)}};$$

$$u_{3,0} = \frac{\epsilon Re(4A+B)}{(1-\epsilon z)^{(5)}};$$

$$w_{4,0} = \frac{3A+B}{(1-\epsilon z)^{(6)}}; \tag{3.132}$$

$$u_{1,0} = -\frac{\epsilon ReA}{(1-\epsilon z)^{(3)}}. \tag{3.133}$$

Equating the expressions for the stress tensor gives

$$\frac{x_2\tau_a}{1-\epsilon z} = \frac{x_1}{(1+\epsilon^{(2)})^{(1/2)}(1-\epsilon z)^{(4)}}\Big((p-p_e-\epsilon\tau_a)(1-\epsilon z)^{(3)} + 2\epsilon(1+\epsilon^{(2)})(2A+B)\Big); \tag{3.134}$$

$$-\frac{x_1\tau_h}{1-\epsilon z}^{(4)} = \frac{x_2}{(1+\epsilon^{(2)})^{(1/2)}(1-\epsilon z)^{(4)}}\Big((p-p_e-\epsilon\tau_a)(1-\epsilon z)^{(3)} + 2\epsilon(1+\epsilon^{(2)})(2A+B)\Big);$$

$$\frac{1}{(1+\epsilon^{(2)})^{(1/2)}(1-\epsilon z)^{(3)}}\Big((\tau_a+\epsilon p-\epsilon p_e + 2(\epsilon^{(2)}+1)(2A+B)\Big). \tag{3.135}$$

Rearranging Eq. (3.135) gives

$$p = \frac{(\epsilon p_e - \tau_a)(1-\epsilon z)^{(3)} + 2(1+\epsilon^{(2)})(2A+B)}{(1-\epsilon z)^{(3)}(1+\epsilon^{(2)})^{(1/2)}}. \tag{3.136}$$

Substituting this into Eq. (3.134) and equating the coefficients of x_1 gives $\tau_h = 0$. Comparing the coefficients of x_2 gives

$$\tau_a = \frac{2(1+\epsilon^{(2)})(2A+B)}{(1-\epsilon z)^{(3)}}.$$

Then substituting this expression back into Eq. (3.136), it is found that $p = p_e$. Recalling the expressions for p_I and q_I for the case of the tapered pipe given by Eqs. (3.127) and (3.128) and also considering from Eq. (3.44) that for a tapered pipe h_I is given by

$$h_I = \frac{\pi p_e(1-\epsilon z)^{(6)}}{8}. \tag{3.137}$$

Substituting the velocity and pressure terms into Eq. (3.56) and rearranging gives an expression for $\frac{dp_e}{dz}$ which can be integrated to give the following

$$p_e = \frac{4(2A+B)(1+\epsilon^{(2)})}{3\epsilon(1-\epsilon z)^{(3)}} - \frac{Re(4A^{(2)}+B^{(2)}+AB)}{60(1-\epsilon z)^{(4)}} + k, \tag{3.138}$$

where k is a constant of integration. Substituting this into Eqs. (3.127), (3.128), (3.137) gives

$$p_I = \frac{4\pi(2A+B)(1+\epsilon^{(2)})}{3\epsilon(1-\epsilon z)} - \frac{Re\pi(4A^{(2)}+B^{(2)}+AB)}{60(1-\epsilon z)^{(2)}} + k\pi(1-\epsilon z)^{(2)};$$

$$q_I = \frac{\pi(2A+B)(1+\epsilon^{(2)})(1-\epsilon z)}{3\epsilon} - \frac{Re\pi(4A^{(2)}+B^{(2)}+AB)}{240} + \frac{k\pi(1-\epsilon z)^{(4)}}{4};$$

$$h_I = \frac{\pi(2A+B)(1+\epsilon^{(2)})(1-\epsilon z)^{(3)}}{6\epsilon} - \frac{Re\pi(4A^{(2)}+B^{(2)}+AB)(1-\epsilon z)}{480} + \frac{k\pi(1-\epsilon z)^{(6)}}{8}.$$

Substituting the velocity and pressure terms into Eq. (3.59) and bringing everything to one side gives

$$-\frac{\pi}{60(1-\epsilon z)}(55z(B-\frac{23}{11}A)\epsilon^{(3)}-(55B+115A)\epsilon^{(2)}+(-2ReA^{(2)}+(ReB+120z)A+40Bz)\epsilon-40B-120A)=0. \tag{3.139}$$

Expressing B as an ϵ expansion of the form

$$B=b_0+b_1\epsilon+O(\epsilon^{(2)}),$$

where b_0 and b_1 are coefficients to be found so as to minimise the error in Eq. (3.139), it is found that $b_0=-3A$ and $b_1=-\frac{ReA^{(2)}}{8}$. That is

$$B=-3A-\frac{ReA^{(2)}}{8}\epsilon. \tag{3.140}$$

Further terms do not reduce the order of the error due to the approximation of other terms already introducing an error of $O(\epsilon^{(2)})$. Substituting Eq. (3.140) into Eqs. (3.131)-(3.132) gives

$$u_{5,0}=\frac{\epsilon^{(2)}Re^{(2)}A^{(2)}}{8(1-\epsilon z)^{(7)}}; \tag{3.141}$$

$$w_{4,0}=-\frac{\epsilon ReA^{(2)}}{8(1-\epsilon z)^{(6)}}; \tag{3.142}$$

$$u_{3,0}=\frac{\epsilon ReA(8-\epsilon ReA)}{8(1-\epsilon z)^{(5)}}; \tag{3.143}$$

$$w_{2,0}=\frac{A(A\epsilon Re-8)}{8(1-\epsilon z)^{(4)}}. \tag{3.144}$$

So substituting Eqs. (3.130),(3.133),(3.141),(3.144) back into Eq. (3.14) gives the velocity form

$$\begin{aligned}\mathbf{v}=\epsilon Ax&\left[-\frac{1}{(1-\epsilon z)^{(3)}}+\frac{(8-\epsilon ReA)(x^{(2)}+y^{(2)})}{8(1-\epsilon z)^{(5)}}+\frac{\epsilon ReA(x^{(2)}+y^{(2)})}{8(1-\epsilon z)^{(7)}}\right]\mathbf{e}_1\\ +\epsilon Ay&\left[-\frac{1}{(1-\epsilon z)^{(3)}}+\frac{(8-\epsilon ReA)(x^{(2)}+y^{(2)})}{8(1-\epsilon z)^{(5)}}+\frac{\epsilon ReA(x^{(2)}+y^{(2)})}{8(1-\epsilon z)^{(7)}}\right]\mathbf{e}_2\\ +A&\left[\frac{1}{(1-\epsilon z)^{(2)}}+\frac{A(\epsilon ReA-8)(x^{(2)}+y^{(2)})}{8(1-\epsilon z)^{(4)}}-\frac{\epsilon ReA^{(2)}(x^{(2)}+y^{(2)})^{(2)}}{8(1-\epsilon)^{(6)}}\right]\mathbf{e}_3.\end{aligned} \tag{3.145}$$

Given that $p=p_e$, from Eq. (3.138), the pressure is given by

$$p=-\frac{A(1+\epsilon^{(2)})(4+\frac{\epsilon ReA}{2})}{3\epsilon(1-\epsilon z)^{(3)}}-\frac{A^{(2)}(10+\frac{\epsilon^{(2)}Re^{(2)}A^{(2)}}{64}+\frac{5\epsilon ReA}{8})}{60(1-\epsilon z)^{(4)}}.$$

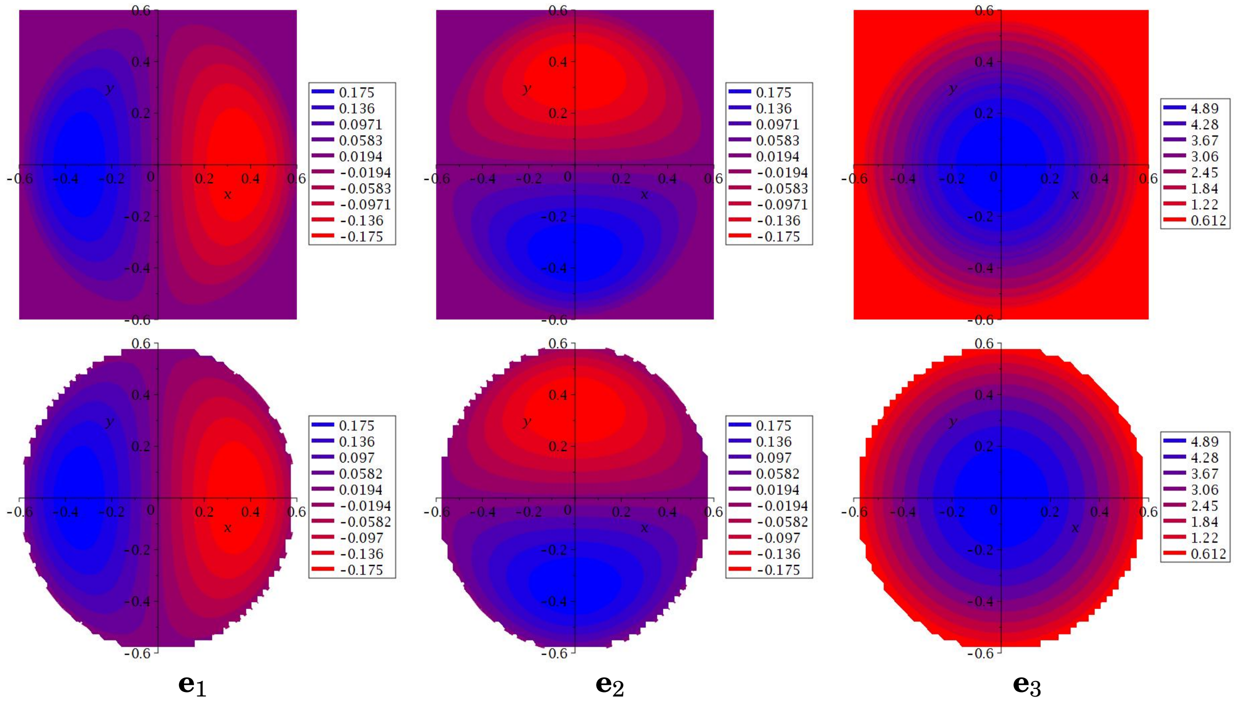


Figure 3.11: Contour plots of flow through a tapered pipe cross-section, given by Eq. (3.145) for $K = 5$ (top) and from a corresponding simulation using STAR-CCM+ (bottom), at $z = 4$, with $Re = 2$, $\epsilon = 0.1$, in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

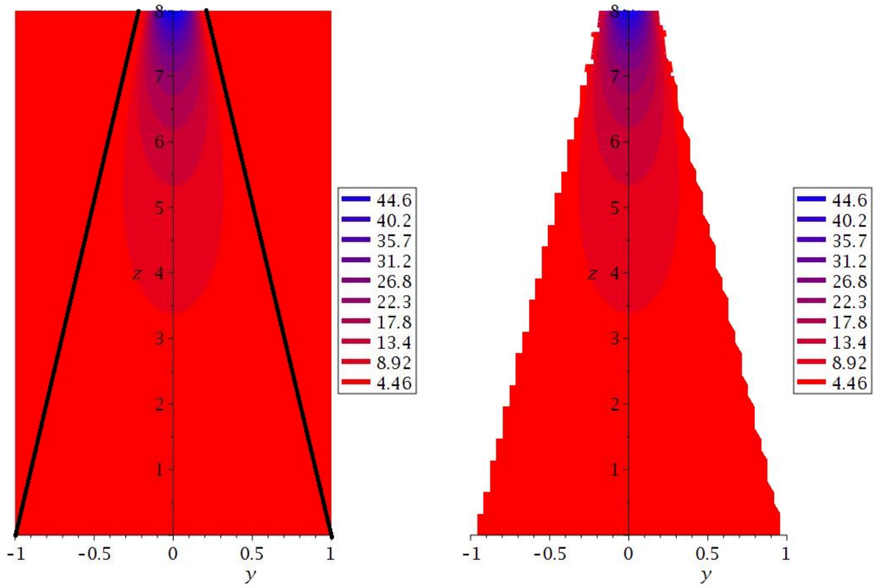


Figure 3.12: Contour plots of coaxial flow ($\mathbf{e}_3$ direction) along a tapered pipe cross-section, given by Eq. (3.129) for $K = 5$ (left) and from a simulation in STAR-CCM+ (right), with $Re = 2$ at the inflow section.

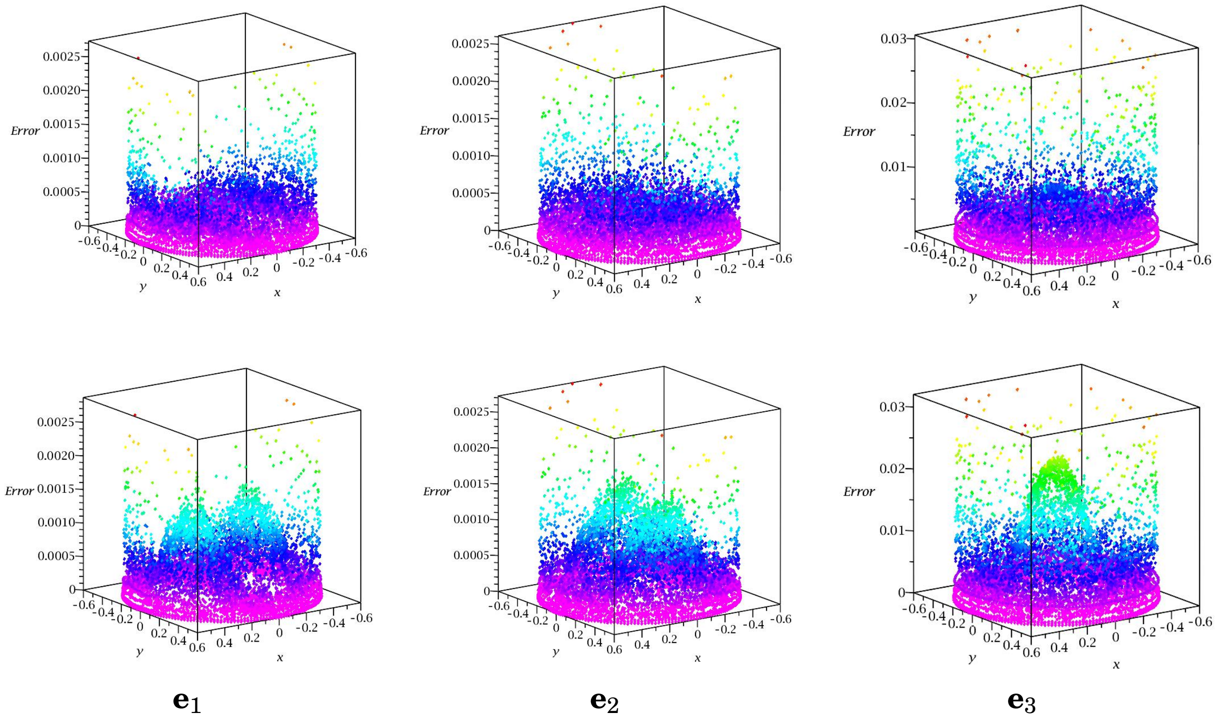


Figure 3.13: Plots of the relative error, calculated by Eq. (2.18), in a cross-section of the tapered pipe, between the 3D finite volume simulation and the 1D solution given by Eq. (3.145) for $K = 5$, with $Re = 2$ (top) and $Re = 4$ (bottom), $\epsilon = 0.1$, in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

As was the case with order $K = 3$, at order $K = 5$ the results from the director theory model agree well with the CFD simulation, the comparison of the velocity contours can be seen in Figs. 3.11 and 3.12. There is a maximum relative error of approximately 3%, as can be seen in Fig. 3.13. The error of the in-plane velocity looks very similar at both orders, but the error of the coaxial velocity appears to reduce slightly with the higher order case. Some higher error close to the walls of the pipe remains present for all cases investigated. However as the velocity tends to zero close to the wall, it is unsurprising that the magnitude of the relative error may be larger at these points. There could also be factors from the simulation settings, such as the number of prism layers, affecting how well the solution is resolved close to the wall. Additionally, artificially thicker boundary layers are known to be the result of using the segregated solver (based on the projection method). Note also the finite volume setup requires the solution to develop fully from the artificial inflow boundary (uniform velocity is prescribed), and this may not be entirely achieved at the location the cross section was investigated. Apart from the error apparent

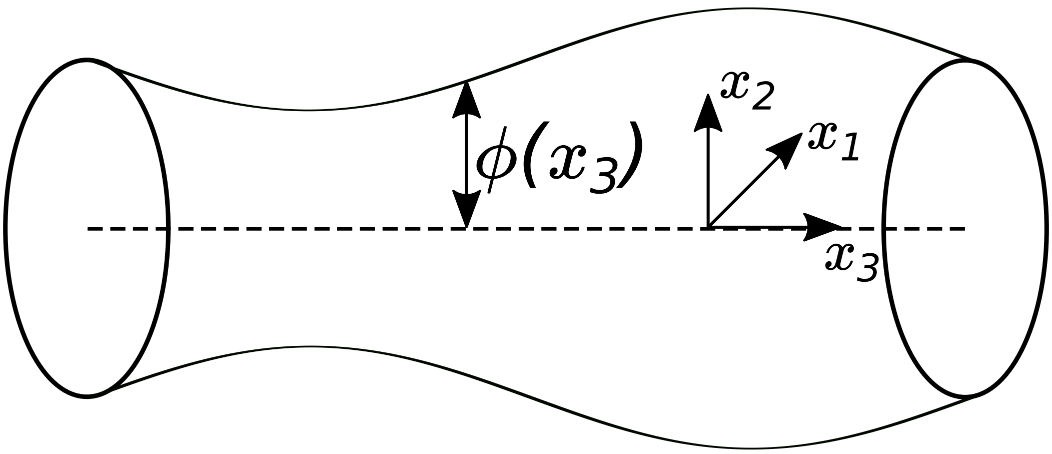


Figure 3.14: A straight axisymmetric sinusoidal pipe with radius $\phi(z)$ given by Eq. (3.146), with lateral surface H.

closer to the no-slip wall, we observe a profile which contains two peaks within the free-stream domain. This error appears due to the low order expansion assumed the solution would take, hence the basis functions of the 1D director theory solution. Higher order expansions will result in more accurate 1D solutions.

3.5.2 Sinusoidal pipe

This subsection will consider the case of a sinusoidal (wavy walled) pipe, as shown in Fig. 3.14. Specific orders of the velocity expansion of orders $K = 3$ and $K = 5$ will be considered. For a sinusoidal pipe, the radius is given by

$$\phi(z) = 1 - \epsilon \sin(cz), \tag{3.146}$$

where ϵ and c are constants, with ϵ relating to the amplitude and c relating to the wavelength of the sinusoidal wall.

3.5.2.1 K=3

With the description of the wall given by Eq. (3.146), Eqs (3.103) - (3.108) now become:

$$2\pi(1-\epsilon\sin(cz))(\tau_a - p_e\epsilon c\cos(cz)) = \frac{\partial p}{\partial z} + \pi Re\left(\frac{1}{2}\frac{dA}{dt} - \frac{A^{(2)}}{(1-\epsilon\sin(cz))^{(3)}} + \frac{4A^{(2)}}{3(1-\epsilon\sin(cz))^{(5)}}\right);$$

$$-\frac{\pi A}{4}\left[-\epsilon^{(2)}c^{(3)}\cos(cz)\sin(cz)+\epsilon c^{(3)}\cos(cz)(1-\epsilon\sin(cz))\right]+\pi(1-\epsilon\sin(cz))^{(2)}(-\epsilon c\cos(cz)\tau_a-p_e)$$
$$=-p_I-\frac{Re\pi}{12}\left(-\epsilon c\cos(cz)(1-\epsilon\sin(cz))\frac{dA}{dt}-\frac{5A^{(2)}\epsilon^{(3)}c^{(3)}\cos^{(3)}(cz)}{2(1-\epsilon\sin(cz))^{(2)}}+\frac{A^{(2)}\epsilon c^{(2)}\sin(cz)}{2(1-\epsilon\sin(cz))}\right);$$

$$-\phi^{(4)}\left[\frac{5}{2Re}(-\epsilon c\cos(cz))^{(2)}u_{0,3}+\frac{-\epsilon c\cos(cz)(1-\epsilon\sin(cz))}{Re}\frac{\partial u_{0,3}}{\partial z}+\frac{(1-\epsilon\sin(cz))^{(2)}}{12Re}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}\right]$$
$$+\phi^{(2)}\tau_h\left(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)\right)^{(1/2)}=-\frac{Re(1-\epsilon\sin(cz))^{(3)}}{6}\left(\frac{(1-\epsilon\sin(cz))^{(3)}}{2Re}\frac{\partial u_{0,3}}{\partial z}+\frac{A\epsilon c\cos(cz)}{Re}u_{0,3}\right.$$
$$\left.-\frac{A}{4Re}\frac{\partial u_{0,3}}{\partial z}\right);$$

$$\frac{\pi A}{2}\left[\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)+\epsilon c^{(2)}\sin(cz)(1-\epsilon\sin(cz))\right]+\pi A+\pi(1-\epsilon\sin(cz))^{(3)}(\tau_a-\epsilon c p_e\cos(cz))=\frac{\partial q_I}{\partial z}$$
$$+\frac{\pi Re}{4}\left(\phi^{(2)}\frac{dA}{dt}+\frac{7A^{(2)}\epsilon c\cos(cz)}{3(1-\epsilon\sin(cz))}\right);$$

$$\frac{-\pi A(1-\epsilon\sin(cz))}{24}\left[-\frac{3\epsilon^{(3)}c^{(3)}\cos^{(3)}(cz)}{2}\right.$$
$$\left.-\frac{9\epsilon^{(2)}c^{(3)}\cos(cz)\sin(cz)}{2}(1-\epsilon\sin(cz))+\epsilon c^{(3)}\cos(cz)(1-\epsilon\sin(cz))^{(2)}\right]$$
$$+\frac{\pi A\epsilon c\cos(cz)(1-\epsilon\sin(cz))}{2}+\frac{\pi(1-\epsilon\sin(cz))^{(4)}}{4}(-\epsilon c\tau_a\cos(cz)-p_e)$$
$$=-q_I-\frac{\pi Re}{24}\left(\frac{-\epsilon c\cos(cz)(1-\epsilon\sin(cz))^{(3)}}{4}\frac{dA}{dt}\right.$$
$$\left.+\frac{A^{(2)}\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{5}+A^{(2)}\epsilon c^{(2)}\sin(cz)(1-\epsilon\sin(cz))+\frac{(1-\epsilon\sin(cz))^{(10)}}{10Re}u_{0,3}^{(2)}\right);$$

$$-\frac{(1-\epsilon\sin(cz))^{(6)}}{12}\left[\frac{7\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{Re}u_{0,3}+\frac{\epsilon c^{(2)}\sin(cz)(1-\epsilon\sin(cz))}{Re}u_{0,3}\right.$$
$$\left.-\frac{2\epsilon c\cos(cz)(1-\epsilon\sin(cz))}{Re}\frac{\partial u_{0,3}}{\partial z}+\frac{(1-\epsilon\sin(cz))^{(2)}}{8Re}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}\right]$$
$$-\frac{(1-\epsilon\sin(cz))^{(6)}}{6Re}u_{0,3}+\frac{(1-\epsilon\sin(cz))\tau_h}{4}\left(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)\right)^{(1/2)}$$
$$=-\frac{Re(1-\epsilon\sin(cz))^{(6)}}{8}\left(\frac{(1-\epsilon\sin(cz))^{(2)}}{12Re}\frac{\partial u_{0,3}}{\partial t}+\frac{A}{30Re}\frac{\partial u_{0,3}}{\partial z}-\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))Re}u_{0,3}\right).$$

If a steady solution is sought, these equations reduce to:

(3.147)

$$2\pi(1-\epsilon\sin(cz))(\tau_a-p_e\epsilon c\cos(cz))=\frac{\partial p_I}{\partial z}+\pi Re\left(-\frac{A^{(2)}}{(1-\epsilon\sin(cz))^{(3)}}+\frac{4A^{(2)}}{3(1-\epsilon\sin(cz))^{(5)}}\right);$$

$$-\frac{\pi A}{4}\left[-\epsilon^{(2)}c^{(3)}\cos(cz)\sin(cz)+\epsilon c^{(3)}\cos(cz)(1-\epsilon\sin(cz))\right]+\pi(1-\epsilon\sin(cz))^{(2)}(-\epsilon c\cos(cz)\tau_a-p_e)$$
$$=-p_I-\frac{Re\pi}{12}\left(-\frac{5A^{(2)}\epsilon^{(3)}c^{(3)}\cos^{(3)}(cz)}{2(1-\epsilon\sin(cz))^{(2)}}+\frac{A^{(2)}\epsilon c^{(2)}\sin(cz)}{2(1-\epsilon\sin(cz))}\right);$$

$$-\phi^{(4)}\left[\frac{5}{2Re}(-\epsilon c\cos(cz))^{(2)}u_{0,3}+\frac{-\epsilon c\cos(cz)(1-\epsilon\sin(cz))}{Re}\frac{\partial u_{0,3}}{\partial z}+\frac{(1-\epsilon\sin(cz))^{(2)}}{12Re}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}\right]$$
$$+\phi^{(2)}\tau_h\left(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)\right)^{(1/2)}=-\frac{Re(1-\epsilon\sin(cz))^{(3)}}{6}\left(\frac{(1-\epsilon\sin(cz))^{(3)}}{2Re}\frac{\partial u_{0,3}}{\partial z}+\frac{A\epsilon c\cos(cz)}{Re}u_{0,3}\right.$$
$$\left.-\frac{A}{4Re}\frac{\partial u_{0,3}}{\partial z}\right);$$

$$\frac{\pi A}{2}\left[\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)+\epsilon c^{(2)}\sin(cz)(1-\epsilon\sin(cz))\right]+\pi A+\pi(1-\epsilon\sin(cz))^{(3)}(\tau_a-\epsilon c p_e\cos(cz))=\frac{\partial q_I}{\partial z}$$
$$+\frac{\pi Re}{4}\left(+\frac{7A^{(2)}\epsilon c\cos(cz)}{3(1-\epsilon\sin(cz))}\right);$$

$$\frac{-\pi A(1-\epsilon\sin(cz))}{24}\left[-\frac{3\epsilon^{(3)}c^{(3)}\cos^{(3)}(cz)}{2}-\frac{9\epsilon^{(2)}c^{(3)}\cos(cz)\sin(cz)}{2}(1-\epsilon\sin(cz))+\epsilon c^{(3)}\cos(cz)(1-\epsilon\sin(cz))^{(2)}\right]$$
$$+\frac{\pi A\epsilon c\cos(cz)(1-\epsilon\sin(cz))}{2}+\frac{\pi(1-\epsilon\sin(cz))^{(4)}}{4}(-\epsilon c\tau_a\cos(cz)-p_e)=-q_I$$
$$-\frac{\pi Re}{24}\left(\frac{A^{(2)}\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{5}+A^{(2)}\epsilon c^{(2)}\sin(cz)(1-\epsilon\sin(cz))+\frac{(1-\epsilon\sin(cz))^{(10)}}{10Re}u_{0,3}^{(2)}\right);$$

$$-\frac{(1-\epsilon\sin(cz))^{(6)}}{12}\left[\frac{7\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{Re}u_{0,3}+\frac{\epsilon c^{(2)}\sin(cz)(1-\epsilon\sin(cz))}{Re}u_{0,3}-\frac{2\epsilon c\cos(cz)(1-\epsilon\sin(cz))}{Re}\frac{\partial u_{0,3}}{\partial z}\right.$$
$$\left.+\frac{(1-\epsilon\sin(cz))^{(2)}}{8Re}\frac{\partial^{(2)}u_{0,3}}{\partial z^{(2)}}\right]-\frac{(1-\epsilon\sin(cz))^{(6)}}{6Re}u_{0,3}+\frac{(1-\epsilon\sin(cz))\tau_h}{4}\left(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)\right)^{(1/2)}$$
$$=-\frac{Re(1-\epsilon\sin(cz))^{(6)}}{8}\left(\frac{(1-\epsilon\sin(cz))^{(2)}}{12Re}\frac{\partial u_{0,3}}{\partial t}+\frac{A}{30Re}\frac{\partial u_{0,3}}{\partial z}-\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))Re}u_{0,3}\right).$$

Substituting Eq. (3.146) into Eq. (3.98) gives

$$w_{0,0}=\frac{A}{(1-\epsilon\sin(cz))^{(2)}}. \tag{3.148}$$

Then from Eq. (3.17),

$$u_{1,0}=-\frac{ARe\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}. \tag{3.149}$$

From Eq. (3.65),

$$u_{3,0}=\frac{ARe\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(5)}}. \tag{3.150}$$

From Eq. (3.66),

$$u_{0,1} = -u_{0,3}(1-\epsilon\sin(cz))^{(2)}. \tag{3.151}$$

From Eq. (3.67),

$$w_{2,0} = -\frac{A}{(1-\epsilon\sin(cz))^{(4)}}. \tag{3.152}$$

Substituting Eq. (3.146) into Eq. (3.29) gives

$$\mathbf{t} = \left[\frac{1}{1-\epsilon\sin(cz)}\left(\frac{x_1(\tau_a\epsilon c\cos(cz)-p_e)}{\left(1+(\epsilon c\cos(cz))^{(2)}\right)^{(1/2)}} - x_2\tau_h\right)\right]\mathbf{e}_1 + \left[\frac{1}{1-\epsilon\sin(cz)}\left(\frac{x_2(\tau_a\epsilon c\cos(cz)-p_e)}{\left(1+(\epsilon c\cos(cz))^{(2)}\right)^{(1/2)}} + x_1\tau_h\right)\right]\mathbf{e}_2 + \left[\frac{\tau_a+p_e\epsilon c\cos(cz)}{\left(1+(\epsilon c\cos(cz))^{(2)}\right)^{(1/2)}}\right]\mathbf{e}_3.$$

For the sinusoidal pipe, the components of the outward unit normal to the surface are given by

$$v_1 = \frac{x_1}{(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}};$$

$$v_2 = \frac{x_2}{(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}};$$

$$v_3 = \frac{\epsilon c\cos(cz)}{[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}.$$

Then substituting Eqs. (3.30) - (3.32) into Eq. (3.4) gives

$$
\begin{aligned}
\mathbf{t} = & \frac{x_1}{(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}[-p+\frac{2}{R}(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})]\mathbf{e}_1 \\
& + \frac{x_1}{Re(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}[4x_1x_2u_{3,0}+(2x_1^{(2)}-2x_2^{(2)})u_{0,3}]\mathbf{e}_2 \\
& + \frac{x_1}{Re(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right. \\
& \left.+2Rex_1w_{2,0}\right]\mathbf{e}_3 + \frac{x_2}{Re(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}[4x_1x_2u_{3,0}+(2x_1^{(2)}-2x_2^{(2)})u_{0,3}]\mathbf{e}_1 \\
& + \frac{x_2}{(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}[-p+\frac{2}{Re}(u_{1,0}+(x_1^{(2)}+3x_2^{(2)})u_{3,0}+2x_1x_2u_{0,3})]\mathbf{e}_2 \\
& + \frac{x_2}{Re(1-\epsilon\sin(cz))[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_2+\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_1\right. \\
& \left.+2Rex_2w_{2,0}\right]\mathbf{e}_3 \\
& + \frac{\epsilon c\cos(cz)}{R[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right. \\
& \left.+2Rex_1w_{2,0}\right]\mathbf{e}_1 \\
& + \frac{\epsilon c\cos(cz)}{Re[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_2+\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_1\right. \\
& \left.+2Rex_2w_{2,0}\right]\mathbf{e}_2 + \frac{\epsilon c\cos(cz)}{[1+(\epsilon c\cos(cz))^{(2)}]^{(1/2)}}\left[-p+2\left(\frac{\partial v}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]\mathbf{e}_3.
\end{aligned}
$$

Collecting together the unit vectors and using the boundary condition given by Eq. (3.5) leads to

$$
\begin{aligned}
\mathbf{t} = & \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(-px_1+\frac{2}{Re}x_1u_{1,0}+\frac{6}{Re}x_1(1-\epsilon\sin(cz))^{(2)}u_{3,0}\right. \\
& -\frac{2}{Re}(1-\epsilon\sin(cz))^{(2)}u_{0,3}+\frac{\epsilon c}{Re}\cos(cz)(1-\epsilon\sin(cz))\left(\left(\frac{\partial u_{1,0}}{\partial z}+(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)x_1\right. \\
& \left.\left.-\left(\frac{\partial u_{0,1}}{\partial z}+(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)x_2+2Rex_1w_{2,0}\right)\right)\mathbf{e}_1 \\
& + \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(\frac{6}{Re}x_2(1-\epsilon\sin(cz))^{(2)}u_{3,0}+\frac{2}{Re}x_1(1-\epsilon\sin(cz))^{(2)}u_{0,3}-px_2\right. \\
& -\frac{2}{Re}x_2u_{1,0}+\frac{\epsilon c}{Re}\cos(cz)(1-\epsilon\sin(cz))\left(\left(\frac{\partial u_{1,0}}{\partial z}+(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)x_2\right. \\
& \left.\left.+\left(\frac{\partial u_{0,1}}{\partial z}+(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)x_1+2Rex_2w_{2,0}\right)\right)\mathbf{e}_2 \\
& + \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(\frac{1}{Re}(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{1,0}}{\partial z}+\frac{1}{Re}(1-\epsilon\sin(cz))^{(4)}\frac{\partial u_{3,0}}{\partial z}\right. \\
& \left.+2(1-\epsilon\sin(cz))^{(2)}w_{2,0}+\epsilon c\cos(cz)(1-\epsilon\sin(cz))\left(-p+2\left(\frac{\partial v}{\partial z}+(1-\epsilon\sin(cz))^{(2)}\frac{\partial w_{2,0}}{\partial z}\right)\right)\right)\mathbf{e}_3.
\end{aligned}
$$

Then substituting in Eqs. (3.148) - (3.152) gives

$$
\begin{aligned}
\mathbf{t} = & \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(-px_1 - 2x_1\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.\\
& + 6x_1\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}\\
& - \frac{2}{Re}(1-\epsilon\sin(cz))^{(2)}u_{0,3} + \epsilon c\cos(cz)\left(\left(\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}} - \frac{A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.\right.\\
& + \left.\left.\left(-\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}} + \frac{5A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{1-\epsilon\sin(cz))^{(3)}}\right)\right)x_1\\
& - \frac{1}{Re}\left(-\frac{\partial u_{0,3}}{\partial z}(1-\epsilon\sin(cz))^{(2)} + 2u_{0,3}\epsilon c\cos(cz)(1-\epsilon\sin(cz)) + (1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)x_2\\
& \left.-2x_1\frac{A}{(1-\epsilon\sin(cz))^{(4)}}\right)\Bigg)\mathbf{e}_1\\
& + \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(6x_2\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.\\
& + \frac{2}{Re}x_1(1-\epsilon\sin(cz))^{(2)}u_{0,3} - px_2\\
& + 2x_2\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}} + \epsilon c\cos(cz)\left(\left(\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}} - \frac{A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.\right.\\
& + \left.\left.\left(-\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}} + \frac{5A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right)\right)x_2\\
& + \frac{1}{Re}\left(-\frac{\partial u_{0,3}}{\partial z}(1-\epsilon\sin(cz))^{(2)} + u_{0,3}\epsilon c\cos(cz)(1-\epsilon\sin(cz)) + (1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)x_1\\
& \left.-2x_2\frac{A}{(1-\epsilon\sin(cz))^{(4)}}\right)\Bigg)\mathbf{e}_2\\
& + \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(\left(\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))} - \frac{A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(2)}}\right)\right.\\
& + \left(-\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))} + \frac{5A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(2)}}\right)\\
& - 2\frac{A}{(1-\epsilon\sin(cz))^{(2)}} + \epsilon c\cos(cz)(1-\epsilon\sin(cz))(-p\\
& \left.+2\left(\frac{2A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}} - \frac{4A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right)\right)\Bigg)\mathbf{e}_3.
\end{aligned}
$$

Equating the two expressions for the stress vector gives

(3.153)

$$\left[\frac{1}{1-\epsilon\sin(cz)}\left(\frac{x_1(\tau_a\epsilon c\cos(cz)-p_e)}{\left(1+(\epsilon c\cos(cz))^{(2)}\right)^{(1/2)}}-x_2\tau_h\right)\right]$$
$$=\frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(-px_1-2x_1\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}+6x_1\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.$$
$$-\frac{2}{Re}(1-\epsilon\sin(cz))^{(2)}u_{0,3}+\epsilon c\cos(cz)\left(\left(\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}}-\frac{A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.\right.$$
$$\left.\left.+\left(-\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}}+\frac{5A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{1-\epsilon\sin(cz))^{(3)}}\right)\right)x_1\right.$$
$$-\frac{1}{Re}\left(-\frac{\partial u_{0,3}}{\partial z}(1-\epsilon\sin(cz))^{(2)}+2u_{0,3}\epsilon c\cos(cz)(1-\epsilon\sin(cz))+(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)x_2$$
$$\left.\left.-2x_1\frac{A}{(1-\epsilon\sin(cz))^{(4)}}\right)\right);$$

(3.154)

$$\left[\frac{1}{1-\epsilon\sin(cz)}\left(\frac{x_2(\tau_a\epsilon c\cos(cz)-p_e)}{\left(1+(\epsilon c\cos(cz))^{(2)}\right)^{(1/2)}}+x_1\tau_h\right)\right]$$
$$=\frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(6x_2\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}+\frac{2}{Re}x_1(1-\epsilon\sin(cz))^{(2)}u_{0,3}-px_2\right.$$
$$+2x_2\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}+\epsilon c\cos(cz)\left(\left(\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}}-\frac{A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right.\right.$$
$$\left.\left.+\left(-\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))^{(2)}}+\frac{5A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right)\right)x_2\right.$$
$$+\frac{1}{Re}\left(-\frac{\partial u_{0,3}}{\partial z}(1-\epsilon\sin(cz))^{(2)}+u_{0,3}\epsilon c\cos(cz)(1-\epsilon\sin(cz))+(1-\epsilon\sin(cz))^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)x_1$$
$$\left.\left.-2x_2\frac{A}{(1-\epsilon\sin(cz))^{(4)}}\right)\right);$$

(3.155)
$$
\begin{aligned}
&\left[\frac{\tau_a + p_e \epsilon c\cos(cz)}{\left(1+(\epsilon c\cos(cz))^{(2)}\right)^{(1/2)}}\right] \\
&= \frac{1}{(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}}\left(\left(\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))} - \frac{A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(2)}}\right)\right. \\
&+\left(-\frac{A\epsilon c^{(2)}\sin(cz)}{(1-\epsilon\sin(cz))} + \frac{5A\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)}{(1-\epsilon\sin(cz))^{(2)}}\right) \\
&-2\frac{A}{(1-\epsilon\sin(cz))^{(2)}} + \epsilon c\cos(cz)(1-\epsilon\sin(cz))(-p \\
&\left.+2\left(\frac{2A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}} - \frac{4A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}\right)\right)\bigg).
\end{aligned}
$$

Then rearranging Eq. (3.155) and simplifying gives

(3.156)
$$
\begin{aligned}
p &= \frac{1}{\epsilon c\cos(cz)(1-\epsilon^{(2)}\sin(cz))^{(3)}}\Big(\epsilon^{(7)}cp_e\cos^{(7)}(cz) - \epsilon^{(6)}\tau_a\cos^{(6)}(cz) + 3\epsilon^{(5)}cp_e(1-\epsilon^{(2)})\cos^{(5)}(cz) \\
&+\epsilon^{(4)}(6Ac^{(2)} + 2\epsilon c^{(2)}\sin(cz) - 3\tau_a + 3\epsilon^{(2)}\tau_a)\cos^{(4)}(cz) + 3\epsilon^{(3)}cp_e(\epsilon^{(2)}-1)^{(2)}\cos^{(3)}(cz) \\
&-\epsilon^{(2)}\Big(2\epsilon A(\epsilon^{(2)}c^{(2)} + 3c^{(2)} - 1)\sin(cz) + 3\tau_a(\epsilon^{(2)}-1)^{(2)} + 2A(2\epsilon^{(2)}c^{(2)} + c^{(2)} - 3)\Big)\cos^{(2)}(cz) \\
&-\epsilon c p_e(\epsilon^{(2)}-1)^{(3)}\cos(cz) - 2A\epsilon(\epsilon^{(2)}+3)\sin(cz) + \tau_a(\epsilon^{(2)}-1)^{(3)} - 2A(3\epsilon^{(2)}-1)\Big).
\end{aligned}
$$

Substituting this back into Eqs. (3.153) and (3.154) and setting $u_{0,3} = 0$, the following expressions for τ_a and τ_h can be obtained:

$$\tau_a = -\frac{2A(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))}{(1-\epsilon\sin(cz))^{(3)}};$$

$$\tau_h = 0.$$

Substituting these into Eq. (3.156) and simplifying gives

(3.157)
$$p = p_e.$$

From Eq. (3.42),

(3.158)
$$p_I = \pi p_e(1-\epsilon\sin(cz))^{(2)}.$$

From Eq. (3.43),

(3.159)
$$q_I = \frac{\pi}{4}p_e(1-\epsilon\sin(cz))^{(4)}.$$

Substituting into Eq. (3.147) and rearranging, then integrating with respect to z, an expression for p_e can be found as follows:

$$p_e = \frac{A\left[2(-18-\epsilon c^{(2)}\sin(cz)-13\epsilon^{(2)}c^{(2)}+14\epsilon^{(2)}c^{(2)}\sin^{(2)}(cz))(1-\epsilon\sin(cz))-7ReA\epsilon c\cos(cz)\right]}{36\epsilon c\cos(cz)(1-\epsilon\sin(cz))^{(4)}}.$$

This can be substituted into Eq. (3.157) to give the expression for the pressure p. Substituting Eqs. (3.148)-(3.152) into Eq. (3.45) gives the velocity as

$$\begin{aligned}\mathbf{v} = &\Bigg(\Bigg(-\frac{A\epsilon c\cos(cz)((-\epsilon^{(3)}\sin(cz)-3\epsilon^{(2)})\cos^{(2)}(cz)+(3\epsilon+\epsilon^{(3)})\sin(cz)+1+3\epsilon^{(2)})}{(\epsilon^{(2)}\cos^{(2)}(cz)+1-\epsilon^{(2)})^{(3)}}\\
&+\frac{1}{(\epsilon^{(2)}\cos^{(2)}(cz)+1-\epsilon^{(2)})^{(5)}}((x^{(2)}+y^{(2)})A\epsilon c\cos(cz)((\epsilon^{(5)}\sin(cz)+5\epsilon^{(4)})\cos^{(4)}(cz)\\
&+((-10\epsilon^{(3)}-2\epsilon^{(5)})\sin(cz)-10\epsilon 2-10\epsilon^{(4)})\cos^{(2)}(cz)+(5\epsilon+10\epsilon^{(3)}+\epsilon^{(5)})\sin(cz)\\
&+(1+10\epsilon^{(2)}+5\epsilon^{(4)})))\Bigg)x\Bigg)\mathbf{e}_1\\
&+\Bigg(\Bigg(-\frac{A\epsilon c\cos(cz)((-\epsilon^{(3)}\sin(cz)-3\epsilon^{(2)})\cos^{(2)}(cz)+(3\epsilon+\epsilon^{(3)})\sin(cz)+1+3\epsilon^{(2)})}{(\epsilon^{(2)}\cos^{(2)}(cz)+1-\epsilon^{(2)})^{(3)}}\\
&+\frac{1}{(\epsilon^{(2)}\cos^{(2)}(cz)+1-\epsilon^{(2)})^{(5)}}((x^{(2)}+y^{(2)})A\epsilon c\cos(cz)((\epsilon^{(5)}\sin(cz)+5\epsilon^{(4)})\cos^{(4)}(cz)\\
&+((-10\epsilon^{(3)}-2\epsilon^{(5)})\sin(cz)-10\epsilon^{(2)}-10\epsilon^{(4)})\cos^{(2)}(cz)+(5\epsilon+10\epsilon^{(3)}+\epsilon^{(5)})\sin(cz)\\
&+(1+10\epsilon^{(2)}+5\epsilon^{(4)})))\Bigg)y\Bigg)\mathbf{e}_2\\
&+\Bigg(-\frac{2A}{4\epsilon\sin(cz)+\epsilon^{(2)}\cos(2cz)-2-\epsilon^{(2)}}\\
&+\frac{2A(x^{(2)}+y^{(2)})}{4\epsilon\sin(cz)+\epsilon^{(2)}\cos(2cz)-2-\epsilon^{(2)})(1-\epsilon\sin(cz))^{(2)}}\Bigg)\mathbf{e}_3.\end{aligned} \tag{3.160}$$

As can be seen from Fig. 3.15, the shape of the velocity contours matches well between the $K=3$ director theory solution and the STAR-CCM+ simulation. Comparisons of the contours of the coaxial velocity are given in Fig. 3.16. In the case of a sinusoidal flow, a periodic condition could be set at the inlet for the simulation, rather than having to wait for the flow to become fully developed. In the finite volume simulation, μ and ρ were set to 1. The pipe had an inlet diameter of $2m$ and the finite volume mesh base size was set to $0.03m$

The error plots in Fig. 3.17 show reasonably good agreement between the order $K=3$ solution and the simulation with a maximum relative error of approximately 6.5% in the $Re=2$ case and 4.5% in the $Re=4$ case.

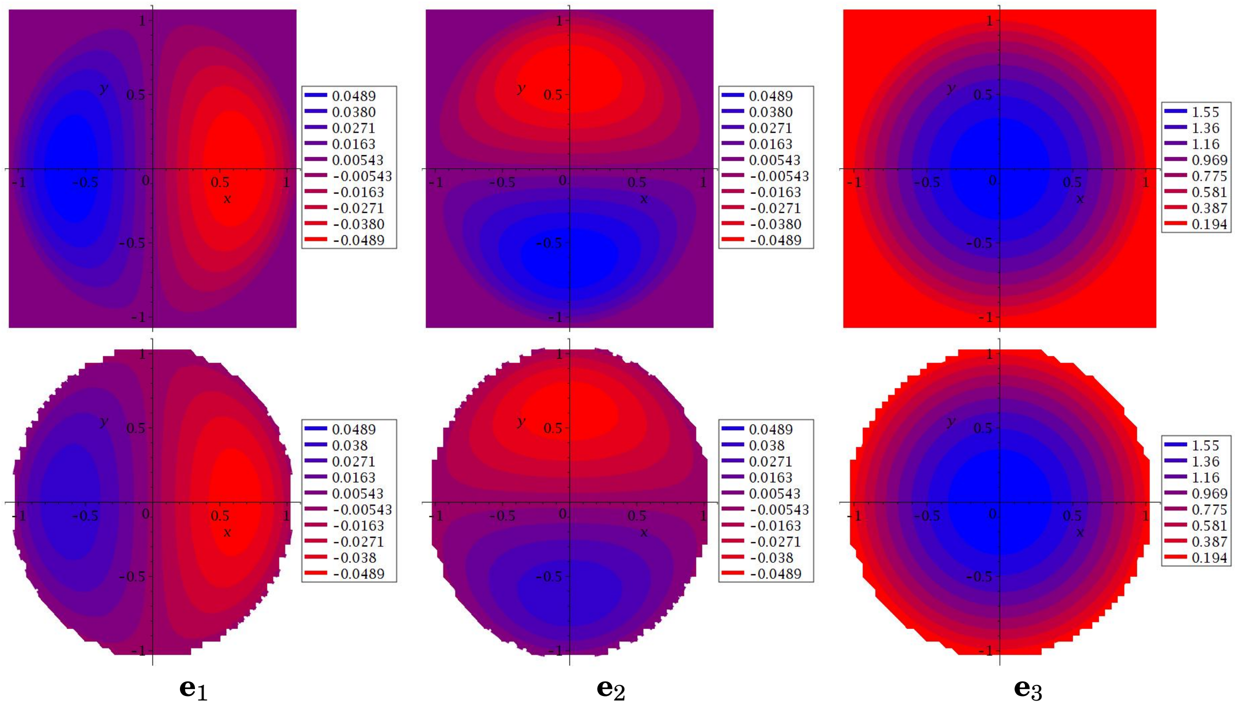


Figure 3.15: Contour plots of flow through a sinusoidal pipe cross-section, given by Eq. (3.160) for order $K = 3$ (top) and from a corresponding simulation in STAR-CCM+ (bottom), at location $z = 4.3768$, with $Re = 2$ at the inflow section, $\epsilon = 0.1$, $c = 2\pi/5$, in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

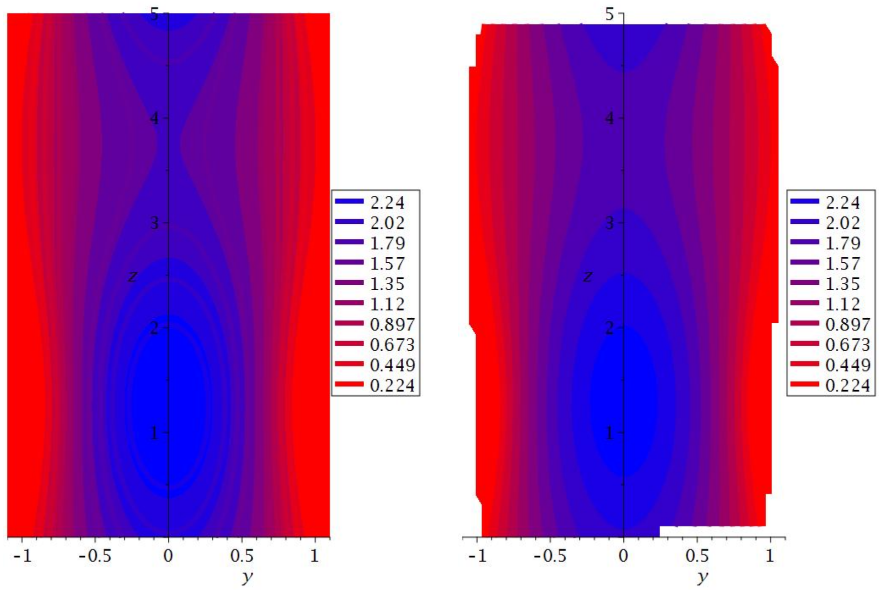


Figure 3.16: Contour plots of the coaxial flow ($\mathbf{e}_3$ direction) along a sinusoidal pipe cross-section, given by Eq. (3.160) for $K = 3$ (left) and from a corresponding simulation in STAR-CCM+ (right), with $Re = 2$ at the inflow section.

