Author:
Bell, Nick S

Title:
Arcs on hyperbolic surfaces

a view towards counting

Arcs on hyperbolic surfaces: a view towards counting

Nick Bell

School of Mathematics
University of Bristol

A dissertation submitted to the University of Bristol in accordance with the requirements of the degree of Doctor of Philosophy in the Faculty of Science.

April 11th 2022

Word count: 24812 words

Abstract

We give the asymptotic growth of the number of arcs of bounded length between boundary components on hyperbolic surfaces with boundary, analogous to a result of Mirzakhani for curves [25,26]. Specifically, if S has genus g, n boundary components and p punctures, then the number of orthogeodesic arcs in each pure mapping class group orbit of length at most L is asymptotic to $L^{6g-6+2(n+p)}$ times a constant. We prove an analogous result for arcs between cusps, where we define the length of such an arc to be the length of the sub-arc obtained by removing certain cuspidal regions from the surface. In demonstrating these results, we develop tools to examine a well-known association between arcs and curves. We demonstrate that this association is not injective but is uniformly k-to-1 across arcs of the same type, and on the pair of pants P it maps at most two two-ended arcs to the same curve. We also derive arbitrarily large families of arcs whose lengths are equal under any hyperbolic metric.

Disclaimer: Much of Chapter 2 as well as large parts of the introduction and preliminaries in this thesis appear in the publication [5] by the author.

Acknowledgements

I am deeply grateful to my supervisor Viveka Erlandsson for her constant and generous support; her advice and guidance have been truly invaluable. I feel very lucky to have been her student.

My thanks to EPSRC for funding my position, and to the University of Bristol and the staff in the School of Mathematics for their support.

My time in Bristol was defined by the people who made it more than just a time of study. Ayesha, Harry, Jordan, Peter, everyone both in and above the Pit to list a few; I'm glad to have shared this with you.

I would also like to thank Juan Souto, Hugo Parlier and John Mackay for their helpful comments, and Lars Louder who provided a counterexample which sparked much of the work in the latter half of this thesis.

My deepest thanks go to my family and friends who have supported me through everything. My wife Emma, who deserves credit for her help with string and salt and pepper shakers. Mum, Dad and Fynn, thank you for always being there.

Author's declaration

I declare that the work in this dissertation was carried out in accordance with the requirements of the University's Regulations and Code of Practice for Research Degree Programmes and that it has not been submitted for any other academic award. Except where indicated by specific reference in the text, the work is the candidate's own work. Work done in collaboration with, or with the assistance of, others, is indicated as such. Any views expressed in the dissertation are those of the author.

SIGNED: ...Nicholas Bell... DATE: ......11/04/2022......

Contents

Introduction **1**

1 Preliminaries **7**

1.1 Hyperbolic surfaces, curves and arcs 7

1.2 Cuspidal regions 10

1.3 Counting curves 13

1.4 Infinite arcs and their lengths 15

1.5 Words associated to curves and arcs 16

2 Counting arcs **19**

2.1 Associating arcs and curves 19

2.2 Counting arcs 24

2.3 Counting infinite arcs 32

2.4 Simple two-ended arcs have no associates 38

3 Pairs of associate arcs **45**

3.1 Characterising arcs in P using $\pi_1(P)$ 45

3.2 Signatures and Frames 50

3.3 Relating frames and pairs of associate arcs 65

3.4 Implications of the relation 77

4 Length-equivalent arcs **81**

4.1 An example of a family of length-equivalent arcs 82

4.2 Trace-equivalent words 83

4.3 Constructing new families of length-equivalent arcs 87

A Computation of the error function limit **95**

List of Figures

2.1 Examples of simple arcs and the curves associated to them 20

2.2 Examples of associate arcs of the first kind 22

2.3 An example of associate arcs of the second kind 23

2.4 The curve associated to the arcs from Figure 2.3 24

2.5 An infinite arc on a generalised pair of pants 33

2.6 Quadrilaterals made by dividing P along geodesics 34

3.1 Examples of words written in cyclical notation 54

3.2 Labelling positions in cyclical notation 55

3.3 Decomposing a wheel 56

3.4 Constructing a wheel from a signature 57

3.5 Collapsing a wheel constructed by an unfocused signature 62

3.6 Proving Lemma 3.3.2 66

3.7 Proving Proposition 3.3.3 69

3.8 The arcs from Figure 2.3 75

4.1 The "figure-eight" curve from Buser's example of length equivalent curves 83

Notation

Symbol	Meaning
S	A surface of negative Euler characteristic, typically with boundary
X	A hyperbolic metric on S
∂S	The collection of boundary components of S
S^o	The interior of a surface S
$\mathfrak{C}$	The set of punctures of S
γ	A homotopy class of a curve, or its geodesic representative
$\mathscr{C}(S)$	The set of homotopy classes of curves on S
α	A homotopy class of an (infinite) arc, or its (ortho)geodesic rep.
$\mathcal{A}(S)$	The set of homotopy classes of arcs on S
$\mathcal{A}_{i,j}(S)$	The set of homotopy classes of arcs on S between δ_i and δ_j
γ_α, $I(\alpha)$	The curve associated to an arc α
PMod(S), Mod(S)	The (pure) mapping class group of the interior of S
φ	A mapping class, or a representative of a mapping class
$\ell_X(\cdot)$	The length of an object under a hyperbolic metric X
$\ell_X^t(\alpha)$	The t-length of an infinite arc α
$k(\alpha)$	The number of arcs of type α associated to the same curve
H_t^p	The cuspidal region at the puncture p of volume t
$\mathscr{H}_t$	The union of all cuspidal regions of volume t
A_c^δ	The annular region at the boundary component δ of width c
$\mathscr{A}_c$	The union of all annular regions of width c
$\iota(\cdot,\cdot)$	The intersection number between two objects
$\mathcal{ML}(S)$	The space of measured laminations on S
$\pi_1(S)$	The fundamental group of S
F_n	The free group on n generators
D_{2n}	The dihedral group of order $2n$
P	A pair of pants, or three-holed sphere
α_P	A seam on P
δ_0^P, δ_1^P	The initial and terminal boundary components of α_P
δ_0^α, δ_1^α	The initial and terminal boundary components of an (oriented) arc α
ι_α	The immersion of P determined by α
ϕ_α	The homomorphism of $\pi_1(P)$ determined by α
w_α	The conjugator corresponding to α
$\mathfrak{W}$	The set of conjugators
(m,r)	A signature, where m and r are even and $r \leq m$
(m,r,c)	A frame, where (m,r) is a signature and c is a conjugator

W	A wheel; a word written in cyclical notation
Θ_m	A set of $2m+2$ positions uniformly distributed around a circle
L^q	A line of symmetry in Θ_m
ρ_q	The reflection in the line L^q
$x_W(p)$	The syllable of W in position p
ψ^*	The self-homomorphism of $\pi_1(S)$ induced by $\psi\colon S \to S$
$tr_\mu(\cdot)$	The trace of a group element under the representation μ
$\lvert w \rvert$	The syllable length of a word w

Introduction

Given a hyperbolic metric X on a surface S, it is a much studied problem to "count" curves (also known as closed geodesics) of bounded length; that is, find the growth of the number of curves of length at most L as L grows. Work by Huber [18], later generalised by Margulis [21], demonstrates that the growth is asymptotially exponential:

$$\lim_{L\to\infty} \frac{|\{\gamma \text{ unoriented closed geodesic on } S \mid \ell_X(\gamma) \leq L\}|}{\frac{1}{2L}e^L} = 1.$$

However, if we only consider *simple* curves - those which do not self-intersect - or more generally, curves which share a mapping class group orbit, the story is very different. It was first observed by Rees [33] and Birman-Series [6] that the number of simple curves grows polynomially in the length. That is, there exist constants $C_1, C_2 > 0$ such that

$$C_1 L^{6g-6+2r} \leq |\{\gamma \text{ simple curve on } S \mid \ell_X(\gamma) \leq L\}| \leq C_2 L^{6g-6+2r},$$

where g is the genus and r is the number of ends.

The first asymptotic result in this direction is due to McShane-Rivin [24], who gave the asymptotic growth of the number of simple curves on the punctured torus T as

$$\lim_{L\to\infty} \frac{|\{\gamma \text{ simple curve on } T \mid \ell_X(\gamma) \leq L\}|}{L^2} = C$$

for some $C > 0$.

The punctured torus is a very special case of a hyperbolic surface, and as such the methods used to prove this result do not generalise to other surfaces. However, celebrated results by Mirzakhani [25, 26] show that the same asymptotic result does in fact hold in general, even when restricting to a mapping class group orbit. Before giving the precise statement, we will introduce some notation.

Throughout the following, let S be an orientable surface of negative Euler characteristic of genus g with n boundary components and p punctures, where we assume $(g, n+p) \neq (0, 3)$. Let $\mathrm{Mod}(S)$ be the mapping class group and let $\mathrm{PMod}(S)$ be the pure mapping class group: the finite-index subgroup of $\mathrm{Mod}(S)$ consisting of exactly those elements which fix each boundary component and each puncture of S. See [12] for a thorough treatment of mapping class groups. Here we will say that two multicurves, by which we mean formal sums of finitely many weighted curves, are *of the same type* if they share a $\mathrm{PMod}(S)$-orbit. With this notation, Mirzakhani's result is as follows.

Mirzakhani's Theorem ([25, 26])**.** *Let Y be a complete hyperbolic metric on the interior S^o and γ_0 be a multi-curve on S. Then we have that*

$$\lim_{L\to\infty} \frac{|\{\gamma \text{ of type } \gamma_0 \mid \ell_Y(\gamma) \leq L\}|}{L^{6g-6+2(n+p)}} = \mathfrak{c}(\gamma_0)\mathfrak{m}(Y),$$

where $\mathfrak{c}(\gamma_0)$ is a constant depending on the type γ_0 and $\mathfrak{m}(Y)$ is a constant depending on Y.

We refer the reader to [11], [25] and [26] for details of the constants. Here, $\ell_Y(\gamma)$ denotes the Y-length of the geodesic representative of γ. Mirzakhani first proved the above result for simple multicurves in [25] and then for general multicurves in [26]; see [10] and [11] for an alternative proof of both cases of this theorem. In fact, Mirzakhani's Theorem holds if we redefine the type of a multicurve to correspond to the orbit of *any* finite-index subgroup of $\mathrm{Mod}(S)$.

In this thesis, we show that Mirzakhani's Theorem holds when we replace curves with arcs, also known as orthogeodesics (see Definition 1.1.2). The question of adapting Mirzakhani's original proof for simple curves to arcs was first raised in a talk by Wolpert, in the case of so-called *lariats* (simple arcs from a cusp to itself). Here, we take a different approach, which also allows us to consider both simple arcs and general arcs. We first prove the following.

Theorem 1. *Let X be a complete, finite-area, hyperbolic metric on S with non-empty geodesic boundary. Let α_0 be an arc on S. Then there exist positive constants $\mathfrak{c}(\alpha_0)$ and $\mathfrak{m}(X)$ such that*

$$\lim_{L\to\infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0)\mathfrak{c}(\alpha_0)\mathfrak{m}(X),$$

where $k(\alpha_0)$ is some positive integer.

Remark: We differentiate between the constants $k(\alpha_0)$ and $\mathfrak{c}(\alpha_0)$ as the latter comes from Mirzakhani's Theorem, whereas the former is the result of work in this thesis. See Proposition 2.2.4 for the definition of the constant $k(\alpha_0)$, and (1) for the definition of $\mathfrak{c}(\alpha_0)$.

Remark: Theorem 1 also holds for multi-arcs, but the majority of the work goes into demonstrating it for single arcs. We remark throughout Section 2.2 on how to prove the analogues of the various results needed to do this, and give the statement in Theorem 2.2.6.

We also consider *infinite arcs*, that is, arcs whose endpoints are at punctures of S. As implied by the name, infinite arcs have infinite length as they descend infinitely far down the cusps. Hence we must define a suitable notion of the length of infinite arcs to allow us to derive an analogue of Theorem 1. A natural way to do this is to cut off the cusps (of volume t) and consider the length $\ell^t(\alpha)$ of the segment of the arc which remains (we refer to Section 1.4 for the precise definition). There are other natural choices of length to assign to infinite arcs, such as the truncated length (see [28]). As we will explain in Section 2.3, Theorem 2 also holds for the truncated length and a length closely related to the λ-length (see [29,30]). We prove the following result.

Theorem 2. *Let X be a complete, finite-area, hyperbolic metric on S with (possibly empty) geodesic boundary. Let α_0 be an infinite arc on S. Then for any positive $t \leq 1$, we have*

$$\lim_{L\to\infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell^t_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0)\mathfrak{c}(\alpha_0)\mathfrak{m}(X)$$

where $k(\alpha_0)$, $\mathfrak{c}(\alpha_0)$ and $\mathfrak{m}(X)$ are as in Theorem 1. In particular, the limit does not depend on t.

Remark: Our arguments can be easily modified to apply to rays, by which we mean arcs between a boundary component and a puncture. We could equally adapt this argument for any collection of arcs, infinite arcs, and rays.

The study of orthogeodesics on hyperbolic surfaces has a rich history. For example, if one counts *all* orthogeodesics of length at most L, Basmajian's Identity

[2] gives an upper bound exponential in L for this number. The actual asymptotic growth was shown to be exponential by Parkonnen and Paulin in [27] (see [15] for a generalisation).

These results can be viewed as analogues to Huber's [18] and Margulis' [21] Prime Geodesic theorem mentioned above. Our results are instead analogues of Mirzakhani's Theorem, counting arcs in each (pure) mapping class group orbit and giving polynomial asymptotic growth.

As mentioned above, instead of modifying Mirzakhani's original proof we take a different, much simpler approach. The main idea is to associate a curve γ_α to each arc α in a way which respects length, up to a well-behaved error, and then use Mirzakhani's Theorem to deduce Theorems 1 and 2. In fact, $\mathfrak{c}(\alpha_0)$ will be shown in each case to be closely related to $\mathfrak{c}(\gamma_{\alpha_0})$; we will get that

$$\mathfrak{c}(\alpha_0) = 2^{6g-6+2(n+p)}\mathfrak{c}(\gamma_{\alpha_0}) \tag{1}$$

where $\mathfrak{c}(\gamma_{\alpha_0})$ is as in Mirzakhani's Theorem and γ_{α_0} is the curve associated to α_0 as defined in Sections 2.1 and 2.3.

Given an arc α between two boundary components, there is a natural way to associate an immersed pair of pants P in S to α such that α is a seam of P. See Section 2.1 for a detailed discussion of this. The cuff of P which is not a boundary component of S is what we refer to as the curve associated to α, denoted γ_α. In this way, we get a well-defined map

$$I\colon \{\text{arcs on } S\} \to \{\text{curves on } S\}.$$

The precise definition is given in Section 2.1. We show that this map is Mod(S)-equivariant and preserves the length of an arc up to a well-behaved error.

However, somewhat surprisingly this process is not reversible; there exist immersed pairs of pants in S which share all three boundary components yet are not homotopic. See Example 3.1.3 for an example of this. In other words, the map I is not one-to-one. In Chapter 3, we develop techniques to study how many arcs can be associated to the same curve, and prove the following theorem.

Theorem 3. *Let P be a pair of pants, and let α be a two-ended arc on P. Let $\gamma_\alpha = I(\alpha)$. Then $|I^{-1}(\gamma_\alpha)| \leq 2$.*

As mentioned above, there are cases when two arcs are associated to the same curve, an explicit example of which is given in Example 3.1.3. By construction, the length of a curve γ_α associated to an arc α is given by the length of α; this is demonstrated in Section 2.2. We therefore get that if two arcs are associated to the same curve, they must have the same length under any hyperbolic metric we put on S (excluding arcs of the first kind; see Section 2.1). That is, they are *length-equivalent*. As an application of our study of arcs associated to the same curve, we are able to create arbitrarily large families of length-equivalent arcs. The existence of such families is already known due to the work of Buser [8], using a different method.

This thesis is organised as follows. In Chapter 1, we will discuss some necessary background and introduce various terms. In Chapter 2 we demonstrate various qualities of the association between arcs and curves, before proving Theorem 1 and Theorem 2. Chapter 3 looks at the possible values for $k(\alpha_0)$ in these theorems, ultimately proving Theorem 3. We conclude by discussing length-equivalent arcs in Chapter 4.

Chapter 1

Preliminaries

1.1 Hyperbolic surfaces, curves and arcs

Let S be a connected orientable surface of negative Euler characteristic of genus g with n boundary components and p punctures, with $(g, n+p) \neq (0,3)$. By ∂S we shall mean the boundary of S, consisting of the n boundary components. The p punctures correspond to ends of S, and we denote the collection of punctures as $\mathfrak{C}$. When convenient, we may consider the punctures as marked points on (the closure of) S. Let X be a complete, finite-area, hyperbolic metric on S such that ∂S is geodesic. We will consider S to be endowed with such a metric throughout the following. We define the mapping class group of S to be the group of orientation-preserving homeomorphisms of the interior of S up to homotopy; that is,

$$\mathrm{Mod}(S) \coloneqq \mathrm{Mod}(S^o) = {}^{\mathrm{Homeo}^+(S^o)}\!\big/\!{}_{\mathrm{Homeo}_0^+(S^o)},$$

where $S^o = S \setminus \partial S$ is the interior of S, $\mathrm{Homeo}^+(S^o)$ is the space of orientation-preserving homeomorphisms of S^o and $\mathrm{Homeo}_0^+(S^o)$ is the subgroup of homeomorphisms properly homotopic to the identity. We refer the reader to [12] for more background on mapping class groups.

Definition 1.1.1: We define a *curve* to be (the image of) a continuous map $\gamma \colon \mathbb{S}^1 \to S$. We consider curves up to free homotopy. We assume curves to be essential, meaning not homotopic to a point or a puncture, and non-peripheral, meaning not homotopic to a boundary component.

We identify curves which differ by an orientation. Recall that each homotopy class of curves has a unique geodesic representative. By abuse of notation, we will use γ to refer to a curve, its homotopy class and its geodesic representative under X. If a curve can be realised by an embedding, we call it simple.

We define the *geometric intersection number* between curves γ_1 and γ_2 to be

$$\iota(\gamma_1, \gamma_2) = \min\{|\gamma_1' \cap \gamma_2'| \mid \gamma_i' \text{ homotopic to } \gamma_i,\ \gamma_1' \text{ and } \gamma_2' \text{ intersect transversely.}\}.$$

Note that the intersection number between two distinct curves is realised by their geodesic representatives, and a curve γ is simple exactly when $\iota(\gamma, \gamma) = 0$.

Definition 1.1.2: An *arc* is (the image of) a continuous map $\alpha\colon [0,1] \to S$ such that $\alpha(0), \alpha(1) \in \partial S$ and $\alpha\big((0,1)\big) \subset S^o$. We consider arcs up to homotopy relative to ∂S, where we allow the endpoints to move along ∂S, and we assume that they are not homotopic into the boundary.

Each homotopy class of arcs has a unique geodesic representative which meets the boundary orthogonally, which we refer to as an *orthogeodesic*.

Definition 1.1.3: We define an *infinite arc* to be (the image of) a continuous map $\alpha\colon (0,1) \to S$ such that $\lim_{t\to 0} \alpha(t) \in \mathfrak{C}$ and $\lim_{t\to 1} \alpha(t) \in \mathfrak{C}$. We consider infinite arcs up to homotopy relative to $\mathfrak{C}$, and we assume that they are not homotopic into $\mathfrak{C}$.

We identify (infinite) arcs which differ by an orientation, and again by abuse of notation, we refer to both an (infinite) arc and its homotopy class by α. If an (infinite) arc can be realised as an embedding, then we call it simple. We stress that throughout, we allow (infinite) arcs to have self-intersections; we do not only consider simple arcs. Note that the geometric intersection number between (infinite) arcs is defined analogously to curves, and the intersection number between two (infinite) arcs is realised by their (ortho)geodesic representatives.

We define the length of (a homotopy class of) a curve or arc to be the length of its geodesic or orthogeodesic representative, which we denote by $\ell_X(\cdot)$. We will discuss how to assign appropriate finite lengths to infinite arcs in Section 1.4.

Definition 1.1.4: We call an arc *two-ended* if $\alpha(0)$ and $\alpha(1)$ are on distinct boundary components, and *one-ended* otherwise. This is defined for infinite arcs analogously.

A *multicurve* or a *multi-arc* is a finite formal sum of weighted curves or arcs respectively. Explicitly, if ω is a multicurve (resp. multi-arc), then

$$\omega = \sum_{i=1}^{m} a^i \omega^i$$

for some $a^i \in \mathbb{R}_+$ and $m \in \mathbb{Z}_+$, where each ω^i is a curve (resp. arc) and for all $i \neq j$, ω^i and ω^j are not homotopic. We will refer to the ω^i as the components of ω. As before, the geometric intersection number between two multicurves or multi-arcs is defined to be the minimum size of their intersection across all of their homotopic representatives. The length of a multicurve or multi-arc is defined to be the weighted sum of the lengths of its components: for $\omega = \sum_{i=1}^{m} a^i \omega^i$, we have $\ell_X(\omega) = \sum_{i=1}^{m} a^i \ell_X(\omega^i)$.

The pure mapping class group $\mathrm{PMod}(S)$ acts naturally on curves and arcs in S. If φ is a mapping class and ω is either a geodesic curve, an orthogeodesic arc or a geodesic infinite arc, then we define $\varphi \cdot \omega$ to be the (ortho-)geodesic representative of $f(\omega)$, where f is any representative of φ.

Let ω_0 be a curve or arc. For any curve or arc ω, we say that ω is *of type* ω_0 if they share an orbit in the pure mapping class group; that is, there exists some $\varphi \in \mathrm{PMod}(S)$ such that $\varphi \cdot \omega_0 = \omega$. We note that the action of $\mathrm{PMod}(S)$ preserves the self-intersection number of a curve or arc; thus if two curves or arcs are of the same type, they have the same self-intersection number. Curves or arcs with the same self-intersection number need not be of the same type, but there are only finitely many types of curve or arc with a given self-intersection number. See for example [11] for a demonstration of this.

The action of $\mathrm{PMod}(S)$ on multicurves and multi-arcs is defined analogously to the above: if $\omega = \sum_{i=1}^{m} a^i \omega^i$ is a multicurve or multi-arc, then

$$\varphi \cdot \omega = \sum_{i=1}^{m} a^i (\varphi \cdot \omega^i).$$

We say that a multicurve or multi-arc ω is of type ω_0 if ω and ω_0 share a $\mathrm{PMod}(S)$-orbit. As a result, we have that if $\omega = \sum_{i=1}^{m} a^i \omega^i$ and $\omega_0 = \sum_{j=1}^{n} a_0^j \omega_0^j$ are of the same type then $m = n$ and, up to relabelling, for all $i \in \{1, \ldots, m\}$ $a^i = a_0^i$ and ω^i is of type ω_0^i.

1.2 Cuspidal regions

In this section we discuss the behaviour of curves and arcs in regions called *cusps*. We refer the reader to [8] for a general background on hyperbolic geometry.

Let X be a complete, finite-area hyperbolic metric on S. Then each puncture $p \in \mathfrak{C}$ corresponds to a cusp. Recall that a cusp is an end which has a neighbourhood H_t isometric to

$$\left\{ z \in \mathbb{H}^2 \middle| \operatorname{Im}(z) > \frac{1}{t} \right\} \Big/ \langle z \mapsto z+1 \rangle$$

for some $t > 0$, where we have identified the hyperbolic plane $\mathbb{H}^2$ with the Poincaré upper half-plane. Such a region has volume t, and we refer to H_t as a *cuspidal region (of volume t)*. The ends of any infinite arc escape down cusps, and the unique geodesic representative of its homotopy class eventually intersects the horocyclic foliation of the corresponding cusps orthogonally.

Definition 1.2.1: For $t > 0$ and $p \in \mathfrak{C}$, let H_t^p denote the cuspidal region at p of volume t. Denote the union of these regions over all p by $\mathscr{H}_t = \cup_{p \in \mathfrak{C}} H_t^p$.

The following statement is well-known to experts, and can be seen as a result of the Collar Lemma (see for example Theorem 4.4.6 of [8], and [19]).

Lemma 1.2.2. *For any $t < 2$, the cuspidal regions H_t^p are embedded and pairwise disjoint.*

For each boundary curve δ in ∂S and for any $c > 0$, define the annulus A_c^δ to be the set of points at a distance less than c from δ. That is,

$$A_c^\delta = \{x \in S \mid d_X(x, \delta) < c\}.$$

Denote by $\mathscr{A}_c = \cup_\delta A_c^\delta$ the union of these annuli over all δ in ∂S. It again follows from the Collar Lemma, applied to the boundary curves, that there exists $c' > 0$ depending on X such that the annuli $A_{c'}^\delta$ are embedded and pairwise disjoint. Thus we have the following, and we refer the reader to [4] for more details.

Lemma 1.2.3. *There exists $c' > 0$ such that for all $t < 2$, $\mathscr{H}_t \cap \mathscr{A}_{c'} = \emptyset$, and in particular, $S \setminus \overline{(\mathscr{H}_t \cup \mathscr{A}_{c'})}$ is homeomorphic to S^o.*

For any p and $t < 2$, and for any complete geodesic γ intersecting ∂H_t^p transversely, $\gamma \cap H_t^p$ takes one of two forms. Either it never leaves the cuspidal region and so intersects every horocycle in H_t^p orthogonally, or it winds around the cusp before leaving the region, and hence, when long enough, creates self-intersections. In the latter case, we call the segment *returning*. In fact, the deeper into H_t^p a returning segment goes the more times it must self-intersect, and there is a direct relationship between the length of a returning segment and its self-intersection number which we record below for future reference. Recalling that $\iota(\cdot,\cdot)$ denotes the (geometric) intersection number, we have the following.

Lemma 1.2.4. *Let p be a puncture on S equipped with a hyperbolic metric X, and let $d > 0$. Suppose β is a geodesic segment in H_1^p with both endpoints on ∂H_1^p such that $\iota(\beta, \beta) \leq d$. Then there exists some positive $B = B(d)$ such that*

$$\ell_X(\beta) \leq B.$$

In particular, any geodesic curve γ with at most d self-intersections never enters $\mathscr{H}_{e^{-B(d)}}$.

We refer to [3] and [4] for more details about the behaviour of returning segments and for the proof of Lemma 1.2.4. In particular, Proposition 3.4 of [4] gives a much more precise description of the relationship between how far an arc goes into a cusp and its self-intersection number.

We can make a similar observation regarding boundary components on S. Whenever a complete geodesic enters a small annulus around a boundary curve δ, it spirals towards δ, and unless it is asymptotic to δ it eventually leaves the annulus, creating self-intersections if long enough. It follows that if γ is a geodesic curve with at most d self-intersections, there exists some $c < c'$ depending on d (and X) such that γ never enters $\mathscr{A}_c$. Putting this together with the above gives us that γ is contained in the compact subsurface $S \setminus (\mathscr{H}_{e^{-B(d)}} \cup \mathscr{A}_c) \subset S^o$. Note that as before, $S \setminus \overline{(\mathscr{H}_{e^{-B(d)}} \cup \mathscr{A}_c)}$ is homeomorphic to S^o. Furthermore, since $\mathrm{Mod}(S)$ preserves the self-intersection number of curves and arcs, the above is true for any curve of type γ. We summarise this fact below for reference; for a more precise description see Proposition 3.4 of [4].

Lemma 1.2.5. *Let γ_0 be a curve. Then there exists a compact subsurface $K \subset S^o$ with K^o homeomorphic to S^o such that for any γ of type γ_0, the geodesic representative of γ is contained in K.*

Since multicurves have finitely many components, this lemma holds for multicurves by taking the union of the compact subsurfaces given for (the support of) each component.

Let d be some non-negative integer, and let α be an infinite arc such that $\iota(\alpha, \alpha) = d$. Then similarly to the above, $\alpha \cap \mathscr{H}_{e^{-B(d)}}$ consists of exactly 2 components, which are simple geodesic rays. Equivalently, $\alpha \cap (S \setminus \mathscr{H}_{e^{-B(d)}})$ has exactly one component. We state this here for reference.

Lemma 1.2.6. *Let α be an infinite arc. Then there exists some positive $t_\alpha < 1$, depending only on $\iota(\alpha, \alpha)$, such that $\alpha \cap (S \setminus \mathscr{H}_{t_\alpha})$ has exactly one component.*

We also need the fact that if a geodesic goes far enough into a cusp then it must intersect itself inside $\mathscr{H}_2$. To see this, suppose β is a returning geodesic segment in $\mathscr{H}_2$ that enters H_t^p for some $t \leq 1$ and some $p \in \mathfrak{C}$. Consider the cuspidal region H_2^p and identify it with

$$\left\{ z \in \mathbb{H}^2 \middle| \operatorname{Im}(z) > \frac{1}{2} \right\} \Big/ \langle z \mapsto z+1 \rangle .$$

A fundamental domain for the action of $z \mapsto z+1$ is the region in $\mathbb{H}^2$ bounded by $x = 0$ and $x = 1$. Note that any geodesic in $\mathbb{H}^2$ neither of whose endpoints are at ∞ which intersects the line $y = \frac{1}{t}$ also intersects its translate under the map $z \mapsto z+1$, and this intersection occurs above the line $y = \frac{1}{2}$. Hence β intersects itself inside the embedded cuspidal region H_2^p. Moreover, any segment entering a cusp in $\mathscr{H}_1$ must intersect itself in a slightly larger cusp; for example, a cusp in $\mathscr{H}_2$. We record this here for reference, and refer to [23] and Proposition 3.2 of [4] for more details.

