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# Measure and Statistical Attractors for Nonautonomous Dynamical Systems

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**Abstract**

Various inequivalent notions of attraction for autonomous dynamical systems have been proposed, each of them useful to understand specific aspects of attraction. Milnor's notion of a *measure* attractor considers invariant sets with positive measure basin of attraction, while Ilyashenko's weaker notion of a *statistical* attractor considers positive measure points that approach the invariant set in terms of averages. In this paper we propose generalisations of these notions to nonautonomous evolution processes in continuous time. We demonstrate that pullback/forward measure/statistical attractors can be defined in an analogous manner and relate these to the respective autonomous notions when an autonomous system is considered as nonautonomous. There are some subtleties even in this special case–we illustrate an example of a two-dimensional flow with a one-dimensional measure attractor containing a single point statistical attractor. We show that the single point can be a pullback measure attractor for this system. Finally, for the particular case of an asymptotically autonomous system (where there are autonomous future and past limit systems) we relate pullback (respectively, forward) attractors to the past (respectively, future) limit systems.

**Keywords** Nonautonomous dynamical system · Measure attractor · Statistical attractor

**Mathematics Subject Classification** 37C60 · 34D45

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## 1 Introduction

Dissipative dynamical systems are remarkable in that they allow one to gain a clear picture of the asymptotic (long time behaviour) by considering the typically much smaller subset of phase space that is repeatedly visited over a long timescale. These comprise the "attractors" for the system. It is well known that such attractors may not only be chaotic but also "strange", namely they can be highly anisotropic in structure and fractional in dimension [23].

There is however no universally agreed definition of an attractor, even for an autonomous system. The most commonly considered definition is the topological notion of an asymptotically stable invariant set due to Lyapunov. However, weaker measure based notions (that allow one to ignore exceptional sets of initial conditions) are probably closer to what is needed in many applications. This motivated Milnor's definition of a measure attractor [19] as well as Ilyashenko's even weaker notion of a statistical attractor [13]. A succession of successively weaker notions of attraction for autonomous systems can be summarised in the implications:

$$\boxed{\text{measure attractor} \Rightarrow \text{weak measure attractor} \Rightarrow \text{statistical attractor.}}$$

Non autonomous dynamical systems with an explicit time dependence, pose many challenges. A notion of a measure attractor for random dynamical systems was presented in [3] where it was related to measure attractors for the corresponding skew product system. More generally, there are several inequivalent notions of time limit corresponding to different ways of choosing start and end times such that the time between these becomes unbounded; the *pullback* limit fixes the end time and compares forward trajectories starting further in the past, while the *forward* limit fixes a start time and considers forward trajectories from there. The monograph [16] discusses this in some detail and in various contexts, and gives various topological notions of attraction.

In this paper we propose notions of measure and statistical attractor that are suitable for general non autonomous dynamical systems. We relate these to each other with various results and examples, and look at special cases. In particular, we give some nontrivial examples of nonautonomous measure attractors that are not topological attractors. Using the pullback (pb) notion we define notions of attraction with implications:

$$\boxed{\text{pb measure attractor} \Rightarrow \text{pb weak measure attractor} \Rightarrow \text{pb statistical attractor.}}$$

while for the forward (fw) notion we define notions of attraction with implications:

fw measure attractor ⇒ fw weak measure attractor ⇒ fw statistical attractor.

We review some important definitions and results concerning autonomous attractors in Sect. 1.1 and nonautonomous attractors in Sect. 1.2.

We propose and investigate a definition for measure attractors in a nonautonomous system in Sect. 2, in both the forward (Sect. 2.1) and pullback (Sect. 2.2) senses before discussing their properties in more depth in Sect. 2.3. In Sect. 2.4 we present some more subtle results (Theorems 2.20 and 2.19) that relate measure attractors for autonomous systems to pullback and forward measure attractors respectively, when they are viewed as nonautonomous.

In Sect. 3 we propose definitions for statistical attractors for nonautonomous systems in the pullback and forward senses (Sect. 3.1) before relating pullback statistical attractors to autonomous statistical attractors in Sect. 3.2: in particular we show in Theorem 3.6 that a statistical attractor for an autonomous system is a pullback and forward statistical attractor if the system is viewed as nonautonomous. Sect. 3.3 presents an explicit example to show that an autonomous statistical attractor that is not a measure attractor can be a pullback measure attractor in the nonautonomous sense. This is proven for our example in Proposition 3.8 which furthermore shows that an autonomous measure attractor may only be a pullback weak measure attractor when viewed in the nonautonomous setting.

In general, pullback and forward notions of attraction are independent of each other, but there are cases where they can be related. We discuss the case of asymptotically autonomous systems in Sect. 4 where these notions relate to properties of the limiting autonomous systems. Finally, we discuss outlook and various remaining issues in Sect. 5.

### 1.1 Autonomous Attractors

We consider a dynamical system given by a flow $\phi_t : \mathbb{R}^d \to \mathbb{R}^d$ for $t \in \mathbb{R}$, which satisfies the *initial value property* $\phi_0 = \text{Id}$ and the *group property* $\phi_{t+s} = \phi_t \circ \phi_s$ for all $t, s \in \mathbb{R}$. We assume throughout the paper that $\phi_t(x)$ is continuously differentiable in $t \in \mathbb{R}$ and $x \in \mathbb{R}^d$, and note that sometimes, we consider flows that are only defined on compact subsets of $\mathbb{R}^d$.

The omega limit set of a point $x \in \mathbb{R}^d$ is defined by

$$\omega(x) := \bigcap_{t>0} \overline{\{\phi_s(x) : s > t\}}.$$

Omega limit sets are invariant, i.e. $\phi_t(\omega(x)) = \omega(x)$ for all $t \in \mathbb{R}$. The *basin of attraction* of a compact and invariant set $A \subset \mathbb{R}^d$ is defined by

$$\mathcal{B}(A) = \{x \in \mathbb{R}^d : \omega(x) \subset A \text{ and } \omega(x) \neq \emptyset\}.$$

Note that invariance of $A$ implies that $A \subset \mathcal{B}(A)$. The most important classical notion of attractor is that of asymptotic stability. A nonempty, compact and invariant set $A \subset \mathbb{R}^d$ is said to be an *asymptotically stable attractor* if $\mathcal{B}(A)$ is a neighbourhood of $A$, and $A$ is Lyapunov stable, i.e. for all neighbourhoods $U$ of $A$, there is a neighbourhood $V$ of $A$ such that $\phi_t(V) \subset U$ for all $t > 0$. It follows that for an asymptotically stable attractor $A$, there exists an $\varepsilon > 0$ such that[1] $B_\varepsilon(A)$ is attracted uniformly forward in time, i.e.

$$\lim_{t\to\infty} d(\phi_t(B_\varepsilon(A)), A) = 0\,.$$

On the other hand, an attractor that attracts a neighbourhood uniformly will be asymptotically stable. One can distinguish a global attractor that attracts all sets, and local attractors that do not attract all sets but are indecomposable in some way - for example one can assume that $A$ contains a dense orbit. We refer to [19, 22] for more of a discussion and references.

We deal with two important generalisations of these ideas in this paper: measure attractors and statistical attractors.

Let $\ell$ denote the Lebesgue measure on $\mathbb{R}^d$. A nonempty, compact and invariant set $A \subset \mathbb{R}^d$ is said to be a *measure attractor* if $\ell(\mathcal{B}(A)) > 0$ and for any nonempty, compact and invariant set $A' \subsetneq A$, we have $\ell(\mathcal{B}(A) \setminus \mathcal{B}(A')) > 0$ [19]. A compact and invariant set $A$ satisfying only the first condition of a positive measure basin of attraction is called a *weak* measure attractor [5]. The second condition ensures that all of $A$ is necessary to attract all the basin points, that is, $A$ is minimal with respect to the basin. We note that a weak measure attractor is not necessarily a measure attractor. However, existence of a weak measure attractor implies existence of a measure attractor [5, Lemma 3.2], whereas any compact invariant set containing a measure attractor is a weak measure attractor. If no proper subset of a measure attractor $A$ is a measure attractor, then we say $A$ is a *minimal* measure attractor.

A measure attractor does not have to attract a neighbourhood or be Lyapunov stable. Instead it is a weaker, point-wise notion of attraction that ensures there is a positive probability of observing the attractor by randomly choosing an initial condition. A related concept is that of the *likely limit set*, namely the smallest closed subset $\Lambda_M$ such that $\omega(x) \subset \Lambda_M$ for a full measure set of $x$. The likely limit set is then the maximal measure attractor for the flow.

More generally, following [5], we can define a *likely limit set* of a positive measure set $X \subset \mathbb{R}^d$, denoted as $\Lambda_M(X)$ as follows. Let

$$\Omega_M(X) := \overline{\bigcup_{x\in X} \omega(x)} \quad \text{and} \quad \Lambda_M(X) := \bigcap_{Y=_0 X} \Omega_M(Y)\,,$$

where $W =_0 V$ means that $W$ and $V$ differ on a set of zero Lebesgue measure, i.e. $\ell((W \setminus V) \cup (V \setminus W)) = 0$. As discussed in [5, 19], for a positive measure set $X \subset \mathbb{R}^d$, compactness of $\Lambda_M(X)$ implies that it is a measure attractor.

The following proposition states that (weak) measure attractors attract large measure subsets of their basins uniformly. We will use this result later in our discussion of nonautonomous generalisations of measure attraction, but we state and prove this result now as it is of more general interest. Note that one can only expect a proper subset of $\mathcal{B}(A)$ to be attracted uniformly: there will be points in $\mathcal{B}(A)$ starting arbitrarily close to the boundary that is also an invariant set.

**Proposition 1.1** *Suppose that $\phi_t$ is a smooth flow defined on some compact*[2] *$X \subset \mathbb{R}^d$ with $0 < \ell(X) < \infty$. Suppose that $A$ is a weak measure attractor for $\phi_t$, i.e.*

$$\ell(\mathcal{B}(A)) > 0.$$

[1] We write $d(x, A) := \inf_{a\in A} \|x - a\|$ for the distance of a point $x$ to a set $A$. For two sets $A, B \subset \mathbb{R}^d$, we define the Hausdorff semi-distance by $d(A, B) := \sup_{a\in A} d(a, B)$ and the Hausdorff distance by $d_H(A, B) := \max(d(A, B), d(B, A))$. The open $\eta$-neighbourhood of $A$ is denoted by $B_\eta(A)$.

[2] Note that it is easy to generalise to non compact space if we replace, in the statement of the theorem, the basin of attraction $\mathcal{B}(A)$ with some finite measure subset of it.

*Then for all* $\varepsilon > 0$ *there exists a closed set* $U_\varepsilon \subset \mathcal{B}(A)$ *with* $\ell(U_\varepsilon) > \ell(\mathcal{B}(A)) - \varepsilon$ *that is uniformly attracted to* $A$*, namely*

$$\lim_{t\to\infty} d(\phi_t(U_\varepsilon), A) = 0.$$

***Proof*** Consider the set of measurable functions $f_t(x) := d(\phi_t(x), A)$ and note that for $\ell$-almost all $x \in \mathcal{B}(A)$ we have $\lim_{t\to\infty} f_t(x) = 0$. Because $\mathcal{B}(A)$ has finite measure and the limit is finite, we can apply Egorov's theorem [24, Theorem 4.17] to deduce that there is a closed $U_\varepsilon \subset \mathcal{B}(A)$ with $\ell(\mathcal{B}(A) \setminus U_\varepsilon) < \varepsilon$ where $f_t(x)$ converges uniformly to 0. Hence

$$\lim_{t\to\infty} d(\phi_t(U_\varepsilon), A) = 0,$$

as required. □

An even weaker notion of attractor was formulated by Ilyashenko in [13] (for compact spaces)[3], which requires only almost all in *time* convergence of almost all orbits. Define

$$L(x, U, s) := \frac{1}{s}\ell\left(\{0 \le t \le s;\ \phi_t(x) \in U\}\right). \tag{1}$$

Let $A$ be a nonempty, compact invariant set. We define the *statistical basin of attraction* of $A$

$$\mathcal{B}_{stat}(A) = \{x\ :\ L(x, B_\varepsilon(A), s) \to 1 \text{ as } s \to \infty \text{ for all } \varepsilon > 0\}.$$

We call $A$ a *statistical attractor*[4] if $\ell(\mathcal{B}_{stat}(A)) > 0$.

Note that there is an equivalent definition of a statistical attractor which is analogous to the measure attractor definition above, that is, in terms of *statistical* $\omega$-limit sets (see[5] [14]).

There is a further way to characterise a statistical attractor; in a form that makes clear the almost all in time property and that is closer to the familiar way of defining the attractivity property. We say that a measurable set $M \subset \mathbb{R}$ has *full density at* $-\infty$ if $\lim_{s\to\infty} \frac{1}{s}\ell(M \cap [-s, 0]) = 1$, and $M \subset \mathbb{R}$ has *full density at* $\infty$ if $\lim_{s\to\infty} \frac{1}{s}\ell(M \cap [0, s]) = 1$.

**Proposition 1.2** *Let* $A$ *be a statistical attractor. Then for every* $x \in \mathcal{B}_{stat}(A)$*, there exists a set* $T_\infty$ *of full density at* $\infty$ *such that*

$$\lim_{t\to\infty,\, t\in T_\infty} d(\phi_t(x), A) = 0\,.$$

Before proving Proposition 1.2, we require a lemma to show we can construct a suitable $T_\infty$ set.

[3] Ilyashenko defined the statistical attractor $\Lambda_S$ to be the smallest closed set such that almost all orbits spend almost all time in any open neighbourhood $U$ of $\Lambda_S$. Note that Ilyashenko's definition does not have a notion of a local or *weak* statistical attractor. The definition given in this paper is a more general one, which allows for potentially 'weak' statistical attractors.

[4] One could also distinguish between "weak" and "strong" notions statistical attractor, as in the case of measure attractor. A suitable strong definition would be that $A$ is unique with respect the basin of statistical attraction, up to a Lebesgue null set. We do not do this in this paper, for the sake of simplicity.

[5] In [14] the *essential omega limit set* is defined as $\omega_{ess}(x) := \{y\ :\ \limsup_{s\to\infty} L(x, B_\varepsilon(y), s) > 0,\ \text{for all } \varepsilon > 0\}$ and the *statistical limit set* of $X \subset \mathbb{R}^d$ of positive measure as $\Omega_S(X) := \overline{\cup_{x\in X}\omega_{ess}(x)}$ and $\Lambda_S(X) := \cap_{Y=_0 K}\Omega_S(Y)$. If $A = \Lambda_S(X)$ is compact, $A$ is referred to as the *statistical attractor* with *basin of attraction*

$$\mathcal{B}_{stat}(A) = \{x\ :\ \omega_{ess}(x) \subset A \text{ and } \omega_{ess}(x) \neq \emptyset\}.$$

See [14] for proof that this definition is equivalent to Ilyashenko's definition.

**Lemma 1.3** *Let $\{T_n\}_{n\in\mathbb{N}}$ be a sequence of sets, $T_n \subset \mathbb{R}$ such that $T_n$ has full density at $\infty$. Then, there exists an increasing sequence of times $\{s_n\}_{n\in N}$ such that the set defined by*

$$T_\infty := \bigcup_{N=1}^{\infty} \bar{T}_N$$

*where*

$$\bar{T}_N := \left( \bigcup_{n=1}^{N-1} T_n \cap [s_n, s_{n+1}] \right)$$

*has full density at $\infty$.*

***Proof*** See Appendix A. □

***Proof of Proposition 1.2*** Let $x \in \mathcal{B}_{stat}(A)$. Let $T_n = \{t \in [0,\infty) \ : \ \phi_t(x) \in B_{1/n}(A)\}$ for some $n \in \mathbb{N}$. Then by definition, $T_n$ has full density at $\infty$ for each $n \in \mathbb{N}$.

Let $T_\infty$ be as defined in Lemma 1.3, and note that it has full density at $\infty$. Let $t_k \in T_\infty$ be an increasing sequence of times, then for each $k \in \mathbb{N}$, $t_k \in [s_{n_k}, s_{n_k+1}]$ for some $n_k \in \mathbb{N}$ so that $t_k \in T_{n_k}$. By definition of $T_{n_k}$ we have that

$$d(\phi_{t_k}(x), A) \leq \frac{1}{n_k} \to 0,$$

as $k \to \infty$, which completes the proof. □

### 1.2 Nonautonomous Attractors

In contrast to (autonomous) dynamical systems, where the time evolution $\phi_t$ depends on the elapsed time $t \in \mathbb{R}$, for nonautonomous dynamical systems, the time evolution $\Phi_{t,s}$ depends explicitly on initial time $s \in \mathbb{R}$ and final time $t \in \mathbb{R}$. Such a $\Phi_{t,s}$ is called a *process* [16], and we require that a process satisfies the initial value and cocycle property

$$\Phi_{s,s} = \mathrm{Id} \quad \text{and} \quad \Phi_{t,s} = \Phi_{t,u} \circ \Phi_{u,s} \qquad \text{for all } t, u, s \in \mathbb{R}\,.$$

Note that for simplicity, we assume that the process is invertible, even though many results in this paper generalise easily to non-invertible processes $\Phi_{t,s}$ that are defined only for $t \geq s$. We also assume that $\Phi_{t,s}$ is continuously differentiable in all arguments.