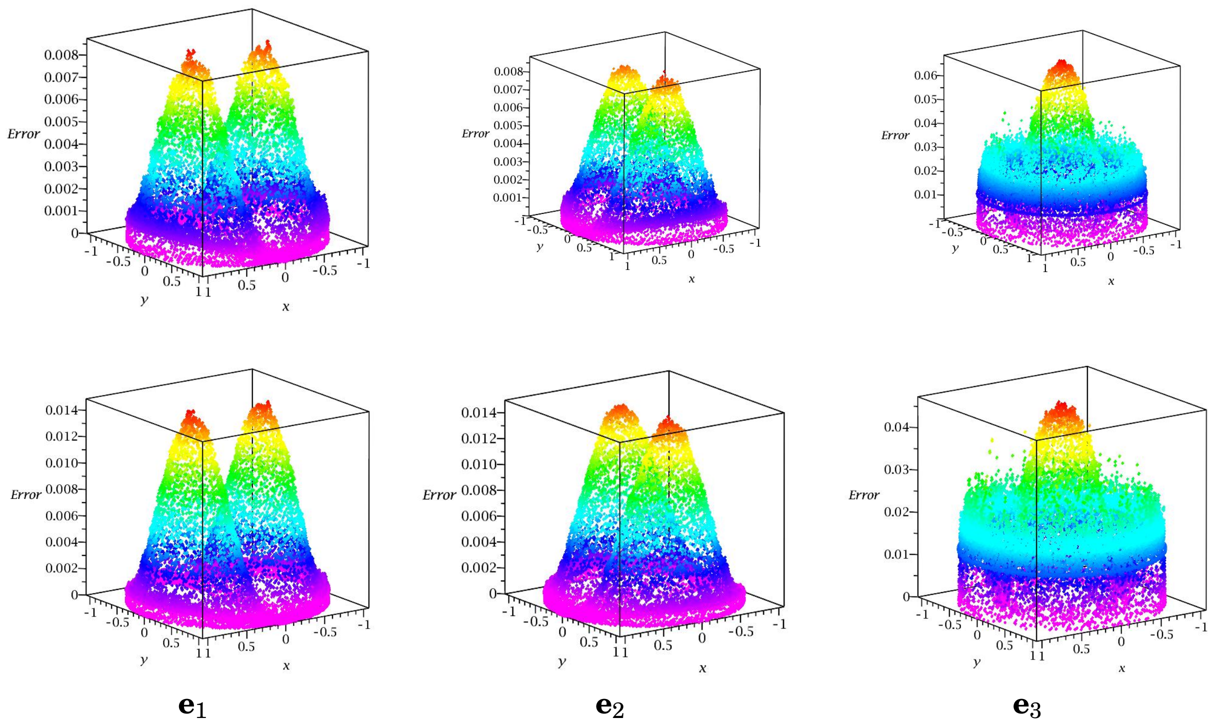


Figure 3.17: Plots of the relative error, calculated by Eq. (2.18), in a cross-section of the sinusoidal pipe, between the 3D finite volume simulation using a periodic boundary condition and the solution given by Eq. (3.160) for $K = 3$. Solution are obtained for $Re = 2$ (top) and $Re = 4$ (bottom), $\epsilon = 0.1$, $c = 2\pi/5$ in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

3.5.2.2 K=5

By assuming there is no swirling flow, $u_{0,1}$, $u_{0,3}$ and $u_{0,5}$ are set to 0. The radius for the sinusoidal pipe is given by Eq. (3.146). Motivated by Poiseuille flow, it is assumed that

$$w_{0,0} = \frac{A}{(1-\epsilon\sin(cz))^{(2)}}, \tag{3.161}$$

where A is a constant. Then, substituting Eq. (3.161) into Eq. (3.47) and rearranging gives

$$u_{1,0} = -\frac{A\epsilon c Re\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}}. \tag{3.162}$$

Then substituting in Eqs. (3.146), (3.161), (3.162) and solving Eqs.(3.48), (3.49), (3.109), (3.111) simultaneously gives

$$u_{3,0} = \frac{\epsilon c Re(A+B)\cos(sz)}{(1-\epsilon\sin(cz))^{(5)}}; \tag{3.163}$$

(3.164)
$$u_{5,0} = -\frac{\epsilon c Re A\cos(cz)}{(1-\epsilon\sin(cz))^{(7)}};$$

(3.165)
$$w_{2,0} = -\frac{A+B}{(1-\epsilon\sin(cz))^{(4)}};$$

(3.166)
$$w_{4,0} = \frac{B}{(1-\epsilon\sin(cz))^{(6)}};$$

where B is a constant of integration. Then equating the expressions for the stress tensor gives

(3.167)
$$\frac{1}{(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}(1-\epsilon\sin(cz))^{(6)}}\Big(-(1-\epsilon\sin(cz))^{(5)}(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}\tau_h y$$
$$+\Big(c\epsilon^{(5)}(\tau_a\epsilon\sin(cz)-5\tau_a+2(A-B)c^{(2)})\cos^{(5)}(cz)$$
$$+\epsilon^{(4)}(p_e-p)(\epsilon\sin(cz)-5)\cos^{(4)}(cz)+2s\epsilon^{(3)}\Big(\epsilon\Big(-(\epsilon^{(2)}+5)\tau_a+2(A-B)c^{(2)}\Big)\sin(cz)+5(\epsilon^{(2)}+1)\tau_a$$
$$-(\epsilon^{(2)}c^{(2)}+c^{(2)}-1)(A-B)\Big)\cos^{(3)}(cz)-2(p_e-p)\epsilon^{(2)}\Big((\epsilon^{(3)}+5\epsilon)\sin(cz)-5(\epsilon^{(2)}+1)\Big)\cos^{(2)}(cz)$$
$$+c\epsilon\Big(\epsilon\Big((\epsilon^{(4)}+10\epsilon^{(2)}+5)\tau_a+4(A-B)\Big)\sin(cz)-(5\epsilon^{(4)}-10\epsilon^{(2)}-1)\tau_a-2(\epsilon^{(2)}+1)(A-B)\Big)\cos(cz)$$
$$+\Big((\epsilon^{(5)}+10\epsilon^{(3)}+5\epsilon)\sin(cz)-5\epsilon^{(4)}-10\epsilon^{(2)}-1\Big)(p_e-p)\Big)x\Big)=0;$$

(3.168)
$$\frac{1}{(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}(1-\epsilon\sin(cz))^{(6)}}\Big(-(1-\epsilon\sin(cz))^{(5)}(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}\tau_h x$$
$$+\Big(c\epsilon^{(5)}(\tau_a\epsilon\sin(cz)-5\tau_a+2(A-B)c^{(2)})\cos^{(5)}(cz)$$
$$+\epsilon^{(4)}(p_e-p)(\epsilon\sin(cz)-5)\cos^{(4)}(cz)+2c\epsilon^{(3)}\Big(\epsilon\Big(-(\epsilon^{(2)}+5)\tau_a+2(A-B)c^{(2)}\Big)\sin(cz)+5(\epsilon^{(2)}+1)\tau_a$$
$$-(\epsilon^{(2)}c^{(2)}+c^{(2)}-1)(A-B)\Big)\cos^{(3)}(cz)-2(p_e-p)\epsilon^{(2)}\Big((\epsilon^{(3)}+5\epsilon)\sin(cz)-5(\epsilon^{(2)}+1)\Big)\cos^{(2)}(cz)$$
$$+c\epsilon\Big(\epsilon\Big((\epsilon^{(4)}+10\epsilon^{(2)}+5)\tau_a+4(A-B)\Big)\sin(cz)-(5\epsilon^{(4)}-10\epsilon^{(2)}-1)\tau_a-2(\epsilon^{(2)}+1)(A-B)\Big)\cos(cz)$$
$$+\Big((\epsilon^{(5)}+10\epsilon^{(3)}+5\epsilon)\sin(cz)-5\epsilon^{(4)}-10\epsilon^{(2)}-1\Big)(p_e-p)\Big)y\Big)=0;$$

(3.169)
$$\frac{1}{(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))^{(1/2)}(1-\epsilon^{(2)}\sin^{(2)}(cz))^{(3)}}\Big(-c\epsilon^{(7)}(p_e-p)\cos^{(7)}(cz)+\tau_a\epsilon^{(6)}\cos^{(6)}(cz)$$
$$+3\epsilon^{(5)}c(\epsilon^{(2)}-1)(p_e-p)\cos^{(5)}(cz)-\Big(2\epsilon c^{(2)}(A-B)\sin(cz)+3(\epsilon^{(2)}-1)\tau_a+6(A-B)c^{(2)}\Big)\epsilon^{(4)}\cos^{(4)}(cz)$$
$$-3c\epsilon^{(3)}(\epsilon^{(2)}-1)^{(2)}(p_e-p)\cos^{(3)}(cz)+2\Big(\epsilon(\epsilon^{(2)}c^{(2)}+3c^{(3)}-1)(A-B)\sin(cz)+(3\epsilon^{(4)}-6\epsilon^{(2)}+3)\tau_a$$
$$+2(A-B)(3\epsilon^{(2)}c^{(2)}+c^{(2)}-3)\Big)\epsilon^{(2)}\cos^{(2)}(cz)+\epsilon c(\epsilon^{(2)}-1)^{(3)}(p_e-p)\cos(cz)+2\epsilon(\epsilon^{(2)}+3)(A-B)\sin(cz)$$
$$-(\epsilon^{(2)}-1)^{(3)}\tau_a+2((3\epsilon^{(2)}+1)(A-B)\Big)=0.$$

Rearranging Eq. (3.169) gives the following expression:

$$
\begin{aligned}
p = {} & \frac{1}{\epsilon c\cos(cz)(1-\epsilon^{(2)}\sin^{(2)}(cz))^{(3)}}\Big(c\epsilon^{(7)}p_e)\cos^{(7)}(cz) - \tau_a\epsilon^{(6)}\cos^{(6)}(cz) \\
& -3\epsilon^{(5)}c(\epsilon^{(2)}-1)p_e\cos^{(5)}(cz) + \Big(2\epsilon c^{(2)}(A-B)\sin(cz) + 3(\epsilon^{(2)}-1)\tau_a + 6(A-B)c^{(2)}\Big)\epsilon^{(4)}\cos^{(4)}(cz) \\
& +3c\epsilon^{(3)}(\epsilon^{(2)}-1)^{(2)}p_e\cos^{(3)}(cz) - 2\Big(\epsilon(\epsilon^{(2)}c^{(2)} + 3c^{(3)} - 1)(A-B)\sin(cz) + (3\epsilon^{(4)} - 6\epsilon^{(2)} + 3)\tau_a \\
& +2(A-B)(3\epsilon^{(2)}c^{(2)} + c^{(2)} - 3)\Big)\epsilon^{(2)}\cos^{(2)}(cz) - \epsilon c(\epsilon^{(2)}-1)^{(3)}p_e\cos(cz) - 2\epsilon(\epsilon^{(2)}+3)(A-B)\sin(cz) \\
& +(\epsilon^{(2)}-1)^{(3)}\tau_a - 2((3\epsilon^{(2)}+1)(A-B)\Big).
\end{aligned}
$$

Substituting this back into Eq. (3.167) and equating the coefficients of y gives $\tau_h = 0$. Substituting back into Eq. (3.167) and equating the coefficients of x gives

$$
\tau_a = -\frac{(A-B)(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz))}{(1-\epsilon\sin(cz))^{(3)}}. \tag{3.170}
$$

Substituting Eq. (3.170) back into the expression for p and simplifying gives $p = p_e$. The expressions for p_I and q_I for a sinusoidal pipe are given by Eqs (3.158) and (3.159). The expression for h_I, defined by Eq. (3.44), for a sinusoidal pipe is given by

$$
h_I = \frac{\pi p_e}{8}(1-\epsilon\sin(cz))^{(6)}.
$$

Substituting the velocity terms and pressure terms into Eq. (3.56) and rearranging gives

$$
\frac{dp_e}{dz} = -\frac{60(A-B)(1-\epsilon\sin(cz))(1+\epsilon^{(2)}c^{(2)}\cos^{(2)}(cz)) + \epsilon c Re(10A^{(2)} - 5AB + B^{(2)})\cos(cz)}{15(1-\epsilon\sin(cz))^{(5)}}.
$$

Integrating gives a complicated expression for p_e. Substituting the velocity and pressure terms into Eq. (3.62) and simplifying gives

(3.171)

$$
\begin{aligned}
& 8\epsilon^{(4)}\left(A - \frac{7}{5}B\right)c^{(2)}\cos^{(8)}(cz/2) - 16\epsilon^{(4)}\left(A - \frac{7}{5}\right)c^{(2)}\cos^{(6)}(cz/2) + \epsilon^{(3)}\left(-\frac{53}{5}B + 7A\right)c^{(2)}\sin(cz/2)\cos^{(5)}(cz/2) \\
& + \frac{3}{2}\left(c^{(2)}\left(7A - \frac{143}{15}B\right)\epsilon^{(2)} + \left(-\frac{2}{3}A + \frac{22}{15}B\right)c^{(2)} - \frac{4}{3}B\right)\epsilon^{(2)}\cos^{(4)}(cz/2) \\
& - \left(\left(-\frac{53}{5}B + 7A\right)c\epsilon + \frac{1}{30}(7A-B)^{(2)}Re\right)\epsilon^{(2)}c\sin(cz/2)\cos^{(3)}(cz/2) \\
& + \epsilon\left(\left(-\left(\frac{5}{2}A - \frac{31}{10}B\right)c^{(2)}\epsilon^{(3)} + \left(A - \frac{11}{5}B\right)c^{(2)} + 2B\right)\epsilon + \frac{1}{60}(7A-B)^{(2)}Res\right)\cos^{(2)}(cz/2) \\
& + \epsilon\left(\left(\frac{5}{2}A - \frac{31}{10}B\right)c^{(2)}\epsilon^{(2)} + \frac{1}{60}(7A-B)^{(2)}Rec\epsilon + \left(\frac{1}{4}A - \frac{9}{60}B\right)c^{(2)} - 2B\right)\sin(cz/2)\cos(cz/2) \\
& - \left(\frac{5}{8}A - \frac{31}{40}B\right)c^{(2)}\epsilon^{(2)} - \frac{1}{120}(7A-B)^{(2)}Rec\epsilon + \frac{1}{2}B = 0.
\end{aligned}
$$

The left hand side is the expression for the error in this equation. Now, express B in the form

$$B = c_0 + c_1\epsilon + c_2\epsilon^{(2)} + O(\epsilon^{(3)}), \tag{3.172}$$

where c_0,c_1,c_2 etc are constants to be chosen such that the error if Eq. (3.171) in minimised.

Then, to eliminate the error at first order in Eq. (3.171), it is required that $c_0 = 0$. To minimise the error at order ϵ, it is required that

$$c_1 = \frac{7A^{(2)}Rec}{60}.$$

As there is an error of order ϵ and ϵ is small, coefficients of orders of higher powers of ϵ will have minimal effect on the error in Eq. (3.171). So the expression for B is now given by

$$B = \frac{7\epsilon A^{(2)}Rec}{60}. \tag{3.173}$$

Substituting Eq.(3.173) back into Eqs. (3.162) - (3.166) gives

$$u_{3,0} = \frac{\epsilon c Re(60A + 7\epsilon A^{(2)}Rec)\cos(cz)}{60(1-\epsilon\sin(cz))^{(5)}};$$
$$u_{5,0} = -\frac{7c^{(2)}\epsilon^{(2)}Re^{(2)}A^{(2)}\cos(cz)}{60(1-\epsilon\sin(cz))^{(7)}};$$
$$w_{2,0} = -\frac{60A + 7A^{(2)}Rec\epsilon}{60(1-\epsilon\sin(cz))^{(4)}};$$
$$w_{4,0} = \frac{7A^{(2)}Rec\epsilon}{60(1-\epsilon\sin(cz))^{(6)}}.$$

Then substituting into the expression for the velocity at order $K = 5$, given by Eq. (3.3) gives the velocity as

$$\begin{aligned}\mathbf{v} = x_1\Bigg[&-\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}} + (x_1^{(2)} + x_2^{(2)})\frac{\epsilon c(60A + 7\epsilon A^{(2)}Rec)\cos(cz)}{60(1-\epsilon\sin(cz))^{(5)}}\\ &-(x_1^{(2)} + x_2^{(2)})^{(2)}\frac{7c^{(2)}\epsilon^{(2)}ReA^{(2)}\cos(cz)}{60(1-\epsilon\sin(cz))^{(7)}}\Bigg]\mathbf{e}_1\\ +x_2\Bigg[&-\frac{A\epsilon c\cos(cz)}{(1-\epsilon\sin(cz))^{(3)}} + (x_1^{(2)} + x_2^{(2)})\frac{\epsilon c(60A + 7\epsilon A^{(2)}Rec)\cos(cz)}{60(1-\epsilon\sin(cz))^{(5)}}\\ &-(x_1^{(2)} + x_2^{(2)})^{(2)}\frac{7c^{(2)}\epsilon^{(2)}ReA^{(2)}\cos(cz)}{60(1-\epsilon\sin(cz))^{(7)}}\Bigg]\mathbf{e}_2\\ +\Bigg[&\frac{A}{(1-\epsilon\sin(cz))^{(2)}} - (x_1^{(2)} + x_2^{(2)})\frac{60A + 7A^{(2)}Rec\epsilon}{60(1-\epsilon\sin(cz))^{(4)}} + (x_1^{(2)} + x_2^{(2)})^{(2)}\frac{7A^{(2)}Rec\epsilon}{60(1-\epsilon\sin(cz))^{(6)}}\Bigg]\mathbf{e}_3.\end{aligned} \tag{3.174}$$

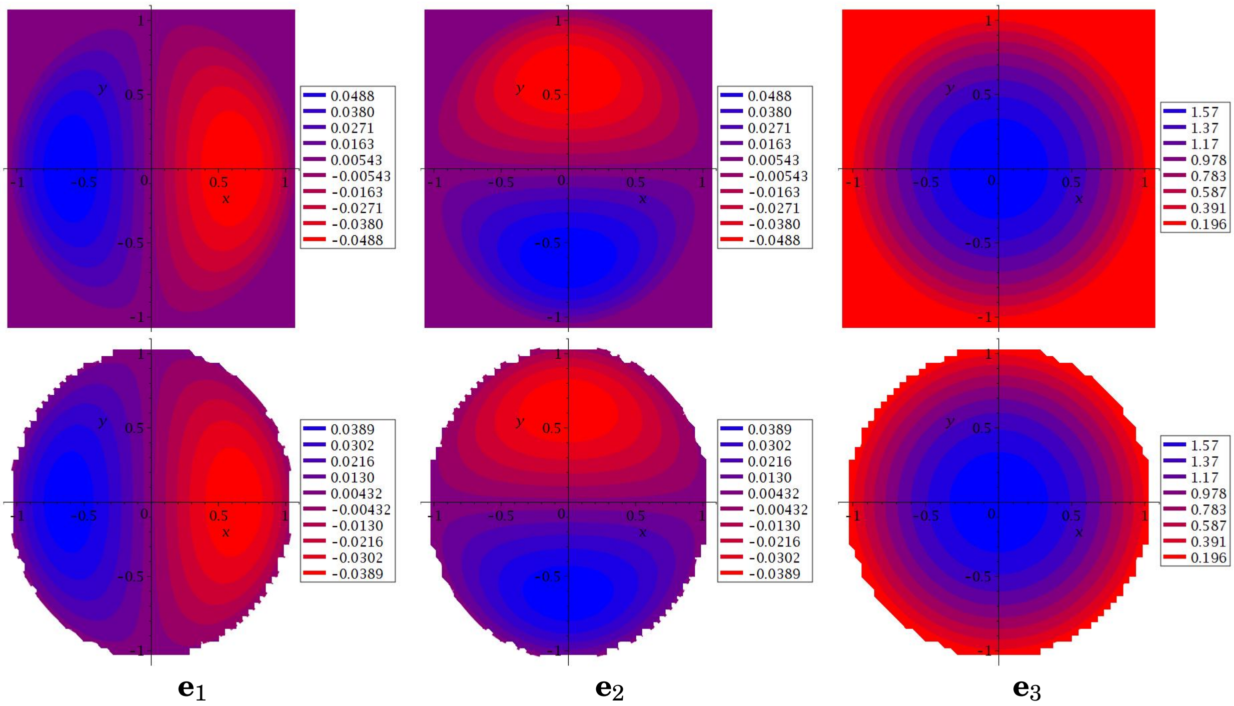


Figure 3.18: Contour plots of flow through a sinusoidal pipe cross-section, given by Eq. (3.174) for $K = 5$ (top) and from a corresponding simulation using STAR-CCM+ (bottom) using a periodic boundary condition, at location $z = 4.3768$, with $Re = 2$ at the inflow section, $\epsilon = 0.1$, $c = 2\pi/5$ in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

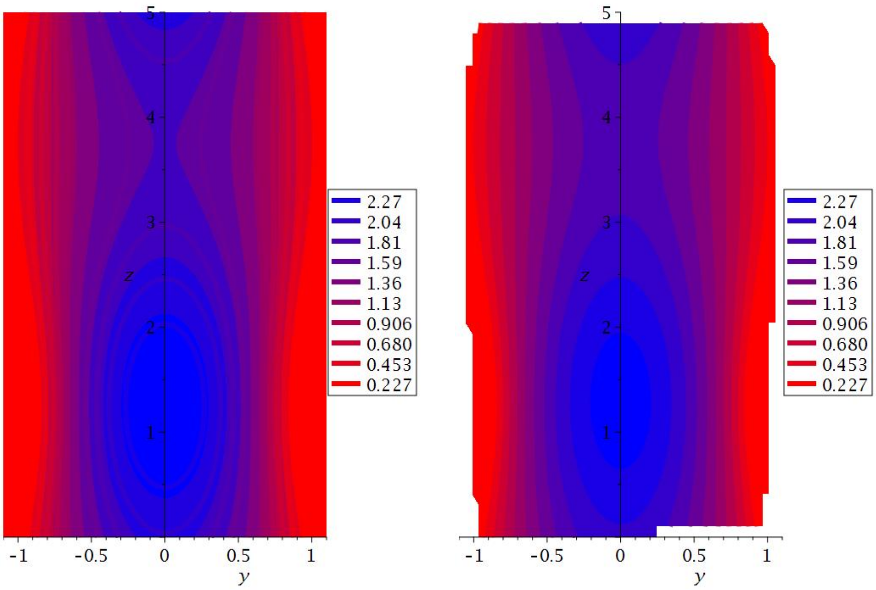
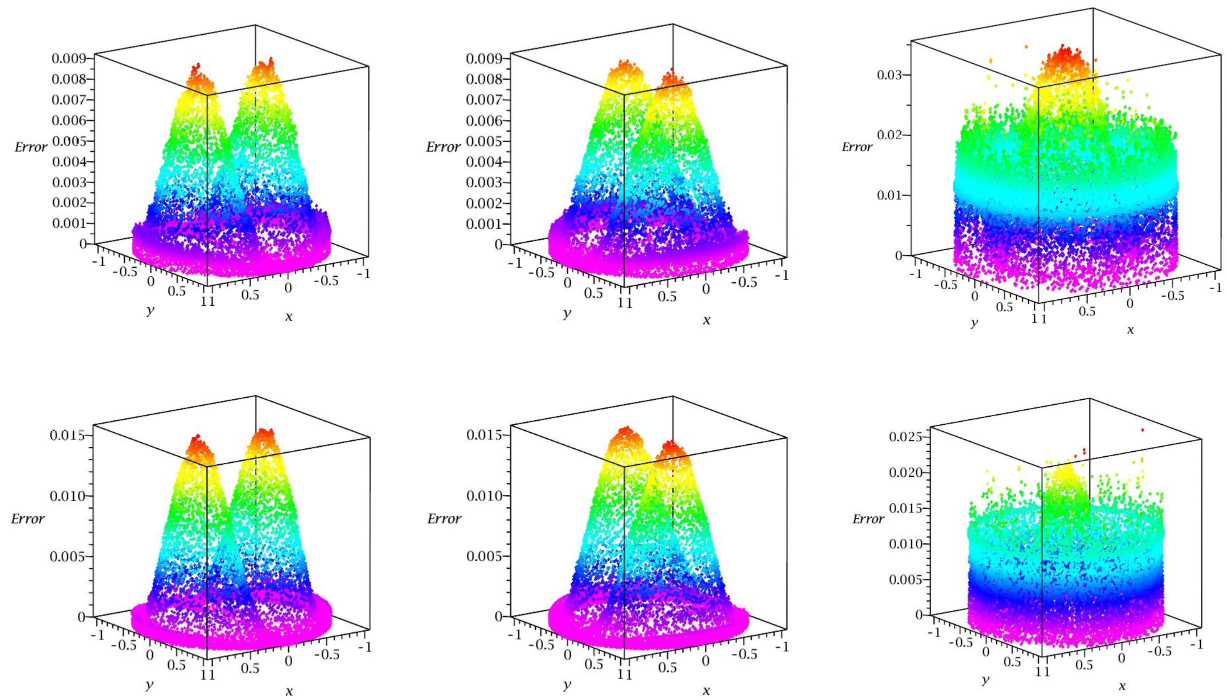


Figure 3.19: Contour plots of coaxial flow ($\mathbf{e}_3$ directions) along a sinusoidal pipe cross-section, given by Eq. (3.174) for $K = 5$ (left) and from a corresponding simulation using STAR-CCM+ (right), with $Re = 2$ at the inflow section.



$\mathbf{e}_1$ $\mathbf{e}_2$ $\mathbf{e}_3$

Figure 3.20: Plots of the relative error, calculated by Eq. (2.18), in a cross-section of the sinusoidal pipe, between the 3D finite volume simulation using a periodic boundary condition and the solution given by Eq. (3.174) for $K = 5$. Solutions are obtained for $Re = 2$ (top) and $Re = 4$ (bottom), $\epsilon = 0.1$, $c = 2\pi/5$ in the $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ directions respectively.

The similarity in the velocity contours can be seen in Figs. 3.18 and 3.20. It can be seen that the error of the coaxial velocity ($\mathbf{e}_3$) direction is reduced when the velocity expansion order is increased from $K = 3$ (Fig. 3.17) to $K = 5$ (Fig. 3.20). In the $Re = 2$ case, the maximum relative error decreases from approximately 6.5% to 3.5%. In the $Re = 4$ case, the maximum relative error decreases from approximately 4.5% to 2.5%. There is also an error close to the wall, as was seen and discussed in the case of the tapered pipe.

3.6 Summary of Chapter

This chapter presented a director theory approach for modelling fluid flow in a straight axisymmetrical pipe. This was done by starting from the standard equations of motion of incompressible and Newtonian fluid flow (Navier-Stokes, continuity), and then assuming that the fluid velocity could be approximated by a summation of weighting function (polynomials of the in-plane coordinates) multiplied by directors (vector functions of the coaxial coordinate and time). By assuming axisymmetry, the number of unknown directors can be reduced. In this case, this was done by converting to cylindrical polar coordinates, assuming the velocity was independent of θ, which put restrictions on the values of some of the directors. Other conditions assumed include zero velocity at the wall.

Then the conservation of momentum equations are integrated over the cross-section of the pipe rather than solved for point-wise. The equations are multiplied by each respective weighting function and integrated. This results in a system of partial differential equations in terms of the coaxial coordinate and time. The number of equation depends on the order K chosen for the velocity expansion. In principle the expansion can be carried out to higher orders of K, however only the cases K=3 and K=5 have been included here. Additional equations arising from boundary and incompressibility conditions complete the system. At this point, the radius of the pipe is only considered a generic function of the coaxial coordinate and time. For specific geometries, the system can be simplified and solved.

Results are presented in this chapter for some specific geometries. First a pipe of constant radius is considered, and it is shown that Poiseuille flow can be recovered if consistent assumptions are made. Then a steady decaying swirling flow is obtained. Then the case where the radius varies along the pipe is considered. Solutions are recovered for flow in a tapered and sinusoidal pipe and compared with results from three-dimensional finite volume simulations. In both cases the director theory solutions matched the simulations well, with typical maximum relative errors of around 5%. Two orders of velocity expansion were considered in these cases, it would be interesting to see results at higher orders and to see whether the error decreases and if flow separation occurs, although the difficulty of obtaining an analytic solution increases with higher orders. The higher relative errors close to the wall due to the velocity tending to zero at these points, it would also be interesting to see if there is any change in the near wall error at higher orders.

This work could be extended by considering a pipe radius that varies with time as well as along the pipe. At the point that system of partial differential equations is derived, this model allows a radius that depends on time, but some of the simplifications that followed in the geometries that were solved in this chapter would not apply, leaving a more complicated system to solve, however this should be possible with numerical techniques. Another extension would be to have a non-circular cross-section, in which case the simplifications made based on axisymmetry would not apply. Considering a non-Newtonian fluid would also be of interest, and again is possible in the approach but would lead to more complex algebraic equations.

CHAPTER 4

TOROIDALLY CURVED PIPE

4.1 Introduction

Having investigated flow in straight axisymmetric pipes, the attention is now turned to curved pipes. To gain an understanding of how director theory can be applied to curved pipes, the work of Green et al [7] was studied, where director theory is applied to fluid flow in a toroidally curved pipe. The same approach is taken in the setup of the relevant local coordinate system, but a different approach is taken in the determination of the equations of motion which are to be solved for. In [7] the equations of motion are expressed in terms of various kinetic variables representing, for example force vectors. It was decided to take a different approach and work directly from the Navier-Stokes equations in terms of the usual variables of velocity and pressure, as this seemed more appropriate and intuitive.

The first step is to consider a coordinate system that followes the centreline of the pipe and then to derive the relevant boundary and incompressibility conditions and equations of motion in this coordinate system. The fluid velocity is again assumed to be well approximated by a series of weighting function (polynomials of the in-plane coordinates) multiplied by directors. The directors are in general vector functions of the coaxial coordinate and time, but in this case of a pipe of constant radius and curvature, a fully developed steady flow was assumed, so they reduce to constants. The number of unknown directors are reduced through condition of planar symmetry, incompressibility and boundary conditions. The equations of motion are multiplied by each weighting

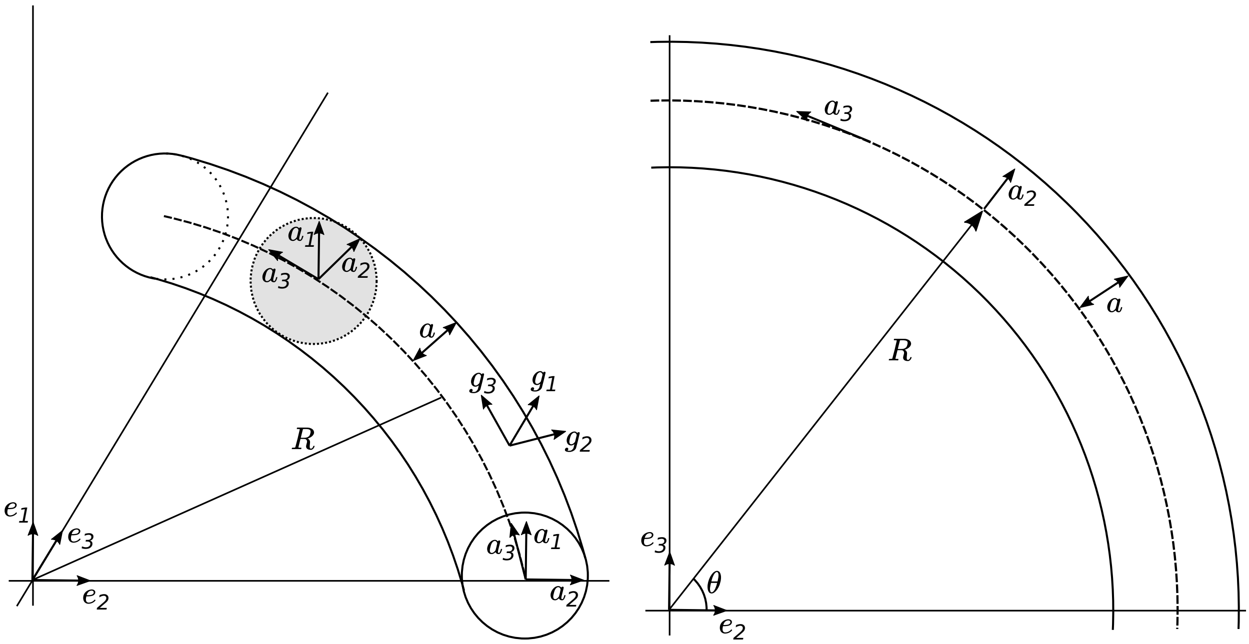


Figure 4.1: Setup for a toroidal pipe, with Cartesian basis ($\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$), curvilinear covariant basis ($\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3$) and corresponding orthonormal basis ($\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$), where a is the radius of the pipe and R is the distance from the centre of curvature to the centreline of the pipe [7].

function in turn and integrated over the cross-section of the pipe, to form a system of quadratic equations where the unknown directors are solved for. Putting these values back in the velocity expansion then gives the velocity profile. The results from the director theory model is compared with that from 3D finite volume simulations at a range of Reynolds numbers.

4.2 Setup

Consider a torus pipe with its centre at the origin, its centreline about $x^1 = 0$ and R is the constant radius from the centre of the torus to the centreline. The first step is to find the curvilinear coordinates (ζ^1,ζ^2,ζ^3) for this case.

Choose the coordinates such that the centreline of the torus is always in the ζ^3 direction and that the ζ^1,ζ^2 coordinates are in the cross-section. With the stated configuration, choosing $\zeta^1 = x^1$, ζ^1 always points upwards in the cross section, as can be seen from Fig 4.1. Choose $\zeta^3 = R\theta$, θ is the angle between the $\mathbf{e}_2$ and $\mathbf{a}_2$ vectors in a clockwise direction, as shown in Fig 4.1. Then the centreline of the torus follows the ζ^3 direction. From this,

$$\tan(\zeta^3/R) = x^3/x^2.$$

So

$$\zeta^3 = R\arctan(x^3/x^2).$$

Then from Fig 4.1,

$$\zeta^2 = (x_2^{(2)} + x_3^{(2)})^{(1/2)} - R.$$

This gives a set of orthogonal curvilinear coordinates. The inverse is given by

$$x^1 = \zeta^1;$$

$$x^2 = (R+\zeta^2)\cos(\zeta^3/R);$$

$$x^3 = (R+\zeta^2)\sin(\zeta^3/R).$$

The position vector to any point in the fluid is given by

$$\begin{aligned}\mathbf{r} &= x^i\mathbf{e}_i\\ &= \zeta^1\mathbf{e}_1 + (R+\zeta^2)\cos(\zeta^3/R)\mathbf{e}_2 + (R+\zeta^2)\sin(\zeta^3/R)\mathbf{e}_3.\end{aligned}$$

The covariant base vectors, given by Eq. (2.5), are found to be

$$\mathbf{g}_1 = \mathbf{e}_1; \tag{4.1}$$

$$\mathbf{g}_2 = \cos(\zeta^3/R)\mathbf{e}_2 + \sin(\zeta^3/R)\mathbf{e}_3; \tag{4.2}$$

$$\mathbf{g}_3 = -\frac{R+\zeta^2}{R}\sin(\zeta^3/R)\mathbf{e}_2 + \frac{R+\zeta^2}{R}\cos(\zeta^3/R)\mathbf{e}_3; \tag{4.3}$$

and the contravariant base vectors, defined by Eq. (2.6), to be

$$\mathbf{g}^1 = \mathbf{e}_1;$$

$$\mathbf{g}^2 = \cos(\zeta^3/R)\mathbf{e}_2 + \sin(\zeta^3/R)\mathbf{e}_3;$$

$$\mathbf{g}^3 = -\frac{R}{R+\zeta^2}\sin(\zeta^3/R)\mathbf{e}_2 + \frac{R}{R+\zeta^2}\cos(\zeta^3/R)\mathbf{e}_3.$$

From Eq. (2.7)

$$g^{(1/2)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\zeta^3/R) & -\frac{R+\zeta^2}{R}\sin(\zeta^3/R) \\ 0 & \sin(\zeta^3/R) & \frac{R+\zeta^2}{R}\cos(\zeta^3/R) \end{vmatrix}$$
$$= \frac{R+\zeta^2}{R}\cos^{(2)}(\zeta^3/R) + \frac{R+\zeta^2}{R}\sin^{(2)}(\zeta^3/R)$$
$$= \frac{R+\zeta^2}{R}.$$

A set of orthonormal basis vectors $\mathbf{a}_i$, as shown in Fig. 4.1, are given by

$$\mathbf{a}_1 = \mathbf{e}_1; \tag{4.4}$$

$$\mathbf{a}_2 = \cos(\zeta^3/R)\mathbf{e}_2 + \sin(\zeta^3/R)\mathbf{e}_3; \tag{4.5}$$

$$\mathbf{a}_3 = -\sin(\zeta^3/R)\mathbf{e}_2 + \cos(\zeta^3/R)\mathbf{e}_3. \tag{4.6}$$

Let ϕ be the fixed radius of the pipe. Then the lateral surface can be represented by

$$\phi^{(2)} - \zeta^{1^2} - \zeta^{2^2} = 0.$$

The fluid velocity, given by Eq. (2.8), can rewritten as

$$\mathbf{v}(\zeta^1,\zeta^2,\zeta^3,t) = \mathbf{v}^*(\zeta^3,t) + \sum_{N=1}^{K} \lambda_N^i(\zeta^1,\zeta^2)\mathbf{w}_N^i(\zeta^3,t), \tag{4.7}$$

recalling that the $\mathbf{w}_N^i$ are the director velocities and the $\lambda_N^i(\zeta^1,\zeta^2)$ could be considered as shape functions. In this case $\mathbf{v}^*(\zeta^3,t)$ is the velocity along the centreline of the toroidal pipe and $\mathbf{w}_N^i(\zeta^3,t) = w_N^i(\zeta^3,t)\mathbf{g}_i$. Expressing this in terms of the orthonormal basis $\mathbf{a}_i$, with $\mathbf{v}^* = v_i^*\mathbf{a}_i$ and $\mathbf{w}_N^i = w_{Nj}^i\mathbf{a}_j$ (To expand on this, $\mathbf{w}_N^i = w_N^i\mathbf{g}_i$, so w_N^i for each $i = 1,2,3$ is the component of the Nth director in the $\mathbf{g}_i$ direction. Now each of these $\mathbf{g}_i$ components of the Nth director are being split into their respective $\mathbf{a}_j$ components for $j = 1,2,3$, so w_{Nj}^i is the component of the Nth director which is in the $\mathbf{g}_i$ direction which is in the $\mathbf{a}_j$ direction.), gives

$$\mathbf{v} = v_j^*(\zeta^3,t)\mathbf{a}_j + \sum_{N=1}^{K} \lambda_N^i(\zeta^1,\zeta^2)w_{Nj}^i(\zeta^3,t)\mathbf{a}_j. \tag{4.8}$$

Assuming an incompressible flow, the incompressibility condition is given by

$$\frac{\partial \mathbf{v}}{\partial x^i}\cdot\mathbf{e}_i = 0.$$

To express this in terms of the curvilinear coordinates ζ^i, consider that

$$\frac{\partial \mathbf{v}}{\partial x^i} = \frac{\partial \mathbf{v}}{\partial \zeta^j}\frac{\partial \zeta^j}{\partial x^i}$$

and the Cartesian basis $\mathbf{e}_i$ can be expressed in terms of the basis $\mathbf{a}_i$ as

$$\mathbf{e}_1 = \mathbf{a}_1;$$

$$\mathbf{e}_2 = \cos(\zeta^3/R)\mathbf{a}_2 - \sin(\zeta^3/R)\mathbf{a}_3;$$

$$\mathbf{e}_3 = \sin(\zeta^3/R)\mathbf{a}_2 + \cos(\zeta^3/R)\mathbf{a}_3.$$

As $\mathbf{a}_2$ and $\mathbf{a}_3$ depend on ζ^3, they will be affected by differentiation with respect to ζ^3 as follows:

$$\begin{aligned}\frac{\partial \mathbf{a}_2}{\partial \zeta^3} &= -\frac{1}{R}\sin(\zeta^3/R)\mathbf{e}_2 + \frac{1}{R}\cos(\zeta^3/R)\mathbf{e}_3\\ &= \frac{1}{R}\mathbf{a}_3;\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{a}_3}{\partial \zeta^3} &= -\frac{1}{R}\cos(\zeta^3/R)\mathbf{e}_2 - \frac{1}{R}\sin(\zeta^3/R)\mathbf{e}_3\\ &= -\frac{1}{R}\mathbf{a}_2.\end{aligned}$$

Then

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial x^1}\cdot\mathbf{e}_1 &= \left(1\frac{\partial \mathbf{v}}{\partial \zeta^1} + 0\frac{\partial \mathbf{v}}{\partial \zeta^2} + 0\frac{\partial \mathbf{v}}{\partial \zeta^3}\right)\cdot\mathbf{a}_1\\ &= \frac{\partial \mathbf{v}}{\partial \zeta^1}\cdot\mathbf{a}_1\\ &= \left(\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^1}w_{Nj}^i\mathbf{a}_j\right)\cdot\mathbf{a}_1\\ &= \sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^1}w_{N1}^i;\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial x^2}\cdot\mathbf{e}_2 &= \left(0\frac{\partial \mathbf{v}}{\partial \zeta^1}+\cos(\zeta^3/R)\frac{\partial \mathbf{v}}{\partial \zeta^2}-\frac{R}{R+\zeta^2}\sin(\zeta^3/R)\frac{\partial \mathbf{v}}{\partial \zeta^3}\right)\cdot\left(\cos(\zeta^3/R)\mathbf{a}_2-\sin(\zeta^3/R)\mathbf{a}_3\right)\\
&= \left(\cos(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^2}w_{Nj}^i\mathbf{a}_j-\frac{R}{R+\zeta^2}\sin(\zeta^3/R)\left[\frac{\partial v_j^*}{\partial \zeta^3}\mathbf{a}_j+\frac{v_2^*}{R}\mathbf{a}_3-\frac{v_3^*}{R}\mathbf{a}_2+\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{Nj}^i}{\partial \zeta^3}\mathbf{a}_j\right.\right.\\
&\quad\left.\left.+\sum_{N=1}^{K}\frac{\lambda_N^i w_{N2}^i}{R}\mathbf{a}_3-\sum_{N=1}^{K}\frac{\lambda_N^i w_{N3}^i}{R}\mathbf{a}_2\right]\right)\cdot\left(\cos(\zeta^3/R)\mathbf{a}_2-\sin(\zeta^3/R)\mathbf{a}_3\right)\\
&= \cos^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^2}w_{N2}^i-\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^2}w_{N3}^i-\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\frac{\partial v_2^*}{\partial \zeta^3}\\
&\quad+\frac{R}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)\frac{\partial v_3^*}{\partial \zeta^3}+\frac{v_2^*}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)+\frac{v_3^*}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\\
&\quad-\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N2}^i}{\partial \zeta^3}+\frac{R}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N3}^i}{\partial \zeta^3}\\
&\quad+\frac{1}{R+\zeta^2}\cos\zeta^3/R\sin\zeta^3/R\sum_{N=1}^{K}\lambda_N^i w_{N3}^i+\frac{1}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i w_{N2}^i;
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \mathbf{v}}{\partial x^3}\cdot\mathbf{e}_3 &= \left(0\frac{\partial \mathbf{v}}{\partial \zeta^1}+\sin(\zeta^3/R)\frac{\partial \mathbf{v}}{\partial \zeta^2}+\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\frac{\partial \mathbf{v}}{\partial \zeta^3}\right)\cdot\left(\sin(\zeta^3/R)\mathbf{a}_2+\cos(\zeta^3/R)\mathbf{a}_3\right)\\
&= \left(\sin(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^2}w_{Nj}^i\mathbf{a}_j+\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\left[\frac{\partial v_j^*}{\partial \zeta^3}\mathbf{a}_j+\frac{v_2^*}{R}\mathbf{a}_3-\frac{v_3^*}{R}\mathbf{a}_2+\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{Nj}^i}{\partial \zeta^3}\mathbf{a}_j\right.\right.\\
&\quad\left.\left.+\sum_{N=1}^{K}\frac{\lambda_N^i w_{N2}^i}{R}\mathbf{a}_3-\sum_{N=1}^{K}\frac{\lambda_N^i w_{N3}^i}{R}\mathbf{a}_2\right]\right)\cdot\left(\sin(\zeta^3/R)\mathbf{a}_2+\cos(\zeta^3/R)\mathbf{a}_3\right)\\
&= \sin^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^2}w_{N2}^i+\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial \lambda_N^i}{\partial \zeta^2}w_{N3}^i+\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\frac{\partial v_2^*}{\partial \zeta^3}\\
&\quad+\frac{R}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)\frac{\partial v_3^*}{\partial \zeta^3}+\frac{v_2^*}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)-\frac{v_3^*}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\\
&\quad+\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N2}^i}{\partial \zeta^3}+\frac{R}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N3}^i}{\partial \zeta^3}\\
&\quad+\frac{1}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i w_{N2}^i-\frac{1}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i w_{N3}^i.
\end{aligned}
$$

So, $\frac{\partial \mathbf{v}}{\partial x^i}\cdot\mathbf{e}_i = 0$ gives

$$
\begin{aligned}
&\sum_{N=1}^{K}\frac{\partial\lambda_N^i}{\partial\zeta^1}w_{N1}^i+\cos^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial\lambda_N^i}{\partial\zeta^2}w_{N2}^i-\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial\lambda_N^i}{\partial\zeta^2}w_{N3}^i\\
&-\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\frac{\partial v_2^*}{\partial\zeta^3}\\
&+\frac{R}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)\frac{\partial v_3^*}{\partial\zeta^3}+\frac{v_2^*}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)+\frac{v_3^*}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\\
&-\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N2}^i}{\partial\zeta^3}+\frac{R}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N3}^i}{\partial\zeta^3}\\
&+\frac{1}{R+\zeta^2}\cos\zeta^3/R\sin\zeta^3/R\sum_{N=1}^{K}\lambda_N^i w_{N3}^i+\frac{1}{R+\zeta^2}\sin^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i w_{N2}^i\\
&+\sin^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial\lambda_N^i}{\partial\zeta^2}w_{N2}^i+\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\frac{\partial\lambda_N^i}{\partial\zeta^2}w_{N3}^i+\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\frac{\partial v_2^*}{\partial\zeta^3}\\
&+\frac{R}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)\frac{\partial v_3^*}{\partial\zeta^3}+\frac{v_2^*}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)-\frac{v_3^*}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\\
&+\frac{R}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N2}^i}{\partial\zeta^3}+\frac{R}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i\frac{\partial w_{N3}^i}{\partial\zeta^3}\\
&+\frac{1}{R+\zeta^2}\cos^{(2)}(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i w_{N2}^i-\frac{1}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\sum_{N=1}^{K}\lambda_N^i w_{N3}^i=0.
\end{aligned}
$$

Simplifying using trigonometric identities gives

$$\frac{v_2^*}{R+\zeta^2}+\frac{R}{R+\zeta^2}\frac{\partial v_3^*}{\partial\zeta^3}+\sum_{N=1}^{K}\left[\frac{\partial\lambda_N^i}{\partial\zeta^1}w_{N1}^i+\frac{\partial\lambda_N^i}{\partial\zeta^2}w_{N2}^i+\frac{\lambda_N^i w_{N2}^i}{R+\zeta^2}+\frac{R\lambda_N^i}{R+\zeta^2}\frac{\partial w_{N3}^i}{\partial\zeta^3}\right]=0.$$

Multiplying by $R+\zeta^2$ gives

$$v_2^*+R\frac{\partial v_3}{\partial\zeta^3}+\sum_{N=1}^{K}\left[\left(R+\zeta^2\right)\frac{\partial\lambda_N^i}{\partial\zeta^1}w_{N1}^i+\frac{\partial}{\partial\zeta^2}\left(\left(R+\zeta^2\right)\lambda_N^i\right)w_{N2}^i+R\lambda_N^i\frac{\partial w_{N3}^i}{\partial\zeta^3}\right]=0.$$

It is noted that this differs from the corresponding equation representing the incompressibility condition in [7], this is specifically Eq. 2.15 in Green et al's paper. However, that equation was found to be dimensionally inconsistent, so the present result provides more confidence. The dimensional inconsistently of Eq. 2.15 in [7] was discovered within the work of this thesis, by performing a dimensionality check, after extensive effort and checks failed to derive the same equation, and continued to yield the equation above.

