EVOLUTIONARY FINANCE AND DYNAMIC GAMES

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CONTENTS

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ABSTRACT

Evolutionary Finance (EF) explores financial markets as evolving biological systems. Investors pursuing diverse investment strategies compete for the market capital. Some survive and some become extinct. A central goal of the study is to identify investment strategies guaranteeing survival. The problem is examined within a non-traditional game-theoretic framework combining stochastic dynamic games and evolutionary game theory. Models analysed in this area employ only objectively observable market data, in contrast to traditional neoclassical settings relying upon unobservable agents' characteristics: individual utilities and beliefs. The main results provide effective constructions of survival strategies. The thesis contributes to EF in three respects: (i) the most general EF model with long-lived dividend-paying assets is developed; (ii) a new model with endogenous asset dividends is proposed; (iii) a systematic study of the notion of an unbeatable strategy (a game solution concept playing a key role in EF) is conducted.

DECLARATION

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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CHAPTER 1. INTRODUCTION

Evolutionary Finance (EF) aims at improving our understanding of the causes and effects of the dynamic nature of financial markets through the application of Darwinian ideas. Market places for risky assets exhibit an unparalleled degree of dynamics and evolution in the behavior and interaction of its participants. The innovations in investment styles, products, and the regulatory framework appear to be limitless. All of these changes can be traced back to human endeavor, which tries to achieve intended aims within an environment characterized by the adaptive, self-organizational, and endogenous dynamics of the decisions and interaction of all market participants. With unintended consequences at times. Shiller, in his book Irrational Exuberance [157], concludes:

> In sum, stock prices clearly have a life of their own; they are not simply responding to earnings or dividends. Nor does it appear that they are determined only by information about future earnings or dividends. In seeking explanations of stock price movements, we must look elsewhere.

It is this "life of their own" of financial markets that the EF approach strives to capture.

EF has two defining characteristics: a descriptive approach to the specification of investors and a focus on the dynamics of the wealth distribution. Investors are allowed to adopt behaviors driven by heuristic reasoning and/or behavioral biases, e.g., myopic optimization, dependence of decisions on past performance, and other forms of bounded rationality. For instance, financial practitioners at the cutting edge of active investment are mainly concerned with beating a benchmark, which is awarded a bonus, rather than in pursuing more elusive goals. The dynamics of investors' wealth is driven by the market interaction of investors and the randomness of asset payoffs. Successful investors gain wealth, while unsuccessful ones lose wealth and disappear in the long-run. In the short-run all investors' wealth levels fluctuate, sometimes violently, depending on the risk appetite of an investor and her clients' patience. The focus of this study will be on investment strategies ensuring long-run stability and survival in crisis environments.

The choice of the equilibrium concept marks another main shift in the paradigm of how markets can be modeled. Rather than following the neoclassical world in which all of the investors share the same opinion about the possible future contingencies (and the price of each asset in every possible state), market equilibrium is only invoked in the short run through market clearing at the current date. The advantage of this approach is twofold: computational and conceptual. Heterogeneity of investors represents the diversity of opinions and types of behavior; short-run goals shift the focus from discounted expected utility to the wealth of investors and its dynamics. A main object of study is the performance of investment styles, in particular within a specific set of strategies. EF opens the door to the study of this line of inquiry without invoking a notion of equilibrium that would require the agreement of market participants about future prices. In EF frameworks, the process of market dynamics is described as a sequence of consecutive short-run equilibria determining equilibrium asset prices over each time period. The notion of a short-run price equilibrium is defined directly via the set of investment strategies/portfolio rules of the market players specifying the patterns of their investment behavior. Moreover, in those EF models that combine evolutionary ideas with behavioural approaches (see Sect. 2.1 Evolutionary Behavioural Finance) no utility maximization or full rationality of market players are involved.

Evolutionary ideas have a long history in the social sciences going back to Malthus, who played an inspirational role for Darwin (see, e.g., Hodgeson [91]). Veblen [172] coined the term “evolutionary economics" and started a systematic use of the evolutionary approach in the social sciences [173]. Schumpeter [151] laid the groundwork for Evolutionary Economics in the 20th century. An important role in the creation of this branch of Economics was played by the works of Alchian [1], Boulding [32], Downie [48], D. Friedman [66, 67], M. Friedman [68], Hodgson [91, 92], Penrose [135], Nelson [132], and Nelson and Winter [133].

A powerful momentum to work in this area was given by the interdisciplinary research conducted in the 1980s and 1990s under the auspices of the Santa Fe Institute, New Mexico, USA, where researchers of different backgrounds—economists, mathematicians, physicists and biologists—combined their efforts to study evolutionary dynamics in biology, economics and finance; see, e.g., the series of volumes "Economy as an Evolving Complex System": Vol. I, Anderson, Arrow and Pines, eds. [9], Vol. II, Arthur, Durlauf and Lane, eds. [12], and Vol. III, Blume and Durlauf, eds., [21] published in the framework of the Economics Program at the Santa Fe Institute. The very term "Evolutionary Finance" seems to appear for the first time in the paper by LeBaron [108] discussing numerical experiments based on the Santa Fe Artificial Stock Market model (see Palmer et al. [134] and Arthur et al. [13]).

Fundamental contributions to EF and in particular to the analysis of dynamic stochastic general equilibrium models were made by Blume and Easley [22], [23], [24], Sandroni [145], Bottazzi et al. [30, 31], Bottazzi and Dindo [28, 29], Brock et al. [35], Coury and Sciubba [42], Farmer [63], Farmer and Lo [64], Guerdijkova and Sciubba [77], Lo [115, 116, 117, 118], Lo et al. [119], Sciubba [153, 154], and Zhang et al. [181]. This branch of literature is focused on the problems of market selection among rational agents with different beliefs. Namely, in [22] Blume and Easley develop an evolutionary model of a financial market and show that (provided all the agents have the same saving rule) those who maximize a logarithmic utility function will accumulate in the long run all the market wealth. However, if no agents are using this rule, then the agents with the most accurate predictions might be not the most prosperous and agents with inaccurate predictions are not necessarily driven out of the market.

Sandroni in [145] examines who survives in the market without the assumption of identical discount factors of all the agents. It turns out that all agents who survive must have the highest level of “fitness”, which depends on the beliefs and the discount factor. In particular, if all agents have the same discount factors, then only the agents whose beliefs are “the closest” to the truth will survive.

Blume and Easley [23] prove that in complete markets survival or extinction of an agent depends on her beliefs and the discount factor. Moreover, controlling for discount factors, only traders with the correct beliefs survive. Hence, the market selection hypothesis holds and supports the observation that “firms behave as if they were seeking to maximize their expected returns" and "unless the behavior of businessmen in some way or other approximated behavior consistent with the maximization of returns, it seems unlikely that they would remain in business for long" (Friedman [68]). Therefore, the ability to predict returns correctly is a key to survival in many DSGE models. A comprehensive review of the seminal papers addressing the market selection problem is given in Blume and Easley [24].

The EF models studied in this dissertation invoke ideas of evolutionary game theory (Weibull [177], Vega-Redondo [174], Samuelson [142], Hofbauer and Sigmund [93], Kojima [103], Gintis [72], Foster and Young [65], Cabrales [37], Germano [71]) and games of survival (Milnor and Shapley [129], Shubik and Thompson [160], Borch [25], Karni and Schmeidler [98], and Amir et al. [6]).

Another important source for the EF models considered in this thesis is capital growth theory, or the theory of growth-optimal investments: Shannon [155], Kelly [100], Latané [107], Breiman [34], Algoet and Cover [2], Hakansson and Ziemba [79], Cover [43], Györfi et al. [78], MacLean et al. [121], Kuhn and Luenberger [104], Ziemba and Vickson [182], MacLean and Ziemba [122], and others. For a textbook presentation of capital growth theory see Evstigneev et al. [56], Ch. 17. The EF models we deal with may be regarded as capital growth models with endogenous (formed in the dynamic equilibrium), rather than exogenous as in the classical theory, asset prices.

A survey describing the state of the art in the field by 2016 and outlining a program for further research is given in Evstigneev et al. [57]. An elementary textbook treatment of the subject can be found in Evstigneev et al. [56], Ch. 20. For a most recent review of studies related to EF, see Holtfort [94].

What has this new paradigm for finance achieved so far? On the one hand, it has improved our understanding of the dynamics of asset prices since many stylized facts, such as for instance excess volatility, can be explained by the endogenous dynamics of wealth. Excess volatility was first pointed out by Shiller [156] who showed that the prices of the S&P 500 index are more volatile than the fundamental values computed with models of expected utility maximization given rational expectations. Boswijk et al. [27] showed that a simple Evolutionary Finance model can explain the excess volatility of the S&P 500. Other stylized facts that are hard to reconcile with utility maximization include stochastic volatility, autocorrelation and heavy tails in the return distribution of asset prices (cf. Cont [41] for a more exhaustive list). For comprehensive treatments of the achievements of Evolutionary Finance in asset pricing we refer to LeBaron [109] and Hommes [95].

On the other hand Evolutionary Finance also contributes to portfolio theory, which is not descriptive but normative. Portfolio theory asks how to invest. The traditional answer (see for example Markowitz [125]) is that one should maximize an objective function given the return expectation one has. In this view, returns are taken as exogenous. However, modeling the financial market via a few investment strategies, the impact of the strategies (not the individual investors) on the market is obvious and a game theoretic approach would be more suitable. One should select a strategy that performs well in competition with the other strategies. Performing well in evolutionary models means at least to stay alive. Thus, in evolutionary portfolio theory there is a focus on so-called survival strategies. Applying this idea to the evolution of relative wealth, survival requires that no other strategy achieve a higher growth rate of wealth. Of course, this criterion has always been criticized by adherents of utility maximization (see e.g. Samuelson [144]), but as Sciubba [154, p. 125] put it eloquently: a survival strategy "might not make you happy, but will definitely keep you alive".

One might suspect that the existence and the characterization of survival strategies depend on the exogenous stochastic process and on the market ecology, i.e. the set of investment strategies competing for wealth. This is indeed the case when one limits the pool of strategies. However, since there is always a potential for innovation, it would be risky to do so. Indeed the most general result on survival strategies that was achieved so far (see Evstigneev et al. [53]) shows the existence of a survival strategy for any ecology of investment strategies and any dividend processes. The survival strategy can be characterized as being a well-diversified fundamental strategy, which is contrarian. As such, it might explain the great success of value investing in equity markets (cf. Gergaud and Ziemba [70]).

Furthermore, most other results in the literature are based on a limited set of strategies – not allowing all innovations. Limiting the market ecology has been a successful strategy to better understand asset prices. For example, the paper of Scholl et al. [150] limits the ecology to a fundamental, a momentum and a noise trader strategy, and is able to explain many interesting stylized facts of asset prices. Surely, models explaining stylized facts of asset pricing get stronger the simpler they are. However, such a limitation is potentially dangerous when one wants to draw general conclusions for portfolio theory. A strategy that is best in a restricted ecology might suffer severe losses when a new strategy from outside the current ecology emerges. A similar remark applies to the famous Brock and Hommes model (see [95]), which is also based on a similar set of three types of strategies but enriches the evolution of wealth by allowing investors to switch between the three strategies. As a result, much richer asset price dynamics may be achieved. But as Hens and Schenk-Hoppé [87] have shown, introducing a strategy that stolidly follows the fundamental strategy of Evstigneev et al. [53] would drive out all other strategies of the Brock and Hommes model.

Finally, results in Evolutionary Finance depend on the market microstructure. In the famous Santa Fe model (Palmer et al. [134], Arthur et al. [13], LeBaron [109]), strategies are generated by genetic algorithms and markets are cleared by a market maker. As was shown in Lensberg and Schenk-Hoppé [110], also using genetic algorithms, the survival strategy of Evstigneev et al. [53] will evolve when one uses a batch auction as in [53] [1].

The remainder of the dissertation consists of three chapters.

Chapter 2. This chapter analyzes a dynamic stochastic equilibrium model of an asset market based on behavioural and evolutionary principles. The core of the model is a non-traditional game-theoretic framework integrating stochastic dynamic games and evolutionary game theory. It relies only on objectively observable market data and does not use unobservable individual agents' characteristics (utilities and beliefs), which makes the model amenable for quantitative applications. A central goal of the study is to find an investment strategy that allows an investor to survive in the market selection process. The main results show that such a strategy exists, is asymptotically unique and easily computable.

Chapter 3. In contrast with the majority of EF models where asset dividends are given exogenously, the model considered in this chapter deals with endogenous dividends. They depend on the fraction of total market wealth invested in each particular asset. The main results establish the existence of an evolutionary stable investment strategy and provide its effective construction.

[1] That the asset price dynamics of Evolutionary Finance models depends on the market microstructure is shown in Bottazzi et al. [31] and Anufriev and Panchenko [10]. The point made in Lensberg and Schenk-Hoppé [110] is to show that also the outcome of the market selection depends on the market microstructure.

Chapter 4 conducts a systematic study of the notion of an unbeatable strategy as a game solution concept. The study is motivated by the applications of this notion in Evolutionary Finance. In the context of EF models, the concepts of a survival strategy and an unbeatable strategy are equivalent (see Section 2.4). A general framework (game with relative preferences) suitable for the analysis of this concept is proposed. Basic facts regarding unbeatable strategies are presented and a number of examples and applications considered.

CHAPTER 2. BEHAVIOURAL EQUILIBRIUM AND EVOLUTIONARY DYNAMICS IN ASSET MARKETS

"Mainstream economic theory is based on the rationality assumption: that people act as best they can to promote their interests. In contrast, behavioural economics holds that people act by behavioural rules of thumb, often with poor results. ... People indeed act by rules, which usually work well, but may work poorly in exceptional or contrived scenarios. The reason is that like physical features, behavioural rules are the product of evolutionary processes; and evolution works on the usual, the common—not the exception, not the contrived scenario."

R.J. Aumann, A synthesis of behavioural and mainstream economics. *Nat. Hum. Behav.* 3, 666–670 (2019).

This chapter, a core part of the thesis, develops a dynamic stochastic equilibrium model of an asset market combining behavioural and evolutionary principles. The basis of the model is a non-traditional game-theoretic framework involving elements of stochastic dynamic games and evolutionary game theory. Its main characteristic feature is that it employs only objectively observable market data and does not use hidden individual agents' characteristics (such as their utilities and beliefs). A central goal of the study is to identify an investment strategy that allows an investor to survive in the market selection process, i.e., to keep with probability one a strictly positive, bounded away from zero share of market wealth over an infinite time horizon, irrespective of the strategies used by the other players. The main results show that under very general assumptions, such a strategy exists, is asymptotically unique and easily computable. The results of this chapter are published in Evstigneev et al. [53].