There is an extensive literature that generalises various notion of attractors to a nonautonomous setting. However, there is a significant problem in that "basin" and "attractor" need to be considered at different time-points where the system may behave quite differently. Hence, in the nonautonomous case, there are more notions of attractor depending on precisely which limiting process is considered.

The concepts of nonautonomous attractor that have been most successfully applied are based on the ideas of forward and pullback attraction. Analogously to the autonomous case, we can define both global and local concepts. We proceed to define local notions, following [16, Chapter 3]. Firstly, we introduce the concept of a *nonautonomous set* $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$, which is a family of sets $A(t) \subset \mathbb{R}^d$, the so-called *fibers* of $\mathcal{A}$. A nonautonomous set $\mathcal{A}$ is said to be *compact* if $A(t)$ is compact for all $t \in \mathbb{R}$, and *invariant* if $\Phi_{t,s}(A(s)) = A(t)$ for all $t, s \in \mathbb{R}$.

We say that a nonempty, compact, nonautonomous set $\mathcal{A}$ is a *local forward attractor* if there is an $\eta > 0$ such that

$$\lim_{t\to\infty} d\big(\Phi_{t,t_0}(B_\eta(A(t_0)), A(t)\big) = 0 \quad \text{for all } t_0 \geq 0\,.$$

$\mathcal{A}$ is said to be a *local pullback attractor* if there exists an $\eta > 0$ such that

$$\lim_{t_0\to-\infty} d\big(\Phi_{t,t_0}(B_\eta(A(t_0)), A(t)\big) = 0 \quad \text{for all } t \leq 0.$$

A more general definition can be made with respect to an *attraction universe*. This includes both local and global types of attractors. See [16] for precise definitions.

It is known, (Lemma 2.15 of [16]), that invariant nonautonomous sets of processes are composed of *entire solutions*, defined as a mapping $\xi : \mathbb{R} \to \mathbb{R}^d$ such that

$$\xi(t) = \Phi_{t,s}(\xi(s))$$

for all $t \geq s$. Hence, all attractors must be composed of such trajectories.

Since forward attraction corresponds to the future and pullback attraction corresponds to the past, these notions are not equivalent [15]. While future asymptotic behaviour depends on the present state, pullback attractors give us the likely present state of a system that has started in some distant past. We note that both types of convergence are important to fully understand the dynamics. This is exemplified in rate-induced tipping, which can be associated with a scenario where a local pullback attractor limits to a repeller depending on the future only [4].

While both notions of attraction are meaningful to understand the dynamics, in some sense, the pullback attractor can be seen as closer to a generalisation of the autonomous attractor [8]. In particular, forward attracting trajectories are typically time-varying, whilst pullback attractors are fixed sets in phase space for any given end-time. In addition, they can be shown to have properties more in line with those of autonomous attractors, for example, uniqueness (see e.g. [16, Proposition 3.8]) and existence given an *absorbing* set (see e.g. [16, Proposition 3.27]). In contrast, forward attractors are generally not unique, see e.g. [15].

Any autonomous system can be thought of as a trivial nonautonomous system and so any notions for nonautonomous systems can be applied to autonomous systems. It might seem strange to use notions of attraction developed for nonautonomous systems on these trivial cases, but we argue that this can be productive as a way of understanding the properties and limitations of these notions.

## 2 Measure Attractors for Nonautonomous Dynamical Systems

Generalisation of the notion of measure attractor to pullback or forward attraction in nonautonomous systems is not trivial, in particular as the measure of an invariant set may vary significantly from one fibre to the next. In this section, we propose a way of doing this.

### 2.1 Forward Measure Attractors

Consider a nonempty, compact and invariant nonautonomous set $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ of a nonautonomous dynamical system $\Phi_{t,s}$. The *basin of forward measure attraction* is defined to be the nonautonomous set $\mathcal{B}^+(\mathcal{A})$ with fibres

$$\mathcal{B}^+(\mathcal{A})(t_0) = \Big\{x \in \mathbb{R}^d : \lim_{t\to\infty} d(\Phi_{t,t_0}(x), A(t)) = 0\Big\} \quad \text{for all } t_0 \geq 0.$$

Note that $\mathcal{A} \subset \mathcal{B}^+(\mathcal{A})$ and $\mathcal{A}$ is a local forward attractor if $\mathcal{B}^+(\mathcal{A})$ contains an open neighbourhood that is attracted uniformly.

**Definition 2.1** (Forward measure attractor) We say that a nonempty, compact, invariant nonautonomous set $\mathcal{A}$ is a *weak forward measure attractor* if there exists a $t_0 \geq 0$ such that

$$\ell(\mathcal{B}^+(\mathcal{A})(t)) > 0 \quad \text{for all } t \geq t_0 \,.$$

If, in addition, for any weak forward measure attractor $\mathcal{A}' \subsetneq \mathcal{A}$, we have

$$\ell(\mathcal{B}^+(\mathcal{A})(t) \setminus \mathcal{B}^+(\mathcal{A}')(t)) > 0 \quad \text{for all } t \geq t_0 \,,$$

we say $\mathcal{A}$ is a *forward measure attractor*.

***Remark 2.2*** Since in this paper we are in the differentiable setting (that is, we assume $\Phi_{t,s}$ is a diffeomorphism) we only need to verify that the conditions of Definition 2.1 hold for one time $t_0 \in \mathbb{R}^+$. We note that $\Phi_{t_1,t_0}\mathcal{B}^+(\mathcal{A})(t_0) \subseteq \mathcal{B}^+(\mathcal{A})(t_1)$ for all $t_1 > t_0$, however the condition in the definition of a weak forward measure attractor is nonetheless required for all $t \geq t_0$ in the more general case where $\Phi_{t,s}$ is not a diffeomorphism, as we cannot guarantee that $\Phi_{t_1,t_0}$ does not map sets of positive measure to zero measure sets (there are homeomorphisms that map positive measure sets to zero measure sets, see e.g. [11]).

We discuss some immediate properties of forward measure attractors. If $x \in A(t_0)$ then, by the invariance of $\mathcal{A}$, $\Phi_{t,t_0}x \in A(t)$ for all $t \in \mathbb{R}$. In particular, this means that

$$A(t) \subset \mathcal{B}^+(\mathcal{A})(t),$$

for all $t \in \mathbb{R}$. It follows that if $\ell(A(t)) > 0$ then $\ell(\mathcal{B}^+(\mathcal{A})(t)) > 0$ and hence we can have that $\ell(\mathcal{B}^+(\mathcal{A})(t) \setminus A(t)) = 0$. That is, $\mathcal{A}$ is not attracting points (of positive measure) outside of itself (the same holds in the autonomous situation), in fact it could be repelling points.

Any forward measure attractor must also be composed of entire solutions. It follows from the definition of the forward measure attractor above, that every entire solution having positive measure forward basin of attraction (i.e. an entire solution that is a weak forward measure attractor) is a forward measure attractor.

Local forward attractors (as defined in Sect. 1.2) are intrinsically non-unique [16, Example 3.6], and we demonstrate this below with a forward measure attractor example.

***Example 2.3*** Consider the ordinary differential equation

$$\dot{x} = x^2(x-1)\,,$$

which has two equilibria: a saddle $x = 0$, and a repeller $x = 1$. All solutions starting in $[0, 1)$ converge to 0, while solutions starting in $(-\infty, 0)$ diverge to $-\infty$. The equilibrium point $x = 0$ corresponds to an invariant nonautonomous set $\mathcal{A}$ with $A(t) = \{0\}$ for all $t \in \mathbb{R}$. We note that $\mathcal{A}$ is not a local forward attractor because negative solutions diverge to $-\infty$. However, $\mathcal{A}$ is a forward measure attractor, and its basin is given by $\mathcal{B}(\mathcal{A})(t) = [0, 1)$ for all $t \in \mathbb{R}$.

A nonautonomous forward attractor is inherently non-unique, which is illustrated by this autonomous situation. Under the definition, the attractor must be a nonautonomous invariant set and hence any entire solution that converges to the attractor in the future can be added to get another attractor. Thus, in the above example, we can take any trajectory passing through the basin. That is, any trajectory such that $x(t_0) \in [0, 1)$, will be a forward measure attractor as well, with basin $[0, 1)$.

The following example is inspired by the nonautonomous logistic equation [8].

***Example 2.4*** Consider the ordinary differential equation

$$\dot{x} = -e^{-t^2} x^2 ,$$

whose process is given by

$$\Phi_{t,t_0}(x_0) = \frac{1}{\frac{1}{x_0} + \int_{t_0}^{t} e^{-s^2}\, \mathrm{d}s} ,$$

and note that we have $\int_{t_0}^{t} e^{-s^2}\, \mathrm{d}s = \frac{1}{2}\sqrt{\pi}(\operatorname{erf}(t) - \operatorname{erf}(t_0))$, where $\operatorname{erf}(t)$ is the Gauss error function. For each $x_0 \in \mathbb{R}$, the pullback limit is given by

$$\xi(t, x_0) = \lim_{t_0 \to -\infty} \Phi_{t,t_0}(x_0) = \frac{x_0}{1 + \frac{1}{2} x_0 \sqrt{\pi}(\operatorname{erf}(t) + 1)} .$$

Note that $x_0 \mapsto \xi(t, x_0)$ is bijective for all $t \in \mathbb{R}$, and for each fixed $x_0 \in \mathbb{R}$, the mapping $t \mapsto \xi(t, x_0)$ is a solution to the above differential equation (and these are all solutions). The forward limit

$$\lim_{t \to \infty} \xi(t, x_0) = \frac{x_0}{1 + x_0\sqrt{\pi}} ,$$

is also bijective in $x_0$, and thus, the distance of any two solutions $\xi(t, x_0)$ and $\xi(t, x_1)$ for $x_0 \neq x_1$ is bounded away from 0 uniformly in $t \in \mathbb{R}$. This means that no solution attracts another solution in the sense of both (classical, i.e. non-measure) pullback and forward attraction. However, for a fixed $x_0 > 0$, the invariant nonautonomous set $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$, defined by

$$A(t) = \big[\xi(t, -x_0), \xi(t, x_0)\big]$$

is a forward measure attractor, since its fibers have positive Lebesgue measure. More precisely, $\ell(\mathcal{B}^+(\mathcal{A})(t)) = \xi(t, x_0) - \xi(t, -x_0) > 0$.

We saw two examples of nonautonomous flows for which the standard definition of a forward attractor does not apply, even though the dynamics clearly exhibits attractive behaviour. The next example illustrates that basins for attracting behaviour can be even more complicated, illustrating what have been named ‘riddled’ basins of attraction (see e.g. [2, 21]), where around every point in the basin there is a point arbitrarily close by that is not in the basin.

***Example 2.5*** We consider the following system of differential equations composed of a scalar system with a subcritical pitchfork coupled to the Lorenz ’84 model [18].

$$\begin{aligned}
\dot{x} &= -y^2 - z^2 - ax + aF \\
\dot{y} &= xy - bxz - y + G \\
\dot{z} &= bxy + xz - z \\
\dot{w} &= (x - \lambda)w + w^3 - cw^5
\end{aligned} \tag{2}$$

with $a = 0.25$, $b = 4$, $F = 8$ and $G = 1$, which are standard parameter values for Lorenz ’84 with chaotic variability (e.g. [18, Fig. 5]). We set $c = 0.1$ and note that $w = 0$ is an invariant subspace for all parameter values, but whether it is attracting or repelling depends on the parameter $\lambda$. Figure 1 illustrates the basin of attraction for the invariant subspace $w = 0$, for two different values of $\lambda$.

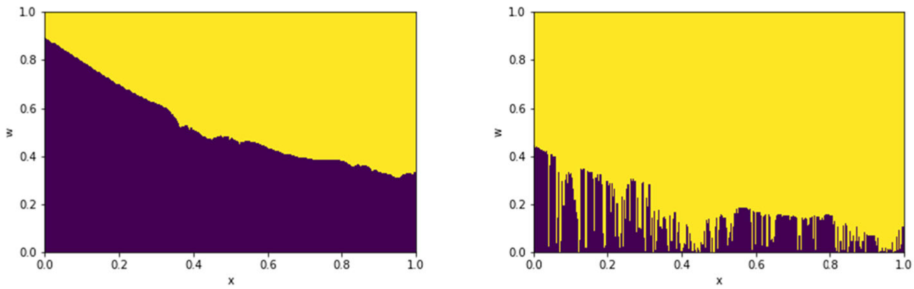


**Fig. 1** Solutions of (2) starting on a grid of $300 \times 300$ uniformly space points in $(x, w) \in [0, 1]^2$ with $y = z = 1$ is integrated forward to time $t = 100$ via Python's *Scipy* module using the *Odeint* solver for ordinary differential equations. If the solution is deemed to *converge* to $w = 0$ (that is, $|w| < 0.01$) then the initial condition is coloured purple, otherwise it is coloured yellow; left: $\lambda = 1.20$ illustrates an asymptotically stable basin of attraction for $w = 0$; right: $\lambda = 1.05$ shows an approximation of the riddled basin of attraction for the $w = 0$ attractor

Next we consider the question of uniqueness and minimality of measure attractors. In the autonomous case, a measure attractor $A$ is unique in the sense that there isn't another measure attractor $A'$ with almost surely the same basin of attraction.[6] Thus, in Example 2.3, $\{0\}$ is the unique autonomous measure attractor. However, when considered as a nonautonomous attractor it is clearly not unique, as already discussed. However, in Example 2.3, the attractor is *minimal*. In [19], a measure attractor is defined to be minimal if no proper subset of it is a measure attractor. In Example 2.4, the attractors are unique for a given basin, but there is no minimal one.

## 2.2 Pullback Measure Attractors

This subsection deals with the definition of pullback measure attractors. A nonautonomous set $\mathcal{N} = \{N(t)\}_{t\in\mathbb{R}}$ is said to be *pullback attracted* to an invariant and compact nonautonomous set $\mathcal{A}$ if

$$\lim_{t_0 \to -\infty} d(\Phi_{t,t_0} N(t_0), A(t)) = 0 \quad \text{for all } t \leq 0$$

and we write

$$\mathcal{N} \underset{pb}{\longrightarrow} \mathcal{A}.$$

We define the *basin of pullback measure attraction* $\mathcal{B}^-(\mathcal{A})$ as the family of nonautonomous sets that are pullback attracted to $\mathcal{A}$, i.e.

$$\mathcal{B}^-(\mathcal{A}) := \left\{ \mathcal{N} \,:\, \mathcal{N} \underset{pb}{\longrightarrow} \mathcal{A} \right\}. \tag{3}$$

**Definition 2.6** *(Pullback measure attractor)* A nonempty, compact and invariant nonautonomous set $\mathcal{A}$ is called a *weak pullback measure attractor* if there exists a nonautonomous

[6] Suppose $A$ is a measure attractor with basin of attraction $\mathcal{B}(A)$ and let $A'$ be another measure attractor with $\mathcal{B}(A') = \mathcal{B}(A)$ almost surely. Then $A \cap A'$ is a closed proper subset of $A$ and $\ell(\mathcal{B}(A) \setminus \mathcal{B}(A \cap A')) = 0$ which is a contradiction because $\mathcal{A}$ is a measure attractor. Hence, there can be only one measure attractor with basin of attraction equal to $\mathcal{B}(A)$ almost surely.

set $\mathcal{N} \in \mathcal{B}^{-}(\mathcal{A})$ such that

$$\liminf_{t \to -\infty} \ell(N(t)) > 0 . \tag{4}$$

A weak pullback measure attractor $\mathcal{A}$ is called a *pullback measure attractor* if for any weak pullback measure attractor $\mathcal{A}' \subsetneq \mathcal{A}$, there exists a nonautonomous set $\mathcal{N} \in \mathcal{B}^{-}(\mathcal{A})$ such that for all $\mathcal{N}' \in \mathcal{B}^{-}(\mathcal{A}')$ with $\mathcal{N}' \subset \mathcal{N}$, we have

$$\liminf_{t \to -\infty} \ell(N(t) \setminus N'(t)) > 0 .$$

***Remark 2.7*** In the definitions above, the conditions on pullback attracted nonautonomous sets are statements about limits as $t \to -\infty$. Note that the finite time structure of the nonautonomous set does not play a role in their pullback attracting behaviour.

***Remark 2.8*** Using the differential equation $\dot{x} = x$, we demonstrate that (4) should not be replaced by the weaker condition $\ell(N(t)) > 0$ for all $t \leq 0$. This differential equation has a repelling equilibrium at 0. We can find a nonautonomous set $\mathcal{N}$, satisfying this weaker condition and defined by $N(t) = [0, e^{2t}]$ for all $t \in \mathbb{R}$, which is attracted by the nonautonomous set $\mathcal{R} = \{0\}_{t \in \mathbb{R}}$ that corresponds to the repelling equilibrium. Fix $\varepsilon > 0$ and using $\Phi_{t,t_0}(x_0) = x_0 e^{t-t_0}$, we get

$$d(\Phi_{t,t_0}(N(t_0)), R(t)) = d([0, e^{t+t_0}], 0) < \varepsilon \quad \text{whenever } t_0 < \ln(\varepsilon) - t .$$

However, $\lim_{t \to -\infty} \ell(N(t)) = 0$, and therefore, $\mathcal{R}$ does not satisfy (4).