A fully developed flow would satisfy

$$\frac{\partial}{\partial\zeta^3}(\mathbf{v}\cdot\mathbf{a}_i)=0.$$

The satisfaction of this condition can be readily guaranteed by insisting on the vanishing of the gradient with respect to ζ^3 of the velocity $\mathbf{v}^*$ and the director velocities $\mathbf{w}^i_N$ [7].

The velocity components in the cross-flow directions should be equal, so there is no flow out of the walls, so set $w^1_{N1} = w^2_{N2}$. From Eqs. (4.1) - (4.6)

$$\mathbf{g}_1 = \mathbf{a}_1;$$

$$\mathbf{g}_2 = \mathbf{a}_2;$$

$$\mathbf{g}_3 = \frac{R+\zeta^2}{R}\mathbf{a}_3.$$

As w^i_{Nj} is the $\mathbf{a}_j$ component of the $\mathbf{g}_i$ vector, this implies that

$$w^1_{N2} = w^1_{N3} = w^2_{N1} = w^2_{N3} = w^3_{N1} = w^3_{N2} = 0.$$

It is noted that a different justification is offered for why these terms are set to zero in [7], namely that setting some of the director velocities equal to zero still allows for the specification on a fairly general set of weighting function in the velocity expansion and at the same time will result in a simple system of equations of motion.

Given that a term such as $\mathbf{v}^*(\zeta^3, t)$ can be represented by any one of the series terms (when ζ^1 and ζ^2 are 0), without loss of generality, set $\mathbf{v}^* = 0$ [7]. Note that, for a steady flow and with the previous simplifications, the expression for velocity can now be written in the reduced form

$$\mathbf{v} = \sum_{N=1}^{K} \lambda^i_N(\zeta^1, \zeta^2) w^i_{Ni} \mathbf{a}_i. \tag{4.9}$$

4.3 Choice of Weighting Functions

Now the focus is on choosing the weighting functions $\lambda^i_N(\zeta^1, \zeta^2)$. Green et al [7] suggest considering a particular polynomial form for the velocity of the form

$$\mathbf{v} = \sum_{j=0}^{H} \sum_{i=0}^{j} (\zeta^1)^{(i)} (\zeta^2)^{(j-i)} a^k_{ij} \mathbf{a}_k \tag{4.10}$$

for $k = 1,2,3$, where H is the hierarchical order of the theory. So the a^k_{ij} coefficients represent magnitudes of each polynomial term, in the $\mathbf{a}_k$ direction, for $i = 1,2,3$ in the velocity expansion.

This clearly already satisfies the conditions for a steady and fully developed flow, as the expression is independent of t and ζ^3. Restrictions for the coefficients a^k_{ij} are found through the requirements of satisfying other conditions such as no-slip and incompressibility. Another condition that Green et al [7] suggest should be satisfied is flow symmetry about the $\zeta^2 - \zeta^3$ plane (i.e. the osculating plane). This means $\mathbf{v}(\zeta^1, \zeta^2)$ must satisfy the following:

$$\begin{aligned}
\mathbf{v}(\zeta^1,\zeta^2)\cdot\mathbf{a}_1 &= -\mathbf{v}(-\zeta^1,\zeta^2)\cdot\mathbf{a}_1;\\
\mathbf{v}(\zeta^1,\zeta^2)\cdot\mathbf{a}_2 &= \mathbf{v}(-\zeta^1,\zeta^2)\cdot\mathbf{a}_2;\\
\mathbf{v}(\zeta^1,\zeta^2)\cdot\mathbf{a}_3 &= \mathbf{v}(-\zeta^1,\zeta^2)\cdot\mathbf{a}_3.
\end{aligned}$$

For Eq. (4.10) to satisfy these conditions, it is required that

- $a^1_{ij} = 0$ when i is even;
- $a^2_{ij} = 0$ when i is odd;
- $a^3_{ij} = 0$ when i is odd.

Given this, the expression given by Eq. (4.10) can be rewritten as

$$\begin{aligned}
\mathbf{v} = &\frac{1}{2}\sum_{j=0}^{H}\sum_{i=0}^{j}((\zeta^1)^i - (-\zeta^1)^i)(\zeta^2)^{(j-i)}a^1_{ij}\mathbf{a}_1 + \frac{1}{2}\sum_{j=0}^{H}\sum_{i=0}^{j}((\zeta^1)^i + (-\zeta^1)^i)(\zeta^2)^{(j-i)}a^2_{ij}\mathbf{a}_2\\
&+\frac{1}{2}\sum_{j=0}^{H}\sum_{i=0}^{j}((\zeta^1)^i + (-\zeta^1)^i)(\zeta^2)^{(j-i)}a^3_{ij}\mathbf{a}_3.
\end{aligned} \tag{4.11}$$

This will be borne in mind when satisfying the no-slip and incompressibility conditions. The incompressibility condition is given by

$$\begin{aligned}
\frac{\partial\mathbf{v}}{\partial x^i}\cdot\mathbf{e}_i &= \frac{\partial\mathbf{v}}{\partial\zeta^j}\frac{\partial\zeta^j}{\partial x^i}\\
&= \frac{\partial\mathbf{v}}{\partial\zeta^1}\cdot\mathbf{a}_1 + \frac{\partial\mathbf{v}}{\partial\zeta^2}\frac{\partial\zeta^2}{\partial x^2}\cdot(\cos(\zeta^3/R)\mathbf{a}_2 - \sin(\zeta^3/R)\mathbf{a}_3) + \frac{\partial\mathbf{v}}{\partial\zeta^2}\frac{\partial\zeta^2}{\partial x^3}\cdot(\sin(\zeta^3/R)\mathbf{a}_2 + \cos(\zeta^3/R)\mathbf{a}_3)\\
&= \frac{\partial v_1}{\partial\zeta^1} + \cos^{(2)}(\zeta^3/R)\frac{\partial v_2}{\partial\zeta^2} - \cos(\zeta^3/R)\sin(\zeta^3/R)\frac{\partial v_3}{\partial\zeta^2} + \frac{v_2}{R+\zeta^2}\sin^{(2)}(\zeta^3/R) + \frac{v_3}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\\
&+\frac{\partial v_2}{\partial\zeta^2}\sin^{(2)}(\zeta^3/R) + \frac{\partial v_3}{\partial\zeta^2}\sin(\zeta^3/R)\cos(\zeta^3/R) + \frac{v_2}{R+\zeta^2}\cos^{(2)}(\zeta^3/R) - \frac{v_3}{R+\zeta^2}\cos(\zeta^3/R)\sin(\zeta^3/R)\\
&= \frac{\partial v_1}{\partial\zeta^1} + \frac{\partial v_2}{\partial\zeta^2} + \frac{v_2}{R+\zeta^2} = 0.
\end{aligned}$$

So, in curvilinear coordinates, the incompressibility condition is

$$\frac{\partial v_1}{\partial\zeta^1} + \frac{\partial v_2}{\partial\zeta^2} + \frac{v_2}{R+\zeta^2} = 0. \tag{4.12}$$

The no-slip condition is given by $\mathbf{v}(\zeta^1,\zeta^2) = \mathbf{0}$ on the surface $(\zeta^1)^{(2)} + (\zeta^2)^{(2)} = r^{(2)}$.

It is not easy to identify the restrictions that need to be made on a^k_{ij} in order for Eq. (4.10) to satisfy these conditions for a general H. So to proceed, these conditions are satisfied for specific values of H to see if a pattern emerges. This has been done with the aid of Maple for $H = 1,2,3,4,5$.

First substitute Eq. (4.11) into Eq. (4.12). To satisfy Eq. (4.12), set the coefficients of each of the terms in the resulting polynomial to 0. This gives a system of simultaneous equations for a^1_{ij} and a^2_{ij} terms. Reduce the number of unknown terms by expressing some of them in terms of others. Once this has been done, substitute the terms that there are expressions for back into Eq. (4.11). Next, satisfy the no-slip condition by setting $\mathbf{v}(\zeta^1,(r^{(2)}-(\zeta^1)^{(2)})^{(1/2)}) = \mathbf{0}$, or equivalently $\mathbf{v}((r^{(2)}-(\zeta^2)^{(2)})^{(1/2)},\zeta^2) = 0$. Taylor expand the resulting expression, in order to be able to equate powers of ζ^1. Set the coefficients of the powers of ζ^1 to 0 in each direction. Again, this creates more simultaneous equations, through which the number of unknown a^k_{ij} terms can be reduced. The remaining unknown terms will need to be later solved for by substituting into the equations of motion. For now, an expression for $\mathbf{v}$ has been obtained with fewer numbers of unknown terms.

This process will be shown for the case $H = 2$.

Substituting $H = 2$ into Eq. (4.11) gives

(4.13)
$$\mathbf{v} = (a^1_{11}\zeta^1+a^1_{12}\zeta^1\zeta^2)\mathbf{a}_1+(a^2_{00}+a^2_{01}\zeta^2+a^2_{02}(\zeta^2)^{(2)}+a^2_{22}(\zeta^1)^{(2)})\mathbf{a}_2+(a^3_{00}+a^3_{01}\zeta^2+a^3_{02}(\zeta^2)^{(2)}+a^3_{22}(\zeta^1)^{(2)})\mathbf{a}_3.$$

Substituting this into Eq. (4.12) gives

$$\frac{\partial}{\partial\zeta^1}(a^1_{11}\zeta^1+a^1_{12}\zeta^1\zeta^2)+\frac{\partial}{\partial\zeta^2}(a^2_{00}+a^2_{01}\zeta^2+a^2_{02}(\zeta^2)^{(2)}+a^2_{22}(\zeta^1)^{(2)})+\frac{a^2_{00}+a^2_{01}\zeta^2+a^2_{02}(\zeta^2)^{(2)}+a^2_{22}(\zeta^1)^{(2)}}{R+\zeta^2} = 0.$$

Carrying out the differentiation gives

$$a^1_{11}+a^1_{12}\zeta^2+a^2_{01}+2a^2_{02}\zeta^2+\frac{a^2_{00}+a^2_{01}\zeta^2+a^2_{02}(\zeta^2)^{(2)}+a^2_{22}(\zeta^1)^{(2)}}{R+\zeta^2} = 0.$$

As $(R+\zeta^2) > 0$, this can be rewritten as

$$(R+\zeta^2)(a^1_{11}+a^1_{12}\zeta^2+a^2_{01}+2a^2_{02}\zeta^2)+a^2_{00}+a^2_{01}\zeta^2+a^2_{02}(\zeta^2)^{(2)}+a^2_{22}(\zeta^1)^{(2)} = 0;$$

or equivalently as

$$Ra^1_{11}+Ra^2_{01}+a^2_{00}+(a^1_{11}+Ra^1_{12}+2a^2_{01}+2Ra^2_{02})\zeta^2+(a^1_{12}+3a^2_{02})(\zeta^2)^{(2)}+a^2_{22}(\zeta^1)^{(2)} = 0.$$

This can be satisfied by setting the coefficients of each power of ζ^1 and ζ^2 to 0, giving the following:

$$\begin{aligned} Ra^1_{11} + Ra^2_{01} + a^2_{00} &= 0; \\ a^1_{11} + Ra^1_{12} + 2a^2_{01} + 2Ra^2_{02} &= 0; \\ a^1_{12} + 3a^2_{02} &= 0; \\ a^2_{22} &= 0. \end{aligned}$$

Hence the following relations can be deduced:

$$\begin{aligned} a^1_{12} &= -3a^2_{02}; \\ a^1_{11} &= Ra^2_{02} - 2a^2_{01}; \\ a^2_{00} &= R(3a^2_{01} - Ra^2_{02}). \end{aligned}$$

Now substituting these expressions, along with $a^2_{22} = 0$ back into Eq. (4.13) gives

$$\begin{aligned} \mathbf{v} = &((R + a^2_{02} - 2a^2_{01})\zeta^1 - 2a^2_{02}\zeta^1\zeta^2)\mathbf{a}_1 + (R(3a^2_{01} - Ra^2_{02}) + a^2_{01}\zeta^2 + a^2_{02}(\zeta^2)^{(2)})\mathbf{a}_2 \\ &+ (a^3_{00} + a^3_{01}\zeta^2 + a^3_{02}(\zeta^2)^{(2)} + a^3_{22}(\zeta^1)^{(2)})\mathbf{a}_3. \end{aligned} \tag{4.14}$$

Now, satisfy the no-slip condition by setting $\mathbf{v}((r^{(2)} - (\zeta^2)^{(2)})^{(1/2)}, \zeta^2) = 0$. This gives

$$\begin{aligned} v_1((r^{(2)} - (\zeta^2)^{(2)})^{(1/2)}, \zeta^2) &= (r^{(2)} - (\zeta^2)^{(2)})^{(1/2)}[Ra^2_{02} - 3a^2_{02}\zeta^2 - 2a^2_{01}] = 0; \\ v_2((r^{(2)} - (\zeta^2)^{(2)})^{(1/2)}, \zeta^2) &= 3Ra^2_{01} + a^2_{01}\zeta^2 - r^{(2)}a^2_{02} + a^2_{02}(\zeta^2)^{(2)} = 0; \\ v_3((r^{(2)} - (\zeta^2)^{(2)})^{(1/2)}, \zeta^2) &= a^3_{00} + a^3_{01}\zeta^2 + a^3_{02}(\zeta^2)^{(2)} + a^3_{22}(r^{(2)} - (\zeta^2)^{(2)}) = 0. \end{aligned}$$

Using the expression for v_2 gives

$$a^2_{01} = \frac{R - (\zeta^2)^{(2)}}{3R + \zeta^2} a^2_{02}.$$

Substituting this into the expression for v_1 gives

$$\frac{(r^{(2)} - (\zeta^2)^{(2)})^{(1/2)}}{3R + \zeta^2} a^2_{02}(3r^{(2)} - 2R + R\zeta^2 - (\zeta^2)^{(2)} - 9R(\zeta^2)^{(2)} - 3a^2_{02}(\zeta^2)^{(3)}) = 0.$$

Therefore $a^2_{02} = 0$ and due to the above relations, $a^1_{ij} = 0$ and $a^2_{ij} = 0$ for all i and j. To satisfy the expression for v_3, set the coefficients of the powers of ζ^2 to 0. This gives the following:

$$\begin{aligned} a^3_{00} + r^{(2)}a^3_{22} &= 0; \\ a^3_{01} &= 0; \\ a^3_{02} - a^3_{22} &= 0. \end{aligned}$$

Hence the following relations are obtained:

$$a^3_{00} = -r^{(2)}a^3_{22};$$
$$a^3_{02} = a^3_{22}.$$

Substituting everything back into Eq. (4.14) gives

$$\mathbf{v} = a^3_{22}((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_3.$$

The following results were obtained by considering different H.

When $H = 1$,

$$\mathbf{v} = \mathbf{0}.$$

When $H = 2$,

$$\mathbf{v} = a^3_{22}((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_3.$$

When $H = 3$,

$$\mathbf{v} = (a^3_{22} + a^3_{03}\zeta^2)((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_3.$$

When $H = 4$,

$$\mathbf{v} = [a^3_{22} + a^3_{44}r^{(2)} + a^3_{03}\zeta^2 + a^3_{04}(\zeta^2)^{(2)} + a^3_{44}(\zeta^1)^{(2)}]((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_3.$$

When $H = 5$,

$$\begin{aligned}\mathbf{v} = &\frac{a^1_{35}\zeta^1}{4}(2R\zeta^2 - r^{(2)} + (\zeta^1)^{(2)} + 3(\zeta^2)^{(2)})((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_1\\
&+ \frac{a^1_{35}}{8}(R + \zeta^2)(r^{(2)} - 5(\zeta^1)^{(2)} - (\zeta^2)^{(2)})((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_2\\
&+ (a^3_{22} + a^3_{44}r^{(2)} + (a^3_{25}r^{(2)} - a^3_{45}r^{(2)} + a^3_{03})\zeta^2 + a^3_{04}(\zeta^2)^{(2)} + a^3_{44}(\zeta^1)^{(2)}\\
&+ (a^3_{25} - a^3_{45})(\zeta^2)^{(3)} + a_{45}(\zeta^1)^{(2)}\zeta^2)((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - r^{(2)})\mathbf{a}_3.\end{aligned}$$

Note that for $H = 1$ to 4, $a^1_{ij}, a^2_{ij} = 0$ for all i, j, hence there is no flow in the $\mathbf{a}_1$ and $\mathbf{a}_2$ directions. These expressions are consistent with the results of Green et al [7], who provide forumule for the weighting functions λ^i_N. Green et al [7] state that after these substitutions into the incompressibility and no-slip conditions, the velocity, for general H is given by

(4.15)
$$\mathbf{v} = \sum_{N=1}^{L}(\lambda^1_N c^1_N \mathbf{a}_1 + \lambda^2_N c^2_N \mathbf{a}_2) + \sum_{N=1}^{K}\lambda^3_N c^3_N \mathbf{a}_3$$

where L is the reduced number of directors in the cross-section and K is the number of coaxial directors. The values for L and K are given by

$$L = \begin{cases} 0 & \text{for } H < 5 \\ \frac{1}{4}((H-3)^{(2)} - 1) & \text{for even } H \\ \frac{1}{4}(H-3)^{(2)} & \text{for odd } H \end{cases}$$

and

$$K = \begin{cases} \frac{1}{4}H^2 & \text{for even } H \\ \frac{1}{4}(H^2 - 1) & \text{for odd } H. \end{cases}$$

In [7], formulae are given for λ_N^i where i_N and j_N appear as exponents. An algorithm is given for calculating the values of i_N and j_N from the value of N. Plots of these weighting functions, from Maple, are given in Figs. 4.2 to 4.4, for $N = 1$ to 12. These figures show the individual shapes of the weighting functions, which will be combined together to form the velocity profile. To get back to the form of Eq. (4.9), Green et al [7] set $\lambda_N^1 = \lambda_N^2 = 0$ for $N > L$ and then $c_i^1 = w_{Ni}^i$. Note however, that in using this approach, the form of λ_N^1 and λ_N^2, for each particular N will change based on the value of H. To avoid this, it is instead proposed here that $\lambda_N^1 = \lambda_N^2 = 0$ for $N < 6$. Using this approach, the forms of λ_N^i and w_{Ni}^i for the velocity given above when $H = 5$ are shown in Table 4.1 for clarity, where a_{ij}^k are the coeficients, as represented in Eq. (4.10).

Note that as the order of the expansion is increased in the representation of $\mathbf{v}$, λ_N^i has not changed, however w_{N3}^3 has sometimes been adjusted by a term multiplied by $r^{(2)}$, w_{13}^3 for example, suggesting that for a thin tube, these additional terms represent a minor adjustment.

$\lambda_1^3 = (\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2;$	$w_{13}^3 = a_{22}^3 + a_{44}^3 a^2;$
$\lambda_2^3 = \zeta^2((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{23}^3 = a_{03}^3 + a_{25}^3 a^2 - a_{45}^3 a^2;$
$\lambda_3^3 = (\zeta^2)^{(2)}((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{33}^3 = a_{04}^3;$
$\lambda_4^3 = (\zeta^1)^{(2)}((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{43}^3 = a_{44}^3;$
$\lambda_5^3 = (\zeta^2)^{(3)}((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{53}^3 = a_{25}^3 - a_{45}^3;$
$\lambda_6^3 = (\zeta^1)^{(2)}\zeta^2((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{63}^3 = a_{45}^3;$
$\lambda_6^1 = 2\zeta^1(2R\zeta^2 - a^2 + (\zeta^1)^{(2)} + 3(\zeta^2)^{(2)})((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{61}^1 = \frac{a_{35}^1}{8};$
$\lambda_6^2 = (R + \zeta^2)(a^2 - 5(\zeta^1)^{(2)} - (\zeta^2)^{(2)})((\zeta^1)^{(2)} + (\zeta^2)^{(2)} - a^2);$	$w_{62}^2 = \frac{a_{35}^1}{8}.$

Table 4.1: Forms of the weighting functions for order $H = 5$

4.4 Equations of motion

The equations of motion for fluid flow are generally comprised of the Navier-Stokes equations, which represent conservation of linear momentum and the continuity equation, which represents conservation of mass. The typical representation of the Navier-Stokes equations in vector form is

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^2 \mathbf{v},$$

where p is pressure, ρ is density and ν is kinematic viscosity. An integrated version, over the cross-section, is generally taken in the director theory approach, as is the case here.

If an order of $H = 7$ is chosen for the velocity expansion, this corresponds to $K = 12$ and $L = 4$ in Eq. (4.15). The weighting functions for this order can be deduced in a similar manner as was demonstrated above for lower orders. The velocity for order $H = 7$ is given

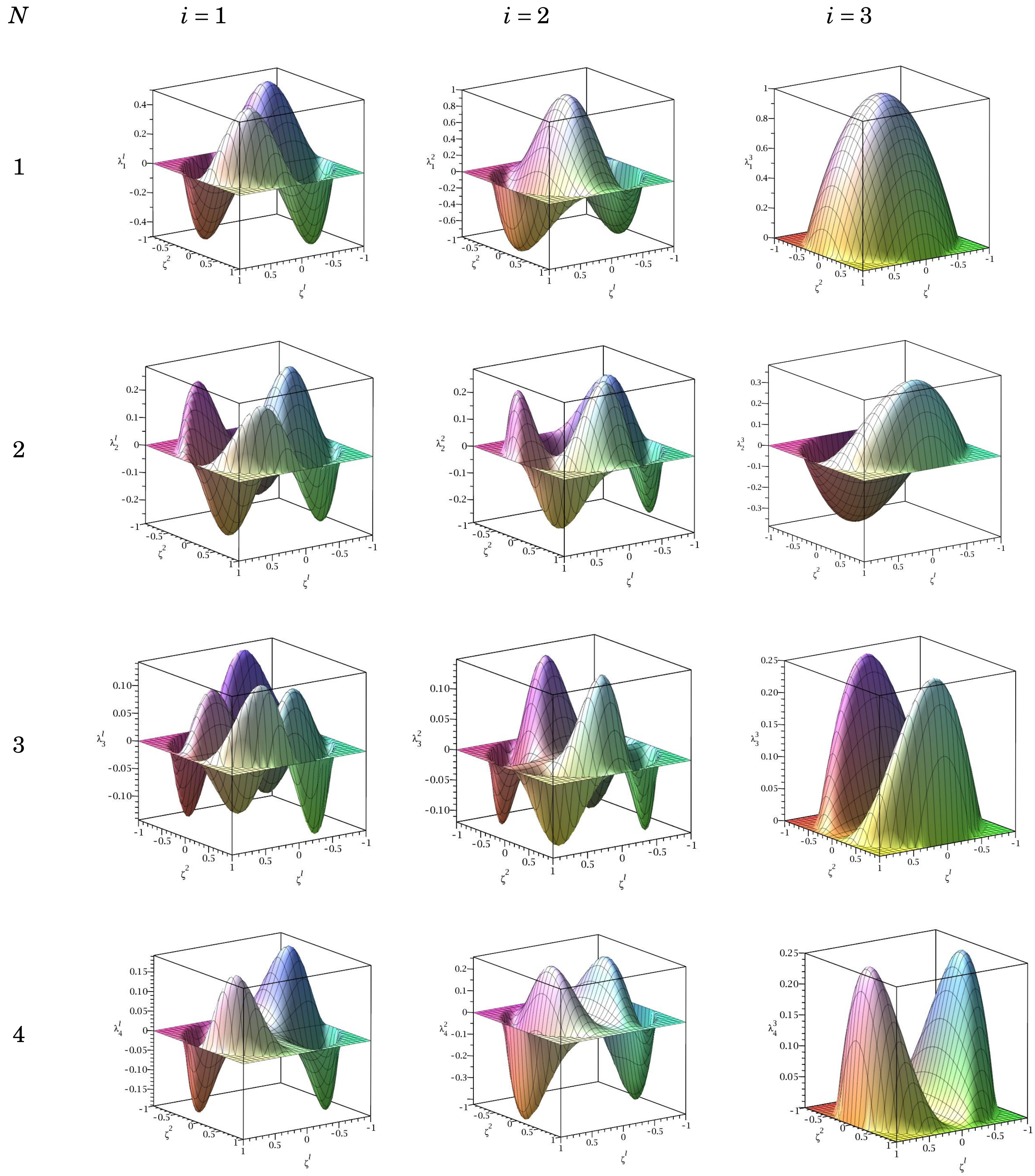


Figure 4.2: Plots of the weighting functions λ_N^i in the representation given by Eq. (4.15), for $N = 1,2,3,4$, with ζ^1 along the bottom right axis, ζ^2 along the bottom left axis and λ_N^i along the vertical axis.

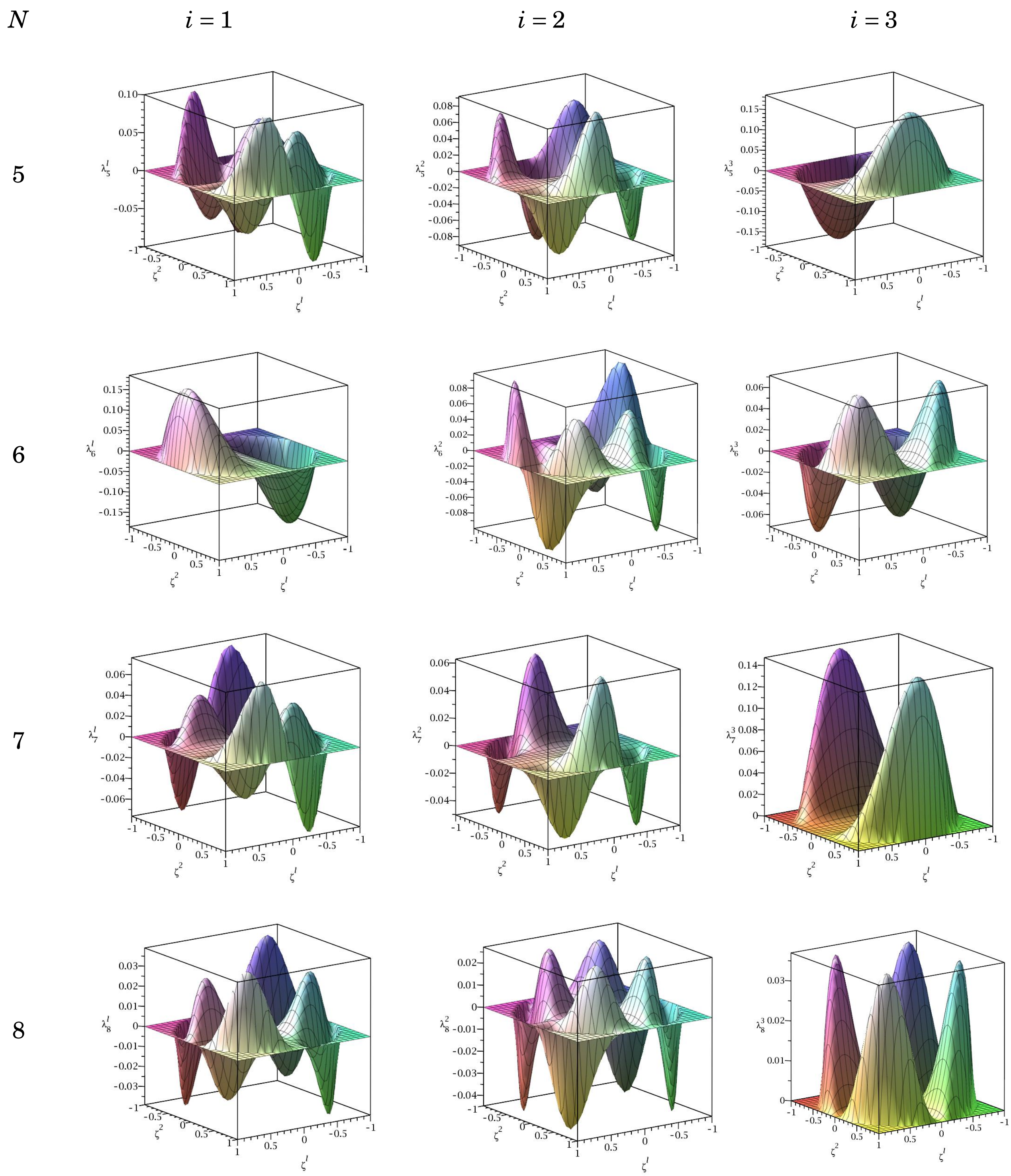


Figure 4.3: Plots of the weighting functions λ_N^i in the representation given by Eq. (4.15), for $N = 5, 6, 7, 8$, with ζ^1 along the bottom right axis, ζ^2 along the bottom left axis and λ_N^i along the vertical axis.

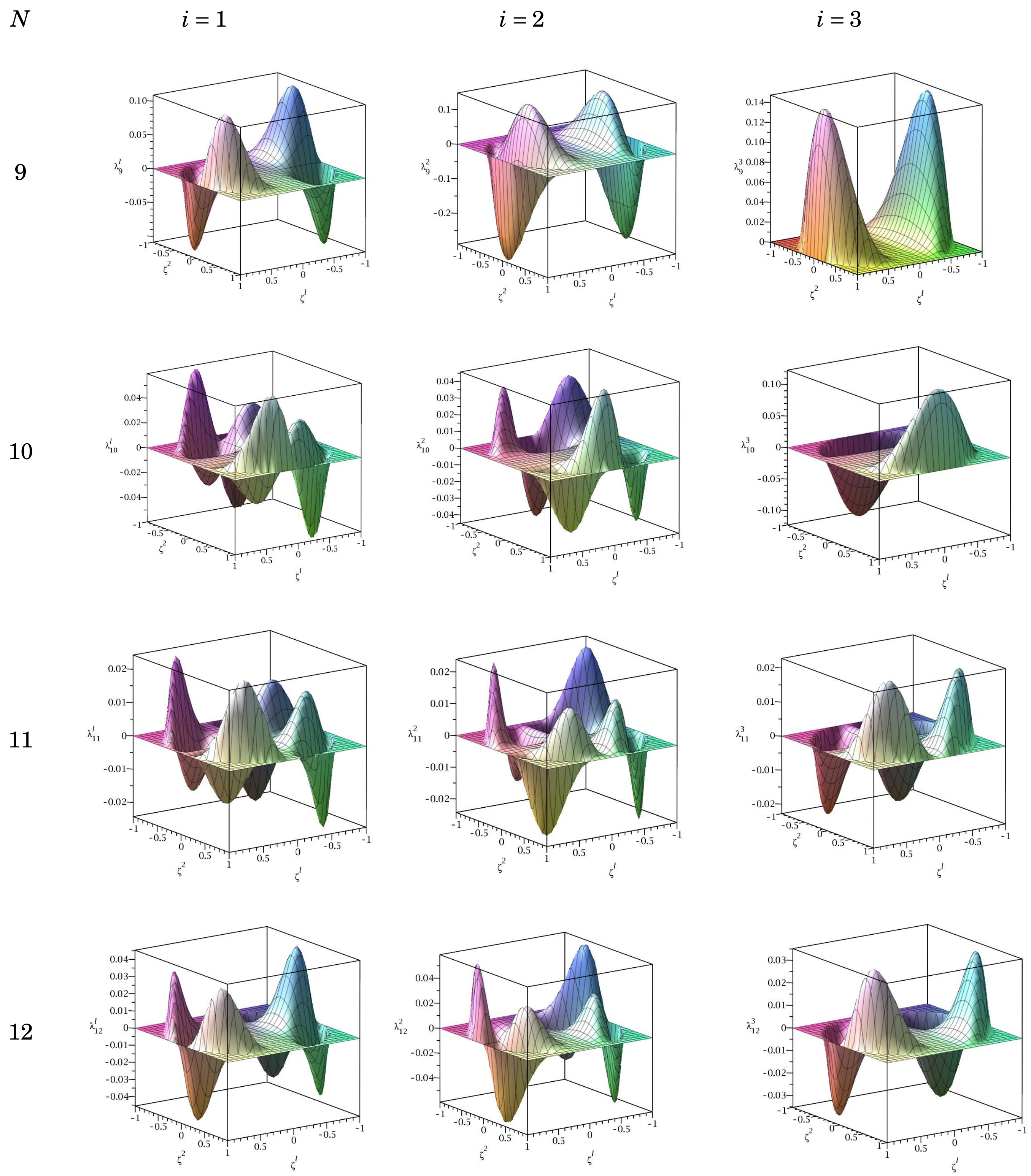


Figure 4.4: Plots of the weighting functions λ_N^i in the representation given by Eq. (4.15), for $N = 9, 10, 11, 12$, with ζ^1 along the bottom right axis, ζ^2 along the bottom left axis and λ_N^i along the vertical axis.

by

$$v_1(\zeta^1,\zeta^2)=\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg(\Bigg(4\frac{\zeta^1\zeta^2}{a^2}\left(1+\frac{\zeta^2}{R}\right)-2\frac{\zeta^1}{R}\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_6+\Bigg(\frac{\zeta^1}{a}\left(1+\frac{\zeta^2}{R}\right)\Bigg(5\frac{(\zeta^2)^{(2)}}{a^2}$$
$$-1+\frac{(\zeta^1)^{(2)}}{a^2}\Bigg)-2\frac{\zeta^1\zeta^2}{Ra}\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_9+\Bigg(\frac{\zeta^1\zeta^2}{a^2}\left(1+\frac{\zeta^2}{R}\right)\left(6\frac{(\zeta^2)^{(2)}}{a^2}-2+2\frac{(\zeta^1)^{(2)}}{a^2}\right)$$
$$-2\frac{\zeta^1(\zeta^2)^{(2)}}{Ra^2}\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_{11}$$
$$+\Bigg(4\frac{(\zeta^1)^{(3)}\zeta^2}{a^{(4)}}\left(1+\frac{\zeta^2}{R}\right)-2\frac{(\zeta^1)^{(3)}}{a^2R}\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_{12}\Bigg);$$

$$v_2(\zeta^1,\zeta^2)=\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg(\Bigg(-\left(1+\frac{\zeta^2}{R}\right)\left(5\frac{(\zeta^1)^{(2)}}{a^2}-1+\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_6$$
$$+\Bigg(-\frac{\zeta^2}{a}\left(1+\frac{\zeta^2}{R}\right)\left(5\frac{(\zeta^1)^{(2)}}{a^2}-1+\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_9$$
$$+\Bigg(-\frac{(\zeta^2)^{(2)}}{a^2}\left(1+\frac{\zeta^2}{R}\right)\left(5\frac{(\zeta^1)^{(2)}}{a^2}-1+\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_{11}$$
$$+\Bigg(-\frac{(\zeta^1)^{(2)}}{a^2}\left(1+\frac{\zeta^2}{R}\right)\left(7\frac{(\zeta^1)^{(2)}}{a^2}-3+3\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg)u_{12}\Bigg);$$

$$v_3(\zeta^1,\zeta^2)=\left(1-\frac{(\zeta^1)^{(2)}}{a^2}-\frac{(\zeta^2)^{(2)}}{a^2}\right)\Bigg(w_1+\frac{\zeta^2}{a}w_2+\frac{(\zeta^2)^{(2)}}{a^2}w_3+\frac{(\zeta^1)^{(2)}}{a^2}w4+\frac{(\zeta^2)^{(3)}}{a^3}w_5+\frac{(\zeta^1)^{(2)}\zeta^2}{a^3}w_6$$
$$+\frac{(\zeta^2)^{(4)}}{a^{(4)}}w_7+\frac{(\zeta^1)^{(2)}(\zeta^2)^{(2)}}{a^{(4)}}w_8+\frac{(\zeta^1)^{(4)}}{a^{(4)}}w_9+\frac{(\zeta^2)^{(5)}}{a^{(5)}}w_{10}+\frac{(\zeta^1)^{(2)}(\zeta^2)^{(3)}}{a^{(5)}}w_{11}+\frac{(\zeta^1)^{(4)}\zeta^2}{a^{(5)}}w_{12}\Bigg).$$

The $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ components of the momentum equation are given respectively by

$$v_1\frac{\partial v_1}{\partial\zeta^1}+v_2\frac{\partial v_1}{\partial\zeta^2}=-\frac{1}{\rho}\frac{\partial p}{\partial\zeta^1}+\frac{\mu}{\rho}\left(\frac{\partial^{(2)}v_1}{\partial(\zeta^1)^{(2)}}+\frac{1}{R+\zeta^2}\frac{\partial v_1}{\partial\zeta^2}+\frac{\partial^{(2)}v_2}{\partial(\zeta^2)^{(2)}}\right);$$

$$v_1\frac{\partial v_2}{\partial\zeta^1}+v_2\frac{\partial v_2}{\partial\zeta^2}-\frac{v_3^2}{R+\zeta^2}=-\frac{1}{\rho}\frac{\partial p}{\partial\zeta^2}+\frac{\mu}{\rho}\left(-\frac{v_2}{(R+\zeta^2)^{(2)}}+\frac{\partial^{(2)}v_2}{\partial(\zeta^2)^{(2)}}+\frac{1}{R+y}\frac{\partial v_2}{\partial\zeta^2}+\frac{\partial^{(2)}v_2}{\partial(\zeta^1)^{(2)}}\right);$$

$$v_1\frac{\partial v_3}{\partial\zeta^1}+v_2\frac{\partial v_3}{\partial\zeta^2}+\frac{v_2v_3}{R+\zeta^2}=-\frac{R}{\rho(R+\zeta^2)}\frac{\partial p}{\partial\zeta^3}+\frac{\mu}{\rho}\Bigg(\frac{(R+\zeta^2)^{(2)}}{r^{(2)}}\frac{\partial^{(2)}v_3}{\partial(\zeta^1)^{(2)}}+\frac{2}{R}\frac{\partial v_3}{\partial\zeta^2}-\frac{4v_3}{R(R+\zeta^2)}$$
$$+\frac{R+\zeta^2}{r^{(2)}}\frac{\partial v_3}{\partial\zeta^2}+\frac{(R+\zeta^2)^{(2)}}{r^{(2)}}\frac{\partial^{(2)}v_3}{\partial(\zeta^2)^{(2)}}\Bigg).$$

4.5 Nondimensionalisation

Before carrying out the integration over the cross-section and solving for the director velocities, first the equations are nondimensionalised. The following dimensionless variables are introduced:

$$X = \zeta^1/a; \qquad Y = \zeta^2/a; \qquad Z = \zeta^3/a;$$
$$V_1 = \frac{Re_0 v_1}{2c}; \qquad V_2 = \frac{Re_0 v_2}{2c}; \qquad V_3 = \frac{v_3}{2c};$$
$$P = \frac{ap}{2\mu c}; \qquad \delta = \frac{a}{R}; \qquad Re_0 = \frac{2ac\rho}{\mu};$$

where c represents the average coaxial velocity and Re_0 the Reynolds number for flow in a corresponding straight pipe, subject to the same coaxial pressure gradient, δ represents the curvature ratio. These choice of scales are chosen with foresight to simplify the algebra that follows.

The nondimensionalised velocities are expressed by

$$\begin{aligned} V_1(X,Y) = (1-X^{(2)}-Y^{(2)})\Big(&U_6\Big(4XY(1+\delta Y)-2\delta X(1-X^{(2)}-Y^{(2)})\Big)+U_9\Big(X(1+\delta Y)(X^{(2)}+5Y^{(2)}-1)\\ &-2\delta X(1-X^{(2)}-Y^{(2)})\Big)+U_{11}\Big(XY(1+\delta Y)(2X^{(2)}+6Y^{(2)}-2)-2\delta XY^{(2)}(1-X^{(2)}-Y^{(2)})\Big)\\ &+U_{12}\Big(4X^{(3)}Y(1+\delta Y)-2\delta X^{(3)}(1-X^{(2)}-Y^{(2)})\Big)\Big); \end{aligned}$$

$$\begin{aligned} V_2(X,Y) = -(1-X^{(2)}-Y^{(2)})\Big(&U_6\Big((1+\delta Y)(5X^{(2)}+Y^{(2)}-1)\Big)+U_9\Big(Y(1+\delta Y)(5X^{(2)}+Y^{(2)}-1)\Big)\\ &+U_{11}\Big(Y^{(2)}(1+\delta Y)(5X^{(2)}+Y^{(2)}-1)\Big)+U_{12}\Big(X^{(2)}(1-\delta Y)(7X^{(2)}+3Y^{(2)}-3)\Big)\Big); \end{aligned}$$

$$\begin{aligned} V_3(X,Y) = (1-X^{(2)}-Y^{(2)})(1&+W_2Y+W_3Y^{(2)}+W_4X^{(2)}+W_5Y^{(3)}+W_6X^{(2)}Y+W_7Y^{(4)}+W_8X^{(2)}Y^{(2)}\\ &+W_9X^{(4)}+W_{10}Y^{(5)}+W_{11}X^{(2)}Y^{(3)}+W_{12}X^{(4)}Y). \end{aligned}$$

The $\mathbf{a}_1, \mathbf{a}_2$ and $\mathbf{a}_3$ components of the momentum equation are expressed respectively in terms of the dimensionless variables by

$$\frac{1}{Re_0}V_1\frac{\partial V_1}{\partial X}+\frac{1}{Re_0}V_2\frac{\partial V_1}{\partial Y} = -\frac{\partial P}{\partial X}+\frac{1}{Re_0}\frac{\partial^{(2)}V_1}{\partial X^{(2)}}+\frac{\delta}{Re_0(1+\delta Y)}\frac{\partial V_1}{\partial Y}+\frac{1}{Re_0}\frac{\partial^{(2)}V_2}{\partial Y^{(2)}}; \tag{4.16}$$

$$\begin{aligned} \frac{1}{Re_0}V_1\frac{\partial V_2}{\partial X}+\frac{1}{Re_0}V_2\frac{\partial V_2}{\partial Y}-Re_0\frac{\delta}{(1+\delta Y)}V_3^2 = &-\frac{\partial P}{\partial Y}-\frac{\delta^{(2)}}{Re_0(1+\delta Y)^{(2)}}V_2+\frac{1}{Re_0}\frac{\partial^{(2)}V_2}{\partial Y^{(2)}}\\ &+\frac{\delta}{Re_0(1+\delta Y)}\frac{\partial V_2}{\partial Y}+\frac{1}{Re_0}\frac{\partial^{(2)}V_2}{\partial X^{(2)}}; \end{aligned} \tag{4.17}$$

(4.18)
$$V_1\frac{\partial V_3}{\partial X}+V_2\frac{\partial V_3}{\partial Y}+\frac{\delta}{1+\delta Y}V_2V_3=-\frac{1}{1+\delta Y}\frac{\partial P}{\partial Z}+\frac{\partial V_3}{\partial X^{(2)}}+\frac{\partial V_3}{\partial Y^{(2)}}+\frac{\delta}{1+\delta Y}\frac{\partial V_3}{\partial Y}-\frac{\delta^{(2)}}{(1+\delta Y)^{(2)}}V_3.$$

Multiplying by each respective weighting function λ_N^i and integrating over the cross-section on the pipe A, the integrated equations of motion are

$$\int\int_A \lambda_N^1\left[(1+\delta Y)\left(V_1\frac{\partial V_1}{\partial X}+V_2\frac{\partial V_1}{\partial Y}-\frac{\partial^{(2)}V_1}{\partial X^{(2)}}-\frac{\partial^{(2)}V_2}{\partial Y^{(2)}}\right)-\delta\frac{\partial V_1}{\partial Y}\right]dXdY$$
$$=-Re_0\int\int_A \lambda_N^1(1+\delta Y)\frac{\partial P}{\partial X}dXdY;$$

$$\int\int_A \lambda_N^2\left[(1+\delta Y)\left(V_1\frac{\partial V_2}{\partial X}+V_2\frac{\partial V_2}{\partial Y}-\frac{\partial^{(2)}V_2}{\partial Y^{(2)}}-\frac{\partial^{(2)}V_2}{\partial X^{(2)}}\right)-Re_0^{(2)}\delta V_3^2+\frac{\delta^{(2)}}{(1+\delta Y)}V_2-\delta\frac{\partial V_2}{\partial Y}\right]dXdY$$
$$=-Re_0\int\int_A \lambda_N^2(1+\delta Y)\frac{\partial P}{\partial Y}dXdY;$$

$$\int\int_A \lambda_N^3\left[(1+\delta Y)\left(V_1\frac{\partial V_3}{\partial X}+V_2\frac{\partial V_3}{\partial Y}-\frac{\partial V_3}{\partial X^{(2)}}-\frac{\partial V_3}{\partial Y^{(2)}}\right)+\delta V_2V_3-\delta\frac{\partial V_3}{\partial Y}+\frac{\delta^{(2)}}{(1+\delta Y)}V_3\right]dXdY$$
$$=-\int\int_A \lambda_N^2\frac{\partial P}{\partial Z}dXdY.$$

Writing in summation form with the $i=1$ and $i=2$ component equations added together, these become

(4.19)
$$\sum_{M=1}^{L}\sum_{R=1}^{L}E_{NMR}U_MU_R-\sum_{M=1}^{L}C_{NM}U_M-Re_0^{(2)}\sum_{M=1}^{K}\sum_{R=1}^{K}D_{NMR}W_MW_R=Re_0Q_N;$$

(4.20)
$$\sum_{M=1}^{L}\sum_{R=1}^{K}F_{NMR}U_MW_R-\sum_{M=1}^{K}G_{NM}W_M=P_N;$$

where U_i and W_i are the non-dimensionalised cross-sectional and coaxial director velocities respectively, N is the order of the equation (i.e. which weighting function has multiplied the equation) and shorthand for integral terms has been introduced which are defined as follows:

(4.21)
$$E_{NMR}=\int\int_A\left[(1+\delta Y)\left(\lambda_N^1\lambda_M^1\frac{\partial\lambda_R^1}{\partial X}+\lambda_N^1\lambda_M^2\frac{\partial\lambda_R^1}{\partial Y}+\lambda_N^2\lambda_M^1\frac{\partial\lambda_r^{(2)}}{\partial X}+\lambda_N^2\lambda_M^2\frac{\partial\lambda_r^{(2)}}{\partial Y}\right)\right]dXdY;$$

(4.22)
$$C_{NM}=\int\int_A\left[(1+\delta Y)\left(\lambda_N^1\frac{\partial^{(2)}\lambda_M^1}{\partial X^{(2)}}+\lambda_N^1\frac{\partial^{(2)}\lambda_M^1}{\partial Y^{(2)}}+\lambda_N^2\frac{\partial^{(2)}\lambda_M^2}{\partial X^{(2)}}+\lambda_N^2\frac{\partial^{(2)}\lambda_M^2}{\partial Y^{(2)}}\right)\right.$$
$$\left.+\delta\left(\lambda_N^1\frac{\partial\lambda_M^1}{\partial Y}+\lambda_N^2\frac{\partial\lambda_M^2}{\partial Y}\right)-\frac{\delta^{(2)}\lambda_N^2\lambda_M^2}{1+\delta Y}\right]dXdY;$$

$$D_{NMR} = \int\int_A \delta\lambda_N^2 \lambda_M^3 \lambda_r^{(3)} dXdY; \tag{4.23}$$

$$Q_N = \int\int_A \left[(1+\delta Y)\left(\lambda_N^1 \frac{\partial P}{\partial X} + \lambda_N^2 \frac{\partial P}{\partial Y}\right)\right] dXdY;$$

$$F_{NMR} = \int\int_A \left[(1+\delta Y)\left(\lambda_N^3 \lambda_M^1 \frac{\partial \lambda_r^{(3)}}{\partial X} + \lambda_N^3 \lambda_M^2 \frac{\partial \lambda_r^{(3)}}{\partial Y}\right) + \delta \lambda_N^3 \lambda_M^2 \lambda_r^{(2)}\right] dXdY; \tag{4.24}$$

$$G_{NM} = \int\int_A \left[(1+\delta Y)\left(\lambda_N^3 \frac{\partial^{(2)}\lambda_M^3}{\partial X^{(2)}} + \lambda_N^3 \frac{\partial^{(2)}\lambda_M^3}{\partial Y^{(2)}}\right) + \delta\lambda_N^3 \frac{\partial \lambda_M^3}{\partial Y} - \frac{\delta^{(2)}\lambda_N^3\lambda_M^3}{1+\delta Y}\right] dXdY; \tag{4.25}$$

$$P_N = \int\int_A \lambda_N^3 \frac{\partial P}{\partial Z} dXdY.$$

4.6 Dealing with the pressure

After substituting the velocity expansion into the integrated equations of motion, the only unknowns appearing in the system of equations besides the velocity directors are the pressure gradients. This model will solve the velocity field for the same coaxial pressure gradient as Poiseuille flow in that of a corresponding straight pipe, as carried out by Green et al [7]. By considering Eq. (4.20) for Poiseuille flow in a straight pipe, the cross-sectional velocity directors U_i would all equal zero and the only non-zero coaxial velocity director would be W_1 which would be equal to 1 due to how W_i was defined in the non-dimensionalisation. So Eq. (4.20) for a straight pipe would be

$$P_N = -G_{N1}.$$

Substituting this back into Eq. (4.20) for a curved pipe gives

$$\sum_{M=1}^{L}\sum_{R=1}^{K} F_{NMR} U_M W_R - \sum_{M=1}^{K} G_{NM} W_M = -G_{N1}. \tag{4.26}$$

So, for $N = 1$,

$$G_{11} = -\int\int_A (1 - X^{(2)} - Y^{(2)})\left[-4(1+\delta Y) - 2\delta Y - \frac{\delta^{(2)}}{1+\delta Y}(1 - X^{(2)} - Y^{(2)})\right] dXdY.$$

For $N = 2$,

$$G_{21} = -\int\int_A Y(1 - X^{(2)} - Y^{(2)})\left[-4(1+\delta Y) - 2\delta Y - \frac{\delta^{(2)}}{1+\delta Y}(1 - X^{(2)} - Y^{(2)})\right] dXdY.$$

N	λ_N^3	G_{N1}
1	$(1-X^{(2)}-Y^{(2)})$	6.28
2	$Y(1-X^{(2)}-Y^{(2)})$	1.57×10^{-1}
3	$Y^{(2)}(1-X^{(2)}-Y^{(2)})$	1.05
4	$X^{(2)}(1-X^{(2)}-Y^{(2)})$	1.05
5	$Y^{(3)}(1-X^{(2)}-Y^{(2)})$	5.89×10^{-3}
6	$X^{(2)}Y(1-X^{(2)}-Y^{(2)})$	1.96×10^{-3}
7	$Y^{(4)}(1-X^{(2)}-Y^{(2)})$	3.93×10^{-1}
8	$X^{(2)}Y^{(2)}(1-X^{(2)}-Y^{(2)})$	1.31×10^{-1}
9	$X^{(4)}(1-X^{(2)}-Y^{(2)})$	3.93×10^{-1}
10	$Y^{(5)}(1-X^{(2)}-Y^{(2)})$	2.95×10^{-3}
11	$X^{(2)}Y^{(3)}(1-X^{(2)}-Y^{(2)})$	5.89×10^{-4}
12	$X^{(4)}Y(1-X^{(2)}-Y^{(2)})$	5.89×10^{-4}
13	$Y^{(6)}(1-X^{(2)}-Y^{(2)})$	1.96×10^{-1}
14	$X^{(2)}Y^{(4)}(1-X^{(2)}-Y^{(2)})$	3.93×10^{-2}
15	$X^{(4)}Y^{(2)}(1-X^{(2)}-Y^{(2)})$	3.93×10^{-2}
16	$X^{(6)}(1-X^{(2)}-Y^{(2)})$	1.96×10^{-1}
17	$Y^{(7)}(1-X^{(2)}-Y^{(2)})$	1.72×10^{-3}
18	$X^{(2)}Y^{(5)}(1-X^{(2)}-Y^{(2)})$	2.45×10^{-4}
19	$X^{(4)}Y^{(3)}(1-X^{(2)}-Y^{(2)})$	1.47×10^{-4}
20	$X^{(6)}Y(1-X^{(2)}-Y^{(2)})$	2.45×10^{-4}
21	$Y^{(8)}(1-X^{(2)}-Y^{(2)})$	1.15×10^{-1}
22	$X^{(2)}Y^{(6)}(1-X^{(2)}-Y^{(2)})$	1.64×10^{-2}
23	$X^{(4)}Y^{(4)}(1-X^{(2)}-Y^{(2)})$	9.82×10^{-3}
24	$X^{(6)}Y^{(2)}(1-X^{(2)}-Y^{(2)})$	1.64×10^{-2}

Table 4.2: Values of the integrated coaxial pressure gradient term G_{N1}, for $\delta = 0.01$, for $N = 1$ to 24.