2.1. Evolutionary Behavioural Finance

In this chapter we develop a dynamic stochastic equilibrium model of an asset market combining evolutionary and behavioural approaches. The classical financial DSGE theory going back to Kydland and Prescott [106] and Radner [139, 140] (see Magill and Quinzii [123]) relies upon the hypothesis of full rationality of market players, who are assumed to maximize their utilities or preferences subject to budget constraints, i.e., solve well-defined and precisely stated constrained optimization problems. The model we consider relaxes these assumptions and permits traders/investors to have a whole variety of patterns of behaviour determined by their individual psychology, not necessarily describable in terms of utility maximization. Strategies may involve, for example, mimicking, satisficing, rules of thumb based on experience, etc. Strategies might be interactive—depending on the behaviour of the others. Objectives might be of an evolutionary nature: survival (especially in crisis environments), domination in a market segment, fastest capital growth, etc. They might be relative—taking into account the performance of the others.

Models considered in this field—they are referred to as "EBF" (Evolutionary Behavioural Finance) models—combine elements of the theory of stochastic dynamic games and evolutionary game theory. The former offers the general notion of a strategy and the latter suggests the game solution concept: a *survival strategy*. In EBF frameworks, the process of market dynamics is described as a sequence of consecutive short-run equilibria determining equilibrium asset prices over each time period. The notion of a short-run price equilibrium is defined directly via the set of strategies of the market players specifying the patterns of their investment behaviour (*behavioural equilibrium*).

The main focus of EBF is on investment strategies that survive in the market selection process, i.e., guarantee with probability one a positive, bounded away from zero share of market wealth over an infinite time horizon. Typical results show that such strategies exist, are asymptotically unique and easily computable. The computations do not require, in contrast with the classical DSGE, the knowledge of hidden agents' characteristics such as individual utilities and beliefs.

Financial DSGE models integrating evolutionary and behavioural approaches were proposed in Amir et al. [5, 6]. A survey describing the state of the art in EBF by 2016 and outlining a program for further research was given in Evstigneev et al. [57]. An elementary textbook treatment of the subject can be found in Evstigneev et al. [56], Ch. 20. For a most recent review of the development of studies related to this area see Holtfort [94]. General perspectives of a synthesis of behavioural and mainstream economics based on the evolutionary approach are discussed in a recent paper by Aumann [15].

EBF models invoke ideas related to behavioural economics and finance (Tversky and Kahneman [171], Shiller [158], Bachmann et al. [17]), evolutionary game theory (Weibull [177], Samuelson [142], Gintis [72], Kojima [103]) and games of survival (Milnor and Shapley [129], Shubik and Thompson [160])[2].

The present study draws on the previous work of Amir et al. [5], where a prototype of the model studied here was developed and some versions of the results we get in this chapter were obtained. However, that study was conducted under very restrictive assumptions (equality of growth rates of the total volumes of all the assets and equality of investment rates of the market participants). Relaxing these assumptions required overcoming a number of conceptual and technical difficulties. Even the form of the main result on the existence of a survival strategy in the present, more general, setting differs substantially from that in Amir et al. [5]. Now this strategy is defined as a solution to a certain stochastic equation, in contrast with the previous, more specialized, model where it could be represented in an explicit form as the sum of a convergent series. For the proof of the existence and uniqueness of this solution we needed to develop new mathematical tools related to the ergodic theory of random dynamical systems: non-stationary stochastic Perron-Frobenius theorems (for stationary versions of these results see, e.g., Babaei et al. [16]).

The structure of the chapter is as follows. Section 2 describes the model. Section 3 states the main results. Section 4 discusses the EBF modeling approach, its characteristic features and applications. Section 5 contains some auxiliary propositions needed for the analysis of the

[2]For a comprehensive discussion of game-theoretic aspects of EBF in a different but closely related model see Amir et al. [6], Sections 1 and 6.

model. Section 6 proves the main results. Appendix A includes routine proofs of a number of lemmas formulated in Section 6. Appendix B derives a non-stationary stochastic version of the Perron-Frobenius theorem used in this work.

2.2. The model

We consider a market where $K \geq 2$ *assets* are traded. The market is influenced by random factors modeled in terms of an exogenous stochastic process $s_1, s_2, ...$, where s_t is a random element of a measurable space S_t ("state of the world" at date t). The market opens at date 0 and the assets are traded at all moments of time $t = 0, 1, 2, ...$ At each date $t = 1, 2, ...$ assets $k = 1, 2, ..., K$ pay dividends $D_{t,k}(s^t) \geq 0$ depending on the history $s^t := (s_1, ..., s_t)$ of states of the world up to date t. The functions $D_{t,k}(s^t)$ (as well as all other functions of s^t we will consider) are assumed to be measurable with respect to the product σ-algebra in the space $S_1 \times ... \times S_t$ and satisfy

$$\sum_{k=1}^{K} D_{t,k}(s^t) > 0 \text{ for all } t \geq 1 \text{ and } s^t. \tag{1}$$

This condition means that at each date in each random situation at least one asset yields a strictly positive dividend. The total volume (the number of units) of asset k available in the market at date $t \geq 0$ is $V_{t,k}(s^t) > 0$, where $V_{t,k}(s^t)$ is a measurable function of s^t. For $t = 0$, the number $V_{t,k} = V_{0,k} > 0$ is constant.

We denote by $p_t \in \mathbb{R}^K_+$ the vector of market prices of the assets. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset k at date $t \geq 0$. There are $N \geq 2$ *investors* (*traders*) acting in the market. A *portfolio* of investor i at date $t \geq 0$ is specified by a vector $x^i_t = (x^i_{t,1}, ..., x^i_{t,K}) \in \mathbb{R}^K_+$, where $x^i_{t,k}$ is the amount (the number of units) of asset k in the portfolio x^i_t. The scalar product $\langle p_t, x^i_t \rangle = \sum_{k=1}^{K} p_{t,k} x^i_{t,k}$ expresses the value of the investor i's portfolio x^i_t at date t in terms of the prices $p_{t,k}$. The *state of the market* at each date t is characterized by the set of vectors $(p_t, x^1_t, ..., x^N_t)$, where p_t is the vector of asset prices and $x^1_t, ..., x^N_t$ are the traders' portfolios.

At date $t = 0$ the investors have *initial endowments* $w^i_0 > 0$ $(i = 1, 2, ..., N)$, that form their budgets at date 0. Investor i's budget at date $t \geq 1$ is

$$w_t^i(s^t) = \langle D_t(s^t) + p_t(s^t), x_{t-1}^i(s^{t-1}) \rangle,$$

where $D_t(s^t) = (D_{t,1}(s^t), ..., D_{t,K}(s^t))$. It consists of two components: the dividends $\langle D_t, x_{t-1}^i \rangle$ paid by the portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of x_{t-1}^i expressed in terms of the prices $p_t = (p_{t,1}, ..., p_{t,K})$ at date t.

For each $t \geq 0$, every trader $i = 1, 2, ..., N$ selects a vector of *investment proportions* $\lambda_t^i = (\lambda_{t,1}^i, ..., \lambda_{t,K}^i)$ according to which i plans to distribute the available budget between assets. Vectors λ_t^i belong to the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \geq 0 : \ a_1 + ... + a_K = 1\}.$$

In terms of the game we are going to describe, the vectors λ_t^i represent the players' (investors') *actions* or *control variables.* The investment proportions at each date $t \geq 0$ are selected by the N traders simultaneously and independently, so that we deal here with a simultaneous-move N-person dynamic game. For $t \geq 1$, players' actions might depend, generally, on the history $s^t = (s_1, ..., s_t)$ of the realized states of the world and the *history of the game* $(p^{t-1}, x^{t-1}, \lambda^{t-1})$, where $p^{t-1} = (p_0, ..., p_{t-1})$ is the sequence of asset price vectors up to time $t-1$, and

$$x^{t-1} := (x_0, x_1, ..., x_{t-1}), \ x_l = (x_l^1, ..., x_l^N),$$

$$\lambda^{t-1} = (\lambda_0, \lambda_1, ..., \lambda_{t-1}), \ \lambda_l = (\lambda_l^1, ..., \lambda_l^N),$$

are the sets of vectors describing the portfolios and the investment proportions of all the players at all the dates up to $t-1$. The history of the game contains information about the *market history*—the sequence $(p_0, x_0), ..., (p_{t-1}, x_{t-1})$ of the states of the market—and about the actions λ_l^i of all the players (investors) $i = 1, ..., N$ at all the dates $l = 0, ..., t-1$. A vector $\Lambda_0^i \in \Delta^K$ and a sequence of measurable functions with values in Δ^K

$$\Lambda_t^i(s^t, p^{t-1}, x^{t-1}, \ \lambda^{t-1}), \ t = 1, 2, ...$$

form an *investment (trading) strategy* Λ^i of trader i, specifying a *portfolio rule* according to which trader i selects investment proportions at each date $t \geq 0$. This is a general game-theoretic definition of a strategy, assuming full information about the history of the game, including the players' previous actions, and the knowledge of all the past and present states of the world.

Among general portfolio rules, we will distinguish those for which Λ_t^i depends only on s^t, rather than on the whole market history $(p^{t-1}, x^{t-1}, \lambda^{t-1})$. We will call such portfolio rules *basic*. They play an important role in the present work: the survival strategy we are going to construct will belong to this class. The essence of the main result (Theorem 2) lies in the fact that it indicates a relatively simple basic strategy, requiring a very limited volume of information and guaranteeing survival in competition with *any* other strategies which might use all theoretically possible information.

For each $k = 1, ..., K$, a sequence of functions $\alpha_{0,k}, \alpha_{1,k}(s^1), \alpha_{2,k}(s^2), ...$ is given characterizing transaction costs for buying asset k in the market under consideration. It is assumed that $0 < \alpha_{t,k} \leq 1$. If an investor i allocates wealth $w_{t,k}^i$ to asset k at time t, then the value of the kth position of the i's portfolio will be $p_{t,k}x_{t,k}^i = \alpha_{t,k}w_{t,k}^i$. The amount $(1 - \alpha_{t,k})w_{t,k}^i$ will cover transaction costs.

Suppose that at date 0 each investor i has selected some investment proportions $\lambda_0^i = (\lambda_{0,1}^i, ..., \lambda_{0,K}^i) \in \Delta^K$. Then the amount allocated to asset k by trader i is $\lambda_{0,k}^i w_0^i$, where $w_0^i > 0$ is the i's initial endowment, so that the value of the holding of asset k in the i's portfolio is $\alpha_{0,k}\lambda_{0,k}^i w_0^i$. Thus the value of the total holding of asset k in all the investors' portfolios amounts to $\alpha_{0,k}\sum_{i=1}^N \lambda_{0,k}^i w_0^i$. It is assumed that the market is always in equilibrium (asset supply is equal to asset demand), which makes it possible to determine the equilibrium price $p_{0,k}$ of each asset k from the equations

$$p_{0,k}V_{0,k} = \alpha_{0,k}\sum_{i=1}^{N} \lambda_{0,k}^i w_0^i, \; k = 1, 2, ..., K. \tag{2}$$

On the left-hand side of (2) we have the total value $p_{0,k}V_{0,k}$ of all the assets of the type k in the market (recall that the total amount of asset k at date 0 is $V_{0,k}$). The investment proportions $\lambda_0^i = (\lambda_{0,1}^i, ..., \lambda_{0,K}^i)$ chosen by the traders at date 0 determine their portfolios $x_0^i = (x_{0,1}^i, ..., x_{0,K}^i)$ at date 0 by the formula

$$x_{0,k}^i = \frac{\alpha_{0,k}\lambda_{0,k}^i w_0^i}{p_{0,k}}, \; k = 1, 2, ..., K, \; i = 1, ..., N. \tag{3}$$

Assume now that all the investors have chosen their investment proportion vectors $\lambda_t^i =$

$(\lambda^i_{t,1}, ..., \lambda^i_{t,K})$ at date $t \geq 1$. Then the equilibrium of asset supply and demand determines the market clearing prices

$$p_{t,k} V_{t,k} = \alpha_{t,k} \sum_{i=1}^{N} \lambda^i_{t,k} \langle D_t + p_t, x^i_{t-1} \rangle, \ k = 1, ..., K. \quad (4)$$

The investment budgets $\langle D_t + p_t, x^i_{t-1} \rangle$ of the traders $i = 1, 2, ..., N$ are distributed between assets in the proportions $\lambda^i_{t,k}$, so that the kth position of the trader i's portfolio $x^i_t = (x^i_{t,1}, ..., x^i_{t,K})$ is

$$x^i_{t,k} = \frac{\alpha_{t,k} \lambda^i_{t,k} \langle D_t + p_t, x^i_{t-1} \rangle}{p_{t,k}}, \ k = 1, ..., K, \ i = 1, ..., N. \quad (5)$$

Note that the price vector p_t is determined implicitly as the solution to the system of equations (4).

Define

$$\gamma_{t,k}(s^t) = V_{t,k}(s^t)/V_{t-1,k}(s^{t-1}).$$

The number $\gamma_{t,k}$ characterizes the speed of growth of the total volume $V_{t,k}$ of asset k. It can be shown (see Proposition 1 in Section 5) that a non-negative vector $p_t(s^t)$ satisfying equations (4) exists and is unique (for any s^t and any feasible x^i_{t-1} and λ^i_t) as long as the following condition holds

$$\alpha_{t,k}(s^t) < \gamma_{t,k}(s^t) \text{ for all } t \geq 1 \text{ and all } s^t. \quad (6)$$

This condition is implied by the basic assumptions under which the results of this chapter are obtained (see Section 4). Note that if there are no transaction costs, i.e. $\alpha_{t,k} = 1$, then (6) means that the total volumes of all the assets grow in time at a strictly positive rate. In another extreme case, when $\gamma_{t,k} = 1$, i.e. $V_{t,k}$ is constant in t, condition (6) requires that $\alpha_{t,k} < 1$, i.e. the transaction cost rate is non-zero. This property—termed in Mathematical Finance "efficient market friction" (see, e.g., Kabanov and Safarian [96], p. 117)—plays an important role in various models with transaction costs, excluding phenomena like the Saint Petersburg paradox. In our context it is indispensable since in those cases when this assumption does not hold, a short-run equilibrium might fail to exist.