Lemma 1.2.7. *Let $0 < t \leq 1$. If β is a geodesic segment in $\mathscr{H}_2$ with both endpoints on $\partial \mathscr{H}_2$, and $\beta \cap \mathscr{H}_t \neq \emptyset$, then $\iota(\beta, \beta) \geq 1$.*

In particular, any simple geodesic not asymptotic to a puncture cannot enter the region $\mathscr{H}_1$.

1.3 Counting curves

We now comment very briefly on measured laminations; briefly, because although central to Mirzakhani's work, they somewhat surprisingly play no role here except in order to state a constant below. A measured lamination is a closed subset λ of S foliated by complete simple geodesics together with a transverse measure, which is a measure on arcs transverse to λ invariant under homotopy transverse to λ and under concatenation. We denote the space of compactly supported measured laminations on S^o as $\mathcal{ML}(S)$: for background we refer the reader to [14, 31, 34]. To give some intuitive perspective, we note that $\mathcal{ML}(S)$ is homeomorphic to $\mathbb{R}^{6g-6+2(n+p)}$ and that any measured lamination can be obtained as a limit of weighted simple curves. As the support of any measured lamination is a disjoint union of simple geodesics, we have the following as a consequence of Lemma 1.2.7.

Lemma 1.3.1. *For any $\lambda \in \mathcal{ML}(S)$, the support of λ is contained in $S^o \setminus \mathscr{H}_1$. In fact, there exists a compact subsurface $K \subset S^o \setminus \mathscr{H}_1$ which contains the supports of all elements of $\mathcal{ML}(S)$.*

In the proof of Theorem 1, we will need to use Mirzakhani's Theorem. However, this theorem is stated for complete finite-area hyperbolic metrics on the interior S^o, and we will need to use the result for our metric X on S which has geodesic boundary. This issue is resolved by instead using a generalisation of Mirzakhani's Theorem to complete Riemannian metrics. We state this in full generality below, but note that metrics with variable negative curvature are sufficient for our purposes.

Theorem 1.3.2 ([9], Corollary 1.3)**.** *Let Y be a complete Riemannian metric on $S^o = S \setminus \partial S$. Then for any multicurve γ_0,*

$$\lim_{L\to\infty} \frac{|\{\gamma \text{ of type } \gamma_0 \mid \ell_Y(\gamma) \leq L\}|}{L^{6g-6+2(n+p)}} = \mathfrak{c}(\gamma_0)\mathfrak{m}(Y)$$

where $\mathfrak{c}(\gamma_0)$ is as in Mirzakhani's Theorem, $\mathfrak{m}(Y)$ is a constant depending on Y, and $\ell_Y(\gamma)$ is the length of a shortest curve homotopic to γ.

Remark: We refer the reader to Mirzakhani's original result [26], the survey by Wright [35], and Erlandsson and Souto's book [11, Chapter 8] for details on the constants appearing in Theorem 1.3.2. The constant $\mathfrak{m}(Y)$ can be expressed in terms of the Thurston measure $\mathfrak{m}_{\text{Thu}}$ on $\mathcal{ML}(S)$ as

$$\mathfrak{m}(Y) = \mathfrak{m}_{\text{Thu}}(\{\lambda \in \mathcal{ML}(S) \mid \ell_Y(\lambda) \leq 1\}). \tag{1.1}$$

Following the notation of [11], $\mathfrak{c}(\gamma_0)$ can be written as

$$\mathfrak{c}(\gamma_0) = \frac{\mathfrak{c}^{\text{PMod}(S)}(\gamma_0)}{\mathfrak{b}_{g,n+p}}, \tag{1.2}$$

and both $\mathfrak{c}^{\text{PMod}(S)}(\gamma_0)$ and $\mathfrak{b}_{g,n+p}$ can also be expressed in terms of Thurston measures. The original constants, due to Mirzakhani, were expressed in a different fashion, using integrals over moduli space with respect to the Weil-Petersson metric. See the end of Chapter 8 in [11] for a discussion on the relationship between these constants and those appearing in [11]. For details on the Thurston measure, see [34].

To see that Theorem 1.3.2 implies that we can count curves in our setting, let γ_0 be a multicurve on S and let $K = K(\gamma_0)$ be the compact subsurface of S^o given by Lemma 1.2.5. By Lemma 1.3.1, we may assume that K is such that $\mathcal{ML}(S) \subset K$. Take any complete negatively curved Riemannian metric Y on S^o which agrees with X on K. Since the geodesic representative of every multicurve γ of type γ_0 is contained in K and $\mathcal{ML}(S)$ sits inside K, we have that $\ell_Y(\gamma) = \ell_X(\gamma)$ for all γ of type γ_0 and $\ell_Y(\lambda) = \ell_X(\lambda)$ for all $\lambda \in \mathcal{ML}(S)$. From the latter equality, we get that $\mathfrak{m}(X) = \mathfrak{m}(Y)$ using (1.1). Moreover, we remark that $\mathfrak{c}(\gamma_0)$ is independent of the metric chosen, and so we have the following consequence.

Corollary 1.3.3. *Let X be a complete, finite-area, hyperbolic metric on S such that ∂S is geodesic. Then for any multicurve γ_0,*

$$\lim_{L\to\infty} \frac{|\{\gamma \text{ of type } \gamma_0 \mid \ell_X(\gamma) \leq L\}|}{L^{6g-6+2(n+p)}} = \mathfrak{c}(\gamma_0)\mathfrak{m}(X)$$

where $\mathfrak{c}(\gamma_0)$ and $\mathfrak{m}(X)$ are as in Mirzakhani's Theorem.

1.4 Infinite arcs and their lengths

Here we discuss the assignment of appropriate finite lengths to infinite arcs. For any $t \in (0,1]$, let $\mathscr{H}_t = \cup_{p \in \mathfrak{C}} H_t^p$ be the union of the cuspidal regions of volume t as in Definition 1.2.1.

Definition 1.4.1: We define the *t-length* $\ell_X^t(\alpha)$ of any infinite arc α to be the length of $\alpha^t = \alpha \cap (S \setminus \mathscr{H}_t)$. That is,

$$\ell_X^t(\alpha) = \ell_X(\alpha^t).$$

Note that in general, α^t could consist of multiple connected components and in this case, $\ell_X(\alpha^t)$ is the sum of the lengths of its components. Fix an infinite arc α and let t_α be given by Lemma 1.2.6. Then $\alpha \cap (S \setminus \mathscr{H}_{t_\alpha})$ is connected, and moreover for any $t \leq t_\alpha$, α^t has exactly one component.

Another finite length one can assign to an infinite arc is the *truncated length* as defined by Parlier in [28]. Choose a standard collection of cuspidal regions, which we may take as $\mathscr{H}_1$. For any infinite arc α, the (doubly) truncated length of α is the length of the segment of α between the first and last times α crosses $\partial\mathscr{H}_1$. We denote this length by $\ell_X^{Tr}(\alpha)$.

This length is closely related to the λ*-length* introduced by Penner in [29] and [30]. In our setting, and choosing the appropriate cuspidal regions, we have that

$$\lambda(\alpha) = \mathrm{e}^{\frac{1}{2}\ell_X^{Tr}(\alpha)}. \tag{1.3}$$

As mentioned in the introduction, the t-length of an arc is closely related to the truncated length. Note that for any $\alpha \in \mathcal{A}(S)$, $\ell_X^{t_\alpha}(\alpha)$ and $\ell_X^{Tr}(\alpha)$ differ by a constant, and this constant depends only on $\iota(\alpha,\alpha)$. This is because the t_α-length of α is exactly the truncated length plus the lengths of the two geodesic segments of α between $\partial\mathscr{H}_1$ and $\partial\mathscr{H}_{t_\alpha}$, which each have length $\ln(\frac{1}{t_\alpha})$. Thus we can write

$$|\ell_X^{t_\alpha}(\alpha) - \ell_X^{Tr}(\alpha)| \leq 2\ln\left(\frac{1}{t_\alpha}\right). \tag{1.4}$$

Recall that by Lemma 1.2.6, t_α depends only on $\iota(\alpha, \alpha)$. Using this fact, one can demonstrate the following as a consequence of Theorem 2.

Corollary 1.4.2. *Let X be a complete, finite-area, hyperbolic metric on S with (possibly empty) geodesic boundary. Let α_0 be an infinite arc on S. Then we have*

$$\lim_{L\to\infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X^{Tr}(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0)\mathfrak{c}(\alpha_0)\mathfrak{m}(X)$$

where $k(\alpha_0)$, $\mathfrak{c}(\alpha_0)$ and $\mathfrak{m}(X)$ are as in Theorem 1. By (1.3), we can replace $\ell_X^{Tr}(\alpha)$ with $2\ln(\lambda(\alpha))$ and the same statement holds.

1.5 Words associated to curves and arcs

It will be useful to represent curves as words in the fundamental group of a surface, particularly in the case of a pair of pants, also known as a three-holed sphere $S_{0,3}$. Let P be a pair of pants with boundary components δ_0^P, δ_1^P and δ_2^P, and fix some basepoint $p_0 \in P$. We denote the fundamental group of P with this basepoint as $\pi_1(P, p_0)$. Recall that the fundamental group of a surface consists of homotopy class of loops based at the basepoint, where homotopies must fix the basepoint. Since P is path-connected, a different choice of basepoint gives an isomorphic group, thus for ease of notation we simply write $\pi_1(P)$. Also recall that the free homotopy classes of curves on a surface are in one-to-one correspondence with conjugacy classes of elements in the fundamental group of the surface.

The fundamental group of P is a free group on two generators. From here onwards, we fix the generators to be the loops a and b based at p_o freely homotopic to δ_0^P and δ_1^P respectively. Accordingly, each element in $\pi_1(P)$ is identified with a word in a, b, a^{-1} and b^{-1}. Given a reduced word $w = x_1^{j_1} x_2^{j_2} \dots x_n^{j_n}$ where $x_i = a, b$ and $j_i \in \mathbb{Z}_{\neq 0}$, the *word length* of w is given as

$$|w|_{word} = \sum_{i=1}^{n} |j_i|.$$

Each $x_i^{j_i}$ is referred to as a *syllable* of the word. If x_i is a or b, then we refer to $x_i^{j_i}$

as an a-syllable or a b-syllable respectively. The *syllable length* of w is

$$|w| = n.$$

We call a word *reduced* if no neighbouring syllables are written in the same generator. We call a word *cyclically reduced* if it is reduced and the first syllable is not the inverse of the last syllable: that is, either $x_1 \neq x_n$ or $j_1 j_n > 0$.

Let $\varphi\colon P \to P$ be a continuous map. Then the free homotopy class of φ induces a homomorphism $\varphi^*\colon \pi_1(P) \to \pi_1(P)$; we can choose a representative of φ which fixes the basepoint p_0 and define $\varphi^*(\gamma) = \varphi(\gamma)$, where γ is any (homotopy class of a) loop based at p_0. It turns out that two such continuous maps are homotopic exactly when the homomorphisms they induce are conjugates of one another, by this we mean that there exists some word $\omega \in \pi_1(P)$ such that for all $\gamma \in \pi_1(P)$, $\varphi_2^*(\gamma) = \omega\varphi_1^*(\gamma)\omega^{-1}$. In this case, we write $\varphi_2^* = \omega\varphi_1^*\omega^{-1}$. We will briefly discuss the proof of this below. We refer the reader to Chapter 1 of [13] for a thorough treatment of the fundamental group and homomorphisms induced by continuous maps of the surface.

Theorem 1.5.1. *For a surface S and a basepoint $x_0 \in S$, let $\varphi_1, \varphi_2\colon S \to S$ be continuous maps and $\varphi_1^*, \varphi_2^*\colon \pi_1(S, x_0) \to \pi_1(S, x_0)$ be their induced homomorphisms. Then φ_1 and φ_2 are (freely) homotopic if and only if φ_1^* and φ_2^* are conjugate by some word in $\pi_1(S, x_0)$.*

Proof. Suppose that φ_1 and φ_2 are freely homotopic, and choose (freely) homotopic representatives ${\varphi_1}'$ and ${\varphi_2}'$ of these maps which fix x_0. Let $f_t\colon S \to S$ be a free homotopy for $t \in [0, 1]$, such that $f_0 = {\varphi_1}'$ and $f_1 = {\varphi_2}'$. Consider the closed loop $\omega = \{f_t(x_0) \mid t \in [0, 1]\}$ traced out by x_0 under this homotopy. Then we have that $\varphi_2^* = \omega\varphi_1^*\omega^{-1}$; in other words, the homomorphisms are conjugate (see Lemma 1.19 in [13]).

Suppose now that φ_1^* and φ_2^* are conjugate by some word ω in $\pi_1(S)$; so $\varphi_2^* = \omega\varphi_1^*\omega^{-1}$. As before, we can take homotopic representatives of φ_1 and φ_2 which fix x_0. There exists a homotopy $f_t\colon S \to S$ such that $f_0 = \varphi_1$ and $\{f_t(x_0) \mid t \in [0, 1]\}$ is the loop ω. Let $\varphi_1' = f_1$, and note that φ_1' also fixes x_0. As above, ${\varphi_1}'^* = \omega\varphi_1^*\omega^{-1}$. Hence ${\varphi_1}'^* = \varphi_2^*$, and so ${\varphi_1}'$ and φ_2 are homotopic under a homotopy fixing x_0 (see

Proposition 1B.9 in [13]). Note that the universal cover of S is contractible. Combining this homotopy with the free homotopy f_t, we see that φ_2 is freely homotopic to φ_1. □

As mentioned in the introduction, we will associate a curve to an arc by relating the arc to an immersed pair of pants in the surface. However, we will see (see Example 3.1.3 for instance) that there exist continuous maps ι_α and $\iota_{\alpha'}$ of P into S such that $\iota_\alpha(P)$ and $\iota_{\alpha'}(P)$ share all three boundary components, but are *not* homotopic.

By Theorem 1.5.1, we have that any two continuous maps of P into itself are not homotopic if and only if the induced self-homomorphisms of $\pi_1(P)$ are not conjugate. We record this here for reference.

Corollary 1.5.2. *Let $\iota_1 \colon P \to P$ and $\iota_2 \colon P \to P$ be two continuous maps. Then $\iota_1(P)$ and $\iota_2(P)$ are homotopic if and only if the induced homomorphisms $\iota_1^* \colon \pi_1(P) \to \pi_1(P)$ and $\iota_2^* \colon \pi_1(P) \to \pi_1(P)$ are conjugate.*

Chapter 2

Counting arcs

This chapter contains the proof of the main results of this thesis, and is heavily based on a publication [5] by the author. First in Section 2.1, we study a method for associating curves to arcs, which is well-known to experts. We demonstrate in Section 2.2 that this association treats length nicely, it is equivariant with respect to PMod(S), and when restricted to a type of arc the fibers have constant cardinality. We then lay out the proof of Theorem 1. In Section 2.3 we prove that analogous properties hold for infinite arcs and give the proof of Theorem 2.

2.1 Associating arcs and curves

We begin by focusing on ordinary (compact) arcs, that is, arcs between a pair of boundary components. Recall that an arc is *two-ended* if it starts and ends at distinct boundary components, and *one-ended* if it starts and ends at the same boundary component. To define an association from arcs to curves in a way which only distorts the length of the arc in a bounded manner, we want to construct a curve from an arc. If we consider a simple arc α on a surface S, we find an intuitive way to do this. Take the boundary of a regular neighbourhood of the union of α and the corresponding boundary components of S; in the case that α is simple and two-ended, this neighbourhood is homeomorphic to an embedded pair of pants P, otherwise known as the three-holed sphere $S_{0,3}$. A pair of pants equipped with a hyperbolic metric has useful geometric properties which give us various identities, allowing us to relate the length of α and the length of the (geodesic representative of the) boundary of this neighbourhood. However, if α is not simple or is one-ended then the boundary of this neighbourhood will in fact be a multi-curve. To ensure that we always construct a curve from our arc whilst also constructing a pair of pants to relate their lengths, we must take a slightly different approach. If we instead construct our curve by taking two parallel copies of α and joining their ends around the boundary components, we get the same result as the previous method for simple two-ended arcs but we also get a single curve in more complex cases: here the curve will be the boundary of an immersed pair of pants, or rather the image of a boundary component of a pair of pants under a continuous map. Examples of this can be seen in Figure 2.1, and we define this method concretely below.

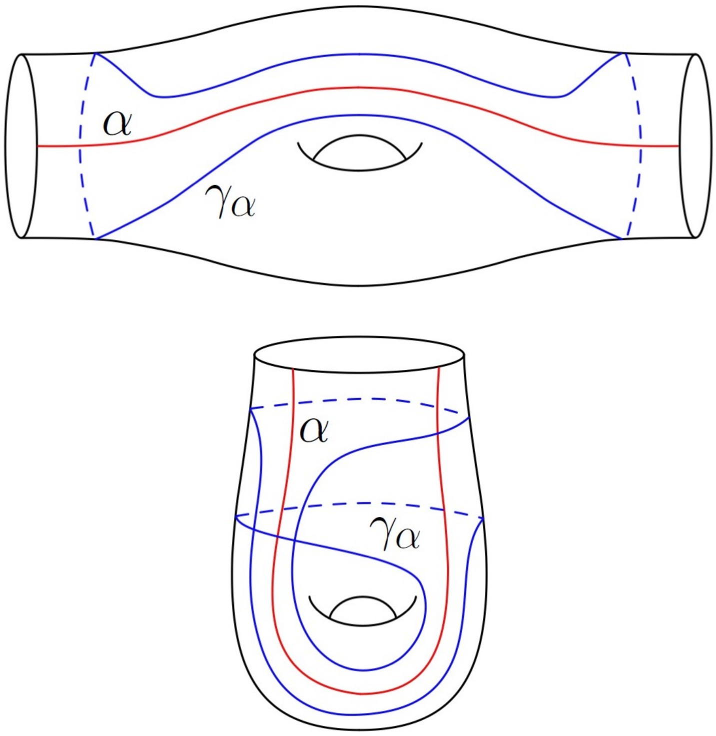

Figure 2.1: *Examples of arcs (in red) and the curves associated to them (in blue). One arc (above) is simple and two-ended, whereas the other (below) is simple and one-ended. Note that the curve associated to the one-ended arc is not simple.*

We fix an orientation on S, which induces an orientation on the boundary components. We will use the "right-hand rule" convention for this; on a boundary component, you are facing the positive direction if the interior of the surface is on your right hand side. Let $\alpha\colon [0,1] \to S$ be an arc on S, oriented from $\alpha(0)$ to $\alpha(1)$. The endpoints $\alpha(0)$ and $\alpha(1)$ each lie on a boundary component which we denote by δ_0^α and δ_1^α respectively: note that these are not necessarily distinct. Pick basepoints p_0 on δ_0^α and p_1 on δ_1^α, and consider these boundary components as loops based at their respective basepoints. Apply a homotopy to α so that $\alpha(0) = p_0$ and $\alpha(1) = p_1$, moving the endpoints no more than halfway around the boundary component to do so. Then we define the *curve associated to* α to be the geodesic curve γ_α (freely) homotopic to the concatenated path

$$\alpha^{-1} \cdot \delta_1^\alpha \cdot \alpha \cdot \delta_0^\alpha$$

which starts and ends at p_0. As mentioned above, in the case that α is simple and $\delta_0^\alpha \neq \delta_1^\alpha$, γ_α is homotopic to the boundary of a regular neighbourhood of the union of α, δ_0^α and δ_1^α. An example of this can be found in Figure 2.1. Recall that we identify arcs and curves that differ by an orientation, and note that the arc α^{-1} which differs from α only in orientation gives rise to exactly the same curve as α, even in orientation.

Let P be a pair of pants and fix an orientation on P. The boundary components of P are referred to as *cuffs*, and for each pair of cuffs the unique homotopy class of simple arcs between them is called a *seam*. We label the cuffs as δ_0^P, δ_1^P and δ_2^P and the seam between δ_0^P and δ_1^P as α_P. For any arc α on S with endpoints on δ_0^α and δ_1^α, there exists a continuous map ι_α such that

$$\begin{aligned} \iota_\alpha &\colon P \to S, \\ \iota_\alpha(\delta_0^P) &= \delta_0^\alpha, \\ \iota_\alpha(\delta_1^P) &= \delta_1^\alpha, \\ \iota_\alpha(\alpha_P) &= \alpha. \end{aligned} \tag{2.1}$$

Note that the images of the two cuffs and the seam under this map determine the image of the third cuff up to homotopy, since this is exactly the (free) homotopy class of $\alpha^{-1} \cdot \delta_1^\alpha \cdot \alpha \cdot \delta_0^\alpha$. That is,

$$\gamma_\alpha = \iota_\alpha(\delta_2^P) \tag{2.2}$$

(up to homotopy). We note here that this is an alternate definition of the curve associated to an arc. Let ι'_α be another continuous map which satisfies (2.1). Then since they agree on δ_0^P, δ_1^P and α_P, we have that the images $\iota_\alpha(\delta_2^P)$ and $\iota'_\alpha(\delta_2^P)$ of the third cuff are homotopic and such a homotopy extends to a homotopy from $\iota_\alpha(P)$ to $\iota'_\alpha(P)$. Thus any two such continuous maps of P are homotopic.

Let $\mathcal{A}(S)$ and $\mathcal{C}(S)$ denote the sets of (homotopy classes of) arcs and curves on

S respectively. We define the *association map* I as

$$I\colon \mathcal{A}(S) \to \mathcal{C}(S), \\ I(\alpha) = \gamma_\alpha. \tag{2.3}$$

In Chapter 3, we will study this association in detail.

2.1.1 Associate arcs on S

Somewhat surprisingly, the map I is *not* one-to-one. That is, we can find curves γ such that $|I^{-1}(\gamma)| > 1$. In a special case, we can see this rather quickly: suppose S is a four-holed sphere and α and β are disjoint simple two-ended arcs between distinct pairs of boundary components. Then γ_α and γ_β are the same (homotopy class of) curve, up to orientation. For simple examples of these arcs, see Figure 2.2.

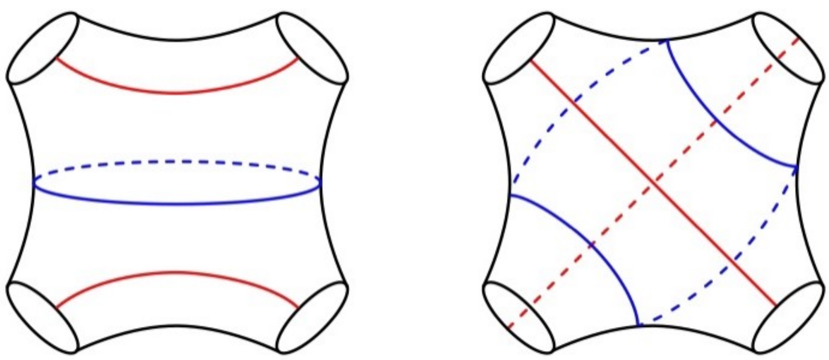

Figure 2.2: *Two examples of pairs of associate arcs of the first kind on $S_{0,4}$. In each example, the two arcs in red are associated to the curve in blue.*

However, there are also much less trivial examples of this where the arcs are between the same boundary components, and these examples can occur on any surface with boundary. For instance, the two arcs in Figure 2.3 represent distinct homotopy classes on P, but their associated curves are homotopic. This can be seen by hand with some work, but in Chapter 3 we will study how to derive this example (see Example 3.1.3) and many more like it. By mapping a pair of pants into a surface S in such a way that we respect the relevant pair of boundary components, we can produce such examples on any surface with boundary.

When a collection of arcs are associated to the same curve under I, we will

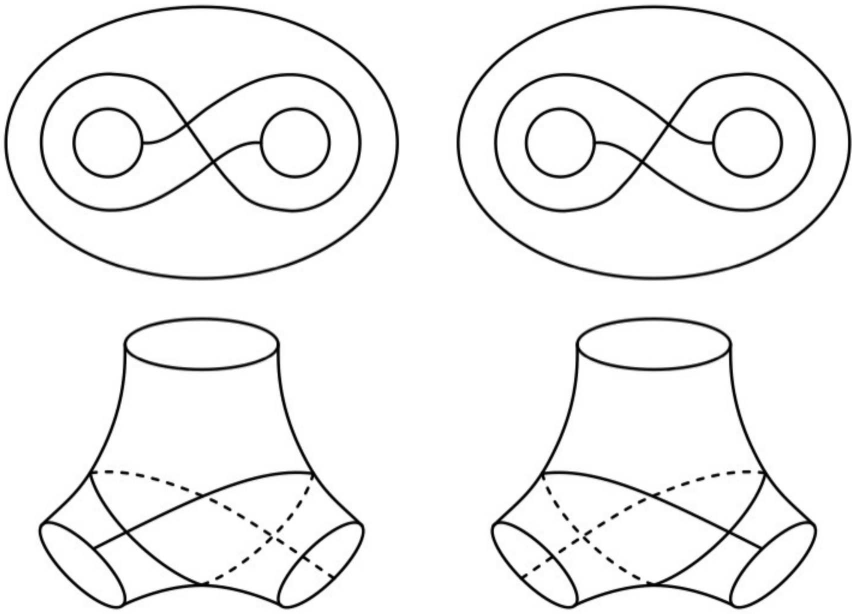

Figure 2.3: *An example of two arcs of the second kind associated to the same curve (shown in Figure 2.4), drawn from two perspectives for clarity. These are described in Example 3.1.3. Using the terminology from Chapter 3, these correspond to the frame* $(2, 2, \emptyset)$.

refer to them as *associate* arcs. We will refer to the curves in $I(\mathcal{A}(S))$ as *curves associated to arcs*. As mentioned, in Chapter 3 we will investigate associate arcs and demonstrate that on the pair of pants P, at most two two-ended arcs can be associated to the same curve. We would expect this statement to generalise to all arcs on any hyperbolic surface.

Take two associate arcs α_1 and α_2 on a surface S, and denote the curve associated to them as γ. The arcs α_1 and α_2 either join different (not necessarily disjoint) pairs of boundary components as in Figure 2.2, or the same pair of boundary components as in Figure 2.3. We will refer to these as arcs of the first kind and arcs of the second kind respectively.

Associate arcs of the first kind are not the focus of this work; we consider them to be trivial examples as they can never be of the same type under $\mathrm{PMod}(S)$ and will not affect the proof of our main theorem. In the following when discussing associate arcs, we will assume them to be of the second kind.

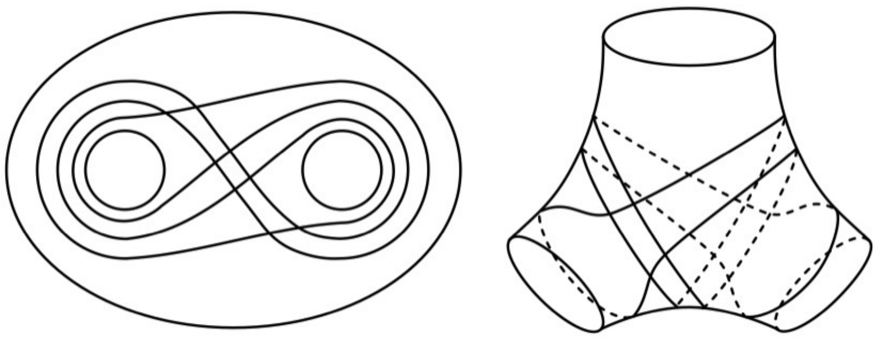

Figure 2.4: *The curve associated to the arcs from Figure 2.3, drawn from two perspectives for clarity.*

2.2 Counting arcs

In this section, we will discuss various properties of the map I needed to prove Theorem 1, which we recall here.

Theorem 1. *Let X be a complete, finite-area, hyperbolic metric on S with non-empty geodesic boundary. Let α_0 be an arc on S. Then there exist positive constants $\mathfrak{c}(\alpha_0)$ and $\mathfrak{m}(X)$ such that*

$$\lim_{L\to\infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \le L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0)\mathfrak{c}(\alpha_0)\mathfrak{m}(X),$$

where $k(\alpha_0)$ is some positive integer.

In full generality, this theorem holds for weighted multi-arcs (see Theorem 2.2.6). In the following, we will mention how to generalise each statement we prove for arcs to multi-arcs; this theorem then holds for multi-arcs using a proof analogous to that of Theorem 1 given in Section 2.2.3.

To begin with, we describe how to extend I to the set of multi-arcs. In the case that $\alpha = \sum_{i=1}^{m} a^i \alpha^i$ is a multi-arc, we define the *multicurve associated to* α to be the weighted sum of the the curves associated to its components. That is,

$$\gamma_\alpha = \sum_{i=1}^{m} a^i \gamma_{\alpha^i} = \sum_{i=1}^{m} a^i \iota_{\alpha^i}(\delta_2^P). \tag{2.4}$$

Let $\mathcal{A}_{\mathrm{multi}}(S)$ and $\mathcal{C}_{\mathrm{multi}}(S)$ be the sets of weighted multi-arcs and weighted multicurves respectively. By abuse of notation, we define the association map on multi-arcs by

$$I\colon \mathcal{A}_{\mathrm{multi}}(S) \to \mathcal{C}_{\mathrm{multi}}(S),$$
$$I(\alpha) = I\Big(\sum_{i=1}^{m} a^i \alpha^i\Big) = \sum_{i=1}^{m} a^i I(\alpha^i).$$

2.2.1 Distortion of the length of an arc under I

In this section, we want to show that the association almost preserves the lengths of arcs. We can prove that I distorts the length of arcs in a controlled way using the geometric properties of a hyperbolic pair of pants P (see also Section 6 of [4] for various expressions relating the lengths of α and γ_α).