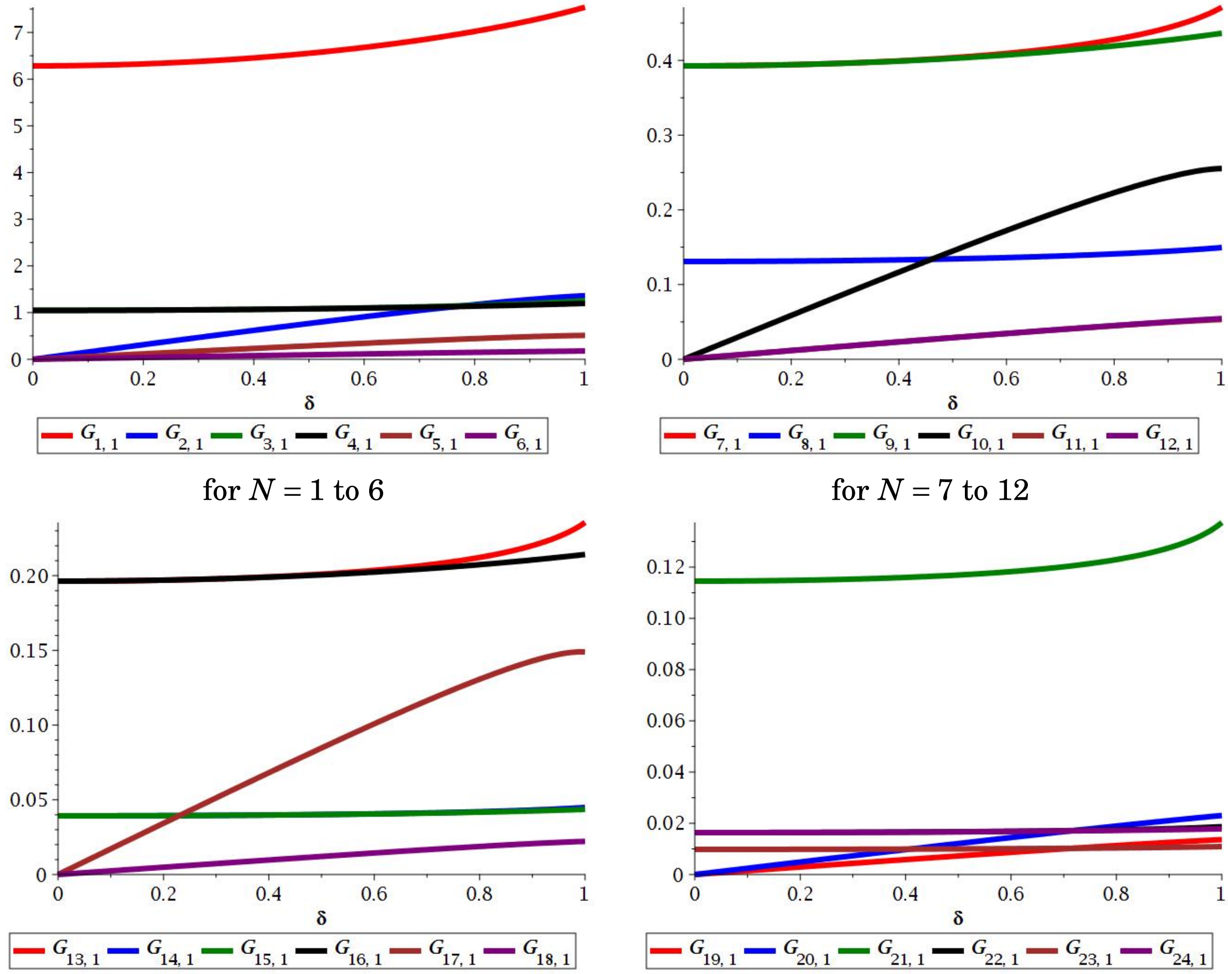
For $N = 3$,

$$G_{31} = -\int\int_A Y^{(2)}(1-X^{(2)}-Y^{(2)})\left[-4(1+\delta Y)-2\delta Y-\frac{\delta^{(2)}}{1+\delta Y}(1-X^{(2)}-Y^{(2)})\right]dXdY.$$

It is observed that the first term is changing, and a table summarising this is given in Table 4.2. The first 24 G_{N1} terms are plotted in Fig. 4.5, showing how they varying with the curvature ratio δ. These terms give the magnitude of the driving force of each director in the velocity profile. Generally the magnitude decreases as N is increased.

This still leaves unknown pressure gradients in the cross-section, in Eq. (4.19). Consider Green's theorem which can be written as

$$\oint_C [-LdX + MdY] = \int\int_D \left[\frac{\partial M}{\partial X}+\frac{\partial L}{\partial Y}\right]dXdY. \tag{4.27}$$



for $N = 1$ to 6

for $N = 7$ to 12

for $N = 13$ to 18

for $N = 19$ to 24

Figure 4.5: Figure showing how the value of G_{N1} varies with δ for $N = 1$ to $N = 24$.

If L and M are set as $L = (1+\delta Y)\lambda_N^2 P$ and $M = (1+\delta Y)\lambda_N^1 P$, then Green's theorem gives

$$\oint_C \left[-(1+\delta Y)\lambda_N^2 P\,dX + (1+\delta Y)\lambda_N^1 P\,dY\right] = \int\int_A \Big[\delta\lambda_N^2 P + (1+\delta Y)\left(\frac{\partial \lambda_N^2}{\partial Y}P + \lambda_N^2\frac{\partial P}{\partial Y} + \frac{\partial \lambda_N^1}{\partial X}P + \lambda_N^1\frac{\partial P}{\partial X}\right)\Big]\,dX\,dY.$$

The line integral is around the circumference of the cross-section where $r^{(2)} = X^{(2)} + Y^{(2)} = 1$, $\lambda_N^i = 0$ at $r = 1$ $\forall N$ as $\lambda_N^i = (1 - X^{(2)} - Y^{(2)})f_N^i(X,Y)$. Therefore

$$\int\int_A \left[(1+\delta Y)\left(\lambda_N^1\frac{\partial P}{\partial X} + \lambda_N^2\frac{\partial P}{\partial Y}\right)\right]dX\,dY = -\int\int_A \left[P\left(\delta\lambda_N^2 + \left(\frac{\partial \lambda_N^1}{\partial X} + \frac{\partial \lambda_N^2}{\partial Y}\right)\right)\right]dX\,dY. \tag{4.28}$$

However, the right hand side of Eq. (4.28) equals zero due to the incompressibility condition. Therefore Eq. (4.19) becomes

$$\sum_{M=1}^{L}\sum_{R=1}^{L} E_{NMR}U_M U_R - \sum_{M=1}^{L} C_{NM}U_M - Re_0^{(2)} \sum_{M=1}^{K}\sum_{R=1}^{K} D_{NMR}W_M W_R = 0, \tag{4.29}$$

and the system of equations is now closed.

The friction loss ratio f is defined as the ratio of the friction factor in the curved pipe to the corresponding friction factor in a straight pipe of the same cross-section and for the same pressure gradient [7]. This can be expressed as a ratio of the Reynolds numbers in the straight (Re_0) and curved pipes (Re). That is

$$f = \frac{Re_0}{Re} = \frac{1}{V_3} = \frac{1}{\sum_{N=1}^{K} \lambda_N^3 W_N}$$

This gives a relation between the two Reynolds numbers. It is possible to reformulate Eq. (4.29) with Re rather Re_0, and this is similarly done by Green et al [7], however this changes the equation from a quadratic to a quartic, in terms of the director velocities (U_N, W_N). So for the ease of solving, in this model Eq. (4.29) is kept in its current form, however when presenting results, the Reynolds number Re related to the curved pipe will be used, as this is the true Reynolds number of the flow.

Equations (4.26) and (4.29) form a square system of equations in which the cross-sectional pressure gradients have been eliminated from Eqs. (4.29) through algebraic manipulation, taking advantage of the symmetry conditions of this model, and in which the coaxial pressure gradient is prescribed in Eqs. (4.26). This sets the system of equation on a good form to solve for the unknown directors, the method of which is detailed in the next section. In a more general case where such manipulation is not available, it is much harder to set the system of equations in a form that can be readily solved for.

4.7 Comparisons and Solutions

A square system of $K+L$ equations is formed from Eq. (4.26) for $N = 1$ to K and Eq. (4.29) for $N = 1$ to L. The unknowns to solve for are the director velocities W_N, for $N = 1$ to K, and U_N for $N = 1$ to L. Maple was used to solve this system. Numerical integration (the *evalf* command in Maple) was used to evaluate Eqs. (4.26) and (4.29) at each order due to the complexities of symbolic integration of the functions involved. Numerical integration does introduce numerical error into the equations, although this will have negligible effect on the solution to the system due to the small magnitudes of error introduced.

Once the equations were evaluated, the system was solved with the *fsolve* command in Maple. Then the values for the director velocities are substituted back into the velocity expansion to recover the velocity field. Increasing the order H of the system increases the accuracy of the model, however it also increases the computational time required to numerically integrate the equations. Solutions from the model were produced at orders $H = 7$ (corresponding to $K = 12$ and $L = 4$), $H = 8$ (corresponding to $K = 16$ and $L = 6$) and $H = 12$ (corresponding to $K = 36$ and $L = 20$). The basis for this model originally came from the work of Green et al [7] whose model also solves for fluid flow in a toroidally curved pipe, with the same choice of coordinate system. However different methods were used for deriving the equations of motion to solve for. Whereas the equations in this model were derived directly from the Navier-Stokes equations, Green et al derive the equations of motions through definitions of various kinematic variables. Comparing the integral expressions of the two models, the equations initially appear to differ from each other. However, it can be shown analytically through the use of Green's theorem and the use of the incompressibility condition that these models turn out to be equivalent.

4.7.1 Comparison of model with Green et al [7]

The model developed by Green et al, as given by Eqs. (4.8) and (4.9) in [7] is

$$\sum_{M=1}^{K} A_{NM}B_M + \sum_{M=1}^{L}\sum_{R=1}^{K} C_{NMR}X_MB_R = A_{N1}; \tag{4.30}$$

$$-\sum_{M=1}^{L} D_{NM}X_M = \sum_{M=1}^{L}\sum_{R=1}^{L} E_{NMR}X_MX_R + \kappa_0 \sum_{M=1}^{K}\sum_{R=1}^{K} F_{NMR}B_MB_R; \tag{4.31}$$

where

- κ_0 is the Dean number of flow through the corresponding straight pipe subject to the same pressure gradient;
- B_i is the ith coaxial velocity director;
- X_i is the ith cross-sectional velocity director;
- K is the number of coaxial directors at the given order;
- L is the number of cross-sectional directors at the given order.

The remaining terms in Eqs. (4.30) and (4.31) are integral expressions defined in Appendix A of [7] as follows:

$$F_{NMR} = -\int\int_A \left[\lambda_N^2 \lambda_M^3 \lambda_r^{(3)}\right] dxdy; \tag{4.32}$$

(4.33)
$$A_{NM} = -\int\int_A \left[-\frac{\delta^{(2)}}{1+\delta y}\lambda_N^3\lambda_M^3 + \delta\lambda_N^3\frac{\partial\lambda_M^3}{\partial y} - (1+\delta y)\frac{\lambda_N^3}{\partial x}\frac{\partial\lambda_M^3}{\partial x} + \delta\frac{\partial\lambda_N^3}{\partial y}\lambda_M^3 - (1+\delta y)\frac{\lambda_N^3}{\partial y}\frac{\partial_M^3}{\partial y} \right] dxdy;$$

(4.34)
$$C_{NMR} = -\int\int_A \left[\delta\lambda_N^3\lambda_M^2\lambda_r^{(3)} + (1+\delta y)\left(\lambda_N^3\lambda_M^1\frac{\partial\lambda_r^{(3)}}{\partial x} + \lambda_N^3\lambda_M^2\frac{\partial\lambda_r^{(3)}}{\partial y} \right) \right] dxdy;$$

(4.35)
$$D_{NM} = -\int\int_A \left[\frac{2\delta^{(2)}}{1+\delta y}\lambda_N^2\lambda_M^2 + (1+\delta y)\left(2\frac{\partial\lambda_N^1}{\partial x}\frac{\partial\lambda_M^1}{\partial x} + \frac{\partial\lambda_N^1}{\partial y}\frac{\partial\lambda_M^1}{\partial y} + \frac{\partial\lambda_N^1}{\partial y}\frac{\partial\lambda_M^2}{\partial x} \right.\right.$$
$$\left.\left. + \frac{\partial\lambda_N^2}{\partial x}\frac{\partial\lambda_M^2}{\partial x} + \frac{\partial\lambda_N^2}{\partial x}\frac{\partial\lambda_M^1}{\partial y} + 2\frac{\partial\lambda_N^2}{\partial y}\frac{\partial\lambda_M^2}{\partial y} \right) \right] dxdy;$$

(4.36)
$$E_{NMR} = \int\int_A (1+\delta y)\left(\lambda_N^1\left(\frac{\partial\lambda_M^1}{\partial x}\lambda_R^1 + \frac{\partial\lambda_M^1}{\partial y}\lambda_r^{(2)} \right) + \lambda_N^2\left(\frac{\lambda_M^2}{\partial x}\lambda_R^1 + \frac{\lambda_M^2}{\partial y}\lambda_r^{(2)} \right) \right) dxdy;$$

where

- δ is the ratio between the radius of the pipe and the radius of curvature;
- λ_i^j is the weighting function of the ith director in the $\mathbf{a}_j$ direction.

Eq. (4.30) and Eq. (4.26) each correspond to conservation of momentum in the coaxial ($\mathbf{a}_3$) direction. Eq. (4.31) and Eq. (4.29) each correspond to conservation of momentum in the cross-section.

- Eq. (4.33) corresponds to Eq. (4.25);
- Eq. (4.34) corresponds to Eq. (4.24);
- Eq. (4.35) corresponds to Eq. (4.22);
- Eq. (4.36) corresponds to Eq. (4.21);
- Eq. (4.32) corresponds to Eq. (4.23).

While it can be seen that Eq. (4.24) is equivalent to Eq. (4.34), Eq. (4.21) is equivalent to Eq. (4.36) and Eq. (4.23) is equivalent to Eq. (4.32), there initially appears to be differences between Eqs. (4.25) and (4.33) and Eqs. (4.22) and (4.35). However the following will show that they are mathematically equivalent.

Making use of Green's Theorem, given by Eq. (4.27), with L and M set as

$$L = (1+\delta Y)\lambda_N^1\frac{\partial\lambda_M^1}{\partial Y};$$

$$M = (1+\delta Y)\lambda_N^1 \frac{\partial \lambda_M^1}{\partial X};$$

then, taking C to be the circumference of the cross-section of the pipe, Eq. (4.27) gives

$$\oint_C (1+\delta Y)\lambda_N^1 \left(\frac{\partial \lambda_M^1}{\partial Y} - \frac{\partial \lambda_M^1}{\partial X} dY \right) = \int\int_A \left[(1+\delta Y)\left(\frac{\partial \lambda_N^1}{\partial X} \partial \lambda_M^1 \partial X + \lambda_N^1 \frac{\partial^{(2)} \lambda_M^1}{\partial X^{(2)}} + \frac{\partial \lambda_N^1}{\partial Y}\frac{\partial \lambda_M^1}{\partial Y} + \lambda_N^1 \frac{\partial \lambda_M^1}{\partial Y^{(2)}} \right) + \delta \lambda_N^1 \frac{\partial \lambda_M^1}{\partial Y} \right] dX dY.$$

Rearranging gives

$$\int\int_A (1+\delta Y)\left(\lambda_N^1 \frac{\partial^{(2)} \lambda_M^1}{\partial X^{(2)}} + \lambda_N^1 \frac{\partial^{(2)} \lambda_M^1}{\partial Y^{(2)}} \right) dX dY = \oint_C (1+\delta Y)\lambda_N^1 \left(\frac{\partial \lambda_M^1}{\partial Y} - \frac{\partial \lambda_M^1}{\partial X} dY \right) - \int\int_A \left[(1+\delta Y)\left(\frac{\partial \lambda_N^1}{\partial X}\frac{\partial \lambda_M^1}{\partial X} + \frac{\partial \lambda_N^1}{\partial Y}\frac{\partial \lambda_M^1}{\partial Y} \right) + \delta \lambda_N^1 \frac{\partial \lambda_M^1}{\partial Y} \right] dX dY.$$

However, over the closed curved C, $r=1$ and as $r^{(2)} = X^{(2)} + Y^{(2)}$, $\lambda_N^1 = 0$ at $r=1$ $\forall N$ because $\lambda_N^i = (1 - X^{(2)} - Y^{(2)}) f_N^i(X,Y)$. Therefore the line integrals around C will equal zero and the above simplifies to

$$\int\int_A (1+\delta Y)\left(\lambda_N^1 \frac{\partial^{(2)} \lambda_M^1}{\partial X^{(2)}} + \lambda_N^1 \frac{\partial^{(2)} \lambda_M^1}{\partial Y^{(2)}} \right) dX dY = -\int\int_A \left[(1+\delta Y)\left(\frac{\partial \lambda_N^1}{\partial X}\frac{\partial \lambda_M^1}{\partial X} + \frac{\partial \lambda_N^1}{\partial Y} \partial \lambda_M^1 \partial Y \right) + \delta \lambda_N^1 \frac{\partial \lambda_M^1}{\partial Y} \right] dX dY.$$

Substituting this into Eq. (4.22) gives

$$C_{NM} = -\int\int_A \left[\frac{\delta^{(2)} \lambda_N^2 \lambda_M^2}{1+\delta Y} + (1+\delta Y)\left(\frac{\partial \lambda_N^1}{\partial X}\frac{\partial \lambda_M^1}{\partial X} + \frac{\partial \lambda_N^1}{\partial Y} \partial \lambda_M^1 \partial Y + \frac{\partial \lambda_N^2}{\partial X}\frac{\partial \lambda_M^2}{\partial X} + \frac{\partial \lambda_N^2}{\partial Y} \partial \lambda_M^2 \partial Y \right) \right] dX dY. \tag{4.37}$$

Comparing Eq. (4.35) with Eq. (4.37), it appears that Eq. (4.35) has more terms. The additional terms in Eq. (4.35) compared with Eq. (4.37) are

$$-\int\int_A \left[\frac{\delta^{(2)}}{1+\delta Y} \lambda_N^2 \lambda_M^2 + (1+\delta Y)\left(\frac{\partial \lambda_N^1}{\partial X}\frac{\partial \lambda_M^1}{\partial X} + \frac{\partial \lambda_N^1}{\partial Y}\frac{\partial \lambda_M^2}{\partial X} + \frac{\partial \lambda_N^2}{\partial X}\frac{\partial \lambda_M^1}{\partial Y} + \frac{\partial \lambda_N^2}{\partial Y}\frac{\partial \lambda_M^2}{\partial Y} \right) \right] dX dY. \tag{4.38}$$

However, it will be shown that this expression equates to zero.

Through the application of Green's Theorem, and considering the closed line integrals will equal zero due to the λ_N^i term, Eq. (4.38) can be converted to

$$\int\int_A \left[\lambda_N^1 \left(\delta \frac{\partial \lambda_M^2}{\partial X} + (1+\delta Y) \left(\frac{\partial^{(2)} \lambda_M^1}{\partial X^{(2)}} + \frac{\partial^{(2)} \lambda_M^2}{\partial X \partial Y} \right) \right) + \lambda_N^2 \left(\delta \frac{\partial \lambda_M^2}{\partial Y} - \frac{\delta^{(2)}}{1+\delta Y} \lambda_M^2 + (1+\delta Y) \left(\frac{\partial^{(2)} \lambda_M^1}{\partial X \partial Y} + \frac{\partial^{(2)} \lambda_M^2}{\partial Y^{(2)}} \right) \right) \right] dX dY.$$

Considering these terms as they appear in the equations, multiplied by the corresponding velocity director U_N and M summed from 1 to L then this can be written in terms of velocity components V_i, where V_i is the velocity in the $\mathbf{a}_i$ direction, as follows:

$$\int\int_A \left[\lambda_N^1 \left(\delta \frac{\partial V_2}{\partial X} + (1+\delta Y) \left(\frac{\partial^{(2)} V_1}{\partial X^{(2)}} + \frac{\partial^{(2)} V_2}{\partial X \partial Y} \right) \right) + \lambda_N^2 \left(\delta \frac{\partial V_2}{\partial Y} - \frac{\delta^{(2)}}{1+\delta Y} V_2 + (1+\delta Y) \left(\frac{\partial^{(2)} V_1}{\partial X \partial Y} + \frac{\partial^{(2)} V_2}{\partial Y^{(2)}} \right) \right) \right] dX dY. \tag{4.39}$$

The non-dimensionalised incompressibility condition is

$$\frac{\partial V_1}{\partial X} + \frac{\partial V_2}{\partial Y} + \frac{\delta}{1+\delta Y} V_2 = 0. \tag{4.40}$$

Differentiating Eq. (4.40) with respect to X and multiplying by $(1+\delta Y)$ gives

$$(1+\delta Y) \left(\frac{\partial^{(2)} V_1}{\partial X^{(2)}} + \frac{\partial^{(2)} V_2}{\partial X \partial Y} \right) + \delta \frac{\partial V_2}{\partial X} = 0. \tag{4.41}$$

Differentiating Eq. (4.40) with respect to Y and multiplying by $(1+\delta Y)$ gives

$$(1+\delta Y) \left(\frac{\partial^{(2)} V_1}{\partial X \partial Y} + \frac{\partial^{(2)} V_2}{\partial Y^{(2)}} \right) + \delta \frac{\partial V_2}{\partial Y} - \frac{\delta^{(2)}}{1+\delta Y} V_2 = 0. \tag{4.42}$$

Hence by Eqs. (4.41) and (4.42), Eq. (4.39) equates to zero and hence Eq. (4.22) and Eq. (4.35) are equivalent. Also, by applying Green's Theorem, it is found that Eq. (4.25) and Eq. (4.33) are equivalent. Hence the systems of equations from the model presented in this chapter, represented by Eqs. (4.26) and (4.29), and the model presented by Green et al [7], represented by Eqs. (4.30) and (4.31), are equivalent. The model of Green et al was also implemented in Maple and solved for to confirm that the two models did produce the same solutions.

4.7.2 Computational fluid dynamics simulations

The models in this chapter were compared to 3D finite volume simulations using STAR-CCM+. A curved pipe of constant radius and constant curvature was created for the finite volume solver, with the pipe extending 120° around the centre of curvature. For comparisons with the $\delta = 0.01$ case, the radius of the pipe was set to 1 m and the radius of curvature to 100 m, while for comparisons with the $\delta = 0.1$ case, the radius of the pipe was set to 1 m and the radius of curvature to 10 m. The boundary conditions were specified to as periodic on the artificial sections, while the no-slip velocity and a zero Neumann pressure condition were assigned to the pipe walls. The average coaxial velocity was set to match that of the director theory model given the prescribed coaxial pressure gradient. The density and dynamic viscosity of the fluid were set to 1 kg m^{-3} and 1 Pa s respectively.

The mesh and solution data was exported as an Ensight Gold File. Subsequently the data was saved as a CSV spreadsheet with the use of ParaView. To create the error plots in Figs. 4.8, 4.10, 4.13 and 4.14, and calculate the error means shown in Figs. 4.15 and 4.16, the simulation data was imported into Maple and the velocity from the director theory model was evaluated at each data point taken from the mesh in the simulation. To create the velocity contour plots of the simulation solutions, shown in Figs. 4.6 and 4.11, interpolation was used on the simulation data and then plotted. Although velocity contours can be viewed directly in Simcentre STAR-CCM+, this method was used so that the contour style was the same as those shown for the director theory model in Figs. 4.7, 4.9 and 4.12, for ease of visual comparison.

4.7.3 Solutions

Solutions will be shown from the model presented in this chapter and compared with those produced with 3D finite volume solver STAR-CCM+. Comparisons of friction loss against Dean number were also found to agree with those presented by Green et al [7].

As can be seen from Fig. 4.15, for a slightly curved pipe (curvature ratio of $\delta = 0.01$, it was found that while the error between the director model and the simulation initially decreases as the order of the assumed velocity expansion in the director model is increased, after around order of $H = 8$, the error remains approximately constant with increasing orders. It can also be seen that the magnitude of the constant error is higher when compared to the simulation with a coarser mesh, suggesting that at least some error comes from the simulation and its mesh discretisation. Another error source could be with how the Reynolds number are compared between the simulation and the director model. In the simulation, an appropriate value of the mass flow rate is set for the required Reynolds number. However, the director theory model, the value of Re_0 (the Reynolds number for the corresponding flow through a straight pipe) is set rather that the true value of Re. The value of Re_0 is calculated by finding the coaxial pressure gradient from the simulation and then calculating Re_0 from this bases of Poiseuille flow. Some error is likely to be introduced when finding the pressure gradient from the simulation. These factors are believed to account for the remaining error.

As can be seen from Fig. 4.16, for a more steeply curved pipe (curvature ratio of $\delta = 0.1$), the magnitude of the error does not settle until after order $H = 11$ and the magnitude of the error is higher compared with the slightly curved pipe (curvature ratio of $\delta = 0.01$) with a simulation using the same mesh base size. The error also does not uniformly decrease at lower orders in this case. This could be related to the truncations at particular orders not satisfying the system of equations as well as others. Turning our attention to the the higher resolution of finite volume solution, it appears that the error at lower orders is due to error in the model, but some of the error at higher orders may be attributed to the finite volume simulation.

Fig. 4.17 shows the error at each order compared with the solution at the highest order of the director theory model that was calculated, which is $H = 15$ for the $\delta = 0.01$ case and $H = 14$ for the $\delta = 0.1$ case. It can be seen that for the $\delta = 0.01$ case, the model appears to be converging, while for the $\delta = 0.1$ case, convergence is less clear. It appears as though their maybe be a periodic pattern. It would be interesting to run the model at higher orders to see if a clear pattern emerges, or if there is a uniform convergence past a certain order, unfortunately this is computationally difficult.

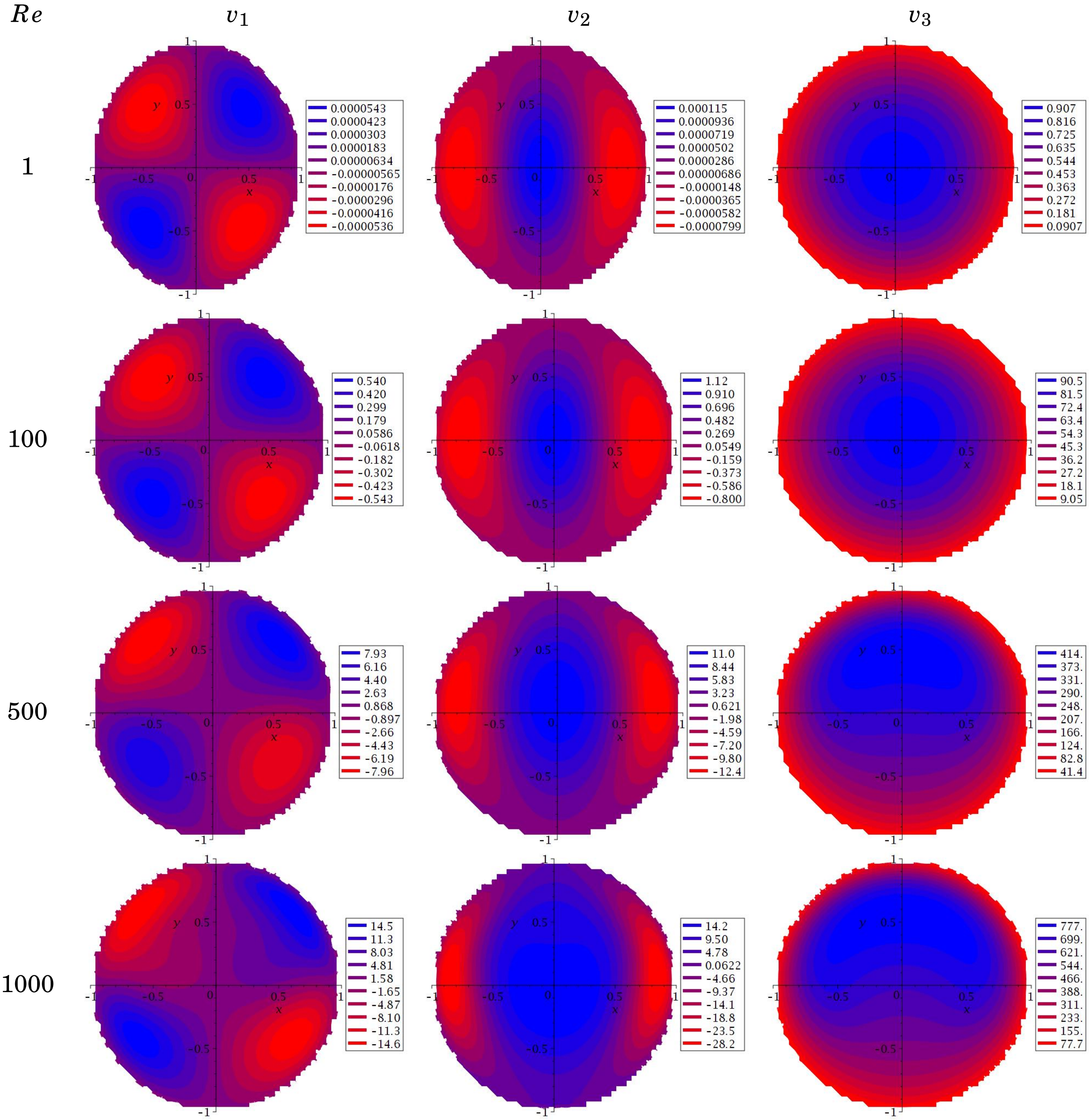


Figure 4.6: Contour plots of flow through a toroidally curved pipe cross-section from a simulation using STAR-CCM+, with curvature ratio $\delta = 0.01$, pipe radius $a = 1m$ and a mesh base size of $0.025m$, at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions respectively.

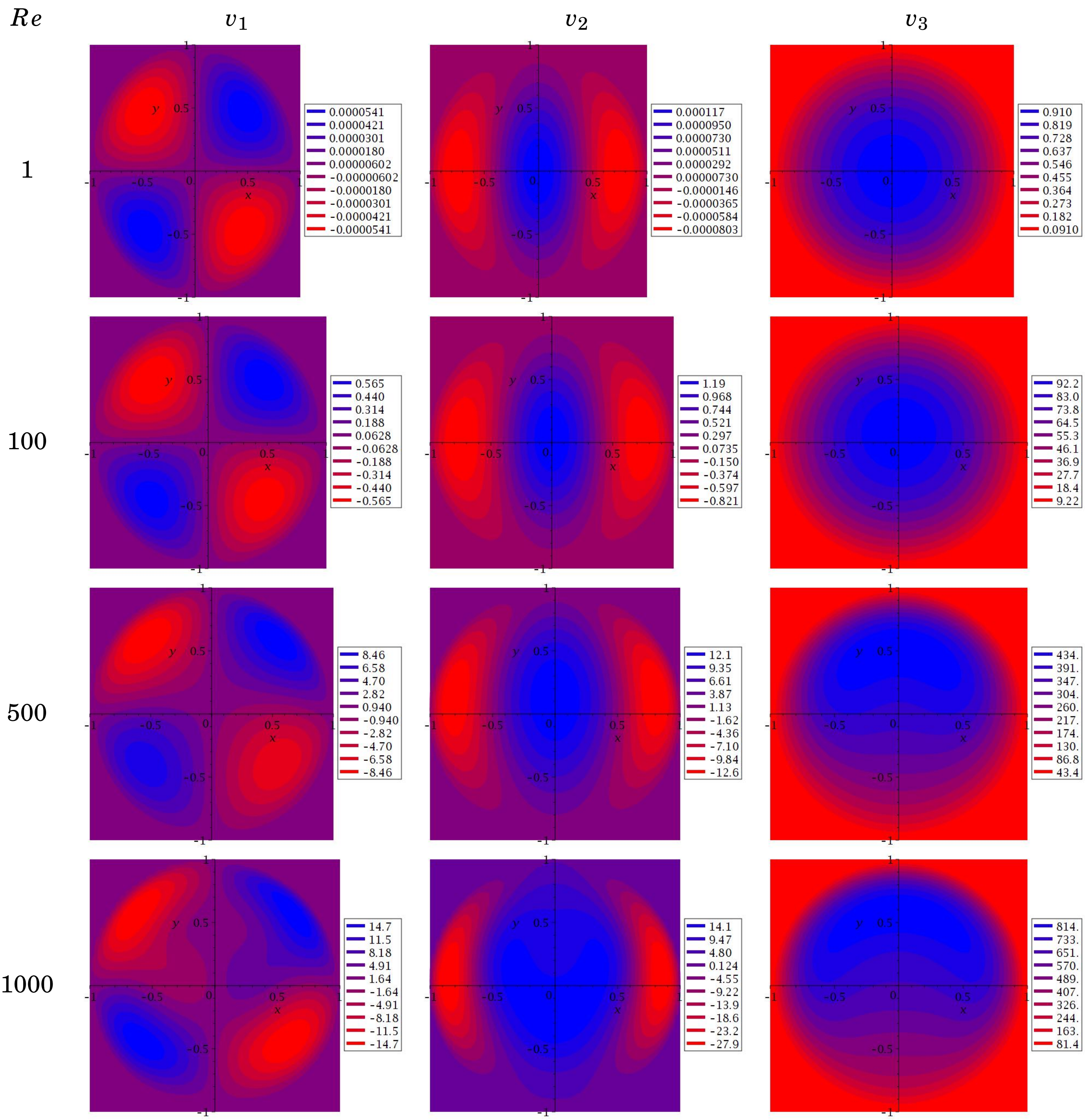


Figure 4.7: Contour plots of flow through a toroidally curved pipe cross-section given by the director theory model for order $H = 8$, with curvature ratio $\delta = 0.01$, at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions respectively.

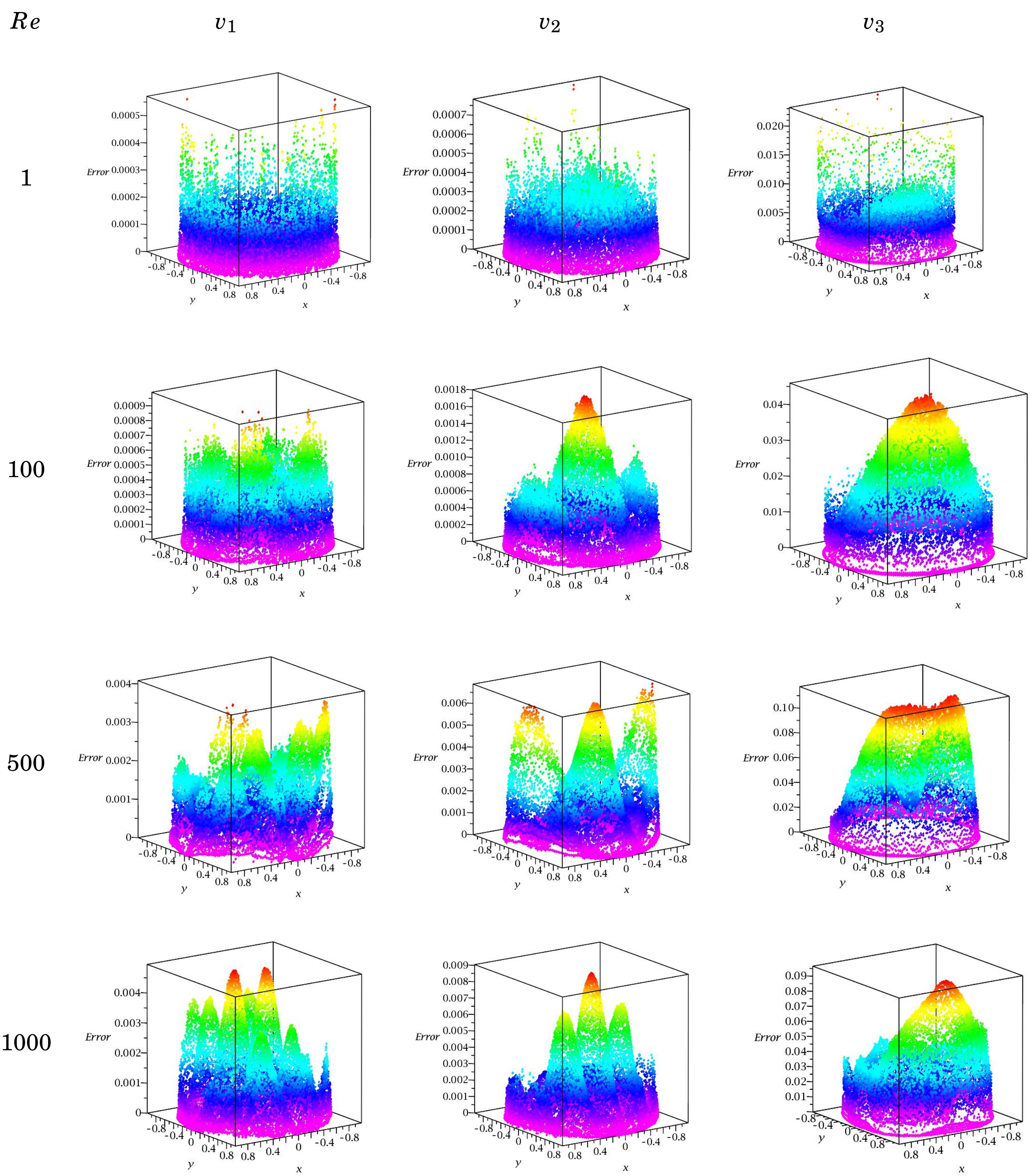


Figure 4.8: Plots of the relative error of the velocity in a cross-section of the toroidally curved pipe, between the 3D finite volume simulation with mesh base size 0.025m and the solution form the director theory model represented by Eqs. (4.26), (4.29), for order $H = 8$. Solutions are obtained for curvature ration $\delta = 0.01$ at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions respectively.

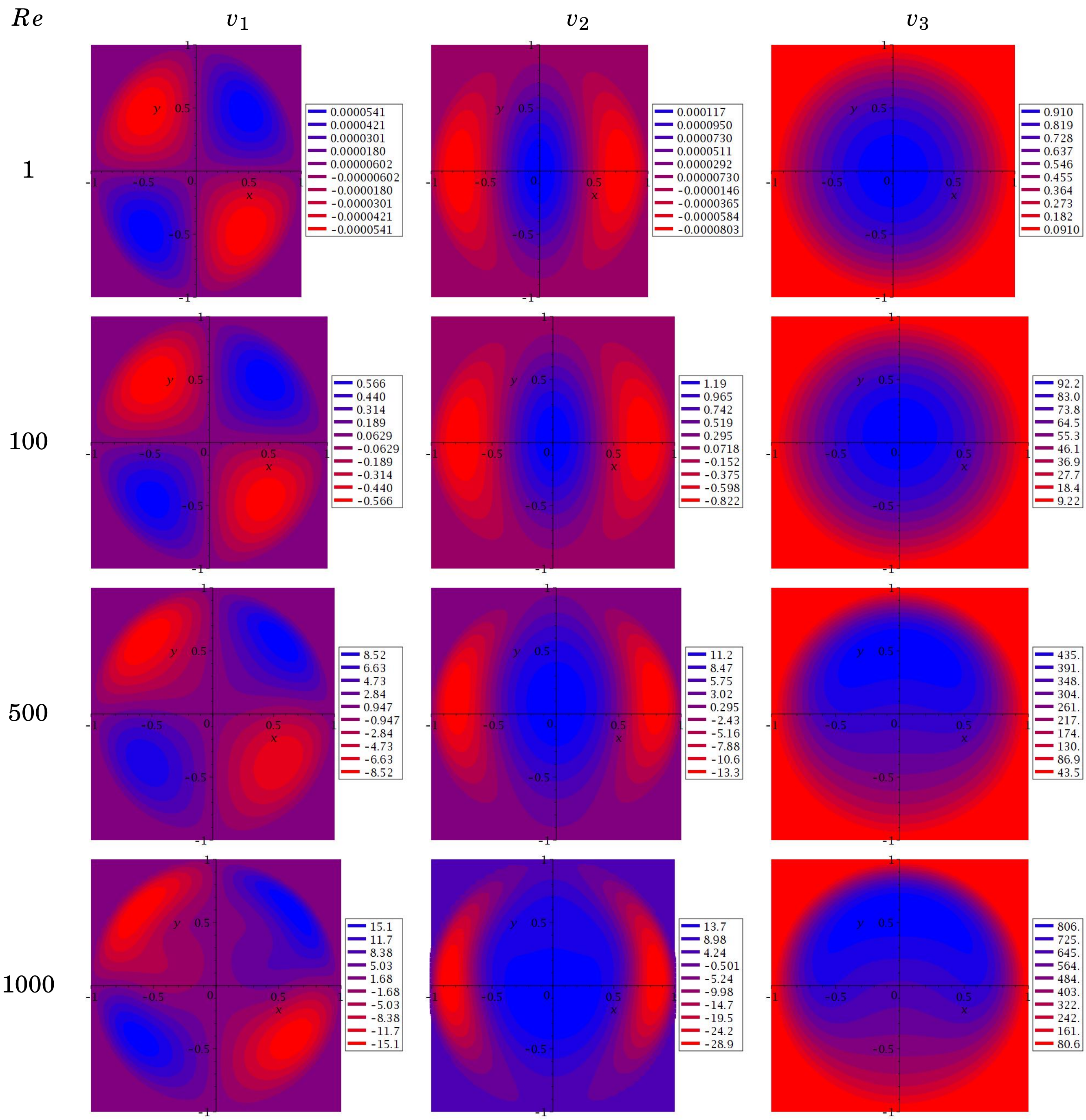


Figure 4.9: Contour plots of flow through a toroidally curved pipe cross-section given by the director theory model for order $H = 12$ with curvature ration $\delta = 0.01$, at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ directions respectively.

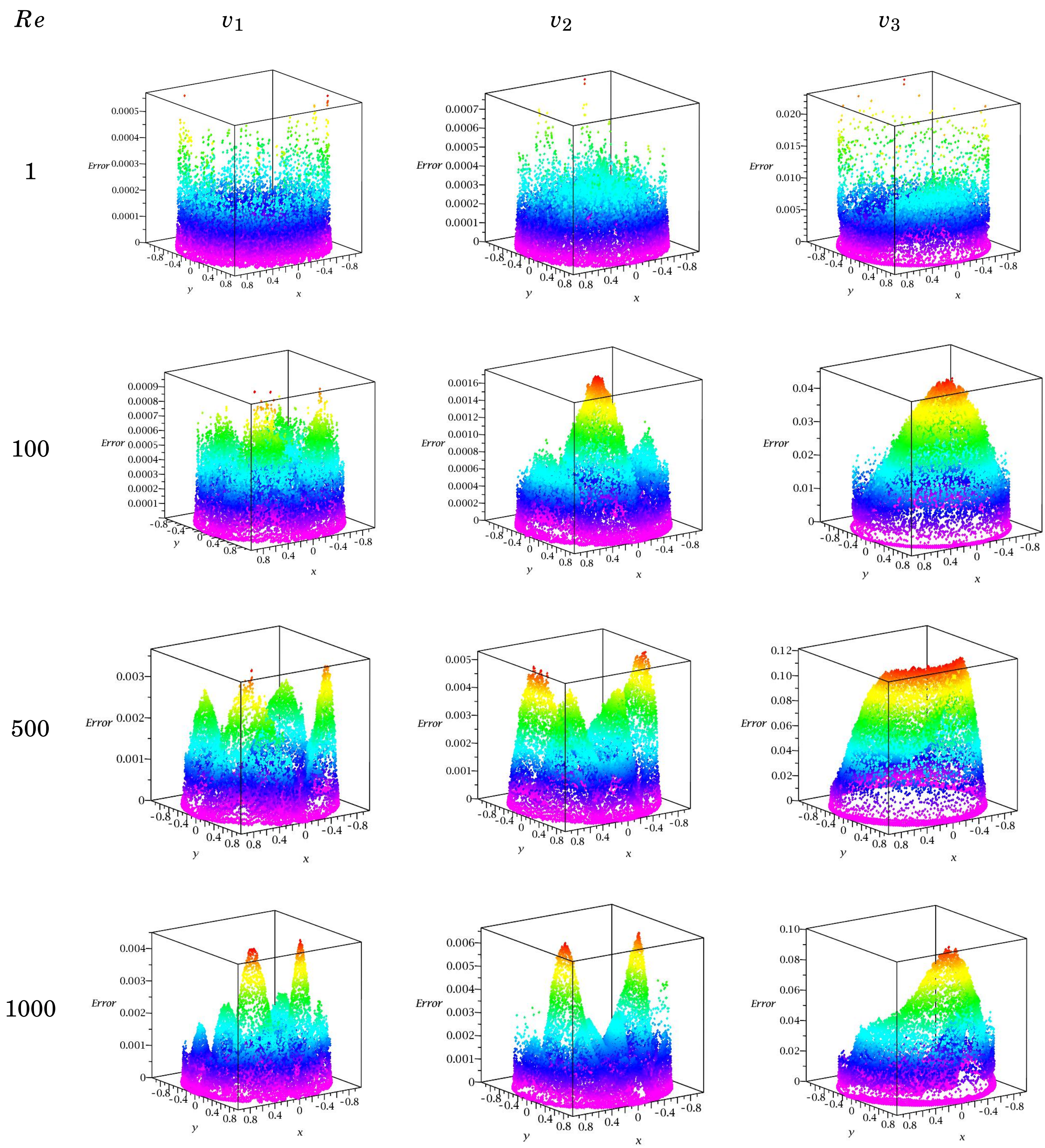


Figure 4.10: Plots of the relative error of the velocity in a cross-section of the toroidally curved pipe, between the 3D finite volume simulation with mesh base size $0.025m$ and the solution from the director theory model represented by Eqs. (4.26), (4.29), for order $H = 12$. Solutions are obtained for curvature ratio $\delta = 0.01$ at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions respectively.