Given a strategy profile $(\Lambda^1, ..., \Lambda^N)$ of investors and their initial endowments $w^1_0, ..., w^N_0$, we can generate a path of the market game by setting

$$\lambda^i_0 = \Lambda^i_0, \ i = 1, ..., N, \quad (7)$$

$$\lambda_t^i = \Lambda_t^i(s^t, p^{t-1}, x^{t-1},\ \lambda^{t-1}),\ t = 1, 2, ...,\ i = 1, ..., N, \tag{8}$$

and by defining p_t and x_t^i recursively according equations (2)–(5). The random dynamical system described defines step by step the vectors of investment proportions $\lambda_t^i(s^t)$, the equilibrium prices $p_t(s^t)$ and the investors' portfolios $x_t^i(s^t)$ as measurable vector functions of s^t for each moment of time $t \geq 0$. Thus we obtain a random path of the game

$$(p_t(s^t); x_t^1(s^t), ..., x_t^N(s^t); \lambda_t^1(s^t), ..., \lambda_t^N(s^t)),\ t \geq 0, \tag{9}$$

as a vector stochastic process in $\mathbb{R}_+^K \times \mathbb{R}_+^{KN} \times \mathbb{R}_+^{KN}$.

The above description of asset market dynamics requires clarification. Equations (3) and (5) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path all the portfolios $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$ are non-zero and the wealth

$$w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle \tag{10}$$

of each investor is strictly positive. Further, by summing up equations (5) over $i = 1, ..., N$, we find that

$$\sum_{i=1}^N x_{t,k}^i = \frac{\alpha_{t,k} \sum_{i=1}^N \lambda_{t,k}^i \langle D_t + p_t, x_{t-1}^i \rangle}{p_{t,k}} = \frac{p_{t,k} V_{t,k}}{p_{t,k}} = V_{t,k} \tag{11}$$

(the market clears) for every asset k and each date $t \geq 1$. The analogous relations for $t = 0$ can be obtained by summing up equations (3). Thus for every equilibrium state of the market $(p_t, x_t^1, ..., x_t^N)$, we have $p_t > 0$, $x_t^i \neq 0$ and (11).

We give a simple sufficient condition for a strategy profile to be admissible. This condition will hold for all the strategy profiles we shall deal with in the present chapter, and in this sense it does not restrict generality. Suppose that some trader, say trader 1, uses a portfolio rule that always prescribes to invest into all the assets in strictly positive proportions $\lambda_{t,k}^1$. Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (2) that $p_{0,k} \geq \alpha_{0,k} V_{0,k}^{-1} \lambda_{0,k}^1 w_0^1 > 0$ and from (3) that $x_0^1 = (x_{0,1}^1, ..., x_{0,K}^1) > 0$ (coordinatewise). Assuming that $x_{t-1}^1 > 0$ and arguing by induction, we obtain

$$\langle D_t + p_t, x_{t-1}^1 \rangle \geq \langle D_t, x_{t-1}^1 \rangle > 0$$

in view of (1), which in turn yields $p_t > 0$ and $x_t^1 > 0$ by virtue of (4) and (5), as long as $\lambda_{t,k}^1 > 0$.

2.3. The main results

Let $(\Lambda^1, ..., \Lambda^N)$ be an admissible strategy profile of the investors. Consider the path (9) of the random dynamical system generated by this strategy profile and the given initial endowments w_0^i. We are primarily interested in the long-run behaviour of the *relative wealth* or the *market shares* $r_t^i := w_t^i / W_t$ of the traders, where w_t^i is the investor i's wealth at date $t \geq 0$ and $W_t := \sum_{i=1}^N w_t^i$ is the *total market wealth*. We shall say that a portfolio rule Λ, or an investor i using it, *survives* with probability one if $\inf_{t \geq 0} r_t^i > 0$ almost surely (a.s.). This means that for almost all realizations of the process of states of the world $s_1, s_2, ...$, the market share of investor i using Λ is bounded away from zero by a strictly positive random variable.

Definition. Let us say that a portfolio rule Λ is a *survival strategy* if any investor using it survives with probability one irrespective of what portfolio rules are used by the other investors.

We will construct a strategy Λ^* which, as we shall prove, will be a survival strategy. Put

$$\rho_{t,k} := \frac{\alpha_{t,k}}{\gamma_{t,k}} = \frac{\alpha_{t,k} V_{t-1,k}}{V_{t,k}}, \; t \geq 1, \; k = 1, ..., K.$$

Define the *relative dividends* of the assets $k = 1, ..., K$ by

$$R_{t,k} = R_{t,k}(s^t) := \frac{D_{t,k}(s^t) V_{t-1,k}(s^{t-1})}{\sum_{m=1}^K D_{t,m}(s^t) V_{t-1,m}(s^{t-1})}, \; k = 1, ..., K, \; t \geq 1, \qquad (12)$$

and put $R_t(s^t) := (R_{t,1}(s^t), ..., R_{t,K}(s^t))$. The strategy $\Lambda^* = (\lambda_t^*(s^t))_{t \geq 0}$, where $\lambda_t^* = (\lambda_{t,1}^*, ..., \lambda_{t,K}^*)$, is defined as the basic strategy satisfying the equation

$$E_t[\rho_{t+1,k} \lambda_{t+1,k}^* + (1 - \sum_{m=1}^K \rho_{t+1,m} \lambda_{t+1,m}^*) R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)}, \; k = 1, ..., K. \qquad (13)$$

Here $E_t(\cdot) = E(\cdot|s^t)$ stands for the conditional expectation given s^t. We will provide conditions under which the strategy Λ^* exists and is unique up to stochastic equivalence, i.e. if $\Lambda = (\lambda_t(s^t))_{t\geq 0}$ is another solution to (13), then $\lambda_t^* = \lambda_t$ (a.s.) for all t.

Throughout the chapter we will assume that the following conditions hold:

(A.1) There exist constants $\upsilon > 0$ and $l \geq 0$ such that for each t and k, we have

$$\max_{1\leq m\leq l} R_{t+m,k} \geq \upsilon. \tag{14}$$

(A.2) There exist strictly positive constants κ and α such that for all k, t

$$\alpha \leq \rho_{t,k} \leq 1 - \kappa. \tag{15}$$

Theorem 1. *Under assumptions (A.1) and (A.2), a solution $(\lambda_t^*)_{t\geq 0}$ to equation (13) exists and is unique up to stochastic equivalence. There exists a constant $\delta > 0$ such that $\lambda_{t,k}^* \geq \delta$.*

For a proof of Theorem 1 see Appendix B, Theorem B.2.

Let us discuss the meaning of equation (13). Suppose for the moment that the growth rates of all the assets are the same, so that

$$\rho_{t,1} = \rho_{t,2} = ... = \rho_{t,K} = \rho_t. \tag{16}$$

In this case, equation (13) takes on the following form

$$E_t[\rho_{t+1}\lambda_{t+1,k}^* + (1-\rho_{t+1})R_{t+1,k}] = \lambda_{t,k}^* \text{ (a.s.)}, \tag{17}$$

and it admits an explicit solution. The kth coordinate $\lambda_{t,k}^*$ of the vector λ_t^* can be represented as the conditional expectation of the sum of the series

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k} \ , \tag{18}$$

where

$$\rho_t^l := \begin{cases} 1-\rho_{t+l}\,, & \text{if } l = 1, \\ \rho_{t+1}\rho_{t+2....}\rho_{t+l-1}(1-\rho_{t+l}), & \text{if } l > 1. \end{cases} \tag{19}$$

Note that in view of (15), the series of random variables

$$\sum_{l=1}^{\infty} \rho_t^l = (1 - \rho_{t+1}) + \rho_{t+1}(1 - \rho_{t+2}) + \rho_{t+1}\rho_{t+2}(1 - \rho_{t+3}) + ...$$

converges uniformly, and its sum is equal to one. Therefore the series of random vectors $\sum_{l=1}^{\infty} \rho_t^l R_{t+l,k}$ in (18) converges uniformly to a random vector belonging the unit simplex Δ^K, so that the right-hand side of (18) is well-defined. The proof of equation (18) will be given in Proposition 5 below.

Assume that $\rho_t = \rho$ is constant. Then formula (18) can be written as

$$\lambda_{t,k}^* = E_t \sum_{l=1}^{\infty} [(1 - \rho)\rho^{l-1} R_{t+l,k}]. \tag{20}$$

Further, if the random elements s_t are independent and identically distributed (i.i.d.) and the relative dividends $R_{t,k}(s^t) = R_k(s_t)$ depend only on the current state s_t and do not explicitly depend on t, then $E_t R_k(s_{t+l}) = ER_k(s_t)$ $(l \geq 1)$, and so

$$\lambda_{t,k}^* = ER_k(s_t), \tag{21}$$

which means that the strategy Λ^* is formed by the sequence of vectors $(ER_1(s_t), ..., ER_K(s_t))$ (constant and independent of t and s^t). Note that in this special case, the formula (21) for Λ^* does not involve the factor ρ.

Formulas (18), (20) and (21) reflect two general principles in Financial Economics:

(a) The strategy Λ^* prescribes the allocation of wealth among assets in the proportions of their fundamental values—the expectations of the future relative (discounted, weighted) dividends.

(b) The portfolio rule Λ^* defined in terms of the relative dividends provides an investment recommendation in line with the CAPM principles, emphasizing the role of the market portfolio (see, e.g., [56], Chapter 7).

In this connection it should be emphasized that instead of the traditional weighing assets according to their prices, the weights in the definition of Λ^* are based on fundamentals, so that Λ^* is an example of *fundamental indexing* (Arnott et al. [11]).

As we have already noted, EBF can be viewed as an extension of the classical capital growth theory (Kelly [100], Breiman [34], Algoet and Cover [2], and others) to the case of endogenous asset prices and returns. In the classical setting, a central role is played by the famous Kelly portfolio rule [100] guaranteeing the fastest asymptotic growth rate of wealth in the long run. The Kelly rule is obtained by the maximization of the expected logarithm of the portfolio return. It can be shown (see the next section) that in the present model survival is equivalent to the fasted relative growth of wealth in the long run. Therefore Λ^* may be viewed as a counterpart of the Kelly portfolio rule in the present model. However, in the game-theoretic model at hand, where the performance of a strategy depends not only on the strategy itself but on the whole strategy profile, Λ^* cannot be obtained as a solution to a single-agent optimization problem with a logarithmic or any other objective functional.

It should be noted that in the case of different $\rho_{t,k}$, when condition (16) does not hold, we cannot provide an explicit formula, like (18), for the strategy Λ^*. However, we can suggest an algorithm for computing Λ^* converging at an exponential rate. This algorithm is actually contained in the proof of the existence and uniqueness of a solution to equation (13), see Appendix B, formulas (71) and (72).

The main results of the chapter are formulated in Theorems 2 and 3.

Theorem 2. *The portfolio rule Λ^* is a survival strategy.*

As we have already noted, the portfolio rule Λ^* belongs to the class of basic portfolio rules: the investment proportions $\lambda_t^*(s^t)$ depend only on the history s^t of the process of states of the world and do not depend on the market history.

Note that the class of basic strategies is *sufficient* in the following sense. Any sequence of vectors $r_t = (r_t^1, ..., r_t^N)$ $(r_t = r_t(s^t))$ of market shares generated by some strategy profile $(\Lambda^1, ..., \Lambda^N)$ can be generated by a strategy profile $(\lambda_t^1(s^t), ..., \lambda_t^N(s^t))$ consisting of basic portfolio rules. The corresponding vector functions $\lambda_t^i(s^t)$ can be defined recursively by (7) and (8), using (2)–(5). Thus it is sufficient to prove Theorem 2 only for basic portfolio rules; this will imply that the portfolio rule (18) survives in competition with any, not necessarily basic, strategies.

The following result shows that the survival portfolio rule Λ^* is unique in the class of all basic strategies.

Theorem 3. *If there exists another basic survival strategy* $\Lambda = (\lambda_t)$, *then:*

$$\sum_{t=0}^{\infty} ||\lambda_t^* - \lambda_t||^2 < \infty \text{ (a.s.)}.$$

Proofs of Theorems 2 and 3 are given in the remainder of the chapter.

2.4. Discussion

In this section we discuss the EBF approach, the model under consideration and the results obtained.

Uniqueness of the surviving portfolio rule Λ^*. Theorem 3 states that the Λ^* strategy is a unique survival strategy in the class of *basic* portfolio rules. Namely, if all the traders use basic strategies, which do not depend on the history of the play, then the only survivals are the traders using strategies "asymptotically equivalent" to Λ^*. However, if traders are allowed to use general, not necessarily basic strategies, it is likely that the portfolio rule Λ^* is not a unique survival strategy. Construction of a non-basic survival *trigger strategy* seems plausible and indicates an interesting direction for further research. Some examples of this kind pertaining to a different, but closely related, model might support this conjecture (see [6], Section 5).

As it was pointed out earlier, the Λ^* strategy can be viewed as an analog of the "betting your beliefs" rule, or the Kelly rule, although it was obtained without maximizing a logarithmic utility function. Blume and Easley [22] claimed the Kelly rule to be a dominant strategy as it guarantees the fastest growth rate. In our model the Λ^* strategy possesses the same property: it is shown that survival strategies are equivalent to unbeatable strategies with asymptotically fastest wealth accumulation. This similarity poses the question whether the portfolio rule Λ^* is prone to the same critique as the "betting your beliefs" rule. Namely, Sandroni [145] showed that the Kelly rule is not a unique survival strategy in the model where investors are optimizing over consumption. According to [145], in complete markets the survival of any strategy depends on the ability of the investor to make accurate predictions about future returns. This observation makes me believe that in the model with endogenous consumption and investment, a trader adopting the Λ^*strategy will not be a unique survivor. Moreover, the ability to evaluate the fundamental value of assets might be a necessary and sufficient condition for survival. However, the question remains open and requires further analysis.

Marshallian temporary equilibrium. In the general methodological perspective, the modeling framework at hand relies upon the Marshallian [126] principle of temporary equilibrium. The dynamics of the asset market in this framework are similar to the dynamics of the commodity market as outlined in the classical treatise by Alfred Marshall [126] (Book V, Chapter II "Temporary Equilibrium of Demand and Supply"). The ideas of Marshall were developed in the framework of mathematical economics by Samuelson [143]. As it was noticed by Samuelson and discussed in detail by Schlicht [149], in order to study the process of market dynamics by using the Marshallian "moving equilibrium method," one needs to distinguish between at least two sets of economic variables changing with different speeds. Then the set of variables changing slower (in our case, the set $x_t = (x_t^1, ..., x_t^N)$ of investors' portfolios) can be temporarily fixed, while the other (in our case, the asset prices p_t) can be assumed to rapidly reach the unique state of partial equilibrium. Samuelson [143], pp. 321–323, writes about this approach:

> I, myself, find it convenient to visualize equilibrium processes of quite different speed, some very slow compared to others. Within each long run there is a shorter run, and within each shorter run there is a still shorter run, and so forth in an infinite regression. For analytic purposes it is often convenient to treat slow processes as data and concentrate upon the processes of interest. For example, in a short run study of the level of investment, income, and employment, it is often convenient to assume that the stock of capital is perfectly or sensibly fixed.