Lemma 2.2.1. *Let X be a complete, finite-area, hyperbolic metric on S such that ∂S is geodesic. There exists a constant $C(X) > 0$ such that for any $\alpha \in \mathcal{A}(S)$,*

$$|\ell_X(I(\alpha)) - 2\ell_X(\alpha)| \leq C(X),$$

where $I(\alpha) = \gamma_\alpha$ is the curve associated to α.

Proof. This will follow from basic hyperbolic geometry. Let $\alpha \in \mathcal{A}(S)$, and let ι_α be the continuous map given by (2.1).

The immersed pair of pants $\iota_\alpha(P)$ consists of a pair of hyperbolic right angled hexagons with side lengths $\frac{\ell_X(\delta_0^\alpha)}{2}, \ell_X(\alpha), \frac{\ell_X(\delta_1^\alpha)}{2}, r, \frac{1}{2}\ell_X(\gamma_\alpha), s$ for some $r, s > 0$. Fixing the lengths of three sides of such a hexagon fixes the lengths of the remaining three sides. As such, choosing the lengths of 2 cuffs and the seam between them on a pair of pants fixes the length of the third cuff; that is, the lengths of δ_0^α, δ_1^α and α determine the length of γ_α. More precisely,

$$\cosh\frac{\ell_X(\gamma_\alpha)}{2} = \sinh\frac{\ell_X(\delta_0^\alpha)}{2}\sinh\frac{\ell_X(\delta_1^\alpha)}{2}\cosh\ell_X(\alpha) - \cosh\frac{\ell_X(\delta_0^\alpha)}{2}\cosh\frac{\ell_X(\delta_1^\alpha)}{2} \tag{2.5}$$

(see Theorem 2.4.1 of [8]). Let $\partial S = \{\delta_1, \dots, \delta_n\}$. Then for some i and j, $\delta_0^\alpha = \delta_i$ and $\delta_1^\alpha = \delta_j$. To simplify notation, we will write

$$\begin{aligned} A_{i,j} &= \sinh \frac{\ell_X(\delta_i)}{2} \sinh \frac{\ell_X(\delta_j)}{2}, \\ B_{i,j} &= \cosh \frac{\ell_X(\delta_i)}{2} \cosh \frac{\ell_X(\delta_j)}{2}. \end{aligned}$$

Rearranging equation (2.5), we can write the length of γ_α as

$$\ell_X(\gamma_\alpha) = 2\cosh^{-1}(A_{i,j} \cosh \ell_X(\alpha) - B_{i,j}),$$

and we want to show that this length is close to $2\ell_X(\alpha)$. To this end, we define the error function $E_{i,j} \colon [m_{i,j}, \infty) \to \mathbb{R}$ by

$$E_{i,j}(\ell) = 2\cosh^{-1}(A_{i,j} \cosh \ell - B_{i,j}) - 2\ell,$$

where $m_{i,j}$ is a lower bound on the lengths of arcs between δ_i and δ_j. This can be taken as $m_{i,j} \coloneqq \cosh^{-1}\left(\frac{B_{i,j}+1}{A_{i,j}}\right)$. This function is continuous, and the limit

$$\lim_{\ell\to\infty} E_{i,j}(\ell) = 2\ln(A_{i,j})$$

exists. We can convince ourselves of this with the following approximations; the concrete computation of this limit is given in Appendix A.

$$\begin{aligned} \lim_{\ell\to\infty} E_{i,j}(\ell) &= \lim_{\ell\to\infty} \left(2\cosh^{-1}(A_{i,j}\cosh \ell - B_{i,j}) - 2\ell\right) \\ &= \lim_{\ell\to\infty} \left(2\cosh^{-1}(A_{i,j}\cosh \ell) - 2\ell\right) \\ &= \lim_{\ell\to\infty} \left(2\cosh^{-1}\left(A_{i,j}\frac{\mathrm{e}^\ell + \mathrm{e}^{-\ell}}{2}\right) - 2\ell\right) \\ &= \lim_{\ell\to\infty} \left(2\cosh^{-1}\left(A_{i,j}\frac{\mathrm{e}^\ell}{2}\right) - 2\ell\right) \\ &= \lim_{\ell\to\infty} \left(2\ln\left(A_{i,j}\frac{\mathrm{e}^\ell}{2} + \sqrt{\left(A_{i,j}\frac{\mathrm{e}^\ell}{2}\right)^2 - 1}\right) - 2\ell\right) \\ &= \lim_{\ell\to\infty} \left(2\ln\left(A_{i,j}\frac{\mathrm{e}^\ell}{2} + \sqrt{\left(A_{i,j}\frac{\mathrm{e}^\ell}{2}\right)^2}\right) - 2\ell\right) \\ &= \lim_{\ell\to\infty} \left(2\ln(A_{i,j}\mathrm{e}^\ell) - 2\ell\right) \end{aligned}$$

$$= \lim_{\ell\to\infty} \Big(2\ln(A_{i,j}) + 2\ln(\mathrm{e}^{\ell}) - 2\ell\Big)$$
$$= 2\ln(A_{i,j}).$$

Hence $|E_{i,j}(\ell)|$ is bounded for all $\ell \in [m_{i,j}, \infty)$, and thus there exists $C(i,j) > 0$ such that for any arc α between δ_i and δ_j we have $|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha)| \leq C(i,j)$. Therefore as S has finitely many boundary components, there exists $C(X) > 0$ such that for any $\alpha \in \mathcal{A}(S)$,

$$|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha)| \leq C(X).$$

□

If $\alpha = \sum_{i=1}^m a^i\alpha^i$ is a multi-arc, then we say the *weight* of α is the sum of the weights of its components, $W(\alpha) = \sum_{i=1}^m |a^i|$. As multi-arcs have finitely many components, Lemma 2.2.1 holds for multi-arcs as a direct consequence if we allow the constant to depend on the weight. In particular, we have:

Corollary 2.2.2. *Let X be a complete, finite-area, hyperbolic metric on S such that ∂S is geodesic. Let $\alpha \in \mathcal{A}_{multi}(S)$. Then there exists a constant $C(X) > 0$ such that*

$$|\ell_X(I(\alpha)) - 2\ell_X(\alpha)| \leq W(\alpha)C(X),$$

where $I(\alpha) = \gamma_\alpha$ is the multicurve associated to α and $W(\alpha)$ is the weight of α.

Proof. Let $\alpha = \sum_{i=1}^m a^i\alpha^i$ be a multi-arc. If γ_α is the multicurve associated to α, then we have $\gamma_\alpha = \sum_{i=1}^m a^i\gamma_\alpha^i$ as in (2.4), where $\gamma_\alpha^i = \gamma_{\alpha^i}$ is the curve associated to α^i. Using the triangle inequality and Lemma 2.2.1, we have

$$\begin{aligned}
|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha)| &= \Big|\sum_{i=1}^m a^i\ell_X(\gamma_\alpha^i) - 2\sum_{j=1}^m a^j\ell_X(\alpha^j)\Big| \\
&= \Big|\sum_{i=1}^m a^i\ell_X(\gamma_{\alpha^i}) - 2\sum_{j=1}^m a^j\ell_X(\alpha^j)\Big| \\
&= \Big|\sum_{i=1}^m a^i\big(\ell_X(\gamma_{\alpha^i}) - 2\ell_X(\alpha^i)\big)\Big| \\
&\leq \sum_{i=1}^m |a^i||\ell_X(\gamma_{\alpha^i}) - 2\ell_X(\alpha^i)| \\
&\leq \sum_{i=1}^m |a^i|C(X)
\end{aligned}$$

$$= C(X) \sum_{i=1}^{m} |a^i|,$$

and so the lemma holds. □

Note that the weight of a multi-arc is invariant under the action of the pure mapping class group, and so this bound is uniform across multi-arcs of the same type.

2.2.2 Arcs of the same type are uniformly k-to-1

Part of the proofs of our main theorems will require understanding how close to injective the map I is when we restrict it to a type of arc. Recall that two arcs are *of the same type* if they share an orbit under the action of the pure mapping class group $\mathrm{PMod}(S)$.

We mentioned in Section 2.1 that I can map multiple arcs to the same curve, which will be studied in detail in Chapter 3. In particular, the number of arcs mapped to a given curve varies across $I(\mathcal{A}(S))$. However, when we restrict I to a type of arc we can prove that this number cannot vary. We will show this using the fact that I is equivariant with respect to $\mathrm{PMod}(S)$, which we now demonstrate. In fact, this map is equivariant with respect to the mapping class group $\mathrm{Mod}(S)$.

Lemma 2.2.3. *Let $\varphi \in \mathrm{PMod}(S)$ and $\alpha \in \mathcal{A}(S)$. Then*

$$I(\varphi \cdot \alpha) = \varphi \cdot I(\alpha).$$

Proof. To see this, write

$$\gamma_{\varphi\cdot\alpha} = \iota_{\varphi\cdot\alpha}(\delta_2^P)$$

using the alternate definition (2.2) given for γ_α in Section 2.1. We have that $\iota_{\varphi\cdot\alpha}(P) \subset S$ is an immersed pair of pants with boundary components δ_0^α and δ_1^α, and the seam between them is $\varphi \cdot \alpha$. Similarly, $\varphi \cdot \iota_\alpha(P) \subset S$ is an immersed pair of pants with boundary components δ_0^α and δ_1^α and seam between them $\varphi \cdot \alpha$, since φ fixes the boundary components of S. Therefore, $\iota_{\varphi\cdot\alpha}(P) = \varphi \cdot \iota_\alpha(P)$ up to homotopy, and in particular $\iota_{\varphi\cdot\alpha}(\delta_2^P) = \varphi \cdot \iota_\alpha(\delta_2^P)$. Since $\gamma_\alpha = \iota_\alpha(\delta_2^P)$ by (2.2), we have

$$\gamma_{\varphi\cdot\alpha} = \varphi \cdot \gamma_\alpha$$

and so the lemma holds. □

Let α_0 be an arc, and let α be an arc of type α_0. Then α and α_0 are related by some pure mapping class φ. If we imagine the map φ acting on the pair of pants $\iota_{\alpha_0}(P)$ of which the associated curve γ_{α_0} is a cuff, we can see that α will be the seam of this new pair of pants and its associated curve will be the image of this cuff. Thus γ_α will be a curve of type γ_{α_0} for any arc α of type α_0. We define the restriction of I to arcs of type α_0

$$I_{\alpha_0} \colon \mathrm{PMod}(S) \cdot \alpha_0 \to \mathrm{PMod}(S) \cdot \gamma_{\alpha_0}$$

by $I_{\alpha_0}(\alpha) = I(\alpha)$ for all arcs α of type α_0. By Lemma 2.2.3, this map is well-defined. We can show that not only is this restriction of I surjective, it is also uniformly k-to-one for some positive integer k.

Proposition 2.2.4. *Let α_0 be an arc on S. Then there exists $k = k(\alpha_0)$ such that I_{α_0} is surjective and k-to-1.*

Proof. Let $\gamma \in \mathrm{PMod}(S) \cdot \gamma_{\alpha_0}$, so $\gamma = \varphi \cdot \gamma_{\alpha_0}$ for some $\varphi \in \mathrm{PMod}(S)$. Let $\alpha = \varphi \cdot \alpha_0$. Then by Lemma 2.2.3,

$$\gamma = \varphi \cdot \gamma_{\alpha_0} = \gamma_{\varphi \cdot \alpha_0} = I_{\alpha_0}(\varphi \cdot \alpha_0) = I_{\alpha_0}(\alpha),$$

thus I_{α_0} is surjective.

Consider the collection of arcs α_i such that $\gamma_{\alpha_i} = \gamma$. This set is finite: by Lemma 2.2.1, the maximum length of such an arc is $\frac{1}{2}\ell_X(\gamma) + \frac{1}{2}C(X)$, and thus there are only finitely many. Suppose that there are exactly k such arcs, and suppose further that for some other curve γ' of type γ_{α_0}, there are exactly k' arcs α_i' such that $\gamma_{\alpha_i'} = \gamma'$. Since γ and γ' are of the same type, there exists some $\psi \in \mathrm{PMod}(S)$ such that $\gamma' = \psi \cdot \gamma$. Again by Lemma 2.2.3, we have that for each $i \in \{1, \ldots, k\}$,

$$I_{\alpha_0}(\psi \cdot \alpha_i) = \gamma_{\psi \cdot \alpha_i} = \psi \cdot \gamma_{\alpha_i} = \psi \cdot \gamma = \gamma'.$$

Thus we have constructed a set of k arcs which are associated to γ', and so $k \leq k'$. Analogously, we can construct a set of k' arcs which are associated to γ, hence $k' \leq k$. Therefore $k = k'$, and so k is uniform across all curves of type γ_{α_0}. □

The map I remains PMod(S)-equivariant when defined on multi-arcs as an immediate corollary to Lemma 2.2.3. Following the proof of Proposition 2.2.4, we can see that the restriction of the association map to multi-arcs of a particular type is surjective and k-to-1, for some k depending only on the type.

Corollary 2.2.5. *Let α_0 be a multi-arc and γ_{α_0} be as in* (2.4)*. Let*

$$I_{\alpha_0}\colon \mathrm{PMod}(S)\cdot\alpha_0 \to \mathrm{PMod}(S)\cdot\gamma_{\alpha_0}$$

be the restriction of I to multi-arcs of type α_0. Then there exists $k = k(\alpha_0)$ such that I_{α_0} is surjective and k-to-1.

2.2.3 Proving Theorem 1

We can now prove that when counting arcs of a given type of bounded length, the growth is polynomial in L.

Proof of Theorem 1. Let α_0 be an arc, and consider the set

$$\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}$$

for some $L > 0$. By Proposition 2.2.4, there exists some k such that the association map I_{α_0} is k-to-1. Thus for each curve in the image of the above set under I_{α_0}, there are exactly k arcs in its pre-image. By Lemma 2.2.1, the maximum length of a curve associated to an arc in this set is $2L + C(X)$. Hence we can write

$$|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}| \leq k|\{\gamma \text{ of type } \gamma_{\alpha_0} \mid \ell_X(\gamma) \leq 2L + C(X)\}|.$$

Then we have

$$\begin{aligned}
&\limsup_{L\to\infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} \\
&\qquad \leq \limsup_{L\to\infty} \frac{k|\{\gamma \text{ of type } \gamma_{\alpha_0} \mid \ell_X(\gamma) \leq 2L + C(X)\}|}{L^{6g-6+2(n+p)}} \\
&\qquad = k \cdot \limsup_{L\to\infty} \frac{|\{\gamma \text{ of type } \gamma_{\alpha_0} \mid \ell_X(\gamma) \leq 2L + C(X)\}|}{(2L + C(X))^{6g-6+2(n+p)}} \frac{(2L + C(X))^{6g-6+2(n+p)}}{L^{6g-6+2(n+p)}}
\end{aligned}$$

$$= k \cdot 2^{6g-6+2(n+p)} \mathfrak{c}(\gamma_{\alpha_0}) \mathfrak{m}(X)$$

using Corollary 1.3.3. Using a similar argument, we have

$$|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}| \geq k |\{\gamma \text{ of type } \gamma_{\alpha_0} \mid \ell_X(\gamma) \leq 2L - C(X)\}|,$$

as every curve of type γ_{α_0} of length at most $2L - C(X)$ must be associated to an arc of type α_0 of length at most L. and therefore

$$\liminf_{L \to \infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} \geq k \cdot 2^{6g-6+2(n+p)} \mathfrak{c}(\gamma_{\alpha_0}) \mathfrak{m}(X).$$

Hence, since the limit superior and inferior both exist and agree, we have that the limit exists and equals the same value. In other words,

$$\lim_{L \to \infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0) \mathfrak{c}(\alpha_0) \mathfrak{m}(X),$$

where $\mathfrak{c}(\alpha_0) \coloneqq 2^{6g-6+2(n+p)} \mathfrak{c}(\gamma_{\alpha_0})$, $k(\alpha_0)$ is as in Proposition 2.2.4, and $\mathfrak{c}(\gamma_{\alpha_0})$ and $\mathfrak{m}(X)$ are as in Mirzakhani's Theorem. □

As mentioned in the introduction, Theorem 1 also holds when we consider multi-arcs instead of arcs. Following the above proof but substituting Lemma 2.2.1 and Proposition 2.2.4 with Corollary 2.2.2 and Corollary 2.2.5 respectively, we have the following result.

Theorem 2.2.6. *Let X be a complete, finite-area, hyperbolic metric on S with non-empty geodesic boundary. Let α_0 be a multi-arc on S. Then there exist positive constants $\mathfrak{c}(\alpha_0)$ and $\mathfrak{m}(X)$ such that*

$$\lim_{L \to \infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0) \mathfrak{c}(\alpha_0) \mathfrak{m}(X),$$

where $k(\alpha_0)$ is some positive integer.

2.3 Counting infinite arcs

The main work in this section is to prove Lemma 2.2.1, Lemma 2.2.3 and Proposition 2.2.4 from Section 2.2 for infinite arcs, with modifications to account for the range of values the t-length of an infinite arc can take. The proof of Theorem 2, which we recall here, will be analogous to that of Theorem 1. Recall from Definition 1.4.1 that the t-length of an infinite arc α is the length of the segment $\alpha^t = \alpha \cap (S \setminus \mathscr{H}_t)$, where $\mathscr{H}_t$ is as in Definition 1.2.1.

Theorem 2. *Let X be a complete, finite-area, hyperbolic metric on S with (possibly empty) geodesic boundary. Let α_0 be an infinite arc on S. Then for any positive $t \leq 1$, we have*

$$\lim_{L \to \infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell^t_X(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} = k(\alpha_0)\mathfrak{c}(\alpha_0)\mathfrak{m}(X)$$

where $k(\alpha_0)$, $\mathfrak{c}(\alpha_0)$ and $\mathfrak{m}(X)$ are as in Theorem 1. In particular, the limit does not depend on t.

In Section 1.4, we discussed how to assign appropriate finite lengths to infinite arcs. The curve γ_α associated to an infinite arc α is defined analogously to the case of arcs between boundary components. Recalling the notation from Section 1.4, we denote by p_0^α and p_1^α the cusps at each end of α, where α is oriented from p_0^α to p_1^α. With t_α as in Lemma 1.2.6, define γ_α to be the geodesic curve (freely) homotopic to the loop given by the concatenation

$$(\alpha^{-1})^{t_\alpha} \cdot h_1^\alpha \cdot \alpha^{t_\alpha} \cdot h_0^\alpha. \tag{2.6}$$

Here, $h_0^\alpha = \partial H_{t_\alpha}^{p_0^\alpha}$ and $h_1^\alpha = \partial H_{t_\alpha}^{p_1^\alpha}$ (see Definition 1.2.1) are the horocycles at p_0^α and p_1^α of length t_α, viewed as loops with appropriate basepoints and orientations. Note that by Lemma 1.2.2, if we replaced t_α with any positive $t < t_\alpha$, we would get the same curve γ_α. Let P be a (generalised) pair of pants with one boundary component and two cusps, labelled δ, p_0 and p_1 respectively. There is a continuous map $\iota_\alpha \colon P \to S$ which sends p_0 and p_1 to p_0^α and p_1^α respectively, and such that (the homotopy class of) the simple infinite arc between them is mapped to α. Then equivalently, γ_α is the geodesic representative of $\iota_\alpha(\delta)$.

Abusing notation, we define the association map I on infinite arcs to be

$$I\colon \mathcal{A}_\infty(S) \to \mathcal{C}(S),$$
$$I(\alpha) = \gamma_\alpha,$$

where $\mathcal{A}_\infty(S)$ is the set of all infinite arcs on S and γ_α is given by (2.6).

We will now prove an analogue of Lemma 2.2.1 for infinite arcs. As t can be taken arbitrarily close to 0, the t-length of an arc can be arbitrarily long, and so any bound on the difference between the t-lengths of infinite arcs and the lengths of their associated curves must depend on t. Furthermore, arcs which self-intersect arbitrarily often will go arbitrarily deep into the cusps, and therefore so will their curves. Thus for a fixed value of t, this difference can become arbitrarily large. Hence, any such bound must also depend on self-intersection number.

Lemma 2.3.1. *Let α be an infinite arc. Then for any positive $t < 1$, there exists $C\big(\iota(\alpha,\alpha),t\big) > 0$ such that*

$$|\ell_X(I(\alpha)) - 2\ell_X(\alpha^t)| \leq C\big(\iota(\alpha,\alpha),t\big)$$

where $I(\alpha) = \gamma_\alpha$ is the curve associated to α.

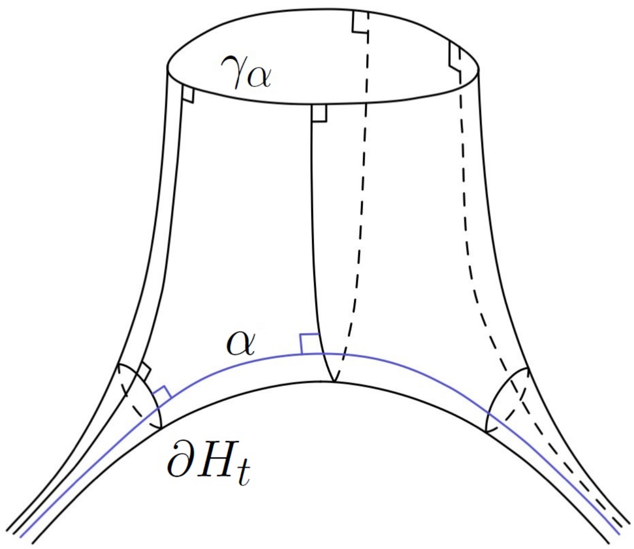

Figure 2.5: *The pre-image of an infinite arc α in the generalised pair of pants P, with the perpendiculars which we cut along.*

Proof. Let α be an infinite arc, and let t_α be given by Lemma 1.2.6. We will start by proving the lemma in the case that $t \leq t_\alpha$. Then we will demonstrate that for $t > t_\alpha$, the difference between $\ell_X^t(\alpha)$ and $\ell_X^{t_\alpha}(\alpha)$ is uniformly bounded across all arcs with the same self-intersection number, and so complete the proof.

Suppose that $t \leq t_\alpha$. Equip the generalised pair of pants P with a metric using the pullback of X through ι_α. Cut P along four geodesic arcs: the pre-image of α, the perpendicular compact simple geodesic arc from the boundary component to itself, and the two simple geodesic rays between the boundary component and the cusps. These geodesics are highlighted in Figure 2.5. We are left with 4 isometric copies of a quadrilateral with three right angles and one ideal vertex, which we label as in Figure 2.6.

Consider this quadrilateral in the upper-half space model for $\mathbb{H}^2$ and normalise it such that the ideal vertex is at ∞ and the edges incident to it are on the lines $x = 0$ and $x = 1$. We will write $\ell_{\mathbb{H}^2}(\cdot)$ for the length of a path in $\mathbb{H}^2$. Since $t \leq t_\alpha$, we have that the edge qw has length $\ell_{\mathbb{H}^2}(qw) = \frac{1}{2}\ell_X(\alpha^t)$, and the edge uv has length $\ell_{\mathbb{H}^2}(uv) = \frac{1}{4}\ell_X(\gamma_\alpha)$. As the length of the boundary of the cuspidal region of volume t is t, the length of the segment which lives in this quadrilateral is $\frac{t}{2}$. Therefore it lies on the line $y = \frac{2}{t}$.

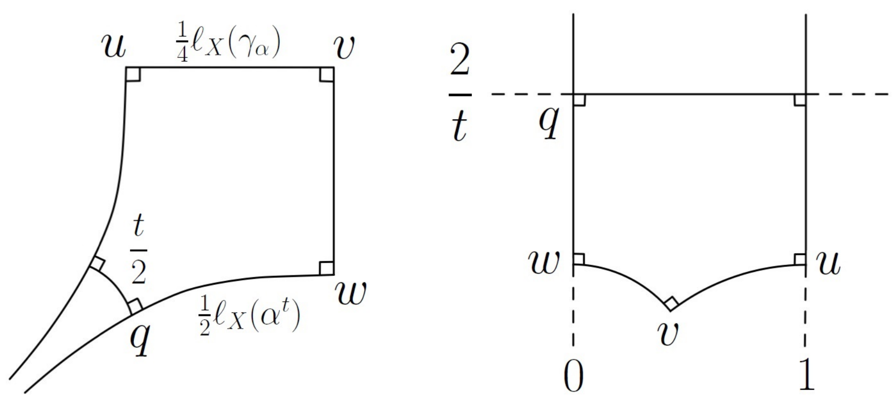

Figure 2.6: *One of the quadrilaterals acquired from cutting P (left), and the same quadrilateral in the upper-half plane model after normalising (right).*

Since q lies on the line $y = \frac{2}{t}$ and $\ell_{\mathbb{H}^2}(qw) = \frac{1}{2}\ell_X(\alpha^t)$, we have that $w = \frac{2}{t}i\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)}$. Hence the edges vw and uv lie on circles C_1 and C_2 defined by the equations

$$\begin{aligned} x^2 + y^2 &= \left(\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)}\right)^2 \\ (x-1)^2 + y^2 &= 1 - \left(\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)}\right)^2 \end{aligned}$$

respectively. We can then compute u as the solution to the simultaneous equations given by C_2 and $x = 1$ with positive y-coordinate. Similarly, we can compute v from C_1 and C_2. We then have that as complex numbers, $u = 1 + i\sqrt{1 - (\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)})^2}$ and $v = (\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)})^2 + i\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)}\sqrt{1 - (\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)})^2}$. We calculate the distance between u and v by mapping the geodesic on which the edge uv lies isometrically to the imaginary axis. The length of uv after applying this isometry is

$$\begin{aligned} \ell_{\mathbb{H}^2}(uv) &= \ln\left(\frac{\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)}}{1 - \sqrt{1 - (\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)})^2}}\right) \\ &= \cosh^{-1}\left(\frac{t}{2}\mathrm{e}^{\frac{1}{2}\ell_X(\alpha^t)}\right). \end{aligned}$$

as $(\frac{2}{t}\mathrm{e}^{-\frac{1}{2}\ell_X(\alpha^t)})^{-1} \geq 1$. Moreover, as $\ell_{\mathbb{H}^2}(uv) = \frac{1}{4}\ell_X(\gamma_\alpha)$ by construction, we can thus express the length of γ_α as

$$\ell_X(\gamma_\alpha) = 4\cosh^{-1}\left(\frac{t}{2}\mathrm{e}^{\frac{1}{2}\ell_X(\alpha^t)}\right).$$

Hence, the difference $\ell_X(\gamma_\alpha) - 2\ell_X(\alpha^t)$ can be written as

$$4\cosh^{-1}\left(\frac{t}{2}\mathrm{e}^{\frac{1}{2}\ell_X(\alpha^t)}\right) - 2\ell_X(\alpha^t).$$

The function $E_t(\ell) = 4\cosh^{-1}(\frac{t}{2}\mathrm{e}^{\frac{1}{2}\ell}) - 2\ell$ is continuous on $[m_t, \infty)$, where $m_t := 2\ln(\frac{2}{t})$ is a lower bound on the length of α^t, and $\lim_{\ell\to\infty} E_t(\ell) = 4\ln(t)$. It follows that there exists $C_1(t) > 0$ such that

$$|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha^t)| \leq C_1(t). \tag{2.7}$$

Now suppose that $t > t_\alpha$. Note that α^t is contained in α^{t_α}; thus we have that $\ell_X^t(\alpha) < \ell_X^{t_\alpha}(\alpha)$. Consider $\alpha^{t_\alpha} \setminus \alpha^t$, which lies in $\mathscr{H}_t \setminus \mathscr{H}_{t_\alpha}$. Exactly two of the components of $\alpha^{t_\alpha} \setminus \alpha^t$ are simple geodesic arcs from $\partial\mathscr{H}_t$ to $\partial\mathscr{H}_{t_\alpha}$ which meet each boundary orthogonally. Hence, these two components each have length $\ln(\frac{t}{t_\alpha}) \leq \ln(\frac{1}{t_\alpha})$. The other components, if any, are returning segments in $\mathscr{H}_t$ with both endpoints on $\partial\mathscr{H}_t$. Let β be some such segment of α, and let $d = \iota(\alpha, \alpha)$. Then we must have $\iota(\beta, \beta) \leq d$. Thus, as $t \leq 1$, $\ell_X(\beta) \leq B(d)$ where B is given by Lemma 1.2.4. As B only depends on d, this holds for any such segment. Now we need to show that there are only finitely many segments of α in $\mathscr{H}_t \setminus \mathscr{H}_{t_\alpha}$. From Lemma 1.2.7, we have that the self-intersection number of each segment is at least 1, and indeed the sum of the self-intersection numbers of these segments is at most d. Hence, $\alpha \cap (\mathscr{H}_t \setminus \mathscr{H}_{t_\alpha})$ has at most $d + 2$ components and so

$$\ell_X^{t_\alpha}(\alpha) - \ell_X^t(\alpha) \leq dB(d) + 2\ln\left(\frac{t}{t_\alpha}\right) \leq dB(d) + 2\ln\left(\frac{1}{t_\alpha}\right).$$

Note that by Lemma 1.2.6, t_α depends only on $d = \iota(\alpha, \alpha)$. Thus there exists some $C_2\big(\iota(\alpha, \alpha)\big) > 0$ such that

$$|\ell_X^{t_\alpha}(\alpha) - \ell_X^t(\alpha)| \leq C_2\big(\iota(\alpha, \alpha)\big). \tag{2.8}$$

Now by applying (2.7) to t_α, we have that $|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha^{t_\alpha})| \leq C_1(t_\alpha)$. Combining this with (2.8), we can write

$$|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha^t)| \leq C_1(t_\alpha) + 2C_2\big(\iota(\alpha, \alpha)\big).$$

Therefore, for any $t \leq 1$,

$$|\ell_X(\gamma_\alpha) - 2\ell_X(\alpha^t)| \leq C\big(\iota(\alpha, \alpha), t\big)$$

where

$$C\big(\iota(\alpha, \alpha), t\big) = \begin{cases} C_1(t) & \text{if } t \leq t_\alpha, \\ C_1(t_\alpha) + 2C_2\big(\iota(\alpha, \alpha)\big) & \text{if } t > t_\alpha. \end{cases}$$

□

The fact that I defined on infinite arcs is $\mathrm{PMod}(S)$-equivariant holds by an argument analogous to the proof of Lemma 2.2.3. That is, for any infinite arc α and any $\varphi \in \mathrm{PMod}(S)$,

$$I(\varphi \cdot \alpha) = \varphi \cdot I(\alpha). \tag{2.9}$$

For any infinite arc α_0, $I_{\alpha_0} \colon \mathrm{PMod}(S) \cdot \alpha_0 \to \mathrm{PMod}(S) \cdot \gamma_{\alpha_0}$ is the restriction of I to $\mathrm{PMod}(S) \cdot \alpha_0$. Given any curve γ of type γ_{α_0} and a fixed value of t, there are only finitely many arcs α of type α_0 such that $\ell_X(\alpha^t) \leq \frac{1}{2}\ell_X(\gamma) + \frac{1}{2}C\big(\iota(\alpha_0, \alpha_0), t\big)$. By Lemma 2.3.1, this means that there are only finitely many arcs α of type α_0 such that $\gamma_\alpha = \gamma$. Using this together with (2.9), Proposition 2.2.4 holds for I_{α_0} by an analogous argument.