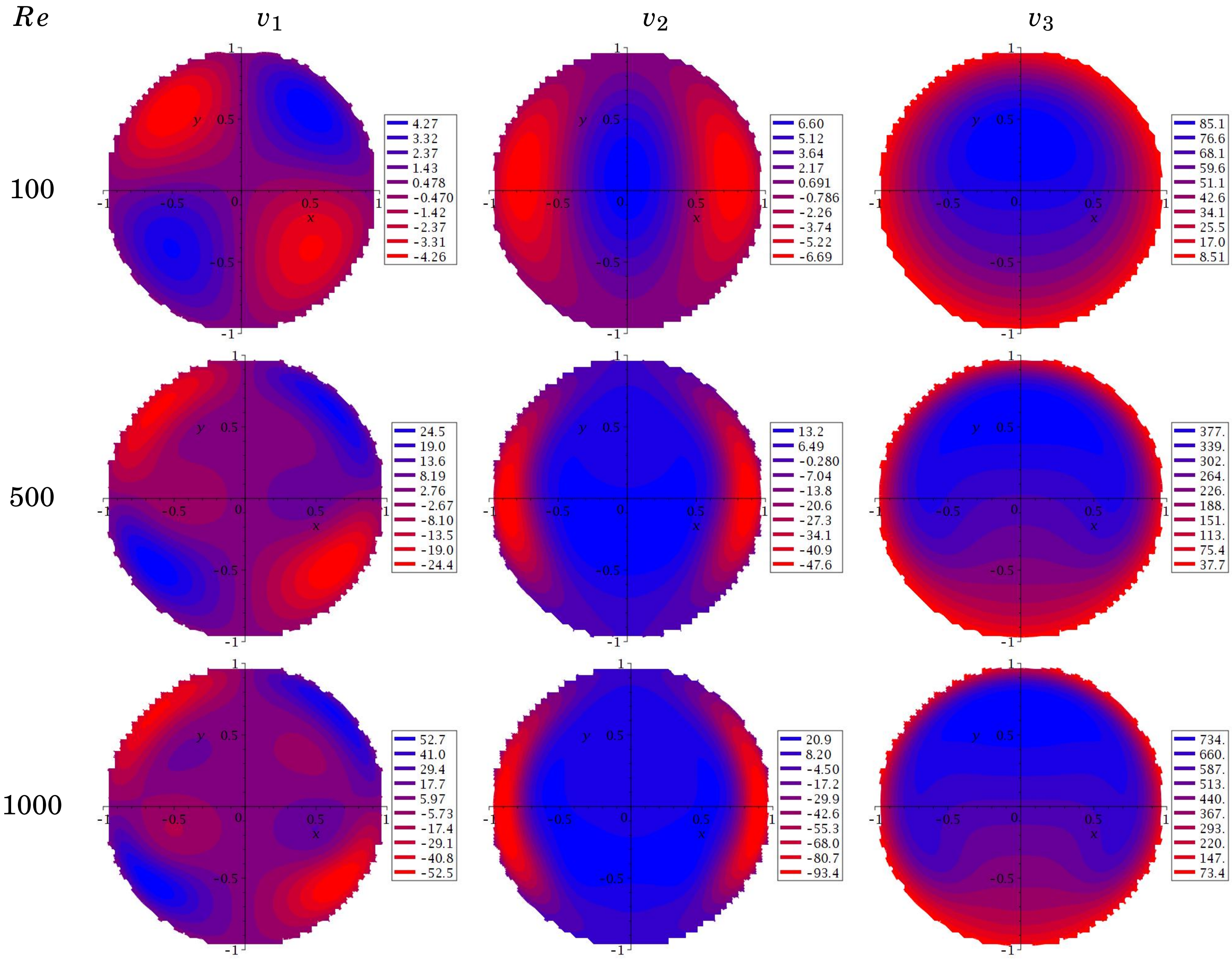


Figure 4.11: Contour plots of flow through a toroidally curved pipe cross-section from a simulation using STAR-CCM+, with curvature ratio $\delta = 0.1$, pipe radius $a = 1m$ and mesh base size of $0.025m$, at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}+3$ directions respectively.

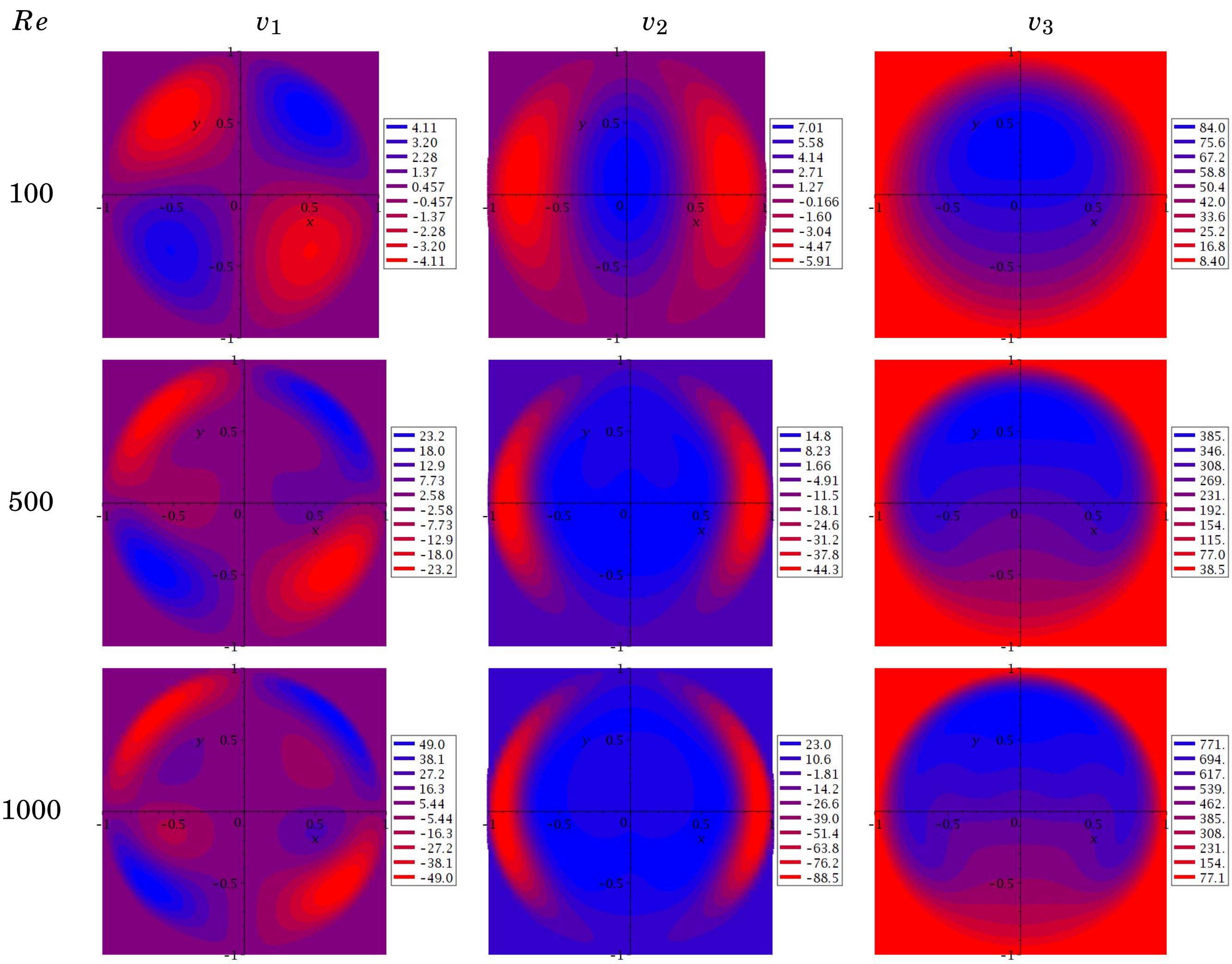


Figure 4.12: Contour plots of flow through a toroidally curved pipe cross-section given by the director theory model for order $H = 12$ with curvature ration $\delta = 0.1$, at varying Reynolds numbers, in the $\mathbf{a}_1$, $\mathbf{a}_2$, $\mathbf{a}_3$ directions respectively.

4.8 Summary of Chapter

This chapter presented a director theory approach to modelling fluid flow in a toroidally curved pipe of constant radius. This was done by setting up a curvilinear coordinate system that followed along the centreline of the pipe and subsequently deriving the equations of motion in this coordinate system. The velocity was assumed to be approximated by a series expansion of weighting functions,which depend on the cross-sectional coordinates multiplied by directors, which are vectors that in general depend on the coaxial coordinate and time (however in this specific case they are independent of both). The weighting functions were assumed to be polynomials of the in-plane coordinates, then

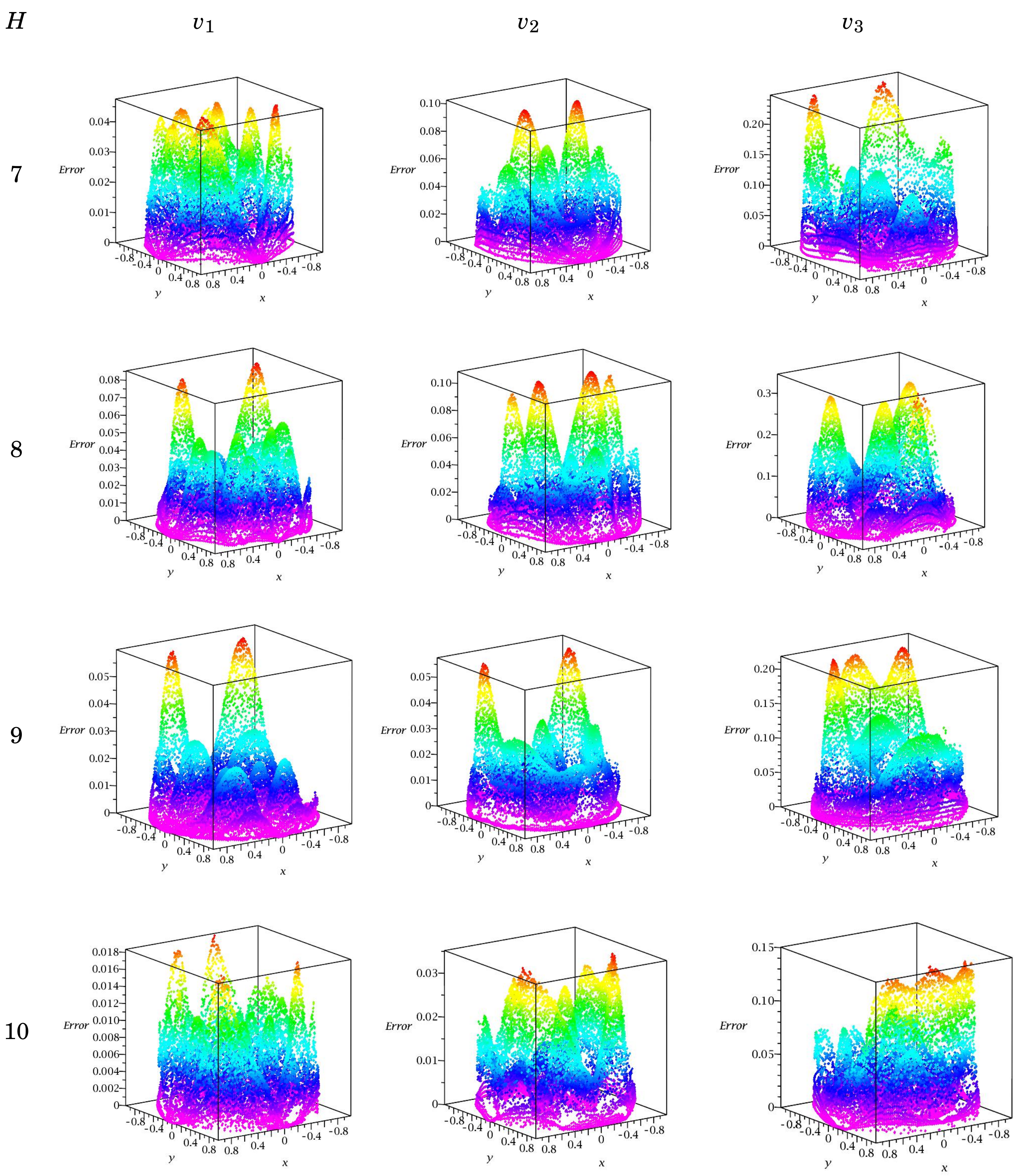


Figure 4.13: Plots of the relative error of the velocity in a cross-section of the toroidally curved pipe, between the 3D finite volume simulation with mesh base size $0.025m$ and the solution from the director theory model represented by Eqs. (4.26), (4.29) at varying orders H. Solutions are obtained for curvature ratio $\delta = 0.1$ at Reynolds number $Re = 1000$, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions respectively.

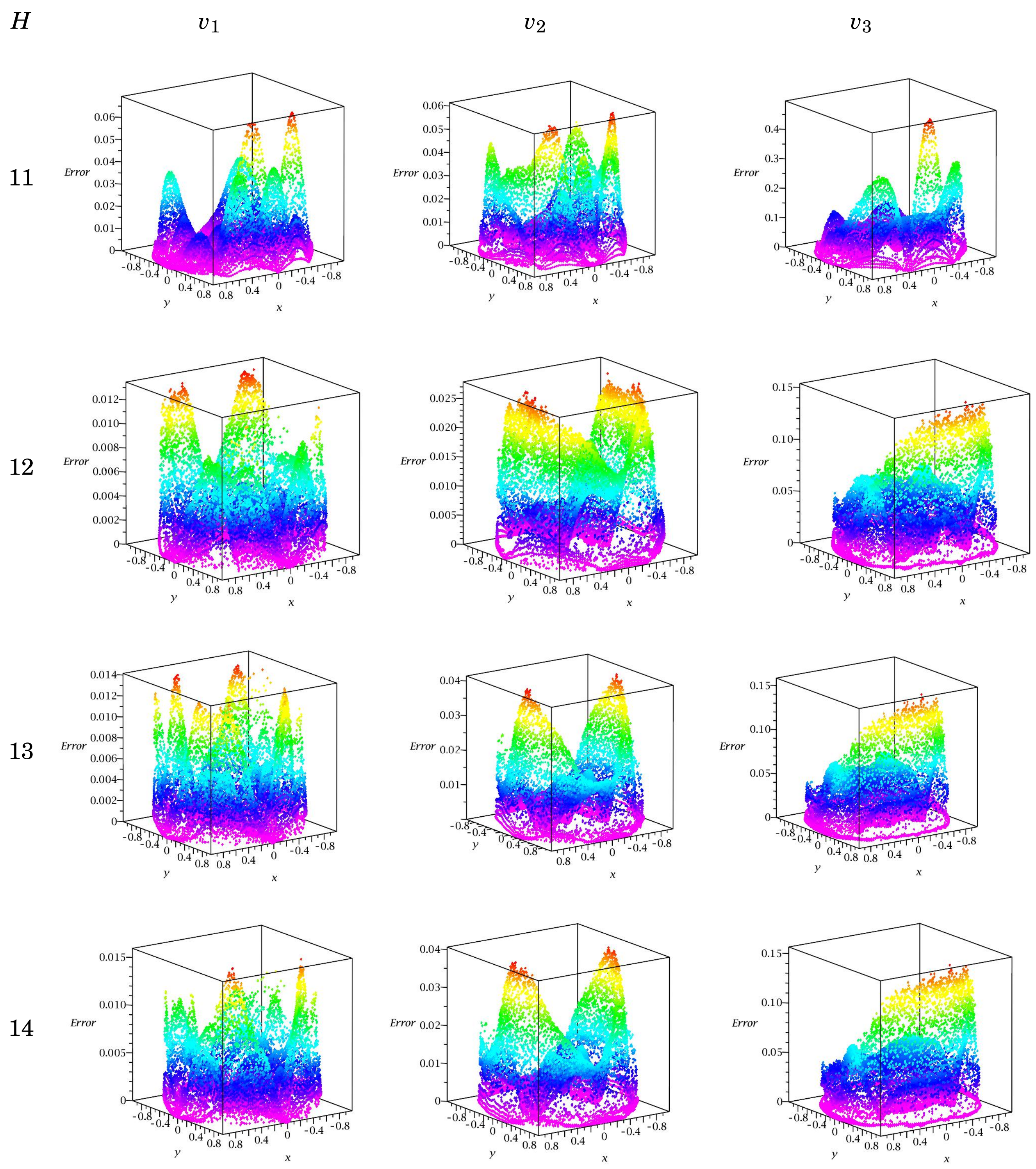
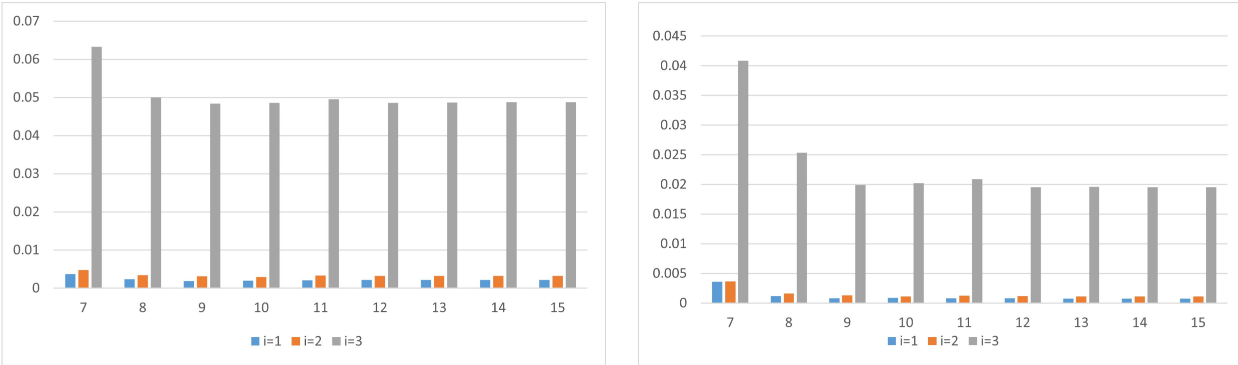
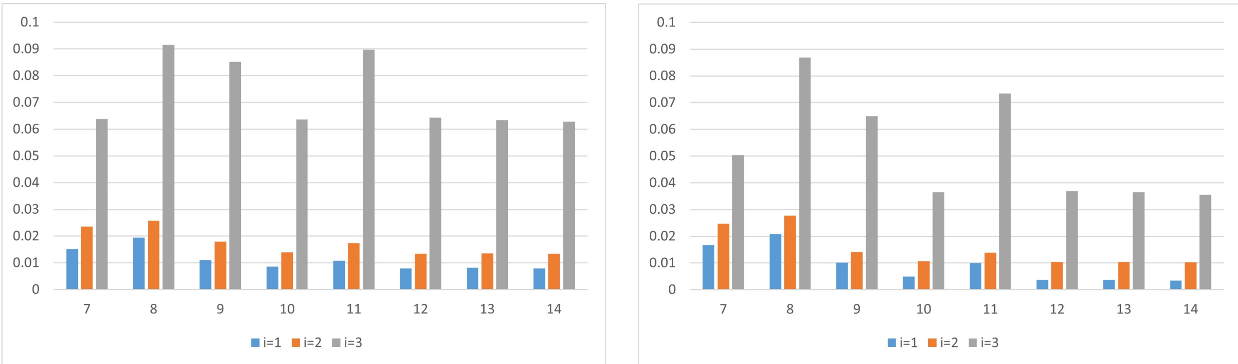


Figure 4.14: Plots of the relative error of the velocity in a cross-section of the toroidally curved pipe, between the 3D finite volume simulation with mesh base size $0.025m$ and the solution from the director theory model represented by Eqs. (4.26), (4.29) at varying orders H. Solutions are obtained for curvature ratio $\delta = 0.1$ at Reynolds number $Re = 1000$, in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions respectively.



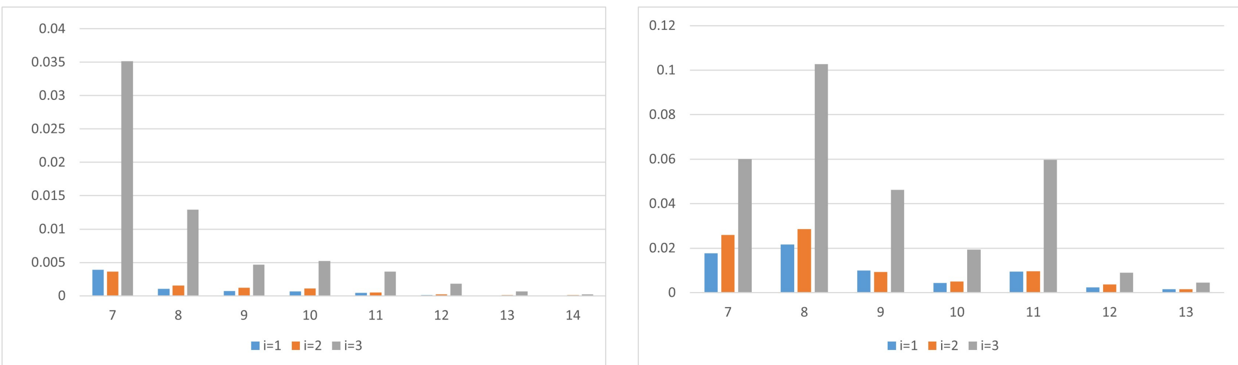
base size 0.05m base size 0.025m

Figure 4.15: Graph of the mean error of the velocity for $Re = 1000$ and $\delta = 0.01$, in the $\mathbf{a}_i$ direction for $i = 1,2,3$ of the solutions from the model presented in this chapter compared with a 3D simulation, for different mesh base sizes, in STAR-CCM+ normalised by the mean velocity, against the order H of the model for $H = 7-15$.



base size 0.05m base size 0.025m

Figure 4.16: Graph of the mean error of the velocity for $Re = 1000$ and $\delta = 0.1$, in the $\mathbf{a}_i$ direction for $i = 1,2,3$ of the solutions from the model presented in this chapter compared with a 3D simulation, for different mesh base sizes, in STAR-CCM+ normalised by the mean velocity, against the order H of the model for $H = 7-14$.



$\delta = 0.01$, order $H = 7-14$ compared to $H = 15$ $\delta = 0.1$, orders $H = 7-13$ compared to $H = 14$

Figure 4.17: Graph of the mean error of the velocity for $Re = 1000$, in the $\mathbf{a}_i$ direction for $i = 1,2,3$ of the solution from the model presented in this chapter for varying orders and is compared with the solution at the highest order.

boundary, incompressibility and symmetry conditions were applied to further restrict their form.

The equations of motion are then multiplied by each respective weighting function and integrated over the cross-section to give a system of algebraic nonlinear equations (with the $i = 1, 2$ components added together), where the unknowns are the director velocities. Through the use of Green's theorem and the incompressibility condition, the in-plane pressure gradients are eliminated from the equations. The coxial pressure gradient is prescribed to be the same as that of Poiseuille flow in a corresponding straight pipe. The equations were then solved numerically in Maple for chosen values of the curvature ratio δ and Reynolds number Re.

The solutions from the director theory approach were compared with the results from corresponding CFD simulations created in STAR-CCM+. The solutions matched reasonably well, although increasing the value of the Reynolds number or the radius of curvature (hence increasing the Dean number) tended to result in a higher relative error between the solutions.

This work could be extended by considering a pipe of varying radius, a pipe with varying curvature or torsion and time dependent or non-Newtonian flows. Some of these cases will be considered in the next chapter.

CHAPTER 5

CURVED PIPES WITH DEPENDENCE ON THE COAXIAL DIRECTION

5.1 Introduction

This chapter will be an extension of the work in Chapter 4, in which fluid flow in a toroidally curved pipe of constant radius was modelled. In the present chapter, first will be considered the case where the radius of the pipe varies. Then will be considered the case where the radius of curvature R varies. The big change from the previous chapters is that here, not only will be allowed geometries that vary along the centreline of the pipe, but also unsteady flow will be allowed for. Thus the equations describing the fluid motion will now have a dependence in the co-axial direction (ζ^3) and time (t), generalising the previous derivations.

5.2 Varying Pipe Radius

5.2.1 Equations of motion

Consider the equations of motion for a toroidally curved pipe, using the coordinate basis defined in the previous chapter, but in this case the radius of the pipe can vary, so at this stage no coaxial or time independence will be assumed. In this more general case, the Navier-Stokes equations, corresponding to Eqs. (4.16), (4.17), (4.18) in the previous chapter, are given by

(5.1)
$$\frac{\partial V_1}{\partial T}+\frac{1}{Re_0}V_1\frac{\partial V_1}{\partial X}+\frac{1}{Re_0}V_2\frac{\partial V_1}{\partial Y}+\frac{V_3}{1+\delta Y}\frac{\partial V_1}{\partial Z}=-\frac{\partial P}{\partial X}+\frac{1}{Re_0}\frac{\partial^{(2)}V_1}{\partial X^{(2)}}+\frac{1}{Re_0}\frac{\partial^{(2)}V_1}{\partial Y^{(2)}}+\frac{1}{Re_0}\frac{\delta}{1+\delta Y}\frac{\partial V_1}{\partial Y}$$
$$+\frac{1}{Re_0}\frac{1}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}V_1}{\partial Z^{(2)}};$$

(5.2)
$$\frac{\partial V_2}{\partial T}+\frac{1}{Re_0}V_1\frac{\partial V_2}{\partial X}+\frac{1}{Re_0}V_2\frac{\partial V_2}{\partial Y}+\frac{V_3}{1+\delta Y}\frac{\partial V_2}{\partial Z}-\frac{Re_0\delta}{1+\delta Y}V_3^{(2)}=-\frac{\partial P}{\partial Y}+\frac{1}{Re_0}\frac{\partial^{(2)}V_2}{\partial X^{(2)}}+\frac{1}{Re_0}\frac{\partial^{(2)}V_2}{\partial Y^{(2)}}$$
$$+\frac{1}{Re_0}\frac{\delta}{1+\delta Y}\frac{\partial V_2}{\partial Y}+\frac{1}{Re_0}\frac{1}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}V_2}{\partial Z^{(2)}}-\frac{1}{Re_0}\frac{\delta^{(2)}}{(1+\delta Y)^{(2)}}V_2-\frac{\delta}{(1+\delta Y)^{(2)}}\frac{\partial V_3}{\partial Z};$$

(5.3)
$$Re_0\frac{\partial V_3}{\partial T}+V_1\frac{\partial V_3}{\partial X}+V_2\frac{\partial V_3}{\partial Y}+\frac{\delta}{1+\delta Y}V_2V_3+Re_0\frac{V_3}{1+\delta Y}\frac{\partial V_3}{\partial Z}=-\frac{1}{1+\delta Y}\frac{\partial P}{\partial Z}+\frac{1}{Re_0}\frac{\delta}{(1+\delta Y)^{(2)}}\frac{\partial V_2}{\partial Z}$$
$$+\frac{\partial^{(2)}V_3}{\partial X^{(2)}}+\frac{\partial^{(2)}V_3}{\partial Y^{(2)}}+\frac{\delta}{1+\delta Y}\frac{\partial V_3}{\partial Y}+\frac{1}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}V_3}{\partial Z^{(2)}}-\frac{\delta^{(2)}}{(1+\delta Y)^{(2)}}V_3;$$

where T is non-dimensionalised time given by

$$T=\frac{2U}{a},$$

recalling that a is the radius of the pipe.

By integrating Eqs. (5.1) - (5.3) over the cross-section of the pipe and adding together Eq. (5.1) and Eq. (5.2), a more general form of the integrated equations of motion, given by Eqs. (4.19), (4.20), where the radius of the pipe can vary along the pipe and with time (in this case $U_N=U_N(Z,t)$, $W_N=W_N(Z,t)$ is given by

$$Re\sum_{M=1}^{L}G_{NM}\frac{\partial U_M}{\partial T}+\sum_{M=1}^{L}\sum_{R=1}^{L}H_{NMR}U_MU_R+Re\sum_{M=1}^{L}\sum_{R=1}^{K}I_{NMR}\frac{\partial U_M}{\partial Z}W_R-Re^{(2)}\sum_{M=1}^{K}\sum_{R=1}^{K}J_{NMR}W_MW_R$$
$$=-ReQ_N+\sum_{M=1}^{L}S_{NM}U_M+\sum_{M=1}^{L}O_{NM}\frac{\partial^{(2)}U_M}{\partial Z^{(2)}}-Re\sum_{M=1}^{K}T_{NM}\frac{\partial W_M}{\partial Z};$$

$$Re\sum_{M=1}^{K}A_{NM}\frac{\partial W_M}{\partial T}+\sum_{M=1}^{L}\sum_{R=1}^{K}B_{NMR}U_MW_R+\sum_{M=1}^{K}\sum_{R=1}^{K}C_{NMR}W_M\frac{\partial W_R}{\partial Z}$$
$$=-P_N+\sum_{M=1}^{L}D_{NM}\frac{\partial U_M}{\partial Z}+\sum_{M=1}^{K}E_{NM}W_M+\sum_{M=1}^{K}F_{NM}\frac{\partial^{(2)}W_M}{\partial Z^{(2)}};$$

where

$$A_{NM}=\int\int_A(1+\delta Y)\lambda_N^3\lambda_M^3\,dX\,dY;$$

$$B_{NMR} = \int\int_A \lambda_N^3 \left[(1+\delta Y)\left(\lambda_M^2 \frac{\partial \lambda_R^3}{\partial X} + \lambda_M^2 \frac{\partial \lambda_R^3}{\partial Y}\right) + \delta \lambda_M^2 \lambda_R^3\right] dX dY;$$

$$C_{NMR} = \int\int_A \lambda_N^3 \lambda_M^3 \lambda_R^3 dX dY;$$

$$P_N = \int\int_A \lambda_N^2 \frac{\partial P}{\partial Z};$$

$$D_{NM} = \int\int_A \frac{\delta}{1+\delta Y} \lambda_N^3 \lambda_M^2 dX dY;$$

$$E_{NM} = \int\int_A \lambda_N^3 \left[(1+\delta Y)\left(\frac{\partial^{(2)} \lambda_M^3}{\partial X^{(2)}} + \frac{\partial^{(2)} \lambda_M^3}{\partial Y^{(2)}}\right) + \delta \frac{\partial \lambda_M^3}{\partial Y} - \frac{\delta^{(2)}}{1+\delta Y} \lambda_M^3\right] dX dY;$$

$$F_{NM} = \int\int_A \frac{\lambda_N^3 \lambda_M^3}{1+\delta Y} dX dY;$$

$$G_{NM} = \int\int_A (1+\delta Y)(\lambda_N^1 \lambda_M^1 + \lambda_N^2 \lambda_M^2) dX dY;$$

$$H_{NMR} = \int\int_A (1+\delta Y)\left[\lambda_N^1 \lambda_M^1 \frac{\partial \lambda_R^1}{\partial X} + \lambda_N^1 \lambda_M^2 \frac{\partial \lambda_R^1}{\partial Y} + \lambda_N^2 \lambda_M^1 \frac{\partial \lambda_R^2}{\partial X} + \lambda_N^2 \lambda_M^2 \frac{\partial \lambda_R^2}{\partial Y}\right] dX dY;$$

$$I_{NMR} = \int\int_A [\lambda_N^1 \lambda_M^1 \lambda_R^3 + \lambda_N^2 \lambda_M^2 \lambda_R^3] dX dY;$$

$$J_{NMR} = \int\int_A \delta \lambda_N^2 \lambda_M^3 \lambda_R^3 dX dY;$$

$$Q_N = \int\int_A (1+\delta Y)\left(\lambda_N^1 \frac{\partial P}{\partial X} + \lambda_N^2 \frac{\partial P}{\partial Y}\right) dX dY;$$

$$S_{NM} = \int\int_A \left[(1+\delta Y)\left(\lambda_N^1 \frac{\partial^{(2)} \lambda_M^1}{\partial X^{(2)}} + \lambda_N^1 \frac{\partial \lambda_M^1}{\partial Y^{(2)}} + \lambda_N^2 \frac{\partial \lambda_M^2}{\partial X^{(2)}} + \lambda_N^2 \frac{\partial^{(2)} \lambda_M^2}{\partial Y^{(2)}}\right) + \delta\left(\lambda_N^1 \frac{\partial \lambda_M^1}{\partial Y} + \lambda_N^2 \frac{\partial \lambda_M^2}{\partial Y}\right)\right.$$
$$\left. + \frac{\delta^{(2)}}{1+\delta Y} \lambda_N^2 \lambda_M^2\right] dX dY;$$

$$O_{NM} = \int\int_A \frac{1}{1+\delta Y} (\lambda_N^1 \lambda_M^1 + \lambda_N^2 \lambda_M^2) dX dY;$$

$$T_{NM} = \int\int_A \frac{2\delta}{1+\delta Y} \lambda_N^2 \lambda_M^2 dX dY.$$

This is the case with the two cross-flow velocities sharing the same director velocities U_N. If the relevant simplifications are made to these equations, the Eqs. (4.19) and (4.20) from the previous chapter can be recovered. The equation for the incompressibility condition for the toroidally curved pipe of constant radius, namely, Eq. (4.12), provided a simple expression between V_1 and V_2 and it was a simple matter to express them in terms of the same velocity directors. However, in this case with a varying pipe radius, the incompressibility condition now includes V_3, its non-dimensional form being given by

$$\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+Re\frac{1}{1+\delta Y}\frac{\partial V_3}{\partial Z}=0, \tag{5.4}$$

and it is not trivial to find weighting functions such that V_1 and V_2 can be expressed in terms of the same velocity directors. If the velocity is expressed as a more general expansion, akin to Eq. (4.11), where the cross-flow velocities do not share the same weighting functions, the integrated equations of motion can be expressed as three separate equations, with U_N, C_N and W_N the respective director velocities in the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ directions, for the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ components respectively as

$$\begin{aligned} &Re\sum_{M=1}^{L}A1_{NM}\frac{\partial U_M}{\partial T}+\sum_{M=1}^{L}\sum_{R=1}^{L}B1_{NMR}U_MU_R+\sum_{M=1}^{K}\sum_{R=1}^{L}D1_{NMR}C_MU_R\\ &+Re\sum_{M=1}^{K}\sum_{R=1}^{L}E1_{NMR}W_M\frac{\partial U_R}{\partial Z}=-ReP1_N+\sum_{M=1}^{L}F1_{NM}U_M+\sum_{M=1}^{L}H1_{NM}\frac{\partial^{(2)}U_M}{\partial Z^{(2)}}; \end{aligned} \tag{5.5}$$

where

$$A1_{NM}=\int\int_A(1+\delta Y)\lambda_N^1\lambda_M^1dXdY;$$

$$B1_{NMR}=\int\int_A(1+\delta Y)\lambda_N^1\lambda_M^1\frac{\partial\lambda_R^1}{\partial X}dXdY;$$

$$D1_{NMR}=\int\int_A(1+\delta Y)\lambda_N^1\lambda_M^2\frac{\partial\lambda_R^1}{\partial Y}dXdY;$$

$$E1_{NMR}=\int\int_A\lambda_N^1\lambda_M^3\lambda_R^1dXdY;$$

$$P1_N=\int\int_A(1+\delta Y)\lambda_N^1\frac{\partial P}{\partial X}dXdY;$$

$$F1_{NM}=\int\int_A\left[(1+\delta Y)\left(\lambda_N^1\frac{\partial^{(2)}\lambda_M^1}{\partial X^{(2)}}+\lambda_N^1\frac{\partial^{(2)}\lambda_M^1}{\partial Y^{(2)}}\right)+\delta\lambda_N^1\frac{\partial\lambda_M^1}{\partial Y}\right]dXdY;$$

$$H1_{NM}=\int\int_A\frac{\lambda_N^1\lambda_M^1}{1+\delta Y}dXdY;$$

(5.6)

$$\begin{aligned} &Re\sum_{M=1}^{K}A2_{NM}\frac{\partial C_M}{\partial T}+\sum_{M=1}^{L}\sum_{R=1}^{K}B2_{NMR}U_MC_R+\sum_{M=1}^{K}\sum_{R=1}^{K}D2_{NMR}C_MC_R+Re\sum_{M=1}^{K}\sum_{R=1}^{K}E2_{NMR}W_M\frac{\partial C_R}{\partial Z}\\ &-Re^{(2)}\sum_{M=1}^{K}\sum_{R=1}^{K}J2_{NMR}W_MW_R=-ReP2_N+\sum_{M=1}^{K}F2_{NM}C_M+\sum_{M=1}^{K}H2_{NM}\frac{\partial^{(2)}C_M}{\partial Z^{(2)}}-Re\sum_{M=1}^{K}S2_{NM}\frac{\partial W_M}{\partial Z}; \end{aligned}$$

$$A2_{NM} = \int\int_A (1+\delta Y)\lambda_N^2\lambda_M^2 dXdY;$$

$$B2_{NMR} = \int\int_A (1+\delta Y)\lambda_N^2\lambda_M^1 \frac{\partial\lambda_R^2}{\partial X} dXdY;$$

$$D2_{NMR} = \int\int_A (1+\delta Y)\lambda_N^2\lambda_M^2 \frac{\partial\lambda_R^2}{\partial Y} dXdY;$$

$$E2_{NMR} = \int\int_A \lambda_N^2\lambda_M^3\lambda_R^2 dXdY;$$

$$J2_{NMR} = \int\int_A \delta\lambda_N^2\lambda_M^3\lambda_R^3 dXdY;$$

$$P2_N = \int\int_A (1+\delta Y)\lambda_N^2 \frac{\partial P}{\partial Y} dXdY;$$

$$F2_{NM} = \int\int_A \left[(1+\delta Y)\left(\lambda_N^2 \frac{\partial^{(2)}\lambda_M^2}{\partial X^{(2)}} + \lambda_N^2 \frac{\partial^{(2)}\lambda_M^2}{\partial Y^{(2)}}\right) + \delta\lambda_N^2 \frac{\partial\lambda_M^2}{\partial Y} - \frac{\delta^{(2)}\lambda_N^2\lambda_M^2}{1+\delta Y}\right] dXdY;$$

$$H2_{NM} = \int\int_A \frac{\lambda_N^2\lambda_M^2}{1+\delta Y} dXdY;$$

$$S2_{NM} = \int\int_A \frac{\delta\lambda_N^2\lambda_M^3}{1+\delta Y} dXdY;$$

(5.7)

$$Re^{(2)}\sum_{M=1}^{K} A3_{NM}\frac{\partial W_M}{\partial T} + Re\sum_{M=1}^{L}\sum_{R=1}^{K} B3_{NMR}U_M W_R + Re\sum_{M=1}^{K}\sum_{R=1}^{K} D3_{NMR}C_M W_R$$
$$+Re^{(2)}\sum_{M=1}^{K}\sum_{R=1}^{K} E3_{NMR}W_M\frac{\partial W_R}{\partial Z} = -ReP3_N + \sum_{M=1}^{K} S3_{NM}\frac{\partial C_M}{\partial Z} + Re\sum_{M=1}^{K} F3_{NM}W_M + Re\sum_{M=1}^{K} H3_{NM}\frac{\partial^{(2)}W_M}{\partial Z^{(2)}};$$

where

$$A3_{NM} = \int\int_A (1+\delta Y)\lambda_N^3\lambda_M^3 dXdY;$$

$$B3_{NMR} = \int\int_A (1+\delta Y)\lambda_N^3\lambda_M^1 \frac{\partial\lambda_R^3}{\partial X} dXdY;$$

$$D3_{NMR} = \int\int_A \left[(1+\delta Y)\lambda_N^3\lambda_M^2 \frac{\partial\lambda_R^3}{\partial Y} + \delta\lambda_N^3\lambda_M^2\lambda_R^3\right] dXdY;$$

$$E3_{NMR} = \int\int_A \lambda_N^3\lambda_M^3\lambda_R^3 dXdY;$$

$$P3_N = \int\int_A \lambda_N^3 \frac{\partial P}{\partial Z} dXdY;$$

$$S3_{NM} = \int\int_A \frac{\delta\lambda_N^3\lambda_M^2}{1+\delta Y} dXdY;$$

$$F3_{NM} = \int\int_A \left[(1+\delta Y)\left(\lambda_N^3 \frac{\partial^{(2)}\lambda_M^3}{\partial X^{(2)}} + \lambda_N^3 \frac{\partial^{(2)}\lambda_M^3}{\partial Y^{(2)}}\right) + \delta\lambda_N^3 \frac{\partial \lambda_M^3}{\partial Y} - \frac{\delta^{(2)}\lambda_N^3\lambda_M^3}{1+\delta Y}\right] dXdY;$$

$$H3_{NM} = \int\int_A \frac{\lambda_N^3\lambda_M^3}{1+\delta Y} dXdY.$$

The equation of motion given by Eqs. (5.5)-(5.7) can be summarised in operator notation as

$$\mathscr{M}(\frac{\partial\Psi}{\partial T}) = \mathscr{A}(\Psi) + \mathscr{B}(\Psi,\Psi) + \mathscr{C}\left(\frac{\partial\Psi}{\partial Z}\right) + \mathscr{E}\left(\Psi, \frac{\partial\Psi}{\partial Z}\right) + \mathscr{F}\left(\frac{\partial^{(2)}\Psi}{\partial Z^{(2)}}\right) - \mathscr{Q}(Z,t),$$

where $\Psi = (U_1(Z,T),...,U_L(Z,T),C_1(Z,T),...,C_K(Z,T),W_1(Z,T),...W_K(Z,T))$, $\mathscr{A}$ is a linear operator, $\mathscr{B}$ is a homogeneous quadratic operator, $\mathscr{M}$ and $\mathscr{C}$ are linear first order differential operators, $\mathscr{E}$ is a homogeneous quadratic first order differential operator, $\mathscr{F}$ is a linear second order operator and $\mathscr{Q}$ is the operator denoting the integrated pressure gradient.

To illustrate more clearly what the system of equations look like, taking the order $H = 3$ which corresponds to $L = 2$ and $K = 4$, the $\mathbf{a}_1$ component of the equations are

$$\begin{aligned} Re\left(A1_{11}\frac{\partial U_1}{\partial T} + A1_{12}\frac{\partial U_2}{\partial T}\right) &= F1_{11}U_1 + F1_{12}U_2 - B1_{111}U_1^{(2)} - B1_{112}U_1U_2 - B1_{121}U_2U_1 - B1_{122}U_2^{(2)} \\ &- D1_{111}C_1U_1 - D1_{112}C_1U_2 - D1_{121}C_2U_1 - D1_{122}C_2U_2 - D1_{131}C_3U_1 - D1_{132}C_3U_2 \\ &- D1_{141}C_4U_1 - D1_{142}C_4U_2 - Re\left(E1_{111}W_1\frac{\partial U_1}{\partial Z} + E1_{112}W_1\frac{\partial U_2}{\partial Z} + E1_{121}W_2\frac{\partial U_1}{\partial Z}\right. \\ &\left. + E1_{122}W_2\frac{\partial U_2}{\partial Z} + E1_{131}W_3\frac{\partial U_1}{\partial Z} + E1_{132}W_3\frac{\partial U_2}{\partial Z} + E1_{141}W_4\frac{\partial U_1}{\partial Z} + E1_{142}W_4\frac{\partial U_2}{\partial Z}\right) \\ &+ H1_{11}\frac{\partial^{(2)}U_1}{\partial Z^{(2)}} + H1_{12}\frac{\partial^{(2)}U_2}{\partial Z^{(2)}} - ReP1_1; \end{aligned}$$

$$\begin{aligned} Re\left(A1_{21}\frac{\partial U_1}{\partial T} + A1_{22}\frac{\partial U_2}{\partial T}\right) &= F1_{21}U_1 + F1_{22}U_2 - B1_{211}U_1^{(2)} - B1_{212}U_1U_2 - B1_{221}U_2U_1 - B1_{222}U_2^{(2)} \\ &- D1_{211}C_1U_1 - D1_{212}C_1U_2 - D1_{221}C_2U_1 - D1_{222}C_2U_2 - D1_{231}C_3U_1 - D1_{232}C_3U_2 \\ &- D1_{241}C_4U_1 - D1_{242}C_4U_2 - Re\left(E1_{211}W_1\frac{\partial U_1}{\partial Z} + E1_{212}W_1\frac{\partial U_2}{\partial Z} + E1_{221}W_2\frac{\partial U_1}{\partial Z}\right. \\ &\left. + E1_{222}W_2\frac{\partial U_2}{\partial Z} + E1_{231}W_3\frac{\partial U_1}{\partial Z} + E1_{232}W_3\frac{\partial U_2}{\partial Z} + E1_{241}W_4\frac{\partial U_1}{\partial Z} + E1_{242}W_4\frac{\partial U_2}{\partial Z}\right) \\ &+ H1_{21}\frac{\partial^{(2)}U_1}{\partial Z^{(2)}} + H1_{22}\frac{\partial^{(2)}U_2}{\partial Z^{(2)}} - ReP1_2. \end{aligned}$$