As it follows from the above citation, Samuelson thinks about a hierarchy of various equilibrium processes with different speeds. In our model, it is sufficient to deal with only two levels of such a hierarchy. We leave the price adjustment process leading to the solution of the partial equilibrium problem (4) beyond the scope of the model. It can be shown, however, that this equilibrium will be reached at an exponential rate in the course of a naturally defined *tâtonnement* procedure. This can be demonstrated by using the contraction property of the operator

(23) involved in the equilibrium pricing equation (4). Our framework makes it possible to admit a whole spectrum of mechanisms leading to an equilibrium in the short run. In reality, various auction-type mechanisms are used for the purpose of equilibrating bids and offers, resulting in market clearing. An analysis of several types of such mechanisms and their implications for the structure of trading in financial markets has been performed by Bottazzi et al. [31].

A rigorous mathematical treatment of the above multiscale approach, involving “rapid” and “slow” variables, is provided within continuous-time settings in the theory of *singular perturbations*, see e.g. Smith [164] and Kevorkian and Cole [101]. In connection with economic modelling, questions of this kind are considered in detail in the monograph by Schlicht [149]. The equations on pp. 29–30 in [149] are direct continuous-time (deterministic) counterparts of our equations (4) and (5).

The term "temporary equilibrium" was apparently coined for the first time by Marshall. However, in the last decades this term has been associated basically with a different, non-Marshallian notion, going back to Lindahl [113] and Hicks [89]. This notion was developed in formal settings by Grandmont, Hildenbrand and others, see [74, 75, 76]. The characteristic feature of the Lindahl-Hicks temporary equilibrium is the idea of *forecasts* or *beliefs* about the future states of the world, which the market participants possess and which are formalized in terms of stochastic kernels (transition functions) conditioning the distributions of future states of the world upon the agents’ private information. A comprehensive discussion of this direction of research is provided by Magill and Quinzii [124]. In this work, we pursue a completely different approach. Our model might indirectly take into account agents’ forecasts or beliefs, but they can be only implicitly reflected in the agents’ investment strategies. We do not need to model in formal terms how the market players form, update and use these beliefs in their investment decisions.

For further comments on the comparison of the financial DSGE models based on the traditional Walrasian paradigm and those relying upon the EBF approach, see Amir et al. [3], Section 7.

In order to survive you have to win! One might think that the focus on survival substantially restricts the scope of the analysis, since "one should care about survival only if things go wrong". It turns out, however, that the class of survival strategies in most of the EBF models coincides with the class of unbeatable strategies performing in the long run not worse in terms of wealth accumulation than any other strategies competing in the market. To demonstrate this let us reformulate the notion of a survival strategy in terms of the wealth processes w_t^i of the market players $i = 1, 2, ..., N$. Survival of a portfolio rule Λ^1 used by player 1 means that $w_t^1 \geq c \sum_{i=1}^N w_t^i$, where c is a strictly positive random variable. The last inequality holds if and only if

$$w_t^i \leq C w_t^1, \;\; i = 1, ..., N, \tag{22}$$

where C is some random variable. Property (22) expresses the fact that the wealth of any player i using any strategy Λ^i cannot grow asymptotically faster than the wealth of player 1 who uses the strategy Λ^1. If this is the case, the portfolio rule Λ^1 is called *unbeatable.* Thus survival strategies are those and only those that are unbeatable: in order to survive, you have to win! For a general definition and discussion of the notion of an unbeatable strategy as a game solution concept see Chapter 4.

Evolutionary portfolio theory. One of the sources of motivation for EBF has always been related to quantitative applications of the results to portfolio selection problems. The data of EBF models needed for quantitative financial analysis are essentially the same as those needed for the applications of the theory of derivative securities pricing (e.g. the Black-Scholes formula) in Mathematical Finance/Financial Engineering. They do not need the knowledge, or the algorithms for revealing, hidden agents' characteristics such as their utilities and beliefs. The model and the results are described in operational terms and require only statistical estimates of objectively observable asset data.

A crucial role in the applications of EBF to portfolio selection is played by the discovery of investment factors that deliver returns in excess of the market. For example, Basu [19] found the so-called value factor, according to which investing into equities with a high book-to-market ratio delivers higher returns than the market. Banz [18] found that the same is true if one invests into equities with small market capitalization. Carhart [38] found the momentum factor according to which investing in equities that have recently gone up delivers excess returns.

Moreover even though by now hundreds of investment factors have been proposed, Harvey et al. [84] have shown that only a few factors are needed to understand the dynamics of equity returns. The current state of these discoveries is summarized in the Fama-French [61] five-factor model. According to these empirical results, the return of every portfolio selection strategy can be decomposed into its allocation to a few investment factors. Thus, it is natural to model the dynamics of equity markets by modelling the dynamic interaction of those investment factors. And this is what EBF is perfectly suited for. In the EBF framework, an investment factor defines a strategy determining the corresponding investment proportions. Note that investment factors are not based on individuals' utility functions and subjective probabilities! EBF can then be used to compute what impact the increase in relative wealth corresponding to one factor has on any other factor. In particular, the impact of a factor on itself gives a model-based measure of the capacity of the factor. This is very practical information since investors should avoid being stuck in crowded strategies. Also, when a certain investment factor gets fashionable this has cross impacts on other factors that one can compute based on the EBF model. For example, in recent years investing according to ESG (Environmental, Social, and Corporate Governance) criteria has become fashionable, and the EBF approach shows that this has a strong negative impact on the momentum factor. Finally, based on this approach one can compute the dynamics of the relative wealth, so that one can use the EBF model to determine which investment factors survive in the long run. A first paper systematically developing these ideas and opening up a new realm of fruitful applications of EBF to portfolio selection problems has recently been published in the Journal of Portfolio Management (Hens et al. [88]).

2.5. Auxiliary propositions

In this section we prove several auxiliary propositions needed for the analysis of the model at hand. The first proposition establishes the existence and uniqueness of an equilibrium price vector at each date $t \geq 0$.

Proposition 1. *Let assumption (6) hold. Let* $x_{t-1} = (x_{t-1}^1, ..., x_{t-1}^N)$ *be a set of vectors* $x_{t-1}^i \in \mathbb{R}_+^K$ *satisfying (11). Then for any* s^t *there exists a unique solution* $p_t \in \mathbb{R}_+^K$ *to equations*

(4). This solution is measurable with respect to all the parameters involved in (4).

Proof of Proposition 1. Fix some t and s^t and consider the operator transforming a vector $p = (p_1, ..., p_K) \in \mathbb{R}^K_+$ into the vector $q = (q_1, ..., q_K) \in \mathbb{R}^K_+$ with coordinates

$$q_k = \alpha_{t,k} V_{t,k}^{-1} \sum_{i=1}^{N} \lambda^i_{t,k} \langle D_t + p, x^i_{t-1} \rangle. \tag{23}$$

This operator is contracting in the norm $||p||_V := \sum_k |p_k| V_{t-1,k}$. Indeed, by virtue of (6) we have

$$\beta := \max_{k=1,...,K} \{\alpha_{t,k} V_{t-1,k} V_{t,k}^{-1}\} < 1,$$

and so

$$||q - q'||_V = \sum_{k=1}^{K} |q_k - q'_k| V_{t-1,k} \leq$$

$$\sum_{k=1}^{K} \alpha_{t,k} V_{t-1,k} V_{t,k}^{-1} \sum_{i=1}^{N} \lambda^i_{t,k} |\langle p - p', x^i_{t-1} \rangle| \leq \beta \sum_{i=1}^{N} \sum_{k=1}^{K} \lambda^i_{t,k} |\langle p - p', x^i_{t-1} \rangle| =$$

$$\beta \sum_{i=1}^{N} |\langle p - p', x^i_{t-1} \rangle| \leq \beta \sum_{i=1}^{N} \sum_{m=1}^{K} |p_m - p'_m| x^i_{t-1,m} =$$

$$\beta \sum_{m=1}^{K} \sum_{i=1}^{N} |p_m - p'_m| x^i_{t-1,m} = \beta \sum_{m=1}^{K} |p_m - p'_m| V_{t-1,m} = \beta ||p - p'||_V \ ,$$

where the last but one equality follows from (11). By using the contraction principle, we obtain the existence, uniqueness and measurability of the solution to (4). □

In the next proposition, we derive a system of equations governing the dynamics of the market shares of the investors given their admissible strategy profile $(\Lambda^1, ..., \Lambda^N)$. Consider the path (9) of the random dynamical system generated by $(\Lambda^1, ..., \Lambda^N)$ and the sequence of vectors $r_t = (r^1_t, ..., r^N_t)$, where r^i_t is the investor i's market share at date t.

Proposition 2. *The following equations hold:*

$$w^i_{t+1} = \sum_{k=1}^{K} (\rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda^i_{t,k} w^i_t}{\langle \lambda_{t,k}, w_t \rangle}, \tag{24}$$

$i = 1, ..., N$, $t \geq 0$.

Proof of Proposition 2. From (4) and (5) we get

$$p_{t,k} = \alpha_{t,k} V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^{i} \langle p_t + D_t, x_{t-1}^{i} \rangle = \alpha_{t,k} V_{t,k}^{-1} \sum_{i=1}^{N} \lambda_{t,k}^{i} w_t^i = \alpha_{t,k} V_{t,k}^{-1} \langle \lambda_{t,k}, w_t \rangle,$$

$$x_{t,k}^{i} = \frac{\alpha_{t,k} \lambda_{t,k}^{i} w_t^i}{p_{t,k}} = \frac{\alpha_{t,k} \lambda_{t,k}^{i} w_t^i}{\alpha_{t,k} V_{t,k}^{-1} \langle \lambda_{t,k}, w_t \rangle} = \frac{V_{t,k} \lambda_{t,k}^{i} w_t^i}{\langle \lambda_{t,k}, w_t \rangle},$$

where $t \geq 1$, $w_t := (w_t^1, ..., w_t^N)$ and $\lambda_{t,k} := (\lambda_{t,k}^1, ..., \lambda_{t,k}^N)$. Consequently, we have

$$w_{t+1}^{i} = \sum_{k=1}^{K} (p_{t+1,k} + D_{t+1,k}) x_{t,k}^{i} = \sum_{k=1}^{K} \left(\frac{\alpha_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle}{V_{t+1,k}} + D_{t+1,k} \right) \frac{V_{t,k} \lambda_{t,k}^{i} w_t^i}{\langle \lambda_{t,k}, w_t \rangle} =$$

$$\sum_{k=1}^{K} \left(\frac{\alpha_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle V_{t,k}}{V_{t+1,k}} + D_{t+1,k} V_{t,k} \right) \frac{\lambda_{t,k}^{i} w_t^i}{\langle \lambda_{t,k}, w_t \rangle} =$$

$$\sum_{k=1}^{K} (\langle \lambda_{t+1,k}, w_{t+1} \rangle \rho_{t+1,k} + D_{t+1,k} V_{t,k}) \frac{\lambda_{t,k}^{i} w_t^i}{\langle \lambda_{t,k}, w_t \rangle},$$

where, we recall, $\rho_{t+1,k} = \alpha_{t+1,k} V_{t,k} / V_{t+1,k}$. □

Consider the model with two traders ($N = 2$) using strategies $\Lambda^i = (\lambda_{t,k}^i(s^t))$, $i = 1, 2$, and denote by x_t the ratio of their market shares:

$$x_t = \frac{r_t^1}{r_t^2} = \frac{w_t^1}{w_t^2}.$$

Recall that the *relative dividends* $R_{t,k}(s^t)$ of the assets $k = 1, ..., K$ are defined by (12), and $R_t(s^t)$ denotes the vector $(R_{t,1}(s^t), ..., R_{t,K}(s^t))$. Further, let us define for $i = 1, 2$,

$$U_{t+1}^{i} := 1 - \sum_{k=1}^{K} \rho_{t+1,k} \lambda_{t+1,k}^{i} = \sum_{k=1}^{K} (1 - \rho_{t+1,k}) \lambda_{t+1,k}^{i} \ . \tag{25}$$

Proposition 3. *The sequence x_t is generated by the following random dynamical system*

$$x_{t+1} = x_t \frac{\sum_{k=1}^{K} [\rho_{t+1,k} \lambda_{t+1,k}^{2} + R_{t+1,k} U_{t+1}^{2}] \dfrac{\lambda_{t,k}^{1}}{\lambda_{t,k}^{1} x_t + \lambda_{t,k}^{2}}}{\sum_{k=1}^{K} [\rho_{t+1,k} \lambda_{t+1,k}^{1} + R_{t+1,k} U_{t+1}^{1}] \dfrac{\lambda_{t,k}^{2}}{\lambda_{t,k}^{1} x_t + \lambda_{t,k}^{2}}} \quad (t = 0, 1, ...). \tag{26}$$

Proof of Proposition 3. Let $i \in \{1, 2\}$ and $j \in \{1, 2\}$, $j \neq i$. By virtue of Proposition 2, we have (24). Then

$$W_{t+1} = w_{t+1}^{1} + w_{t+1}^{2} = \sum_{k=1}^{K} \rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + \sum_{k=1}^{K} D_{t+1,k} V_{t,k} \ ,$$

and so

$$\sum_{k=1}^{K} D_{t+1,k}V_{t,k} = W_{t+1} - \sum_{k=1}^{K} \rho_{t+1,k} \left\langle \lambda_{t+1,k}, w_{t+1} \right\rangle = \left\langle w_{t+1}, U_{t+1} \right\rangle . \tag{27}$$

Indeed,

$$\langle w_{t+1}, U_{t+1} \rangle = \sum_{l=1}^{2} w_{t+1}^{l} (1 - \sum_{k=1}^{K} \rho_{t+1,k} \lambda_{t+1,k}^{l}) = W_{t+1} - \sum_{k=1}^{K} \rho_{t+1,k} \sum_{l=1}^{2} w_{t+1}^{l} \lambda_{t+1,k}^{l}$$

$$= W_{t+1} - \sum_{k=1}^{K} \rho_{t+1,k} \langle w_{t+1}, \lambda_{t+1,k} \rangle.$$