***Remark 2.9*** As in the case of forward measure attractors, pullback measure attractors must also be composed of entire solutions. It follows that an entire solution that is a weak pullback measure attractor must be a pullback measure attractor.

We define the *deterministic basin of pullback attraction* to be the nonautonomous set $\mathcal{B}_d^{-}(\mathcal{A})$ with fibres

$$\mathcal{B}_d^{-}(\mathcal{A})(t) = \left\{ x \in \mathbb{R}^d : \lim_{t_0 \to -\infty} d(\Phi_{t,t_0}(x), A(t)) = 0 \right\} .$$

The deterministic basin of pullback attraction consists of fibres $\mathcal{B}_d^{-}(\mathcal{A})(t)$ of points that converge to $A(t)$, however the points in the fibre actually 'live' in the infinite past. Note that in the case of an invertible process, $\mathcal{B}_d^{-}(\mathcal{A})(t) = \mathcal{B}_d^{-}(\mathcal{A})(s)$ for all $t, s \in \mathbb{R}$, that is, the basin does not depend on time.

Using the above deterministic basin of attraction we could have defined the pullback measure attractor analogously to the forward one. That is, by requiring that, for a compact invariant set $\mathcal{A}$,

$$\ell(\mathcal{B}_d^{-}(\mathcal{A}))(t)) > 0$$

for all $t$ and in addition, for any invariant $\mathcal{A}' \subset \mathcal{A}$, that we have

$$\ell(\mathcal{B}_d^{-}(\mathcal{A}))(t) \setminus \mathcal{B}_d^{-}(\mathcal{A})(t)) > 0$$

for all $t$. However, as we will see in Example 2.11 below; in the nonautonomous case it is possible that the *deterministic* basin of pullback attraction is empty, while there are well defined non autonomous sets that are pullback attracted. First we give an example where both deterministic and nonautonomous basins exist, followed by an example where the deterministic basin is empty.

***Example 2.10*** We consider the autonomous Bernoulli equation

$$\dot{x} = x^3 - x \tag{5}$$

with three equilibrium points; an attracting one at $x = 0$ and two repellers at $x = \pm 1$. The flow is given by

$$\phi_t(x) = \frac{\operatorname{sign}(x)}{\sqrt{1 + (x^{-2} - 1)e^{2t}}}.$$

Consider for the $t \in \mathbb{R}$ the homeomorphism

$$h_t(x) = (\sin^2(t) + 1)^{-1} x\,,$$

and define a process $\Phi_{t,s}$ via the nonautonomous coordinate transformation $h_t$, i.e. for $x > 0$

$$\begin{aligned}\Phi_{t,t_0}(x) &:= h_t \circ \phi_{t-t_0} \circ h_{t_0}^{-1}(x)\\ &= \frac{(\sin^2(t) + 1)^{-1}}{\sqrt{1 + ((\sin^2(t_0) + 1)x)^{-2} - 1)e^{2(t-t_0)}}}.\end{aligned}$$

The function $\Phi_{t,s}$ is a process since $\Phi_{t_0,t_0} x = x$ and

$$\begin{aligned}\Phi_{t,s} \circ \Phi_{s,t_0} &= h_t \circ \phi_{t-s} \circ h_s^{-1} \circ h_s \circ \phi_{s-t_0} \circ h_{t_0}^{-1}\\ &= h_t \circ \phi_{t-t_0} \circ h_{t_0}^{-1} = \Phi_{t,t_0}\,.\end{aligned}$$

We note that if $((\sin^2(t_0) + 1)x)^{-2} > 1$ for all $t_0$, i.e. $x \in (-\frac{1}{2}, \frac{1}{2})$, then $\Phi_{t,t_0} x \to 0$ as $t_0 \to -\infty$. Therefore $(-\frac{1}{2}, \frac{1}{2})$ is the deterministic pullback basin of attractor $A_0(t) = 0$ for all $t \in \mathbb{R}$.

However, there are also nonautonomous sets that are pullback attracted. For example, $N(t) = (-\frac{1}{4}, \frac{1}{4})$ for all $t \in \mathbb{R}$ is pullback attracted to $\mathcal{A}_0$. However, more interesting nonautonomous sets are as well. Fix some $0 < \delta << 1$ and let

$$N(t_0) = \left\{ x \;:\; |x| \le \frac{1}{\sin^2(t_0) + 1} - \delta \right\},$$

which is pullback attracted to $A_0(t) = 0$. To see this, note that if $x \in N(t_0)$ then

$$|x(\sin^2(t_0) + 1)| \le 1 - \delta.$$

Let $y := x(\sin^2(t_0) + 1) = h_{t_0}^{-1}(x) \in [-1 + \delta, 1 - \delta]$. Fix any $\varepsilon > 0$. Since $\phi_t$ is a flow and $\{0\}$ is an attracting fixed point with $(-1, 1)$ as the basin of attraction, there exists a $T_0 < 0$ such that for all $t_0 < T_0$ it holds that

$$d(\phi_{t-t_0}([-1 + \delta, 1 - \delta]), \{0\}) < \varepsilon.$$

Then it is clear that also $d(h_t \circ \phi_{t-t_0} \circ h_{t_0}^{-1}(N(t_0))) < \varepsilon$, since $h_t$ is contracting. It is straightforward to see that $\ell(N(t)) > 0$ for all $t \in \mathbb{R}$ and therefore that $A_0(t)$ satisfies the conditions of a pullback measure attractor as per Definition 2.6.

***Example 2.11*** We consider again the Bernoulli Eq. (5) from Example 2.10, with the nonautonomous coordinate transformation

$$h_t(x) = x - a \sin t\,,$$

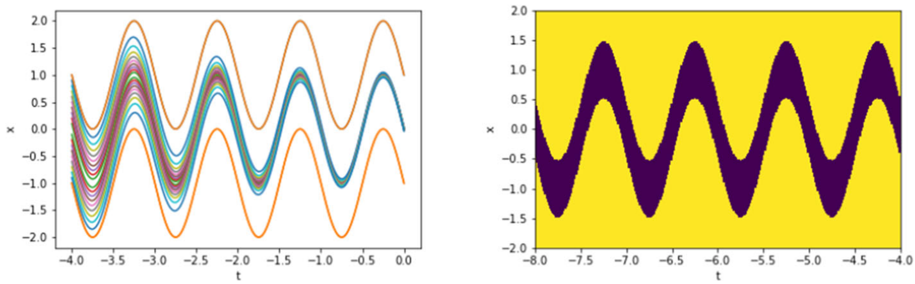


**Fig. 2** Left: Trajectories of (6) with $a = 1$, starting at $t_0 = -8\pi$ and integrated forward to time 0, for initial conditions in the range $[-1, 1]$. All trajectories, apart from those with initial conditions $x_0 = \pm 1$ converge to $-a \sin t$. Starting at different initial times would change the range of initial conditions that converge. This is shown in the second plot (right). Initial conditions at different starting times $t_0 \in [-16\pi, -8\pi]$ are integrated forward to $t_0 + 8\pi$. Initial points that converge to $-a \sin t$ are coloured in purple, while those that diverge are yellow. Note that there is no deterministic basin of attraction, i.e. there is no set of starting points that is attracted to $-a \sin t$ for all $t_0$

where $a > 0$, and this transforms the flow of (5) into the process

$$\Phi_{t,t_0}(x) := \frac{\operatorname{sign}(x + a \sin t_0)}{\sqrt{1 + ((x + a \sin t_0)^{-2} - 1)e^{2(t-t_0)}}} - a \sin t. \tag{6}$$

For $\phi_{t-t_0} h_{t_0}^{-1}(x)$ to converge we require

$$-1 - a \sin t_0 < x < 1 - a \sin t_0.$$

If $a < 1$ then the set $(-1 + a, 1 - a)$ is a deterministic basin of pullback attraction. If $a \geq 1$ then there is no deterministic set of values of $x$ which satisfies the above for all $t_0$, however, the nonautonomous set $D(t_0) = \{x \ : \ |x + a \sin t_0| \leq 1 - \delta\}$ is pullback attracted to $A(t) = -a \sin t$ for all $a > 0$ and any $1 > \delta > 0$, as illustrated in Fig. 2.

**Lemma 2.12** *Let $\mathcal{A}$ be a pullback measure attractor that attracts $\mathcal{N} = \{N(t)\}_{t \in \mathbb{R}}$ for the process $\Phi_{t,s}$, and let $\mathcal{B}_d^-(\mathcal{A})$ denote the deterministic basin of pullback attraction. Then*

$$\mathcal{B}_d^-(\mathcal{A})(t) = \bigcup_{\mathcal{N} \in \mathcal{B}^-(\mathcal{A})} \bigcap_{t_0 \leq t} N(t_0),$$

*for all $t \leq 0$.*

***Proof*** Let $\hat{N}(t) := \bigcup_{\mathcal{N} \in \mathcal{B}^-(\mathcal{A})} \bigcap_{t_0 \leq t} N(t_0)$. Suppose that $x \in \mathcal{B}_d^-(\mathcal{A})(t)$. We can define a nonautonomous set $\tilde{\mathcal{N}} := \left\{\{x\}\right\}_{t \in \mathbb{R}}$. Clearly $\tilde{\mathcal{N}} \in \mathcal{B}^-(\mathcal{A})$ and $x \in \tilde{N}(t)$ for all $t_0 \leq t$ so that $x \in \cap_{t_0 \leq t} \tilde{N}(t_0)$. Hence, $\mathcal{B}_d^-(\mathcal{A})(t) \subset \hat{N}(t)$. Suppose next that $x \in \hat{N}(t)$ so that $x \in \cap_{t_0 \leq t} N(t_0)$ for some $\mathcal{N} \in \mathcal{B}^-(\mathcal{A})$. Then $x \in N(t_0)$ for all $t_0 \leq t$. By definition it follows that,

$$d(\Phi_{t,t_0} x, A(t)) \leq d(\Phi_{t,t_0} N(t_0), A(t)) \to 0,$$

as $t_0 \to -\infty$. □

We remark that the proof of Lemma 2.12 does not require the process $\Phi_{t,s}$ to be invertible.

***Remark 2.13*** It follows from the proof of the above lemma that

$$\bigcup_{\mathcal{N}\in\mathcal{B}^-(\mathcal{A})} \bigcap_{t_0\le t} N(t_0) = \bigcup_{\mathcal{N}\in\mathcal{B}^-(\mathcal{A})} \overline{\bigcap_{t_0\le t} N(t_0)},$$

since if $x \in \overline{\bigcap_{t_0\le t} N(t_0)}$, then the nonautonomous set $\mathcal{N}'$ defined by $N'(t) = N(t) \cup \{x\}$ belongs to the basin of pullback attraction $\mathcal{B}^-(\mathcal{A})$.

### 2.3 Properties of Forward and Pullback Measure Attractors

In this section, we state some results giving properties of pullback measure attractors. The next result applies both to forward and pullback attractors.

**Proposition 2.14** *Let $\Phi_{t,s}$ be a process with a weak pullback (resp. forward) measure attractor $\mathcal{A}$ and let $h_t : \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism on $\mathbb{R}^d$ for all $t \in \mathbb{R}$ such that the maps $(t, x) \mapsto h_t(x)$ and $(t, x) \mapsto h_t^{-1}(x)$ are continuous. Furthermore, suppose that $h_t^{-1}$ is Lipschitz continuous for all $t \in \mathbb{R}$, with a Lipschitz constant $K > 0$ that does not depend on $t \in \mathbb{R}$. Then the process $\Psi_{t,t_0}$, given by*

$$\Psi_{t,t_0} := h_t \circ \Phi_{t,t_0} \circ h_{t_0}^{-1}, \tag{7}$$

*has a weak pullback (resp. forward) measure attractor $\mathcal{A}_h = \{A_h(t)\}_{t\in\mathbb{R}}$ given by $A_h(t) = h_t(A(t))$.*

***Proof*** As seen in Example 2.10, $\Psi_{t,t_0}$ is a process, and continuity in $(t, t_0, x)$ follows from the continuity assumptions on $(t, x) \to h_t(x)$ and $(t, x) \to h_t^{-1}(x)$. Note that $\mathcal{A}_h$ is compact since continuous functions preserve compactness. $\mathcal{A}_h$ is invariant for $\Psi$ since $\Psi_{t,t_0} A_h(t_0) = h_t \circ \Phi_{t,t_0} \circ A(t_0) = h_t \circ A(t) = A_h(t)$.

Let $\mathcal{N}$ be a nonautonomous set in $\mathcal{B}^-(\mathcal{A})$, the basin of pullback measure attraction of $\mathcal{A}$ and let $N_h(t) := h_t N(t)$. We note that

$$\Phi_{t,t_0} \circ h_{t_0}^{-1} N_h(t_0) = \Phi_{t,t_0} N(t_0),$$

so that, since $\lim_{t_0\to-\infty} d(\Phi_{t,t_0} N(t_0), A(t)) = 0$ for all $t \le 0$, it follows that

$$\lim_{t_0\to-\infty} d(\Psi_{t,t_0} N_h(t_0), A_h(t)) = \lim_{t_0-\infty} d(h_t \circ \Phi_{t,t_0} \circ h_{t_0}^{-1} N_h(t_0), h_t A(t)) = 0,$$

where we have used the continuity of $h_t$ in the state space variable.

By [12, Prop 2.2], it holds that

$$\ell(h_t^{-1}(N_h(t))) \le K^d \ell(N_h(t)),$$

so that

$$K^{-d} \ell(N(t)) \le \ell(N_h(t)).$$

Taking lim inf in $t$ on both sides shows that $\liminf_{t\to-\infty} \ell(\mathcal{N}_h(t)) > 0$. This shows that $\mathcal{A}_h$ is a weak pullback measure attractor for $\psi_{t,s}$.

Next we show that it is a weak forward measure attractor as well. Let $x \in h_s \circ \mathcal{B}^+(\mathcal{A})(s)$ for some $s \in \mathbb{R}$. Then, $y = h_s^{-1}(x) \in \mathcal{B}^+(\mathcal{A})(s)$ so that

$$d(\Psi_{t,s} x, A_h(t)) = d(h_t \circ \Phi_{t,s} y, h_t \circ A(t)) \to 0,$$

as $t \to \infty$ and hence $h_s \circ \mathcal{B}^+(\mathcal{A})(s) \subset \mathcal{B}^+(\mathcal{A}_h)(s)$.

It remains to show that $\ell(h_s \circ \mathcal{B}^+(\mathcal{A})(s)) > 0$. By [12], Proposition 2.2, it holds that

$$\ell(h_s^{-1}(h_s \circ \mathcal{B}^+(\mathcal{A})(s))) \leq K^d \ell(h_s \circ \mathcal{B}^+(\mathcal{A})(s)),$$

so that

$$0 < K^{-d} \ell(\mathcal{B}^+(\mathcal{A})(s)) \leq \ell(h_s \circ \mathcal{B}^+(\mathcal{A})(s)).$$

Hence $\mathcal{A}_h$ is a weak forward measure attractor for $\psi_{t,s}$. □

We can use Proposition 2.14 to show there is an equivalence between attractors under suitable nonautonomous coordinate changes.

**Corollary 2.15** *Let $\Phi_{t,s}$ be a process and let $h_t : \mathbb{R}^d \to \mathbb{R}^d$ be a bi-Lipschitz homeomorphism on $\mathbb{R}^d$ for all $t \in \mathbb{R}$ with a Lipschitz constant $K > 0$ that does not depend on $t \in \mathbb{R}$, such that the maps $t \to h_t(x)$ and $t \to h_t^{-1}(x)$ are continuous for all $x \in \mathbb{R}^d$. Then the nonautonomous set $\mathcal{A}_h$, given by $A_h(t) := h_t A(t)$, is a weak pullback (forward, resp.) measure attractor for $\Psi$, defined by (7)) if and only if $\mathcal{A}$ is weak pullback (forward, resp.) measure attractor for $\Phi_{t,s}$.*

***Proof*** Note that $\Phi_{t,t_0} = h_t^{-1} \circ \Psi_{t,t_0} \circ h_{t_0}$ and the result follows from Proposition 2.14. □

**Definition 2.16** *(Measure pullback absorbing set)* A nonempty, compact nonautonomous set $\mathcal{B} = \{B(t)\}_{t\in\mathbb{R}}$ with fibres in $\mathbb{R}^d$ is called *measure pullback absorbing* if there exists a nonautonomous set $\mathcal{N} = \{N(t)\}_{t\in\mathbb{R}}$ with $\liminf_{t\to-\infty} \ell(N(t)) > 0$ such that for each $t \in \mathbb{R}$ there exists a $T = T(t, \mathcal{N}) > 0$ such that $\Phi_{t,t_0} N(t_0) \subset B(t)$ for all $t_0 \leq t - T$.