The $\mathbf{a}_2$ component of the equations are

$$
\begin{aligned}
&Re\left(A2_{11}\frac{\partial C_1}{\partial T}+A2_{12}\frac{\partial C_2}{\partial T}+A2_{13}\frac{\partial C_3}{\partial T}+A2_{14}\frac{\partial C_4}{\partial T}\right)=F2_{11}C_1+F2_{12}C_2+F2_{13}C_3+F2_{14}C_4\\
&\quad -B2_{111}U_1C_1-B2_{112}U_1C_2-B2_{113}U_1C_3-B2_{114}U_1C_4-B2_{121}U_2C_1-B2_{122}U_2C_2\\
&\quad -B2_{123}U_2C_3-B2_{124}U_2C_4-D2_{111}C_1^{(2)}-D2_{112}C_1C_2-D2_{113}C_1C_3-D2_{114}C_1C_4\\
&\quad -D2_{121}C_2C_1-D2_{122}C_2^{(2)}-D2_{123}C_2C_3-D2_{124}C_2C_4-D2_{131}C_3C_1-D2_{132}C_3C_2-D2_{133}C_3^{(2)}\\
&\quad -D2_{134}C_3C_4-D2_{141}C_4C_1-D2_{142}C_4C_2-D2_{143}C_4C_3-D2_{144}C_4^{(2)}-Re^{(2)}\Big(J2_{111}W_1^{(2)}\\
&\quad +J2_{112}W_1W_2+J2_{113}W_1W_3+J2_{114}W_1W_4+J2_{121}W_2W_1+J2_{122}W_2^{(2)}+J2_{123}W_2W_3\\
&\quad +J2_{124}W_2W_4+J2_{131}W_3W_1+J2_{132}W_3W_2+J2_{133}W_3^{(2)}+J2_{134}W_3W_4+J2_{141}W_4W_1\\
&\quad +J2_{142}W_4W_2+J2_{143}W_4W_3+J2_{144}W_4^{(2)}\Big)-2Re\left(S2_{11}\frac{\partial W_1}{\partial Z}+S2_{12}\frac{\partial W_2}{\partial Z}+S2_{13}\frac{\partial W_3}{\partial Z}+S2_{14}\frac{\partial W_4}{\partial Z}\right)\\
&\quad -E2_{111}W_1\frac{\partial C_1}{\partial Z}-E2_{112}W_1\frac{\partial C_2}{\partial Z}-E2_{113}W_1\frac{\partial C_3}{\partial Z}-E2_{114}W_1\frac{\partial C_4}{\partial Z}-E2_{121}W_2\frac{\partial C_1}{\partial Z}-E2_{122}W_2\frac{\partial C_2}{\partial Z}\\
&\quad -E2_{123}W_2\frac{\partial C_3}{\partial Z}-E2_{124}W_2\frac{\partial C_4}{\partial Z}-E2_{131}W_3\frac{\partial C_1}{\partial Z}-E2_{132}W_3\frac{\partial C_2}{\partial Z}-E2_{133}W_3\frac{\partial C_3}{\partial Z}-E2_{134}W_3\frac{\partial C_4}{\partial Z}\\
&\quad -E2_{141}W_4\frac{\partial C_1}{\partial Z}-E2_{142}W_4\frac{\partial C_2}{\partial Z}-E2_{143}W_4\frac{\partial C_3}{\partial Z}-E2_{144}W_4\frac{\partial C_4}{\partial Z}+H2_{11}\frac{\partial^{(2)}C_1}{\partial Z^{(2)}}+H2_{12}\frac{\partial^{(2)}C_2}{\partial Z^{(2)}}\\
&\quad +H2_{13}\frac{\partial^{(2)}C_3}{\partial Z^{(2)}}+H2_{14}\frac{\partial^{(2)}C_4}{\partial Z^{(2)}}-ReP2_1;
\end{aligned}
$$

$$
\begin{aligned}
&Re\left(A2_{21}\frac{\partial C_1}{\partial T}+A2_{22}\frac{\partial C_2}{\partial T}+A2_{23}\frac{\partial C_3}{\partial T}+A2_{24}\frac{\partial C_4}{\partial T}\right)=F2_{21}C_1+F2_{22}C_2+F2_{23}C_3+F2_{24}C_4\\
&\quad -B2_{211}U_1C_1-B2_{212}U_1C_2-B2_{213}U_1C_3-B2_{214}U_1C_4-B2_{221}U_2C_1-B2_{222}U_2C_2\\
&\quad -B2_{223}U_2C_3-B2_{224}U_2C_4-D2_{211}C_1^{(2)}-D2_{212}C_1C_2-D2_{213}C_1C_3-D2_{214}C_1C_4\\
&\quad -D2_{221}C_2C_1-D2_{222}C_2^{(2)}-D2_{223}C_2C_3-D2_{224}C_2C_4-D2_{231}C_3C_1-D2_{232}C_3C_2-D2_{233}C_3^{(2)}\\
&\quad -D2_{234}C_3C_4-D2_{241}C_4C_1-D2_{242}C_4C_2-D2_{243}C_4C_3-D2_{244}C_4^{(2)}-Re^{(2)}\Big(J2_{211}W_1^{(2)}\\
&\quad +J2_{212}W_1W_2+J2_{213}W_1W_3+J2_{214}W_1W_4+J2_{221}W_2W_1+J2_{222}W_2^{(2)}+J2_{223}W_2W_3\\
&\quad +J2_{224}W_2W_4+J2_{231}W_3W_1+J2_{232}W_3W_2+J2_{233}W_3^{(2)}+J2_{234}W_3W_4+J2_{241}W_4W_1\\
&\quad +J2_{242}W_4W_2+J2_{243}W_4W_3+J2_{244}W_4^{(2)}\Big)-2Re\left(S2_{21}\frac{\partial W_1}{\partial Z}+S2_{22}\frac{\partial W_2}{\partial Z}+S2_{23}\frac{\partial W_3}{\partial Z}+S2_{24}\frac{\partial W_4}{\partial Z}\right)\\
&\quad -E2_{211}W_1\frac{\partial C_1}{\partial Z}-E2_{212}W_1\frac{\partial C_2}{\partial Z}-E2_{213}W_1\frac{\partial C_3}{\partial Z}-E2_{214}W_1\frac{\partial C_4}{\partial Z}-E2_{221}W_2\frac{\partial C_1}{\partial Z}-E2_{222}W_2\frac{\partial C_2}{\partial Z}\\
&\quad -E2_{223}W_2\frac{\partial C_3}{\partial Z}-E2_{224}W_2\frac{\partial C_4}{\partial Z}-E2_{231}W_3\frac{\partial C_1}{\partial Z}-E2_{232}W_3\frac{\partial C_2}{\partial Z}-E2_{233}W_3\frac{\partial C_3}{\partial Z}-E2_{234}W_3\frac{\partial C_4}{\partial Z}\\
&\quad -E2_{241}W_4\frac{\partial C_1}{\partial Z}-E2_{242}W_4\frac{\partial C_2}{\partial Z}-E2_{243}W_4\frac{\partial C_3}{\partial Z}-E2_{244}W_4\frac{\partial C_4}{\partial Z}+H2_{21}\frac{\partial^{(2)}C_1}{\partial Z^{(2)}}+H2_{22}\frac{\partial^{(2)}C_2}{\partial Z^{(2)}}\\
&\quad +H2_{23}\frac{\partial^{(2)}C_3}{\partial Z^{(2)}}+H2_{24}\frac{\partial^{(2)}C_4}{\partial Z^{(2)}}-ReP2_2;
\end{aligned}
$$

$$
\begin{aligned}
&Re\left(A2_{31}\frac{\partial C_1}{\partial T}+A2_{32}\frac{\partial C_2}{\partial T}+A2_{33}\frac{\partial C_3}{\partial T}+A2_{34}\frac{\partial C_4}{\partial T}\right)=F2_{31}C_1+F2_{32}C_2+F2_{33}C_3+F2_{34}C_4\\
&\quad -B2_{311}U_1C_1-B2_{312}U_1C_2-B2_{313}U_1C_3-B2_{314}U_1C_4-B2_{321}U_2C_1-B2_{322}U_2C_2\\
&\quad -B2_{323}U_2C_3-B2_{324}U_2C_4-D2_{311}C_1^{(2)}-D2_{312}C_1C_2-D2_{313}C_1C_3-D2_{314}C_1C_4\\
&\quad -D2_{321}C_2C_1-D2_{322}C_2^{(2)}-D2_{323}C_2C_3-D2_{324}C_2C_4-D2_{331}C_3C_1-D2_{332}C_3C_2-D2_{333}C_3^{(2)}\\
&\quad -D2_{334}C_3C_4-D2_{341}C_4C_1-D2_{342}C_4C_2-D2_{343}C_4C_3-D2_{344}C_4^{(2)}-Re^{(2)}\Big(J2_{311}W_1^{(2)}\\
&\quad +J2_{312}W_1W_2+J2_{313}W_1W_3+J2_{314}W_1W_4+J2_{321}W_2W_1+J2_{322}W_2^{(2)}+J2_{323}W_2W_3\\
&\quad +J2_{324}W_2W_4+J2_{331}W_3W_1+J2_{332}W_3W_2+J2_{333}W_3^{(2)}+J2_{334}W_3W_4+J2_{341}W_4W_1\\
&\quad +J2_{342}W_4W_2+J2_{343}W_4W_3+J2_{344}W_4^{(2)}\Big)-2Re\left(S2_{31}\frac{\partial W_1}{\partial Z}+S2_{32}\frac{\partial W_2}{\partial Z}+S2_{33}\frac{\partial W_3}{\partial Z}+S2_{34}\frac{\partial W_4}{\partial Z}\right)\\
&\quad -E2_{311}W_1\frac{\partial C_1}{\partial Z}-E2_{312}W_1\frac{\partial C_2}{\partial Z}-E2_{313}W_1\frac{\partial C_3}{\partial Z}-E2_{314}W_1\frac{\partial C_4}{\partial Z}-E2_{321}W_2\frac{\partial C_1}{\partial Z}-E2_{322}W_2\frac{\partial C_2}{\partial Z}\\
&\quad -E2_{323}W_2\frac{\partial C_3}{\partial Z}-E2_{324}W_2\frac{\partial C_4}{\partial Z}-E2_{331}W_3\frac{\partial C_1}{\partial Z}-E2_{332}W_3\frac{\partial C_2}{\partial Z}-E2_{333}W_3\frac{\partial C_3}{\partial Z}-E2_{334}W_3\frac{\partial C_4}{\partial Z}\\
&\quad -E2_{341}W_4\frac{\partial C_1}{\partial Z}-E2_{342}W_4\frac{\partial C_2}{\partial Z}-E2_{343}W_4\frac{\partial C_3}{\partial Z}-E2_{344}W_4\frac{\partial C_4}{\partial Z}+H2_{31}\frac{\partial^{(2)}C_1}{\partial Z^{(2)}}+H2_{32}\frac{\partial^{(2)}C_2}{\partial Z^{(2)}}\\
&\quad +H2_{33}\frac{\partial^{(2)}C_3}{\partial Z^{(2)}}+H2_{34}\frac{\partial^{(2)}C_4}{\partial Z^{(2)}}-ReP2_3;
\end{aligned}
$$

$$
\begin{aligned}
&Re\left(A2_{41}\frac{\partial C_1}{\partial T}+A2_{42}\frac{\partial C_2}{\partial T}+A2_{43}\frac{\partial C_3}{\partial T}+A2_{44}\frac{\partial C_4}{\partial T}\right)=F2_{41}C_1+F2_{42}C_2+F2_{43}C_3+F2_{44}C_4\\
&\quad -B2_{411}U_1C_1-B2_{412}U_1C_2-B2_{413}U_1C_3-B2_{414}U_1C_4-B2_{421}U_2C_1-B2_{422}U_2C_2\\
&\quad -B2_{423}U_2C_3-B2_{424}U_2C_4-D2_{411}C_1^{(2)}-D2_{412}C_1C_2-D2_{413}C_1C_3-D2_{414}C_1C_4\\
&\quad -D2_{421}C_2C_1-D2_{422}C_2^{(2)}-D2_{423}C_2C_3-D2_{424}C_2C_4-D2_{431}C_3C_1-D2_{432}C_3C_2-D2_{433}C_3^{(2)}\\
&\quad -D2_{434}C_3C_4-D2_{441}C_4C_1-D2_{442}C_4C_2-D2_{443}C_4C_3-D2_{444}C_4^{(2)}-Re^{(2)}\Big(J2_{411}W_1^{(2)}\\
&\quad +J2_{412}W_1W_2+J2_{413}W_1W_3+J2_{414}W_1W_4+J2_{421}W_2W_1+J2_{422}W_2^{(2)}+J2_{423}W_2W_3\\
&\quad +J2_{424}W_2W_4+J2_{431}W_3W_1+J2_{432}W_3W_2+J2_{433}W_3^{(2)}+J2_{434}W_3W_4+J2_{441}W_4W_1\\
&\quad +J2_{442}W_4W_2+J2_{443}W_4W_3+J2_{444}W_4^{(2)}\Big)-2Re\left(S2_{41}\frac{\partial W_1}{\partial Z}+S2_{42}\frac{\partial W_2}{\partial Z}+S2_{43}\frac{\partial W_3}{\partial Z}+S2_{44}\frac{\partial W_4}{\partial Z}\right)\\
&\quad -E2_{411}W_1\frac{\partial C_1}{\partial Z}-E2_{412}W_1\frac{\partial C_2}{\partial Z}-E2_{413}W_1\frac{\partial C_3}{\partial Z}-E2_{414}W_1\frac{\partial C_4}{\partial Z}-E2_{421}W_2\frac{\partial C_1}{\partial Z}-E2_{422}W_2\frac{\partial C_2}{\partial Z}\\
&\quad -E2_{423}W_2\frac{\partial C_3}{\partial Z}-E2_{424}W_2\frac{\partial C_4}{\partial Z}-E2_{431}W_3\frac{\partial C_1}{\partial Z}-E2_{432}W_3\frac{\partial C_2}{\partial Z}-E2_{433}W_3\frac{\partial C_3}{\partial Z}-E2_{434}W_3\frac{\partial C_4}{\partial Z}\\
&\quad -E2_{441}W_4\frac{\partial C_1}{\partial Z}-E2_{442}W_4\frac{\partial C_2}{\partial Z}-E2_{443}W_4\frac{\partial C_3}{\partial Z}-E2_{444}W_4\frac{\partial C_4}{\partial Z}+H2_{41}\frac{\partial^{(2)}C_1}{\partial Z^{(2)}}+H2_{42}\frac{\partial^{(2)}C_2}{\partial Z^{(2)}}\\
&\quad +H2_{43}\frac{\partial^{(2)}C_3}{\partial Z^{(2)}}+H2_{44}\frac{\partial^{(2)}C_4}{\partial Z^{(2)}}-ReP2_4.
\end{aligned}
$$

The $\mathbf{a}_3$ component of the equations are

$$
\begin{aligned}
&Re\left(A3_{11}\frac{\partial W_1}{\partial T}+A3_{12}\frac{\partial W_2}{\partial T}+A3_{13}\frac{\partial W_3}{\partial T}+A3_{14}\frac{\partial W_4}{\partial T}\right)=F3_{11}W_1+F3_{12}W_2+F3_{13}W_3+F3_{14}W_4\\
&\quad -B3_{111}U_1W_1-B3_{112}U_1W_2-B3_{113}U_1W_3-B3_{114}U_1W_4-B3_{121}U_2W_1-B3_{122}U_2W_2\\
&\quad -B3_{123}U_2W_3-B3_{124}U_2W_4-D3_{111}C_1W_1-D3_{112}C_1W_2-D3_{113}C_1W_3-D3_{114}C_1W_4\\
&\quad -D3_{121}C_2W_1-D3_{122}C_2W_2-D3_{123}C_2W_3-D3_{124}C_2W_4-D3_{131}C_3W_1-D3_{132}C_3W_2\\
&\quad -D3_{133}C_3W_3-D3_{134}C_3W_4-D3_{141}C_4W_1-D3_{142}C_4W_2-D3_{143}C_4W_3-D3_{144}C_4W_4\\
&\quad +S3_{11}\frac{\partial C_1}{\partial Z}+S3_{12}\frac{\partial C_2}{\partial Z}+S3_{13}\frac{\partial C_3}{\partial Z}+S3_{14}\frac{\partial C_4}{\partial Z}-Re\left(E3_{111}W_1\frac{\partial W_1}{\partial Z}+E3_{112}W_1\frac{\partial W_2}{\partial Z}\right.\\
&\quad +E3_{113}W_1\frac{\partial W_3}{\partial Z}+E3_{114}W_1\frac{\partial W_4}{\partial Z}+E3_{121}W_2\frac{\partial W_1}{\partial Z}+E3_{122}W_2\frac{\partial W_2}{\partial Z}+E3_{123}W_2\frac{\partial W_3}{\partial Z}\\
&\quad +E3_{124}W_2\frac{\partial W_4}{\partial Z}-E3_{131}W_3\frac{\partial W_1}{\partial Z}-E3_{132}W_3\frac{\partial W_3}{\partial Z}+E3_{133}W_3\frac{\partial W_3}{\partial Z}+E3_{134}W_3\frac{\partial W_4}{\partial Z}+E3_{141}W_4\frac{\partial W_1}{\partial Z}\\
&\quad \left.-E3_{142}W_4\frac{\partial W_2}{\partial Z}+E3_{143}W_4\frac{\partial W_3}{\partial Z}+E3_{144}W_4\frac{\partial W_4}{\partial Z}\right)+H3_{11}\frac{\partial^{(2)}W_1}{\partial Z^{(2)}}+H3_{12}\frac{\partial^{(2)}W_2}{\partial Z^{(2)}}+H3_{13}\frac{\partial^{(2)}W_3}{\partial Z^{(2)}}\\
&\quad +H3_{14}\frac{\partial^{(2)}W_4}{\partial Z^{(2)}}-P3_1;
\end{aligned}
$$

$$
\begin{aligned}
Re&\left(A3_{21}\frac{\partial W_1}{\partial T}+A3_{22}\frac{\partial W_2}{\partial T}+A3_{23}\frac{\partial W_3}{\partial T}+A3_{24}\frac{\partial W_4}{\partial T}\right)=F3_{21}W_1+F3_{22}W_2+F3_{23}W_3+F3_{24}W_4\\
&-B3_{211}U_1W_1-B3_{212}U_1W_2-B3_{213}U_1W_3-B3_{214}U_1W_4-B3_{221}U_2W_1-B3_{222}U_2W_2\\
&-B3_{223}U_2W_3-B3_{224}U_2W_4-D3_{211}C_1W_1-D3_{212}C_1W_2-D3_{213}C_1W_3-D3_{214}C_1W_4\\
&-D3_{221}C_2W_1-D3_{222}C_2W_2-D3_{223}C_2W_3-D3_{224}C_2W_4-D3_{231}C_3W_1-D3_{232}C_3W_2\\
&-D3_{233}C_3W_3-D3_{234}C_3W_4-D3_{241}C_4W_1-D3_{242}C_4W_2-D3_{243}C_4W_3-D3_{244}C_4W_4\\
&+S3_{21}\frac{\partial C_1}{\partial Z}+S3_{22}\frac{\partial C_2}{\partial Z}+S3_{23}\frac{\partial C_3}{\partial Z}+S3_{24}\frac{\partial C_4}{\partial Z}-Re\left(E3_{211}W_1\frac{\partial W_1}{\partial Z}+E3_{212}W_1\frac{\partial W_2}{\partial Z}\right.\\
&+E3_{213}W_1\frac{\partial W_3}{\partial Z}+E3_{214}W_1\frac{\partial W_4}{\partial Z}+E3_{221}W_2\frac{\partial W_1}{\partial Z}+E3_{222}W_2\frac{\partial W_2}{\partial Z}+E3_{223}W_2\frac{\partial W_3}{\partial Z}\\
&+E3_{224}W_2\frac{\partial W_4}{\partial Z}-E3_{231}W_3\frac{\partial W_1}{\partial Z}-E3_{232}W_3\frac{\partial W_3}{\partial Z}+E3_{233}W_3\frac{\partial W_3}{\partial Z}+E3_{234}W_3\frac{\partial W_4}{\partial Z}+E3_{241}W_4\frac{\partial W_1}{\partial Z}\\
&\left.-E3_{242}W_4\frac{\partial W_2}{\partial Z}+E3_{243}W_4\frac{\partial W_3}{\partial Z}+E3_{244}W_4\frac{\partial W_4}{\partial Z}\right)+H3_{21}\frac{\partial^{(2)}W_1}{\partial Z^{(2)}}+H3_{22}\frac{\partial^{(2)}W_2}{\partial Z^{(2)}}+H3_{23}\frac{\partial^{(2)}W_3}{\partial Z^{(2)}}\\
&+H3_{24}\frac{\partial^{(2)}W_4}{\partial Z^{(2)}}-P3_2;
\end{aligned}
$$

$$
\begin{aligned}
Re&\left(A3_{31}\frac{\partial W_1}{\partial T}+A3_{32}\frac{\partial W_2}{\partial T}+A3_{33}\frac{\partial W_3}{\partial T}+A3_{34}\frac{\partial W_4}{\partial T}\right)=F3_{31}W_1+F3_{32}W_2+F3_{33}W_3+F3_{34}W_4\\
&-B3_{311}U_1W_1-B3_{312}U_1W_2-B3_{313}U_1W_3-B3_{314}U_1W_4-B3_{321}U_2W_1-B3_{322}U_2W_2\\
&-B3_{323}U_2W_3-B3_{324}U_2W_4-D3_{311}C_1W_1-D3_{312}C_1W_2-D3_{313}C_1W_3-D3_{314}C_1W_4\\
&-D3_{321}C_2W_1-D3_{322}C_2W_2-D3_{323}C_2W_3-D3_{324}C_2W_4-D3_{331}C_3W_1-D3_{332}C_3W_2\\
&-D3_{333}C_3W_3-D3_{334}C_3W_4-D3_{341}C_4W_1-D3_{342}C_4W_2-D3_{343}C_4W_3-D3_{344}C_4W_4\\
&+S3_{31}\frac{\partial C_1}{\partial Z}+S3_{32}\frac{\partial C_2}{\partial Z}+S3_{33}\frac{\partial C_3}{\partial Z}+S3_{34}\frac{\partial C_4}{\partial Z}-Re\left(E3_{311}W_1\frac{\partial W_1}{\partial Z}+E3_{312}W_1\frac{\partial W_2}{\partial Z}\right.\\
&+E3_{313}W_1\frac{\partial W_3}{\partial Z}+E3_{314}W_1\frac{\partial W_4}{\partial Z}+E3_{321}W_2\frac{\partial W_1}{\partial Z}+E3_{322}W_2\frac{\partial W_2}{\partial Z}+E3_{323}W_2\frac{\partial W_3}{\partial Z}\\
&+E3_{324}W_2\frac{\partial W_4}{\partial Z}-E3_{331}W_3\frac{\partial W_1}{\partial Z}-E3_{332}W_3\frac{\partial W_3}{\partial Z}+E3_{333}W_3\frac{\partial W_3}{\partial Z}+E3_{334}W_3\frac{\partial W_4}{\partial Z}+E3_{341}W_4\frac{\partial W_1}{\partial Z}\\
&\left.-E3_{342}W_4\frac{\partial W_2}{\partial Z}+E3_{343}W_4\frac{\partial W_3}{\partial Z}+E3_{344}W_4\frac{\partial W_4}{\partial Z}\right)+H3_{31}\frac{\partial^{(2)}W_1}{\partial Z^{(2)}}+H3_{32}\frac{\partial^{(2)}W_2}{\partial Z^{(2)}}+H3_{33}\frac{\partial^{(2)}W_3}{\partial Z^{(2)}}\\
&+H3_{34}\frac{\partial^{(2)}W_4}{\partial Z^{(2)}}-P3_3;
\end{aligned}
$$

$$
\begin{aligned}
&Re\left(A3_{41}\frac{\partial W_1}{\partial T}+A3_{42}\frac{\partial W_2}{\partial T}+A3_{43}\frac{\partial W_3}{\partial T}+A3_{44}\frac{\partial W_4}{\partial T}\right)=F3_{41}W_1+F3_{42}W_2+F3_{43}W_3+F3_{44}W_4\\
&\quad -B3_{411}U_1W_1-B3_{412}U_1W_2-B3_{413}U_1W_3-B3_{414}U_1W_4-B3_{421}U_2W_1-B3_{422}U_2W_2\\
&\quad -B3_{423}U_2W_3-B3_{424}U_2W_4-D3_{411}C_1W_1-D3_{412}C_1W_2-D3_{413}C_1W_3-D3_{414}C_1W_4\\
&\quad -D3_{421}C_2W_1-D3_{422}C_2W_2-D3_{423}C_2W_3-D3_{424}C_2W_4-D3_{431}C_3W_1-D3_{432}C_3W_2\\
&\quad -D3_{433}C_3W_3-D3_{434}C_3W_4-D3_{441}C_4W_1-D3_{442}C_4W_2-D3_{443}C_4W_3-D3_{444}C_4W_4\\
&\quad +S3_{41}\frac{\partial C_1}{\partial Z}+S3_{42}\frac{\partial C_2}{\partial Z}+S3_{43}\frac{\partial C_3}{\partial Z}+S3_{44}\frac{\partial C_4}{\partial Z}-Re\left(E3_{411}W_1\frac{\partial W_1}{\partial Z}+E3_{412}W_1\frac{\partial W_2}{\partial Z}\right.\\
&\quad +E3_{413}W_1\frac{\partial W_3}{\partial Z}+E3_{414}W_1\frac{\partial W_4}{\partial Z}+E3_{421}W_2\frac{\partial W_1}{\partial Z}+E3_{422}W_2\frac{\partial W_2}{\partial Z}+E3_{423}W_2\frac{\partial W_3}{\partial Z}\\
&\quad +E3_{424}W_2\frac{\partial W_4}{\partial Z}-E3_{431}W_3\frac{\partial W_1}{\partial Z}-E3_{432}W_3\frac{\partial W_3}{\partial Z}+E3_{433}W_3\frac{\partial W_3}{\partial Z}+E3_{434}W_3\frac{\partial W_4}{\partial Z}+E3_{441}W_4\frac{\partial W_1}{\partial Z}\\
&\quad \left.-E3_{442}W_4\frac{\partial W_2}{\partial Z}+E3_{443}W_4\frac{\partial W_3}{\partial Z}+E3_{444}W_4\frac{\partial W_4}{\partial Z}\right)+H3_{41}\frac{\partial^{(2)}W_1}{\partial Z^{(2)}}+H3_{42}\frac{\partial^{(2)}W_2}{\partial Z^{(2)}}+H3_{43}\frac{\partial^{(2)}W_3}{\partial Z^{(2)}}\\
&\quad +H3_{44}\frac{\partial^{(2)}W_4}{\partial Z^{(2)}}-P3_4.
\end{aligned}
$$

5.2.2 Pressure relation

In the present coordinate system, which was set up in Chapter 4, the differential operator ∇ is given by

$$
\nabla=\mathbf{a}_1\frac{\partial}{\partial X}+\mathbf{a}_2\frac{\partial}{\partial Y}+\mathbf{a}_3\frac{1}{1+\delta Y}\frac{\partial}{\partial Z}.
$$

Then taking the divergence of the Navier-Stokes equation, given by Eqs. (5.1) - (5.3) gives

(5.8)

$$
\begin{aligned}
&Re_0\frac{\partial}{\partial T}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]+V_1\frac{\partial}{\partial X}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]\\
&+V_2\frac{\partial}{\partial Y}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]+\frac{\delta^{(2)}V_2^{(2)}}{(1+\delta Y)^{(2)}}+\frac{Re_0\delta V_2}{(1+\delta Y)^{(2)}}\frac{\partial V_3}{\partial Z}+\frac{2Re_0}{1+\delta Y}\frac{\partial V_3}{\partial X}\frac{\partial V_1}{\partial Z}\\
&+2\frac{\partial V_2}{\partial X}\frac{\partial V_1}{\partial Y}+\frac{Re_0V_3}{1+\delta Y}\frac{\partial}{\partial Z}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]+\left(\frac{\partial V_1}{\partial X}\right)^{(2)}+\left(\frac{\partial V_2}{\partial Y}\right)^{(2)}\\
&-\frac{2Re_0^{(2)}\delta V_3}{1+\delta Y}\frac{\partial V_3}{\partial Y}+\frac{Re_0\delta V_3}{(1+\delta Y)^{(2)}}\frac{\partial V_2}{\partial Z}+\frac{2Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Y}\frac{\partial V_2}{\partial Z}+\frac{Re_0^{(2)}}{(1+\delta Y)^{(2)}}\left(\frac{\partial V_3}{\partial Z}\right)^{(2)}\\
&=-Re_0\left[\frac{\partial^{(2)}P}{\partial X^{(2)}}+\frac{\partial^{(2)}P}{\partial Y^{(2)}}+\frac{\delta}{1+\delta Y}\frac{\partial P}{\partial Y}+\frac{1}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}P}{\partial Z^{(2)}}\right]+\frac{\partial^{(2)}}{\partial X^{(2)}}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]\\
&+\frac{\partial^{(2)}}{\partial Y^{(2)}}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]+\frac{\delta}{1+\delta Y}\frac{\partial}{\partial Y}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]\\
&+\frac{1}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}}{\partial Z^{(2)}}\left[\frac{\partial V_1}{\partial X}+\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{1+\delta Y}+\frac{Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Z}\right]+\frac{2\delta^{(2)}}{(1+\delta Y)^{(2)}}\frac{\partial V_2}{\partial Y}\\
&-\frac{Re_0\delta}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}V_3}{\partial Y\partial Z}+\frac{\delta}{(1+\delta Y)^{(3)}}\frac{\partial^{(2)}V_2}{\partial Z^{(2)}}.
\end{aligned}
$$

In this coordinate system, the Laplacian operator is

$$\nabla^{(2)}=\frac{\partial^{(2)}}{\partial X^{(2)}}+\frac{\partial^{(2)}}{\partial Y^{(2)}}+\frac{\delta}{1+\delta Y}\frac{\partial}{\partial Y}+\frac{1}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}}{\partial Z^{(2)}}.$$

Hence, recalling the incompressibility condition given by Eq. (5.4), Eq. (5.8) simplifies and after rearrangement gives

(5.9)

$$
\begin{aligned}
Re_0\nabla^{(2)}P=&\frac{2\delta^{(2)}}{(1+\delta Y)^{(2)}}\frac{\partial V_2}{\partial Y}-\frac{Re_0\delta}{(1+\delta Y)^{(2)}}\frac{\partial^{(2)}V_3}{\partial Y\partial Z}+\frac{\delta}{(1+\delta Y)^{(3)}}\frac{\partial^{(2)}V_2}{\partial Z^{(2)}}-\frac{\delta^{(2)}V_2^{(2)}}{(1+\delta Y)^{(2)}}-\frac{Re_0\delta V_2}{(1+\delta Y)^{(2)}}\frac{\partial V_3}{\partial Z}\\
&-\frac{2Re_0}{1+\delta Y}\frac{\partial V_3}{\partial X}\frac{\partial V_1}{\partial Z}-2\frac{\partial V_2}{\partial X}\frac{\partial V_1}{\partial Y}-\left(\frac{\partial V_1}{\partial X}\right)^{(2)}-\left(\frac{\partial V_2}{\partial Y}\right)^{(2)}+\frac{2Re_0^{(2)}\delta V_3}{1+\delta Y}\frac{\partial V_3}{\partial Y}-\frac{Re_0\delta V_3}{(1+\delta Y)^{(2)}}\frac{\partial V_2}{\partial Z}\\
&-\frac{2Re_0}{1+\delta Y}\frac{\partial V_3}{\partial Y}\frac{\partial V_2}{\partial Z}-\frac{Re_0^{(2)}}{(1+\delta Y)^{(2)}}\left(\frac{\partial V_3}{\partial Z}\right)^{(2)}.
\end{aligned}
$$

Eq. (5.9) is known as the pressure Poisson equation and can be coupled with Eqs. (5.5) - (5.7) in an iterative solver where the velocity and pressure is updated in turn.

5.2.3 Solution strategies for the reduced equations

In the case of a toroidally curved pipe with constant curvature and constant radius, the coaxial pressure gradients appearing in the equations of motion could be removed from the system of equations to solve for through the use of Green's Theorem and the incompressibility equation (see Section 4.6 in Chapter 4). However, in this case where the directors in the two cross-sectional directions are no longer assumed to be the same, this approach can no longer be taken. Therefore an explicit expression for the pressure is required to be able to solve the system. Hence the pressure Poisson equation, in this case given by Eq. (5.9), is used.

One strategy for solving the coupled system (Eqs. (5.5) - (5.7) and Eq. (5.9)) that has been investigated is to assume a polynomial expansion for the pressure, in terms of the cross-sectional coordinates, at each point along the pipe. Then discretising along the pipe and in time, the system of integrated equations, represented by Eqs. (5.5) - (5.7), along with equations representing the no-slip boundary condition, is firstly solved for with a trivial pressure. Then the Laplacian of the pressure is calculated from the velocity terms by Eq. (5.9). (Here, any number of different solution methods could be used, as well as an explicit analytical solution based on Green's functions. This is used to calculate the coefficients of the pressure expansion. Finally the system of equations is resolved with the pressure terms included to update the velocity. This is done for each Z-step within each T-step, first going along the pipe and then moving forward in time.

A method using simple finite differences in time and space has been investigated for this system. Another approach would be to do the spatial discretisation using finite differences and then use a stiff ODE solver in time. One of the difficulties of this approach is that the 3D pressure equation and the system of 1D equations for the director weights for the fluid velocity live in different domains and have to be solved in the iterative coupled manner. A simple implementation of the finite difference approach has been set up in Matlab. The results are promising in the representation of the initial data, however it suffers from numerical instability problems. A complete numerical implementation is beyond the scope of this thesis.

5.3 Varying Curvature

5.3.1 Setup

Consider the case where the radius of curvature varies along the pipe. In this case a similar coordinate transformation to that described in Chapter 4 can be used, that is

$$\zeta^1 = x_1;$$

$$\zeta^2 = (x_2^{(2)} + x_3^{(2)})^{(1/2)} - R;$$

$$\zeta^3 = R \arctan(x_3/x_2);$$

where (x_1, x_2, x_3) are the Cartesian coordinates and $(\zeta^1, \zeta^2, \zeta^3)$ are a set of orthogonal curvilinear coordinates. The inverse is given by

$$x_1 = \zeta^1;$$

$$x_2 = (R + \zeta^1)\cos(\zeta^3/R);$$

$$x_3 = (R + \zeta^2)\sin(\zeta^3/R).$$

The difference with Chapter 4 is that R is no longer assumed to be a constant, specifically R now admits the representation $R = R(\zeta^3)$. The covariant base vectors for $(\zeta^1, \zeta^2, \zeta^3)$ are defined by

$$\mathbf{g_i} = \frac{\partial x_1}{\partial \zeta^i}\mathbf{e}_1 + \frac{\partial x_2}{\partial \zeta^i}\mathbf{e}_2 + \frac{\partial x_3}{\partial \zeta^i}\mathbf{e}_3,$$

for $i = 1, 2, 3$, where $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are the Cartesian unit base vectors. Hence

$$\mathbf{g}_1 = \mathbf{e}_1;$$

$$\mathbf{g}_2 = \cos(\zeta^3/R)\mathbf{e}_2 + \sin(\zeta^3/R)\mathbf{e}_3;$$

$$\mathbf{g}_3 = \left[\frac{\partial R}{\partial \zeta^3}\cos(\zeta^3/R) - (R+\zeta^2)\left(\frac{1}{R} - \frac{\zeta^3}{R^{(2)}}\frac{dR}{d\zeta^3}\right)\sin(\zeta^3/R)\right]\mathbf{e}_2 + \left[\frac{\partial R}{\partial \zeta^3}\sin(\zeta^3/R) + (R+\zeta^2)\left(\frac{1}{R} - \frac{\zeta^3}{R^{(2)}}\frac{dR}{d\zeta^3}\right)\cos(\zeta^3/R)\right]\mathbf{e}_3.$$

The contravariant base vectors $\mathbf{g}^i$ are defined by

$$\mathbf{g}^i \cdot \mathbf{g}_j = \delta^i_j.$$

The determinant, as defined in Eq. (2.7) is given by

$$g^{(1/2)} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos(\zeta^3/R) & \frac{\partial R}{\partial \zeta^3}\cos(\zeta^3/R) - (R+\zeta^2)\left(\frac{1}{R} - \frac{\zeta^3}{R^{(2)}}\frac{dR}{d\zeta^3}\right)\sin(\zeta^3/R) \\ 0 & \sin(\zeta^3/R) & \frac{\partial R}{\partial \zeta^3}\sin(\zeta^3/R) + (R+\zeta^2)\left(\frac{1}{R} - \frac{\zeta^3}{R^{(2)}}\frac{dR}{d\zeta^3}\right)\cos(\zeta^3/R) \end{vmatrix} = (R+\zeta^2)\left(\frac{1}{R} - \frac{\zeta^3}{R^{(2)}}\frac{dR}{d\zeta^3}\right).$$

Consider the set of orthonormal basis $\mathbf{a}_i$ given by

$$\mathbf{a}_1 = \mathbf{e}_1;$$

$$\mathbf{a}_2 = \cos(\zeta^3/R)\mathbf{e}_2 + \sin(\zeta^3/R)\mathbf{e}_3;$$

$$\mathbf{a}_3 = -\sin(\zeta^3/R)\mathbf{e}_2 + \cos(\zeta^3/R)\mathbf{e}_3.$$

5.3.2 Equations of motion

The incompressibility condition is given by

$$\frac{\partial \mathbf{v}}{\partial x_i} \cdot \mathbf{e}_i = 0,$$

for $i = 1,2,3$. This can be expressed in terms of the curvilinear coordinates $(\zeta^1, \zeta^2, \zeta^3)$ considering that

$$\frac{\partial \mathbf{v}}{\partial x_i} = \frac{\partial \mathbf{v}}{\partial \zeta^1}\frac{\partial \zeta^1}{\partial x_i} + \frac{\partial \mathbf{v}}{\partial \zeta^2}\frac{\partial \zeta^2}{\partial x_i} + \frac{\partial \mathbf{v}}{\partial \zeta^3}\frac{\partial \zeta^3}{\partial x_i}.$$

The derivation of the incompressibility condition is similar to that carried out in Section 4.2 of Chapter 4, but with the added complexity that the radius of curvature R is no longer a constant but a function of ζ^3. The derivatives of R with respect to x_2 and x_3 will appear in the derivation, and their calculations are presented as follows:

$$\begin{aligned}\frac{\partial R}{\partial x_2} &= \frac{\partial R}{\partial \zeta^3}\frac{\partial \zeta^3}{\partial x_2}\\ &= \frac{\partial R}{\partial \zeta^3}\left(\frac{\partial R}{\partial x_2}\arctan x_3/x_2 - \frac{R x_3}{x_2^{(2)} + x_3^{(2)}}\right)\\ &= \frac{\partial R}{\partial \zeta^3}\left(\frac{\partial R}{\partial x_2}\frac{\zeta^3}{R} - \frac{R\sin\zeta^3/R}{\zeta^2 + R}\right);\end{aligned}$$

rearranging gives

$$\frac{\partial R}{\partial x_2} = -\frac{\partial R}{\partial \zeta^3}\frac{R\sin\zeta^3/R}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}.$$

Similarly,

$$\begin{aligned}\frac{\partial R}{\partial x_3} &= \frac{\partial R}{\partial \zeta^3}\frac{\partial \zeta^3}{\partial x_3}\\ &= \frac{\partial R}{\partial \zeta^3}\left(\frac{\partial R}{\partial x_3}\arctan x_3/x_2 + \frac{R x_2}{x_2^{(2)} + x_3^{(2)}}\right)\\ &= \frac{\partial R}{\partial \zeta^3}\left(\frac{\partial R}{\partial x_3}\frac{\zeta^3}{R} + \frac{R\cos\zeta^3/R}{R + \zeta^2}\right);\end{aligned}$$

rearranging gives

$$\frac{\partial R}{\partial x_3}=\frac{\partial R}{\partial \zeta^3}\frac{R\cos\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}.$$

Expressing the velocity as $\mathbf{v}=v_1\mathbf{a}_1+v_2\mathbf{a}_2+v_3\mathbf{a}_3$, carrying out the calculations, it is found that

$$\frac{\partial \mathbf{v}}{\partial x_1}\cdot\mathbf{e}_1=\frac{\partial v_1}{\partial \zeta^1};$$

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial x_2}\cdot\mathbf{e}_2&=\cos^{(2)}\zeta^3/R\frac{\partial v_2}{\partial \zeta^2}+\frac{\partial R}{\partial \zeta^3}\frac{\partial v_2}{\partial \zeta^2}\frac{R\cos\zeta^3/R\sin\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^2}\frac{\zeta^3}{R}\right)}-\frac{\partial R}{\partial \zeta^3}\frac{\partial v_2}{\partial \zeta^3}\frac{\zeta^3\cos\zeta^3/R\sin\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\\
&-\frac{\partial v_2}{\partial \zeta^3}\frac{R\cos\zeta^3/R\sin\zeta^3/R}{R+\zeta^2}+\frac{v_3}{R}\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3\cos\zeta^3/R\sin\zeta^3/R}{(R+\zeta^2)}+\frac{v_3\cos\zeta^3/R\sin\zeta^3/R}{R+\zeta^2}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)\\
&-\frac{\partial v_3}{\partial \zeta^2}\cos\zeta^3/R\sin\zeta^3/R-\frac{\partial v_3}{\partial \zeta^2}\frac{\partial R}{\partial \zeta^3}\frac{R\sin^{(2)}\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}+\frac{\partial R}{\partial \zeta^3}\frac{\partial v_3}{\partial \zeta^3}\frac{\zeta^3\sin^{(2)}\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\\
&+\frac{\partial v_3}{\partial \zeta^3}\frac{R\sin^{(2)}\zeta^3/R}{R+\zeta^2}+\frac{\partial R}{\partial \zeta^3}\frac{v_2}{R}\frac{\zeta^3\sin^{(2)}\zeta^3/R}{(R+\zeta^2)}+\frac{v_2\sin^{(2)}\zeta^3/R}{R+\zeta^2}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right);\end{aligned}$$

$$\begin{aligned}\frac{\partial \mathbf{v}}{\partial x_3}\cdot\mathbf{e}_3&=\sin^{(2)}\zeta^3/R\frac{\partial v_2}{\partial \zeta^2}-\frac{\partial R}{\partial \zeta^3}\frac{\partial v_2}{\partial \zeta^2}\frac{R\sin\zeta^3/R\cos\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}+\cos\zeta^3/R\sin\zeta^3/R\frac{\partial v_3}{\partial \zeta^2}\\
&-\frac{\partial R}{\partial \zeta^3}\frac{\partial v_3}{\partial \zeta^2}\frac{R\cos^{(2)}\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}+\frac{\partial R}{\partial \zeta^3}\frac{\partial v_2}{\partial \zeta^3}\frac{\zeta^3\cos\zeta^3/R\sin\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}-\frac{\partial R}{\partial \zeta^3}\frac{v_3}{R}\frac{\zeta^3\cos\zeta^3/R\sin\zeta^3/R}{(R+\zeta^2)}\\
&+\frac{\partial v_2}{\partial \zeta^3}\frac{R\cos\zeta^3/R\sin\zeta^3/R}{R+\zeta^2}-\frac{v_3\cos\zeta^3/R\sin\zeta^3/R}{R+\zeta^2}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)+\frac{\partial R}{\partial \zeta^3}\frac{\partial v_3}{\partial \zeta^3}\frac{\zeta^3\cos^{(2)}\zeta^3/R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\\
&+\frac{\partial R}{\partial \zeta^3}\frac{v_2}{R}\frac{\zeta^3\cos^{(2)}\zeta^3/R}{(R+\zeta^2)}+\frac{\partial v_3}{\partial \zeta^3}\frac{R\cos^{(2)}\zeta^3/R}{R+\zeta^2}+\frac{v_2\cos^{(2)}\zeta^3/R}{R+\zeta^2}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right).\end{aligned}$$

Therefore the incompressibility condition is given by

$$\frac{\partial v_1}{\partial \zeta^1}+\frac{\partial v_2}{\partial \zeta^2}+\frac{v_2}{R+\zeta^2}+\frac{R\frac{\partial v_3}{\partial \zeta^3}-R\frac{\partial R}{\partial \zeta^3}\frac{\partial v_3}{\partial \zeta^2}}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}=0.$$

This can be written in non-dimensional form as

$$\begin{aligned}&\frac{1}{Re_0}\frac{\partial V_1}{\partial X}+\frac{1}{Re_0}\frac{\partial V_2}{\partial Y}+\frac{\delta V_2}{Re_0(1+\delta Y)}+\frac{1}{\delta(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_3}{\partial Y}\frac{\partial \delta}{\partial Z}\\
&+\frac{\delta}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_3}{\partial Z}=0.\end{aligned}\tag{5.10}$$

The incompressible and Newtonian Navier-Stokes equation is given in vector form by

$$\frac{\partial \mathbf{v}}{\partial t} + \mathbf{v} \cdot \nabla \mathbf{v} = -\frac{\nabla p}{\rho} + \nu \nabla^{(2)} \mathbf{v}.$$

In this coordinate system, it is found that the differential operator ∇ is given by

$$\nabla = \mathbf{a}_1 \frac{\partial}{\partial \zeta^1} + \mathbf{a}_2 \frac{\partial}{\partial \zeta^2} + \mathbf{a}_3 \left(\frac{-\partial R}{\partial \zeta^3} \frac{R}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial}{\partial \zeta^2} + \frac{R}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial}{\partial \zeta^3} \right). \tag{5.11}$$

After carrying out the algebra and simplifying, it is found that

$$\begin{aligned}
\mathbf{v} \cdot \nabla \mathbf{v} = & \left(v_1 \frac{\partial v_1}{\partial \zeta^1} + v_2 \frac{\partial v_1}{\partial \zeta^2} - \frac{\partial R}{\partial \zeta^3} \frac{R v_3}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial v_1}{\partial \zeta^2} + \frac{R v_3}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial v_1}{\partial \zeta^3} \right) \mathbf{a}_1 \\
& + \left(v_1 \frac{\partial v_2}{\partial \zeta^1} + v_2 \frac{\partial v_2}{\partial \zeta^2} - \frac{\partial R}{\partial \zeta^3} \frac{R v_3}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial v_2}{\partial \zeta^2} + \frac{R v_3}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial v_2}{\partial \zeta^3} - \frac{v_3^{(2)}}{R+\zeta^2} \right) \mathbf{a}_2 \\
& + \left(v_1 \frac{\partial v_3}{\partial \zeta^1} + v_2 \frac{\partial v_3}{\partial \zeta^2} - \frac{\partial R}{\partial \zeta^3} \frac{R v_3}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial v_3}{\partial \zeta^2} + \frac{R v_3}{(R+\zeta^2)\left(1 - \frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)} \frac{\partial v_3}{\partial \zeta^2} + \frac{v_2 v_3}{R+\zeta^2} \right) \mathbf{a}_3;
\end{aligned}$$

$$\begin{aligned}
\nabla^{(2)}\mathbf{v} = \Bigg[& \frac{\partial^{(2)}v_1}{\partial\zeta^{1(2)}} + \left(1+\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)} \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\right)\frac{\partial^{(2)}v_1}{\partial\zeta^{2(2)}} - 2\frac{\partial R}{\partial\zeta^3}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)}v_1}{\partial\zeta^2\zeta^3} \\
& + \left(\frac{\partial^{(2)}R}{\partial\zeta^{3(2)}}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(3)}} - 2\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} + \frac{1}{R}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)\right)\frac{\partial v_1}{\partial\zeta^2} \\
& + \frac{1}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(3)}}\left(\zeta^3R\frac{\partial^{(2)}R}{\partial\zeta^{3(2)}} + 2R\frac{\partial R}{\partial\zeta^3} - 3\zeta^3\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\right)\frac{\partial v_1}{\partial\zeta^3} + \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)}v_1}{\partial\zeta^{3(2)}}\Bigg]\mathbf{a}_1 \\
& + \Bigg[\frac{\partial^{(2)}v_2}{\partial\zeta^{1(2)}} + \left(1+\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)} \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\right)\frac{\partial^{(2)}v_2}{\partial\zeta^{2(2)}} - 2\frac{\partial R}{\partial\zeta^3}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)}v_2}{\partial\zeta^2\partial\zeta^3} \\
& + \left(\frac{\partial^{(2)}R}{\partial\zeta^{3(2)}}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(3)}} - 2\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} + \frac{1}{R}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)\right)\frac{\partial v_2}{\partial\zeta^2} \\
& + \frac{1}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(3)}}\left(\zeta^3R\frac{\partial^{(2)}R}{\partial\zeta^{3(2)}} + 2R\frac{\partial R}{\partial\zeta^3} - 3\zeta^3\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\right)\frac{\partial v_2}{\partial\zeta^3} + \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)}v_2}{\partial\zeta^{3(2)}} \\
& + 2\frac{\partial R}{\partial\zeta^3}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_3}{\partial\zeta^2} + \left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\frac{v_3\zeta^3}{R(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} - \frac{v_2}{(R+\zeta^2)^{(2)}}\Bigg]\mathbf{a}_2 \\
& + \Bigg[\frac{\partial^{(2)}v_3}{\partial\zeta^{1(2)}} + \left(1+\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)} \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\right)\frac{\partial^{(2)}v_3}{\partial\zeta^{2(2)}} - 2\frac{\partial R}{\partial\zeta^3}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)}v_3}{\partial\zeta^2\partial\zeta^3} \\
& + \left(\frac{\partial^{(2)}R}{\partial\zeta^{3(2)}}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(3)}} - 2\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} + \frac{1}{R}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)\right)\frac{\partial v_3}{\partial\zeta^2} \\
& + \frac{1}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(3)}}\left(\zeta^3R\frac{\partial^{(2)}R}{\partial\zeta^{3(2)}} + 2R\frac{\partial R}{\partial\zeta^3} - 3\zeta^3\left(\frac{\partial R}{\partial\zeta^3}\right)^{(2)}\right)\frac{\partial v_3}{\partial\zeta^3} + \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)}v_3}{\partial\zeta^{3(2)}} \\
& - \frac{\partial R}{\partial\zeta^3}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_2}{\partial\zeta^2} - \frac{v_2\zeta^3}{R(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial\zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} - \frac{v_3}{(R+\zeta^2)^{(2)}}\Bigg]\mathbf{a}_3.
\end{aligned}$$

Hence the $\mathbf{a}_1$, $\mathbf{a}_2$ and $\mathbf{a}_3$ components of the equation of motion are given respectively by

$$
\begin{aligned}
&\frac{\partial v_1}{\partial t} + v_1\frac{\partial v_1}{\partial \zeta^1} + v_2\frac{\partial v_1}{\partial \zeta^2} - \frac{\partial R}{\partial \zeta^3}\frac{R v_3}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_1}{\partial \zeta^2} + \frac{R v_3}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_1}{\partial \zeta^3} = -\frac{1}{\rho}\frac{\partial p}{\partial \zeta^1} \\
&+\nu\Bigg[\frac{\partial^{(2)} v_1}{\partial \zeta^{1(2)}} + \left(1+\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\right)\frac{\partial^{(2)} v_1}{\partial \zeta^{2(2)}} - 2\frac{\partial R}{\partial \zeta^3}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)} v_1}{\partial \zeta^2 \zeta^3} \\
&+\left(\frac{\partial^{(2)} R}{\partial \zeta^{3(2)}}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(3)}} - 2\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} + \frac{1}{R}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)\right)\frac{\partial v_1}{\partial \zeta^2} \\
&+\frac{1}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(3)}}\left(\zeta^3 R\frac{\partial^{(2)} R}{\partial \zeta^{3(2)}} + 2R\frac{\partial R}{\partial \zeta^3} - 3\zeta^3\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\right)\frac{\partial v_1}{\partial \zeta^3} + \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)} v_1}{\partial \zeta^{3(2)}}\Bigg];
\end{aligned}
$$

$$
\begin{aligned}
&\frac{\partial v_2}{\partial t} + v_1\frac{\partial v_2}{\partial \zeta^1} + v_2\frac{\partial v_2}{\partial \zeta^2} - \frac{\partial R}{\partial \zeta^3}\frac{R v_3}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_2}{\partial \zeta^2} + \frac{R v_3}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_2}{\partial \zeta^3} - \frac{v_3^{(2)}}{R+\zeta^2} = -\frac{1}{\rho}\frac{\partial p}{\partial \zeta^2} \\
&+\nu\Bigg[\frac{\partial^{(2)} v_2}{\partial \zeta^{1(2)}} + \left(1+\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\right)\frac{\partial^{(2)} v_2}{\partial \zeta^{2(2)}} - 2\frac{\partial R}{\partial \zeta^3}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)} v_2}{\partial \zeta^2 \partial \zeta^3} \\
&+\left(\frac{\partial^{(2)} R}{\partial \zeta^{3(2)}}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(3)}} - 2\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} + \frac{1}{R}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)\right)\frac{\partial v_2}{\partial \zeta^2} \\
&+\frac{1}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(3)}}\left(\zeta^3 R\frac{\partial^{(2)} R}{\partial \zeta^{3(2)}} + 2R\frac{\partial R}{\partial \zeta^3} - 3\zeta^3\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\right)\frac{\partial v_2}{\partial \zeta^3} + \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)} v_2}{\partial \zeta^{3(2)}} \\
&+2\frac{\partial R}{\partial \zeta^3}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_3}{\partial \zeta^2} + \left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{v_3\zeta^3}{R(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} - \frac{v_2}{(R+\zeta^2)^{(2)}}\Bigg];
\end{aligned}
$$

$$\frac{\partial v_3}{\partial t} + v_1\frac{\partial v_3}{\partial \zeta^1} + v_2\frac{\partial v_3}{\partial \zeta^2} - \frac{\partial R}{\partial \zeta^3}\frac{R v_3}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_3}{\partial \zeta^2} + \frac{R v_3}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_3}{\partial \zeta^2} + \frac{v_2 v_3}{R+\zeta^2}$$

$$= -\frac{1}{\rho}\left(\frac{-\partial R}{\partial \zeta^3}\frac{R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial p}{\partial \zeta^2} + \frac{R}{(R+\zeta^2)\left(1-\frac{\partial R}{\partial \zeta^3}\zeta^3 R\right)}\frac{\partial p}{\partial \zeta^3}\right)$$

$$+\nu\left[\frac{\partial^{(2)} v_3}{\partial \zeta^{1(2)}} + \left(1+\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\right)\frac{\partial^{(2)} v_3}{\partial \zeta^{2(2)}} - 2\frac{\partial R}{\partial \zeta^3}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)} v_3}{\partial \zeta^2 \partial \zeta^3}\right.$$

$$+\left(\frac{\partial^{(2)} R}{\partial \zeta^{3(2)}}\frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(3)}} - 2\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} + \frac{1}{R}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)\right)\frac{\partial v_3}{\partial \zeta^2}$$

$$+\frac{1}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(3)}}\left(\zeta^3 R\frac{\partial^{(2)} R}{\partial \zeta^{3(2)}} + 2R\frac{\partial R}{\partial \zeta^3} - 3\zeta^3\left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\right)\frac{\partial v_3}{\partial \zeta^3} + \frac{R^{(2)}}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}}\frac{\partial^{(2)} v_3}{\partial \zeta^{3(2)}}$$

$$\left.-\frac{\partial R}{\partial \zeta^3}\frac{R}{(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)}\frac{\partial v_2}{\partial \zeta^2} - \left(\frac{\partial R}{\partial \zeta^3}\right)^{(2)}\frac{v_2\zeta^3}{R(R+\zeta^2)^{(2)}\left(1-\frac{\partial R}{\partial \zeta^3}\frac{\zeta^3}{R}\right)^{(2)}} - \frac{v_3}{(R+\zeta^2)^{(2)}}\right].$$