By using the definition of the relative dividends $R_{t+1,k}$, we can rewrite formula (24) as follows:

$$w_{t+1}^{i} = \sum_{k=1}^{K} \left[\rho_{t+1,k} \left\langle \lambda_{t+1,k}, w_{t+1} \right\rangle + \left\langle w_{t+1}, U_{t+1} \right\rangle R_{t+1,k} \right] \frac{\lambda_{t,k}^{i} w_{t}^{i}}{\langle \lambda_{t,k}, w_{t} \rangle}.$$

Further, consider the expression in the brackets above:

$$\rho_{t+1,k} \left\langle \lambda_{t+1,k}, w_{t+1} \right\rangle + \left\langle w_{t+1}, U_{t+1} \right\rangle R_{t+1,k} = w_{t+1}^{i} \left(\rho_{t+1,k} \lambda_{t+1,k}^{i} + R_{t+1,k} U_{t+1}^{i} \right) +$$

$$w_{t+1}^{j} \left(\rho_{t+1,k} \lambda_{t+1,k}^{j} + R_{t+1,k} U_{t+1}^{j} \right) . \tag{28}$$

This, combined with (24), yields

$$w_{t+1}^{i} = w_{t+1}^{i} \sum_{k=1}^{K} \left[\rho_{t+1,k} \lambda_{t+1,k}^{i} + R_{t+1,k} U_{t+1}^{i} \right] \frac{\lambda_{t,k}^{i} w_{t}^{i}}{\langle \lambda_{t,k}, w_{t} \rangle} +$$

$$w_{t+1}^{j} \sum_{k=1}^{K} \left[\rho_{t+1,k} \lambda_{t+1,k}^{j} + R_{t+1,k} U_{t+1}^{j} \right] \frac{\lambda_{t,k}^{i} w_{t}^{i}}{\langle \lambda_{t,k}, w_{t} \rangle},$$

and so

$$w_{t+1}^{i} \left(1 - \sum_{k=1}^{K} \left[\rho_{t+1,k} \lambda_{t+1,k}^{i} + R_{t+1,k} U_{t+1}^{i} \right] \frac{\lambda_{t,k}^{i} w_{t}^{i}}{\langle \lambda_{t,k}, w_{t} \rangle} \right) =$$

$$w_{t+1}^{j} \sum_{k=1}^{K} \left[\rho_{t+1,k} \lambda_{t+1,k}^{j} + R_{t+1,k} U_{t+1}^{j} \right] \frac{\lambda_{t,k}^{i} w_{t}^{i}}{\langle \lambda_{t,k}, w_{t} \rangle}. \tag{29}$$

Finally, note that

$$\frac{\lambda_{t,k}^{i} w_{t}^{i}}{\langle \lambda_{t,k}, w_{t} \rangle} = 1 - \frac{\lambda_{t,k}^{j} w_{t}^{j}}{\langle \lambda_{t,k}, w_{t} \rangle},$$

and consequently, the expression in the parentheses in equation (29) can be written as:

$$1-\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i}+R_{t+1,k}U_{t+1}^{i}\right]\left(1-\frac{\lambda_{t,k}^{j}w_{t}^{j}}{\langle\lambda_{t,k},w_{t}\rangle}\right)=$$

$$1-\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i}+R_{t+1,k}U_{t+1}^{i}\right]+\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i}+R_{t+1,k}U_{t+1}^{i}\right]\frac{\lambda_{t,k}^{j}w_{t}^{j}}{\langle\lambda_{t,k},w_{t}\rangle}=$$

$$=\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i}+R_{t+1,k}U_{t+1}^{i}\right]\frac{\lambda_{t,k}^{j}w_{t}^{j}}{\langle\lambda_{t,k},w_{t}\rangle},$$

where the last equality follows from (25). Thus we obtain from (29)

$$\frac{w_{t+1}^{i}}{w_{t+1}^{j}}=\frac{w_{t}^{i}}{w_{t}^{j}}\frac{\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{j}+R_{t+1,k}U_{t+1}^{j}\right]\dfrac{\lambda_{t,k}^{i}}{\langle\lambda_{t,k},w_{t}\rangle}}{\sum_{k=1}^{K}\left[\rho_{t+1,k}\lambda_{t+1,k}^{i}+R_{t+1,k}U_{t+1}^{i}\right]\dfrac{\lambda_{t,k}^{j}}{\langle\lambda_{t,k},w_{t}\rangle}},$$

which completes the proof. □

The next proposition shows that it is sufficient to consider the case when $N=2$, i.e., the general model can be reduced to the case of two investors.

Proposition 4. *In the model with two investors $i=1,2$ using the strategies Λ and $\tilde{\Lambda}$, respectively, the wealth w_t^1 of the first player coincides with the wealth w_t^1 of the first player in the original model, and the wealth $\tilde{w}_t^2$ of the second "aggregate" investor coincides with the total wealth $w_t^2+...+w_t^N$ of the group of $N-1$ investors $i=2,...,N$ in the original model.*

Proof of Proposition 4. Define

$$\tilde{w}_t^2=w_t^2+...+w_t^N, \tag{30}$$

$$\tilde{\lambda}_{t,k}^2=\frac{\lambda_{t,k}^2w_t^2+...+\lambda_{t,k}^Nw_t^N}{\tilde{w}_t^2}\quad(k=1,2,...K). \tag{31}$$

We have

$$\tilde{\lambda}_{t,k}^2\geq0,\ \sum_{k=1}^{K}\tilde{\lambda}_{t,k}^2=\frac{w_t^2+...+w_t^N}{w_t^2+...+w_t^N}=1,$$

which means that the vector $\tilde{\lambda}_t^2:=(\tilde{\lambda}_{t,1}^2,...,\tilde{\lambda}_{t,K}^2)$ belongs to the unit simplex Δ^K. Let us regard $\tilde{\lambda}_t^2=(\tilde{\lambda}_{t,1}^2,...,\tilde{\lambda}_{t,K}^2)$ as the vector of investment proportions of an "aggregate investor", whose wealth is $\tilde{w}_t^2 = w_t^2 + ... + w_t^N$. The sequence of vectors $\tilde{\lambda}_t^2 = \tilde{\lambda}_t^2(s^t)$ defines a portfolio rule, which will be denoted by $\tilde{\Lambda}$. Note that

$$\tilde{\lambda}_{t,k}^2 \tilde{w}_t^2 = \lambda_{t,k}^2 w_t^2 + ... + \lambda_{t,k}^N w_t^N, \tag{32}$$

and so

$$\langle \lambda_{t,k}, w_t \rangle = \lambda_{t,k}^1 w_t^1 + \tilde{\lambda}_{t,k}^2 \tilde{w}_t^2, \ \langle \lambda_{t+1,k}, w_{t+1} \rangle = \lambda_{t+1,k}^1 w_{t+1}^1 + \tilde{\lambda}_{t+1,k}^2 \tilde{w}_{t+1}^2. \tag{33}$$

Recall that the dynamics of wealth of N investors is governed by the system of equations

$$w_{t+1}^i = \sum_{k=1}^{K} (\rho_{t+1,k} \langle \lambda_{t+1,k}, w_{t+1} \rangle + D_{t+1,k} V_{t,k}) \frac{\lambda_{t,k}^i w_t^i}{\langle \lambda_{t,k}, w_t \rangle}, \ i = 1, 2, ..., N$$

(see (24)). By summing up these equations over $i = 2, 3, ..., N$ and using (30), (32) and (33), we get

$$w_{t+1}^1 = \sum_{k=1}^{K} [\rho_{t+1,k} (\lambda_{t+1,k}^1 w_{t+1}^i + \tilde{\lambda}_{t+1,k}^2 \tilde{w}_{t+1}^2) + D_{t+1,k} V_{t,k}] \frac{\lambda_{t,k}^1 w_t^1}{\lambda_{t,k}^1 w_t^1 + \tilde{\lambda}_{t,k}^2 \tilde{w}_t^2},$$

$$\tilde{w}_{t+1}^2 = \sum_{k=1}^{K} [\rho_{t+1,k} (\lambda_{t+1,k}^1 w_{t+1}^1 + \tilde{\lambda}_{t+1,k}^2 \tilde{w}_{t+1}^2) + D_{t+1,k} V_{t,k}] \frac{\tilde{\lambda}_{t,k}^2 \tilde{w}_t^2}{\lambda_{t,k}^1 w_t^1 + \tilde{\lambda}_{t,k}^2 \tilde{w}_t^2},$$

which completes the proof. □

Proposition 5. *Under assumption (16), the portfolio rule $\Lambda^* = (\lambda_{t,k}^*)$ can be computed by formula (18).*

Proof of Proposition 5. Suppose (18) holds. Let us verify (17). We have

$$E_t(\rho_{t+1} \lambda_{t+1,k}^*) = E_t(\rho_{t+1} E_{t+1} \sum_{l=1}^{\infty} \rho_{t+1}^l R_{t+1+l,k}) =$$

$$E_t(E_{t+1} \sum_{l=1}^{\infty} \rho_{t+1} \rho_{t+1}^l R_{t+1+l,k}) = E_t(\sum_{l=1}^{\infty} \rho_{t+1} \rho_{t+1}^l R_{t+1+l,k}),$$

and so

$$E_t[\rho_{t+1} \lambda_{t+1,k}^* + (1 - \rho_{t+1}) R_{t+1,k}] = E_t[\sum_{l=1}^{\infty} \rho_{t+1} \rho_{t+1}^l R_{t+1+l,k} + (1 - \rho_{t+1}) R_{t+1,k}] =$$

$$E_t(\sum_{l=1}^{\infty} \rho_t^{l+1} R_{t+l+1,k} + \rho_t^1 R_{t+1,k}) = E_t \sum_{l=1}^{\infty} \rho_t^l R_{t+l,k} = \lambda_{t,k}^*$$

because $1 - \rho_{t+1} = \rho_t^1$ and

$$\rho_t^{l+1} = \rho_{t+1}\rho_{t+2}....\rho_{t+l}(1 - \rho_{t+l+1}) = \rho_{t+1}\rho_{t+1}^l$$

for $l \geq 1$. □

2.6. Proofs of the main results

In this section, proofs of Theorems 2 and 3 are given. The plan of the proofs is as follows. Proposition 4 shows that we can consider, without loss of generality, the case of two investors. This reduces the dimension of the original random dynamical system from a general N to $N = 2$. Proposition 3 describes a one-dimensional system which governs the evolution of the ratio $x_t = r_t^1/r_t^2$ of the market shares of the two investors, and thus reduces the dimension of the problem to 1. Our goal is to show that the random sequence (x_t) defined recursively by (26) is bounded away from zero almost surely. To this end it turns out to be convenient to take a "step back" and to increase the dimension to K (the number of assets). Assuming that the first trader uses the investment proportions $\lambda_{t,k}^1 = \lambda_{t,k}^*(s^t)$ prescribed by the portfolio rule Λ^* and the second trader employs investment proportions $\lambda_{t,k}^2 = \lambda_{t,k}(s^t)$ specified by some other portfolio rule Λ, we introduce the following change of variables

$$y_t^k = \lambda_{t,k}/x_t\,,\ \ k = 1, ..., K, \tag{34}$$

and define $y_t := (y_t^1, ..., y_t^K)$. We examine the dynamics of the random vectors $y_t = y_t(s^t)$ implied by the system (26). The norm $|y_t| := \sum_k |y_t^k|$ of the vector $y_t \geq 0$ is equal to $\sum_k(\lambda_{t,k}/x_t) = 1/x_t$, and what we need is to show that $1/|y_t|$ is bounded away from zero (a.s.). To prove this, we construct a stochastic Lyapunov function — a function of y_t which forms a non-negative supermartingale (ζ_t) along a path (y_t) of the system at hand (see Lemma 3 below). By using the supermartingale convergence theorem, we prove that the stochastic process ζ_t converges (a.s.), which implies that it is bounded (a.s.). We complete the proof of Theorem 2 by showing that the boundedness of ζ_t implies that $x_t = 1/|y_t|$ is bounded away from zero. By using the above techniques, together with some additional considerations, we complete this section with a proof of Theorem 3.

We begin the realization of the plan outlined with two lemmas containing inequalities involving the variables y_t^k defined by (34). Define the non-negative random variables

$$Y_t := \ln(1 + |y_t|) = -\ln r_t^1, \tag{35}$$

$$Z_{t,k} := \ln(1 + \frac{y_t^k}{\lambda_{t,k}^*}) = \ln\left(1 + \frac{r_t^2 \lambda_{t,k}}{r_t^1 \lambda_{t,k}^*}\right), \tag{36}$$

$$\gamma_{k,m}^{t+1} = \frac{1 + y_{t+1}^m/\lambda_{t+1,m}^*}{1 + y_t^k/\lambda_{t,k}^*}. \tag{37}$$

In particular, we have $\gamma_{k,k}^{t+1} = (1 + y_{t+1}^k/\lambda_{t+1,k}^*)/(1 + y_t^k/\lambda_{t,k}^*)$.

Later in the proofs, the following two equalities will be employed:

$$\ln \gamma_{k,m}^{t+1} = Z_{t+1,m} - Z_{t,k} \tag{38}$$

and

$$\frac{\lambda_{t+1,m}\lambda_{t,k}^*|y_{t+1}| - \lambda_{t+1,m}^* y_t^k}{\lambda_{t,k}^* + y_t^k} = \lambda_{t+1,m}^* \left(\gamma_{k,m}^{t+1} - 1\right). \tag{39}$$

The algebraic identity (39) can be checked quite easily, and we leave its proof to the reader. In particular, if $m = k$ then (39) takes on the following form:

$$\frac{\lambda_{t+1,k}\lambda_{t,k}^*|y_{t+1}| - \lambda_{t+1,k}^* y_t^k}{\lambda_{t,k}^* + y_t^k} = \lambda_{t+1,k}^* \left(\gamma_{k,k}^{t+1} - 1\right).$$

Proofs of Theorems 2 and 3 are based on Lemmas 1–4 which we formulate below and prove in Appendix A.

Let us define a function

$$f(x) = \frac{(x-1)\ln x}{x+2}, \tag{40}$$

which will be helpful in estimating some logarithmic expressions.