Proposition 2.3.2. *Let α_0 be an infinite arc. Then there exists some $k = k(\alpha_0)$ such that I_{α_0} is surjective and k-to-1.*

Armed with this, we can follow the argument from the proof of Theorem 1 to prove Theorem 2. That is, we can demonstrate that the growth of the number of infinite arcs of a given type of bounded t-length is polynomial in the length.

Proof of Theorem 2. Let α_0 be an infinite arc, and fix some positive $t \leq 1$. Let γ_{α_0} be the curve associated to α_0, as in (2.6). Using the same argument as in the proof of Theorem 1, replacing Lemma 2.2.1 and Proposition 2.2.4 with Lemma 2.3.1 and Proposition 2.3.2, we have

$$\begin{aligned}
&\lim_{L\to\infty} \frac{|\{\alpha \text{ of type } \alpha_0 \mid \ell_X^t(\alpha) \leq L\}|}{L^{6g-6+2(n+p)}} \\
&\qquad = k \cdot 2^{6g-6+2(n+p)} \lim_{L\to\infty} \frac{|\{\gamma \text{ of type } \gamma_{\alpha_0} \mid \ell_X(\gamma) \leq L\}|}{L^{6g-6+2(n+p)}} \\
&\qquad = k \cdot 2^{6g-6+2(n+p)} \mathfrak{c}(\gamma_{\alpha_0}) \mathfrak{m}(X) \\
&\qquad = k(\alpha_0) \mathfrak{c}(\alpha_0) \mathfrak{m}(X)
\end{aligned}$$

where $\mathfrak{c}(\alpha_0) = 2^{6g-6+2(n+p)}\mathfrak{c}(\gamma_{\alpha_0})$, $k(\alpha_0)$ is as in Proposition 2.3.2, and $\mathfrak{c}(\gamma_{\alpha_0})$ and $\mathfrak{m}(X)$ are as in Mirzakhani's Theorem. □

Remark: As previously mentioned, Theorem 2 holds when we replace the t-length by the truncated length ℓ_X^{Tr} (see Corollary 1.4.2). This can be seen by applying Theorem 2 in the case that $t = t_{\alpha_0}$, and using the bound on the difference in the t_{α_0}-length and the truncated length from (1.4).

2.4 Simple two-ended arcs have no associates

As a final comment before we leave this chapter, we document a proof that $k(\alpha_0) = 1$ in a special case which requires no further technical work. Recall that we call an arc *two-ended* if it is between two distinct boundary components on S, that is, such that $\delta_0^\alpha \neq \delta_1^\alpha$. Otherwise, we call it *one-ended*. Working directly from the definition of arcs and their associated curves, we can demonstrate that if α is simple and two-ended, then it has no associate arc. In other words, there is no other arc on S associated to the same curve as α, and thus the association map I_α is 1-to-1 (see Theorem 2.4.4). In Theorem 1, this will correspond to $k(\alpha) = 1$. Our first goal is to demonstrate that if α is simple and two-ended then the curve associated to it is simple, by relating the self-intersection number of an arc to that of the curve associated to it. This is done by giving a more hands-on construction of a representative of γ_α and finding its self-intersection number.

To begin, let α be any (not necessarily simple) orthogeodesic arc. The representative of γ_α will be constructed using the following method, which we refer to as the "*hoop-and-stick*" construction. Choose two positive parameters ε and δ and imagine standing on the arc α at the endpoint $\alpha(0)$ with two objects; a stick of length ε and a hoop covered with paint. If $\delta_0^\alpha = \delta_1^\alpha$, take a step of length δ forwards before proceeding. Balance the hoop at the end of your stick on your left and push it away from you, following the orientation of δ_0^α. As it comes back around from your right, stop it at the end of your stick, leaving a 2ε gap between the endpoints of the path up to this point. Now walk along α to $\alpha(1)$, pushing the hoop along at the end of your stick. In this manner, we draw out a copy of α which maintains a constant distance of ε from the original. Upon reaching $\alpha(1)$, stop the hoop and turn around, before rolling the hoop around δ_1^α as you did in the first step. Note that we do not take a step forward at this stage. Finally, proceed to walk back along α to $\alpha(0)$, rolling the hoop on your right again. Once you reach $\alpha(0)$, the hoop has reached its original start point and we have drawn out a closed loop on S. This loop, denoted γ_α^{hs}, is homotopic to γ_α, and we call this the *hoop-and-stick representative* of γ_α. In other words, we have formed an immersed version of a regular neighbourhood of the arc together with its boundary components. Note that choosing large values of ε

or δ could cause the curve to behave unexpectedly; we always consider sufficiently small values of each parameter.

Consider any self-intersection point of α. In a small neighbourhood around this point, we see that γ_α^{hs} draws out two strips of width 2ε around each segment of α in the neighbourhood, which cross exactly once. Thus at each self-intersection point of α, the hoop-and-stick curve self-intersects exactly 4 times. If δ_0^α and δ_1^α are distinct, this is the only case in which γ_α^{hs} self-intersects. Otherwise, γ_α^{hs} crosses itself exactly twice more; this occurs once when the first copy of α drawn out crosses the original copy of δ_0^α, then again when the second copy of α crosses near the first point of self-intersection. Thus we have that

$$\iota(\gamma_\alpha^{hs}, \gamma_\alpha^{hs}) = 4\iota(\alpha, \alpha) + 2\Delta$$

where

$$\Delta = \begin{cases} 1 & \text{if } \delta_0^\alpha = \delta_1^\alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, the self-intersection number of γ_α is bounded from above by this value. We record this here for reference.

Proposition 2.4.1. *Let α be an arc on S. Then*

$$\iota(\gamma_\alpha, \gamma_\alpha) \leq 4\iota(\alpha, \alpha) + 2\Delta$$

where $\Delta = 1$ if $\delta_0^\alpha = \delta_1^\alpha$ and $\Delta = 0$ otherwise.

From this we can deduce that the curve associated to a simple two-ended arc is itself simple. We can in fact demonstrate that the converse also holds, which we outline here.

Proposition 2.4.2. *Let α be a two-ended arc on S. Then γ_α is simple if and only if α is simple.*

Proof. As a result of Proposition 2.4.1, we have that if α is simple then γ_α must be simple.

For the other direction, suppose that α is two-ended and γ_α is simple. We will use some standard facts about homology, and refer the reader to any textbook on the subject, such as [7, 13]. We denote by $\iota^*(\cdot,\cdot)$ the algebraic intersection number between two objects, and recall that the algebraic intersection number depends only on homology.

As the curves $\gamma_\alpha, \delta_0^\alpha$ and δ_1^α bound some immersed subsurface X, we have that γ_α is homologous to the sum $\delta_0^\alpha + \delta_1^\alpha$, with the orientation inherited from the surface. We claim that X is in fact an embedded subsurface.

To see this, first note that γ_α is separating. If it were not, we could cut along γ_α and find a simple arc in the resulting cut surface which would define a curve η in S such that $\iota^*(\gamma_\alpha, \eta) = 1$ but $\iota^*(\delta_0^\alpha + \delta_1^\alpha, \eta) = 0$. Suppose that δ_0^α and δ_1^α are not in the same connected component of the cut surface $S \setminus \gamma_\alpha$. Then we can find an arc β in S which joins δ_0^α and δ_1^α and crosses γ_α exactly once, such that $\iota^*(\gamma_\alpha, \beta) = 1$ but $\iota^*(\delta_0^\alpha + \delta_1^\alpha, \beta) = 0$. Thus γ_α, δ_0^α and δ_1^α are boundary components of the same component of $S \setminus \gamma_\alpha$, which is X. Suppose X has a boundary component θ which is none of these. Then we can find an arc β joining δ_0^α and θ such that $\iota^*(\gamma_\alpha, \beta) = 0$ but $\iota^*(\delta_0^\alpha + \delta_1^\alpha, \beta) = 1$. So no such boundary component θ exists. As a result, we can say that the subsurface X is embedded.

Here, we specify that α is the orthogeodesic representative of its homotopy class. We know that part of α lies in X, as it joins δ_0^α and δ_1^α. Suppose α does not lie entirely within X. Then α must intersect γ_α. Recall the hoop-and-stick representative γ_α^{hs} of γ_α from the discussion above Proposition 2.4.1. As α and γ_α intersect, so too do γ_α^{hs} and γ_α. Suppose there exist bigons between γ_α^{hs} and γ_α. Then each such bigon implies the existence of a bigon between the geodesics α and γ_α, which is impossible. Thus γ_α^{hs} and γ_α must have non-trivial geometric intersection number, and therefore γ_α has non-trivial geometric self-intersection number. However, γ_α is a simple curve, so we have a contradiction. Thus α must lie entirely within X.

We can therefore restrict our focus to the subsurface X, and possibly by applying a homotopy we have that the map $\iota_\alpha \colon P \to X$ is surjective. In fact, this map ι_α is an immersion, and thus has a well-defined degree $s \in \mathbb{Z}_+$. We can express this degree as

$$s = \frac{\text{area of } P}{\text{area of } X}$$

and as the hyperbolic surface with three boundary components of minimal area is the pair of pants, we must have that $s = 1$. Therefore, ι_α is a proper immersion of degree 1, and hence is an embedding. In particular, α must be simple as it is the image of the seam between δ_0^P and δ_1^P. □

The argument that we will set out relies on the Isotopy Extension Theorem, which we state here for reference.

Isotopy Extension Theorem (Theorem 1.3, [16])**.** *Let γ be an embedding of the circle into S, and consider some isotopy h of γ. Then h can be extended to the entire surface S.*

We need to verify a short lemma in which we apply the Isotopy Extension Theorem.

Lemma 2.4.3. *Let γ be a simple curve on S, and let $\varphi \in \mathrm{PMod}(S)$ such that $\varphi \cdot \gamma = \gamma$. Then for any simple representative γ' of the homotopy class of γ, there exists a representative φ' of the homotopy class of φ such that $\varphi'(\gamma') = \gamma'$ point-wise.*

Proof. Abusing notation, we choose some representative of the mapping class φ and denote it by φ. By the definition of $\varphi \cdot \gamma = \gamma$, there exists a homotopy h_t from $\varphi(\gamma')$ to γ' for any representative γ' of the homotopy class of γ. We can choose this homotopy such that $h_1 \circ \varphi$ fixes γ' point-wise.

Note that since γ is simple, then as long as we choose a representative which does not self-intersect, we can take h_t to in fact be an isotopy. Therefore, we can apply the Isotopy Extension Theorem to γ' and h_t, extending h_t to an isotopy of S, denoted $\tilde{h_t}$. Now, the map φ' given by

$$\varphi' := \tilde{h_1} \circ \varphi$$

applies φ before moving the image of γ' back onto itself by isotopy. Explicitly, $\varphi'(\gamma') = \tilde{h_1}(\varphi(\gamma')) = \gamma'$ point-wise. Furthermore, φ' is indeed homotopic to φ via the homotopy $\tilde{h}'_t := \tilde{h_t} \circ \varphi$, and so we are done. □

Using this, we can demonstrate that no two simple two-ended arcs of the same type are associated to the same curve under I on any surface S. Note that associate arcs of the first kind (as in Figure 2.2) are never of the same type under the action of $\mathrm{PMod}(S)$.

Theorem 2.4.4. *Let α be a simple two-ended arc on some hyperbolic surface S. Then the association map I_α is 1-to-1.*

Proof. Recall from Lemma 2.2.3 that the association map I_α is PMod(S)-equivariant. That is, $I_\alpha \circ \varphi = \varphi \circ I_\alpha$ for any pure mapping class φ. Let γ_α be the curve associated to α, and recall that by Proposition 2.4.2 it is simple.

Let α_1, α_2 be associate arcs of type α such that $\gamma_{\alpha_1} = \gamma_{\alpha_2}$; equivalently, $I_\alpha(\alpha_1) = I_\alpha(\alpha_2)$. Since they are of type α, by definition we can write

$$\alpha_1 = \varphi_1 \cdot \alpha$$
$$\alpha_2 = \varphi_2 \cdot \alpha$$

for some pure mapping classes φ_1, φ_2. Thus we have

$$\begin{aligned} I_\alpha(\alpha_1) = I_\alpha(\alpha_2) &\implies I_\alpha(\varphi_1 \cdot \alpha) = I_\alpha(\varphi_2 \cdot \alpha) \\ &\implies \varphi_1 \cdot I_\alpha(\alpha) = \varphi_2 \cdot I_\alpha(\alpha) \\ &\implies \varphi_1 \cdot \gamma_\alpha = \varphi_2 \cdot \gamma_\alpha \end{aligned}$$

using the equivariance of I_α with respect to PMod(S). We can rearrange this last statement to give $(\varphi_2^{-1} \circ \varphi_1) \cdot \gamma_\alpha = \gamma_\alpha$, leaving us in a position to apply Lemma 2.4.3. The goal here is to find a representative of $\varphi_2^{-1} \circ \varphi_1$ which fixes a well-chosen representative of γ_α which contains α as a segment, and thus this representative will also fix α. To construct our representative of γ_α, follow the hoop-and-stick construction detailed above Proposition 2.4.1 with a slight alteration. After drawing out the first copy of δ_0^α, allow the hoop to come all the way to α before pushing it along in front of you along α. Upon reaching $\alpha(1)$, continue as in the original method. We now have a homotopic representative of γ_α which contains α as a segment, which we will denote γ_α^{hs*}. We know that γ_α is simple by Proposition 2.4.1, and γ_α^{hs*} does not self-intersect. Hence we can apply Lemma 2.4.3 to construct a representative $(\varphi_2^{-1} \circ \varphi_1)'$ of the mapping class $\varphi_2^{-1} \circ \varphi_1$ such that

$$(\varphi_2^{-1} \circ \varphi_1)'(\gamma_\alpha^{hs*}) = \gamma_\alpha^{hs*}$$

point-wise, and as such we also have that $(\varphi_2^{-1} \circ \varphi_1)'(\alpha) = \alpha$ point-wise. Therefore we can write

$$(\varphi_2^{-1} \circ \varphi_1) \cdot \alpha = \alpha$$

which further implies that $\varphi_1 \cdot \alpha = \varphi_2 \cdot \alpha$. Thus by definition, $\alpha_1 = \alpha_2$ and the map I_α is 1-to-1. □

This proof does not extend to more general arcs, as the Isotopy Extension Theorem plays a key role here and a more general statement does not hold for strict homotopies. As mentioned previously, in Chapter 3 we lay out the tools required to study this problem in a general setting.

Chapter 3

Pairs of associate arcs

This chapter is dedicated to the study of associate arcs; arcs whose images under the association map I are equal. Whilst the definition of the association is purely topological, we will find ourselves using more discrete and combinatorial methods to study such arcs. Using these, we describe all instances of two-ended associate arcs on the pair of pants P, and demonstrate that no more than 2 two-ended arcs are mapped to the same curve.

Theorem 3. *Let P be a pair of pants, and let α be a two-ended arc on P. Let $\gamma_\alpha = I(\alpha)$. Then $|I^{-1}(\gamma_\alpha)| \leq 2$.*

Future work will focus on demonstrating that Theorem 3 implies that the same fact holds when α is one-ended, or on any surface with boundary.

First, in Section 3.1 we define a correspondence between arcs and words in generators of the fundamental group of the pair of pants P, which is the crux of the rest of the chapter. In Section 3.2, we lay out the key definitions and terminology we will need moving forward, culminating in a proposed system for labelling instances of associate arcs on P. Section 3.3 contains the bulk of the work, using the tools we gave ourselves in Section 3.2 to demonstrate that our labels - what we will define as *frames* - are in one-to-one correspondence with pairs of associate arcs. Finally in Section 3.4, we discuss what this labelling tells us about associate arcs, as well as a geometric point of view which corroborates our findings. We will also discuss the potential for generalising this work to all surfaces with boundary.

3.1 Characterising arcs in P using $\pi_1(P)$

Recall from Section 1.5 that P is a pair of pants with boundary components δ_0^P, δ_1^P and δ_2^P. We fix a basepoint $p_0 \in P$ and consider $\pi_1(P) \coloneqq \pi_1(P, p_0)$, which is isomorphic to the free group F_2.

To start, we comment that arcs between different pairs of boundary components cannot be associated to the same curve. The bulk of the chapter will then deal with the case of arcs joining the same pair of boundary components.

A curve associated to a two-ended arc on P is homologous to the third boundary component. If the arc is instead one-ended, then the associated curve is homologous to twice the corresponding boundary component. As a result, we have the following.

Lemma 3.1.1. *Let α and β be arcs on P which do not join the same pair of boundary components. Then $\gamma_\alpha \neq \gamma_\beta$.*

In the following, we will restrict our focus to two-ended arcs between the same pair of boundary components.

Definition 3.1.2: For any $i, j \in \{0, 1, 2\}$ where $i \leq j$, let $\mathcal{A}_{i,j}(P)$ be the set of homotopy classes of (unoriented) arcs between δ_i^P and δ_j^P, and let $I_{i,j}$ be the map $I_{i,j} \colon \mathcal{A}_{i,j}(P) \to \mathscr{C}(P)$ defined by $I_{i,j}(\alpha) = I(\alpha)$.

We now introduce a method for identifying each arc in $\mathcal{A}_{i,j}(P)$ for $i \neq j$ with a word in generators of $\pi_1(P)$ of a particular form. This will prove consistently useful in studying arcs associated to the same curve. As discussed in Section 2.1 (see (2.1)), an arc on a surface S corresponds to some continuous map of the pair of pants P inside S. Thus when $S = P$, we have a continuous map $P \to P$.

Let $i, j \in \{0, 1, 2\}$ such that $i \neq j$. Throughout the following, α will be a (two-ended) arc in $\mathcal{A}_{i,j}(P)$. For any arc $\alpha \in \mathcal{A}_{i,j}(P)$, we have a continuous map $P \to P$ fixing δ_i^P and δ_j^P, up to homotopy. Fix the generators a and b of $\pi_1(P)$, where a and b are loops on P freely homotopic to δ_i^P and δ_j^P respectively. The images of a and b under these continuous maps must be homotopic to a and b respectively, and hence their images under the induced homomorphisms are conjugate to a and b respectively. For any arc α between δ_i^P and δ_j^P, we denote the homomorphism ι_α^* induced by ι_α as ϕ_α for ease of notation. We can then write

$$\begin{aligned} \phi_\alpha \colon a &\mapsto w_a a w_a^{-1} \\ b &\mapsto w_b b w_b^{-1} \end{aligned} \tag{3.1}$$

for some $w_a, w_b \in \pi_1(P)$; that is, w_a and w_b are both words in a and b. Since ϕ_α is a homomorphism, its action on the generators determines its action on the entirety of $\pi_1(P)$, and thus these two words w_a and w_b fully describe ϕ_α. Recall that by Corollary 1.5.2, two continuous maps ι_1 and ι_2 of P are homotopic if and only if their induced homomorphisms are conjugate. We can use this to give an example of two arcs in P which are associated to the same curve.

Example 3.1.3: Consider the arcs α_1 and α_2 such that the corresponding continuous maps ι_{α_1} and ι_{α_2} induce the homomorphisms ϕ_1 and ϕ_2 respectively, given as

$$\begin{aligned} \phi_1 \colon a &\mapsto a & \qquad \phi_2 \colon a &\mapsto a \\ b &\mapsto b^{-1}aba^{-1}b & b &\mapsto ba^{-1}bab^{-1}. \end{aligned}$$

These two maps are not conjugate by any word, hence by Corollary 1.5.2 the corresponding pairs of pants are not homotopic. Thus α_1 and α_2 are distinct arcs. Recall that we defined the association of a curve γ_α to an arc α in Section 2.1. As in (2.2), the curves associated to α_1 and α_2 are given by $\iota_{\alpha_1}(\delta_2^P)$ and $\iota_{\alpha_1}(\delta_2^P)$. As δ_2^P corresponds to the word $b^{-1}a^{-1}$ in our chosen generators, these curves correspond to the elements $\phi_1(b^{-1}a^{-1})$ and $\phi_2(b^{-1}a^{-1})$. We can compute these as

$$\begin{aligned} \phi_1(b^{-1}a^{-1}) &= b^{-1}ab^{-1}a^{-1}ba^{-1}, \\ \phi_2(b^{-1}a^{-1}) &= ba^{-1}b^{-1}ab^{-1}a^{-1}. \end{aligned}$$

Recall that curves are freely homotopic if their corresponding words in the fundamental group are conjugate. We can see that conjugating $\phi_1(b^{-1}a^{-1})$ by ba^{-1} gives us

$$(ba^{-1})b^{-1}ab^{-1}a^{-1}ba^{-1}(ba^{-1})^{-1} = ba^{-1}b^{-1}ab^{-1}a^{-1}$$

which is $\phi_2(b^{-1}a^{-1})$. Hence $\iota_{\alpha_1}(\delta_2^P)$ and $\iota_{\alpha_2}(\delta_2^P)$ are homotopic, and so $\gamma_{\alpha_1} = \gamma_{\alpha_2}$.

Note that α_1 and α_2 are given in Figure 2.3, and their associated curve is in Figure 2.4. The reader may find it illuminating to draw stages of the homotopy between γ_{α_1} and γ_{α_2}, noting that this does not give rise to a homotopy between the corresponding immersed pairs of pants.

As described above, the homomorphism ϕ_α induced by the continuous map corresponding to an arc α can be fully described by the two words w_a and w_b, as in

(3.1). However, we can improve this by combining them to form a single word which also fully describes ϕ_α.

By Corollary 1.5.2, we can conjugate ϕ_α by any word and the resulting homomorphism will be induced by a continuous map of a pair of pants freely homotopic to ι_α, and hence by an arc homotopic to α relative to the boundary. We can therefore conjugate ϕ_α by w_α^{-1} to obtain a simpler map which fixes a and conjugates b by the word $w_a^{-1}w_b$ but still corresponds to α. Moreover, there is always such a word of minimal word length which is unique by construction. Thus we need only consider maps which have been *reduced* in this way.

To demonstrate this more concretely, consider all maps ϕ of the form given in (3.1) and define an equivalence relation on the set of these maps by $\phi_1 \sim \phi_2$ if and only if for some $w \in \pi_1(P)$, $\phi_2 = w\phi_1 w^{-1}$. As homomorphisms are equivalent under this relation exactly when they are conjugate, each homotopy class of arcs corresponds to a unique equivalence class. Let ϕ be of the form given in (3.1). Then we can reduce ϕ by conjugating by w_a^{-1} and performing any necessary cancellation in the words $w_a^{-1}w_b$ and $w_b^{-1}w_a$. In any equivalence class, there is a representative which fixes a and conjugates b by a word of minimal syllable length.

Definition 3.1.4: A homomorphism $\phi\colon \pi_1(P) \to \pi_1(P)$ is *reduced* if it takes the form

$$\begin{aligned} \phi\colon a &\mapsto a \\ b &\mapsto wbw^{-1} \end{aligned}$$

for some word w which has minimal syllable length across homomorphisms in the equivalence class of ϕ which fix a.

We will see below that there is a unique reduced map in each equivalence class. However, we note that we have made a choice here; one could equally define reduced maps to fix b, in which case we would focus on the word w^{-1} instead of w. Our results would remain the same with this change.

These reduced maps must in fact take a particular form, which we will then use to see they are unique.

Lemma 3.1.5. *Let ϕ be a reduced homomorphism which conjugates b by w. Then w is cyclically reduced, and takes the form*

$$w = b^{k_1} a^{l_1} \dots b^{k_n} a^{l_n}$$

for some positive integer n and some non-zero integers $k_1, \dots, k_n, l_1, \dots, l_n$.

Proof. First, we demonstrate that w must begin with a b-syllable and end with an a-syllable. If w does not end with an a-syllable then it ends with a b-syllable; for some word w' and integer K, $w = w'b^K$. We can write

$$\begin{aligned}\phi(b) &= wbw^{-1} = w'b^K b b^{-K} w'^{-1} \\ &= w'bw'^{-1},\end{aligned}$$

hence the map ϕ' which fixes a and conjugates b by w' is a reduced map equivalent to ϕ, as $\phi = \phi'$. However, w' is shorter in syllable length than w, contradicting the fact that ϕ is reduced. Similarly, if w does not begin with a b-syllable then it begins with an a-syllable, and conjugating ϕ by the inverse of this syllable gives an equivalent map which fixes a and conjugates b by a shorter word, so again we find a contradiction.

Suppose that w is not cyclically reduced. If w is not a reduced word, then after performing all cancellations we have a shorter word which defines exactly the same map, contradicting the fact that ϕ was a reduced map. Since w must start and end with different letters as shown above, there is no cyclic cancellation to perform and we are done. □

Consider an equivalence class of these maps and let ϕ and ϕ' be reduced maps from this class. As they are equivalent there is some word z such that $\phi' = z\phi z^{-1}$. However, as both maps are reduced, they both fix a. Therefore $z = a^l$ for some integer l. This means that the word by which ϕ' conjugates b begins with an a-syllable, which cannot happen by Lemma 3.1.5. So the only possibility for the word z is the empty word, and hence $\phi' = \phi$. We have proved the following.

Lemma 3.1.6. *There is a unique reduced map in each equivalence class of homomorphisms.*

As a consequence, we can relate these reduced maps, and therefore the words defining them, bijectively with arcs. Beginning with an arc α on P, we can find the corresponding continuous map ι_α of P inside itself which is unique up to homotopy. This in turn gives a unique reduced homomorphism ϕ_α on the fundamental group, defined by a unique word w_α of the form given by Lemma 3.1.5. This word is called the *conjugator* corresponding to α. For future reference we define the set of conjugators

$$\mathfrak{W} := \{w = b^{k_1}a^{l_1} \dots b^{k_n}a^{l_n} \mid n \in \mathbb{Z}_{>0}, k_1, \dots, k_n, l_1, \dots, l_n \in \mathbb{Z}_{\neq 0}\}. \tag{3.2}$$

Note that all $w \in \mathfrak{W}$ are cyclically reduced.

Definition 3.1.7: Let α be an arc on P from δ_i^P to δ_j^P. The *conjugator corresponding to* α is the word $w_\alpha \in \mathfrak{W}$ which defines the unique reduced homomorphism ϕ_α induced by the continuous map ι_α.

By Corollary 1.5.2, arcs which correspond to the same conjugator w are homotopic relative to the boundary, thus no two distinct homotopy classes of arcs can correspond to the same word. Furthermore, every conjugator w defines a homomorphism of the fundamental group and every homomorphism $\pi_1(P) \to \pi_1(P)$ is induced by some continuous map $P \to P$ (see for example Proposition 1B.9 in [13]). The map which induces the homomorphism defined by w will fix δ_i^P and δ_j^P and thus defines an arc on P. We have proved the following.

Theorem 3.1.8. *There is a bijective correspondence between arcs in $\mathcal{A}_{i,j}(P)$ and conjugators $w \in \mathfrak{W}$.*

We will freely swap between an arc α and its corresponding word w_α.

3.2 Signatures and Frames

In Section 2.1, we defined the association map I from arcs to curves, as in (2.3). Recall that the (homotopy class of the) curve associated to an arc α between boundary components δ_i^α and δ_j^α is given by the concatenated path

$$\alpha^{-1} \cdot \delta_j^\alpha \cdot \alpha \cdot \delta_i^\alpha.$$

In this section, we will use the characterisation of arcs as conjugators from Theorem 3.1.8 to study when two arcs in $\mathcal{A}_{i,j}(P)$ can be associated to the same curve under I. Let P be a pair of pants. As in Section 3.1, we fix generators a and b for $\pi_1(P)$ homotopic to the boundary components δ_i^P and δ_j^P respectively. The principal use of this tool will be directly computing the word in $\pi_1(P)$ which gives the curve associated to an arc. We will then be able to directly check whether the curves associated to two different arcs are homotopic. Throughout the following, if α is an arc, then the homomorphism $\phi_\alpha\colon \pi_1(P) \to \pi_1(P)$ is defined as

$$\begin{aligned} \phi_\alpha\colon a &\mapsto a \\ b &\mapsto w_\alpha b w_\alpha^{-1} \end{aligned}$$

where w_α is given by Theorem 3.1.8.