These can be written in dimensionless form as

(5.12)

$$Re_0\delta^{(2)}\frac{\partial V_1}{\partial T} + \delta^{(2)}V_1\frac{\partial V_1}{\partial X} + \delta^{(2)}V_2\frac{\partial V_1}{\partial Y} + Re_0\delta\frac{\partial \delta}{\partial Z}\frac{V_3}{(1+\delta Y)\left(\delta + Z\frac{\partial \delta}{\partial Z}\right)}\frac{\partial V_1}{\partial Y}$$

$$+Re_0\delta^{(2)}\frac{V_3}{(1+\delta Y)\left(\delta+Z\frac{\partial \delta}{\partial Z}\right)}\frac{\partial V_1}{\partial Z} = -Re_0\delta^{(2)}\frac{\partial P}{\partial X} + 2\delta^{(2)}\frac{\partial^{(2)} V_1}{\partial X^{(2)}}$$

$$+\left(\delta^{(2)} + \left(\frac{\partial \delta}{\partial Z}\right)^{(2)}\frac{1}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial \delta}{\partial Z}\right)^{(2)}}\right)\frac{\partial^{(2)} V_1}{\partial Y^{(2)}} + 2Re_0\delta^{(2)}\frac{\partial \delta}{\partial Z}\frac{1}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial \delta}{\partial Z}\right)^{(2)}}\frac{\partial^{(2)} V_1}{\partial Y\partial Z}$$

$$+Re_0\delta\left(\frac{2\delta\left(\frac{\partial \delta}{\partial Z}\right)^{(2)} - \delta^{(2)}\frac{\partial^{(2)}\delta}{\partial Z^{(2)}}}{(1+\delta Y)^{(2)}\left(1+Z\frac{\partial \delta}{\partial Z}\right)^{(3)}} - \frac{2\frac{\partial \delta}{\partial Z}}{(1+\delta Y)^{(2)}\left(1+Z\frac{\partial \delta}{\partial Z}\right)^{(2)}} + \delta^{(2)} + \delta Z\frac{\partial \delta}{\partial Z}\right)\frac{\partial V_1}{\partial Y}$$

$$-\frac{\delta^{(3)}}{(1+\delta Y)\left(1+Z\frac{\partial \delta}{\partial Z}\right)^{(3)}}\left(\delta Z\frac{\partial^{(2)}\delta}{\partial Z^{(2)}} + 2\delta\frac{\partial \delta}{\partial Z} + Z\left(\frac{\partial \delta}{\partial Z}\right)^{(2)}\right)\frac{\partial V_1}{\partial Z} + \frac{\delta^{(4)}}{(1+\delta Y)^{(2)}\left(1-Z\frac{\partial \delta}{\partial Z}\right)^{(2)}}\frac{\partial^{(2)} V_1}{\partial Z^{(2)}};$$

(5.13)

$$Re_0\delta^{(2)}\frac{\partial V_2}{\partial T}+\delta^{(2)}V_1\frac{\partial V_2}{\partial X}+\delta^{(2)}V_2\frac{\partial V_2}{\partial Y}+Re_0\delta\frac{\partial\delta}{\partial Z}\frac{V_3}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_2}{\partial Y}+\frac{Re_0\delta^{(2)}V_3}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_2}{\partial Z}$$
$$-\frac{Re_0^{(2)}\delta^{(3)}V_3^{(2)}}{1+\delta Y}=-Re_0\delta^{(2)}\frac{\partial P}{\partial Y}+\delta^{(2)}\frac{\partial^{(2)}V_2}{\partial X^{(2)}}+\left(\delta^{(2)}+\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}\frac{1}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}\right)\frac{\partial^{(2)}V_2}{\partial Y^{(2)}}$$
$$+\frac{\partial\delta}{\partial Z}\frac{2\delta^{(2)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}\frac{\partial^{(2)}V_2}{\partial Y\partial Z}$$
$$+\left(\frac{2\delta^{(2)}\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}-\delta^{(3)}\frac{\partial^{(2)}\delta}{\partial Z^{(2)}}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(3)}}-\frac{\partial\delta}{\partial Z}\frac{2\delta}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}+\delta^{(3)}+\delta^{(2)}Z\frac{\partial\delta}{\partial Z}\right)\frac{\partial V_2}{\partial Y}$$
$$-\frac{\delta^{(3)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(3)}}\left(\delta\frac{\partial^{(2)}\delta}{\partial Z^{(2)}}+2\delta\frac{\partial\delta}{\partial Z}+Z\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}\right)\frac{\partial V_2}{\partial Z}+\frac{\delta^{(4)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}\frac{\partial^{(2)}V_2}{\partial Z^{(2)}}$$
$$-\frac{\partial\delta}{\partial Z}\frac{2Re_0\delta^{(2)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_3}{\partial Y}+\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}\frac{Re_0\delta^{(3)}ZV_3}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}-\frac{\delta^{(4)}V_2}{(1+\delta Y)^{(2)}};$$

(5.14)

$$Re_0^{(2)}\delta^{(2)}\frac{\partial V_3}{\partial T}+Re_0\delta^{(2)}V_1\frac{\partial V_3}{\partial X}+Re_0\delta^{(2)}V_2\frac{\partial V_3}{\partial Y}+Re_0^{(2)}\delta\frac{\partial\delta}{\partial Z}\frac{V_3}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_3}{\partial Y}$$
$$+\frac{Re_0^{(2)}\delta^{(3)}V_3}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_3}{\partial Z}+\frac{Re_0\delta^{(3)}V_2V_3}{1+\delta Y}$$
$$=Re_0\delta\left(\frac{\partial\delta}{\partial Z}\frac{1}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial P}{\partial Y}-\frac{\delta^{(2)}}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial P}{\partial Z}\right)+Re_0\delta^{(2)}\frac{\partial^{(2)}V_3}{\partial X^{(2)}}$$
$$+Re_0\left(\delta^{(2)}+\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}\frac{1}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}\right)\frac{\partial^{(2)}V_3}{\partial Y^{(2)}}+\frac{\partial\delta}{\partial Z}\frac{2Re_0\delta^{(2)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}\frac{\partial^{(2)}V_3}{\partial Y\partial Z}$$
$$+Re_0\delta\left(\frac{2\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}-\delta\frac{\partial^{(2)}\delta}{\partial Z^{(2)}}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}-\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}\frac{2}{(1+\delta Y)^{(2)}\left(1+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}+\delta^{(2)}+\delta Z\frac{\partial\delta}{\partial Z}\right)\frac{\partial V_3}{\partial Y}$$
$$-\frac{Re_0\delta^{(3)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(3)}}\left(Z\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}+\delta Z\frac{\partial^{(2)}\delta}{\partial Z^{(2)}}+2\delta\frac{\partial\delta}{\partial Z}\right)\frac{\partial V_3}{\partial Z}+\frac{Re_0\delta^{(4)}}{(1+\delta Y)\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}\frac{\partial^{(2)}V_3}{\partial Z^{(2)}}$$
$$+\frac{\partial\delta}{\partial Z}\frac{\delta^{(2)}}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial V_2}{\partial Y}-\left(\frac{\partial\delta}{\partial Z}\right)^{(2)}\frac{\delta^{(3)}ZV_2}{(1+\delta Y)^{(2)}\left(\delta+Z\frac{\partial\delta}{\partial Z}\right)^{(2)}}-\frac{\delta^{(4)}V_3}{(1+\delta Y)^{(2)}}.$$

Considering Eq. (5.11), the differential operator ∇ is given, in this coordinate system, in dimensionless form by

$$\nabla = \mathbf{a}_1\delta\frac{\partial}{\partial X} + \mathbf{a}_2\delta\frac{\partial}{\partial Y} + \mathbf{a}_3\left(\frac{\partial\delta}{\partial Z}\frac{1}{(1+\delta Y)\left(\delta + Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial}{\partial Y} + \frac{\delta^{(2)}}{(1+\delta Y)\left(\delta + Z\frac{\partial\delta}{\partial Z}\right)}\frac{\partial}{\partial Z}\right).$$

An expression for the Laplacian of the pressure could then be obtained by taking the divergence of the equation of motion and then simplifying with the incompressibility condition, similar to as was done for the varying pipe radius case. The expression of the Laplacian of the pressure in this case however will be more complicated than Eq. (5.9) due to the curvature ratio δ now having dependence on z. In principal, the numerical solution strategy will then be the same as discussed in Section 5.2.3.

5.4 Summary of Chapter

In this chapter, the modelling of pipes with more complicated geometry was considered. The particular cases presented were a pipe of constant curvature with a radius that can vary along the pipe and in time, and a pipe of varying planar curvature, where the curvature varies a long the pipe and with time. In these cases, the systems of equations were derived, along with a proposed method for numerically solving them.

It was found that in these cases of more complicated geometry, it was not trivial to reduce the number of directors and relate the directors in the two cross-sectional coordinates together through the equations arising through the conditions of incompressibility and no-slip on the boundary, as was done in Chapter 4. This was due to the directors now having co-axial dependence. This also meant that the cross-sectional pressure gradient terms could not be removed from system of equations through the use of Green's theorem and the equation for incompressibility. This resulted in a different approach to solving the system of equations being considered, of discretising along the pipe and in time and using the pressure Poisson equation to calculate the pressure and update the velocity at each step. While tentative work was put into the solving the system in this way, the numerical solutions require further investigation and this is left as future work.

Another avenue for future work would be to see if a relationship, although more complicated, could be derived between the weighting functions which would allow the directors in the $\mathbf{a}_1$ and $\mathbf{a}_2$ directions to be considered equal and then allow the the coaxial pressure gradients to be taken out of the system of equations through algebraic manipulation. Whether this is possible for more complex geometries than the model presented in Chapter 4 remains an open question.

Chapter 6

Conclusion

6.1 Contributions

The first chapter of this thesis gave an overview of cardiovascular modelling, its uses and the current challenges in producing accurate models. A review of literature on the subjects of director theory and cardiovascular modelling was conducted and the aims of this thesis were presented.

The second chapter gave an introduction to and outline of the development of director theory. This included the setup of the relevant coordinate systems and deriving the relations between the equations of conservation laws in Eulerian and Lagrangian form. It also gave a background to the computational fluid dynamics simulations that would be used in later chapters to compare with the results produced by the director theory models, the subject of this thesis. The next three substantive chapters then work out the detail of the theory for geometries of increasing complexity, starting with steady flows and straight pipes, building up to the general problem of unsteady flows in pipes of arbitrary curvature and cross-sectional variation.

The third chapter focused on using director theory to model flow in a straight pipe, following a similar approach to Caulk and Naghdi [18]. The novelty in this chapter, was also developing the model for $K = 5$ in addition to $K = 3$ and producing solutions for geometries such as a tapering pipe and a wavy walled (sinusoidal) pipe, as well as swirling flow in a cylindrical pipe. The solutions produced were found to match well with 3D simulations carried out using STAR-CCM+ [45], with maximum relative errors between the two of around 5%.

The fourth chapter looked at using director theory to model fluid flow through a toroidally curved pipe, similar to the work of Green et al [7]. The novelty in this chapter was the way the system of equations (integrated equations of motion) were derived, in this case directly from the Navier-Stokes equations, as opposed to in terms of kinetical quantities defined by Green and Naghdi [6]. While the two methods result in the derivation of equivalent systems of equations, the novel approach presented in this thesis is considered a more intuitive approach within the field of fluid dynamics. Solutions were found for orders ranging from $H = 7$ to $H = 15$, for Reynolds numbers ranging from $Re = 1$ to $Re = 1000$ and for two different curvature ratios, namely $\delta = 0.01$ and $\delta = 0.1$. The results were found to compare well with 3D simulations carried out using STAR-CCM+ and comparisons of friction loss against Dean number were found to agree with those presented by Green et al [7] for curvature ratio 0.01 and varying Reynolds number.

The fifth chapter further generalises the application of director theory to the case of non-steady internal flows. Here, equations were derived for unsteady flow in pipes of varying radius and curvature. The numerical implementation to solve these equations was beyond the reach of this thesis and is left as future work.

Another contribution of this thesis has been to modernise the topic of director theory in the application of fluid mechanics and present the method in a more accessible way. It has also been demonstrate that the derivation of the integrated equations of motion can be presented in the form of a rational asymptotic expansion of the Navier-Stokes equations directly, rather than indirectly from conservation laws. Also, explicit expressions have been derived for the various equations that are suitable for future researchers to implement as part of new, efficient numerical schemes for 1D pipe flow in arbitrary geometries.

6.2 Limitations

One limitation of this work applying director theory to fluid flow modelling is that the higher the order of expansion used for the velocity series approximation, the more computationally expensive the model becomes. This primarily effects the computational time taken to derive the coefficients of the system of equations themselves rather than the solving of the system. This is due to the numeric integration over the cross-section of the pipe that is carried out after multiplying each equation of motion component by each respective weighting function. This impacted on how high the order could be reasonably taken for the models, which could have provided further insights, such as whether the magnitude of error between the director theory models and the 3D simulations continues to decrease as the order is increased.

A major limitation of the theory is how to deal with pressure, which appears as a Lagrange multiplier in the Navier-Stokes equations. A difficulty within this thesis has been to find a consistent strategy for incorporating pressure in each of the asymptotic expansions.

Another limitation was the lack of exact solution to compare the director theory models to for most geometries considered in this thesis. For validation, 3D finite volume simulations carried out in STAR-CCM+ were used. However these simulations are likely to have errors themselves, particularly close to the boundaries, despite the finely resolved meshes adopted. This can make it difficult to be sure where the error in the differences in solutions between the director theory model and the simulation is originating from. Although, there are methods to help look into this, such as looking at how the error is effected by different mesh resolutions in the simulations and different orders of velocity in the director theory model.

Another weakness of the present work is that, even the general formulation in Chapter 5 does not consider arbitrary geometries with curvature and torsion. So far only the case of arbitrary planar curvature has been considered. In theory the director theory approach allows for modelling of flow through complicated geometries such as pipes with curvature and torsion and movement in time. However, the more complicated the geometry, the more complicated the initial setup of the model is, with the coordinate system being based on the centreline of the pipe and the more complicated the derivation of the equations of motion become in the appropriate coordinate system. Also it is likely that the more complicated the geometry, the higher the order of velocity expansion that is required to accurately model the flow, which, as mentioned above, becomes more computationally expensive. This could be a barrier for applying director theory to practical uses such as cardiovascular modelling.

In summary, the main challenges with the use of director theory for flow in curved pipes of arbitrary geometry only appear in the setup of the model. As such, with more work and developments, the modelling of more complicated geometries should be feasible, although larger computer resources may be required. Generally, there is an expensive pre-processing stage to obtain the equations for a specific setup. However, this should only need to be done once for a given geometry setup and then solving the system and obtained a CFD results should be comparatively straight forward.

6.3 Future Research Directions

There are a number of directions future research into the application of director theory to fluid flow modelling could be taken. A more in depth study of the error and convergence of solutions as the order of the velocity expansions increases would be of interest. Particular points that could be examined include whether there is an order at which the error no longer reduces/the solutions produced do not change as the order is increased and whether/how this depends on the geometry and dependencies of the pipe. Similarly, a more in-depth study of how the 3D simulation solutions, to which the director theory models were compared, are affected by the meshing, such as whether there is a point at which further refining of the mesh no longer affects the solutions/differences with the director theory solutions. Also whether changing meshing properties such as the number of prism layers affects the solution close to the wall, as an error between the simulations and director theory solutions is often seen close to the wall, which was assumed to be likely originated from the simulations.

Another interesting research direction would be to look at how well director theory can be used to model fluid flow through more complicated geometries. This could in particular include pipes that have torsion in addition to curvature and pulsing or elastic walls. Other considerations could be the modelling of non-circular sections of pipe and of non-Newtonian compressible fluids. It would also be of interest to investigate branching of vessels (pipe networks). The main challenge as the model becomes more complex is dealing with the pressure. It is possible that this may be resolved with a projection approach or by reformulating the equations with the vorticity transport equations. As mentioned above, the main difficulty of such approaches is the increasing complexity in setting up the model, so it would be interesting to look into to whether this initial setup stage of deriving the coordinate system and equations of motion could be approached in a more automated form. If this could be addressed and the model was capable of producing accurate solutions for these more complicated geometries, then this approach could have practical uses such as in the modelling of cardiovascular, lymphatic and pulmonary systems, or other real-world piping networks such as ventilation systems or city water mains, improving accuracy over classical 1D models.

APPENDIX

A

APPENDIX A

A.1 Derivation of Integrated Equations of Motion for a Straight Axisymmetric Pipe

This Appendix details the steps involved in evaluating Eqs. (3.23) and (3.24) to obtain Eqs. (3.36) to (3.41).

Substituting Eqs. (3.32), (3.29) and (3.13) into Eq. (3.23) and evaluating this equation in component form gives the following in the $\mathbf{e}_1$ direction:

$$\frac{\partial}{\partial x_3}\int_A \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2+2x_1w_{2,0}\right]da$$
$$+\int_{\partial A}\phi^{-1}\left(x_1\tau_1\frac{\partial\phi}{\partial x_3}-x_1p_e-x_2\tau_2\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}\right)ds$$
$$=\int_A\rho\left[\left(\frac{\partial u_{1,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial t}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial t}\right)x_2\right.$$
$$+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})$$
$$+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)(2x_1x_2u_{3,0}-(u_{0,1}+(x_1^{(2)}+3x_2^{(2)})u_{0,3}))$$
$$\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right)\right]da.$$

Rewriting this equation in cylindrical polar coordinates (r,θ,z) gives

$$\frac{\partial}{\partial z}\int_0^{\phi}\int_0^{2\pi}\mu\left[\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)r\sin\theta+2rw_{2,0}\cos\theta\right]rd\theta dr$$
$$+\int_0^{2\pi}\phi^{-1}\left[\tau_1\frac{\partial\phi}{\partial z}\phi\cos\theta-p_e\phi\cos\theta-w_{0,0}\tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}r\sin\theta\right]\phi d\theta$$
$$=\int_0^{\phi}\int_0^{2\pi}\rho\left[\left(\frac{\partial u_{1,0}}{\partial t}+r^{(2)}\frac{\partial u_{3,0}}{\partial t}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial t}+r^{(2)}\frac{\partial u_{0,3}}{\partial t}\right)r\sin\theta\right.$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\cos\theta-(u_{0,1}+r^{(2)}u_{0,3})r\sin\theta)(u_{1,0}+r^{(2)}u_{3,0}(1+2\cos^{(2)}\theta)-2r^{(2)}u_{0,3}\cos\theta\sin\theta)$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\sin\theta+(u_{0,1}+r^{(2)}u_{0,3})r\cos\theta)(2r^{(2)}u_{3,0}\cos\theta\sin\theta-(u_{0,1}-r^{(2)}u_{0,3}(1+2\sin^{(2)}\theta)))$$
$$\left.+(w_{0,0}+r^{(2)}w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)r\sin\theta\right)\right]rd\theta dr.$$

As every term in this equation will cancel out when integrated over 0 to 2π. This is similarly the case when evaluating Eq. (3.23) in the $\mathbf{e}_2$ direction.

Finally, evaluating Eq. (3.23) in the $\mathbf{e}_3$ direction gives

$$\frac{\partial}{\partial x_3}\int_A\left[-p+2\mu\left(\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]da$$
$$+\int_{\partial A}\left(1+\left(\frac{\partial\phi}{\partial x_3}^{(2)}\right)\right)^{(1/2)}\left[\frac{\tau_1+p_e\frac{\partial\phi}{\partial x_3}}{\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}}\right]ds$$
$$=\int_A\rho\left[\frac{\partial w_{0,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial t}+2x_1w_{2,0}((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)\right.$$
$$+2x_2w_{2,0}((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)$$
$$\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]da.$$

Simplifying, this becomes

$$\frac{\partial}{\partial x_3}\int_A\left[-p+2\mu\left(\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]da$$
$$+\int_{\partial A}\left(\tau_1+p_e\frac{\partial\phi}{\partial x_3}\right)ds$$
$$=\int_A\rho\left[\frac{\partial w_{0,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial t}+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right.$$
$$\left.+2w_{2,0}(x_1^{(2)}+x_2^{(2)})(u_{1,0}+\phi^{(2)}u_{3,0})\right]da.$$

Converting this equation into cylindrical polar coordinates gives

$$-\frac{\partial p_I}{\partial z}+\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}2\mu\left(\frac{\partial w_{0,0}}{\partial z}+r^{(2)}\frac{\partial w_{2,0}}{\partial z}\right)rdrd\theta+\int_0^{2\pi}\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)\phi d\theta$$
$$=\int_0^{2\pi}\int_0^{\phi}\rho\left[\frac{\partial w_{0,0}}{\partial t}+r^{(2)}\frac{\partial w_{2,0}}{\partial t}+(w_{0,0}+r^{(2)}w_{2,0})\left(\frac{\partial w_{0,0}}{\partial z}+r^{(2)}\frac{\partial w_{2,0}}{\partial z}\right)+2w_{2,0}r^{(2)}(u_{1,0}+r^{(2)}u_{3,0})\right]rdrd\theta,$$

where p_I is the pressure resultant defined by

$$p_I=\int_A p da. \tag{A.1}$$

Carrying out the integration with respect to r gives

$$-\frac{\partial p_I}{\partial z}+\frac{\partial}{\partial z}\int_0^{2\pi}2\mu\left(\frac{\phi^{(2)}}{2}\frac{\partial w_{0,0}}{\partial z}+\frac{\phi^{(4)}}{4}\frac{\partial w_{2,0}}{\partial z}\right)d\theta+\int_0^{2\pi}\phi\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)d\theta$$
$$=\int_0^{2\pi}\rho\left(\frac{\phi^{(2)}}{2}\frac{\partial w_{0,0}}{\partial t}+\frac{\phi^{(4)}}{4}\frac{\partial w_{2,0}}{\partial t}+\frac{\phi^{(2)}}{2}w_{0,0}\frac{\partial w_{0,0}}{\partial z}+\frac{\phi^{(4)}}{4}w_{0,0}\frac{\partial w_{2,0}}{\partial z}+\frac{\phi^{(4)}}{4}w_{2,0}\frac{\partial w_{0,0}}{\partial z}+\frac{\phi^{(6)}}{6}w_{2,0}\frac{\partial w_{2,0}}{\partial z}\right.$$
$$\left.+2w_{2,0}\frac{\phi^{(4)}}{4}u_{1,0}+2w_{2,0}\frac{\phi^{(6)}}{6}u_{3,0}\right)d\theta.$$

After carrying out the integration with respect to θ, the equation can be expressed in the following form:

(A.2)

$$2\pi\mu\frac{\partial}{\partial z}\left[\phi^{(2)}\left(\frac{\partial w_{0,0}}{\partial z}+\frac{\phi^{(2)}}{2}\frac{\partial w_{2,0}}{\partial z}\right)\right]+2\pi\phi\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)=\frac{\partial p_I}{\partial z}+\pi\rho\phi^{(2)}\left[\frac{\partial w_{0,0}}{\partial t}\right.$$
$$\left.+\frac{\partial w_{0,0}}{\partial z}\left(w_{0,0}+\frac{\phi^{(2)}}{2}w_{2,0}\right)+\frac{\phi^{(2)}}{2}\left(\frac{\partial w_{2,0}}{\partial t}+\frac{\partial w_{2,0}}{\partial z}\left(w_{0,0}+\frac{2}{3}\phi^{(2)}w_{2,0}\right)+2w_{2,0}\left(u_{1,0}+\frac{2}{3}\phi^{(2)}u_{3,0}\right)\right)\right].$$

This is Eq. (3.36) in the main text.

Now examining Eq. (3.25) and taking the case when $\alpha_1=1$, substituting in Eqs. (3.32), (3.29), (3.30) and (3.13) gives

$$\frac{\partial}{\partial x_3}\int_A\mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2+2x_1w_{2,0}\right]x_1da$$
$$+\int_{\partial A}\phi^{-1}\left[x_1^{(2)}\left(\tau_1\frac{\partial\phi}{\partial x_3}-p_e\right)-x_1x_2\tau_2\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}\right]ds$$
$$=\int_A(-p+2\mu(u_{1,0}+3x_1^{(2)}u_{3,0}+x_2^{(2)}u_{3,0}-2x_1x_2u_{0,3}))da$$
$$+\int_A\rho\left[\left(\frac{\partial u_{1,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial t}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial t}\right)x_2\right.$$
$$+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)(2x_1^{(2)}u_{3,0}+u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})$$
$$+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)(2x_1x_2u_{3,0}-2x_2^{(2)}u_{0,3}-u_{0,1}-(x_1^{(2)}+x_2^{(2)})u_{0,3})$$
$$\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right)\right]x_1da.$$

Rewriting this equation in cylindrical polar coordinates (r,θ,z) gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\mu r^{(2)}\cos\theta\left[\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)r\sin\theta+2w_{2,0}r\cos\theta\right]drd\theta$$
$$+\int_0^{2\pi}\left[\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)\phi^{(2)}\cos^{(2)}\theta-\tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\phi^{(2)}\cos\theta\sin\theta\right]d\theta$$
$$=-p_I+\int_0^{2\pi}\int_0^{\phi}2\mu r(u_{1,0}+r^{(2)}u_{3,0}(2\cos^{(2)}\theta+1)-2r^{(2)}u_{0,3}\cos\theta\sin\theta)drd\theta$$
$$+\int_0^{2\pi}\int_0^{\phi}\rho r^{(2)}\cos\theta\left[\left(\frac{\partial u_{1,0}}{\partial t}+r^{(2)}\frac{\partial u_{3,0}}{\partial t}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial t}+r^{(2)}\frac{\partial u_{0,3}}{\partial t}\right)r\sin\theta\right.$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\cos\theta-(u_{0,1}+r^{(2)}u_{0,3})r\sin\theta)(2u_{3,0}r^{(2)}\cos^{(2)}\theta+u_{1,0}+r^{(2)}u_{3,0}-2u_{0,3}r^{(2)}\cos\theta\sin\theta)$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\sin\theta+(u_{0,1}+r^{(2)}u_{0,3})r\cos\theta)(2u_{3,0}r^{(2)}\cos\theta\sin\theta-2u_{0,3}r^{(2)}\sin^{(2)}-u_{0,1}-r^{(2)}u_{0,3})$$
$$\left.(w_{0,0}+r^{(2)}w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)r\sin\theta\right)\right]drd\theta.$$

By reducing powers of the trigonometric functions, this can be rewritten as

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu}{2}\left[r^{(5)}\frac{\partial u_{3,0}}{\partial z}\cos2\theta-r^{(5)}\frac{\partial u_{0,3}}{\partial z}\sin2\theta+r^{(5)}\frac{\partial u_{3,0}}{\partial z}+2r^{(3)}w_{2,0}\cos2\theta\right.$$
$$\left.+r^{(3)}\frac{\partial u_{1,0}}{\partial z}\cos2\theta-r^{(3)}\frac{\partial u_{0,1}}{\partial z}\sin2\theta+2r^{(3)}w_{2,0}+r^{(3)}\frac{\partial u_{1,0}}{\partial z}\right]drd\theta$$
$$+\int_0^{2\pi}\left[\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)\frac{\phi^{(2)}}{2}(\cos2\theta+1)-\tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\frac{\phi^{(2)}}{2}\sin2\theta\right]d\theta$$
$$=-p_I+\int_0^{2\pi}\int_0^{\phi}2\mu\left(ru_{1,0}+r^{(3)}u_{3,0}+r^{(3)}u_{3,0}(\cos2\theta+1)-r^{(3)}u_{0,3}\sin2\theta\right)drd\theta$$
$$+\int_0^{2\pi}\int_0^{\phi}\frac{\rho}{2}\left[-4u_{3,0}u_{0,3}r^{(7)}\sin2\theta+w_{2,0}r^{(7)}\frac{\partial u_{3,0}}{\partial z}\cos2\theta-w_{2,0}r^{(7)}\frac{\partial u_{0,3}}{\partial z}\sin2\theta-4u_{1,0}u_{0,3}r^{(5)}\sin2\theta\right.$$
$$-2u_{0,1}u_{0,3}r^{(5)}\cos2\theta+4u_{1,0}u_{3,0}r^{(5)}\cos2\theta-2u_{3,0}u_{0,1}r^{(5)}\sin2\theta$$
$$+w_{0,0}r^{(5)}\frac{\partial u_{3,0}}{\partial z}\cos2\theta-w_{0,0}r^{(5)}\frac{\partial u_{0,3}}{\partial z}\sin2\theta+w_{2,0}r^{(5)}\frac{\partial u_{1,0}}{\partial z}\cos2\theta$$
$$-w_{2,0}r^{(5)}\frac{\partial u_{0,1}}{\partial z}\sin2\theta-u_{0,3}^{(2)}r^{(7)}\cos2\theta+3u_{3,0}^{(2)}r^{(7)}\cos2\theta+w_{2,0}r^{(7)}\frac{\partial u_{3,0}}{\partial z}-2u_{0,1}u_{0,3}r^{(5)}$$
$$+4u_{1,0}u_{3,0}r^{(5)}+w_{0,0}r^{(5)}\frac{\partial u_{3,0}}{\partial z}+r^{(5)}\frac{\partial u_{3,0}}{\partial t}\cos2\theta-r^{(5)}\frac{\partial u_{0,3}}{\partial t}\sin2\theta+w_{2,0}r^{(5)}\frac{\partial u_{1,0}}{\partial z}-2u_{1,0}u_{0,1}r^{(3)}\sin2\theta$$
$$+w_{0,0}r^{(3)}\frac{\partial u_{1,0}}{\partial z}\cos2\theta-w_{0,0}r^{(3)}\frac{\partial u_{0,1}}{\partial z}\sin2\theta+u_{1,0}^{(2)}r^{(3)}-u_{0,1}^{(2)}r^{(3)}+r^{(3)}\frac{\partial u_{1,0}}{\partial t}-u_{0,3}^{(2)}r^{(7)}+3u_{3,0}^{(2)}r^{(7)}$$
$$\left.+u_{1,0}^{(2)}r^{(3)}\cos2\theta-u_{0,1}^{(2)}r^{(3)}\cos2\theta+r^{(5)}\frac{\partial u_{3,0}}{\partial t}+r^{(3)}\frac{\partial u_{1,0}}{\partial t}\cos2\theta-r^{(3)}\frac{\partial u_{0,1}}{\partial t}\sin2\theta+w_{0,0}r^{(3)}\frac{\partial u_{1,0}}{\partial z}\right]drd\theta.$$

The terms containing $\cos2\theta$ or $\sin2\theta$ will cancel when integrated over 0 to 2π. Dropping these terms now, for simplification, gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu}{2}\left(2w_{2,0}r^{(3)}+r^{(3)}\frac{\partial u_{1,0}}{\partial z}+r^{(5)}\frac{\partial u_{3,0}}{\partial z}\right)drd\theta$$
$$+\int_0^{2\pi}\frac{\phi^{(2)}}{2}\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)d\theta$$
$$=-p_I+\int_0^{2\pi}\int_0^{\phi}2\mu\left(u_{1,0}r+2r^{(3)}u_{3,0}\right)drd\theta$$
$$+\int_0^{2\pi}\int_0^{\phi}\frac{\rho}{2}\left[w_{2,0}r^{(7)}\frac{\partial u_{3,0}}{\partial z}-2u_{0,1}u_{0,3}r^{(5)}+4u_{1,0}u_{3,0}r^{(5)}+w_{0,0}r^{(5)}\frac{\partial u_{3,0}}{\partial z}+w_{2,0}r^{(5)}\frac{\partial u_{1,0}}{\partial z}+u_{1,0}^{(3)}r^{(3)}\right.$$
$$\left.-u_{0,1}^{(2)}r^{(3)}+r^{(3)}\frac{\partial u_{1,0}}{\partial t}-u_{0,3}^{(2)}r^{(7)}+2u_{3,0}^{(2)}r^{(7)}+r^{(5)}\frac{\partial u_{3,0}}{\partial t}+w_{0,0}r^{(3)}\frac{\partial u_{1,0}}{\partial z}\right]drd\theta.$$

Carrying out the integration with respect to r gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\frac{\mu}{2}\left(\frac{\phi^{(4)}}{2}w_{2,0}+\frac{\phi^{(4)}}{4}\frac{\partial u_{1,0}}{\partial z}+\frac{\phi^{(6)}}{6}\frac{\partial u_{3,0}}{\partial z}\right)d\theta$$
$$\int_0^{2\pi}\frac{\phi^{(2)}}{2}\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)d\theta$$
$$=-p_I+\int_0^{2\pi}2\mu\left(\frac{\phi^{(2)}}{2}u_{1,0}+\frac{\phi^{(4)}}{2}u_{3,0}\right)d\theta$$
$$\int_0^{2\pi}\frac{\rho}{2}\left[\frac{\phi^{(4)}}{4}\frac{\partial u_{1,0}}{\partial t}+\frac{\phi^{(6)}}{3}u_{3,0}u_{1,0}+\frac{\phi^{(8)}}{4}u_{3,0}^{(2)}+\frac{\phi^{(6)}}{6}\frac{\partial u_{3,0}}{\partial t}+\frac{\phi^{(4)}}{4}u_{1,0}^{(2)}+\frac{\phi^{(6)}}{3}u_{1,0}u_{3,0}+\frac{\phi^{(8)}}{8}u_{3,0}^{(2)}-\frac{\phi^{(4)}}{4}u_{0,1}^{(2)}\right.$$
$$\left.-\frac{\phi^{(6)}}{3}u_{0,1}u_{0,3}-\frac{\phi^{(8)}}{8}u_{0,3}^{(2)}+\frac{\phi^{(4)}}{4}w_{0,0}\frac{\partial u_{1,0}}{\partial z}+\frac{\phi^{(6)}}{6}w_{2,0}\frac{\partial u_{1,0}}{\partial z}+\frac{\phi^{(6)}}{6}w_{0,0}\frac{\partial u_{3,0}}{\partial z}+\frac{\phi^{(8)}}{8}w_{2,0}\frac{\partial u_{3,0}}{\partial z}\right]d\theta.$$

Carrying out the integration with respect to θ gives

$$\frac{\partial}{\partial z}\left[\frac{\mu\pi\phi^{(4)}}{4}\left(2w_{2,0}+\frac{\partial u_{1,0}}{\partial z}+\frac{2}{3}\phi^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)\right]+\pi\phi^{(2)}\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)$$
$$=-p_I+2\pi\mu\phi^{(2)}(u_{1,0}+\phi^{(2)}u_{3,0})$$
$$+\frac{\rho\pi\phi^{(4)}}{4}\left[\frac{\partial u_{1,0}}{\partial t}+\frac{\partial u_{1,0}}{\partial z}\left(w_{0,0}+\frac{2}{3}w_{2,0}\phi^{(2)}\right)+u_{1,0}\left(u_{1,0}+\frac{8}{3}\phi^{(2)}u_{3,0}\right)-u_{0,1}\left(u_{0,1}+\frac{4}{3}u_{0,3}\phi^{(2)}\right)\right.$$
$$\left.+\frac{2}{3}\phi^{(2)}\left(\frac{\partial u_{3,0}}{\partial t}+\frac{\partial u_{3,0}}{\partial z}\left(w_{0,0}+\frac{3}{4}\phi^{(2)}w_{2,0}\right)+3u_{3,0}\left(\frac{3}{4}\phi^{(2)}u_{3,0}\right)-u_{0,3}\left(\frac{3}{4}\phi^{(2)}u_{0,3}\right)\right)\right].$$

After some manipulation, the equation can be rewritten in the form

(A.3)

$$\frac{\pi\mu}{4}\frac{\partial}{\partial z}\left[\phi^{(4)}\left(\frac{\partial u_{1,0}}{\partial z}+\frac{2}{3}\phi^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)\right]+\frac{\pi\mu}{2}\frac{\partial}{\partial z}(w_{2,0}\phi^{(4)})-2\pi\mu\phi^{(2)}(u_{1,0}+\phi^{(2)}u_{3,0})+\pi\phi^{(2)}\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)$$
$$=-p_I+\frac{\rho\pi\phi^{(4)}}{4}\left[\frac{\partial u_{1,0}}{\partial t}+\frac{\partial u_{1,0}}{\partial z}\left(w_{0,0}+\frac{2}{3}\phi^{(2)}w_{2,0}\right)+u_{1,0}\left(u_{1,0}+\frac{2}{3}\phi^{(2)}u_{3,0}\right)-u_{0,1}\left(u_{0,1}+\frac{2}{3}u_{0,3}\phi^{(2)}\right)\right.$$
$$\left.+\frac{2}{3}\phi^{(2)}\left[\frac{\partial u_{3,0}}{\partial t}+\frac{\partial u_{3,0}}{\partial z}\left(w_{0,0}+\frac{3}{4}\phi^{(2)}w_{2,0}\right)+3u_{3,0}\left(u_{1,0}+\frac{3}{4}\phi^{(2)}u_{3,0}\right)-u_{0,3}\left(u_{0,1}+\frac{3}{4}\phi^{(2)}u_{0,3}\right)\right]\right].$$

This is Eq. (3.37) in the main text.

Due to symmetry of the flow, the $\mathbf{e}_2$ component should yield the same equation. Now, examining the $\mathbf{e}_3$ component of the equation gives

$$\frac{\partial}{\partial x_3}\int_A \left[-p_I + 2\mu\left(\frac{\partial w_{0,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right] x_1 da + \int_{\partial A}\left(\tau_1 + p_e \frac{\partial \phi}{\partial x_3}\right) x_1 ds$$
$$= \int_A \mu \left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right] da$$
$$+ \int_A \rho \left[\frac{\partial w_{0,0}}{\partial t} + (x_1^{(2)} + x_2^{(2)})\frac{\partial w_{2,0}}{\partial t} + 2x_1 w_{2,0}((w_{2,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0})x_1 - (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3})x_2)\right.$$
$$+2x_2 w_{2,0}((u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0})x_2 + (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3})x_1)+$$
$$\left.(w_{0,0} + (x_1^{(2)} + x_2^{(2)})w_{2,0})\left(\frac{\partial w_{0,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right] x_1 da.$$

After being converted to cylindrical polar coordinates, every term in this equation will contain either $\cos\theta$, $\sin\theta$, $\cos^{(3)}\theta$, $\cos^{(2)}\theta\sin\theta$ or $\cos\theta\sin^{(2)}\theta$. Noting that

$$\cos^{(3)}\theta = \frac{1}{4}(\cos 3\theta + 3\cos\theta);$$

$$\cos^{(2)}\theta\sin\theta = \frac{1}{4}(\sin 3\theta + \sin\theta);$$

$$\cos\theta\sin^{(2)}\theta = \frac{1}{4}(\cos\theta - \cos 3\theta);$$

all terms will cancel when integrated from 0 to 2π. Now, taking the case for Eq. (3.25) where $\alpha_1 = 2$, substituting in Eqs. (3.32), (3.29), (3.31) and (3.13) gives the following in the $\mathbf{e}_1$ direction

$$\frac{\partial}{\partial x_3}\int_A \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2 + 2x_1 w_{2,0}\right] x_2 da$$
$$+ \int_{\partial A} \phi^{-1}\left[x_1 x_2\left(\tau_1 \frac{\partial \phi}{\partial x_3} - p_e\right) - x_2^{(2)}\tau_2\left(1 + \left(\frac{\partial \phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}\right] ds$$
$$= \int_A \mu[4x_1 x_2 u_{3,0} + (2x_1^{(2)} - 2x_2^{(2)})u_{0,3}]da$$
$$+ \int_A \rho\left[\left(\frac{\partial u_{1,0}}{\partial t} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial t}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial t} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial t}\right)x_2\right.$$
$$+((u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0})x_1 - (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3})x_2)(2x_1^{(2)}u_{3,0} + u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0} - 2x_1 x_2 u_{0,3})$$
$$+((u_{1,0} + (x_1^{(2)} + x_2^{(2)})u_{3,0})x_2 + (u_{0,1} + (x_1^{(2)} + x_2^{(2)})u_{0,3})x_1)(2x_1 x_2 u_{3,0} - 2x_2^{(2)}u_{0,3} - u_{0,1} - (x_1^{(2)} + x_2^{(2)})u_{0,3})$$
$$\left.+(w_{0,0} + (x_1^{(2)} + x_2^{(2)})w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1 - \left(\frac{\partial u_{0,1}}{\partial x_3} + (x_1^{(2)} + x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right)\right] x_2 da.$$

Converting to cylindrical polar coordinates gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\mu r^{(3)}\left[r^{(2)}\frac{\partial u_{3,0}}{\partial z}\cos\theta - r^{(2)}\frac{\partial u_{0,3}}{\partial z}\sin\theta + \frac{\partial u_{1,0}}{\partial z}\cos\theta + 2w_{2,0}\cos\theta - \frac{\partial u_{0,1}}{\partial z}\sin\theta\right]\sin\theta drd\theta$$
$$+\int_0^{2\pi}\left[\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)\phi\cos\theta - \tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\sin\theta\right]\phi\sin\theta d\theta$$
$$=\int_0^{2\pi}\int_0^{\phi}\mu\left[4u_{3,0}r^{(3)}\cos\theta\sin\theta + 2u_{0,3}r^{(3)}(\cos^{(2)}\theta-\sin^{(2)}\theta)\right]drd\theta$$
$$+\int_0^{2\pi}\int_0^{\phi}\rho r^{(3)}\left[w_{2,0}r^{(4)}\frac{\partial u_{3,0}}{\partial z}\cos\theta - w_{2,0}r^{(2)}\frac{\partial u_{0,1}}{\partial z}\sin\theta\right.$$
$$-2u_{3,0}u_{0,1}r^{(2)}\sin\theta - u_{0,1}r^{(2)}\frac{\partial u_{0,3}}{\partial z}\sin\theta + u_{1,0}^{(2)}\cos\theta - u_{0,1}^{(2)}\cos\theta + \frac{\partial u_{1,0}}{\partial t}\cos\theta$$
$$-\frac{\partial u_{0,1}}{\partial t}\sin\theta + r^{(2)}\frac{\partial u_{3,0}}{\partial t}\cos\theta - r^{(2)}\frac{\partial u_{0,3}}{\partial t}\sin\theta - 4u_{1,0}u_{0,3}r^{(2)}\sin\theta + w_{0,0}r^{(2)}\frac{\partial u_{3,0}}{\partial z}\cos\theta$$
$$+w_{2,0}r^{(2)}\frac{\partial u_{1,0}}{\partial z}\cos\theta - 2u_{0,1}u_{0,3}r^{(2)}\cos\theta - w_{2,0}r^{(4)}\frac{\partial u_{0,3}}{\partial z}\sin\theta + 4u_{1,0}u_{3,0}r^{(2)}\cos\theta - 4u_{3,0}u_{0,3}r^{(4)}\sin\theta$$
$$\left.+w_{0,0}\frac{\partial u_{1,0}}{\partial z}\cos\theta - 2u_{1,0}u_{0,1}\sin\theta - w_{0,0}\frac{\partial u_{0,1}}{\partial z}\sin\theta + 2u_{3,0}^{(2)}r^{(4)}\cos\theta - u_{0,3}^{(2)}r^{(4)}\cos\theta\right]\sin\theta drd\theta.$$

Reducing powers of the trigonometric functions gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu r^{(3)}}{2}\left[r^{(2)}\frac{\partial u_{3,0}}{\partial z}\sin 2\theta + r^{(2)}\frac{\partial u_{0,3}}{\partial z}\cos 2\theta - r^{(2)}\frac{\partial u_{0,3}}{\partial z} + \frac{\partial u_{1,0}}{\partial z}\sin 2\theta + \frac{\partial u_{0,1}}{\partial z}\cos 2\theta + 2w_{2,0}\sin 2\theta\right.$$
$$\left.-\frac{\partial u_{0,1}}{\partial z}\right]drd\theta + \int_0^{2\pi}\frac{\phi^{(2)}}{2}\left[\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)\sin 2\theta + \tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}(\cos 2\theta - 1)\right]d\theta$$
$$=\int_0^{2\pi}\int_0^{\phi}2\mu(u_{3,0}r^{(3)}\sin 2\theta + u_{0,3}r^{(3)}\cos 2\theta)drd\theta + \int_0^{2\pi}\int_0^{\phi}\frac{\rho r^{(3)}}{2}\left[4u_{3,0}u_{0,3}r^{(4)}\cos 2\theta\right.$$
$$+w_{2,0}r^{(4)}\frac{\partial u_{3,0}}{\partial z}\sin 2\theta + w_{2,0}r^{(4)}\frac{\partial u_{0,3}}{\partial z}\cos 2\theta + 4u_{1,0}u_{3,0}r^{(2)}\sin 2\theta + 2u_{3,0}u_{0,1}r^{(2)}\cos 2\theta - 2u_{0,1}u_{0,3}r^{(2)}\sin 2\theta$$
$$+4u_{1,0}u_{0,3}r^{(2)}\cos 2\theta + w_{0,0}r^{(2)}\frac{\partial u_{3,0}}{\partial z}\sin 2\theta + w_{0,0}r^{(2)}\frac{\partial u_{0,3}}{\partial z}\cos 2\theta + w_{2,0}r^{(2)}\frac{\partial u_{1,0}}{\partial z}\sin 2\theta + w_{2,0}r^{(2)}\frac{\partial u_{0,1}}{\partial z}\cos 2\theta$$
$$+3u_{3,0}^{(2)}r^{(4)}\sin 2\theta - u_{0,3}^{(2)}r^{(4)}\sin 2\theta - 4u_{3,0}u_{0,3}r^{(4)} - w_{2,0}r^{(4)}\frac{\partial u_{0,3}}{\partial z} - 2u_{3,0}u_{0,1}r^{(2)} - 4u_{1,0}u_{0,3}r^{(2)} - w_{0,0}r^{(2)}\frac{\partial u_{0,3}}{\partial z}$$
$$-w_{2,0}r^{(2)}\frac{\partial u_{0,1}}{\partial z} + r^{(2)}\frac{\partial u_{3,0}}{\partial t}\sin 2\theta + r^{(2)}\frac{\partial u_{0,3}}{\partial t}\cos 2\theta + w_{0,0}\frac{\partial u_{1,0}}{\partial z}\sin 2\theta + w_{0,0}\frac{\partial u_{0,1}}{\partial z}\cos 2\theta + 2u_{1,0}u_{0,1}\cos 2\theta$$
$$\left.-\frac{\partial u_{0,1}}{\partial t} + u_{1,0}^{(2)}\sin 2\theta - u_{0,1}^{(2)}\sin 2\theta - \frac{\partial u_{0,3}}{\partial t}r^{(2)} - w_{0,0}\frac{\partial u_{0,1}}{\partial z} + \frac{\partial u_{1,0}}{\partial t}\sin 2\theta + \frac{\partial u_{0,1}}{\partial t}\cos 2\theta - 2u_{1,0}u_{0,1}\right]drd\theta.$$

Dropping the terms containing $\cos 2\theta$ and $\sin 2\theta$, as they will cancel when integrated from 0 to 2π gives

$$-\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu}{2}\left[r^{(5)}\frac{\partial u_{0,3}}{\partial z}+r^{(3)}\frac{\partial u_{0,1}}{\partial z}\right]drd\theta$$
$$-\int_0^{2\pi}\frac{\tau_2\phi^{(2)}}{2}\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}d\theta$$
$$=-\int_0^{2\pi}\int_0^{\phi}\frac{\rho}{2}\left[4u_{3,0}u_{0,3}r^{(7)}+w_{2,0}r^{(7)}\frac{\partial u_{0,3}}{\partial z}+2u_{3,0}u_{0,1}r^{(5)}+4u_{1,0}u_{0,3}r^{(5)}+w_{0,0}r^{(5)}\frac{\partial u_{0,3}}{\partial z}\right.$$
$$\left.+w_{2,0}r^{(5)}\frac{\partial u_{0,1}}{\partial z}+r^{(3)}\frac{\partial u_{0,1}}{\partial t}+r^{(5)}\frac{\partial u_{0,3}}{\partial t}+w_{0,0}r^{(3)}\frac{\partial u_{0,1}}{\partial z}+2u_{1,0}u_{0,1}r^{(3)}\right]drd\theta.$$

Carrying out the integration with respect to r gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\frac{\mu}{2}\left[\frac{\phi^{(6)}}{6}\frac{\partial u_{0,3}}{\partial z}+\frac{\phi^{(4)}}{4}\frac{\partial u_{0,1}}{\partial z}\right]d\theta$$
$$+\int_0^{2\pi}\frac{\tau_2\phi^{(2)}}{2}\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}d\theta$$
$$=\int_0^{2\pi}\frac{\rho}{2}\left[\frac{u_{3,0}u_{0,3}\phi^{(8)}}{2}+\frac{w_{2,0}\phi^{(8)}}{8}\frac{\partial u_{0,3}}{\partial z}+\frac{u_{3,0}u_{0,1}\phi^{(6)}}{3}+\frac{2u_{1,0}u_{0,3}\phi^{(6)}}{3}+\frac{w_{0,0}\phi^{(6)}}{6}\frac{\partial u_{0,3}}{\partial z}\right.$$
$$\left.+\frac{w_{2,0}\phi^{(6)}}{6}\frac{\partial u_{0,1}}{\partial z}+\frac{\phi^{(4)}}{4}\frac{\partial u_{0,1}}{\partial t}+\frac{\phi^{(6)}}{6}\frac{\partial u_{0,3}}{\partial t}+\frac{w_{0,0}\phi^{(4)}}{4}\frac{\partial u_{0,1}}{\partial z}+\frac{u_{1,0}u_{0,1}\phi^{(4)}}{2}\right]d\theta.$$

Carrying out the integration with respect to θ and cancelling by π gives

$$\frac{\mu}{4}\frac{\partial}{\partial z}\left[\phi^{(4)}\left(\frac{\partial u_{0,1}}{\partial z}+\frac{2\phi^{(2)}}{3}\frac{\partial u_{0,3}}{\partial z}\right)\right]+\phi^{(2)}\tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}$$
$$=\frac{\rho\phi^{(4)}}{4}\left[\frac{\partial u_{0,1}}{\partial t}+\frac{\partial u_{0,1}}{\partial z}\left(w_{0,0}+\frac{2\phi^{(2)}w_{2,0}}{3}\right)+2u_{0,1}\left(u_{1,0}+\frac{2\phi^{(2)}u_{3,0}}{3}\right)\right.$$
$$\left.\frac{2\phi^{(2)}}{3}\left(\frac{\partial u_{0,3}}{\partial t}+\frac{\partial u_{0,3}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+4u_{0,3}\left(u_{1,0}+\frac{3\phi^{(2)}u_{3,0}}{4}\right)\right)\right]. \quad \text{(A.4)}$$

This is Eq. (3.38) in the main text.