Lemma 1. *The function $f(x)$ is non-negative, has a unique root $x = 1$ and satisfies*

$$x - 1 \geq \ln x + f(x), \; x \in (-\infty, +\infty). \tag{41}$$

Lemma 2. *The following inequality holds:*

$$\sum_{k=1}^{K} \lambda_{t+1,k}^* Z_{t+1,k} + \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t+1,k}(1 - \rho_{t+1,m})\lambda_{t+1,m}^* f\left(\gamma_{k,m}^{t+1}\right) \leq$$

$$\sum_{k=1}^{K} \rho_{t+1,k}\lambda^*_{t+1,k}Z_{t,k} + U^*_{t+1}\sum_{k=1}^{K} R_{t+1,k}Z_{t,k} \ . \tag{42}$$

Put

$$\zeta_t := \sum_{k=1}^{K} \lambda^*_{t,k}Z_{t,k} + \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m} f\left(\gamma^t_{k,m}\right) . \tag{43}$$

Lemma 3. *The sequence of random variables ζ_t $(t \geq 1)$ is a non-negative supermartingale, and we have*

$$\zeta_t - E_t\zeta_{t+1} \geq \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m} f\left(\gamma^t_{k,m}\right) \geq 0 \text{ (a.s.)}. \tag{44}$$

Lemma 4. *Let ζ_t be a supermartingale such that $\inf_t E\zeta_t > -\infty$. Then the series of non-negative random variables $\sum_{t=0}^{\infty}(\zeta_t - E_t\zeta_{t+1})$ converges (a.s.).*

In what follows, in the proofs of Theorems 2 and 3 as well as Lemmas 1-4, we will sometimes omit "a.s." where it does not lead to ambiguity.

Proof of Theorem 2. By Lemma 4, the sequence ζ_t defined in (43) is a non-negative supermartingale. Therefore it converges (a.s.), and hence it is bounded above (a.s.) by some random constant C:

$$C \geq \zeta_t = \sum_{k=1}^{K} \lambda^*_{t,k}Z_{t,k} + \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m} f\left(\gamma^t_{k,m}\right) \geq$$
$$\sum_{k=1}^{K} \lambda^*_{t,k}Z_{t,k} = \sum_{k=1}^{K} \lambda^*_{t,k} \ln\left(1 + \frac{r^2_t\lambda_{t,k}}{r^1_t\lambda^*_{t,k}}\right).$$

Here, we used the non-negativity of the function f established in Lemma 1 and the non-negativity of the relative dividends $R_{t,k}$, and $\lambda^*_{t,m}$, and (A.2).

Recall that by virtue of Theorem 1, $\lambda^*_{t,k} \geq \delta$ for any t, k. Therefore $C/\delta \geq \ln\left(1 + r^2_t\lambda_{t,k}/r^1_t\lambda^*_{t,k}\right)$ for all t, k, and there exists some random variable H such that $H \geq 1 + r^2_t\lambda_{t,k}/r^1_t\lambda^*_{t,k}$ for all t, k. Furthermore, there exists some k such that $\lambda_{t,k} \geq 1/K$ (since $\sum_{k=1}^{K}\lambda_{t,k} = 1$). For this k the following inequality holds:

$$H \geq 1 + \frac{r^2_t\lambda_{t,k}}{r^1_t\lambda^*_{t,k}} \geq 1 + \frac{r^2_t}{r^1_t\lambda^*_{t,k}K} \geq 1 + \frac{r^2_t}{r^1_t K} = 1 + \frac{(1-r^1_t)}{r^1_t K},$$

which implies $r^1_t \geq (K(H-1)+1)^{-1} = \tau$. □

Proof of Theorem 3. The proof of this theorem consists in several steps. We outline these steps here and provide details of the arguments in Appendix A.

1st step. We first show that

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}(\gamma_{k,m}^{t}-1)^2 < \infty. \tag{45}$$

2nd step. From (45) we deduce that

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}\left(y_t^m/\lambda_{t,m}^{*} - y_{t-1}^k/\lambda_{t-1,k}^{*}\right)^2 < \infty. \tag{46}$$

3rd step. At this step, by using (46), we obtain:

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K} \left(y_t^m/\lambda_{t,m}^{*} - y_t^k/\lambda_{t,k}^{*}\right)^2 < \infty.$$

This series can be estimated as:

$$\infty > \sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}\left(\frac{y_t^m}{\lambda_{t,m}^{*}} - \frac{y_t^k}{\lambda_{t,k}^{*}}\right)^2 = \sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 \sum_{m=1}^{K}\sum_{k=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda_{t,m}^{*}} - \frac{\lambda_{t,k}}{\lambda_{t,k}^{*}}\right)^2 \geq$$
$$\sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 \sum_{k=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda_{t,m}^{*}} - \frac{\lambda_{t,k}}{\lambda_{t,k}^{*}}\right)^2 \tag{47}$$

for each m. This fact will be used at the next step.

4th step. Next we prove the following estimate for the sum involved in (47):

$$\sum_{k=1}^{K}\left(\lambda_{t,m}/\lambda_{t,m}^{*} - \lambda_{t,k}/\lambda_{t,k}^{*}\right)^2 \geq \left(\lambda_{t,m}/\lambda_{t,m}^{*} - 1\right)^2. \tag{48}$$

Finally, by using (47) and inequality (48), we conclude

$$\infty > \sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 \sum_{m=1}^{K}\sum_{k=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda_{t,m}^{*}} - \frac{\lambda_{t,k}}{\lambda_{t,k}^{*}}\right)^2 \geq \sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 \sum_{m=1}^{K}\left(\frac{\lambda_{t,m}}{\lambda_{t,m}^{*}} - 1\right)^2 =$$
$$\sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 \sum_{m=1}^{K}\left(\frac{\lambda_{t,m}-\lambda_{t,m}^{*}}{\lambda_{t,m}^{*}}\right)^2 \geq \sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 \sum_{m=1}^{K}\left(\lambda_{t,m}-\lambda_{t,m}^{*}\right)^2 =$$
$$\sum_{t=0}^{\infty}\left(\frac{r_t^2}{r_t^1}\right)^2 ||\lambda_t - \lambda_t^{*}||^2 \geq \sum_{t=0}^{\infty}\phi^2 ||\lambda_t - \lambda_t^{*}||^2,$$

where $\phi > 0$ is a random variable such that $r_t^2/r_t^1 \geq \phi$, which exists because $\Lambda = (\lambda_t)$ is a survival strategy. □

2.7. Conclusion

Traditional DSGE models for financial markets (Radner [139, 140], Magill and Quinzii [123]) are based on the Arrow-Debreu general equilibrium theory (GET), which has its roots in the ideas of Leon Walras, one of the classics of economic thought of the 19th century. Models of this kind describe the world of economic agents whose ultimate goal is consumption, and who strive to maximize their individual utilities subject to budget constraints. Equilibrium is understood as a situation in which the goals and interests of the market actors are equilibrated by market clearing prices. Starting from the 1950s, GET became and remained for decades a cornerstone of mainstream economics.

The modelling approach pursued here marks a departure from the conventional general equilibrium paradigm. In the model we deal with, the asset market dynamics are determined by the dynamic interaction of strategies of the traders, rather than by the maximization of utilities of consumption. These strategies are taken as fundamental characteristics of the agents, while the optimality of individual behaviour and the coordination of beliefs (or the lack of it) are not reflected in formal terms but are rather left to the interpretation of the observed behaviour.

In the classical GET models it is typically presumed that the impact on prices of every individual is negligible. A culmination of this idea is the theory of "large" economies with atomless measure spaces whose elements represent "small" agents—one of the highlights of Mathematical Economics of all times: see Aumann [14] and Hildenbrand [90]. EBF describes an absolutely different, globalized world, the world of large, even super large (primarily institutional) investors, who may act at the global level, and whose fundamental objectives are of an evolutionary nature: e.g. survival, domination and fastest growth. In fact fastest growth is often related, and in our models is equivalent, to survival—see Section 2.4. These factors, rather than the utilities of individual consumption come to the fore. In this framework, investment decisions made by each of the market actors might substantially affect the equilibrium prices, in contrast to the above mentioned models with a continuum of small agents.

A specific feature of the present approach is that it rests only on model components that are observable and can be estimated empirically. This aspect, combined with the generality of the behavioural strategies emphasized above makes the theory closer to practical applications. Evidence of successes of this approach is available—see the UK Research Excellence Framework, 2014 Impact Case Study "Mathematical Behavioural Finance"[3], joint research of R. Amir, I.V. Evstigneev, T. Hens and K.R. Schenk-Hoppé.

Appendix A

Proof of Lemma 1. We first observe that for $0 < x \leq 1$ we have $2(x-1)(x+1)^{-1} \geq \ln x$. Hence

$$(x-1) \geq \frac{(x+1)\ln x}{2} \geq \ln x + \frac{(x-1)\ln x}{x+2} = \ln x + f(x). \tag{49}$$

On the other hand, for $x \geq 1$ we have $(x-1)(x+1)/(2x) \geq \ln x$. Therefore

$$x - 1 \geq \frac{2x}{x+1}\ln x \geq \ln x + \frac{(x-1)\ln x}{x+2} = \ln x + f(x). \tag{50}$$

By combining (49) and (50) we obtain (41). Clearly $f(x)$ is non-negative and if $x \neq 1$, then $f(x) = (x-1)\ln x/(x+2) \neq 0$, and so if $x \neq 1$, then $f(x) > 0$. □

Proof of Lemma 2. From formula (26) and (25) with $\lambda^1_{t,k} = \lambda^*_{t,k}$ and $\lambda^2_{t,k} = \lambda_{t,k}$, we obtain

$$x_{t+1} = x_t \frac{\sum_{k=1}^{K}[\rho_{t+1,k}\lambda_{t+1,k} + R_{t+1,k}U_{t+1}]\dfrac{\lambda^*_{t,k}}{\lambda^*_{t,k}x_t + \lambda_{t,k}}}{\sum_{k=1}^{K}[\rho_{t+1,k}\lambda^*_{t+1,k} + R_{t+1,k}U^*_{t+1}]\dfrac{\lambda_{t,k}}{\lambda^*_{t,k}x_t + \lambda_{t,k}}}.$$

Consequently,

$$\sum_{k=1}^{K}[\rho_{t+1,k}\lambda^*_{t+1,k} + R_{t+1,k}U^*_{t+1}]\frac{\lambda_{t,k}}{\lambda^*_{t,k}x_t + \lambda_{t,k}} = \sum_{k=1}^{K}[\rho_{t+1,k}\frac{\lambda_{t+1,k}}{x_{t+1}} + \frac{R_{t+1,k}}{x_{t+1}}U_{t+1}]\frac{\lambda^*_{t,k}x_t}{\lambda^*_{t,k}x_t + \lambda_{t,k}}.$$

By using the notation $y^k_t = \lambda_{t,k}/x_t$ and the fact that $|y_t| = 1/x_t$, we rewrite the above formula as

$$\sum_{k=1}^{K}[\rho_{t+1,k}\lambda^*_{t+1,k} + R_{t+1,k}U^*_{t+1}]\frac{y^k_t}{\lambda^*_{t,k} + y^k_t} = \sum_{k=1}^{K}[\rho_{t+1,k}y^k_{t+1} + R_{t+1,k}\,|y_{t+1}|\,U_{t+1}]\frac{\lambda^*_{t,k}}{\lambda^*_{t,k} + y^k_t},$$

[3] https://impact.ref.ac.uk/casestudies/CaseStudy.aspx?Id=28045

which yields

$$\sum_{k=1}^{K} \rho_{t+1,k} \frac{\lambda^*_{t,k} y^k_{t+1} - \lambda^*_{t+1,k} y^k_t}{\lambda^*_{t,k} + y^k_t} + \sum_{k=1}^{K} R_{t+1,k} \frac{U_{t+1}\lambda^*_{t,k}|y_{t+1}| - U^*_{t+1} y^k_t}{\lambda^*_{t,k} + y^k_t} = 0. \tag{51}$$

Recalling the definition of U_{t+1} (25), we notice that

$$\frac{U_{t+1}\lambda^*_{t,k}|y_{t+1}| - U^*_{t+1} y^k_t}{\lambda^*_{t,k} + y^k_t} = \sum_{m=1}^{K} (1-\rho_{t+1,m})\lambda^*_{t+1,m}\left(\gamma^{t+1}_{k,m} - 1\right),$$

where $\gamma^{t+1}_{k,m}$ comes from (37). Then using (37) and (39), we write (51) as

$$\sum_{k=1}^{K} \rho_{t+1,k}\lambda^*_{t+1,k}\left(\gamma^{t+1}_{k,k} - 1\right) + \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}\left(\gamma^{t+1}_{k,m} - 1\right) = 0. \tag{52}$$

The first sum in (52) can be estimated by using the well-known inequality $\gamma^{t+1}_{k,k} - 1 \geq \ln \gamma^{t+1}_{k,k}$. To estimate the second sum let us employ Lemma 1: $\gamma^{t+1}_{k,m} - 1 \geq \ln \gamma^{t+1}_{k,m} + f(\gamma^{t+1}_{k,m})$. Then we have

$$\sum_{k=1}^{K} \rho_{t+1,k}\lambda^*_{t+1,k} \ln \gamma^{t+1}_{k,k} + \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}(\ln \gamma^{t+1}_{k,m} + f(\gamma^{t+1}_{k,m})) \leq 0.$$

Recall that $\ln \gamma^{t+1}_{k,m} = Z_{t+1,m} - Z_{t,k}$, and so

$$\sum_{k=1}^{K} \rho_{t+1,k}\lambda^*_{t+1,k}(Z_{t+1,k} - Z_{t,k}) +$$

$$\sum_{m=1}^{K}\sum_{k=1}^{K} R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}(Z_{t+1,m} - Z_{t,k} + f(\gamma^{t+1}_{k,m})) \leq 0.$$

This implies

$$\sum_{k=1}^{K} \lambda^*_{t+1,k} Z_{t+1,k} + \sum_{m=1}^{K}\sum_{k=1}^{K} R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m} f\left(\gamma^{t+1}_{k,m}\right) \leq$$

$$\sum_{k=1}^{K} \rho_{t+1,k}\lambda^*_{t+1,k} Z_{t,k} + U^*_{t+1}\sum_{k=1}^{K} R_{t+1,k} Z_{t,k}.$$

This inequality obtained is nothing but the one in (42), which completes the proof. □

Proof of Lemma 3. By virtue of Lemma 1, the function f is non-negative, and so we have

$$\sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m} f\left(\gamma^{t}_{k,m}\right) \geq 0. \tag{53}$$