Consider the simple arc α_P on P given by the seam between δ_i^P and δ_j^P. This arc is associated to the third cuff of P, which is freely homotopic to the element $b^{-1}a^{-1}$ in $\pi_1(P)$. Similarly, an arc α is associated to the curve corresponding to $\phi_\alpha(b^{-1}a^{-1})$. The map ϕ_α is a *reduced* map of the form found in Definition 3.1.4; it fixes a and conjugates b by the word w_α. Recall that two curves on a surface are (freely) homotopic if their corresponding words in the fundamental group are conjugate. Thus for any pair of arcs α_1 and α_2, we can encode the statement $I(\alpha_1) = I(\alpha_2)$ (or equivalently $\gamma_{\alpha_1} = \gamma_{\alpha_2}$) as

$$[\phi_{\alpha_1}(b^{-1}a^{-1})] = [\phi_{\alpha_2}(b^{-1}a^{-1})],$$

where $[\cdot]$ stands for the conjugacy class of an element in $\pi_1(P)$. As these maps are homomorphisms and $\phi_{\alpha_i}(a) = a$, we can write

$$\phi_{\alpha_i}(b^{-1}a^{-1}) = \phi_{\alpha_i}(b^{-1})a^{-1} = \phi_{\alpha_i}(b)^{-1}a^{-1}.$$

By Lemma 3.1.5, the conjugators w_{α_1} and w_{α_2} are cyclically reduced; in particular, they are reduced words beginning with b and ending with a. Thus the words

$$\phi_{\alpha_1}(b^{-1}a^{-1}) = w_{\alpha_1} b^{-1} w_{\alpha_1}^{-1} a^{-1}$$

$$\phi_{\alpha_2}(b^{-1}a^{-1}) = w_{\alpha_2}b^{-1}w_{\alpha_2}^{-1}a^{-1}$$

take the same form. Recall that $\pi_1(P) \simeq F_2$, and that in free groups, cyclically reduced words are conjugate if and only if they are cyclic permutations of each other. A cyclic permutation of a word permutes the letters in a cyclic fashion; for example, we remove a letter from the end of the word and replace it at the beginning. Since $\phi_{\alpha_1}(b^{-1}a^{-1})$ and $\phi_{\alpha_2}(b^{-1}a^{-1})$ both start with a b-syllable, the cyclic permutation from one to the other must be by an even number of syllables. Moreover, the words $\phi_{\alpha_1}(b^{-1}a^{-1})$ and $\phi_{\alpha_2}(b^{-1}a^{-1})$ must have the same word length, and they must have the same set of syllables and therefore the same syllable length. This implies that w_{α_1} and w_{α_2} also have the same syllable length. We record this important observation here.

Lemma 3.2.1. *Let $\alpha_1, \alpha_2 \in \mathcal{A}_{i,j}(P)$ be arcs corresponding to the conjugators w_{α_1} and w_{α_2} such that $I(\alpha_1) = I(\alpha_2)$. Then there exist positive even integers m, r such that their syllable lengths satisfy $|w_{\alpha_1}| = |w_{\alpha_2}| = m$, and by shifting each syllable of $\phi_{\alpha_1}(b^{-1}a^{-1})$ cyclically to the right by r places, we obtain $\phi_{\alpha_2}(b^{-1}a^{-1})$.*

We want to use Lemma 3.2.1 to help us derive conjugators corresponding to associate arcs. First, we note that to encompass all possible examples of pairs of two-ended associate arcs, we need not consider values of r greater than m. To see this, first note that if $|w_\alpha| = m$, then $|\phi_\alpha(b^{-1}a^{-1})| = |w_\alpha b^{-1} w_\alpha^{-1} a^{-1}| = 2m + 2$. Let A and B be words of length $2m+2$. Indeed, permuting A cyclically to the right by at least $2m+2$ syllables is equivalent to permuting instead by the residue of that number modulo $2m+2$. Moreover, if cyclically permuting A by some number of syllables from $m+2$ to $2m$ gives us B, then we can swap the roles of A and B, noting that permuting B cyclically to the right by some number of syllables from 2 to m gives us A. Thus every example of a pair of associate arcs corresponds to some pair (m, r), where $r \leq m$. We call these pairs *signatures.*

Definition 3.2.2: A *signature* is a pair of positive integers (m, r), where m and r are even, and $r \leq m$.

Thus a more accurate statement of Lemma 3.2.1 is the following.

Proposition 3.2.3. *Let $\alpha_1, \alpha_2 \in \mathcal{A}_{i,j}(P)$ be arcs such that $I(\alpha_1) = I(\alpha_2)$. Then there exists some signature (m, r) such that the syllable length of w_{α_1} and w_{α_2} is m, and one of $\phi_{\alpha_1}(b^{-1}a^{-1})$ and $\phi_{\alpha_2}(b^{-1}a^{-1})$ can be cyclically permuted to the right by r places to form the other.*

This description of all possible cases of pairs of two-ended associate arcs between δ_i^P and δ_j^P will be a very useful tool which we can use to reverse-engineer such examples, by deriving words to fit a given signature. Ideally, exactly one pair of arcs would match each signature so that we can use signatures to accurately label each instance of associate arcs. It turns out that whilst this is often true, there are also many examples of signatures with multiple corresponding pairs of associate arcs, such as in Example 3.2.4 below. However, we will be able to acquire such a labelling with a little more work.

Example 3.2.4: Consider $(m, r) = (8, 6)$. This signature corresponds to pairs of conjugators w and w' of syllable length 8 such that a cyclic permutation of $wb^{-1}w^{-1}a^{-1}$ by 6 places to the right gives us $w'b^{-1}w'^{-1}a^{-1}$. The words

$$w_1 = bab^{-1}a^{-1}b^{-1}aba$$
$$w_2 = baba^{-1}b^{-1}a^{-1}ba$$

meet this condition, as can be seen here:

$$w_1 b^{-1} w_1^{-1} a^{-1} = \underbrace{bab^{-1}a^{-1}b^{-1}abab^{-1}a^{-1}b^{-1}a^{-1}}\underbrace{baba^{-1}b^{-1}a^{-1}},$$
$$w_2 b^{-1} w_2^{-1} a^{-1} = \underbrace{baba^{-1}b^{-1}a^{-1}}\underbrace{bab^{-1}a^{-1}b^{-1}abab^{-1}a^{-1}b^{-1}a^{-1}}.$$

Furthermore, the words

$$w_3 = ba^{-1}b^{-1}ab^{-1}aba^{-1}$$
$$w_4 = ba^{-1}bab^{-1}a^{-1}ba^{-1}$$

also meet these criteria. However, it is worth noting here that $w_1b^{-1}w_1^{-1}a^{-1}$ and $w_2b^{-1}w_2^{-1}a^{-1}$ cannot be cyclically permuted to make $w_3b^{-1}w_3^{-1}a^{-1}$ or $w_4b^{-1}w_4^{-1}a^{-1}$. Rather, these examples give two distinct pairs of associate arcs which correspond to the same signature $(8,6)$, and the pairs correspond to different curves.

To start describing conjugators which can satisfy a given signature, we need to view the words $\phi_\alpha(b^{-1}a^{-1})$ from a new perspective. In the spirit of considering cyclic permutations, we write the syllables of these words clockwise in a circle, which we refer to as *cyclical notation*.

Definition 3.2.5: A word $w \in \pi_1(P)$ in a and b written in cyclical notation is called a *wheel*, which we commonly denote as W. The positions in the wheel correspond to syllables of w. We consider wheels up to rotation, so that each wheel corresponds to a conjugacy class in $\pi_1(P)$. See Figure 3.1 for examples.

$$\begin{array}{ccc} & b^{-1} & \\ a^2 & & a^3 \\ b & & b^4 \\ & a^{-1} & \end{array} \qquad\qquad \begin{array}{ccc} & b^{-1} & \\ a & & a^{-1} \\ b^{-1} & & b \\ & a^{-1} & \end{array}$$

Figure 3.1: *Two words of syllable length* 6 *written in cyclical notation;* $ba^2b^{-1}a^3b^4a^{-1}$ *(left), and* $b^{-1}ab^{-1}a^{-1}ba^{-1}$ *(right).*

Recall that if $|w_\alpha| = m$ then $|\phi_\alpha(b^{-1}a^{-1})| = 2m+2$. Thus when writing a word $\phi_\alpha(b^{-1}a^{-1})$ in cyclical notation, there are $2m+2$ positions. We will denote the set of these positions as Θ_m and we label them with the integers $-m$ through $m+1$ modulo $2m+2$. These could equivalently be seen as the vertices of a regular polygon with $2m+2$ sides. The labelling is as in Figure 3.2, with 0 at the top and positive numbers proceeding clockwise.

For each position $p \in \Theta_m$ we denote the syllable of a wheel W in that position by $x_W(p)$. We then say that a wheel W is *based* at a particular syllable b^k if that syllable is in position 0, so that $x_W(0) = b^k$. For consistency, we will only consider wheels to be based at b-syllables. When rotating a wheel, we will assume the rotation to be clockwise - that is, in the positive direction - unless otherwise stated. Furthermore, rotating a wheel by n *places* will always correspond to rotating it by n syllables, rather than letters.

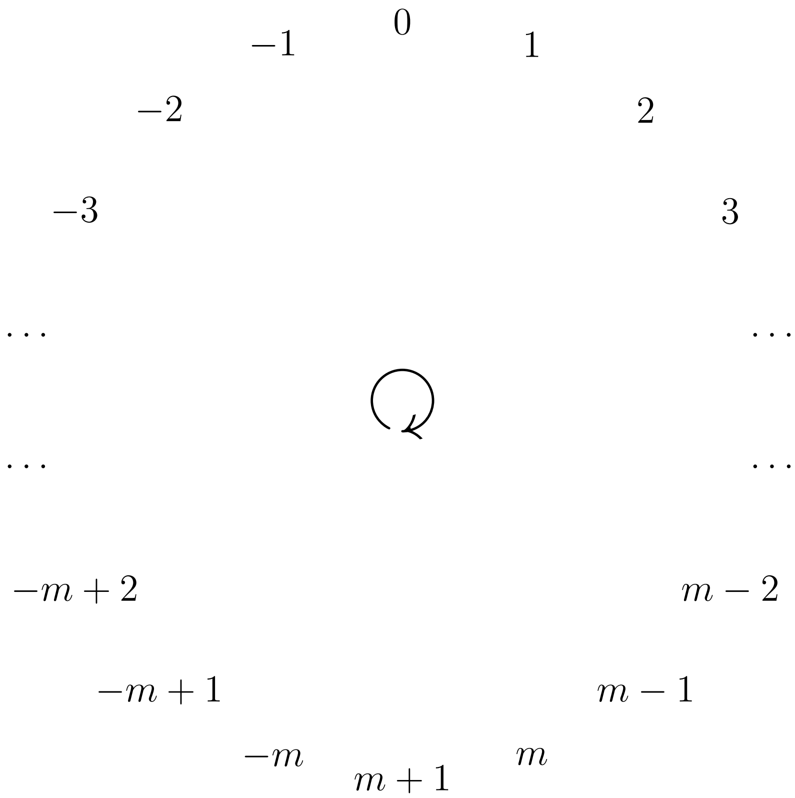

Figure 3.2: *The convention for labelling positions of Θ_m in cyclical notation.*

Definition 3.2.6: We will say that a conjugator w *splits* a wheel if, beginning at some position and reading clockwise until we have read every syllable, we read off the word $wb^{-1}w^{-1}a^{-1}$.

If w splits a wheel, this is equivalent to the existence of a line of anti-symmetry through the wheel which passes through a copy of b^{-1} and a copy of a^{-1}. For any $q \in \Theta_m$, let L^q denote the line between the positions q and $q+m+1$, which bisects Θ_m. Denote the reflection in a line L^q by $\rho_q \colon \Theta_m \to \Theta_m$.

Definition 3.2.7: Given a based wheel W on Θ_m, L^q is a *line of anti-symmetry* if the following hold:

- $x_W(q) = b^{-1}$,
- $x_W(q+m+1) = a^{-1}$,
- for all $p \in \Theta_m$ with $p \neq q$ and $p \neq q+m+1$, $x_W(\rho_q(p)) = (x_W(p))^{-1}$.

In other words, the image of a word in W disjoint from the endpoints of the line is its inverse.

Definition 3.2.8: Let W be a based wheel. We say we *decompose* W with this base by reading off the syllables from $-m$ and proceeding clockwise. Thus a based wheel decomposes as the word

$$x_W(-m)x_W(-m+1)\dots x_W(m)x_W(m+1).$$

See Figure 3.3 for examples of wheels decomposing as words.

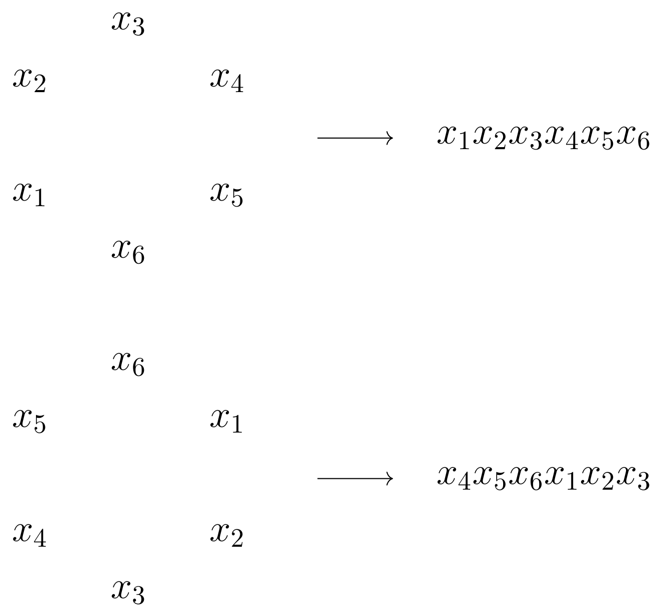

Figure 3.3: *Decomposing the wheel given by $x_1x_2x_3x_4x_5x_6$ with two different bases. In the first case (above), it is based at x_3, and in the second case (below) it is based at x_6.*

In a word of the form $wb^{-1}w^{-1}a^{-1}$, we refer to the copies of b^{-1} and a^{-1} disjoint from w and w^{-1} as the *cores* of the word. If w splits a wheel W, then if we base the wheel at the core b^{-1} it decomposes as $wb^{-1}w^{-1}a^{-1}$.

Let α be an arbitrary arc with conjugator w_α and consider the wheel formed by $\phi_\alpha(b^{-1}a^{-1}) = w_\alpha b^{-1}w_\alpha^{-1}a^{-1}$. Indeed this wheel is split by w_α by construction, but there is not necessarily a second word which splits it. For example, let $w_\alpha = b^5a^4$. Then the wheel given by $b^5a^4b^{-1}a^{-4}b^{-5}a^{-1}$ does not split for any word other than b^5a^4. We can see this by basing the wheel, for example at the core b^{-1}, and checking each line L^q to see if it is a line of anti-symmetry. With this base, L^0 will be a line of anti-symmetry but no other line will.

Let (m, r) be a signature, and suppose that α_1 and α_2 are arcs corresponding to (m, r). This turns out to be a relatively strong assumption on the words w_{α_1} and

w_{α_2}. By assumption, the wheel W formed by writing $\phi_{\alpha_1}(b^{-1}a^{-1})$ (or $\phi_{\alpha_2}(b^{-1}a^{-1})$) in cyclical notation is split by both w_{α_1} and w_{α_2}. Basing the wheel at the core b^{-1} of $\phi_{\alpha_1}(b^{-1}a^{-1})$, we therefore know that L^0 is a line of anti-symmetry, as is the line L^{-r}. This is because if we rotated W from this base clockwise by r syllables, it would decompose as $\phi_{\alpha_2}(b^{-1}a^{-1})$. Moreover, the syllable $x_W(-r)$ would become the new base and L^0 would be line of anti-symmetry with this base. The pre-image of this line under the rotation by r places is L^{-r}, hence L^{-r} is also a line of anti-symmetry when the wheel is based at the core b^{-1} of $\phi_{\alpha_1}(b^{-1}a^{-1})$.

In this way, a signature (m, r) gives rise to two lines of anti-symmetry in Θ_m which differ by a rotation by r places. We will adopt the convention that these two lines are always given as L^0 and L^{-r}, which are highlighted in Figure 3.4.

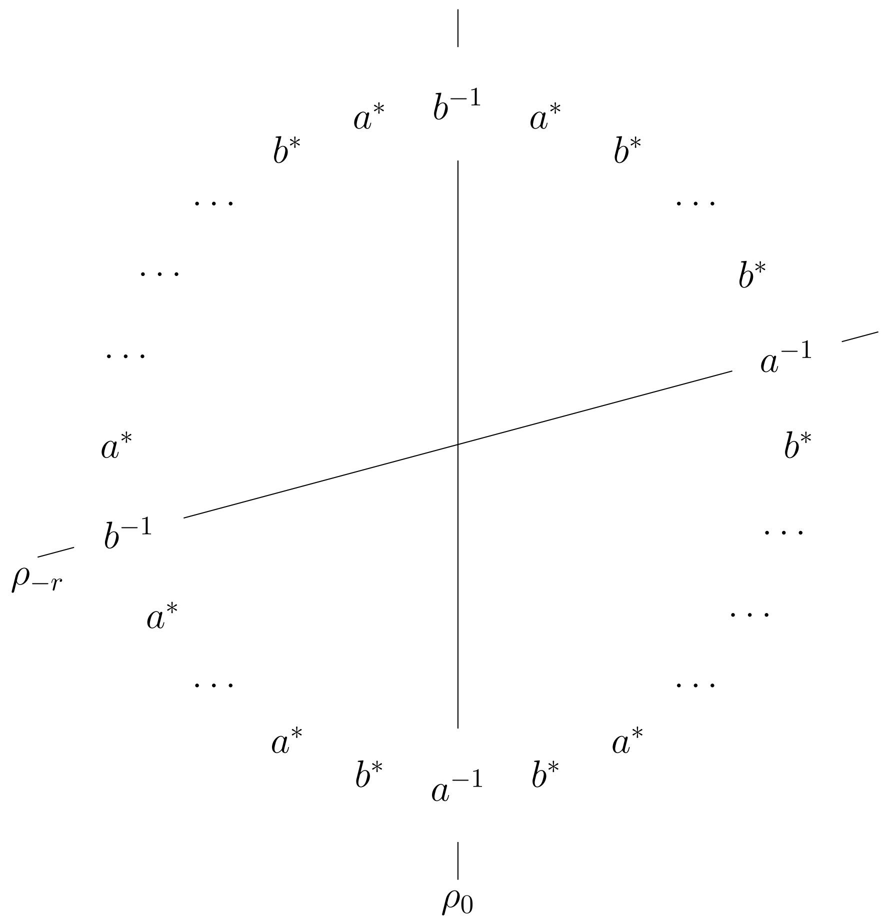

Figure 3.4: *The starting point for constructing a wheel from a signature (m, r). Note that the symbol "$*$" is used as a placeholder for powers yet to be determined. See Section 3.2.1 for a thorough description of the construction.*

The reflections in these lines are ρ_0 and ρ_{-r} respectively, which can be stated as

$$\rho_0(p) = -p$$
$$\rho_{-r}(p) = -p - 2r$$

for all $p \in \Theta_m$. Recall that we consider positions $p \in \Theta_m$ modulo $2m+2$. From the definition of lines of anti-symmetry, we can begin to determine what the syllables of a wheel W which matches the signature (m, r) must be; for instance, $x_W(-r) = b^{-1}$ since it is an end of the line of anti-symmetry L^{-r}, and $x_W(\rho_0(-r)) = x_W(r) = (b^{-1})^{-1} = b$ as L^0 is a line of anti-symmetry. We can then use $x_W(r)$ to determine $x_W(\rho_{-r}(r))$ and so on, alternating which reflection we apply. We can see that there must be at least two orbits in Θ_m under these maps, as neither maps an a-syllable to a b-syllable or vice versa. More precisely, let W be a wheel on Θ_m given by some $\phi_\alpha(b^{-1}a^{-1})$, based at the core b^{-1}. Then W divides into two sub-wheels of a-syllables and the b-syllables, corresponding to the positions

$$\Theta_m^a = \{p \in \Theta_m \mid p \text{ is odd}\},$$
$$\Theta_m^b = \{p \in \Theta_m \mid p \text{ is even}\}.$$

The reflections ρ_0 and ρ_{-r} both fix these sub-wheels, thus $\langle \rho_0, \rho_{-r} \rangle$ has at least two orbits when acting on Θ_m.

We have already seen in Example 3.2.4 that the signature alone is not always enough to determine all syllables in a wheel. This is because $\langle \rho_0, \rho_{-r} \rangle$ may not be the full symmetry group of each of Θ_m^a and Θ_m^b, and so some positions cannot be reached from a core syllable by iterating the two reflections.

The symmetry group of a regular polygon with n sides is isomorphic to the *dihedral group* D_{2n} of order $2n$. We define D_{2n} as

$$D_{2n} := \langle \sigma, \rho \mid \sigma^n, \rho^2, (\rho\sigma)^2 \rangle.$$

Note that the symmetry group of each of Θ_m^a and Θ_m^b is isomorphic to the dihedral group D_{2m+2} as they each have size $m+1$.

Observe that for any p, $\rho_0\rho_{-r}(p) = p + 2r$ and so $\rho_0\rho_{-r}$ is the rotation of Θ_m

clockwise by $2r$ places. We will denote this by σ^{2r}. As a rotation of a regular polygon with $2m+2$ sides, σ^{2r} has order equal to

$$\frac{2m+2}{\text{hcf}(2m+2,2r)} = \frac{m+1}{\text{hcf}(m+1,r)},$$

where $\text{hcf}(\cdot,\cdot)$ is the highest common factor. Thus the subgroup generated by the rotation σ^{2r} and the reflection ρ_0 is a dihedral group of order $\frac{2m+2}{\text{hcf}(m+1,r)}$. Moreover, as $\rho_{-r} = \rho_0\sigma^{2r}$ we have that $\langle \rho_0, \sigma^{2r}\rangle = \langle \rho_0, \rho_{-r}\rangle$. We have proved the following.

Proposition 3.2.9. *Let (m,r) be a signature. Then the corresponding reflections ρ_0 and ρ_{-r} acting on the positions in Θ_m generate*

$$\langle \rho_0, \rho_{-r}\rangle \simeq D_{\frac{2m+2}{\text{hcf}(m+1,r)}}.$$

In particular, they generate the full symmetry group of Θ_m^a and Θ_m^b if and only if $\text{hcf}(m+1,r)=1$.

Given a finite group G acting on some finite set X and an element $x \in X$, let $G_x = \{gx \mid g \in G\}$ be the orbit of x under G, and let $Stab(x) = \{g \in G \mid gx = x\}$ be the stabiliser of x in G. Then the Orbit-Stabiliser Theorem says that

$$|G_x| = \frac{|G|}{|Stab(x)|}. \tag{3.3}$$

As both ρ_0 and the trivial symmetry fix the position 0 in Θ_m, the size of the orbit of 0 under the action of $\langle \rho_0, \rho_{-r}\rangle$ is $\frac{1}{2}|\langle \rho_0, \rho_{-r}\rangle| = \frac{m+1}{\text{hcf}(m+1,r)}$. Since Θ_m^b has size $m+1$, we know that we can reach each position in Θ_m^b from 0 exactly when $m+1$ and r are co-prime. The same holds when considering the orbit of the position $m+1$ in Θ_m^a. This motivates us to classify signatures into two types; those where $m+1$ and r are co-prime, and those where they are not. We will refer to these as *focused* and *unfocused* signatures respectively.

Definition 3.2.10: A signature (m,r) is said to be *focused* if $\text{hcf}(m+1,r)=1$, and *unfocused* if $\text{hcf}(m+1,r) \neq 1$.

If (m,r) is focused, then by Proposition 3.2.9 ρ_0 and ρ_{-r} generate all symmetries of Θ_m^a and Θ_m^b; thus every syllable can be determined by starting at 0 or $m+1$

and alternating reflections until all positions have been reached as described above. Therefore, there is at most one wheel corresponding to each focused signature. It remains to show that each focused signature has at least one corresponding wheel.

Lemma 3.2.11. *Let (m, r) be a focused signature. Then there exists a wheel corresponding to (m, r).*

Proof. To verify that there exists at least one wheel for each focused signature, we need to verify that the lines of anti-symmetry act consistently on the powers around the wheel. To do this, consider the alternate presentation for the dihedral group

$$\langle \rho_0, \rho_{-r} \mid \rho_0^2, \rho_{-r}^2, (\rho_0\rho_{-r})^{m+1} \rangle. \tag{3.4}$$

We need to verify that when applying the relations which fix each position, we also fix each power. This is immediate for the relations ρ_0^2 and ρ_{-r}^2. For the final relation in the presentation given in (3.4), recall from above that $\rho_0\rho_{-r}$ is the rotation clockwise by $2r$ places. Let W be some wheel of size $2m+2$ and base W on Θ_m. We then define the set of *labelled positions* to be $\{(p, x_W(p)) \mid p \in \Theta_m\}$. The lines of anti-symmetry ρ_0 and ρ_{-r} act on this set as follows:

$$\rho_0(p, x_W(p)) = \begin{cases} (-p, x_W(p)^{-1}) & p \neq 0, m+1 \\ (p, x_W(p)) & p = 0, m+1 \end{cases}$$

$$\rho_{-r}(p, x_W(p)) = \begin{cases} (-p-2r, x_W(p)^{-1}) & p \neq -r, -r+m+1 \\ (p, x_W(p)) & p = -r, -r+m+1 \end{cases}.$$

We can combine these expressions to describe the rotation $\rho_0\rho_{-r}$ in a similar manner;

$$\rho_0\rho_{-r}(p, x_W(p)) = \begin{cases} (p+2r, x_W(p)) & p \neq -r, -r+m+1, -2r, -2r+m+1 \\ (p+2r, x_W(p)^{-1}) & p = -r, -r+m+1, -2r, -2r+m+1 \end{cases}. \tag{3.5}$$

Simply stated, if p is either on the line L_{-r} or will be mapped to the line L_0, one of the reflections through these lines does not affect the position or its label and so the resulting label is inverted. If p sits away from these positions, its label will be inverted by both reflections and thus will be fixed.

The rotation $\rho_0\rho_{-r}$ has two orbits, consisting of the even positions and odd positions, denoted Θ_m^b and Θ_m^a respectively as above. Each orbit contains exactly two of the positions $-r, -r+m+1, -2r, -2r+m+1$, thus as we apply successive powers of $\rho_0\rho_{-r}$ up to $\rho_0\rho_{-r}^{m+1}$ the label will be inverted exactly twice. Thus $\rho_0\rho_{-r}^{m+1}(p, x_W(p)) = (p, x_W(p))$. □

As a consequence of Proposition 3.2.9 and Lemma 3.2.11, for each focused signature there is exactly one corresponding wheel.

When (m, r) is unfocused, only the syllables in a sub-wheel are determined. As the positions 0 and $m+1$ are each stabilised by exactly one non-trivial element of $\langle \rho_0, \rho_{-r} \rangle$, we have by (3.3) that their orbits have size $\frac{m+1}{\text{hcf}(m+1,r)}$. Thus the sub-wheel determined by these reflections has size $\frac{2m+2}{\text{hcf}(m+1,r)}$. The positions making up this sub-wheel, which we will refer to as the *focused sub-wheel* constructed by an unfocused signature, are uniformly distributed around Θ_m. By this we mean that the spaces between positions on the sub-wheel are all of the same size. We can calculate by hand that were we to write the syllables of the focused sub-wheel in cyclical notation, the result is the wheel is constructed by the signature $\left(\frac{m+1}{\text{hcf}(m+1,r)} - 1, \frac{r}{\text{hcf}(m+1,r)}\right)$. Note that this signature is always focused.

To determine the remaining syllables in the wheel corresponding to (m, r), consider the syllables in the positions $-m$ through $-m + \text{hcf}(m+1, r) - 2$ clockwise. These lie between the core a^{-1} of $\phi_{\alpha_1}(b^{-1}a^{-1})$ and the first syllable of $\phi_{\alpha_1}(b^{-1}a^{-1})$ which lies on the focused sub-wheel. Note that they are the first $\text{hcf}(m+1, r) - 1$ syllables in the decomposition of the wheel with this base. For simplicity, we label the syllables in these positions as the word c_1. Continuing clockwise from here, we label the syllables between $-m + \text{hcf}(m+1, r) - 2$ and the next syllable on the focused sub-wheel as c_2, which will have the same syllable length as c_1. This continues until we have collated all syllables not present on the focused sub-wheel into $c_1, c_2, \ldots, c_{\frac{2m+2}{\text{hcf}(m+1,r)}}$. Figure 3.5 gives a visual representation of this process. We can now collapse the wheel W on Θ_m into a smaller wheel W_c on $\Theta_{\frac{2m+2}{\text{hcf}(m+1,r)}-1}$, which has size $\frac{4m+4}{\text{hcf}(m+1,r)}$. After collapsing, each c_i acts like a single syllable and takes up only one position. The action of ρ_0 and ρ_{-r} on the positions around $\Theta_{\frac{2m+2}{\text{hcf}(m+1,r)}-1}$ is well-defined, and they still generate a subgroup isomorphic to $D_{\frac{2m+2}{\text{hcf}(m+1,r)}}$ and therefore preserve the focused sub-wheel. Moreover, by the Orbit-Stabiliser Theorem (see

(3.3)), as no non-trivial element of $\langle \rho_0, \rho_{-r} \rangle$ fixes the position of c_1, the size of its orbit is equal to $\frac{2m+2}{\text{hcf}(m+1,r)}$. Hence all c_i share an orbit; in particular, each c_i is equal to c_1 or c_1^{-1}. In the following, we will refer to the word c_1 simply as c.