**Theorem 2.17** *Let $\mathcal{B} = \{B(t)\}$ be measure pullback absorbing and forward invariant for $\Phi_{t,s}$. Then there exists a unique weak pullback measure attractor $\mathcal{A}$ determined by*

$$A(t) := \bigcap_{t_0 \leq t} \Phi_{t,t_0} B(t_0),$$

*for all $t \leq 0$.*

***Proof*** It can be shown (see proof of [16, Lemma 2.20]) that the nonautonomous set defined by

$$A(t) := \bigcap_{t_0 \leq t} \Phi_{t,t_0} B(t_0),$$

is non-empty, compact and $\Phi_{t,s}$ invariant. Note that $A(t) \subset B(t)$ so we can assume that $A(t) \subset N(t)$ for all $t \in \mathbb{R}$. If not, we can replace $N(t)$ with $N(t) \cup B(t)$.

We just have to show that $\mathcal{N}$ is pullback attracted to $\mathcal{A}$. Fix any $t$. By the forward invariance and compactness of $\mathcal{B}$ it is easy to show that $\lim_{t_0\to-\infty} d(\Phi_{t,t_0} B(t_0), A(t)) = 0$ (see e.g. [17, Thm 7.1]). Hence, for all $\varepsilon > 0$, there exists a time $T_{t,\varepsilon} \geq 0$ such that for all $t_0 < -T_{t,\varepsilon}$ we have

$$d(\Phi_{t,t_0} B(t_0), A(t)) < \varepsilon.$$

Since $\mathcal{B}$ is absorbing, there exists a $T(t_0, \mathcal{N})$ such that $\Phi_{t_0,s_0} N(s_0) \subset B(t_0)$ for all $s_0 \leq t_0 - T_{t_0,\mathcal{N}}$. Since $\Phi_{t,s}$ is a process, by the evolution property it holds that

$$\Phi_{t,s_0} N(s_0) = \Phi_{t,t_0} \Phi_{t_0,s_0} N(s_0) \subset \Phi_{t,t_0} B(t_0),$$

hence

$$d(\Phi_{t,s_0}N(s_0), A(t)) \le d(\Phi_{t,t_0}B(t_0), A(t)) \le \varepsilon,$$

and hence $\mathcal{A}$ is a weak pullback measure attractor.

Suppose $\mathcal{A}'$ is another weak measure attractor that attracts the same nonautonomous set with $A'(t) \subset N(t)$. Then

$$d(A'(t), A(t)) = \lim_{t_0\to-\infty} d(\Phi_{t,t_0}A'(t_0), A(t)) \le \lim_{t_0\to-\infty} d(\Phi_{t,t_0}N(t_0), A(t)) = 0$$

and similarly $d(A(t), A'(t)) = 0$, hence $\mathcal{A}' = \mathcal{A}$. □

***Remark 2.18*** If $\Phi_{t,s}$ is a continuous, invertible process on a compact space $X$, then a weak pullback measure attractor always exists, but may be the whole space. To see this, consider any $I \subset X$, a non-empty set with positive measure and define a nonautonomous set $\mathcal{A} := \{A(t)\}_{t\in\mathbb{R}}$ by

$$A(t) := \bigcap_{\tau>0} \overline{\bigcup_{s\le-\tau} \Phi_{t,s}I},$$

for all $t \in \mathbb{R}$. Since $X$ is compact, it is easy to see that $\mathcal{A}$ is compact and non-empty. Invariance follows by continuity and invertibility.

Let $B_{\tau,t} = \overline{\bigcup_{k\le-\tau} \Phi_{t,k}I}$ and note that $A(t) = \bigcap_{\tau>0} B_{\tau,t}$ and $B_{s,t} \subset B_{\tau,t}$ for any $\tau < s$, so that $B_{\tau,t}$ is a non-increasing sequence of sets. Hence $d(B_{\tau,t}, A(t)) \to 0$ as $\tau \to \infty$. Note that $\Phi_{t,s}I \subset B_{\tau,t}$ for all $s \le -\tau$. Hence,

$$\lim_{s\to-\infty} d(\Phi_{t,s}I, A(t)) = 0, \tag{8}$$

and hence $\mathcal{A}$ is a weak pullback attractor with $\{I\}_{t\in\mathbb{R}} \in \mathcal{B}^-(\mathcal{A})$.

## 2.4 Autonomous Measure Attractors as Forward and Pullback Measure Attractors

Processes generalise flows in the following sense. Given a flow $\phi$, there is a corresponding process $\Phi$, defined by $\Phi_{t,s}(x) = \phi_{t-s}(x)$. In this subsection, we consider processes corresponding to flows that have a measure attractor, and we analyse consequences for the existence of forward and pullback measure attractors.

Let $\mathcal{A}$ be a nonautonomous set. We define the *limit inferior* and *limit superior* of $\mathcal{A}$ at $\infty$ (see e.g. [6, Definition 1.1.1]) as

$$\liminf_{t\to\infty} A(t) := \left\{x : \lim_{t\to\infty} d(x, A(t)) = 0\right\}$$

and

$$\limsup_{t\to\infty} A(t) := \left\{x : \liminf_{t\to\infty} d(x, A(t)) = 0\right\},$$

respectively.

**Theorem 2.19** *Let $\phi$ be a flow, and consider the corresponding process $\Phi$. Then the following statements hold.*

(i) *$A$ is a weak measure attractor if and only if $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ is weak forward measure attractor.*
(ii) *If $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ is a forward measure attractor, then $A$ is a measure attractor.*

(iii) *Let $A$ be a measure attractor, and assume that for all weak forward measure attractors $\mathcal{A}'$ with $\mathcal{A}' \subsetneq \mathcal{A} = \{A\}_{t\in\mathbb{R}}$, the set $\bigcup_{t>t_0} A'(t)$ is a proper subset of $A$. Then $\mathcal{A}$ is a forward measure attractor.*

(iv) *Let $A$ be a measure attractor such that $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ is a forward measure attractor. Then for any weak forward measure attractor $\mathcal{A}' = \{A'(t)\}_{t\in\mathbb{R}} \subsetneq \mathcal{A}$, the set $\liminf_{t\to\infty} A'(t)$ is a proper subset of $A$.*

***Proof*** (i) The definition of the basin of forward measure attraction of $\mathcal{A}$ implies

$$\begin{aligned}\mathcal{B}^+(\mathcal{A})(t_0) &= \left\{x \in \mathbb{R}^d : \lim_{t\to\infty} d(\Phi_{t,t_0}(x), A(t_0)) = 0\right\}\\ &= \left\{x \in \mathbb{R}^d : \lim_{t\to\infty} d(\phi_{t-t_0}(x), A) = 0\right\} = \mathcal{B}(A)\,,\end{aligned}$$

and hence $\ell(\mathcal{B}^+(\mathcal{A})(t_0)) = \ell(\mathcal{B}(A)) > 0$ for all $t_0 \in \mathbb{R}$.

(ii) Suppose that $\mathcal{A}$ is a forward measure attractor. Let $A' \subset A$ be a proper invariant subset of $A$ with respect to $\phi$. Then $\mathcal{A}' = \{A'\}_{t\in\mathbb{R}}$ is a proper invariant subset of $\mathcal{A}$ with respect to $\Phi_{t,s}$. Then it follows using the proof of (i) above that for any $t \in \mathbb{R}$, we have

$$0 < \ell(\mathcal{B}^+(\mathcal{A})(t) \setminus \mathcal{B}^+(\mathcal{A}')(t)) = \ell(\mathcal{B}(A) \setminus \mathcal{B}(A')),$$

and this means that $A$ is a measure attractor.

(iii) Let $\mathcal{A}' \subsetneq \mathcal{A}$ be a weak forward measure attractor. By assumption we have that $\tilde{A} := \bigcup_{t\geq t_0} A'(t)$ is a proper subset of $A$ and we can see that it is compact and invariant with respect to $\phi$. Since $A$ is a measure attractor, we get $\ell(\mathcal{B}(A) \setminus \mathcal{B}(\tilde{A})) > 0$. Note that $A'(t) \subset \tilde{A}$ for all $t \geq t_0$ and consequently $\mathcal{B}^+(\mathcal{A}')(t) \subset \mathcal{B}(\tilde{A})$ so that

$$\ell(\mathcal{B}^+(\mathcal{A})(t) \setminus \mathcal{B}^+(\mathcal{A}')(t)) = \ell(\mathcal{B}(A) \setminus \mathcal{B}^+(\mathcal{A}')(t)) > 0$$

for all $t \geq t_0$. This implies that $\mathcal{A}$ is a forward measure attractor.

(iv) Suppose $\hat{A} := \liminf_{t\to\infty} A'(t)$ is not a proper subset of $A$, that is, $\hat{A} = A$. Note that $\hat{A}$ is closed.

Since $A$ is a measure attractor, we have

$$\lim_{t\to\infty} d(\phi_{t-t_0}x, A) = 0$$

for all $x \in \mathcal{B}(A)$ and hence,

$$\bigcap_{t>0}\overline{\bigcup_{s>t}\phi_{t-t_0}x} \subset A = \hat{A}.$$

It follows that,

$$\lim_{t\to\infty} d(y, A'(t)) = 0,$$

for all $y \in \bigcap_{t>0}\overline{\bigcup_{s>t}\phi_{t-t_0}x}$ and hence, since $y$ belongs to a compact set it follows that[7]

$$\lim_{t\to\infty} d\Big(\phi_{t-t_0}x, A'(t)\Big) = 0.$$

This means that $x \in \mathcal{B}(\mathcal{A}')(t_0)$. It follows that $\ell(\mathcal{B}^+(\mathcal{A})(t_0) \setminus (\mathcal{B}^+(\mathcal{A}')(t_0)) = 0$ and hence $\mathcal{A}$ is not a forward measure attractor. □

Next, we compare autonomous and nonautonomous pullback measure definitions.

[7] $d(\phi_{t-t_0}x, A'(t)) \leq d(\phi_{t-t_0}x, \omega(x)) + d(\omega(x), A'(t))$, where $\omega(x)$ is the $\omega$-limit set of $x$.

**Theorem 2.20** *Suppose $\Phi_{t,t_0} = \phi_{t-t_0}$ is a process induced by the flow $\phi_t$ on a compact space $X \subset \mathbb{R}^d$.*

(i) *Suppose that A is a weak measure attractor for $\phi_t$. Then $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ is a nonautonomous weak pullback measure attractor.*
(ii) *Suppose that $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ is a weak pullback measure attractor such that $\ell(\bigcap_{t_0<t} N(t_0)) > 0$ for some $N \in \mathcal{B}^-(\mathcal{A})$. Then A is a weak measure attractor.*

***Proof*** (i) Fix some $\varepsilon > 0$. Let $U_\varepsilon \subset \mathcal{B}(A)$ be as in Proposition 1.1 and let $\mathcal{N}(t) = U_\varepsilon$ for all $t \leq 0$. Recall that $l(U_\varepsilon) \geq \ell(\mathcal{B}(A)) - \varepsilon$ so that we have $\liminf_{t\to-\infty} \ell(N(t)) > 0$. Furthermore $d(\Phi_{t,t_0} N(t_0), A(t)) = d(\phi_{t-t_0} U_\varepsilon, A) \to 0$ as $t_0 \to -\infty$. Hence $\mathcal{A}$ is a weak pullback measure attractor.
(ii) Note that $\mathcal{B}(A) = \mathcal{B}_d^-(\mathcal{A})$. Since $\cap_{t_0<t} N(t_0) \subset \mathcal{B}_d^-(\mathcal{A})$, by Lemma 2.12, we have that $\ell(\mathcal{B}(A)) > 0$ and hence $A$ is a weak measure attractor.

□

***Example 2.21*** Consider the flow $\phi_t$ generated by the differential equation

$$\dot{\theta} = \omega$$
$$\dot{y} = -y$$

on $(\theta, y) \in S^1 \times [-1, 1]$, where $\omega > 0$. This differential equation has a hyperbolic attracting limit cycle $A = S^1 \times \{0\}$ that is the only measure attractor for the flow on this space. Although this is therefore a minimal measure attractor, we claim that it contains infinitely many nonautonomous pullback measure attractors. To verify this claim, consider any proper compact subset $A' \subset A$ that has positive measure. Denote by $\phi_t$ the solution flow for the dynamics on $S^1$ and note that $\phi_t(\theta_0) = \omega t + \theta_0$ preserves Lebesgue measure on $S^1$. Then the nonautonomous set

$$\tilde{A}(t) = \{(\theta, 0) \ : \ \theta - \omega t \in A'\} = \{\theta \in \phi_t^{-1}(A')\} \times \{0\}$$

is compact and invariant and pullback attracts the nonautonomous set

$$N(t) = \tilde{A}(t) \times [-1, 1].$$

Note that

$$\liminf_{t\to-\infty} \ell(N(t)) = 2\ell(A') > 0,$$

since $\phi_t$ preserves Lebesgue measure. Moreover any proper compact subset $A'' \subset A'$ can only pullback attract nonautonomous sets whose Lebesgue measure is strictly smaller than this in the limit. Hence $\{\tilde{A}(t)\}_{t\in\mathbb{R}}$ is a nonautonomous pullback measure attractor.

## 3 Statistical Attractors for Nonautonomous Dynamical Systems

In this section, we propose notions of statistical attraction for nonautonomous dynamical systems. Similarly to the autonomous situation (cf. Proposition 1.2), we require that points in the basin of attraction converge to the attractor only on a relatively dense subset of times.

### 3.1 Forward and Pullback Statistical Attractors

Let $\Phi$ be a process and $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ be an invariant nonautonomous set. Motivated by the autonomous quantity $L$ introduced in Sect. 1.1, we consider

$$\tilde{L}(t_0, x, B_\varepsilon(\mathcal{A}), s) := \frac{1}{s}\ell\Big(\big\{t \in [0, s] : \Phi_{t+t_0,t_0}(x) \in B_\varepsilon(A(t + t_0))\big\}\Big)\,,$$

for $t_0 \in \mathbb{R}$, $s \in \mathbb{R}^+$ and we define the basin of forward statistical attraction of $\mathcal{A}$ as the nonautonomous set

$$\mathcal{B}^+_{stat}(\mathcal{A})(t_0) = \Big\{x \in \mathbb{R}^d : \lim_{s\to\infty} \tilde{L}(t_0, x, B_\varepsilon(\mathcal{A}), s) = 1 \text{ for all } \varepsilon > 0\Big\}\,.$$

Analogously to Proposition 1.2, one can show that $x \in \mathcal{B}^+_{stat}(\mathcal{A})(t)$ if and only if there exists a set $T_{x,t} \subset \mathbb{R}^+$ of full density at $\infty$ such that

$$\lim_{s\to\infty,\, s\in T_{x,t}} d(\Phi_{s,t}(x), A(s)) = 0\,.$$

Forward statistical attractors have the property that they attract a set of positive measure statistically.

**Definition 3.1** *(Forward statistical attractor)* We say that a nonempty, compact and invariant nonautonomous set $\mathcal{A}$ is a *forward statistical attractor* if there exists a $t_0 \geq 0$ such that for all $t \geq t_0$, $\ell(\mathcal{B}^+_{stat}(\mathcal{A})(t)) > 0$.

**Definition 3.2** *(Pullback statistical attractor)* We say that a nonempty, compact and invariant nonautonomous set $\mathcal{A}$ is a *pullback statistical attractor* if there is a nonautonomous set $\mathcal{N} = \{N(t)\}_{t\in\mathbb{R}}$ and a set $T$ of full density at $-\infty$ such that for all $t \leq 0$

(i) $\liminf_{\substack{s\to-\infty\\ s\in T}} \ell(N(s)) > 0$,

(ii) and

$$\lim_{\substack{t_0\to-\infty\\ t_0\in T}} d(\Phi_{t,t_0}N(t_0), A(t)) = 0.$$

Analogously to the basin of pullback measure attraction we can define the *basin of pullback statistical attraction*, $\mathcal{B}^-_{stat}(\mathcal{A})$ as a family of all nonautonomous sets that satisfy conditions (i) and (ii) of Definition 3.2.

***Remark 3.3*** It is clear that $\mathcal{B}^-(\mathcal{A}) \subset \mathcal{B}^-_{stat}(\mathcal{A})$ and $\mathcal{B}^+(\mathcal{A}) \subset \mathcal{B}^+_{stat}(\mathcal{A})$, hence it follows that nonautonomous measure attractors are nonautonomous statistical attractors.