This implies that $\zeta_t \geq 0$. By taking the conditional expectation $E_t(\cdot)$ of both sides of (42), we obtain the following chain of relations:

$$E_t\zeta_{t+1} = E_t\left[\sum_{k=1}^{K}\lambda^*_{t+1,k}Z_{t+1,k} + \sum_{m=1}^{K}\sum_{k=1}^{K}R_{t+1,k}(1-\rho_{t+1,m})\lambda^*_{t+1,m}f\left(\gamma^{t+1}_{k,m}\right)\right] \leq$$

$$E_t\sum_{k=1}^{K}[\rho_{t+1,k}\lambda^*_{t+1,k}Z_{t,k} + U^*_{t+1}R_{t+1,k}Z_{t,k}] =$$

$$\sum_{k=1}^{K}Z_{t,k}E_t\left[\rho_{t+1,k}\lambda^*_{t+1,k} + U^*_{t+1}R_{t+1,k}\right] = \sum_{k=1}^{K}Z_{t,k}\lambda^*_{t,k} \tag{54}$$

where the last equality follows from the definition of $\lambda^*_{t,k}$. By using (53) and (54), we find

$$E_t\zeta_{t+1} \leq E_t\zeta_{t+1} + \sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^{t}_{k,m}\right) \leq \sum_{k=1}^{K}Z_{t,k}\lambda^*_{t,k} +$$

$$\sum_{m=1}^{K}\sum_{k=1}^{K}R_{t,k}(1-\rho_{t,m})\lambda^*_{t,m}f\left(\gamma^{t}_{k,m}\right) = \zeta_t. \tag{55}$$

To complete the proof that ζ_t is a supermartingale it sufficient to prove that the random variable

$$\zeta_1 = \sum_{k=1}^{K}\lambda^*_{1,k}Z_{1,k} + \sum_{m=1}^{K}\sum_{k=1}^{K}R_{1,k}(1-\rho_{1,m})\lambda^*_{1,m}f\left(\gamma^{1}_{k,m}\right)$$

is bounded. To this end we notice that

$$\gamma^1_{k,m} = (1 + y^m_1/\lambda^*_{1,m})/(1 + y^k_0/\lambda^*_{0,k}) \leq 1 + y^m_1/\lambda^*_{1,m} \leq 1 + r^2_1/r^1_1\delta \text{ (a.s.)}$$

because $\lambda^*_{1,k} \geq \delta$ (see Theorem 1). Now it remains only to show that r^1_1 is bounded away from zero by a strictly positive constant.

From equation (26) we get

$$\frac{r^1_1}{r^2_1} = \frac{r^1_0}{r^2_0}\frac{\sum_{k=1}^{K}[\rho_{1,k}\lambda_{1,k} + R_{1,k}U_1]\dfrac{\lambda^*_{0,k}}{\lambda^*_{0,k}x_0 + \lambda_{0,k}}}{\sum_{k=1}^{K}[\rho_{1,k}\lambda^*_{1,k} + R_{1,k}U^*_1]\dfrac{\lambda_{0,k}}{\lambda^*_{0,k}x_0 + \lambda_{0,k}}} =: \frac{r^1_0}{r^2_0}\frac{A}{B}. \tag{56}$$

Since r^2_0 is a strictly positive constant, it sufficient to show that the nominator of the above fraction, which we denote by A, is bounded away from zero and the denominator, denoted by

B, is bounded above. We have

$$A \geq U_1 \sum_{k=1}^{K} R_{1,k} \frac{\lambda^*_{0,k}}{\lambda^*_{0,k} x_0 + \lambda_{0,k}} \geq \kappa \sum_{k=1}^{K} R_{1,k} \frac{\lambda^*_{0,k}}{\lambda^*_{0,k} x_0 + \lambda_{0,k}}$$

$$\geq \kappa \lambda^*_{0,k} / (K \left(\lambda^*_{0,k} x_0 + \lambda_{0,k}\right)) \geq \kappa\delta/(K(\delta x_0 + 1)) =: \overline{A}.$$

The second inequality holds because $U_1 \geq \kappa$ by the definition of U_i (see (25)) and assumption (A.2). The third inequality is valid since there exists k such that $R_{1,k} \geq 1/K$ (because $\sum_{k=1}^{K} R_{1,k} = 1$) and the whole sum is not less than one summand. It remains only to observe that B is bounded above:

$$B \leq \sum_{k=1}^{K} [\rho_{1,k} \lambda^*_{1,k} + R_{1,k} U^*_1] = 1.$$

Finally,

$r^1_1 = r^2_1 A x_0 / B = (1 - r^1_1) A x_0 / B \geq (1 - r^1_1) \overline{A} x_0$, which yields $r^1_1 \geq \overline{A} x_0 / \left(\overline{A} x_0 + 1\right)$. Thus, r^1_1 is bounded away from zero, therefore $Z_{1,k}$ and $\gamma^1_{k,m}$ are bounded above, which implies the boundedness of $f\left(\gamma^1_{k,m}\right)$ and hence the boundedness of ζ_1. The proof is complete. □

Proof of Lemma 4. The random variables $\eta_t := \zeta_t - E_t \zeta_{t+1}$ are non-negative by the definition of a supermartingale. Further, we have

$$\sum_{t=0}^{T-1} E\eta_t = \sum_{t=0}^{T-1} (E\zeta_t - E\zeta_{t+1}) = E\zeta_0 - E\zeta_T\,,$$

and so the sequence $\sum_{t=0}^{T-1} E\, \eta_t$ is bounded because $\sup_T(-E\zeta_T) = -\inf_T E\, \zeta_T < +\infty$. Therefore the series of the expectations $\sum_{t=0}^{\infty} E\, \eta_t$ of the non-negative random variables η_t converges, which implies $\sum_{t=0}^{\infty} \eta_t < \infty$ a.s. because $E \sum_{t=0}^{\infty} \eta_t = \sum_{t=0}^{\infty} E\, \eta_t$ (the last equality holds for any sequence $\eta_t \geq 0$). The proof is complete. □

The remainder of the Appendix provides details of the proof of Theorem 3 (steps 1 to 4).

1st step. Since investor 1 uses the strategy Λ^*, by virtue of Lemma 3 the sequence ζ_t defined by (43) is a non-negative supermartingale. By using inequality (44) and Lemma 4 we obtain

$$\sum_{t=0}^{\infty} \sum_{m=1}^{K} \sum_{k=1}^{K} R_{t,k}(1 - \rho_{t,m}) \lambda^*_{t,m} f\left(\gamma^t_{k,m}\right) < \infty \text{ (a.s.).}$$

By assumption (A.2), we have $(1-\rho_{t,m}) \geq \varkappa > 0$, and since $\lambda^*_{t,m} \geq \delta$, the above inequality implies

$$\sum_{t=0}^{\infty} \sum_{m=1}^{K} \sum_{k=1}^{K} R_{t,k} f\left(\gamma^t_{k,m}\right) < \infty \text{ (a.s.)}. \tag{57}$$

Let us show that if (57) converges, then the following series converges as well:

$$\sum_{t=0}^{\infty} \sum_{m=1}^{K} \sum_{k=1}^{K} R_{t,k} (\gamma^t_{k,m} - 1)^2 < \infty \text{ (a.s.)}. \tag{58}$$

To this end it is sufficient to verify that for some random variable $\theta > 0$, we have

$$G_t := \frac{\sum_{m=1}^{K} \sum_{k=1}^{K} R_{t,k} f\left(\gamma^t_{k,m}\right)}{\sum_{m=1}^{K} \sum_{k=1}^{K} R_{t,k} (\gamma^t_{k,m} - 1)^2} \geq \theta > 0.$$

To prove this we observe that

$$\min_{m,k} \frac{\ln \gamma^t_{k,m}}{(\gamma^t_{k,m}+2)(\gamma^t_{k,m}-1)} = \min_{m,k} \frac{(\gamma^t_{k,m}-1)\ln \gamma^t_{k,m}}{(\gamma^t_{k,m}-1)^2(\gamma^t_{k,m}+2)} \leq G_t.$$

Note that the function $\ln x (x-1)^{-1}(x+2)^{-1}$ is non-increasing and hence it achieves its minimum on $(0, M]$ at M. Furthermore,

$$\gamma^t_{k,m} = \frac{1 + y^m_{t+1}/\lambda^*_{t+1,m}}{1 + y^k_t/\lambda^*_{t,k}} \leq 1 + \frac{y^m_{t+1}}{\lambda^*_{t+1,m}} = 1 + \frac{r^2_{t+1}\lambda_{t+1,m}}{r^1_{t+1}\lambda^*_{t+1,m}} \leq 1 + \frac{1-\tau}{\tau\delta} =: M, \tag{59}$$

where the last inequality holds by virtue of Theorem 2 and because $\lambda^*_{t,m} \geq \delta$. Since $\gamma^t_{k,m} \leq M$ for each t, k, m, we get

$$G_t \geq \min_{t,k,m} \frac{\ln \gamma^t_{k,m}}{(\gamma^t_{k,m}-1)(\gamma^t_{k,m}+2)} \geq \frac{\ln M}{(M-1)(M+2)}.$$

Thus we have proved that the series (58) converges.

2nd step. Using (39) we can see that the following inequality holds:

$$(\gamma^t_{k,m} - 1)^2 = \left(\frac{y^m_t/\lambda^*_{t,m} - y^k_{t-1}/\lambda^*_{t-1,k}}{1 + y^k_{t-1}/\lambda^*_{t-1,k}}\right)^2 \geq \frac{1}{M^2}\left(\frac{y^m_t}{\lambda^*_{t,m}} - \frac{y^k_{t-1}}{\lambda^*_{t-1,k}}\right)^2$$

because $1 + y^k_{t-1}/\lambda^*_{t-1,k} \leq M$ in view of (59). Hence, the series

$$\sum_{t=0}^{\infty} \sum_{m=1}^{K} \sum_{k=1}^{K} R_{t,k} \left(y^m_t/\lambda^*_{t,m} - y^k_{t-1}/\lambda^*_{t-1,k}\right)^2 < \infty \tag{60}$$

converges.

3rd step. Let us denote $a_t^m = y_t^m/\lambda_{t,m}^*$ and $b_{m,k}^t = a_t^m - a_{t-1}^k$. In this new notation,

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K} R_{t,k}(b_{m,k}^t)^2 < \infty$$

for any k, m. Now recall that $\sum_{k=1}^{K} R_{t,k} = 1$ and hence for all t there exists at least one k such that $R_{t,k} \geq 1/K$. Denote this k by k_t^*. Clearly, we have

$$\infty > \sum_{t=0}^{\infty}\sum_{k=1}^{K} R_{t,k}\sum_{m=1}^{K}(b_{m,k}^t)^2 \geq \sum_{t=0}^{\infty}\frac{1}{K}\sum_{m=1}^{K}(b_{m,k_t^*}^t)^2.$$

Fix m and m'. Then it is easy to see that

$$\sum_{t=0}^{\infty}(b_{m,k_t^*}^t)^2 < \infty \text{ and } \sum_{t=0}^{\infty}(b_{m',k_t^*}^t)^2 < \infty. \tag{61}$$

Observe that the following equalities hold

$$b_{m,k_t^*}^t - b_{m',k_t^*}^t = a_t^m - a_{t-1}^{k_t^*} - a_t^{m'} + a_{t-1}^{k_t^*} = a_t^m - a_t^{m'},$$

and so

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{m'=1}^{K}\left(a_t^m - a_t^{m'}\right)^2 = \sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{m'=1}^{K}\left(b_{m,k_t^*}^t - b_{m',k_t^*}^t\right)^2.$$

Furthermore

$$\left(b_{m,k_t^*}^t - b_{m',k_t^*}^t\right)^2 = \left|(b_{m,k_t^*}^t)^2 + (b_{m',k_t^*}^t)^2 - 2b_{m,k_t^*}^t b_{m',k}^t\right| \leq$$

$$\leq (b_{m,k_t^*}^t)^2 + (b_{m',k_t^*}^t)^2 + 2\left|b_{m,k_t^*}^t b_{m',k_t^*}^t\right| \leq 2\left((b_{m,k_t^*}^t)^2 + (b_{m',k_t^*}^t)^2\right)$$

since $2\left|b_{m,k_t^*}^t b_{m',k_t^*}^t\right| \leq (b_{m,k_t^*}^t)^2 + (b_{m',k_t^*}^t)^2$. Therefore, since both series $\sum_{t=0}^{\infty}(b_{m,k_t^*}^t)^2$ and $\sum_{t=0}^{\infty}(b_{m',k_t^*}^t)^2$ converge (see (61)), then for all pairs m, m', we have

$$\sum_{t=0}^{\infty}\left(b_{m,k_t^*}^t - b_{m',k_t^*}^t\right)^2 < \infty,$$

and consequently,

$$\sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{m'=1}^{K}\left(b_{m,k_t^*}^t - b_{m',k_t^*}^t\right)^2 = \sum_{t=0}^{\infty}\sum_{m=1}^{K}\sum_{k=1}^{K}\left(y_t^m/\lambda_{t,m}^* - y_t^k/\lambda_{t,k}^*\right)^2 < \infty. \tag{62}$$

4th step. Consider two cases: (i) $\lambda_{t,m}/\lambda^*_{t,m} \geq 1$ and (ii) $\lambda_{t,m}/\lambda^*_{t,m} \leq 1$. In the first case, among the $K-1$ fractions $\lambda_{t,k}/\lambda^*_{t,k}$ $(k \neq m)$ we can find at least one with $\lambda_{t,m'}/\lambda^*_{t,m'} \leq 1$. Otherwise, $\lambda_{t,m'}/\lambda^*_{t,m'} > 1$ for all $m' \neq m$, i.e., $\lambda_{t,m'} > \lambda^*_{t,m'}$ $(m' \neq m)$ and $\lambda_{t,m} \geq \lambda^*_{t,m}$. Then we get $1 = \sum_{k=1}^K \lambda_{t,k} > \sum_{k=1}^K \lambda^*_{t,k} = 1$, which is a contradiction. By the same argument, we can show in the second case that if $\lambda_{t,m}/\lambda^*_{t,m} \leq 1$, then there exists m' satisfying $\lambda_{t,m'}/\lambda^*_{t,m'} \geq 1$.