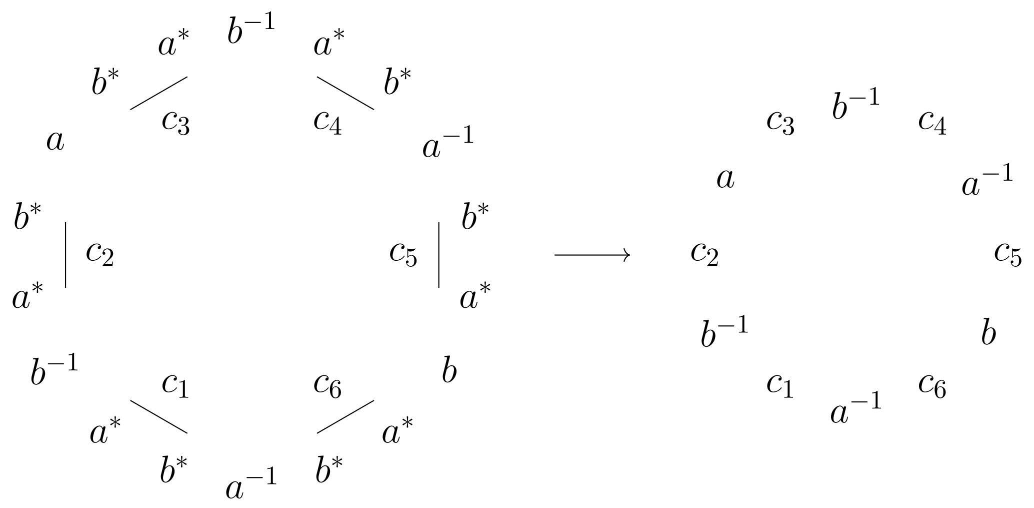

Figure 3.5: *Highlighting the words which we label as c_i and collapse into W_c, when constructing a wheel from* $(8, 6)$.

Therefore, after choosing a length m of conjugator, a rotation r and a word c of syllable length $\text{hcf}(m+1, r) - 1$, we have determined exactly one wheel. This triple will be referred to as a *frame*. We note that as we are choosing c to be an even number of syllables from the start of a conjugator, it is also a conjugator.

Definition 3.2.12: A *frame* is a triple (m, r, c), where (m, r) is a signature and c is a conjugator of syllable length $\text{hcf}(m+1, r) - 1$. If (m, r) is focused then the only choice for c is the empty word.

Whilst each unfocused signature corresponds to infinitely many frames and thus infinitely many wheels, each frame generates exactly one wheel by the above argument. We have proved the following.

Proposition 3.2.13. *Let (m, r, c) be a frame. Then there is exactly one wheel constructed by (m, r, c).*

In Section 3.2.1 below, we describe in detail the process by which we construct a wheel from a frame.

As every pair of two-ended associate arcs corresponds to some signature by Proposition 3.2.3, we can further say that every such pair corresponds to some frame. It remains to be seen that no pair of associate arcs corresponds to multiple frames, which we will demonstrate in Section 3.3. In other words, given a wheel W constructed by a frame, we will see that there are exactly two ways to base W so that it decomposes as $wb^{-1}w^{-1}a^{-1}$ for some conjugator w.

3.2.1 Constructing a wheel from a frame

In this section, we will concretely describe the process by which we construct a unique wheel from a choice of frame (m, r, c).

Given a frame (m, r, c), we take a copy of Θ_m, the circle with $2m+2$ positions distributed uniformly around it such that they lie on the vertices of a regular polygon with $2m+2$ sides. We label these positions with the numbers $-m$ through $m+1$ modulo $2m+2$ as in Figure 3.2. Now we inscribe Θ_m with the two lines L^0 and L^{-r}, which we proclaim are lines of anti-symmetry as in Definition 3.2.7. This places a copy of b^{-1} in positions 0 and $-r$, and a copy of a^{-1} in positions $m+1$ and $-r+m+1$. See Figure 3.4.

We now use this initial information to determine as many syllables around Θ_m as possible, with the aim of constructing the entire wheel. Denote the wheel that we are constructing as W. Note that when constructed, W will be based at the syllable b^{-1} currently in position 0. Recall that ρ_0 and ρ_{-r} are the reflections in the lines L^0 and L^{-r} respectively, given as

$$\rho_0(p) = -p$$
$$\rho_{-r}(p) = -p - 2r.$$

Step 1:

Beginning at $p = 0$, as L^{-r} is a line of anti-symmetry we have that $x_W(\rho_{-r}(0)) = (x_W(0))^{-1}$. Thus as $x_W(0) = b^{-1}$,

$$x_W(-2r) = b.$$

Since L^0 is also a line of anti-symmetry, we have that $x_W(\rho_0(-2r)) = (x_W(-2r))^{-1}$.

Therefore

$$x_W(2r) = b^{-1}.$$

Applying ρ_{-r} again we reach the position $-4r$, then ρ_0 takes us to $4r$, and so on. We terminate this process when we reach $p = -r$, since this position must have been reached by applying ρ_{-r}, and applying the next map ρ_0 fixes this position. Further steps of this process would retrace our path until we reached $p = 0$ again. At each position p in this sequence, if it was reached by applying ρ_{-r} then $x_W(p) = b$. If instead it was reached by applying ρ_0 then $x_W(p) = b^{-1}$. In this manner, we have determined some of the b-syllables around W, on the sub-wheel Θ_m^b consisting of only the even-numbered positions.

We can run this process again, this time starting at the position $m + 1$. As $x_W(m + 1) = a^{-1}$, we instead determine a collection of a-syllables on the sub wheel Θ_m^a consisting of only the odd-numbered positions. Recall that ρ_0 and ρ_{-r} fix Θ_m^a and Θ_m^b, thus these two processes do not overlap.

Suppose $\text{hcf}(m+1, r) = 1$. Then by Proposition 3.2.9, all syllables in the positions on Θ_m^b are determined by the first process and all positions on Θ_m^a are determined by the second. As $\Theta_m = \Theta_m^a \cup \Theta_m^b$, we have constructed W. Note that as in Definition 3.2.12, since $|c| = \text{hcf}(m + 1, r) - 1$ we must have that $(m, r, c) = (m, r, \emptyset)$.

Otherwise, $\text{hcf}(m + 1, r) > 1$ and thus by Proposition 3.2.9 these processes have only determined the syllables on the focused sub-wheel sitting on $\Theta_{\frac{m+1}{\text{hcf}(m+1,r)}-1}$, as described below Definition 3.2.10. This sub-wheel is also based at $x_W(0)$, and its positions are uniformly distributed around Θ_m.

Step 2:

We now proclaim that the syllables in positions $-m$ through $-m+\text{hcf}(m+1, r)-2$ clockwise are given by the word c. Precisely, we write

$$c = x_1 x_2 \dots x_{\text{hcf}(m+1,r)-1}$$

for syllables x_i of c, and state that $x_W(i) = x_{i+m+1}$ for positions $i \in \{-m, -m + 1, \dots, -m + \text{hcf}(m + 1, r) - 2\}$. Thus we have $x_W(-m) = x_1$, $x_W(-m + 1) = x_2$, through to $x_W(-m + \text{hcf}(m + 1, r) - 2) = x_{\text{hcf}(m+1,r)-1}$. We can now follow the process in Step 1 again, beginning this time at $p = -m$. This will determine the syllables in another set of positions on Θ^b_m, as $-m$ is even. Eventually, the process would return to $p = -m$, where we let it terminate. Repeating this process for each position $-m$ through $-m + \text{hcf}(m+1, r) - 2$, we have effectively performed the process on the collapsed wheel W_c on $\Theta_{\frac{2m+2}{\text{hcf}(m+1,r)}-1}$, as described above Definition 3.2.12, beginning at $p = -\frac{2m+2}{\text{hcf}(m+1,r)} - 1$. See Figure 3.5. Again by Proposition 3.2.9, we have that all copies of c around W_c share an orbit under ρ_0 and ρ_{-r}; hence this process determines the remaining syllables of W.

3.3 Relating frames and pairs of associate arcs

In the previous section, we demonstrated that frames are the appropriate system for labelling wheels, and thus pairs of associate arcs. Here we will prove that for any frame, there is exactly one pair of arcs whose words split the wheel that it constructs. The process by which a wheel is constructed from a frame is described in Section 3.2.1. We prove the following.

Theorem 3.3.1. *Let (m, r, c) be a frame. Then for some $s_1, \ldots, s_n, t_1, \ldots, t_n \in \{1, -1\}$, the words*

$$w_1 = cb^{s_1}c^{-1}a^{t_1}cb^{s_2}c^{-1} \ldots a^{t_{n-1}}cb^{s_n}c^{-1}a^{t_n}c$$
$$w_2 = cb^{-s_1}c^{-1}a^{-t_1}cb^{-s_2}c^{-1} \ldots a^{-t_{n-1}}cb^{-s_n}c^{-1}a^{-t_n}c$$

split the wheel generated by (m, r, c), where $n = \frac{1}{2}\left(\frac{m+1}{hcf(m+1,r)} - 1\right)$. Moreover, this choice of powers is unique, and w_1 and w_2 do not split the wheel generated by any other frame.

This theorem will follow from Proposition 3.3.3 and Proposition 3.3.4. We will first derive the structures of w_1 and w_2, before showing that they are unique to this frame.

The reader is advised to keep Figure 3.6 in mind throughout the following. Consider the case that (m, r) is focused. Then the only corresponding frame is $(m, r, \emptyset)$, as the powers around the wheel are determined by the two lines of anti-symmetry. If we construct a wheel from $(m, r, \emptyset)$, then by construction it will decompose as $w_1 b^{-1} w_1^{-1} a^{-1}$ with the initial base for some conjugator w_1. After rotating by r

places, it will instead decompose as $w_2 b^{-1} w_2^{-1} a^{-1}$ for some conjugator w_2. The lines of anti-symmetry L^0 and L^{-r} are mapped to the lines L^r and L^0 respectively by this rotation; in other words, they are the lines of anti-symmetry corresponding to the "signature" $(m, 2m+2-r)$, if we allowed such values in our definition.

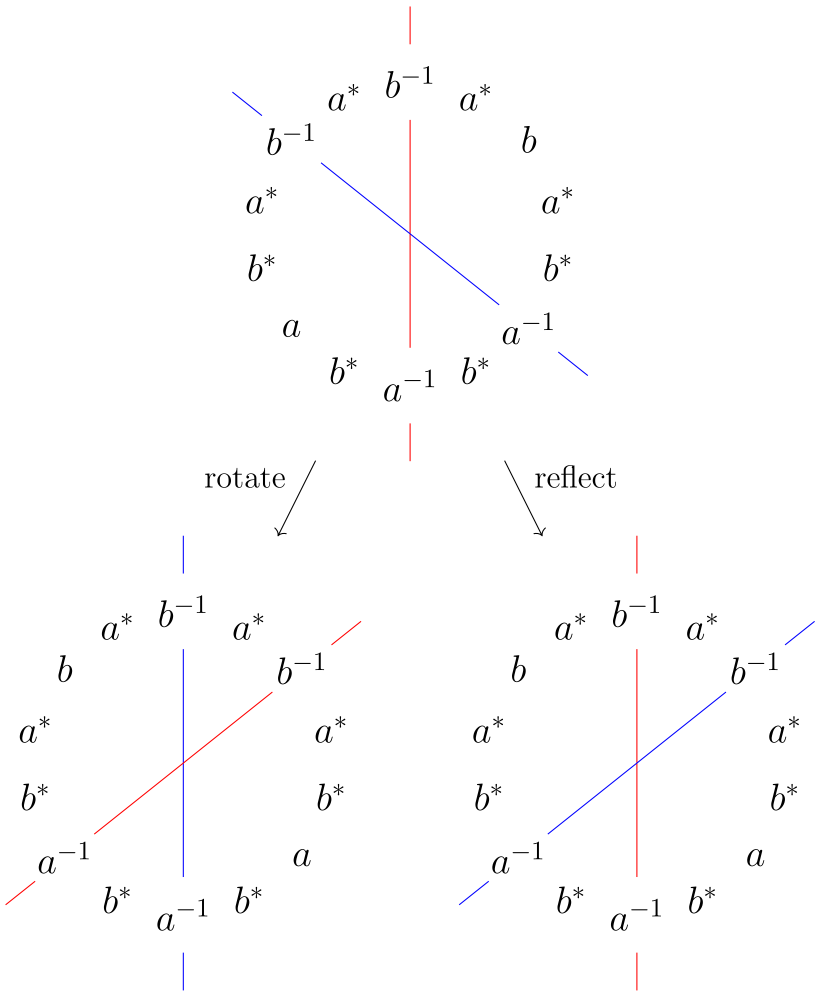

Figure 3.6: *Demonstrating that rotating the lines of anti-symmetry L^0 and L^{-r} by r and reflecting them in the vertical line give the same result, in signature $(6, 2)$. Whilst the symmetries are considered in a different order, their position is identical in either case.*

Instead of rotating by r places to obtain these two new lines, we could have reflected the original pair in L^0. This is simply a reflection, we are not considering it a line of anti-symmetry at this stage. As the signature is focused, the only information required to generate the wheel is the positions of the two lines. Thus, whether we rotate this wheel by r places or reflect it in L^0, the wheel will decompose as the same word. Figure 3.6 demonstrates this visually. Another way to see this is by considering the expressions for these symmetries using cyclical notation. Applying the reflections ρ_0 and ρ_{-r} after rotating their axes by r places gives the reflections $\sigma^r \circ \rho_0 \circ \sigma^{-r}$ and $\sigma^r \circ \rho_{-r} \circ \sigma^{-r}$ respectively, where $\sigma^r(p) = p + r$ for all p. Similarly, the reflections through L^r and L^0 are given by $\rho_0 \circ \rho_{-r} \circ \rho_0$ and $\rho_0 \circ \rho_0 \circ \rho_0 = \rho_0$. For any $p \in \Theta_m$, we can compute

$$\begin{aligned}
\sigma^r \circ \rho_0 \circ \sigma^{-r}(p) &= -(p - r) + r \\
&= -p + 2r \\
\sigma^r \circ \rho_{-r} \circ \sigma^{-r}(p) &= -(p - r) - 2r + r \\
&= -p \\
\rho_0 \circ \rho_{-r} \circ \rho_0(p) &= -(-(-p) - 2r) \\
&= -p + 2r \\
\rho_0(p) &= -p.
\end{aligned}$$

As a result, we can see that $\sigma^r \circ \rho_0 \circ \sigma^{-r} = \rho_0 \circ \rho_{-r} \circ \rho_0$, corresponding to L^r, and $\sigma^r \circ \rho_{-r} \circ \sigma^{-r} = \rho_0$, corresponding to L^0.

Moreover, we know that after reflecting the wheel in L^0 it will decompose as $\overline{w_1} b^{-1} \overline{w_1}^{-1} a^{-1}$, where $\overline{w}$ is w with the exponent of each syllable replaced by its negative. As the wheel after rotating decomposes as $w_2 b^{-1} w_2^{-1} a^{-1}$, we have that $w_2 = \overline{w_1}$.

Furthermore, as every b-syllable and a-syllable around the wheel shares an orbit under $\langle \rho_0, \rho_{-r} \rangle$ with the core b^{-1} or a^{-1} respectively, we have that every syllable must have a power of 1 or -1 as they can be reached by some sequence of reflections in the lines of anti-symmetry from a core. This argument proves the following.

Lemma 3.3.2. *Let (m, r) be a focused signature, and $(m, r, \emptyset)$ be the corresponding frame. Then for some $s_1, \dots, s_{\frac{m}{2}}, t_1, \dots, t_{\frac{m}{2}} \in \{1, -1\}$, the words*

$$\begin{aligned}
w_1 &= b^{s_1} a^{t_1} \dots b^{s_{\frac{m}{2}}} a^{t_{\frac{m}{2}}} \\
w_2 &= b^{-s_1} a^{-t_1} \dots b^{-s_{\frac{m}{2}}} a^{-t_{\frac{m}{2}}}
\end{aligned}$$

split the wheel constructed by $(m, r, \emptyset)$, and they begin at $-m$ and $-m-r$ respectively with the wheel's initial base.

This statement is the first convincing piece of evidence that we should have at most two arcs associated to any given curve. Since we have found that in this case the two words are so closely related, it feels intuitive that there can be no other words defining arcs which are associated to the same curve as these two. This will indeed be the case, as we demonstrate concretely later; for now we generalise Lemma 3.3.2 to the unfocused case.

Proposition 3.3.3. *Let* (m, r, c) *be a frame. Then for some* $s_1, \ldots, s_n, t_1, \ldots, t_n \in \{1, -1\}$*, the words*

$$w_1 = cb^{s_1}c^{-1}a^{t_1}cb^{s_2}c^{-1} \ldots a^{t_{n-1}}cb^{s_n}c^{-1}a^{t_n}c$$
$$w_2 = cb^{-s_1}c^{-1}a^{-t_1}cb^{-s_2}c^{-1} \ldots a^{-t_{n-1}}cb^{-s_n}c^{-1}a^{-t_n}c$$

split the wheel constructed by (m, r, c)*, where* $n = \frac{1}{2}\big(\frac{m+1}{hcf(m+1,r)} - 1\big)$*. These words begin at* $-m$ *and* $-m-r$ *respectively with the wheel's initial base.*

Proof. Let w_1 be the conjugator such that the wheel constructed by (m, r, c) decomposes as $w_1b^{-1}w_1^{-1}a^{-1}$ before rotating. Recall from the discussion following Definition 3.2.10 that there is a focused sub-wheel determined by the unfocused signature (m, r). By Lemma 3.3.2, we know that for each syllable of w_1 which is on the focused sub-wheel, the syllable in its position after rotating by r places clockwise is its inverse. It remains to demonstrate what the remaining syllables must be.

As discussed in the motivation for the definition of a frame, each word between the syllables of the focused sub-wheel is either c or c^{-1}. In particular, we know that the word strictly between $m+1$ and $(m+1) + \text{hcf}(m+1, r) - 1$ clockwise is c. Consider the string "$a^{-1}cb^{s_1}$" between these positions on the wheel, read clockwise. Applying either line of anti-symmetry, the resulting string is "$b^{-s_1}c^{-1}a^{t_0}$", again read clockwise. Here, $t_0 = -1$ under the reflection ρ but $t_0 = 1$ under ρ_{-r}; see Figure 3.7. In particular, we note that the power of each copy of the word c only depends on its position within the wheel; if the preceding syllable clockwise in the focused sub-wheel is an a-syllable, then it is 1, but if the preceding syllable is a b-syllable, it is -1. Therefore, as the syllables on the focused sub-wheel alternate between a and b, the powers of c alternate accordingly. We can therefore say that for some

$s_1, \ldots, s_n, t_1, \ldots, t_n \in \{1, -1\}$,

$$w_1 = cb^{s_1}c^{-1}a^{t_1}cb^{s_2}c^{-1} \ldots a^{t_{n-1}}cb^{s_n}c^{-1}a^{t_n}c.$$

Here, $n = \frac{1}{2}\left(\frac{m+1}{\mathrm{hcf}(m+1,r)} - 1\right)$ as the size of the focused sub-wheel is $\frac{2m+2}{\mathrm{hcf}(m+1,r)} + 2$ and each of a and b make up half of the syllables in the word.

Moreover, after rotating the wheel clockwise by r places, each a-syllable is replaced with an a-syllable and each b-syllable is replaced with a b-syllable, since r is even. Therefore the resulting powers of c after rotating the wheel have not changed. Thus the wheel now decomposes as $w_2b^{-1}w_2^{-1}a^{-1}$, where

$$w_2 = cb^{-s_1}c^{-1}a^{-t_1}cb^{-s_2}c^{-1} \ldots a^{-t_{n-1}}cb^{-s_n}c^{-1}a^{-t_n}c.$$

□

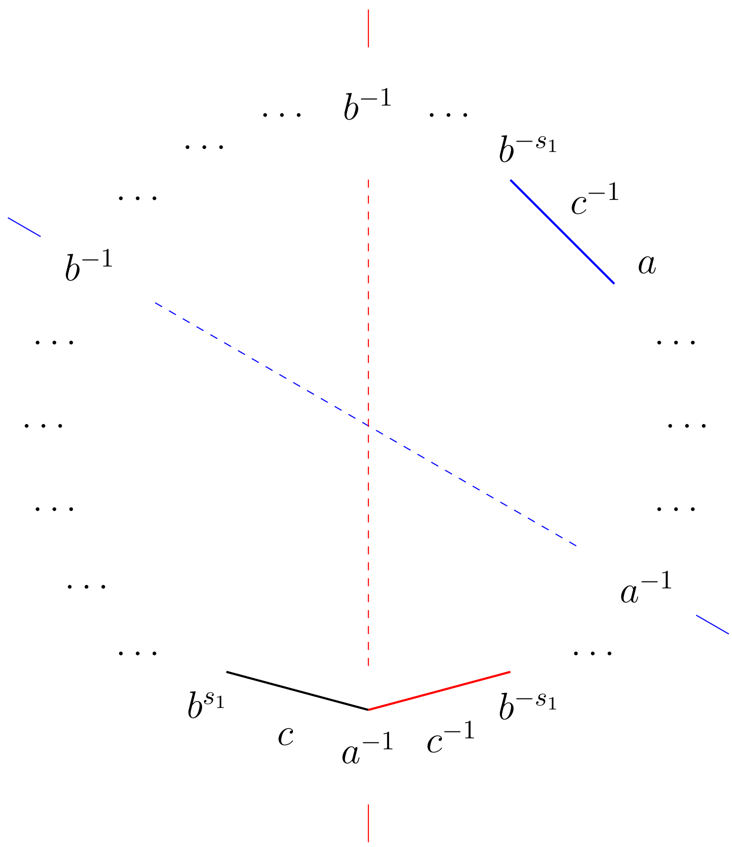

Figure 3.7: *Demonstrating that the power of a copy of the word c depends only on its preceding syllable on the focused sub-wheel.*

As a result of Proposition 3.3.3, we can see that whilst in the focused case we must invert every power in one word to obtain the other, in the unfocused case we must only invert the powers of a particular sub-word. In either case, this demonstrates that there is a strong relationship between the structures of two words which split the same wheel. We will now use this relationship to show that each pair of words is unique to a frame.

Proposition 3.3.4. *Let (m, r, c) be a frame, and let w_1 and w_2 be the words given by Proposition 3.3.3. Then w_1 and w_2 are the only words which split the wheel constructed by (m, r, c), and the only rotations after which this wheel decomposes as $wb^{-1}w^{-1}a^{-1}$ for some conjugator w are 0 and r.*

Proof. We will prove this theorem in two parts, addressing each statement in turn.

By Proposition 3.3.3, we have that w_1 and w_2 split the wheel W constructed by (m, r, c) and begin at $-m$ and $-m-r$ respectively with the initial base. Suppose there exists a third word w' which is neither w_1 nor w_2 and splits the wheel. It must therefore begin at a different position, say $-m-r'$ for some $r' \neq r$ and $r' \neq 0$. This immediately tells us that this wheel can be constructed by multiple frames; these correspond to the signatures (m, r), (m, r') and $(m, r'-r)$, where $r'-r$ is modulo $2m+2$. We can consider each frame in turn and apply Proposition 3.3.3 to tell us what the structure of w_1, w_2 and w' must be, aiming to find a contradiction.

To demonstrate that in fact the only words which split the wheel W are w_1 and w_2, we begin by concerning ourselves with the frame (m, r, c) which corresponds to w_1 and w_2, and the frame (m, r', d) which corresponds to w_1 and w'. Here d is the word formed by the first $\operatorname{hcf}(m+1, r')-1$ syllables of w_1. We will break this situation down by cases, considering whether or not the underlying signatures are focused.

Case 1: (m, r) is focused and (m, r') is focused.

In this case, $c = d = \emptyset$. Proposition 3.3.3 applied to $(m, r, \emptyset)$ gives us that for some powers $s_1, \ldots, s_{\frac{m}{2}}, t_1, \ldots, t_{\frac{m}{2}} \in \{1, -1\}$,

$$w_1 = b^{s_1}a^{t_1}b^{s_2} \ldots a^{t_{\frac{m}{2}-1}}b^{s_{\frac{m}{2}}}a^{t_{\frac{m}{2}}},$$

$$w_2 = b^{-s_1}a^{-t_1}b^{-s_2} \ldots a^{-t_{\frac{m}{2}-1}}b^{-s_{\frac{m}{2}}}a^{-t_{\frac{m}{2}}}.$$

Applying Proposition 3.3.3 again to the frame $(m, r', \emptyset)$, we have that for some powers $s'_1, \dots, s'_{\frac{m}{2}}, t'_1, \dots, t'_{\frac{m}{2}} \in \{1, -1\}$

$$\begin{aligned} w_1 &= b^{s'_1} a^{t'_1} b^{s'_2} \dots a^{t'_{\frac{m}{2}-1}} b^{s'_{\frac{m}{2}}} a^{t'_{\frac{m}{2}}}, \\ w' &= b^{-s'_1} a^{-t'_1} b^{-s'_2} \dots a^{-t'_{\frac{m}{2}-1}} b^{-s'_{\frac{m}{2}}} a^{-t'_{\frac{m}{2}}}. \end{aligned}$$

Combining these, we see that for all i, $s_i = s'_i$ and $t_i = t'_i$, and moreover that $w' = w_2$. Consequently, w_1 and w_2 are the only words which split the wheel W.

Case 2: (m, r) is focused and (m, r') is unfocused.

As before, we have

$$\begin{aligned} w_1 &= b^{s_1} a^{t_1} b^{s_2} \dots a^{t_{\frac{m}{2}-1}} b^{s_{\frac{m}{2}}} a^{t_{\frac{m}{2}}}, \\ w_2 &= b^{-s_1} a^{-t_1} b^{-s_2} \dots a^{-t_{\frac{m}{2}-1}} b^{-s_{\frac{m}{2}}} a^{-t_{\frac{m}{2}}}. \end{aligned} \tag{3.6}$$

In this instance, applying Proposition 3.3.3 to the frame (m, r', d), we have that for some $s'_1, \dots, s'_{n'}, t'_1, \dots, t'_{n'} \in \{1, -1\}$,

$$\begin{aligned} w_1 &= d b^{s'_1} d^{-1} a^{t'_1} d b^{s'_2} d^{-1} \dots a^{t'_{n'-1}} d b^{s'_{n'}} d^{-1} a^{t'_{n'}} d, \\ w' &= d b^{-s'_1} d^{-1} a^{-t'_1} d b^{-s'_2} d^{-1} \dots a^{-t'_{n'-1}} d b^{-s'_{n'}} d^{-1} a^{-t'_{n'}} d. \end{aligned} \tag{3.7}$$

We do not have enough information here to complete the picture. We need to consider two sub-cases concerning the frame $(m, r' - r, e)$, which corresponds to w_2 and w'. Here, e is the first $\text{hcf}(m+1, r'-r) - 1$ syllables of w_2. Note that we are considering $r' - r$ modulo $2m + 2$ to ensure this is well-defined. This frame comes from forgetting w_1; it constructs the wheel with the base such that it decomposes as either $w_2 b^{-1} w_2^{-1} a^{-1}$ or $w' b^{-1} w'^{-1} a^{-1}$, depending on whether $r' < r$ or $r' > r$.

Case 2a: $(m, r' - r)$ is focused.

In this case, we have that for some $s''_1, \dots, s''_{\frac{m}{2}}, t''_1, \dots, t''_{\frac{m}{2}} \in \{1, -1\}$,

$$\begin{aligned} w_2 &= b^{s''_1} a^{t''_1} b^{s''_2} \dots a^{t''_{\frac{m}{2}-1}} b^{s''_{\frac{m}{2}}} a^{t''_{\frac{m}{2}}}, \\ w' &= b^{-s''_1} a^{-t''_1} b^{-s''_2} \dots a^{-t''_{\frac{m}{2}-1}} b^{-s''_{\frac{m}{2}}} a^{-t''_{\frac{m}{2}}}. \end{aligned}$$

Combining this with (3.6), we have that $s''_i = -s_i$ and $t''_i = -t_i$, hence $w' = w_1$. But by (3.7) we have $w' \neq w_1$. This is a contradiction, so we are done.

Case 2b: $(m, r' - r)$ is unfocused.

Here, we have that

$$\begin{aligned} w_2 &= eb^{s''_1}e^{-1}a^{t''_1}eb^{s''_2}e^{-1}\dots a^{t''_{n''-1}}eb^{s''_{n''}}e^{-1}a^{t''_{n''}}e, \\ w' &= eb^{-s''_1}e^{-1}a^{-t''_1}eb^{-s''_2}e^{-1}\dots a^{-t''_{n''-1}}eb^{-s''_{n''}}e^{-1}a^{-t''_{n''}}e. \end{aligned} \tag{3.8}$$

In particular, consider the first syllable of e, which is also the first syllable of w_2. By the expressions for w' in (3.7) and (3.8), we have the first syllable of e is the same as the first syllable of d. Thus w_1, w_2 and w' all share the same first syllable. However, in (3.6) we can see that this syllable is b^{s_1} in w_1 but b^{-s_1} in w_2. This is a contradiction, so we are done.

Case 3: (m, r) is unfocused and (m, r') is focused.

Up to relabelling, this case is identical to Case 2, as we can repeat the argument but swap the roles of r' and r. Thus here too we reach a contradiction.

Case 4: (m, r) is unfocused and (m, r') is unfocused.