***Remark 3.4*** Note that to show $\mathcal{A}$ is a weak pullback statistical attractor, it is enough to construct a sequence of sets $T_{\varepsilon_n,t}$, with $\varepsilon_n \to 0$ as $n \to \infty$, such that $d(\phi_{t,t_0}N(t_0), A(t)) < \varepsilon_n$ for all $t_0 \in T_{\varepsilon_n,t}$. This is because we can use Lemma 1.3 to find a set $T_t$, of full density at $-\infty$, on which

$$\lim_{\substack{t_0\to-\infty\\ t_0\in T_t}} d(\Phi_{t,t_0}N(t_0), A(t)) = 0.$$

***Example 3.5*** We give an example of a statistical pullback attractor which is not a measure pullback attractor. We consider the linear differential equation

$$\dot{x} = a(t)x,$$

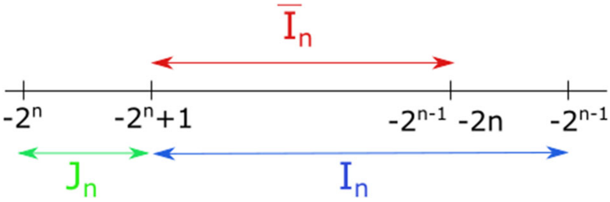


**Fig. 3** A schematic of the intervals $I_k$, $J_k$ and $\bar{I}_k$. Any positive $s$ is in the interval $[2^{n-1}, 2^n]$ for some $n \in \mathbb{N}$. If we want to bound the quantity $\frac{1}{s}\ell(T \cap [-s, 0])$ from below we can note that this will be smallest when $-s = -2^{n-1} - 2n$, since no measure from $\bar{I}_n$ is entering the measure of $\ell(T \cap [-s, 0])$

where

$$a(t) = \begin{cases} -1 & t \in I_k \\ 2^{k-1} & t \in J_k, \end{cases}$$

with sequences of intervals $\{I_k\}_{k\in\mathbb{Z}^+}$ and $\{J_k\}_{k\in\mathbb{Z}_0^+}$ of the form

$$I_k = [-2^k + 1, -2^{k-1}) \quad \text{and } J_k = [-2^k, -2^k + 1)\,,$$

and let $\Phi_{t,t_0}$ denote the associated process. Then for all $k \in \mathbb{N}$ and $x_0 \in \mathbb{R}$, we have

$$\begin{aligned}\Phi_{0,-2^k}(x_0) &= x_0 e^{\int_{-2^k}^0 a(s)\,\mathrm{d}s} = x_0 e^{\sum_{i=1}^k \int_{I_i} -1\,\mathrm{d}s + \sum_{i=0}^k \int_{J_i} 2^{i-1}\,\mathrm{d}s} \\ &= x_0 e^{\sum_{i=1}^k 1-2^{i-1} + \sum_{i=0}^k 2^{i-1}} = x_0 e^{k-(2^k-1)+2^k-\frac{1}{2}} = x_0 e^{k+\frac{1}{2}}\,.\end{aligned}$$

Now define $\bar{I}_k = [-2^k + 1, -2^{k-1} - 2k]$ for $k \geq 5$. Suppose that $t_k \in \bar{I}_k$. Then there exists $\lambda_k \in [0, 1)$ such that the representation $t_k = (1 - \lambda_k)(-2^k + 1) + \lambda_k(-2^{k-1} - 2k)$ holds. Then

$$\begin{aligned}\Phi_{0,t_k}(x_0) &= x_0 e^{\int_{-2^{k-1}}^0 a(s)\,\mathrm{d}s + \int_{t_k}^{-2^{k-1}} a(s)\,\mathrm{d}s} \\ &= x_0 e^{k-\frac{1}{2}+2^{k-1}+t_k} \\ &= x_0 e^{k-\frac{1}{2}+(1-\lambda_k)(1-2^{k-1})-2k\lambda_k} \\ &\to 0,\end{aligned}$$

as $k \to \infty$.

Let $T = \bigcup_{i=5}^{\infty} \bar{I}_i$ and note that this set has full density at $-\infty$, since, for any $s \in \mathbb{R}^+$, there is a $n \in \mathbb{N}$ such that $-s \in [-2^n, -2^{n-1}]$, so that

$$\begin{aligned}\frac{1}{s}\ell(T \cap [-s, 0]) &\geq \frac{1}{2^n + 2(n+1)} \sum_{k=5}^{n} -2^{k-1} - 2k - (-2^k + 1) \\ &= \frac{1}{2^n + 2(n+1)} \sum_{k=5}^{n} 2^{k-1} - 2k - 1 \\ &= \frac{-n^2 - 2n + 2^n + 8}{2^n + 2(n+1)} \to 1,\end{aligned}$$

as $n \to \infty$. See Fig. 3 for an illustration of the intervals $I_k$, $J_k$ and $\bar{I}_k$ which helps to see the inequality above holds.

From the above calculation it follows that for any fixed $x_0 \in \mathbb{R}$, if $t_k \in T$ and $t_k \to -\infty$ as $k \to \infty$, then $\Phi_{0,t_k}(x_0) \to 0$. In fact, we can see that any bounded nonautonomous set will be statistically pullback attracted to 0.

However, we can pick a sequence of times not in $T$, e.g. $t_j = -2^j$ so that no bounded set converges to 0.

### 3.2 Properties of Forward and Pullback Statistical Attractors

Next we establish that a statistical attractor of an autonomous dynamical system is also a pullback statistical attractor. We return to this via an example in Sect. 3.3.

**Theorem 3.6** (Statistical attractors are pullback and forward statistical attractors) *Suppose that $\Phi_{t,t_0} = \phi_{t-t_0}$ is a process that is induced by flow $\phi_t$ defined on a compact set $X \subset \mathbb{R}^d$. Let $A$ be a statistical attractor of $\phi_t$. Then the following statements hold.*

(i) $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ *is a pullback statistical attractor*
(ii) $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$ *is a forward statistical attractor.*

***Proof*** (i) Since $A$ is a statistical attractor, we have $b := \ell(\mathcal{B}_{stat}(A)) > 0$. For any $s > 0$ and $\eta > 0$, we define

$$\tau(x, s, \eta) := \{t \in [0, s] : \phi_t(x) \in B_\eta(A)\}.$$

Since $A$ is a statistical attractor, we have

$$\lim_{s\to\infty} \frac{\ell(\tau(x, s, \eta))}{s} = 1 \quad \text{for all } x \in \mathcal{B}_{stat}(A) \text{ and } \eta > 0\,, \tag{9}$$

i.e. the proportion of the time orbits spent away from any neighbourhood of $A$ goes to zero. Let

$$M_\eta(t) := \left\{x \in \mathcal{B}_{stat}(A) : \phi_t(x) \in B_\eta(A)\right\}$$

and note that $\ell(M_\eta(t)) \leq b$.

We have that

$$\begin{aligned}\int_0^s \ell(M_\eta(t))\,\mathrm{d}t &= \int_0^s \int_{\mathcal{B}_{stat}(A)} \mathbb{1}_{\tau(x,s,\eta)}(t)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{\mathcal{B}_{stat}(A)} \int_0^s \mathbb{1}_{\tau(x,s,\eta)}(t)\,\mathrm{d}t\,\mathrm{d}x \\ &= \int_{\mathcal{B}_{stat}(A)} \ell(\tau(x, s, \eta))\,\mathrm{d}x,\end{aligned}$$

and hence, using Fatou's lemma and (9), we get

$$\begin{aligned}\liminf_{s\to\infty} \frac{1}{s}\int_0^s \ell(M_\eta(t))\,\mathrm{d}t &\geq \int_{\mathcal{B}_{stat}(A)} \liminf_{s\to\infty} \frac{\ell(\tau(x, s, \eta))}{s}\,\mathrm{d}x \\ &= \ell(\mathcal{B}_{stat}(A)) = b,\end{aligned}$$

and since $\ell(M_\eta(t)) \leq b$, this implies

$$\lim_{s\to\infty} \frac{1}{s}\int_0^s \ell(M_\eta(t))\,\mathrm{d}t = b.$$

Hence, for all $\varepsilon > 0$, there exists a time $\tilde{S}(\varepsilon)$ such that for all $s \geq \tilde{S}(\varepsilon)$, $\frac{1}{s}\int_0^s \ell(M_\varepsilon(t))\,\mathrm{d}t \geq b-\varepsilon$. We can choose $\tilde{S}$ to be strictly monotonically decreasing in $\varepsilon$ and such that $\tilde{S}(\varepsilon) \to \infty$ as $\varepsilon \to 0$ [8].

Define the function $E(t) := \tilde{S}^{-1}(t)$, and note that $E(t) \to 0$ as $t \to \infty$. Let $M(t) = M_{E(t)}(t)$. Then,

$$\begin{aligned}\liminf_{s\to\infty} \frac{1}{s}\int_0^s \ell(M(t))\,\mathrm{d}t &\geq \liminf_{s\to\infty} \frac{1}{s}\int_0^s \ell(M_{E(s)}(t))\,\mathrm{d}t \\ &\geq \liminf_{s\to\infty}(b - E(s)) = b. \end{aligned} \tag{10}$$

We consider the set $T_d = \{t \geq 0 : \ell(M(t)) > d\}$ for some $d \in (0, b)$ and show that $T_d$ is a set of full density at $\infty$, i.e. $\lim_{s\to\infty} \frac{1}{s}\ell(T_d \cap [0, s]) = 1$ . Suppose to obtain a contradiction that $\limsup_{s\to\infty} \frac{1}{s}\ell(T_d^c \cap [0, s]) = \xi > 0$ so that $\liminf_{s\to\infty} \frac{1}{s}\ell(T_d \cap [0, s]) = 1 - \xi$. Fix $\varepsilon = \xi(b - d)$. Then,

$$\begin{aligned}\liminf_{s\to\infty} \frac{1}{s}\int_0^s \ell(M(t))\,\mathrm{d}t &= \liminf_{s\to\infty} \frac{1}{s}\Big(\int_{T_d\cap[0,s]} \ell(M(t))\,\mathrm{d}t + \int_{T_d^c\cap[0,s]} \ell(M(t))\,\mathrm{d}t\Big) \\ &\leq \liminf_{s\to\infty} \frac{1}{s}\Big(\ell(T_d \cap [0, s])b\Big) + \limsup_{s\to\infty} \frac{1}{s}\Big(\ell(T_d^c \cap [0, s])d\Big) \\ &= (1 - \xi)b + \xi d = b - \xi(b - d) = b - \varepsilon,\end{aligned}$$

which is a contradiction of (10).

Let $N(t) = M(-t)$. Then it holds that

$$\liminf_{\substack{t\to-\infty,\\ t\in -T_d}} \ell(N(t)) > 0.$$

Finally, let $T_0 = -T_d$. We have that that

$$\lim_{\substack{t_0\to-\infty\\ t_0\in T_0}} d(\Phi_{0,t_0} N(t_0), A) = \lim_{\substack{t_0\to-\infty\\ t_0\in T_0}} d(\phi_{-t_0} M_{E(-t_0)}(-t_0), A) = \lim_{\substack{t_0\to-\infty\\ t_0\in T_0}} E(-t_0) = 0.$$

It remains to show that the above holds for all $t \in \mathbb{R}$, that is,

$$\lim_{\substack{t_0\to-\infty\\ t_0\in T_0}} d(\Phi_{t,t_0} N(t_0), A) = 0. \tag{11}$$

Let $t \in \mathbb{R}$ and fix some $\varepsilon > 0$; continuity of $\Phi_{t,0}$ implies that there exists a $\delta > 0$ such that

$$\Phi_{t,0} B_\delta(A(0)) \subset B_\varepsilon(A(t))\,. \tag{12}$$

Since

$$\lim_{\substack{t_0\to-\infty\\ t_0\in T_0}} d(\Phi_{0,t_0} N(t_0), A(0)) = 0\,,$$

there exists a $\tau = \tau(\varepsilon) < 0$ with

$$d(\Phi_{0,t_0} N(t_0), A(0)) < \delta \iff \Phi_{0,t_0} N(t_0) \subset B_\delta(A(0)) \quad \text{for all } t_0 \in T_0 \text{ and } t_0 < \tau$$

[8] To see this, note that for $\varepsilon' > \varepsilon$, it holds that $\frac{1}{s}\int_0^s \ell(M_{\varepsilon'}(t))\,\mathrm{d}t > \frac{1}{s}\int_0^s \ell(M_\varepsilon(t))\,\mathrm{d}t$. Let $S(\varepsilon) := \inf\{s > 0;\ \frac{1}{\tilde{s}}\int_0^{\tilde{s}} \ell(M_\varepsilon(t))\,\mathrm{d}t > b - \varepsilon \ \forall\, \tilde{s} \geq s\}$. Then $S(\varepsilon') \leq S(\varepsilon)$ so that $S$ is monotone decreasing function. Note that for some large enough $\hat{\varepsilon}$, we must have $S(\hat{\varepsilon}) = 0$. Let $g(\varepsilon)$ be a strictly monotone decreasing function such that $g(\hat{\varepsilon}) = 0$ and define $\tilde{S}(\varepsilon) = S(\varepsilon) + g(\varepsilon)$, so that $\tilde{S}$ is strictly monotone decreasing. Note furthermore that $\tilde{S}(\varepsilon) \to \infty$ as $\varepsilon \to 0$.

This implies using the cocycle property and (12) that

$$\Phi_{t,t_0} N(t_0) \subset B_\varepsilon(A(t)) \quad \text{for all } t_0 \in T_0 \text{ and } t_0 < \tau\,,$$

and we get (11) as required.

(ii) We need to show that

$$\mathcal{B}^+_{stat}(\mathcal{A})(t_0) = \left\{ x \in \mathbb{R}^d : \lim_{s\to\infty} \tilde{L}(t_0, x, B_\varepsilon(\mathcal{A}), s) = 1 \text{ for all } \varepsilon > 0 \right\},$$

has positive measure for all $t_0 \in \mathbb{R}$.

Let $x \in \mathcal{B}_{stat}(A)$. Then,

$$\begin{aligned}\tilde{L}(t_0, x, B_\varepsilon(\mathcal{A}), s) &:= \frac{1}{s}\ell\Big(\Big\{t \in [0, s] : \Phi_{t+t_0,t_0}(x) \in B_\varepsilon(A(t + t_0))\Big\}\Big)\\ &= \frac{1}{s}\ell\Big(\Big\{t \in [0, s] : \phi_t(x) \in B_\varepsilon(A)\Big\}\Big)\,,\end{aligned}$$

and hence $\lim_{s\to\infty} \tilde{L}(t_0, x, B_\varepsilon(\mathcal{A}), s) = 1$ by definition of $\mathcal{B}_{stat}(A)$. Hence, $\mathcal{B}_{stat}(A) \subset \mathcal{B}^+_{stat}(A)(t_0)$ so that $\ell(\mathcal{B}^+_{stat}(A)(t_0)) > 0$ for all $t_0 \in \mathbb{R}$, which completes the proof. □

**Proposition 3.7** *Let $\Phi_{t,s}$ be a process with a pullback (forward, resp.) statistical attractor $\mathcal{A}$ and let $h_t : \mathbb{R}^d \to \mathbb{R}^d$ be a homeomorphism on $\mathbb{R}^d$ for all $t \in \mathbb{R}$ such that the maps $(t, x) \mapsto h_t(x)$ and $(t, x) \mapsto h_t^{-1}(x)$ are continuous. Furthermore, suppose that $h_t^{-1}$ is Lipschitz continuous for all $t \in \mathbb{R}$, with a Lipschitz constant $K > 0$ that does not depend on $t \in \mathbb{R}$. Then $\Psi$, given by the nonautonomous coordinate transformation (7) is a process and has a pullback (forward, resp.) statistical attractor $\mathcal{A}_h = \{A_h(t)\}_{t\in\mathbb{R}}$ given by $A_h(t) = h_t(A(t))$.*

***Proof*** The proof is very similar to Proposition 2.14. □

We note that $\mathcal{A} = \{A\}$ from Theorem 2.20 (i) may not be a pullback measure attractor even if $A$ is a measure attractor (that is, we can only be sure that $\mathcal{A}$ is a weak pullback attractor). This is because it is possible to find examples of measure attractors that contain a proper subset $A'$ which is a statistical attractor such that $\mathcal{B}(A) =_0 \mathcal{B}_{stat}(A')$ and for which it is possible to construct nonautonomous sets which are pullback attracted to $A'$. This is illustrated in the next subsection.

### 3.3 Example Illustrating Pullback Measure Attraction of an Autonomous Statistical Attractor

Theorem 3.6 demonstrates that a statistical attractor for an autonomous system on a compact subset of $\mathbb{R}^d$ is in fact a pullback statistical attractor when considered as a nonautonomous system. It is our conjecture that it is in fact a pullback measure attractor as well. In this section we present an explicit example that illustrates how a suitable nonautonomous basin of pullback measure attraction can be constructed: its geometry is quite subtle as different points need to be avoided in different past fibres.

We identify the unit circle $S^1$ with the unit interval $[0, 1]$ and consider the piecewise smooth autonomous differential equation

$$\left.\begin{aligned}\dot{\theta} &= \begin{cases} y^3 & : \ \theta \le y \\ 1 & : \ \theta > y \end{cases}\\ \dot{y} &= -y^2\,,\end{aligned}\right\} \tag{13}$$

defined on the compact phase space $(\theta, y) \in S^1 \times [0, 1]$. See Fig. 4 for an illustration of trajectories of the system described by (13).