As with the $\alpha_1 = 1$ case, the equation in the $\mathbf{e}_2$ direction should be the same due to symmetry and all the terms in the $\mathbf{e}_3$ direction should cancel.

Now examining Eq. (3.26) and firstly taking the case where $\alpha_1 = \alpha_2 = 1$, the $\mathbf{e}_1$

component of the equation is given by

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\int_A \mu\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^1+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right]x_1^{(2)}da\\
&+\int_{\partial A}\phi^-1\left[x_1\left(\tau_1\frac{\partial\phi}{\partial x_3}-p_e\right)-x_2\tau_2\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}\right]x_1^{(2)}ds\\
&=\int_A 2x_1[-p_I+2\mu(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})]da\\
&=\int_A\rho\left[\left(\frac{\partial u_{1,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial t}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial t}\right)x_2\right.\\
&+(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)\\
&+(2x_1x_2u_{3,0}-(u_{0,1}+(x_1^{(2)}+3x_2^{(2)})u_{0,3}))((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)\\
&\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right)\right].
\end{aligned}
$$

When converted to cylindrical polar coordinates, every term in this equation will contain $\cos^n\theta\sin^m\theta$ where $n+m$ is odd. Therefore all terms will cancel when integrated from 0 to 2π. This should also be the case in the $\mathbf{e}_2$ direction. In the $\mathbf{e}_3$ direction, the equation is

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\int_A\left[-p+2\mu\left(\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]x_1^{(2)}da\\
&+\int_{\partial A}\left(\tau 1+p_e\frac{\partial\phi}{\partial x_3}\right)x_1^{(2)}ds\\
&=\int_A 2\mu x_1\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{1,0}}{\partial x_3}\right)x_2+2w_{2,0}x_1\right]da\\
&+\int_A\rho x_1^{(2)}\left[\frac{\partial w_{0,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial t}+2w_{2,0}x_1((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)\right.\\
&+2w_{2,0}x_2((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)\\
&\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\frac{\partial w_{0,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial w_{2,0}}{\partial x_3}\right)\right]da.
\end{aligned}
$$

Converting to cylindrical polar coordinates gives

$$
\begin{aligned}
&-\frac{\partial q_I}{\partial z}+\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}2\mu r^{(3)}\cos^{(2)}\theta\left[\frac{\partial w_{0,0}}{\partial z}+r^{(2)}\frac{\partial w_{2,0}}{\partial z}\right]drd\theta\\
&+\int_0^{2\pi}\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)\phi^{(3)}\cos^{(2)}\theta d\theta\\
&=\int_0^{2\pi}\int_0^{\phi}2\mu r^{(3)}\cos\theta\left[-r^{(2)}\frac{\partial u_{0,3}}{\partial z}\sin\theta+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\cos\theta+\frac{\partial u_{1,0}}{\partial z}\cos\theta-\frac{\partial u_{0,1}}{\partial z}\sin\theta+2w_{2,0}\cos\theta\right]drd\theta\\
&+\int_0^{2\pi}\int_0^{\phi}\rho r^{(2)}\cos^{(2)}\theta\left[r^{(2)}\frac{\partial w_{2,0}}{\partial t}+\frac{\partial w_{0,0}}{\partial t}+2u_{3,0}w_{2,0}r^{(4)}+w_{2,0}r^{(4)}\frac{\partial w_{2,0}}{\partial z}\right.\\
&\left.2u_{1,0}w_{2,0}r^{(2)}+w_{0,0}r^{(2)}\frac{\partial w_{2,0}}{\partial z}+w_{2,0}r^{(2)}\frac{\partial w_{0,0}}{\partial z}+w_{0,0}\frac{\partial w_{0,0}}{\partial z}\right]drd\theta,
\end{aligned}
$$

where q_I is the pressure resultant defined by

$$
q_I=\int\int_A px_1^{(2)}da=\int\int_A px_2^{(2)}da. \tag{A.5}
$$

Reducing the powers of the trigonometric functions gives

$$
\begin{aligned}
&-\frac{\partial q_I}{\partial z}+\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\mu r^{(3)}(1+\cos 2\theta)\left[\frac{\partial w_{0,0}}{\partial z}+r^{(2)}\frac{\partial w_{2,0}}{\partial z}\right]drd\theta\\
&+\int_0^{2\pi}\frac{\phi^{(3)}}{2}(1+\cos 2\theta)\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)d\theta\\
&=\int_0^{2\pi}\int_0^{\phi}\mu\left[-r^{(5)}\frac{\partial u_{0,3}}{\partial z}\sin 2\theta+r^{(5)}\frac{\partial u_{3,0}}{\partial z}\cos 2\theta+r^{(5)}\frac{\partial u_{3,0}}{\partial z}-r^{(3)}\frac{\partial u_{0,1}}{\partial z}\sin 2\theta\right.\\
&\left.+r^{(3)}\frac{\partial u_{1,0}}{\partial z}\cos 2\theta+2r^{(3)}w_{2,0}\cos 2\theta+r^{(3)}\frac{\partial u_{1,0}}{\partial z}+2r^{(3)}w_{2,0}\right]drd\theta\\
&+\int_0^{2\pi}\int_0^{\phi}\frac{\rho}{2}(1+\cos 2\theta)\left[r^{(5)}\frac{\partial w_{2,0}}{\partial t}+r^{(3)}\frac{\partial w_{0,0}}{\partial t}+2u_{3,0}w_{2,0}r^{(7)}+w_{2,0}r^{(7)}\frac{\partial w_{2,0}}{\partial z}\right.\\
&\left.+2u_{1,0}w_{2,0}r^{(5)}+w_{0,0}r^{(5)}\frac{\partial w_{2,0}}{\partial z}+w_{2,0}r^{(5)}\frac{\partial w_{0,0}}{\partial z}+w_{0,0}\frac{\partial w_{0,0}}{\partial z}\right]drd\theta.
\end{aligned}
$$

Dropping the terms containing $\cos 2\theta$ or $\sin 2\theta$, which will cancel when integrated from 0 to 2π gives

$$
\begin{aligned}
&-\frac{\partial q_I}{\partial z}+\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\mu\left[r^{(3)}\frac{\partial w_{0,0}}{\partial z}+r^{(5)}\frac{\partial w_{2,0}}{\partial z}\right]drd\theta\\
&+\int_0^{2\pi}\frac{\phi^{(3)}}{2}\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)d\theta\\
&=\int_0^{2\pi}\int_0^{\phi}\mu\left[r^{(5)}\frac{\partial u_{3,0}}{\partial z}+r^{(3)}\frac{\partial u_{1,0}}{\partial z}+2r^{(3)}w_{2,0}\right]drd\theta\\
&+\int_0^{2\pi}\int_0^{\phi}\frac{\rho}{2}\left[r^{(5)}\frac{\partial w_{2,0}}{\partial t}+r^{(3)}\frac{\partial w_{0,0}}{\partial t}+2u_{3,0}w_{2,0}r^{(7)}+w_{2,0}r^{(7)}\frac{\partial w_{2,0}}{\partial z}\right.\\
&\left.+2u_{1,0}w_{2,0}r^{(5)}+w_{0,0}r^{(5)}\frac{\partial w_{2,0}}{\partial z}+w_{2,0}r^{(5)}\frac{\partial w_{0,0}}{\partial z}+r^{(3)}c\frac{\partial w_{0,0}}{\partial z}\right]drd\theta.
\end{aligned}
$$

Carrying out the integration with respect to r gives

$$
\begin{aligned}
&-\frac{\partial q_I}{\partial z}+\frac{\partial}{\partial z}\int_0^{2\pi}\mu\left[\frac{\phi^{(4)}}{4}\frac{\partial w_{0,0}}{\partial z}+\frac{\phi^{(6)}}{6}\frac{\partial w_{2,0}}{\partial z}\right]d\theta\\
&+\int_0^{2\pi}\frac{\phi^{(3)}}{2}\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)d\theta\\
&=\int_0^{2\pi}\mu\left[\frac{\phi^{(6)}}{6}\frac{\partial u_{3,0}}{\partial z}+\frac{\phi^{(4)}}{4}\frac{\partial u_{1,0}}{\partial z}+\frac{\phi^{(4)}w_{2,0}}{2}\right]d\theta\\
&+\int_0^{2\pi}\frac{\rho}{2}\left[\frac{\phi^{(6)}}{6}\frac{\partial w_{2,0}}{\partial t}+\frac{\phi^{(4)}}{4}\frac{\partial w_{0,0}}{\partial t}+\frac{u_{3,0}w_{2,0}\phi^{(8)}}{4}+\frac{w_{2,0}\phi^{(8)}}{8}\frac{\partial w_{2,0}}{\partial z}\right.\\
&\left.\frac{w_{2,0}u_{1,0}\phi^{(6)}}{3}+\frac{w_{0,0}\phi^{(6)}}{6}\frac{\partial w_{2,0}}{\partial z}+\frac{w_{2,0}\phi^{(6)}}{6}\frac{\partial w_{0,0}}{\partial z}+\frac{w_{0,0}\phi^{(4)}}{4}\frac{\partial w_{0,0}}{\partial z}\right]d\theta.
\end{aligned}
$$

Carrying out the integration with respect to θ gives

$$
\begin{aligned}
&-\frac{\partial q_I}{\partial z}+\frac{\partial}{\partial z}\left[\mu\pi\phi^{(4)}\left(\frac{1}{2}\frac{\partial w_{0,0}}{\partial z}+\frac{\phi^{(2)}}{3}\frac{\partial w_{2,0}}{\partial z}\right)\right]\\
&+\pi\phi^{(3)}\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)\\
&=\pi\mu\phi^{(4)}\left[\frac{\phi^{(2)}}{3}\frac{\partial u_{3,0}}{\partial z}+\frac{1}{2}\frac{\partial u_{1,0}}{\partial z}+w_{2,0}\right]\\
&+\frac{\pi\rho\phi^{(4)}}{4}\left[\frac{2\phi^{(2)}}{3}\frac{\partial w_{2,0}}{\partial t}+\frac{\partial w_{0,0}}{\partial t}+u_{3,0}w_{2,0}\phi^{(4)}+\frac{w_{2,0}\phi^{(4)}}{2}\frac{\partial w_{2,0}}{\partial z}\right.\\
&\left.+\frac{4w_{2,0}u_{1,0}\phi^{(2)}}{3}+\frac{2w_{0,0}\phi^{(2)}}{3}\frac{\partial w_{2,0}}{\partial z}+\frac{2w_{2,0}\phi^{(2)}}{3}\frac{\partial w_{0,0}}{\partial z}+w_{0,0}\frac{\partial w_{0,0}}{\partial z}\right].
\end{aligned}
$$

Some rearranging gives

$$
\begin{aligned}
&\frac{\pi\mu}{2}\frac{\partial}{\partial z}\left[\phi^{(4)}\left(\frac{\partial w_{0,0}}{\partial z}+\frac{2\phi^{(2)}}{3}\frac{\partial w_{2,0}}{\partial z}\right)\right]-\pi\mu\phi^{(4)}w_{2,0}\\
&-\frac{\pi\mu\phi^{(4)}}{2}\left(\frac{\partial u_{1,0}}{\partial z}+\frac{2\phi^{(2)}}{3}\frac{\partial u_{3,0}}{\partial z}\right)+\pi\phi^{(3)}\left(\tau_1+p_e\frac{\partial\phi}{\partial z}\right)\\
&=\frac{\partial q_I}{\partial z}+\frac{\pi\rho\phi^{(4)}}{4}\left[\frac{\partial w_{0,0}}{\partial t}+\frac{\partial w_{0,0}}{\partial z}\left(w_{0,0}+\frac{2\phi^{(2)}w_{2,0}}{3}\right)\right.\\
&\left.\frac{2\phi^{(2)}}{3}\left(\frac{\partial w_{2,0}}{\partial t}+\frac{\partial w_{2,0}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+2w_{2,0}\left(u_{1,0}+\frac{2\phi^{(2)}u_{3,0}}{4}\right)\right)\right].
\end{aligned}
\tag{A.6}
$$

This is Eq. (3.39) in the main text.

Now examining Eq. (3.27) and taking the case where $\alpha_1=\alpha_2=\alpha_3=1$, after making the usual substitutions, the equation in the $\mathbf{e}_1$ direction is given by

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\int_A \mu x_1^{(3)}\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{1,0}}{\partial x_3}\right)x_2+2x_1w_{2,0}\right]da\\
&+\int_{\partial A} x_1^{(3)}\left[x_1\left(\tau_1\frac{\partial\phi}{\partial x_3}-p_e\right)-x_2\tau_2\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}\right]ds\\
&=\int_A 3x_1^{(2)}[-p_I+2\mu(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})]da\\
&+\int_A \rho x_1^{(3)}\left[\left(\frac{\partial u_{1,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial t}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{1,0}}{\partial t}\right)x_2\right.\\
&+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})\\
&+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)(2x_1x_2u_{3,0}-(u_{0,1}+(x_1^{(2)}+3x_2^{(2)})u_{0,3}))\\
&\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right)\right]da.
\end{aligned}
$$

Converting to cylindrical polar coordinates gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\mu r^{(4)}\cos^{(3)}\theta\left[\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{1,0}}{\partial z}\right)r\sin\theta+2w_{2,0}r\cos\theta\right]drdr\theta$$
$$+\int_0^{2\pi}\phi^{(4)}\cos^{(3)}\theta\left[\left(\frac{\partial\phi}{\partial z}-p_e\right)\cos\theta-\tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\sin\theta\right]d\theta$$
$$=-3q_I+\int_0^{2\pi}\int_0^{\phi}6\mu r^{(3)}\cos^{(2)}\theta[u_{1,0}+(r^{(2)}+2r^{(2)}\cos\theta)u_{3,0}-2u_{0,3}\cos\theta\sin\theta]drd\theta$$
$$+\int_0^{2\pi}\int_0^{\phi}\rho r^{(4)}\cos^{(3)}\theta\left[\left(\frac{\partial u_{1,0}}{\partial t}+r^{(2)}\frac{\partial u_{3,0}}{\partial t}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial t}+r^{(2)}\frac{\partial u_{0,3}}{\partial t}\right)r\sin\theta\right.$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\cos\theta-(u_{0,1}+r^{(2)}u_{0,3})r\sin\theta)(u_{1,0}+r^{(2)}u_{3,0}(1+2\cos^{(2)}\theta)-2r^{(2)}u_{0,3}\cos\theta\sin\theta)$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\sin\theta+(u_{0,1}+r^{(2)}u_{0,3})r\cos\theta)(2u_{3,0}r^{(2)}\cos\theta\sin\theta-(u_{0,1}+r^{(2)}(1+2\sin^{(2)}\theta)u_{0,3}))$$
$$\left.(w_{0,0}+r^{(2)}w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)r\sin\theta\right)\right].$$

Reducing the powers of the trigonometric functions gives

$$
\begin{aligned}
&\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu r^{(5)}}{8}\Bigg[4r^{(2)}\frac{\partial u_{3,0}}{\partial z}\cos 2\theta + r^{(2)}\frac{\partial u_{3,0}}{\partial z}\cos 4\theta - r^{(2)}\frac{\partial u_{0,3}}{\partial z}\sin 4\theta - 2r^{(2)}\frac{\partial u_{0,3}}{\partial z}\sin 2\theta + 3r^{(2)}\frac{\partial u_{3,0}}{\partial z}\\
&+4\frac{\partial u_{0,3}}{\partial z}\cos 2\theta + 8w_{2,0}\cos 2\theta + \frac{\partial u_{1,0}}{\partial z}\cos 4\theta + 2w_{2,0}\cos 4\theta - \frac{\partial u_{0,1}}{\partial z}\sin 4\theta - 2\frac{\partial u_{0,1}}{\partial z}\sin 2\theta + 3\frac{\partial u_{1,0}}{\partial z}\\
&+6w_{2,0}\Bigg]\, dr d\theta + \int_0^{2\pi}\frac{\phi^{(4)}}{8}\Bigg[4\tau_1\frac{\partial\phi}{\partial z}\cos 2\theta + \tau_1\frac{\partial\phi}{\partial z}\cos 4\theta - \tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}(\sin 4\theta + \sin 2\theta)\\
&-4p_e\cos 2\theta - p_e\cos 4\theta + 3\tau_1 - 3p_e\Bigg] d\theta = -3q_I\int_0^{2\pi}\int_0^{\phi}\frac{3\mu r^{(3)}}{2}\Big[6r^{(2)}u_{3,0}\cos 2\theta\\
&+u_{3,0}r^{(2)}\cos 4\theta - u_{0,3}r^{(2)}\sin 4\theta - 2r^{(2)}u_{0,3}\sin 2\theta + 4r^{(2)}u_{3,0} + 2u_{1,0}\cos 2\theta\\
&+2u_{1,0}\Big] dr d\theta + \int_0^{2\pi}\int_0^{\phi}\frac{\rho r^{(5)}}{8}\Bigg[3u_{1,0}^{(2)} - 2u_{0,1}^{(2)} + 3\frac{\partial u_{1,0}}{\partial t} + 4w_{2,0}r^{(4)}\frac{\partial u_{3,0}}{\partial z}\cos 2\theta + w_{2,0}r^{(4)}\frac{\partial u_{3,0}}{\partial z}\cos 4\theta\\
&-w_{2,0}r^{(4)}\frac{\partial u_{0,3}}{\partial z}\sin 4\theta - 2w_{2,0}r^{(4)}\frac{\partial u_{0,3}}{\partial z}\sin 2\theta - 4r^{(4)}u_{0,3}u_{3,0}\sin 4\theta - 8u_{0,3}u_{3,0}r^{(4)}\sin 2\theta\\
&+4r^{(2)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}\cos 2\theta + r^{(2)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}\cos 4\theta\\
&+4w_{2,0}r^{(2)}\frac{\partial u_{1,0}}{\partial z} - 8r^{(2)}u_{0,3}u_{0,1}\cos 2\theta + 16r^{(2)}u_{3,0}u_{1,0}\cos 2\theta + w_{2,0}r^{(2)}\frac{\partial u_{1,0}}{\partial z}\cos 4\theta\\
&-2r^{(2)}u_{0,3}u_{0,1}\cos 4\theta + 4r^{(2)}u_{3,0}u_{1,0}\cos 4\theta - r^{(2)}w_{0,0}\frac{\partial u_{0,3}}{\partial z}\sin 4\theta - 2r^{(2)}w_{0,0}\frac{\partial u_{0,3}}{\partial z}\sin 2\theta\\
&-w_{2,0}r^{(2)}\frac{\partial u_{0,1}}{\partial z}\sin 4\theta - 2r^{(2)}u_{3,0}u_{0,1}\sin 4\theta - 4r^{(2)}u_{0,3}u_{1,0}\sin 4\theta\\
&-2w_{2,0}r^{(2)}\frac{\partial u_{0,1}}{\partial z}\sin 2\theta - 4r^{(2)}u_{3,0}u_{0,1}\sin 2\theta - 8r^{(2)}u_{0,3}u_{1,0}\sin 2\theta\\
&+9r^{(4)}u_{3,0}^{(2)} - 3r^{(4)}u_{0,3}^{(2)} - 4u_{0,1}^{(2)}\cos 2\theta + 4u_{1,0}^{(2)}\cos 2\theta - u_{0,1}^{(2)}\cos 4\theta + u_{1,0}^{(2)}\cos 4\theta + 3r^{(2)}\frac{\partial u_{3,0}}{\partial t}\\
&+4\frac{\partial u_{1,0}}{\partial t}\cos 2\theta + \frac{\partial u_{1,0}}{\partial t}\cos 4\theta - \frac{\partial u_{0,1}}{\partial t}\sin 4\theta - 2\frac{\partial u_{0,1}}{\partial t}\sin 2\theta + 3w_{0,0}\frac{\partial u_{1,0}}{\partial z}\\
&-6r^{(2)}u_{0,3}u_{0,1} + 12r^{(2)}u_{3,0}u_{1,0} + 4w_{0,0}\frac{\partial u_{1,0}}{\partial z}\cos 2\theta + w_{0,0}\frac{\partial u_{1,0}}{\partial z}\cos 4\theta - w_{0,0}\frac{\partial u_{0,1}}{\partial z}\sin 4\theta\\
&-2u_{1,0}u_{0,1}\sin 4\theta - 2w_{0,0}\frac{\partial u_{0,1}}{\partial z}\sin 2\theta - 4u_{1,0}u_{0,1}\sin 2\theta + 12r^{(4)}u_{3,0}^{(2)}\cos 2\theta\\
&-4r^{(4)}u_{0,3}^{(2)}\cos 2\theta + 3r^{(2)}u_{3,0}^{(2)}\cos 4\theta - r^{(4)}u_{0,3}^{(2)}\cos 4\theta + 3w_{2,0}r^{(4)}\frac{\partial u_{3,0}}{\partial z} + 3r^{(2)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}\\
&+4r^{(2)}\frac{\partial u_{3,0}}{\partial t}\cos 2\theta + r^{(2)}\frac{\partial u_{3,0}}{\partial t}\cos 4\theta - r^{(2)}\frac{\partial u_{0,3}}{\partial t}\sin 4\theta - 2r^{(2)}\frac{\partial u_{0,3}}{\partial t}\sin 2\theta + 3w_{2,0}r^{(2)}\frac{\partial u_{1,0}}{\partial z}\Bigg] dr d\theta.
\end{aligned}
$$

Dropping the terms containing $\cos n\theta$ or $\sin n\theta$, as they will cancel when integrated from

0 to 2π, gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu r^{(5)}}{8}\left[3r^{(2)}\frac{\partial u_{3,0}}{\partial z}+3\frac{\partial u_{1,0}}{\partial z}+6w_{2,0}\right]drd\theta$$
$$+\int_0^{2\pi}\frac{3\phi^{(4)}}{8}\left[\tau_1\frac{\partial\phi}{\partial z}-p_e\right]d\theta$$
$$=-3q_I+\int_0^{2\pi}\int_0^{\phi}\frac{3\mu r^{(3)}}{2}[5r^{(2)}u_{3,0}+2u_{1,0}]drd\theta$$
$$+\int_0^{2\pi}\int 0^{\phi}\frac{\rho r^{(5)}}{8}\left[3u_{1,0}^{(2)}-3u_{0,1}^{(2)}+3\frac{\partial u_{1,0}}{\partial t}+9r^{(4)}u_{3,0}^{(2)}-3r^{(4)}u_{0,3}^{(2)}+3r^{(2)}\frac{\partial u_{3,0}}{\partial t}+3w_{0,0}\frac{\partial u_{1,0}}{\partial z}\right.$$
$$\left.-6r^{(2)}u_{0,3}u_{0,1}+12r^{(2)}u_{3,0}u_{1,0}+3w_{2,0}r^{(4)}\frac{\partial u_{3,0}}{\partial z}+3r^{(2)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}+3w_{2,0}r^{(2)}\frac{\partial u_{1,0}}{\partial z}\right]drd\theta.$$

Integrating with respect to r gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\frac{\mu}{8}\left[\frac{3\phi^{(8)}}{8}\frac{\partial u_{3,0}}{\partial z}+\frac{\phi^{(6)}}{2}\frac{\partial u_{1,0}}{\partial z}+\phi^{(6)}w_{2,0}\right]d\theta$$
$$+\int_0^{2\pi}\frac{3\phi^{(4)}}{8}\left[\tau_1\frac{\partial\phi}{\partial z}-p_e\right]d\theta$$
$$=-3q_I+\int_0^{2\pi}\frac{2\mu}{2}\left[\frac{5\phi^{(6)}u_{3,0}}{6}+\frac{\phi^{(4)}u_{1,0}}{2}\right]d\theta$$
$$+\int_0^{2\pi}\frac{\rho}{8}\left[\frac{u_{1,0}^{(2)}\phi^{(6)}}{2}-\frac{u_{0,1}^{(2)}\phi^{(6)}}{2}+\frac{\phi^{(6)}}{2}\frac{\partial u_{1,0}}{\partial t}+\frac{9\phi^{1}0u_{3,0}^{(2)}}{10}-\frac{3\phi^{1}0u_{0,3}^{(2)}}{10}+\frac{3\phi^{(8)}}{8}\frac{\partial u_{3,0}}{\partial t}+\frac{\phi^{(6)}w_{0,0}}{2}\frac{\partial u_{1,0}}{\partial z}\right.$$
$$\left.-\frac{3\phi^{(8)}u_{0,3}u_{0,1}}{4}+\frac{3\phi^{(8)}u_{3,0}u_{1,0}}{2}+\frac{3\phi^{1}0w_{2,0}}{10}\frac{\partial u_{3,0}}{\partial z}+\frac{3\phi^{(8)}}{8}\frac{\partial u_{3,0}}{\partial z}+3\phi^{(8)}w_{2,0}8\frac{\partial u_{1,0}}{\partial z}\right]d\theta.$$

Integrating with respect to θ gives

$$\frac{\partial}{\partial z}\left[\frac{\pi\mu}{8}\left(\frac{3\phi^{(8)}}{4}\frac{\partial u_{3,0}}{\partial z}+\phi^{(6)}\frac{\partial u_{1,0}}{\partial z}+2\phi^{(6)}w_{2,0}\right)\right]$$
$$+\frac{3\phi^{(4)}}{4}\left[\tau_1\frac{\partial\phi}{\partial z}-p_e\right]$$
$$=-3q_I+3\mu\phi^{(4)}\left[\frac{5\phi^{(2)}u_{3,0}}{6}+\frac{u_{1,0}}{2}\right]$$
$$+\frac{\pi\rho\phi^{(6)}}{8}\left[u_{1,0}^{(2)}-u_{0,1}^{(2)}+\frac{\partial u_{1,0}}{\partial t}+\frac{9\phi^{(4)}u_{3,0}^{(2)}}{5}-\frac{3\phi^{(4)}u_{0,3}^{(2)}}{5}+\frac{3\phi^{(2)}}{4}\frac{\partial u_{3,0}}{\partial t}+w_{0,0}\frac{\partial u_{1,0}}{\partial z}-\frac{3\phi^{(2)}u_{0,3}u_{0,1}}{2}\right.$$
$$\left.+3\phi^{(2)}u_{3,0}u_{1,0}+\frac{3\phi^{(4)}w_{2,0}}{5}\frac{\partial u_{3,0}}{\partial z}+\frac{3\phi^{(2)}w_{0,0}}{4}\frac{\partial u_{3,0}}{\partial z}+\frac{3\phi^{(2)}w_{2,0}}{4}\frac{\partial u_{1,0}}{\partial z}\right].$$

After some rearrangement and dividing through by 3, this becomes

$$
\begin{aligned}
&\frac{\pi\mu}{24}\frac{\partial}{\partial z}\left[\phi^{(6)}\left(\frac{\partial u_{1,0}}{\partial z}+\frac{3\phi^{(2)}}{4}\frac{\partial u_{3,0}}{\partial z}\right)\right]+\frac{\pi\mu}{12}\frac{\partial}{\partial z}(\phi^{(6)}w_{2,0})\\
&-\frac{\pi\mu\phi^{(4)}}{2}\left(u_{1,0}+\frac{5\phi^{(2)}u_{3,0}}{3}\right)+\frac{\pi\phi^{(4)}}{4}\left(\tau_1\frac{\partial\phi}{\partial z}-p_e\right)\\
&=-q_I+\frac{\pi\rho\phi^{(6)}}{24}\left[\frac{\partial u_{1,0}}{\partial t}+\frac{\partial u_{1,0}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+u_{1,0}\left(u_{1,0}+\frac{3\phi^{(2)}u_{3,0}}{4}\right)\right.\\
&-u_{0,1}\left(u_{0,1}+\frac{3\phi^{(2)}u_{0,3}}{4}\right)+\frac{3\phi^{(2)}}{4}\left(\frac{\partial u_{3,0}}{\partial t}+\frac{\partial u_{3,0}}{\partial z}\left(w_{0,0}+\frac{4\phi^{(2)}w_{2,0}}{5}\right)\right.\\
&\left.\left.+3u_{3,0}\left(u_{1,0}+\frac{4\phi^{(2)}u_{3,0}}{5}\right)-u_{0,3}\left(u_{0,1}+\frac{4\phi^{(2)}u_{0,3}}{5}\right)\right)\right].
\end{aligned}
\tag{A.7}
$$

This is Eq. (3.40) in the main text.

Now taking the case where $\alpha_1=\alpha_2=\alpha_3=2$ in the $\mathbf{e}_1$ direction, the usual substitutions give

$$
\begin{aligned}
&\frac{\partial}{\partial x_3}\int_A \mu x_2^{(3)}\left[\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{1,0}}{\partial x_3}\right)x_2+2x_1w_{2,0}\right]da\\
&+\int_{\partial A}x_2^{(3)}\left[x_1\left(\tau_1\frac{\partial\phi}{\partial x_3}-p_e\right)-x_2\tau_2\left(1+\left(\frac{\partial\phi}{\partial x_3}\right)^{(2)}\right)^{(1/2)}\right]ds\\
&=\int_A 3x_2^{(2)}\mu[4x_1x_2u_{3,0}+(2x_1^{(2)}-2x_2^{(2)})u_{0,3}]da\\
&+\int_A\rho x_2^{(3)}\left[\left(\frac{\partial u_{1,0}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial t}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial t}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{1,0}}{\partial t}\right)x_2\right.\\
&+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_1-(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_2)(u_{1,0}+(3x_1^{(2)}+x_2^{(2)})u_{3,0}-2x_1x_2u_{0,3})\\
&+((u_{1,0}+(x_1^{(2)}+x_2^{(2)})u_{3,0})x_2+(u_{0,1}+(x_1^{(2)}+x_2^{(2)})u_{0,3})x_1)(2x_1x_2u_{3,0}-(u_{0,1}+(x_1^{(2)}+3x_2^{(2)})u_{0,3}))\\
&\left.+(w_{0,0}+(x_1^{(2)}+x_2^{(2)})w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{3,0}}{\partial x_3}\right)x_1-\left(\frac{\partial u_{0,1}}{\partial x_3}+(x_1^{(2)}+x_2^{(2)})\frac{\partial u_{0,3}}{\partial x_3}\right)x_2\right)\right]da.
\end{aligned}
$$

Converting to cylindrical polar coordinates gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\mu r^{(4)}\sin^{(3)}\theta\left[\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{1,0}}{\partial z}\right)r\sin\theta+2w_{2,0}r\cos\theta\right]drdr\theta$$
$$+\int_0^{2\pi}\phi^{(4)}\sin^{(3)}\theta\left[\left(\frac{\partial\phi}{\partial z}-p_e\right)\cos\theta-\tau_2\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\sin\theta\right]d\theta$$
$$=\int_0^{2\pi}\int_0^{\phi}3\mu r^{(3)}\sin^{(2)}\theta[4u_{3,0}r^{(2)}\cos\theta\sin\theta+2r^{(2)}u_{0,3}(\cos^{(2)}\theta-\sin^{(2)}\theta)]drd\theta$$
$$+\int_0^{2\pi}\int_0^{\phi}\rho r^{(4)}\sin^{(3)}\theta\left[\left(\frac{\partial u_{1,0}}{\partial t}+r^{(2)}\frac{\partial u_{3,0}}{\partial t}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial t}+r^{(2)}\frac{\partial u_{0,3}}{\partial t}\right)r\sin\theta\right.$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\cos\theta-(u_{0,1}+r^{(2)}u_{0,3})r\sin\theta)(u_{1,0}+r^{(2)}u_{3,0}(1+2\cos^{(2)}\theta)-2r^{(2)}u_{0,3}\cos\theta\sin\theta)$$
$$+((u_{1,0}+r^{(2)}u_{3,0})r\sin\theta+(u_{0,1}+r^{(2)}u_{0,3})r\cos\theta)(2u_{3,0}r^{(2)}\cos\theta\sin\theta-(u_{0,1}+r^{(2)}(1+2\sin^{(2)}\theta)u_{0,3}))$$
$$\left.(w_{0,0}+r^{(2)}w_{2,0})\left(\left(\frac{\partial u_{1,0}}{\partial z}+r^{(2)}\frac{\partial u_{3,0}}{\partial z}\right)r\cos\theta-\left(\frac{\partial u_{0,1}}{\partial z}+r^{(2)}\frac{\partial u_{0,3}}{\partial z}\right)r\sin\theta\right)\right].$$

Reducing the powers of the trigonometric functions gives

$$\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{\mu r^{(5)}}{8}\left[2r^{(2)}\frac{\partial u_{3,0}}{\partial z}\sin 2\theta - r^{(2)}\frac{\partial u_{3,0}}{\partial z}\sin 4\theta + r^{(2)}\frac{\partial u_{0,3}}{\partial z}\cos 2\theta - 3r^{(2)}\frac{\partial u_{0,3}}{\partial z} - \frac{\partial u_{1,0}}{\partial z}\sin 4\theta\right.$$
$$\left.+4r^{(2)}\frac{\partial u_{0,3}}{\partial z} - 2w_{2,0}\sin 4\theta + 2\frac{\partial u_{1,0}}{\partial z}\sin 2\theta - \frac{\partial u_{0,1}}{\partial z}\cos 4\theta + 4\frac{\partial u_{0,1}}{\partial z}\cos 2\theta - 3\frac{\partial u_{0,1}}{\partial z}\right] dr d\theta$$
$$+\int_0^{2\pi}\frac{\phi^{(4)}}{8}\left[2\tau_1\frac{\partial \phi}{\partial z} - \tau_1\frac{\partial \phi}{\partial z}\sin 4\theta + \left(1+\tau_2\left(\frac{\partial \phi}{\partial z}\right)^{(2)}\right)^{(1/2)}(4\cos 2\theta - \cos 4\theta - 3)\right.$$
$$+p_e\sin 4\theta - 2p_e\sin 2\theta] d\theta = \int_0^{2\pi}\int_0^{\phi}\frac{3\mu r^{(5)}}{2}[2u_{3,0}\sin 2\theta - u_{3,0}\sin 4\theta - u_{0,3}\cos 4\theta + 2u_{0,3}\cos 2\theta$$
$$-u_{0,3}] dr d\theta + int_0^{2\pi}\int_0^{\phi}\frac{\rho r^{(5)}}{8}\left[6r^{(4)}u_{3,0}^{(2)}\sin 2\theta - 3r^{(2)}u_{3,0}^{(2)}\sin 4\theta + r^{(2)}u_{0,3}^{(2)}\sin 4\theta - 2r^{(4)}u_{0,3}^{(2)}\sin 2\theta\right.$$
$$-12r^{(4)}u_{3,0}u_{0,3} - 3r^{(4)}w_{2,0}\frac{\partial u_{0,3}}{\partial z} - 12r^{(2)}u_{1,0}u_{0,3} - 6r^{(2)}u_{0,1}u_{3,0} - 3r^{(2)}w_{0,0}\frac{\partial u_{0,3}}{\partial z} - r^{(2)}\frac{\partial u_{3,0}}{\partial t}\sin 4\theta$$
$$+2r^{(2)}\frac{\partial u_{3,0}}{\partial t}\sin 2\theta - r^{(2)}\frac{\partial u_{0,3}}{\partial t}\cos 4\theta + 4r^{(2)}\frac{\partial u_{0,3}}{\partial t}\cos 2\theta - 3r^{(2)}w_{2,0}\frac{\partial u_{0,1}}{\partial z} - 2u_{1,0}u_{0,1}\cos 4\theta$$
$$+8u_{1,0}u_{0,1}\cos 2\theta - w_{0,0}\frac{\partial u_{1,0}}{\partial z}\sin 4\theta + 2w_{0,0}\frac{\partial u_{1,0}}{\partial z}\sin 2\theta - w_{0,0}\frac{\partial u_{0,1}}{\partial z}\cos 4\theta + 4w_{0,0}\frac{\partial u_{0,1}}{\partial z}\cos 2\theta$$
$$-r^{(2)}w_{2,0}\frac{\partial u_{1,0}}{\partial z}\sin 4\theta + 2r^{(2)}w_{2,0}\frac{\partial u_{1,0}}{\partial z}\sin 2\theta - r^{(2)}w_{2,0}\frac{\partial u_{0,1}}{\partial z}\cos 4\theta + 4r^{(2)}w_{2,0}\frac{\partial u_{0,1}}{\partial z}\cos 2\theta$$
$$-4r^{(2)}u_{3,0}u_{0,3}\cos 4\theta + 16r^{(2)}u_{3,0}u_{0,3}\cos 2\theta - r^{(4)}w_{2,0}\frac{\partial u_{3,0}}{\partial z}\sin 4\theta + 2r^{(4)}w_{2,0}\frac{\partial u_{3,0}}{\partial z}\sin 2\theta$$
$$-r^{(4)}w_{2,0}\frac{\partial u_{0,3}}{\partial z}\cos 4\theta + 4r^{(4)}w_{2,0}\frac{\partial u_{0,3}}{\partial z}\cos\theta - 4r^{(2)}u_{1,0}u_{3,0}\sin 4\theta + 8r^{(2)}u_{1,0}u_{3,0}\sin 2\theta$$
$$-4r^{(2)}u_{1,0}u_{0,3}\cos 4\theta + 16r^{(2)}u_{1,0}u_{0,3}\cos 2\theta - 2r^{(2)}u_{0,1}u_{3,0}\cos 4\theta$$
$$+8r^{(2)}u_{0,1}u_{3,0}\cos 2\theta + 2r^{(2)}u_{0,1}u_{0,3}\sin 4\theta - 4r^{(2)}u_{0,1}u_{0,3}\sin 2\theta$$
$$-r^{(2)}w_{0,0}\frac{\partial u_{3,0}}{\sin}4\theta + 2r^{(2)}w_{0,0}\frac{\partial u_{3,0}}{\partial z}\sin 2\theta - r^{(2)}w_{0,0}\frac{\partial u_{0,3}}{\partial z}\cos 4\theta + 4r^{(2)}w_{0,0}\frac{\partial u_{0,3}}{\partial t}\cos 2\theta$$
$$-3\frac{\partial u_{0,1}}{\partial t} - u_{1,0}^{(2)}\sin 4\theta + 2u_{1,0}^{(2)}\sin 2\theta + u_{0,1}^{(2)}\sin 4\theta - 2u_{0,1}^{(2)}\sin 2\theta - 3r^{(2)}\frac{\partial u_{0,3}}{\partial t} - 6u_{1,0}u_{0,1} - 3w_{0,0}\frac{\partial u_{0,1}}{\partial z}$$
$$\left.-\frac{\partial u_{1,0}}{\partial t}\sin 4\theta + 2\frac{\partial u_{1,0}}{\partial t}\sin 2\theta - \frac{\partial u_{0,1}}{\partial t}\cos 4\theta + 4\frac{\partial u_{0,1}}{\partial t}\cos 2\theta\right] dr d\theta.$$

Dropping the terms containing $\cos n\theta$ or $\sin n\theta$, as they will cancel when integrated from 0 to 2π, gives

$$-\frac{\partial}{\partial z}\int_0^{2\pi}\int_0^{\phi}\frac{3\mu r^{(5)}}{8}\left[r^{(2)}\frac{\partial u_{0,3}}{\partial z} + \frac{\partial u_{0,1}}{\partial z}\right] dr d\theta - \int_0^{2\pi}\frac{3\phi^{(4)}\tau_2}{8} d\theta$$
$$= -\int_0^{2\pi}\int_0^{\phi}\frac{3\mu r^{(5)}u_{0,3}}{2} dr d\theta - \int_0^{2\pi}\int_0^{\phi}\frac{\rho r^{(5)}}{8}\left[12r^{(4)}u_{3,0}u_{0,3} + 3r^{(4)}w_{2,0}\frac{\partial u_{0,3}}{\partial z} + 12r^{(2)}u_{1,0}u_{0,3}\right.$$
$$\left.+6r^{(2)}u_{0,1}u_{3,0} + 3r^{(2)}w_{0,0}\frac{\partial u_{0,3}}{\partial z} + 3r^{(2)}w_{2,0}\frac{\partial u_{0,1}}{\partial z} + 3\frac{\partial u_{0,1}}{\partial t} + 3r^{(2)}\frac{\partial u_{0,3}}{\partial t} + 6u_{1,0}u_{0,1} + 3w_{0,0}\frac{\partial u_{0,1}}{\partial z}\right] dr d\theta.$$

Integrating with respect to r gives

$$
\begin{aligned}
&-\frac{\partial}{\partial z}\int_0^{2\pi}\frac{3\mu}{8}\left[\frac{\phi^{(8)}}{8}\frac{\partial u_{0,3}}{\partial z}+\frac{\phi^{(6)}}{6}\frac{\partial u_{0,1}}{\partial z}\right]d\theta-\int_0^{2\pi}\frac{3\phi^{(4)}\tau_2}{8}d\theta\\
&=-\int_0^{2\pi}\frac{\mu\phi^{(6)}u_{0,3}}{4}d\theta-\int_0^{2\pi}\frac{3\rho}{8}\left[\frac{2\phi^{(10)}u_{3,0}u_{0,3}}{5}+\frac{\phi^{(10)}w_{2,0}}{10}\frac{\partial u_{0,3}}{\partial z}+\frac{\phi^{(8)}u_{1,0}u_{0,3}}{2}\right.\\
&\left.+\frac{\phi^{(8)}u_{0,1}u_{3,0}}{4}+\frac{\phi^{(8)}w_{0,0}}{8}\frac{\partial u_{0,3}}{\partial z}+\frac{\phi^{(8)}w_{2,0}}{8}\frac{\partial u_{0,1}}{\partial z}+\frac{\phi^{(6)}}{6}\frac{\partial u_{0,1}}{\partial t}+\frac{\phi^{(8)}}{8}\frac{\partial u_{0,3}}{\partial t}+\frac{u_{1,0}u_{0,1}\phi^{(6)}}{3}+\frac{w_{0,0}\phi^{(6)}}{6}\frac{\partial u_{0,1}}{\partial z}\right]d\theta.
\end{aligned}
$$

Integrating with respect to θ gives

$$
\begin{aligned}
&-\frac{\partial}{\partial z}\left[\frac{3\pi\mu\phi^{(6)}}{8}\left(\frac{\phi^{(2)}}{4}\frac{\partial u_{0,3}}{\partial z}+\frac{1}{3}\frac{\partial u_{0,1}}{\partial z}\right)\right]-\frac{3\pi\phi^{(4)}\tau_2}{4}\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\\
&=\frac{\pi\mu\phi^{(6)}u_{0,3}}{2}-\frac{3\pi\rho\phi^{(6)}}{4}\left[\frac{2\phi^{(4)}u_{3,0}u_{0,3}}{5}+\frac{\phi^{(4)}w_{2,0}}{10}\frac{\partial u_{0,3}}{\partial z}+\frac{\phi^{(2)}u_{1,0}u_{0,3}}{2}\right.\\
&\left.\frac{\phi^{(2)}u_{0,1}u_{3,0}}{4}+\frac{\phi^{(2)}w_{0,0}}{8}\frac{\partial u_{0,3}}{\partial z}+\frac{\phi^{(2)}w_{2,0}}{8}\frac{\partial u_{0,1}}{\partial z}+\frac{1}{6}\frac{\partial u_{0,1}}{\partial t}+\phi^{(2)}8\frac{\partial u_{0,3}}{\partial t}+\frac{u_{1,0}u_{0,1}}{3}+\frac{w_{0,0}}{6}\frac{\partial u_{0,1}}{\partial z}\right].
\end{aligned}
$$

After some rearranging and dividing through by 3 this becomes

$$
\begin{aligned}
&\frac{\mu}{24}\frac{\partial}{\partial z}\left[\phi^{(6)}\left(\frac{\partial u_{0,1}}{\partial z}+\frac{3\phi^{(2)}}{4}\frac{\partial u_{0,3}}{\partial z}\right)\right]-\frac{\mu\phi^{(6)}u_{0,3}}{6}+\frac{\phi^{(4)}\tau_2}{4}\left(1+\left(\frac{\partial\phi}{\partial z}\right)^{(2)}\right)^{(1/2)}\\
&=\frac{\rho\phi^{(6)}}{24}\left[\frac{\partial u_{0,1}}{\partial t}+\frac{\partial u_{0,1}}{\partial z}\left(w_{0,0}+\frac{3\phi^{(2)}w_{2,0}}{4}\right)+2u_{0,1}\left(u_{1,0}+\frac{3\phi^{(2)}u_{3,0}}{4}\right)\right.\\
&\left.+\frac{3\phi^{(2)}}{4}\left(\frac{\partial u_{0,3}}{\partial t}+\frac{\partial u_{0,3}}{\partial z}\left(w_{0,0}+\frac{4\phi^{(2)}w_{2,0}}{5}\right)+4u_{0,3}\left(u_{1,0}+\frac{4\phi^{(2)}u_{3,0}}{5}\right)\right)\right].
\end{aligned}
\tag{A.8}
$$

This is Eq. (3.41) in the main text.

APPENDIX B

APPENDIX B

B.1 Nomenclature

- (x_1, x_2, x_3) or (x^2, x^2, x^3) denote the Cartesian coordinates
- $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ denote the Cartesian base vectors
- $\mathbf{x}$ or $\mathbf{r}$ denotes a position vector
- $(\theta^1, \theta^2, \theta^3)$ denote convective coordinates
- t denotes time
- $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$ denote covariant base vectors
- $(\mathbf{g}^1, \mathbf{g}^2, \mathbf{g}^3)$ denote contravariant base vectors
- $(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$ denote the unit base vectors corresponding to $(\mathbf{g}_1, \mathbf{g}_2, \mathbf{g}_3)$
- δ^i_j denotes the Knronecker delta
- $\dot{\eta}$ denotes a time derivative (Chapter 2)
- H denotes the lateral surface
- λ_N denotes the weighting function of the Nth director
- $\mathbf{d}_N$ denotes the Nth director

- $\mathbf{w}_N$ denotes the Nth director velocity
- K denotes the order of expansion (number of director)
- $\mathbf{v}$ denotes fluid velocity
- $e\bar{t}a$ denotes that entity corresponds to the fixed curvilinear coordinate system as opposed to the convective Langrangian coordinate system (Chapter 2)
- $(\zeta^1, \zeta^2, \zeta^3)$ denote curvilinear coordinates
- F denotes an unspecified function associated with the body of fluid (Chapter 2)
- f_N denotes the Nth function in the expansion for F (Chapter 2)
- ρ denotes the mass density of the fluid
- i, j Latin indices take the values $1,2,3$
- α, β Greek indices take the values $1,2$
- p denotes the pressure of the fluid
- $\mathbf{T}_i$ denotes the stress tensor (Chapter 3)
- $\mathbf{t}$ denotes the stress vector on the lateral surface (Chapter 3)
- $\boldsymbol{\nu}$ denotes the unit outward normal on the lateral surface (Chapter 3)
- σ_{ij} denotes the deviatoric stress response
- ϕ denotes the radius of the pipe
- $\mathbf{w}_{j,N-j}$ denotes the director velocity with weighting function $x_1^{(j)} x_2^{N-j}$ (Chapter 3)
- (r, θ, z) denote the cylindrical polar coordinates
- $\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}$ denote the cylindrical polar coordinate base vectors
- μ denotes the dynamic viscosity of the fluid
- Re denotes the Reynold's number of the fluid
- Re_0 denotes the Reynold's number of a corresponding dlow in a straight pipe subject to the same coaxial pressure gradient (Chapters 4, 5)

- R denotes the radius of curvature (Chapters 4, 5)
- L denotes the number of cross-sectional director velocities (Chap 4)
- δ denotes the ratio of curvature (Chapters 4, 5)
- (X, Y, Z) denote dimensionless coordinates (Chapters 4, 5)
- $\mathbf{V}$ denotes dimensionless velocity (Chapters 4, 5)
- P denotes dimensionless pressure (Chapters 4, 5)
- T denotes dimensionless time (Chapter 5)

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