Here we have the following;

$$\begin{aligned} w_1 &= cb^{s_1}c^{-1}a^{t_1}cb^{s_2}c^{-1}\dots a^{t_{n-1}}cb^{s_n}c^{-1}a^{t_n}c, \\ w_2 &= cb^{-s_1}c^{-1}a^{-t_1}cb^{-s_2}c^{-1}\dots a^{-t_{n-1}}cb^{-s_n}c^{-1}a^{-t_n}c. \end{aligned} \tag{3.9}$$

$$\begin{aligned} w_1 &= db^{s'_1}d^{-1}a^{t'_1}db^{s'_2}d^{-1}\dots a^{t'_{n'-1}}db^{s'_{n'}}d^{-1}a^{t'_{n'}}d, \\ w' &= db^{-s'_1}d^{-1}a^{-t'_1}db^{-s'_2}d^{-1}\dots a^{-t'_{n'-1}}db^{-s'_{n'}}d^{-1}a^{-t'_{n'}}d. \end{aligned} \tag{3.10}$$

Again we must proceed to sub-cases.

Case 4a: $(m, r' - r)$ is focused.

Here, we have that

$$\begin{aligned} w_2 &= b^{s''_1}a^{t''_1}b^{s''_2}\dots a^{t''_{\frac{m}{2}-1}}b^{s''_{\frac{m}{2}}}a^{t''_{\frac{m}{2}}}, \\ w' &= b^{-s''_1}a^{-t''_1}b^{-s''_2}\dots a^{-t''_{\frac{m}{2}-1}}b^{-s''_{\frac{m}{2}}}a^{-t''_{\frac{m}{2}}}. \end{aligned}$$

This immediately contradicts the previous equations, as w_2 and w' should both have the same first syllable as w_1, but this implies they are different. Thus we have a contradiction and we are done.

Case 4b: $(m, r' - r)$ is unfocused.

Here, we have that

$$\begin{aligned} w_2 &= eb^{s_1''}e^{-1}a^{t_1''}eb^{s_2''}e^{-1}\dots a^{t_{n''-1}''}eb^{s_{n''}''}e^{-1}a^{t_{n''}''}e, \\ w' &= eb^{-s_1''}e^{-1}a^{-t_1''}eb^{-s_2''}e^{-1}\dots a^{-t_{n''-1}''}eb^{-s_{n''}''}e^{-1}a^{-t_{n''}''}e. \end{aligned} \tag{3.11}$$

Note that if $|c| = |d|$, then we immediately have that $w' = w_2$. However by (3.11) we have that $w' \neq w_2$, so we have a contradiction. Suppose then without loss of generality that $|c| < |d|$. Then the start of w_1 can be written in two ways, considering (3.9) and (3.10);

$$\begin{aligned} cb^{s_1}c^{-1} &= \underbrace{b^*a^*\dots b^*a^*}_{c}\,b^{s_1}\,\underbrace{a^*b^*a^*\dots}_{c^{-1}}, \\ db^{s_1'}d^{-1} &= \underbrace{b^*a^*\dots b^*a^*b^{s_1}a^*b^*a^*\dots b^*a^*}_{d}\,b^{s_1'}\,\underbrace{a^*b^*a^*\dots}_{d^{-1}}. \end{aligned}$$

In the expression for w_2 in (3.9), the syllable b^{s_1} is inverted, but the syllable in the same place in the expression for w' in (3.10) is not, since it is contained within d. In particular, the first syllable in which w_2 and w' differ is b^{s_1}, thus this syllable must also be $b^{s_1''}$ in (3.11). But this means that $e = c$, and the syllables inverted in w_2 to make w' are exactly those inverted to make w_1. Therefore, $w' = w_1$. However, by (3.10), $w' \neq w_1$, and so we have a contradiction.

Thus in every case, either the only words which split the wheel W are w_1 and w_2 or we find a contradiction.

Now suppose that after some rotation $r' \neq r$ and $r' \neq 0$, the wheel decomposes as $w'b^{-1}w'^{-1}a^{-1}$ for some w'. This word therefore splits the wheel, and so we can run through all of the previous cases to reach a contradiction in each case except Case 1, as this case allows for there to be multiple rotations. We now need more information, so we proceed to sub-cases.

Case 1i: $(m, r' - r)$ is focused.

Here we have

$$\begin{aligned} w_2 &= b^{s_1''}a^{t_1''}b^{s_2''}\dots a^{t_{\frac{m}{2}-1}''}b^{s_{\frac{m}{2}}''}a^{t_{\frac{m}{2}}''}, \\ w' &= b^{-s_1''}a^{-t_1''}b^{-s_2''}\dots a^{-t_{\frac{m}{2}-1}''}b^{-s_{\frac{m}{2}}''}a^{-t_{\frac{m}{2}}''}. \end{aligned}$$

We already had that $w' = w_2$, but these equations show that $w' \neq w_2$. Thus we have a contradiction.

Case 1ii: $(m, r' - r)$ is unfocused.

Here we have

$$w_2 = eb^{s''_1}e^{-1}a^{t''_1}eb^{s''_2}e^{-1}\dots a^{t''_{n''-1}}eb^{s''_{n''}}e^{-1}a^{t''_{n''}}e,$$
$$w' = eb^{-s''_1}e^{-1}a^{-t''_1}eb^{-s''_2}e^{-1}\dots a^{-t''_{n''-1}}eb^{-s''_{n''}}e^{-1}a^{-t''_{n''}}e.$$

By the same reasoning as the previous case, we find a contradiction and we are done.

Thus in all cases, the theorem holds. □

As a result of Proposition 3.3.4, we know that only the two words w_1 and w_2 split the wheel constructed by (m, r, c), and moreover that no other frame generates a wheel which w_1 or w_2 split. If there was such a frame, then the wheel it constructed would in fact be the same as the wheel generated by (m, r, c), and thus it would decompose as some $wb^{-1}w^{-1}a^{-1}$ with a different base. Since we have just proven this is impossible, we have that there is a one-to-one correspondence between pairs of associate arcs and frames, and we have hence proved Theorem 3.3.1. Since wheels are given by the words which split them, we have also proved the following.

Corollary 3.3.5. *No two frames generate the same wheel.*

3.3.1 Geometric viewpoint

We now note an observation on the presence of the word "c" in the structure given by Theorem 3.3.1. We demonstrated that this is the correct form of associate arcs by examining wheels and the restrictions that imposing lines of anti-symmetry places on their syllables. However, we can see that this has a geometric correlation as well.

Suppose we began with a pair of arcs α_1 and α_2 with conjugators w_1 and w_2 which are associated to the same curve, and suppose that the signature they correspond to is focused. These arcs sit inside a pair of pants P, and as the signature is focused we have that $w_1 = \overline{w_2}$; the powers in one word are those of the other but inverted. Thus for some powers $s_1, \dots, s_{\frac{m}{2}}, t_1, \dots, t_{\frac{m}{2}} \in \{1, -1\}$, we have

$$w_1 = b^{s_1}a^{t_1}b^{s_2}\dots a^{t_{\frac{m}{2}-1}}b^{s_{\frac{m}{2}}}a^{t_{\frac{m}{2}}},$$

$$w_2 = b^{-s_1} a^{-t_1} b^{-s_2} \dots a^{-t_{\frac{m}{2}-1}} b^{-s_{\frac{m}{2}}} a^{-t_{\frac{m}{2}}}.$$

Recalling the corresponding homomorphisms ϕ_1 and ϕ_2, if we compose ϕ_1 with the homomorphism which inverts a and b, we get $\overline{\phi_1}$ which, up to orientation, corresponds to the same arc as ϕ_2. This homomorphism which inverts a and b corresponds to the map on P which reflects it in a plane bisecting all boundary components. Thus we can see that two focused arcs are in fact symmetric copies of one another, up to orientation. We can see this for example in Figure 2.3, which we give here again for convenience.

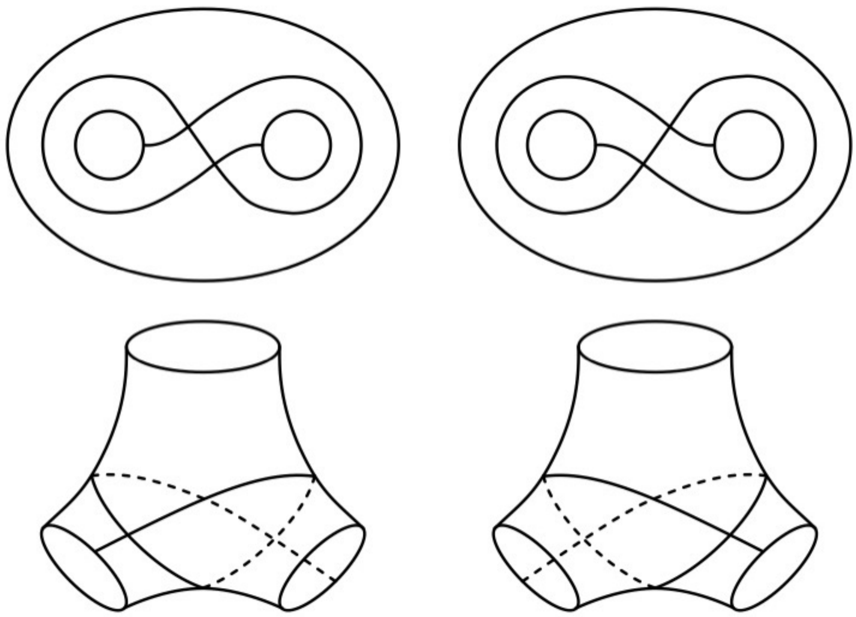

Figure 3.8: *The arcs from Figure 2.3, shown again here for convenience.*

We can imagine mapping this pair of pants inside another copy of itself. Under this continuous map, these arcs lose this symmetry as they are tangled up in the new surface. However, they will still be associated to a common curve, which has also been mapped into the new surface. In this way we can imagine generating many examples of associate arcs distinct from the one we began with, by mapping our pair in ever more complicated ways into P. This is nothing new however, as we can see evidence of this phenomenon in Theorem 3.3.1. Given any continuous map ψ of P inside itself which fixes the boundary components δ_i^P and δ_j^P homotopic to the generators a and b of $\pi_1(P)$, we can consider the corresponding homomorphism of $\pi_1(P)$ inside itself and choose it to be reduced. As a result we have a map

$$\begin{aligned} \psi^* \colon a &\mapsto a \\ b &\mapsto cbc^{-1} \end{aligned}$$

for some conjugator c. The homomorphisms ϕ_1 and ϕ_2 are given by

$$\begin{aligned} \phi_1 \colon a &\mapsto a & \qquad \phi_2 \colon a &\mapsto a \\ b &\mapsto w_1 b w_1^{-1} & b &\mapsto w_2 b w_2^{-1}. \end{aligned}$$

After composing the corresponding continuous maps of P with ψ, the homomorphisms become

$$\begin{aligned} \psi^* \circ \phi_1 \colon a &\mapsto a & \qquad \psi^* \circ \phi_2 \colon a &\mapsto a \\ b &\mapsto \psi^*(w_1)cbc^{-1}\psi^*(w_1^{-1}) & b &\mapsto \psi^*(w_2)cbc^{-1}\psi^*(w_2^{-1}). \end{aligned}$$

In these homomorphisms, b is conjugated by $\psi^*(w_1)c$ and $\psi^*(w_2)c$ respectively. If we expand these expressions, we see that for $n = \frac{m}{2}$,

$$\begin{aligned} \psi^*(w_1)c &= cb^{s_1}c^{-1}a^{t_1}cb^{s_2}c^{-1} \dots a^{t_{n-1}}cb^{s_n}c^{-1}a^{t_n}c \\ \psi^*(w_2)c &= cb^{-s_1}c^{-1}a^{-t_1}cb^{-s_2}c^{-1} \dots a^{-t_{n-1}}cb^{-s_n}c^{-1}a^{-t_n}c, \end{aligned}$$

which is precisely the expression given by Theorem 3.3.1 for the general form of the conjugators corresponding to associate arcs. We also note that the signature corresponding to these words will always be unfocused. If the old signature was (m, r), then the new signature will be $\big(|c|(m+1)+m, |c|r+r\big)$, and we have that $|c|(m+1)+m+1 = (|c|+1)(m+1)$ and $|c|r+r = (|c|+1)r$, thus $\text{hcf}\big(|c|(m+1)+m+1, |c|r+r\big) = |c|+1$, which is never 1. Thus any way in which we choose to map a focused pair of arcs into P results in a pair of arcs related to some unfocused signature.

As a result of this, we can view examples of associate arcs coming from focused signatures as true, irreducible examples of associate arcs in some sense, as there is no simpler example to derive them from. Each focused signature corresponds to a unique way of constructing a pair of inherently associate arcs, whilst unfocused signatures give arcs which are the result of mapping some focused pair of arcs. In this way, we could choose to view frames in a different light; removing the notion of unfocused all together, we could define a frame to be a triple (m, r, c), where $\text{hcf}(m+1, r) = 1$ and c is some conjugator of any length. This perspective highlights the notion of focused signatures being unaltered and original examples of associate arcs.

3.4 Implications of the relation

Following the results from Section 3.2 and Section 3.3 concerning conjugators, we now discuss the implications for the corresponding arcs. We recall that if α is an arc on the pair of pants P, then w_α is the word which conjugates b in the corresponding (reduced) homomorphism $\phi_\alpha\colon \pi_1(P) \to \pi_1(P)$ which fixes a. The correspondence between arcs and conjugators is bijective by Theorem 3.1.8. Recall from Definition 3.1.2 that $I_{i,j}$ is the associate map I restricted to the class of arcs between boundary components δ_i^P and δ_j^P. We can now prove the following theorem, from which we will deduce Theorem 3.

Theorem 3.4.1. *Let $\gamma \in \mathscr{C}(P)$. For $i \neq j$, we have $|I_{i,j}^{-1}(\gamma)| \leq 2$.*

Proof. If no arc joining δ_i^P and δ_j^P is associated to γ, then $|I_{i,j}^{-1}(\gamma)| = 0$. Otherwise, let α be such an arc. Theorem 3.1.8 gives us the corresponding conjugator w_α. We can then write the associated curve $\gamma = \gamma_\alpha$ as the word $w_\alpha b^{-1} w_\alpha^{-1} a^{-1}$. Writing this in cyclical notation, we have a wheel; either this wheel splits for some word other than w_α, or it does not. If not, then there is no other arc in $\mathcal{A}_{i,j}(P)$ which is associated to γ_α as its conjugator must appear in this wheel, and so the pre-image of γ_α under $I_{i,j}$ has size 1. If the wheel does split for some other word w_β, then some frame generates the wheel and by Corollary 3.3.5 we know that this frame is unique. By Theorem 3.3.1, we know that only w_α and w_β split this wheel and thus only α and β are associated to this curve. Hence the pre-image of γ_α has size two. □

Theorem 3 is then an immediate corollary of Lemma 3.1.1 and Theorem 3.4.1.

Theorem 3. *Let P be a pair of pants, and let α be a two-ended arc on P. Let $\gamma_\alpha = I(\alpha)$. Then $|I^{-1}(\gamma_\alpha)| \leq 2$.*

It seems reasonable to believe that a more general version of Theorem 3 should hold. For instance, there does not appear to be any reason why the one-ended case should be more complicated than the two-ended case, though this remains to be studied in detail.

In another direction, we would like to demonstrate that on any surface with boundary, at most two arcs can be associated to the same curve. With more work, it is possible that this will be seen as a direct consequence of Theorem 3. It seems reasonable to believe that on any surface with boundary S, if two arcs are associated to the same curve, then as they are associated to this curve through immersed pairs of pants, they must both exist in some immersed or embedded pair of pants in S. Future work will investigate this possibility, and examine other methods with which we might generalise Theorem 3.

Conjecture 3.4.2. *Let S be a surface, and let γ be a curve on S. Then $|I^{-1}(\gamma)| \leq 2$.*

Conjecture 3.4.2 would imply that in Theorem 1 and Theorem 2, the parameter $k(\alpha_0)$ can only equal 1 or 2. Recall that $k(\alpha_0)$ is the cardinality of $|I^{-1}(\gamma_{\alpha_0})|$, which is uniform across arcs of type α_0. At present, we are not aware of any pairs of associate arcs on a surface which are of the same type. It would be illuminating to find such an example, as this would be a counterexample to the claim that $k(\alpha_0) = 1$ for any arc α_0.

Question: Can we find a method to check whether two associate arcs are of the same type, and hence find a type of arc for which $k = 2$?

We also remark that the cases of arcs which are associated to the same curve are very unique in terms of the conjugators they correspond to. Not only must they have a very particular structure, but the powers $s_1, \ldots, s_n, t_1, \ldots, t_n \in \{1, -1\}$ in Theorem 3.3.1 must also satisfy some deeper condition. Consider for examples the signatures of the form $(6, r)$. As $6 + 1 = 7$ is prime, all such signatures are focused. Labelling the words which split the wheel constructed by $(6, r, \emptyset)$ as $w_{r,1}$ and $w_{r,2}$, we can derive by hand that

$$w_{2,1} = b^{-1}aba^{-1}b^{-1}a, \qquad\qquad w_{2,2} = ba^{-1}b^{-1}aba^{-1},$$

$$w_{4,1} = b^{-1}a^{-1}b^{-1}aba, \qquad w_{4,2} = baba^{-1}b^{-1}a^{-1},$$

$$w_{6,1} = b^{-1}ab^{-1}ab^{-1}a, \qquad w_{6,2} = ba^{-1}ba^{-1}ba^{-1}.$$

At present, no common property which separates these words out from other words of the form given by Theorem 3.3.1 presents itself.

We can easily find many examples of words which fit this structure but do not give associate arcs. For instance, for any conjugator c the words $cb^{-1}c^{-1}a^{-1}c$ and $cb^{-1}c^{-1}a^{-1}c$ are of the structure given by Theorem 3.3.1, but the corresponding arcs are not associated to the same curve. Moreover, if we choose w such that in the word $wb^{-1}w^{-1}a^{-1}$ the only pair of syllables b^{-1} and a^{-1} which are opposite each other in cyclical notation are the cores, then w can have no associates. Beyond this, the criteria which distinguish a word of this form which does indeed correspond to an associate arc are not known, and could be investigated further.

Question: Can we derive all choices of powers in words of the form given in Theorem 3.3.1 which do correspond to associate arcs, and thus describe the full class of associate arcs?

Chapter 4

Length-equivalent arcs

On a surface S of negative Euler characteristic, homotopy classes of paths such as curves and arcs are purely topological objects until we fix a hyperbolic metric on the surface, which assigns a length to each homotopy class of such paths. Different hyperbolic metrics may assign different lengths to the same class of paths, and for any path we can choose a hyperbolic metric which makes it arbitrarily long. See [12] for more details of this.

Interestingly, on any surface there exist collections of curves whose lengths are equal under *any* hyperbolic metric. Such curves are called *length-equivalent*, and have been studied by various authors such as Anderson [1], Buser [8], Horowitz [17], Leininger [20], and Masters [22].

Definition 4.0.1: Let X stand for any hyperbolic metric on S, and $\ell_X(\omega)$ denote the length of the curve or arc ω under X. Then two curves or arcs ω_1 and ω_2 are *length-equivalent* if for all X,

$$\ell_X(\omega_1) = \ell_X(\omega_2).$$

Note that we only consider pairs of curves or pairs of arcs, rather than one of each.

In fact, it has been demonstrated that there are multiple ways to construct families of length-equivalent curves which grow arbitrarily large. These are done using representatives for curves on S in the fundamental group $\pi_1(S)$. Horowitz [17] and Masters [22] each give a method which constructs words in the fundamental group iteratively from fixed generators, based on some integer sequences. We refer to work by Anderson [1] for a thorough survey of such examples. A reader familiar with these methods may recognise similarities with the method we lay out in Section 4.3.

In this chapter, we will demonstrate constructively that such families of arcs exist as well. We will begin by describing an example in Section 4.1 which is given implicitly by Buser [8]. In Section 4.2, we discuss a quality of words in the fundamental group of a surface which can detect whether the corresponding curves are length-equivalent, called the *trace*. Finally, in Section 4.3 we apply this to families of words given by homomorphisms of the form found in Chapter 3 to construct arbitrarily large families of length-equivalent arcs. Our families, unlike that given by Buser, will be made up of pairs of arcs which are associated to the same curve.

4.1 An example of a family of length-equivalent arcs

In Section 3.7 of [8], Buser gives a proof that the spectrum of the lengths of curves on a hyperbolic surface has unbounded multiplicity; by this we mean that there exist values λ such that arbitrarily many curves are of length λ. This is done by constructing arbitrarily large families of length-equivalent curves. In doing this, Buser implicitly constructs a family of length-equivalent arcs which also grows arbitrarily large. The work is done on the once-holed torus $S_{1,1}$, which can be viewed as a subsurface of any hyperbolic surface with genus at least 2.

To demonstrate how Buser's example defines a family of length-equivalent arcs, we view the once-holed torus as a pair of pants P after gluing together two of the cuffs, which we denote as δ_0^P and δ_1^P. Note that in order to perform this gluing map, we require that these boundary curves have the same length; that is, if X is the hyperbolic metric on P then $\ell_X(\delta_0^P) = \ell_X(\delta_1^P)$.

Buser constructs a family of length-equivalent curves by observing that the involution $\rho\colon P \to P$ given by the rotation fixing the third cuff and permuting δ_0^P and δ_1^P is an orientation-preserving isometry. Let ω be a "figure-eight" curve on $S_{1,1}$, with one self-intersection. This curve consists of a loop about the boundary curve and a loop around the genus. Cutting $S_{1,1}$ to retrieve P leaves ω as an arc on P which goes around the third cuff once; see Figure 4.1. We can see that $\rho(\omega) \neq \omega$, but as ρ is an isometry we have that $\ell_X(\rho(\omega)) = \ell_X(\omega)$. Since this is true under any hyperbolic metric X, ω and $\rho(\omega)$ are length-equivalent.

To define further curves, Buser defines a continuous map of $S_{1,1}$ into itself and

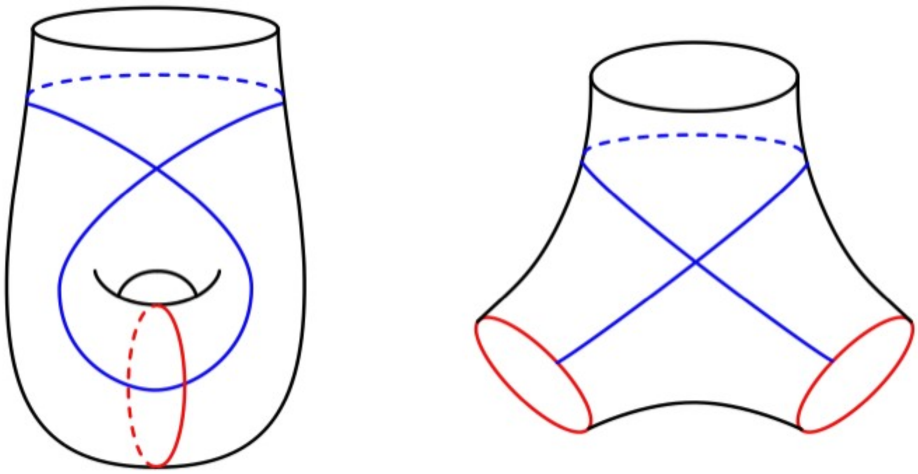

Figure 4.1: *The curve ω on the once-holed torus $S_{1,1}$ (left), and the resulting arc on the pair of pants P (right). The curve on $S_{1,1}$ we cut along to obtain P is highlighted in red.*

iterates this argument. We will present his method here in our language and on the pair of pants P. Consider γ_ω, the curve associated to ω on P. This curve bounds an immersed pair of pants; we denote the continuous map by ι_ω and inscribe the pre-image of this immersed copy of P with the arcs ω and $\rho(\omega)$ before mapping. Applying the involution ρ again to generate another two arcs, we have constructed the 4 arcs $\iota_\omega(\omega), \iota_\omega(\rho(\omega)), \rho(\iota_\omega(\omega))$ and $\rho(\iota_\omega(\rho(\omega))$ on P. These arcs are length-equivalent, as ι_ω preserves length-equivalency and ρ is an isometry. If we continually iterate this procedure, we can generate a collection of 2^κ length-equivalent arcs for any positive integer κ, and thus we are done. See [8] for the details which we omit here.

4.2 Trace-equivalent words

There are many examples of length-equivalent arcs beyond those given by Buser's example. The key observation which justifies this is the fact that associate arcs of the second kind (see Section 2.1) must be length-equivalent. This is because we can express the length of an arc in terms of the length of its associated curve and its corresponding boundary components using (2.5) in the proof of Lemma 2.2.1. Hence, any two arcs of the second kind which are associated to the same curve must always have the same length and hence be length-equivalent. Moreover, if we found two length-equivalent curves which were each associated to two arcs of the second kind, then these four arcs would all be length-equivalent by the same reasoning as above.

Lemma 4.2.1. *Pairs of associate arcs of the second kind are length-equivalent. Moreover, for arcs α_1 and α_2 of the second kind, if γ_{α_1} and γ_{α_2} are length-equivalent then so too are α_1 and α_2.*

We can use this observation to produce more examples of length-equivalent arcs. If we view the four-holed sphere $S_{0,4}$ as two pairs of pants glued together along a cuff, then the two arcs given by the seams of the pairs of pants between the remaining cuffs are associate arcs of the first kind. These arcs are both associated to the simple curve given by the image of the cuffs we glued together, and if we choose a metric on $S_{0,4}$ such that for instance all four boundary components have the same length, these arcs are length-equivalent. More examples can be generated from this pair, such as by applying any power of an appropriately chosen Dehn twist[1]. However, these will only be length-equivalent in pairs. The aim for the rest of this chapter is to construct specially chosen families of pairs of associate arcs of the second kind which we can demonstrate are length-equivalent in arbitrarily large collections. If we do not specify otherwise, associate arcs will be assumed to be of the second kind for the remainder of the chapter.

For some $i \neq j$, take any collection of pairs of associate arcs in $\mathcal{A}_{i,j}(P)$ and consider the collection of the curves associated to them. Each pair is associated to a single curve on P and no two pairs are associated to the same curve by Theorem 3, so we have exactly half as many curves as we do arcs. The literature provides a method to show that our curves are length-equivalent by checking if they are *trace-equivalent*, which would in turn imply by Lemma 4.2.1 that our arcs are length-equivalent.

We document the method to prove that two words in $\pi_1(P)$ are trace-equivalent here, following the work of Anderson[1], Horowitz [17] and Leininger [20]. We refer the reader to these authors for a much more thorough treatment of the notions of trace- and length-equivalency. Consider the fundamental group $\pi_1(P)$ of P. A *representation* of $\pi_1(P)$ is a map from the fundamental group into the group $\mathrm{PSL}(2,\mathbb{R})$

[1] A Dehn twist is a simple example of a mapping class on any surface. See [12] for a definition and thorough treatment.

which assigns to every group element a matrix. These matrices have a well-defined *trace*, which is the sum of the diagonal elements of the matrix. If two group elements are such that no matter what representation is chosen, the corresponding matrices have the same trace, we call them *trace-equivalent*. Note that the trace of a matrix is invariant under conjugation and taking the inverse, thus the same is true for the trace of an element of $\pi_1(P)$. We let $tr_\mu(\cdot)$ denote the trace of a word under a representation μ.

Fact 4.2.2: The trace $tr_\mu(\cdot)$ is invariant under conjugation and taking inverses.

Since conjugate elements of $\pi_1(P)$ have the same trace, the trace of a (homotopy class of a) curve is well-defined. In [20], Leininger proves the following statement using a relationship between the trace and the length function on Teichmüller space, as well as work by Rapinchuck [32] on representation varieties.

Theorem 4.2.3 (Theorem 1.4, [20])**.** *Let γ_1 and γ_2 be curves on a surface S. Then, writing γ_1 and γ_2 for representatives of their conjugacy classes in $\pi_1(S)$, we have*

$$\gamma_1 \text{ and } \gamma_2 \text{ are length-equivalent} \iff \gamma_1 \text{ and } \gamma_2 \text{ are trace-equivalent.}$$

Using the definition of the trace, one can derive the following basic trace relations.

Fact 4.2.4: For group elements u, v, we have that

1. $tr_\mu(u) = tr_\mu(u^{-1})$,
2. $tr_\mu(u) = tr_\mu(vuv^{-1})$,
3. $tr_\mu(uv) = tr_\mu(vu)$,
4. $tr_\mu(uv) = tr_\mu(u)tr_\mu(v) - tr_\mu(uv^{-1})$.

These relations, particularly relation 4, could be used to express the trace of a word in terms of the trace of shorter words which may be simpler to compute. Whilst this method may be sufficient for short words, it is not practical when dealing with longer words. For example, consider the frames $(4, 2, \emptyset)$ and $(4, 4, \emptyset)$ as defined in Definition 3.2.12. If we wanted to check whether the curves they construct are trace-equivalent, we would be examining two 18-syllable words. Using relation 4 in

Fact 4.2.4 to break down the trace of an 18-syllable word results in the product of the traces of two shorter words whose syllable lengths sum to 18, and the trace of another word which could also have 18 syllables. Unless the words have a very particular form which allows for heavy cancellation to reduce the complexity quickly, it would be strenuous to gain any insight on whether they are trace-equivalent. Fortunately, this process can be streamlined using the work of Horowitz [17]. The first observation to make is that if a word can be expressed in terms of some shorter words u and v, we can persistently apply Fact 4.2.4 until we have a polynomial in terms of the traces of u, v and uv. For example, let u and v be (cyclically reduced) words in a and b, and consider uvu. We can use Fact 4.2.4 to write

$$\begin{aligned} tr_\mu(uvu) &= tr_\mu(u)tr_\mu(vu) - tr_\mu(uu^{-1}v^{-1}) \\ &= tr_\mu(u)tr_\mu(uv) - tr_\mu(v). \end{aligned} \tag{4.1}$$

In general, we have the following theorem.