The $y$ component of the solution to this autonomous differential equation is decaying with time ($y \to 0$ as $t \to \infty$), so that the $\theta$ component is getting slower in the region on the left of the line $y = \theta$, while to the right of it, the change it $\theta$ is remaining constant. This means that solutions spend longer and longer time near to $\{(0, 0)\}$. Each individual trajectory will always leave any neighbourhood of $\{(0, 0)\}$ eventually (that is, its limit points are $S^1 \times \{0\}$, however the proportion of time spent outside any neighbourhood of $\{(0, 0)\}$ tends to 0. Hence, $A = S^1 \times \{0\}$, is an (autonomous) measure attractor which contains a singleton set $A' = \{(0, 0)\}$, a saddle-node equilibrium, that is an (autonomous) statistical attractor, for which the statistical basin of attraction is the whole phase space.

Recall that Theorem 2.20, (i) implies that the (autonomous) measure attractor $A$ is a weak pullback measure attractor $\mathcal{A} = \{A\}_{t\in\mathbb{R}}$. We now demonstrate that $\mathcal{A}$ is not a pullback measure attractor, since it turns out that $\mathcal{A}' = \{A'\}_{t\in\mathbb{R}}$ is a pullback measure attractor that attracts nonautonomous sets of maximal measure. This is formulated in Proposition 3.8 below.

**Proposition 3.8** *For the flow generated by (13) $\mathcal{A}'$ is a pullback measure attractor. Moreover for all*

$$\mathcal{N} = \{N(t)\}_{t\in\mathbb{R}} \in \mathcal{B}^-(\mathcal{A}),$$

*there exists $\mathcal{N}' = \{N'(t)\}_{t\in\mathbb{R}} \in \mathcal{B}^-(\mathcal{A}')$ with $\mathcal{N}' \subset \mathcal{N}$ and*

$$\liminf_{t\to-\infty} \ell\big(N(t) \setminus N'(t)\big) = 0.$$

*This means in particular that $\mathcal{A}$ is a weak pullback measure attractor but not a pullback measure attractor.*

Before starting the proof, we explain the strategy briefly. We construct the nonautonomous set $\mathcal{N}'$ by taking $N'(t)$ to contain the preimage of sets of points closer and closer to $(0, 0)$ as $t \to -\infty$. The structure of the differential equation, with a skew product structure and an order-preserving property, allows us to make explicit estimates. It turns out convenient to take as the sets of points close to $(0, 0)$, the points above the line $y = \theta$ where the solution trajectories are slowing down as $y \to 0$ for some small (and decreasing with time) $y = \alpha > 0$ (see Fig. 4; the green intervals illustrate the compliment of the set we consider). We can then show that $N(t)$ has full measure in the limit $t \to -\infty$. Crucially, this does not only hold for sub-sequences of times in a set of full density at $-\infty$.

***Proof*** Denote by $\phi_t(\theta, y) = (\psi_t(\theta, y), \varphi_t(y))$ the flow of (13), and define $L_y := [0, 1] \times \{y\}$. Note that $\phi_t(L_y) = L_{\varphi_t(y)}$ since the time evolution of $y$ does not depend on $\theta$. In fact, this part of the system can be solved explicitly using separation of variables, and we get

$$\varphi_t(y) = \frac{1}{\frac{1}{y} + t} \quad \text{for all } y \in [0, 1] \text{ and } t > 1 - \frac{1}{y}\,. \tag{14}$$

For given $1 \geq \alpha > \beta > 0$, there exists a time $T_{\alpha,\beta} > 0$ with $\varphi_{T_{\alpha,\beta}}(\alpha) = \beta$. It follows from (14) that $T_{\alpha,\beta} = \frac{1}{\beta} - \frac{1}{\alpha}$.

We consider for a given $\alpha > 0$ the set

$$M_\alpha(t) = \big\{\theta \in [0, 1] : \psi_{-t}(\theta, \alpha) \in [0, \varphi_{-t}(\alpha)]\big\} \quad \text{for all } t \leq 0\,,$$

which is the $\theta$-part of the preimage of $[0, \varphi_{-t}(\alpha)] \times \{\varphi_{-t}(\alpha)\}$ under $\phi_{-t}$, and concerns the slow part of the piecewise-defined differential equation. See Fig. 4, which shows, in green,

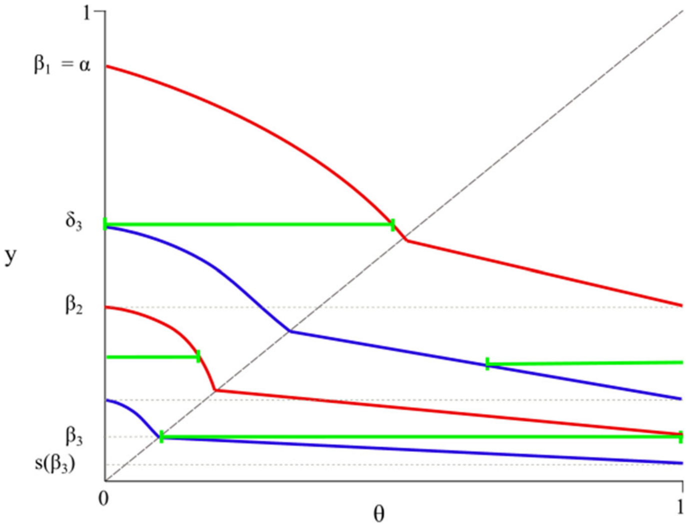


**Fig. 4** Schematic of Step 2. Red lines denotes the trajectory through the point $(1, \beta_3)$, while the blue is the trajectory through $(\beta_3, \beta_3)$. The green interval shows the preimage of the interval $[1, \beta_3] \times \{\beta_3\}$ as it travels back to $\delta_3$. Step 2 of the proof shows that the length of this green interval goes to zero as $k \to \infty$ (and $F^{k-1}(\alpha) = \beta_k \to 0$)

the preimage of the compliment of such an interval. It follows that

$$M_{\varphi_\tau(\alpha)}(t+\tau) = \psi_\tau(M_\alpha(t) \times \{\alpha\}) \quad \text{for all } \alpha > 0, t \leq 0 \text{ and } \tau \in [0, -t]\,.$$

Now it is natural to consider for a given $\mathcal{N} = \{N(t)\}_{t\in\mathbb{R}} \in \mathcal{B}^-(\mathcal{A})$, the nonautonomous set $\mathcal{N}' = \{N'(t)\}_{t\in\mathbb{R}}$, defined by

$$N'(t) := N(t) \cap \bigcup_{\alpha\in(0,1]} M_\alpha(t) \times \{\alpha\} \quad \text{for all } t \leq 0$$

and $N'(t) = N(t)$ for all $t > 0$. It follows that $\mathcal{N}' \in \mathcal{B}^-(\mathcal{A}')$, and we show in the following that

$$\lim_{t\to-\infty} \int_0^1 \ell_1(M_\alpha(t))\, \mathrm{d}\alpha = 1\,,$$

where $\ell_1$ denotes the one-dimensional Lebesgue measure in the fiber $\{y = \alpha\}$, which implies $\liminf_{t\to-\infty} \ell\big(N(t) \setminus N'(t)\big) = 0$.

The remaining proof is divided in four steps.

Step 1. The return map $F : (0, 1] \to (0, 1)$ of the flow $\phi_t$ to the section $\{0\} \times (0, 1]$ can be approximated by

$$F(x) = \frac{x^2}{2} - \frac{3x^4}{8} + O(x^6) \quad \text{for } x \to 0\,. \tag{15}$$

We first note that in the region $\{y \geq \theta\}$, we have $\dot{\theta} = y^3$, and this leads to $\frac{\mathrm{d}\theta}{\mathrm{d}y} = \frac{y^3}{-y^2} = -y$, which implies that trajectories in this region satisfy

$$\theta(y) = -\frac{y^2}{2} + K_1 \tag{16}$$

for some $K_1 \in \mathbb{R}$. Similarly, one can see that trajectories staying in the region $\{y < \theta\}$ satisfy

$$\theta(y) = \frac{1}{y} + K_2 \tag{17}$$

for some constant $K_2 \in \mathbb{R}$.

We now aim at representing the return map $F$ as a composition of two mappings $r$ and $s$. For a given $x \in (0, 1]$, the trajectory starting in $(0, x)$ will eventually cross the identity line $\{y = \theta\}$ line, say at $(u, u)$ for some $u \in (0, 1)$, which, by (16) satisfies $\frac{u^2}{2} + u = K_1$, so that $\theta(y) = -\frac{y^2}{2} + \frac{u^2}{2} + u$. At $(0, x)$, we have $\frac{x^2}{2} = \frac{u^2}{2} + u$, which implies $u = \sqrt{1 + x^2} - 1 =: r(x)$. Using the binomial theorem, we get

$$r(x) = \frac{x^2}{2} - \frac{x^4}{8} + O(x^6)\,. \tag{18}$$

For the next part of the return map, starting at $(u, u) = (r(x), r(x))$, we use (17). We quickly see that $K_2 = u - \frac{1}{u}$ and for the $y$-component at the return to the section $\{0\} \times (0, 1]$, we get $s(u) = \frac{1}{1+u^{-1}-u}$, which satisfies the approximation

$$s(u) = u - u^2 + 2u^3 + O(u^4)\,. \tag{19}$$

Combining both leads to the approximation (15). In summary, the trajectory starting in $(0, x)$ until reaching $(0, F(x))$ is given by

$$\begin{cases} \theta(y) = \frac{1}{2}(x^2 - y^2) & : \ y \in [r(x), x]\,, \\ \theta(y) = \frac{1}{y} - r(x)^{-1} + r(x) & : \ y \in [s(r(x)), r(x)]\,. \end{cases} \tag{20}$$

Step 2. There exists $\alpha_0 > 0$ such that for all $\alpha \in (0, \alpha_0]$, the following holds: With $\beta_k = F^{k-1}(\alpha)$ for $k \in \mathbb{N}$, we show that there exists a $k_0 \in \mathbb{N}$ such that for all $k > k_0$, there exists $\delta_k \in (\beta_2, \beta_1)$ with

$$\phi_{-T_k}\big([\beta_k, 1] \times \{\beta_k\}\big) = \big[0, \tfrac{1}{2}(\alpha^2 - \delta_k^2)\big] \times \{\delta_k\}\,,$$

where $T_k > 0$ is chosen such that $\varphi_{T_k}(\delta_k) = \beta_k$. In addition, $\lim_{k\to\infty} \delta_k = \alpha = \beta_1$.

Note that $F'(0) = 0$, and (15) implies that $F''(x) = 1 - \frac{9x^2}{2} + O(x^4)$, and thus $F''(x) > 0$ in a neighbourhood of 0, which means that $F$ is strictly convex in a neighbourhood of 0. Since $F$ is strictly monotonically increasing, the inverse $F^{-1}$ is strictly concave (in a neighbourhood of 0), and so are iterates of $F^{-1}$. We choose $\alpha_0 > 0$ such that $F^{-1}$ is strictly concave on $[0, \alpha_0]$. We make use of the fact that for strictly concave functions $g : [0, \alpha_0] \to \mathbb{R}$ and $0 < a < b < c < \alpha_0$, we have[9]

$$\frac{g(c) - g(a)}{c - a} > \frac{g(c) - g(b)}{c - b}\,. \tag{21}$$

Choose $\alpha \in (0, \alpha_0)$ and note that then the sequence $(\beta_k)_{n\in\mathbb{N}}$ is defined as above, and we have $\alpha = \beta_1$. For $k \in \mathbb{N}$, define $\delta_k := F^{-k+1}(s(\beta_k)) \in (\beta_2, \beta_1)$.

For any $k \in \mathbb{N}$, we can apply the inequality (21) for the $(-k + 1)$-th iterate of $F^{-1}$, $g = F^{-k+1}$, with $a := \beta_{k+1}$, $b := s(\beta_k)$ and $c := \beta_k$, and we get

$$\frac{F^{-k+1}(\beta_k) - F^{-k+1}(\beta_{k+1})}{\beta_k - \beta_{k+1}} > \frac{F^{-k+1}(\beta_k) - F^{-k+1}(s(\beta_k))}{\beta_k - s(\beta_k)}\,,$$

[9] The definition of strict concavity implies $g(b) > \frac{b-c}{a-c}g(a) + \frac{a-b}{a-c}g(c)$, from which elementary calculations lead to the above inequality.

which is equivalent to

$$\frac{\beta_1 - \beta_2}{\beta_k - \beta_{k+1}} > \frac{\beta_1 - \gamma_1^{(k)}}{\beta_k - s(\beta_k)}.$$

We get

$$\begin{aligned}\beta_1 - \delta_k &< (\beta_1 - \beta_2)\frac{\beta_k - s(\beta_k)}{\beta_k - \beta_{k+1}} \\ &= (\beta_1 - \beta_2)\frac{\beta_k^2 - 2\beta_k^3 + O(\beta_k^4)}{\beta_k - \frac{1}{2}\beta_k^2 + \frac{3}{8}\beta_k^4 + O(\beta_k^6)} \to 0 \quad \text{as } k \to \infty,\end{aligned} \tag{22}$$

where we used (15) and (19).

This implies that there exists a $k_0 \in \mathbb{N}$ such that $\delta_k \geq r(\alpha)$ for all $k > k_0$ (note that $\alpha = \beta_1$). For $k > k_0$, choose $T_k > 0$ such that $\varphi_{T_k}(\delta_k) = \beta_k$. It follows that

$$\phi_{-T_k}(\beta_k, \beta_k) = (0, \delta_k),$$

and we get that

$$\phi_{-T_k}(1, \beta_k) = (\tfrac{1}{2}(\alpha^2 - \delta_k^2), \delta_k).$$

Here, we have applied (20) on the interval $(\beta_1, \beta_2)$, i.e. for $x = \beta_1 = \alpha$, and the trajectory starting at $(1, \beta_k)$ goes backwards in time through $(1, \beta_2)$, and intersects $y = \delta_k$ at $\theta(\delta_k) = \frac{1}{2}(\alpha^2 - \delta_k^2)$ due to (20) since $\delta_k \geq r(\alpha)$. This finishes this step of the proof.

Step 3. For all $\alpha \in (0, \alpha_0]$, we have

$$\lim_{t\to\infty} \ell_1\big(\phi_{-t}([\varphi_t(\alpha), 1] \times \{\varphi_t(\alpha)\})\big) = 0,$$

where $\ell_1$ denotes the one-dimensional Lebesgue measure in the fiber $\{y = \alpha\}$.

Fix $\alpha \in (0, \alpha_0]$. For each $\bar\alpha \in (F(\alpha), \alpha]$, we consider the sequence $(\beta_k^{\bar\alpha})_{k\in\mathbb{N}}$ as in Step 2 with $\beta_1^{\bar\alpha} = \bar\alpha$, and let $\delta_k^{\bar\alpha}$ and $T_k^{\bar\alpha}$ be defined as in Step 2. Due to (22), we get that $\delta_k^{\bar\alpha} \to \bar\alpha$ as $k \to \infty$, uniformly[10] in $\bar\alpha \in (F(\alpha), \alpha]$. Then Step 2 implies that

$$\lim_{k\to\infty} \ell_1\big(\phi_{-T_k^{\bar\alpha}}\big([\beta_k^{\bar\alpha}, 1] \times \{\beta_k^{\bar\alpha}\}\big)\big) = 0,$$

uniformly in $\bar\alpha \in (F(\alpha), \alpha]$.

On the compact set $C := [0, 1] \times [\frac{1}{2}F(\alpha), \alpha]$, consider the continuously differentiable mapping $G : C \to [0, 1]^2$, given by $G(\theta, y) := \phi_{-S(y)}(\theta, y)$, where $S(y)$ is taken to satisfy $\varphi_{S(y)}(\alpha) = y$. Note that continuous differentiability of $G$ implies that this mapping is globally Lipschitz continuous on the compact set $C$, which yields[11]

$$\lim_{k\to\infty} \ell_1\Big( \underbrace{G\big(\phi_{-T_k^{\bar\alpha}}\big([\beta_k^{\bar\alpha}, 1] \times \{\beta_k^{\bar\alpha}\}\big)}_{=\phi_{-S_k^{\bar\alpha}}([\beta_k^{\bar\alpha},1]\times\{\beta_k^{\bar\alpha}\})\subset[0,1]\times\{\alpha\}} \Big) = 0,$$

uniformly in $\bar\alpha \in (F(\alpha), \alpha]$, for certain $S_k^{\bar\alpha} \geq 0$.

[10] Observe that it follows from (22) that $\beta_1^{\bar\alpha} - \delta_k^{\bar\alpha} < (\bar\alpha - F(\bar\alpha))q(\bar\alpha) < (\alpha - F^2(\alpha))q(\alpha)$, for all $\bar\alpha \in (F(\alpha), \alpha]$ where $q$ is a monotone increasing function.

[11] Note that for large enough $k$, the map $\phi_{-T^{\bar\alpha}k}$ maps $y = \beta_k^{\bar\alpha}$ very close to $y = \bar\alpha$ and hence, since $\bar\alpha \in (F(\alpha), \alpha]$, maps it into the compact interval $[\frac{1}{2}F(\alpha), \alpha]$.

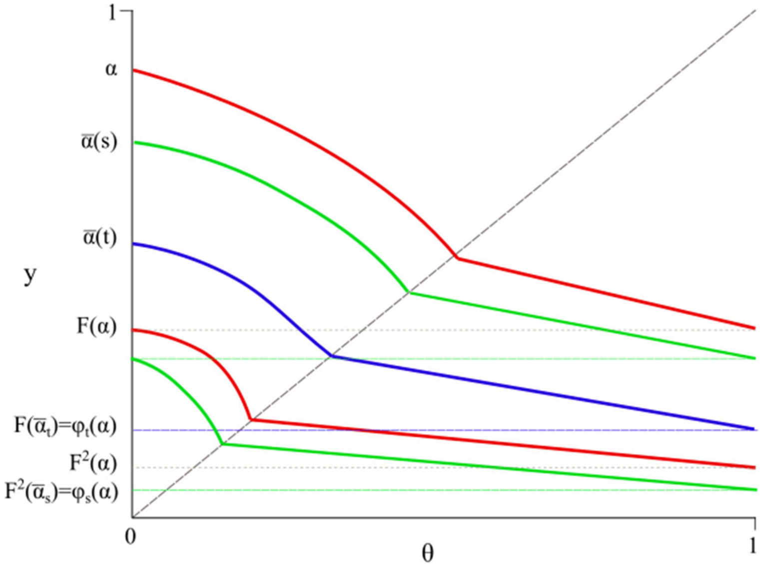


**Fig. 5** Schematic showing Step 3 in the Proof of Proposition 3.8. For some fixed $\alpha \le \alpha_0$, the point $\phi_t(\alpha)$ can be expressed as some iterate of the return map $F$ of some $\bar{\alpha}$, (which depends on $t$), with $\bar{\alpha} \in (F(\alpha), \alpha]$. The figure illustrates two times $s > t$. Green line shows the mapping of $\phi_s(\alpha)$ to $\bar{\alpha}_s$ while the blue shows that of $\phi_t(\alpha)$ to $\bar{\alpha}_t$. The red line represents the trajectory starting at $(0, \alpha)$ at $t = 0$

For any sufficiently large[12] $t$, $\varphi_t(\alpha) \in [F^n(\alpha), F^{n+1}(\alpha)]$ for some $n \ge 1$ so that $\varphi_t(\alpha) = \beta_k^{\bar{\alpha}}$ for some $\bar{\alpha} \in (F(\alpha), \alpha]$ and some $k \in \mathbb{N}$. See Fig. 5 for an illustration. Note that $k$ and $\bar{\alpha}$ depend on $t$. Hence, we can write

$$\phi_{-t}([\varphi_t(\alpha), 1] \times \{\varphi_t(\alpha)\}) = G\big(\phi_{-T_k^{\bar{\alpha}}}\big([\beta_k^{\bar{\alpha}}, 1] \times \{\beta_k^{\bar{\alpha}}\}\big),$$

so that Step 3 follows by Eq. (3.3).

*Step 4. We have*

$$\lim_{t \to -\infty} \int_0^1 \ell_1(M_\alpha(t))\, \mathrm{d}\alpha = 1\,,$$

*and this implies* $\liminf_{t \to -\infty} \ell\big(N(t) \setminus N'(t)\big) = 0$. Using a similar mapping to the mapping $G$ in Step 3, one sees that the statement in Step 3 holds for all $\alpha \in (0, 1]$, and note that this is equivalent to

$$\lim_{t \to \infty} \ell_1(M_\alpha(t)) = 1 \quad \text{for all } \alpha \in (0, 1]\,,$$

and the claimed limit in the statement of Step 4 follows from the dominated convergence theorem.

This finishes the proof of this proposition. □

***Remark 3.9*** One may want to study the differential Eq. (13) on the non-compact phase space $\mathbb{S}^1 \times [0, \infty)$, and we demonstrate now that in this situation, the nonautonomous set $\mathcal{A}'$ (which is still a pullback measure attractor) attracts much less than the weak pullback measure attractor

[12] $t > T_2^\alpha$ is sufficient.

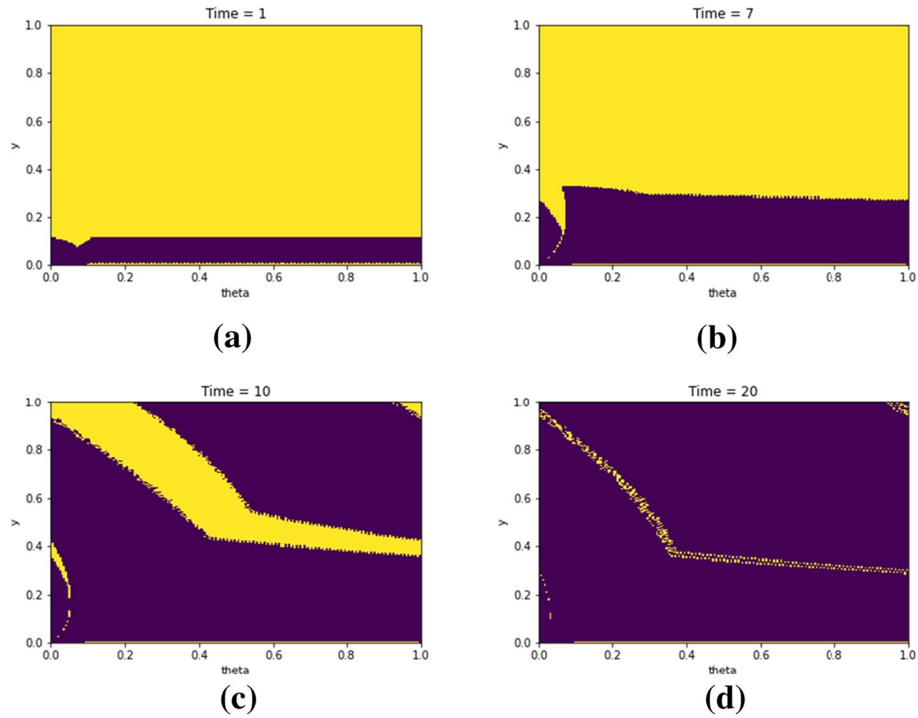


**Fig. 6** A grid of $300 \times 300$ initial points in the domain $(\theta, x) \in [0, 1]^2$ is integrated forward with 4th order Runge–Kutta method. If the solution is found to be a distance less than 0.1 from $(0, 0)$, the initial condition is coloured purple, otherwise it is coloured yellow. The total integration time is (**a**) $t = 1$, (**b**) $t = 7$, (**c**) $t = 10$, (**d**) $t = 20$. Note that the proportion of purple apparently increases to full measure with $t$ but the set of points that are yellow changes with time. This is consistent with Proposition 3.8, where $N(t)$ is approximated by the purple region

$\mathcal{A}$. This means that the second statement of Proposition 3.8 can not be established in this context.

Consider the set

$$B_t := \left[\tfrac{1}{4}, \tfrac{3}{4}\right] \times \left[0, \tfrac{1}{t}\right),$$

and note that with the notation from the proof of Proposition 3.8, one can verify that

$$\ell(\phi_{-\tau}(B_t)) < \infty \quad \text{for all } \tau \in [0, t) \quad \text{and} \quad \ell(\phi_{-t}(B_t)) = \infty\,.$$

This follows, since the $y$-equation is expanding backward in time, and we have a finite escape backward in time of the form $\lim_{\tau \to t} \psi_{-\tau}\left(\frac{1}{t}\right) = \infty$, see (14). Hence, for any $t > 0$, there exists a $K_t > 0$ with

$$\phi_{-t}(B_t) \cap ([0, 1] \times [0, K_t]) = 1\,.$$

Define the nonautonomous set $\mathcal{N} = \{N(t)\}_{t \in \mathbb{R}}$ via $N(t) = [0, 1] \times [0, K_{-t}]$ for $t < 0$ and $N(t) = [0, 1]^2$ for $t \geq 0$. Then $\mathcal{N} \in \mathcal{B}^-(\mathcal{A})$, and for any nonautonomous set $\mathcal{N}' = (N'(t))_{t \in \mathbb{R}} \in \mathcal{B}^-(\mathcal{A}')$, we have

$$\liminf_{t \to -\infty} \ell\big(N(t) \setminus N'(t)\big) \geq 1.$$

## 4 Measure Attractors for Nonautonomous Systems with Autonomous Past and Future Limits

Following [7, Definition 6.3], we define an asymptotically autonomous process as follows.

**Definition 4.1** *(Asymptotically autonomous process)* Let $\Phi_{t,s}$ be a process and $\varphi_t$ a flow. We say $\Phi_{t,s}$ is *asymptotically autonomous to* $\varphi_t$ *in the past* if for all $t > 0$,

$$\lim_{s\to-\infty} \|\Phi_{t+s,s}x_0 - \varphi_t x_0\| = 0 \quad \text{uniformly for } x_0 \text{ in compact subsets}\,.$$

It is *asymptotically autonomous to* $\varphi_t$ *in the future* if for all $t > 0$,

$$\lim_{s\to\infty} \|\Phi_{t+s,s}x_0 - \varphi_t x_0\| = 0 \quad \text{uniformly for } x_0 \text{ in compact subsets}\,.$$

A special case is given by eventually autonomous processes.

**Definition 4.2** *(Eventually autonomous process)* Let $\Phi_{t,s}$ be a process and $\varphi_t$ a flow. We say $\Phi_{t,s}$ is *eventually autonomous to* $\varphi_t$ *in the past* if there exists a $S < 0$ such that

$$\Phi_{s,s-t}x_0 = \varphi_t x_0$$

for all $s \le S$, for all $t < 0$ and all $x_0$. It is *eventually autonomous to* $\varphi_t$ *in the future* if there exists an $S > 0$ such that

$$\Phi_{t+s,s}x_0 = \varphi_t x_0$$

for all $s \ge S$, for all $t > 0$ and all $x_0$.

A *switched system* is a special case of a system that is eventually autonomous in past and future: it is defined by

$$\Phi_{t,s} = \begin{cases} \varphi^-_{t-s} & s < t \le \tau \\ \varphi^+_{t-\tau} \circ \varphi^-_{\tau-s} & s < \tau < t \\ \varphi^+_{t-s} & \tau < s < t, \end{cases} \tag{23}$$

for some $\tau \in \mathbb{R}$, where $\varphi^\pm_t$ are flows.

A possible definition of a nonautonomous $\omega$-limit set is

$$\omega(x_0, t_0) := \{x \in \mathbb{R}^d : \lim_{n\to\infty} \Phi_{t_n,t_0}(x_0) = x \ \text{ for some } t_n \to \infty\} \tag{24}$$

However, such a nonautonomous $\omega$ limit set will typically lack invariance; for an example, see [16, Chapter 3]. In the case of switched systems we have that, if $t_0 < \tau$, then

$$\omega(x_0, t_0) = \omega(\varphi^-_{\tau-t_0}x_0)(\tau) = \omega^+(\varphi^-_{\tau-t_0}x_0),$$

where $\omega^+$ is the limit set of the flow $\varphi^+_t$. If $t_0 > \tau$, then $\omega(x_0)(t_0) = \omega^+(x_0)$ which is invariant under $\varphi^+_t$ and therefore invariant under $\Phi_{t,s}$ for all $\tau < s < t$.

Let $O(t_0, x_0) := \{\Phi_{t,t_0}x_0 \ : \ t > t_0\}$, the orbit of $x_0$ under $\Phi_{t,t_0}$. We say that the orbit is pre-compact if it has compact closure. In the case of more general asymptotically autonomous systems, we have the following result.

**Proposition 4.3** *Consider* $\Omega = \omega(x_0, t_0)$ *for a process* $\Phi_{t,s}$ *that is asymptotically autonomous to* $\varphi_t$ *in the future. Suppose that the orbit* $O(t_0, x_0)$ *is pre-compact. Then,*

(i) *for any* $t > 0$ *we have*

$$\varphi_t \Omega = \Omega$$

*and,*

(ii) *for any* $t > 0$ *and* $\varepsilon > 0$ *there is a* $\rho$ *such that*

$$d_H(\Omega, \Phi_{t+s,s}\Omega) < \varepsilon$$

*for any* $s > \rho$.

***Proof*** (i) Since the orbit $O(t_0, x_0)$ is pre-compact, by [10], Proposition 3.2, we know that the $\omega$ limit set is non-empty and compact.

Fix any positive time $t$ and let $y \in \varphi_t \Omega$. Then there exists an $x \in \Omega$ such that $y = \varphi_t x$. Since $x \in \Omega$, by definition, there exists a sequence of times $\{t_n\}_{n>0}$ such that $t_n \to \infty$ as $n \to \infty$ and $\lim_{n\to\infty} \Phi_{t_n,t_0} x_0 = x$. By the cocycle property

$$\Phi_{t_n+t,t_0} x_0 = \Phi_{t_n+t,t_n} \Phi_{t_n,t_0} x_0,$$

for any $t > 0$. Let $x_n = \Phi_{t_n,t_0} x_0$. We have,

$$\|\Phi_{t_n+t,t_n} x_n - \varphi_t x\| \le \|\Phi_{t_n+t,t_n} x_n - \varphi_t x_n\| + \|\varphi_t x_n - \varphi_t x\|. \tag{25}$$

Since $x_n \in O(t_0, x_0)$ for all $n \ge 0$, which is a precompact set, we can apply the definition of an asymptotically autonomous process (Definition 4.1) on the closure of this set. Then for large enough $n$ we can make the first term on the RHS of Eq. (25) as small as we like uniformly for all times in interval [0,t].

Furthermore, since $x_n \to x$, by continuity of $\varphi_t$ we know that $\varphi_t x_n \to \varphi_t x$. Fix $\varepsilon > 0$. Then, there exists a $N_{\frac{\varepsilon}{2}} \in \mathbb{N}$ and $M_{\frac{\varepsilon}{2}} \in \mathbb{N}$ such that for all $n \ge \max\{N_{\frac{\varepsilon}{2}}, M_{\frac{\varepsilon}{2}}\}$, it holds that

$$\|\Phi_{t_n+t,t_n} x_n - \varphi_t x_n\| \le \frac{\varepsilon}{2}$$

and

$$\|\varphi_t x_n - \varphi_t x\| \le \frac{\varepsilon}{2}.$$

We can therefore set $\tau_n = t_n + t$ and it holds that $\lim_{n\to\infty} \Phi_{\tau_n,t_0} x_0 = \varphi_t x$, and hence $\varphi_t x \in \Omega$. This implies $\varphi_t \Omega \subset \Omega$.

Next we need to show that $\Omega \subset \varphi_t \Omega$. We need to show that for all $x \in \Omega$, there exists a $y \in \Omega$ such that $x = \varphi_t y$. Since $x \in \Omega$, there exists a sequence $\{t_n\}_{n>0}$ such that $\lim_{n\to\infty} \Phi_{t_n,t_0} x_0 = x$. Let $\tau_k = t_{n+k} - t$, $k > 0$, for some $n$ sufficiently large so that $t_n - t > t_0$. Then, suppose that $\lim_{k\to\infty} \Phi_{\tau_k,t_0} x_0$ exists and denote it by $y \in \Omega$ (if not we can choose a convergent sub-sequence).

Using the forward asymptotically autonomous property and arguing as in the first part we have that

$$x = \lim_{k\to\infty} \Phi_{\tau_k+t,t_0} x_0 = \lim_{k\to\infty} \Phi_{\tau_k+t,\tau_k} \Phi_{\tau_k,t_0} x_0 = \lim_{k\to\infty} \varphi_t \Phi_{\tau_k,t_0} x_0 = \varphi_t y, \tag{26}$$

where we have also used continuity of $\varphi$.

(ii) Fix some $t > 0$ and $\varepsilon > 0$. Recall

$$d_H(\Omega, \Phi_{t+s,s}\Omega) := \inf\{\delta > 0 \ : \ \Omega \subset (\Phi_{t+s,s}\Omega)^\delta \text{ and } \Phi_{t+s,s}\Omega \subset \Omega^\delta\},$$

where $X^\delta := \cup_{x\in X}\{z \in M \ : \ \|x - z\| \le \delta\}$.

To show $\Omega \subset (\Phi_{t+s,s}\Omega)^{\varepsilon}$ it suffices to show that for any $x \in \Omega$, there exists a $y \in \Phi_{t+s,s}\Omega$ such that $\|x - y\| \le \varepsilon$. By part (i), we know that there exists $x' \in \Omega$ such that $x = \varphi_t x'$. Take $y = \Phi_{t+s,s}x'$ then since $\Phi_{t,s}$ is asymptotically autonomous we have that for large enough $s$

$$\|x - y\| = \|\varphi_t x' - \Phi_{t+s,s}x'\| < \varepsilon. \tag{27}$$

The other inclusion follows by similar argument.