Theorem 4.2.5 (Horowitz [17])**.** *Let $W(u,v)$ be a word written in the letters u,v. Then there exists some polynomial $\mathcal{P}$ such that*

$$tr_\mu\big(W(u,v)\big) = \mathcal{P}\Big(tr_\mu(u), tr_\mu(v), tr_\mu(uv)\Big).$$

For words u and v, we write $u \equiv_{tr} v$ if they are trace-equivalent. Suppose we have u_1, v_1, u_2, v_2 such that $u_1 \equiv_{tr} u_2, v_1 \equiv_{tr} v_2$ and $u_1v_1 \equiv_{tr} u_2v_2$. Then using (4.1) we can express $tr_\mu(u_iv_iu_i)$ as $tr_\mu(u_i)tr_\mu(u_iv_i) - tr_\mu(v_i)$ for each i. The value of this expression is the same for $i = 1$ and $i = 2$; thus $u_1v_1u_1$ and $u_2v_2u_2$ are trace-equivalent. The general version of this is as follows.

Corollary 4.2.6. *Let $W(u,v)$ be a word written in the letters u,v. Suppose u_1, v_1, u_2, v_2 are words in a and b such that*

$$u_1 \equiv_{tr} u_2,$$

$$v_1 \equiv_{tr} v_2,$$

$$u_1v_1 \equiv_{tr} u_2v_2.$$

Then $W(u_1,v_1) \equiv_{tr} W(u_2,v_2)$.

Armed with this, we will be able to show that large classes of well-chosen curves constructed from frames are trace-equivalent. To choose these curves, we will begin with a pair of maps corresponding to a frame and consider all maps given by applying either map a particular number of times. We will illustrate this by considering the frame of lowest complexity, $(2, 2, \emptyset)$. This corresponds to the two arcs given by the embeddings

$$\begin{aligned} \phi_1 \colon a &\mapsto a & \qquad \phi_2 \colon a &\mapsto a \\ b &\mapsto b^{-1}aba^{-1}b & b &\mapsto ba^{-1}bab^{-1} \end{aligned} \tag{4.2}$$

and their associated curve is given by $b^{-1}ab^{-1}a^{-1}ba^{-1} \in \pi_1(P)$. This is Example 3.1.3 from Chapter 3. From the work in Section 3.1, we have that these homomorphisms each correspond to an continuous map of P within itself. Mapping this example of length-equivalent arcs inside another copy of P in a sensible way will generate another example of length-equivalent arcs, following a similar method to the example in Section 4.1. We describe this in more detail in the following section, and we note that this was discovered independently to Buser's example.

4.3 Constructing new families of length-equivalent arcs

In this section, we will demonstrate that in defining pairs of associate arcs and the frames from which they are derived, we have implicitly constructed new arbitrarily large families of length-equivalent arcs. By taking a collection of associate arcs and showing that the curves associated to them are all length-equivalent using Theorem 4.2.3 and Corollary 4.2.6, we will then have by Lemma 4.2.1 that the associate arcs are all length-equivalent.

Throughout the following, we will say Φ is a *string* of some maps ϕ_1 and ϕ_2 if it takes the form

$$\Phi = \phi_{i_n} \circ \phi_{i_{n-1}} \circ \cdots \circ \phi_{i_1},$$

for some positive integer n, where $i_j = 1, 2$. We say that such a string has length n.

We remark that a composition of reduced maps is reduced, as in Definition 3.1.4. Let ϕ_1 and ϕ_2 be reduced maps, and let w_1 and w_2 be the conjugators defining them.

Then indeed $\phi_1 \circ \phi_2(a) = a$, and we can write

$$\begin{aligned}\phi_1 \circ \phi_2(b) &= \phi_1(w_2 b w_2^{-1}) \\ &= \phi_1(w_2) w_1 b w_1^{-1} \phi_1(w_2)^{-1}.\end{aligned}$$

Consider $\phi_1(w_2)$, in which we replace each syllable b^k in w_2 with $w_1 b^k w_1^{-1}$. Since $w_1 b^k w_1^{-1}$ starts and ends with a b-syllable by Lemma 3.1.5, the resulting word $\phi_1(w_2)$ must be cyclically reduced, begin with b and end with a; the same holds for the word $\phi_1(w_2)w_1$. Therefore $\phi_1 \circ \phi_2$ is a reduced map with conjugator $\phi_1(w_2)w_1$. Moreover, a string Φ of reduced maps of any length is itself reduced.

Recall that by Theorem 3.1.8, each (homotopy class of) arc between a pair of boundary components on P corresponds to a unique conjugator. We first prove the following.

Theorem 4.3.1. *Let α_1 and α_2 be a pair of associate arcs, and w_{α_1} and w_{α_2} their corresponding conjugators. Let ϕ_1 and ϕ_2 be the maps given by*

$$\begin{aligned}\phi_1 \colon a &\mapsto a & \qquad \phi_2 \colon a &\mapsto a \\ b &\mapsto w_{\alpha_1} b w_{\alpha_1}^{-1} & b &\mapsto w_{\alpha_2} b w_{\alpha_2}^{-1}.\end{aligned}$$

Let Φ be any string of ϕ_1 and ϕ_2. Then for some pair of associate arcs β_1 and β_2, the maps $\Phi \circ \phi_1$ and $\Phi \circ \phi_2$ are given as

$$\begin{aligned}\Phi \circ \phi_1 \colon a &\mapsto a & \qquad \Phi \circ \phi_2 \colon a &\mapsto a \\ b &\mapsto w_{\beta_1} b w_{\beta_1}^{-1} & b &\mapsto w_{\beta_2} b w_{\beta_2}^{-1}.\end{aligned}$$

We will verify a simple case of Theorem 4.3.1 by hand before we proceed to the general case. Recall that the maps ϕ_1 and ϕ_2 from Example 3.1.3 derived from the frame $(2, 2, \emptyset)$ are given as follows:

$$\begin{aligned}\phi_1 \colon a &\mapsto a & \qquad \phi_2 \colon a &\mapsto a \\ b &\mapsto b^{-1} a b a^{-1} b & b &\mapsto b a^{-1} b a b^{-1}.\end{aligned}$$

We can check by hand that the pairs $(\phi_1 \circ \phi_1, \phi_1 \circ \phi_2)$ and $(\phi_2 \circ \phi_1, \phi_2 \circ \phi_2)$ each correspond to a pair of associate arcs. Computing them directly, we have

$$\begin{aligned} \phi_1 \circ \phi_1 \colon a &\mapsto a \\ b &\mapsto b^{-1}ab^{-1}a^{-1}bab^{-1}aba^{-1}ba^{-1}b^{-1}aba^{-1}b \\ \phi_1 \circ \phi_2 \colon a &\mapsto a \\ b &\mapsto b^{-1}aba^{-1}ba^{-1}b^{-1}aba^{-1}bab^{-1}ab^{-1}a^{-1}b \end{aligned}$$

$$\begin{aligned} \phi_2 \circ \phi_1 \colon a &\mapsto a \\ b &\mapsto ba^{-1}b^{-1}ab^{-1}aba^{-1}bab^{-1}a^{-1}ba^{-1}bab^{-1} \\ \phi_2 \circ \phi_2 \colon a &\mapsto a \\ b &\mapsto ba^{-1}bab^{-1}a^{-1}ba^{-1}bab^{-1}aba^{-1}b^{-1}ab^{-1} \end{aligned}$$

These four maps are all reduced and distinct. We must verify that each pair is associated to the same curve. The words for the corresponding curves are

$$\begin{aligned} \phi_1 \circ \phi_1(b^{-1}a^{-1}) &= b^{-1}ab^{-1}a^{-1}bab^{-1}ab^{-1}a^{-1}ba^{-1}\underline{b}^{-1}aba^{-1}ba^{-1}, \\ \phi_1 \circ \phi_2(b^{-1}a^{-1}) &= \underline{b}^{-1}aba^{-1}ba^{-1}b^{-1}ab^{-1}a^{-1}bab^{-1}ab^{-1}a^{-1}ba^{-1}, \end{aligned}$$

$$\begin{aligned} \phi_2 \circ \phi_1(b^{-1}a^{-1}) &= ba^{-1}b^{-1}ab^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}\underline{b}a^{-1}bab^{-1}a^{-1}, \\ \phi_2 \circ \phi_2(b^{-1}a^{-1}) &= \underline{b}a^{-1}bab^{-1}a^{-1}ba^{-1}b^{-1}ab^{-1}aba^{-1}b^{-1}ab^{-1}a^{-1}. \end{aligned}$$

In both cases, cyclically permuting the first word of the pair to the right by 6 places gives the second word of the pair. This is made easiest to see by reading each word cyclically beginning at the underlined syllable. Thus these are pairs of conjugate words, and therefore homotopic as curves, so we have two pairs of associate arcs. We can apply a very similar argument to demonstrate that this holds for any such pair of maps and any length of string.

It is worth noting that these pairs of homomorphisms must correspond to the same signature, as they are defined by words of the same length and the cyclic permutations required are by the same amount. Since focused signatures only correspond to a single frame and thus a single pair of homomorphisms, this signature must be unfocused. In fact, this will be the case for all iterations of such maps, as discussed in Section 3.4. Take maps ϕ_1 and ϕ_2 corresponding to a signature (m, r), and a map ψ corresponding to some other signature (m', r'). Then the maps $\phi_i \circ \psi$ correspond to the signature $\big(m'(m+1) + m, r'(m+1)\big)$, which is always unfocused as $m'(m+1) + m + 1$ and $r'(m+1)$ share a factor of $m+1$. In particular, this is true when the map ψ is a string of ϕ_1 and ϕ_2. We now proceed to the general case of Theorem 4.3.1.

Proof of Theorem 4.3.1. As ϕ_1 and ϕ_2 define associate arcs, they correspond to some frame (m, r, c) by the discussion following Proposition 3.2.13. Thus (without loss of generality) cyclically permuting $\phi_1(b^{-1}a^{-1})$ to the right by r places gives $\phi_2(b^{-1}a^{-1})$. Thus we can partition $\phi_1(b^{-1}a^{-1})$ into two words A and B where $|B| = r$ such that

$$\begin{aligned}\phi_1(b^{-1}a^{-1}) &= AB,\\ \phi_2(b^{-1}a^{-1}) &= BA.\end{aligned}$$

Let Φ be any string in ϕ_1 and ϕ_2. We can then write

$$\begin{aligned}\Phi \circ \phi_1(b^{-1}a^{-1}) &= \Phi(AB),\\ &= \Phi(A)\Phi(B),\\ \Phi \circ \phi_2(b^{-1}a^{-1}) &= \Phi(BA),\\ &= \Phi(B)\Phi(A).\end{aligned}$$

Since the map Φ is reduced by the discussion above Theorem 4.3.1, these words are cyclically reduced. Furthermore, cyclically permuting $\Phi \circ \phi_1(b^{-1}a^{-1})$ by the length of $\Phi(B)$ gives $\Phi \circ \phi_2(b^{-1}a^{-1})$, and thus they are conjugate. □

For some pair ϕ_1 and ϕ_2 defining associate arcs, and some $n \in \mathbb{Z}_{>0}$, define the set

$$\Gamma_n := \{\Phi(b^{-1}a^{-1}) \mid \Phi \text{ is a string in } \phi_1 \text{ and } \phi_2 \text{ of length } n\}.$$

These are all of the words defining curves associated to arcs given by strings of length n. We prove the following.

Theorem 4.3.2. *Let ϕ_1 and ϕ_2 be a pair of maps corresponding to a pair of associate arcs. For any n, all arcs corresponding to strings of length n in ϕ_1 and ϕ_2 are length-equivalent.*

Proof. Let Φ be a string of length n of ϕ_1 and ϕ_2. Then by Theorem 4.3.1, Φ corresponds to an associate arc whose associate is given by swapping the first map ϕ_{i_1} in Φ for the other. That is, if $\phi_{i_1} = \phi_1$ then the associate arc is given by the same string with $\phi_{i_1} = \phi_2$. We then know that each such pair of associate arcs is length-equivalent by Lemma 4.2.1. If we can show that the curves associated to these pairs are themselves length-equivalent, this will imply that the arcs are all length-equivalent.

For any n, the collection Γ_n consists of all curves corresponding to strings of length n; if we can demonstrate that all of these curves are trace-equivalent, we will be done by Theorem 4.2.3.

Claim: For any $n \geq 1$, all of the elements of Γ_n are trace-equivalent.

We proceed by induction on n. For $n = 1$ the only strings are ϕ_1 and ϕ_2, thus this case holds by assumption.

Let $n \geq 2$, and suppose all of the words in Γ_{n-1} are trace-equivalent. Take any pair of words $\Phi(b^{-1}a^{-1})$, $\Psi(b^{-1}a^{-1})$ in Γ_n, where Φ and Ψ are strings of length n. Suppose that the curves corresponding to the words $\Phi(b^{-1}a^{-1})$ and $\Psi(b^{-1}a^{-1})$ are homotopic. Then they are conjugate by definition, and hence trace-equivalent.

Suppose instead that $\Phi(b^{-1}a^{-1})$ and $\Psi(b^{-1}a^{-1})$ are not homotopic. Then as conjugate words are trace-equivalent and changing the first map in a string if necessary, without loss of generality we can choose Φ and Ψ to be of the form

$$\Phi = \Phi' \circ \phi_1,$$
$$\Psi = \Psi' \circ \phi_1,$$

where Φ' and Ψ' are strings of length $n - 1$. Thus we can write

$$\Phi(b^{-1}a^{-1}) = \Phi' \circ \phi_1(b^{-1}a^{-1}),$$
$$\Psi(b^{-1}a^{-1}) = \Psi' \circ \phi_1(b^{-1}a^{-1}).$$

By the definition of reduced maps, the word $\phi_1(b^{-1}a^{-1})$ must take the form

$$\begin{aligned}\phi_1(b^{-1}a^{-1}) &= w_1 b^{-1} w_1^{-1} a^{-1} \\ &= b^{k_1} a^{l_1} \dots b^{k_N} a^{l_N}\end{aligned}$$

for some conjugator $w_1 \in \mathfrak{W}$, some positive integer N and non-zero integers $k_1, \dots, k_N$, $l_1, \dots, l_N$. Therefore we have that

$$\Phi(b^{-1}a^{-1}) = \Phi'(b^{k_1} a^{l_1} \dots b^{k_N} a^{l_N}) = \Phi'(b)^{k_1} a^{l_1} \dots \Phi'(b)^{k_N} a^{l_N},$$
$$\Psi(b^{-1}a^{-1}) = \Psi'(b^{k_1} a^{l_1} \dots b^{k_N} a^{l_N}) = \Psi'(b)^{k_1} a^{l_1} \dots \Psi'(b)^{k_N} a^{l_N},$$

as reduced maps fix a. Now that we have two words of the same form, we are led to apply Corollary 4.2.6. Here, the words in question are

$$\begin{aligned}u_1 &= a, \\ v_1 &= \Phi'(b), \\ u_2 &= a, \\ v_2 &= \Psi'(b).\end{aligned}$$

We must verify that $u_1 \equiv_{tr} u_2$, $v_1 \equiv_{tr} v_2$, and $u_1 v_1 \equiv_{tr} u_2 v_2$. We have $u_1 \equiv_{tr} u_2$ as a is trace-equivalent to itself. By assumption, $\Phi'(b)$ and $\Psi'(b)$ are both conjugate to b and hence conjugate to each other, therefore they are trace-equivalent by Fact 4.2.2. Hence $v_1 \equiv_{tr} v_2$. Finally, we write

$$\begin{aligned}u_1 v_1 &= a\Phi'(b) = \Phi'(ab), \\ u_2 v_2 &= a\Psi'(b) = \Psi'(ab),\end{aligned}$$

and note that $\Phi'(ab) \equiv_{tr} \Phi'(b^{-1}a^{-1})$ and $\Psi'(ab) \equiv_{tr} \Psi'(b^{-1}a^{-1})$ as they are inverses. Now Φ' and Ψ' are both strings of length $n-1$, and thus $\Phi'(b^{-1}a^{-1})$ and $\Psi'(b^{-1}a^{-1})$ are trace-equivalent by the induction hypothesis. Hence $\Phi'(ab)$ and $\Psi'(ab)$ are trace-equivalent and thus so too are $\Phi(b^{-1}a^{-1})$ and $\Psi(b^{-1}a^{-1})$ by Corollary 4.2.6. □

As a result, we can now say that beginning with any frame (m, r, c) and its corresponding homomorphisms ϕ_1 and ϕ_2 given by the words w_1 and w_2 from Theorem 3.3.1, we can generate a family of length-equivalent arcs in P between δ_0^P and δ_1^P of size 2^n for any positive integer n. These will be given by strings of ϕ_1 and ϕ_2 of length n. By mapping P into any surface with boundary S such that the images of δ_0^P and δ_1^P are boundary components, we can produce such a family in S.

Appendix A

Computation of the error function limit

We want to show that the limit of the error function

$$E_{i,j}(\ell) = 2\cosh^{-1}(A_{i,j}\cosh(\ell) - B_{i,j}) - 2\ell,$$

as defined in the proof of Lemma 2.2.1, is $2\ln(A_{i,j})$. Cancelling the common factor of 2, our result will follow if we can show that

$$\lim_{\ell\to\infty}\big(\cosh^{-1}(A_{i,j}\cosh(\ell) - B_{i,j}) - \ell\big) = \ln(A_{i,j}).$$

For ease of notation we drop the indices "$\star_{i,j}$". Let $\varepsilon > 0$. We want to show that there exists some L_0 such that for all $\ell \geq L_0$,

$$|\cosh^{-1}(A\cosh(\ell) - B) - \ell - \ln(A)| < \varepsilon.$$

Or equivalently, that

$$-\varepsilon < \cosh^{-1}(A\cosh(\ell) - B) - \ell - \ln(A) < \varepsilon.$$

For $x = A\cosh(\ell) - B$, we can write this as

$$-\varepsilon < \ln(x + \sqrt{x^2 - 1}) - \ell - \ln(A) < \varepsilon$$

since for $x \geq 1$, $\cosh^{-1}(x) = \ln(x + \sqrt{x^2 - 1})$.

Remark: $\lim_{x\to\infty} \dfrac{x + \sqrt{x^2-1}}{2x} = 1.$

Therefore for all $\varepsilon_1 > 0$, there exists L_1 such that for all $\ell \geq L_1$, or equivalently, for all $x \geq A\cosh(L_1) - B$, we have

$$\begin{aligned} x + \sqrt{x^2-1} &< (1+\varepsilon_1)2x \\ x + \sqrt{x^2-1} &> (1-\varepsilon_1)2x. \end{aligned} \tag{A.1}$$

Using the expression $\cosh(z) = \frac{e^z+e^{-z}}{2}$, we can write $x = A\frac{\mathrm{e}^\ell+\mathrm{e}^{-\ell}}{2} - B$. Combining this with (A.1), we have that for all $\ell \geq L_1$,

$$\begin{aligned} \ln(x+\sqrt{x^2-1}) - \ell - \ln(A) &< \ln\Big((1+\varepsilon_1)2A\frac{\mathrm{e}^\ell+\mathrm{e}^{-\ell}}{2} - 2(1+\varepsilon_1)B\Big) - \ell - \ln(A) \\ \ln(x+\sqrt{x^2-1}) - \ell - \ln(A) &> \ln\Big((1-\varepsilon_1)2A\frac{\mathrm{e}^\ell+\mathrm{e}^{-\ell}}{2} - 2(1-\varepsilon_1)B\Big) - \ell - \ln(A) \end{aligned}$$

for some $\varepsilon_1 > 0$.

Remark: $\lim_{\ell\to\infty} \dfrac{\mathrm{e}^\ell+\mathrm{e}^{-\ell}}{\mathrm{e}^\ell} = 1.$

As a result, for all $\varepsilon_2 > 0$, there exists L_2 such that for all $\ell \geq L_2$,

$$\begin{aligned} \mathrm{e}^\ell + \mathrm{e}^{-\ell} &< (1+\varepsilon_2)\mathrm{e}^\ell \\ \mathrm{e}^\ell + \mathrm{e}^{-\ell} &> (1-\varepsilon_2)\mathrm{e}^\ell. \end{aligned}$$

Thus for $\ell \geq \max\{L_1, L_2\}$, we can write

$$\begin{aligned} \ln(x+\sqrt{x^2-1}) - \ell - \ln(A) &< \ln\big((1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell - 2(1+\varepsilon_1)B\big) - \ell - \ln(A) \\ \ln(x+\sqrt{x^2-1}) - \ell - \ln(A) &> \ln\big((1-\varepsilon_1)A(1-\varepsilon_2)\mathrm{e}^\ell - 2(1-\varepsilon_1)B\big) - \ell - \ln(A) \end{aligned}$$

for some $\varepsilon_1, \varepsilon_2 > 0$.

Remark: $\lim_{\ell\to\infty} \dfrac{(1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell - 2(1+\varepsilon_1)B}{(1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell} = 1.$

Hence for all $\varepsilon_3 > 0$, there exists L_3 such that for $\ell \geq L_3$,

$$\begin{aligned} (1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell - 2(1+\varepsilon_1)B) &< (1+\varepsilon_3)(1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell \\ (1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell - 2(1+\varepsilon_1)B) &> (1-\varepsilon_3)(1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^\ell. \end{aligned}$$

So we have that for $\ell \geq \max\{L_1, L_2, L_3\}$,

$$\begin{aligned}
&\ln((1+\varepsilon_1)A(1+\varepsilon_2)\mathrm{e}^{\ell} - 2(1+\varepsilon_1)B) - \ell - \ln(A) \\
&\qquad\qquad < \ln((1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)A\mathrm{e}^{\ell}) - \ell - \ln(A) \\
&\qquad\qquad = \ln((1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)) + \ln(A) + \ell - \ell - \ln(A) \\
&\qquad\qquad = \ln((1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)) \\
&\ln((1-\varepsilon_1)A(1-\varepsilon_2)\mathrm{e}^{\ell} - 2(1-\varepsilon_1)B) - \ell - \ln(A) \\
&\qquad\qquad > \ln((1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon_3)A\mathrm{e}^{\ell}) - \ell - \ln(A) \\
&\qquad\qquad = \ln((1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon_3)) + \ln(A) + \ell - \ell - \ln(A) \\
&\qquad\qquad = \ln((1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon_3))
\end{aligned}$$

for some $\varepsilon_1, \varepsilon_2, \varepsilon_3 > 0$. In particular, recalling the definition of $E_{i,j}$, we have that

$$\begin{aligned}
\frac{1}{2}E_{i,j}(\ell) - \ln(A_{i,j}) &< \ln\big((1+\varepsilon_1)(1+\varepsilon_2)(1+\varepsilon_3)\big) \\
\frac{1}{2}E_{i,j}(\ell) - \ln(A_{i,j}) &> \ln\big((1-\varepsilon_1)(1-\varepsilon_2)(1-\varepsilon_3)\big).
\end{aligned}$$

Let $\varepsilon_0 > 0$ be such that

$$3\ln(1+\varepsilon_0) < \varepsilon$$

and set $\varepsilon_1 = \varepsilon_2 = \varepsilon_3 = \varepsilon_0$. Set $L_0 = \max\{L_1, L_2, L_3\}$. Then by the above working, for all $\ell \geq L_0$,

$$\frac{1}{2}E_{i,j}(\ell) - \ln(A_{i,j}) < \varepsilon.$$

We also have that

$$3\ln(1-\varepsilon_0) > -\varepsilon,$$

and thus

$$\frac{1}{2}E_{i,j}(\ell) - \ln(A_{i,j}) > -\varepsilon$$

and so we are done.

Bibliography

[1] J. W. Anderson, *Variations on a theme of Horowitz*, in Kleinian groups and hyperbolic 3-manifolds (Warwick, 2001), vol. 299 of London Math. Soc. Lecture Note Ser., Cambridge Univ. Press, Cambridge, 2003, pp. 307–341.

[2] A. Basmajian, *The orthogonal spectrum of a hyperbolic manifold*, Amer. J. Math., 115 (1993), pp. 1139–1159.

[3] ——, *Universal length bounds for non-simple closed geodesics on hyperbolic surfaces*, J. Topol., 6 (2013), pp. 513–524.

[4] A. Basmajian, H. Parlier, and S. P. Tan, *Prime orthogeodesics, concave cores and families of identities on hyperbolic surfaces.* Preprint, arXiv:2006.04872, 2020.

[5] N. Bell, *Counting arcs on hyperbolic surfaces.* Groups, Geometry and Dynamics (to appear), Preprint, arXiv:2011.13969, 2020.

[6] J. S. Birman and C. Series, *Geodesics with bounded intersection number on surfaces are sparsely distributed*, Topology, 24 (1985), pp. 217–225.

[7] G. E. Bredon, *Topology and geometry*, vol. 139, Springer Science & Business Media, 2013.

[8] P. Buser, *Geometry and spectra of compact Riemann surfaces*, Modern Birkhäuser Classics, Birkhäuser Boston, Ltd., Boston, MA, 2010. Reprint of the 1992 edition.

[9] V. Erlandsson, H. Parlier, and J. Souto, *Counting curves, and the stable length of currents*, J. Eur. Math. Soc. (JEMS), 22 (2020), pp. 1675–1702.

[10] V. Erlandsson and J. Souto, *Mirzakhani's Curve Counting.* Preprint, arXiv:1904.05091, 2019.

[11] V. Erlandsson and J. Souto, *Geodesic currents and Mirzakhani's curve counting*, Birkhäuser Progress in Mathematics, vol. 345, Birkhäuser Boston, Ltd., Boston, MA, 2022. to be published.

[12] B. Farb and D. Margalit, *A primer on mapping class groups*, vol. 49 of Princeton Mathematical Series, Princeton University Press, Princeton, NJ, 2012.

[13] A. HATCHER, *Algebraic topology*, accessed online at http://www. math. cornell. edu/˜ hatcher/AT/ATpage. html, (2002).

[14] A. E. Hatcher, *Measured lamination spaces for surfaces, from the topological viewpoint*, Topology Appl., 30 (1988), pp. 63–88.

[15] Y. M. He, *Prime number theorems for Basmajian-type identities.* Preprint, arXiv:1811.05367, 2018.

[16] M. W. Hirsch, *Differential topology*, vol. 33, Springer Science & Business Media, 2012.

[17] R. D. Horowitz, *Characters of free groups represented in the special linear group*, Comm. on Pure and App. Math., 25 (1972), pp. 635–649.

[18] H. Huber, *Zur analytischen Theorie hyperbolischen Raumformen und Bewegungsgruppen*, Math. Ann., 138 (1959), pp. 1–26.

[19] L. Keen, *Collars on riemann surfaces*, in Discontinuous groups and Riemann surfaces (Proc. Conf., Univ. Maryland, College Park, Md., 1973), 1974, pp. 263–268.

[20] C. J. Leininger, *Equivalent curves in surfaces*, Geom. Dedicata, 102 (2003), pp. 151–177.

[21] G. A. Margulis, *Certain applications of ergodic theory to the investigation of manifolds of negative curvature*, Funkcional. Anal. i Priložen., 3 (1969), pp. 89–90.

[22] J. D. Masters, *Length multiplicities of hyperbolic 3-manifolds*, Israel J. Math., 119 (2000), pp. 9–28.

[23] G. McShane, *Simple geodesics and a series constant over Teichmuller space*, Invent. Math., 132 (1998), pp. 607–632.

[24] G. McShane and I. Rivin, *Simple curves on hyperbolic tori*, C. R. Acad. Sci. Paris Sér. I Math., 320 (1995), pp. 1523–1528.

[25] M. Mirzakhani, *Growth of the number of simple closed geodesics on hyperbolic surfaces*, Ann. of Math. (2), 168 (2008), pp. 97–125.

[26] M. Mirzakhani, *Counting Mapping Class group orbits on hyperbolic surfaces.* Preprint, `arXiv:1601.03342`, 2016.

[27] J. Parkkonen and F. Paulin, *Counting common perpendicular arcs in negative curvature*, Ergodic Theory Dynam. Systems, 37 (2017), pp. 900–938.

[28] H. Parlier, *Geodesic and orthogeodesic identities on hyperbolic surfaces.* Preprint, `arXiv:2004.09078`, 2020.

[29] R. C. Penner, *The decorated Teichmüller space of punctured surfaces*, Comm. Math. Phys., 113 (1987), pp. 299–339.

[30] ——, *Lambda Lengths*, Aarhus University, Lecture Notes, online at http://www.ctqm.au.dk/research/MCS/lambdalengths.pdf (2006).

[31] R. C. Penner and J. L. Harer, *Combinatorics of train tracks*, vol. 125 of Annals of Mathematics Studies, Princeton University Press, Princeton, NJ, 1992.

[32] A. Rapinchuk, V. Benyash-Krivetz, and V. Chernousov, *Representation varieties of the fundamental groups of compact orientable surfaces*, Israel Journal of Mathematics, 93 (1996), pp. 29–71.

[33] M. Rees, *An alternative approach to the ergodic theory of measured foliations on surfaces*, Ergodic Theory and Dynamical Systems, 1 (1981), pp. 461–488.

[34] W. P. Thurston, *The Geometry and Topology of Three-Manifolds*, Princeton University, Lecture Notes, online at http://www.msri.org/publications/books/gt3m (1986).

[35] A. Wright, *A tour through Mirzakhani's work on moduli spaces of Riemann surfaces*, Bulletin of the American Mathematical Society, 57 (2020), pp. 359–408.