To show $\Phi_{t+s,s}\Omega \subset \Omega^{\varepsilon}$ it suffices to show that for any $x \in \Phi_{t+s,s}\Omega$, there exists a $y \in \Omega$ such that $\|x - y\| \le \varepsilon$. Let $x = \Phi_{t+s,s}z$ some $z \in \Omega$. Let $y = \varphi_t z \in \Omega$ by part (i). Hence, since $\Omega$ is compact,

$$\|x - y\| = \|\varphi_t z - \Phi_{t+s,s}z\| < \varepsilon, \tag{28}$$

holds for $z$ for same $s$. □

**Theorem 4.4** *Let $\Phi_{t,s}$ be an eventually autonomous process to $\varphi_t^+$ in the future.*

(i) *Let $A^+$ be a weak measure attractor for $\varphi_t^+$. Then there exists a weak forward measure attractor $\bar{\mathcal{A}}$. Moreover, if $\mathcal{A}$ is any compact, $\Phi_{t,s}$ invariant nonautonomous set such that $A^+ \subset \liminf_{t\to\infty} A(t)$, then $\mathcal{A}$ is a weak forward measure attractor.*

(ii) *Let $\mathcal{A}$ be a weak forward measure attractor. If $A^+$ is a compact invariant set for $\varphi_t^+$ and $\emptyset \ne \limsup_{t\to\infty} A(t) \subset A^+$, then $A^+$ is a weak measure attractor.*

***Proof*** (i) Since $\Phi_{t,s}$ is eventually autonomous in the future to a flow $\varphi_t^+$, there exists a $\tau > 0$ such that $\Phi_{t+s,s}x_0 = \varphi_t^+ x_0$ for all $s \ge \tau$ and for all $x_0, t \ge 0$. Define the nonautonomous set $\bar{\mathcal{A}}$ given by

$$\bar{A}(s) = \begin{cases} \Phi_{s,\tau}A^+ & s < \tau \\ A^+ & s \ge \tau. \end{cases}$$

It can be checked that $\bar{\mathcal{A}}$ is a compact and $\Phi_{t,s}$ invariant nonautonomous set. We can write down the basin of forward measure attraction for $\Phi_{t,s}$ as

$$\mathcal{B}^+(\bar{\mathcal{A}})(s) = \begin{cases} \Phi_{s,\tau}B(A^+) & s < \tau \\ B(A^+) & s \ge \tau. \end{cases}$$

If $x \in \mathcal{B}^+(\bar{\mathcal{A}})(s)$ for some $s < \tau$ then $x = \Phi_{s,\tau}y$, for some $y \in B(A^+)$ so that

$$\lim_{t\to\infty} d(\Phi_{t,s}x, \bar{A}(t)) = \lim_{t\to\infty} d(\varphi_{t-\tau}^+ y, A^+) = 0,$$

since $y \in B(A^+)$. Furthermore, since $\ell(B(A^+)) > 0$ and $\Phi_{s,\tau}$ is a diffeomorphism, $\Phi_{s,\tau}B(A^+)$ must have positive measure. Hence, $\bar{\mathcal{A}}$ is weak forward measure attractor.

Let $\mathcal{A}$ be any compact invariant nonautonomous set with

$$A^+ \subset \liminf_{t\to\infty} A(t).$$

This implies $\lim_{t\to\infty} d(A^+, A(t)) = 0$ and hence $\mathcal{A}$ forward attracts all the points in $\mathcal{B}^+(\bar{\mathcal{A}})$ by triangle inequality.

(ii) Since $\limsup_{t\to\infty} A(t) \subset A^+$ it implies that $\lim_{t\to\infty} d(A(t), A^+) = 0$. Since $\Phi_{t,s}$ is eventually autonomous for the future, there exists a $s \in \mathbb{R}$ such that $\Phi_{t+s,s} = \varphi_t^+$. It follows that for all $x \in \mathcal{B}(\mathcal{A})(s)$, $d(\varphi_t^+ x, A^+) \le d(\Phi_{t+s,s}x, A(t)) + d(A(t), A^+) \to 0$ as $t \to \infty$. □

***Remark 4.5*** For a given forward measure attractor of a process $\Phi_{t,s}$ that is asymptotically autonomous in the future to a flow $\varphi^+$, there does not necessarily exist a corresponding measure attractor for $\varphi^+$. For example, consider the differential equation

$$\dot{x} = x\left(x^2 - \frac{1}{t}\right).$$

It can be checked readily that the nonautonomous set $\{\mathcal{A}\}_{t>0}$, defined by $A(t) := [-\frac{1}{\sqrt{2t}}, \frac{1}{\sqrt{2t}}]$, is invariant for $\Phi_{t,s}$ and hence a weak forward measure attractor. However, the limiting flow generated by $\dot{x} = x^3$ does not have any attractors.

In the below theorem we relate pullback measure attractors to autonomous attractors of the past limit system. We note that, it is not guaranteed, as is the case for when past limit system attractor is asymptotically stable (see [9] and [1]), that the nonautonomous attractor limits to the autonomous one in the past and we require a further condition in order for that to be the case.

**Theorem 4.6** *Let $X$ be compact and $\Phi_{t,s}$ an invertible process on $X$ which is asymptotically autonomous in the past to a flow $\varphi_t^-$ with an invariant set $A^-$. Suppose $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is a nonempty, compact, invariant nonautonomous set for $\Phi_{t,s}$ such that* $\limsup_{t\to-\infty} A(t)$ *is uniformly attracted to $A^-$ under $\varphi_t^-$. Then,*

$$\lim_{t\to-\infty} d(A(t), A^-) = 0.$$

*In particular, the above holds if $A^-$ is a weak measure attractor for $\varphi^-$ and $\mathcal{A}$ is a weak pullback measure attractor for $\Phi_{t,s}$.*

***Proof*** Suppose to obtain a contradiction that $d(A(t), A^-) \nrightarrow 0$ as $t \to -\infty$. That is, there exists a $\xi > 0$ such that there exist a sequence of times $t_n \to -\infty$ and a sequence of $x_n \in A(t_n)$ such that

$$d(A(t_n), A^-) \geq d(x_n, A^-) > \xi. \tag{29}$$

Let $S := \limsup_{t\to-\infty} A(t)$. By assumption, $S \subset \mathcal{B}(A^-)$ is attracted uniformly. Hence, there exists a $T_\xi > 0$, such that for all $t > T_\xi$

$$d(\varphi_t^- S, A^-) < \xi/2. \tag{30}$$

Let $B(s) = \overline{\bigcup_{t\leq s} A(t)}$ and note that it is compact. By invariance of $\mathcal{A}$ it holds that $d(x_n, A^-) = d(\Phi_{t_n,t_n-T_\xi} y_n, A^-)$, with $y_n \in A(t_n - T_\xi) \subset B(t_k - T_\xi)$ for all $k \geq n$. By compactness of $B(t_n - T_\xi)$, there is a convergent sub-sequence, $y_{n_l} \to y_0$ as $l \to \infty$. Furthermore, we have that,

$\liminf_{t\to-\infty} d(y_0, A(t)) \leq \liminf_{l\to\infty} \Big(d(y_0, y_{n_l}) + d(y_{n_l}, A(t_{n_l} - T_\xi))\Big) = 0$, and hence it follows that $y_0 \in S$ by definition.

Since $\Phi_{t,s}$ is asymptotically autonomous in the past to $\varphi^-$, by Definition 4.1 it holds that

$$\begin{aligned} \|\Phi_{t_{n_k},t_{n_k}-T_\xi} y_{n_k} - \varphi_{T_\xi}^- y_0\| &\leq \|\Phi_{t_{n_k},t_{n_k}-T_\xi} y_{n_k} - \varphi_{T_\xi}^- y_{n_k}\| + \|\varphi_{T_\xi}^- y_{n_k} - \varphi_{T_\xi}^- y_0\| \\ &< \xi/2, \end{aligned} \tag{31}$$

for large enough $k$. Hence,

$$\begin{aligned} d(x_{n_k}, A^-) &= d(\Phi_{t_{n_k},t_{n_k}-T_\xi} y_{n_k}, A^-) \\ &\leq \|\Phi_{t_{n_k},t_{n_k}-T_\xi} y_{n_k} - \varphi_{T_\xi}^- y_0\| + d(\varphi_{T_\xi}^- y_0, A^-) \end{aligned}$$

$$< \xi,$$

by Inequalities (30) and (31), which is a contradiction of (29). □

We give an example that shows it is possible to have an attractor for a process that is asymptotically autonomous in the past, which is unrelated to the attractors of the past limit flow.

***Example 4.7*** Consider a differentiable function $\chi : \mathbb{R} \to \mathbb{R}$ that satisfies

$$\lim_{t\to-\infty} \chi(t) - \sqrt{-t} = \lim_{t\to\infty} \chi(t) + \sqrt{t} = 0\,.$$

Then

$$\Phi_{t,s}(x) = xe^{\chi(t)-\chi(s)} \quad \text{for all } t, s, x \in \mathbb{R}$$

defines a process, and for any fixed $t \in \mathbb{R}$, we have

$$\begin{aligned}\lim_{s\to-\infty} \Phi_{t+s,s}(x) = \lim_{s\to-\infty} xe^{\chi(t+s)-\chi(s)} &= \lim_{s\to\infty} xe^{\sqrt{t+s}-\sqrt{s}} \\ &= \lim_{s\to\infty} xe^{t/(2\sqrt{s})} = x,\end{aligned}$$

and hence $\Phi_{t,s}$ is an asymptotically autonomous process in the past to the identity. Note that all positive measure compact sets are measure attractors for the identity flow. On the other hand, $\{0\}_{t\in\mathbb{R}}$ is an invariant nonautonomous set for $\Phi_{t,s}$ and $\lim_{s\to-\infty} \Phi_{t,s}x = 0$ for all $x \in \mathbb{R}$, which implies that $\{B\}_{t\in\mathbb{R}}$ is pullback attracted by $\{0\}_{t\in\mathbb{R}}$ for any compact set $B \subset \mathbb{R}$. This implies that $\{0\}_{t\in\mathbb{R}}$ is a pullback measure attractor. Similarly, one can show that the process $\Phi_{t,s}$ is asymptotically autonomous in the future to the identity, and $\{0\}_{t\in\mathbb{R}}$ is a forward measure attractor, while 0 is not a measure attractor of the future limiting flow.

**Theorem 4.8** *Let $X$ be compact and $\Phi_{t,s}$ an invertible process on $X$ which is asymptotically autonomous in the future to a flow $\varphi_t^+$ with an invariant set $A^+$. Suppose $\mathcal{A} = \{A(t)\}_{t\in\mathbb{R}}$ is a nonempty, compact, invariant nonautonomous set for $\Phi_{t,s}$ such that* $\limsup_{t\to\infty} A(t)$ *is uniformly attracted to $A^+$ under $\varphi_t^+$. Then,*

$$\lim_{t\to\infty} d(A(t), A^+) = 0.$$

*In particular, the above holds if $A^+$ is a weak measure attractor for $\varphi^+$ and $\mathcal{A}$ is any weak forward measure attractor for $\Phi_{t,s}$.*

***Proof*** The proof is very similar to Theorem 4.6. □

## 5 Discussion and Outlook

There are many inequivalent notions of attraction for autonomous dynamical systems. Two of the weakest (but still useful) notions from a physical point of view are the measure attractor (which makes no assumption about the topology of the basin of attraction but requires it to have positive measure) and the Ilyashenko statistical attractor (which makes no assumption about pointwise convergence, but requires convergence of averaged observables). In this paper, we state and prove some elementary properties of autonomous measure and statistical attractors. In particular, Proposition 1.1 shows that measure attractors have a large degree of uniformity within their basin, while Proposition 1.2 shows that an apparently stronger form of convergence follows from the definition of statistical attraction: these results inspire some of the examples and results stated later in the paper.

In the paper we propose some natural generalisations of measure and statistical attractors to nonautonomous dynamical systems, defined as continuous time processes. These notions are complicated by the fact that convergence in the nonautonomous setting requires a choice of pullback or forward notions, and we address this in our proposed definitions. To test the definitions we consider autonomous systems in a nonautonomous viewpoint. In particular:

- In Theorems 2.19 and 2.20 we illustrate how our notions of weak forward measure and weak pullback measure attraction can be related to autonomous notions.
- Theorem 3.6 relates forward and pullback notions of statistical attractor to the autonomous notion.
- Probably the most surprising result is Proposition 3.8 which illustrates for a specific example that a statistical attractor that is NOT a measure attractor in the autonomous setting may be a pullback measure attractor in the nonautonomous setting. We suspect this result may be true in a more general context, but are unable to prove this.
- In Sect. 4 we turn to asymptotically autonomous systems where in Theorem 4.4 we relate the forward measure attractor to the measure attractor for the future limit system, and Theorem 4.6 where we relate the pullback measure attractor to the measure attractor for the past limit system.

One of the significant barriers to understanding nonautonomous systems is that few dynamical properties are invariant under general nonautonomous coordinate changes [20]: attractors may even become repellers when subjected to an extreme time-dependent coordinate change. Restricting to a suitably constrained set of coordinate changes can however preserve some attraction properties. We give some results on these lines, for example Proposition 2.14 gives sufficient conditions (Lipschitz uniform in $t$) that weak forward measure attraction is retained under coordinate change.

Although we test the proposed definitions in autonomous and asymptotically autonomous contexts, it will be important to test and apply these definitions in more general contexts. We have not addressed the question of when and whether nonautonomous measure attractors can be decomposed, but there is clearly work to be done to clarify when nonautonomous measure attractors are strong/weak and/or minimal. We have also not addressed questions of how attractor structure changes at bifurcation on changing a parameter in a nonautonomous system. In this context we expect various types of bifurcation that will depend on the notion of nonautonomous attraction that is considered and this can be linked to bifurcation of past/future limits in some cases. There will also be true nonautonomous bifurcations (such as rate-induced tipping effects) where past and future limit systems are both involved. It will be a significant challenge to generalise such results for asymptotically stable attractors [1, 4, 25] to weaker notions of attraction.

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**Data Availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study

## Declarations

**Conflict of interest** The authors declare that they have no conflict of interest.

## Appendix

## A Proof of Lemma 1.3

Let $\delta_n = \frac{1}{2^n}, n \in \mathbb{N}_{\geq 1}$ and define $\{\tau_n\}_{n\geq 1}$ as the sequence of times such that $\ell(T_n^c \cap [0, \tau]) < \delta_n \tau$, for all $\tau > \tau_n$.

Let $s_1 = 0$ and $s_2 = \max(\tau_1, \tau_2)$. Further, let $\rho_1 = (\delta_1 + \delta_2)/2 = \frac{3}{2^3}$. Let $\tilde{T}_1 = (T_1 \cap [0, s_2]) \cup (T_2 \cap [s_2, \infty))$. Then,

$$\ell(\tilde{T}_2^c \cap [0, \tau]) = \ell(\tilde{T}_2^c \cap [0, \tilde{t}_2]) + \ell(\tilde{T}_2^c \cap [s_2, \tau)) \leq \delta_1 s_2 + \delta_2 \tau \leq \rho_1 \tau,$$

for all $\tau > S_2$ where $S_2 = \frac{\delta_1 s_2}{\rho_1 - \delta_2} = 2^2 s_2$.

Let $s_3 = \max(S_2, \tau_3)$ and let $\tilde{T}_2 = (\tilde{T}_1 \cap [0, s_3]) \cup (T_3 \cap [s_3, \infty))$ so that

$$\ell(\tilde{T}_2^c \cap [0, \tau]) = \ell(\tilde{T}_2^c \cap [0, s_3]) + \ell(\tilde{T}_2^c \cap [s_3, \tau)) \leq \delta_2 s_3 + \delta_3 \tau \leq \rho_2 \tau,$$

for all $\tau > S_3$ where $S_3 = \frac{\delta_2 s_3}{\rho_2 - \delta_3}$ and $\rho_2 = \frac{3}{2^4}$.

Let $\tilde{T}_n = (\tilde{T}_{n-1} \cap [0, s_{n+1}]) \cup (T_{n+1} \cap [s_{n+1}, \infty])$ with $s_n = \max(S_{n-1}, \tau_n)$ where $S_{n-1}$ is the time such that $\ell(\tilde{T}_{n-1}^c \cap [0, \tau]) \leq \rho_{n-1}\tau$ for all $\tau > S_{n-1}$ and with $\rho_n = (\delta_n + \delta_{n+1})/2 = \frac{3}{2^{n+2}}$, so that

$$\ell(\tilde{T}_n^c \cap [0, \tau]) = \ell(\tilde{T}_n^c \cap [0, s_{n+1}]) + \ell(\tilde{T}_n^c \cap [s_{n+1}, \tau)) \leq \delta_n s_{n+1} + \delta_{n+1}\tau \leq \rho_n \tau,$$

for all $\tau \geq S_{n+1}$ where $S_{n+1} = \frac{\delta_n s_3}{\rho_n - \delta_{n+1}}$.

Note that

$$\tilde{T}_N = \left( \bigcup_{n=1}^{N-1} T_n \cap [s_n, s_{n+1}] \right) \cup (T_N \cap [s_N, \infty])$$

and hence $\tilde{T}_N \cap [0, s_N]$ is a non-decreasing sequence of sets so that $T_\infty := \bigcup_{N=1}^{\infty} \tilde{T}_N \cap [0, s_N]$ is well defined. We have that for some large enough $N$

$$\ell((T_\infty)^c \cap [0, \tau]) \leq \ell((\tilde{T}_N)^c \cap [0, \tau]) \leq \rho_{N-1}\tau$$

and hence, since $\rho_N \to 0$ as $N \to \infty$, $T_\infty$ has full density at $\infty$.

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