Thus, we have proved that for each m there exists m' such that either $\lambda_{t,m}/\lambda^*_{t,m} \geq 1 \geq \lambda_{t,m'}/\lambda^*_{t,m'}$ or $\lambda_{t,m'}/\lambda^*_{t,m'} \geq 1 \geq \lambda_{t,m}/\lambda^*_{t,m}$, Consequently,

$$\left|\lambda_{t,m'}/\lambda^*_{t,m'} - \lambda_{t,m}/\lambda^*_{t,m}\right| \geq \left|\lambda_{t,m}/\lambda^*_{t,m} - 1\right|,$$

which implies

$$\sum_{k=1}^K \left(\lambda_{t,m}/\lambda^*_{t,m} - \lambda_{t,k}/\lambda^*_{t,k}\right)^2 \geq \left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - \frac{\lambda_{t,m'}}{\lambda^*_{t,m'}}\right)^2 \geq \left(\frac{\lambda_{t,m}}{\lambda^*_{t,m}} - 1\right)^2.$$

Appendix B

The purpose of this Appendix is to prove Theorem B.2, which implies the existence and uniqueness of the Λ^* strategy playing a central role in this work (see the definition in Section 3). We will deduce Theorem B.2 from Theorem B.1, which represents a non-stationary version of the stochastic Perron-Frobenius theorem, see Babaei et al. [16] and references therein. In turn, Theorem B.1 will be obtained as a consequence of a chain of auxiliary results formulated in Lemmas B.1–B.2 and Propositions B.1–B.3 below.

Denote by $\mathcal{M}^n$ $(n > 1)$ the set of $n \times n$ matrices $B \geq 0$ such that $Bx \neq 0$ for all $x \in Q := \{x : 0 \neq x \geq 0\}$. For $x = (x^1, ..., x^n) \in R^n$, define $|x| = |x^1| + ... + |x^n|$, $x^0 = x/|x|$, and, for $B \in \mathcal{M}^n$, put

$$\kappa(B) = \max_{x,y \in Q} |(Bx)^0 - (By)^0|.$$

Let $\phi(B)$ denote the ratio of the smallest and the greatest elements of the matrix B.

Lemma B.1. *Let $B_1, B_2, ...B_k \in \mathcal{M}^n$. If $B_i > 0$ and $n > 1$, then*

$$\kappa(B_k...B_1) \leq \rho_1^{-1}\delta_1...\delta_{k-1}, \tag{63}$$

where

$$\rho_i = n^{-2}\phi(B_i)\phi(B_{i+1}),\ \delta_i = (1-2\rho_i).$$

For a proof of this result see [52], Lemma 1.

Put $\Delta = \{x = (x_1, ..., x_n) : x_j \geq 0, \sum x_j = 1\}$. Let $\mathcal{D}^n$ denote the set of matrices B in $\mathcal{M}^n$ representing linear transformations of R^n that map Δ into itself. For $\delta > 0$ we will denote by $\mathcal{D}^n_\delta$ the set of matrices $B \in \mathcal{D}^n$ whose elements are not less than δ.

Lemma B.2. *Let $B_1, B_2, ..., B_k \in \mathcal{D}^n_\delta$. Then*

$$\kappa(B_k...B_1) \leq M\rho^{k-1}, \tag{64}$$

where $M = n^2\delta^{-2}$ and $\rho = 1 - n^{-2}\delta^2$.

Proof. This is immediate from (63) because

$$1 \geq \phi(B_i) \geq \delta,\ \rho_i = n^{-2}\phi(B_i)\phi(B_{i+1}) \geq n^{-2}\delta^2,\ \rho_1^{-1} \leq n^2\delta^{-2} = M,$$

$$\delta_i = 1 - 2\rho_i \leq 1 - n^{-2}\delta^2 = \rho.$$

□

Let $B_1, B_2, ...$ be a sequence of matrices in $\mathcal{D}^n$.

Proposition B.1. *There exists a sequence $(y_t^*)_{\geq 0}$ such that $y_t^* \in \Delta$ and*

$$y_t^* = B_{t+1}y_{t+1}^*,\ t \geq 0. \tag{65}$$

Proof. Put $\Delta^\infty = \Delta \times \Delta \times ...$ and $\mathcal{Y} = R^n \times R^n \times ....$ Let us introduce in $\mathcal{Y}$ the product topology: $(y_t^m)_{t\geq 0} \to (y_t)_{t\geq 0}$ if and only if $y_t^m \to y_t$ for all t. Then $\mathcal{Y}$ is a topological locally convex vector space and Δ^∞ is a compact convex set in $\mathcal{Y}$. Consider the mapping $\mathfrak{B} : \mathcal{Y} \to \mathcal{Y}$ transforming $(y_t)_{t\geq 0}$ into $(B_{t+1}y_{t+1})_{t\geq 0}$. This mapping is continuous and transforms Δ^∞ into itself. Consequently, by the Schauder-Tychonoff theorem (e.g. [179]) it has a fixed point $y^* = \mathfrak{B}y^*$, which proves the proposition. □

For each $t \geq 1$ and $j \geq 0$ denote $B_t^{t+j} = B_t...B_{t+j}$. For any $y = (y_t) \in \Delta^\infty$ denote by $\mathfrak{B}_t^m(y)$ the tth term of the sequence $\mathfrak{B}^m(y) \in \Delta^\infty$, where $\mathfrak{B}^m(y)$ is the mth iterate of the mapping $\mathfrak{B}$. Clearly, if we put $y_t^m = \mathfrak{B}_t^m(y)$ $(t \geq 0)$, then

$$y_t^1 = B_{t+1}y_{t+1} = B_{t+1}^{t+1}y_{t+1},\ y_t^2 = B_{t+1}y_{t+1}^1 = B_{t+1}B_{t+2}y_{t+2} = B_{t+1}^{t+2}y_{t+2}, ...,$$

$$y_t^m = B_{t+1}B_{t+2}...B_{t+m}y_{t+m} = B_{t+1}^{t+m}y_{t+m},\ t \geq 0.$$

Proposition B.2. *Suppose there exist an integer $l \geq 0$ and a real number $\delta > 0$ such that for any $t \geq 1$ the matrix B_t^{t+l} belongs to $\mathcal{D}_\delta^n$. Then the solution $y^* = (y_t^*)_{t\geq 0}$ to equation (65) is unique, and for every $t \geq 0$, the sequence $y_t^m = \mathfrak{B}_t^m(y)$ converges to y_t^* uniformly in $y \in \Delta^\infty$.*

Proof. Uniqueness follows from convergence. To prove the uniform convergence of y_t^m we estimate the distance between y_t^m and y_t^* by using (64). Define

$$H_j = B_{t+(j-1)l+j}^{t+jl+j},\ j \geq 1.$$

For $m \geq l+1$ denote by $k = k(m)$ the greatest natural number such that $kl + k \leq m$ and put

$$C_t^{t+m} = \begin{cases} B_{t+kl+k+1}^{t+m}, & kl+k < m \\ Id, & kl+k = m \end{cases}.$$

Then we have

$$B_{t+1}^{t+m} = B_{t+1}^{t+l+1}B_{t+l+2}^{t+2l+2}B_{t+2l+3}^{t+3l+3}...B_{t+(k-1)l+k}^{t+kl+k}B_{t+kl+k+1}^{t+m} = H_1...H_kC_t^{t+m}$$

and

$$y_t^* = B_{t+1}^{t+m}y_{t+m}^* = H_1...H_kC_t^{t+m}y_{t+m}^*,\ y_t^m = B_{t+1}^{t+m}y_{t+m} = H_1...H_kC_t^{t+m}y_{t+m}.$$

Thus, in view of (64),

$$|y_t^* - y_t^m| \leq M\rho^{k-1}$$

because $H_j \in D_\delta^n$. Therefore $y_t^m = \mathfrak{B}_t^m(y) \to y_t^*$ uniformly in y, since $k = k(m) \to \infty$ as $m \to \infty$. □

Suppose that the matrices $B_t = B_t(\omega) \in \mathcal{D}^n$ are random, i.e., $B_t(\omega)$ for each $t = 1, 2, ...$ is a measurable matrix function on the probability space $(\Omega, \mathcal{F}, P)$. Assume the following condition holds:

($\mathcal{B}$) For some $l \geq 0$ and $\delta > 0$, the matrix $B_t^{t+l}(\omega)$ belongs to $\mathcal{D}_\delta^n$ a.s. for all $t \geq 1$.

Proposition B.3. *Under assumption ($\mathcal{B}$), there exists a sequence $(y_t^*)_{t\geq 0}$ of measurable vector functions $y_t^*(\omega)$ with values in Δ such that*

$$B_{t+1}y_{t+1}^* = y_t^*,\ t \geq 0\ (a.s.). \tag{66}$$

The solution $(y_t^*)_{t\geq 0}$ to equation (66) is unique, and we have $y_t^*(\omega) \geq \delta e$ (a.s.). There exists a set $\Omega_1 \in \mathcal{F}$ with $P(\Omega_1) = 1$ such that for every $t \geq 0$ and $\omega \in \Omega_1$ the sequence $y_t^m(\omega) = \mathfrak{B}_t^m(y)(\omega)$ converges to $y_t^*(\omega)$ uniformly in $y \in \Delta^\infty$.

The uniqueness is understood in terms of stochastic equivalence: if $(y_t^{**}(\omega))_{t\geq 0}$ is another such sequence, then $y_t^{**}(\omega) = y_t^*(\omega)$ (a.s.) for all t.

Proof of Proposition B.3. By assumption, there exists a set $\Omega_1 \in \mathcal{F}$ of full measure such that for $\omega \in \Omega_1$ the matrix $B_t^{t+l}(\omega) \in \mathcal{D}_\delta^n$ for all $t \geq 1$. Take any $\omega \in \Omega_1$ and apply Proposition B.2. We obtain that there exists a sequence of vector functions $y_t^*(\omega)$ with values in Δ satisfying (65) for $\omega \in \Omega_1$. Fix some $d \in \Delta$ and define $y_t^*(\omega)$ as d for $\omega \in \Omega \backslash \Omega_1$. Then (*66*) will hold almost surely. Observe that the functions $y_t^*(\omega)$ are measurable because according to Proposition B.2 (applied to $y := (d, d, ...) \in \Delta^\infty$), we have

$$y_t^*(\omega) = \lim_{m\to\infty} B_{t+1}^{t+m}(\omega)d \text{ for } \omega \in \Omega_1.$$

To prove uniqueness suppose there is another sequence $\hat{y} = (\hat{y}_t(\omega))_{t\geq 0}$ satisfying (*66*) almost surely. Then by virtue of Proposition B.2, $\mathfrak{B}_t^m(\hat{y})(\omega) \to y_t^*(\omega)$ for $\omega \in \Omega_1$. On the other hand, $\mathfrak{B}_t^m(\hat{y})(\omega) = \hat{y}_t(\omega)$ (a.s.), and so $y_t^*(\omega) = \hat{y}_t(\omega)$ (a.s.).

Finally, $y_t^*(\omega) \geq \delta e$ (a.s.) because $y_t^*(\omega) = B_{t+1}^{t+l+1}(\omega) y_{t+l+1}^*(\omega)$, where $y_{t+l+1}^*(\omega) \in \Delta$ and $B_{t+1}^{t+l+1}(\omega) \in \mathcal{D}_\delta^n$ (a.s.). □

Let $A_1(\omega), A_2(\omega), ...$ be a sequence of random matrices. Consider the following condition:

($\mathcal{A}$) For each $t \geq 1$, the matrix $A_t(\omega)$ depends $\mathcal{F}_t$-measurably on ω, and there exist $l \geq 0$ and $\delta > 0$, such that the matrix $A_t^{t+l}(\omega) := A_t(\omega)...A_{t+l}(\omega)$ belongs to $\mathcal{D}_\delta^n$ a.s. for all $t \geq 1$.

Theorem B.1. *Under assumption ($\mathcal{A}$), there exists a sequence $(x_t^*(\omega))_{t\geq 0}$ of vector functions with values in Δ such that $x_t^*(\omega)$ is $\mathcal{F}_t$-measurable and*

$$E_t A_{t+1} x_{t+1}^* = x_t^* \text{ (a.s.)}, \ t \geq 0. \tag{67}$$

This sequence is unique up to stochastic equivalence, and we have

$$x_t^* \geq \delta e \text{ (a.s.)}. \tag{68}$$

Proof. Fix any (non-random) matrix $B^1 \in \mathcal{D}_\delta^n$ and define

$$B_1 = B^1, \ B_t := A_{t-1}, \ t \geq 2. \tag{69}$$

By applying Proposition B.3 to the sequence of matrices $(B_t)_{t\geq 1}$ defined by (69), we obtain that there exists a sequence $y_t^*(\omega)$, $t \geq 1$, of measurable vector functions with values in Δ such that

$$A_t y_{t+1}^* = y_t^*, \ t \geq 1. \tag{70}$$

Define

$$x_t^* = E_t y_{t+1}^*, \ t \geq 0.$$

From (70) we get

$$E_t A_{t+1} y_{t+2}^* = E_t y_{t+1}^*, \ t \geq 0.$$

Therefore

$$E_t A_{t+1} x_{t+1}^* = E_t A_{t+1} E_{t+1} y_{t+2}^* = E_t E_{t+1} A_{t+1} y_{t+2}^* = E_t y_{t+1}^* = x_t^*,$$

and so the sequence $(x_t^*)_{t\geq 0}$ satisfies (67).

Suppose there is another sequence $(\hat{x}_t)_{t\geq 0}$ satisfying $E_t A_{t+1} \hat{x}_{t+1} = \hat{x}_t$ (a.s.) for all $t \geq 0$. Then we have

$$\hat{x}_t = E_t A_{t+1} \hat{x}_{t+1} = E_t A_{t+1} E_{t+1} A_{t+2} \hat{x}_{t+2}$$

$$= E_t E_{t+1} A_{t+1} A_{t+2} \hat{x}_{t+2} = E_t A_{t+1} A_{t+2} \hat{x}_{t+2} \text{ (a.s.)}.$$

Continuing this process, we get

$$\hat{x}_t = E_t A_{t+1} ... A_{t+m} \hat{x}_{t+m} = E_t A_{t+1}^{t+m} \hat{x}_{t+m} \text{ (a.s.)}. \tag{71}$$

By using Proposition B.3, we obtain $A_{t+1}^{t+m} \hat{x}_{t+m} = B_{t+2}^{t+1+m} \hat{x}_{t+m} \to y_{t+1}^*$ (a.s.), consequently,

$$\hat{x}_t = E_t A_{t+1}^{t+m} \hat{x}_{t+m} \to E_t y_{t+1}^* = x_t^* \text{ (a.s.)}, \tag{72}$$

and so $\hat{x}_t = x_t^*$ (a.s.). □

We conclude this Appendix by formulating and proving Theorem B.2—the result on the existence and uniqueness of the Λ^* strategy in the model studied in the present chapter. Let $(\rho_t)_{t\geq 1}$ be a sequence of $\mathcal{F}_t$-measurable random vectors $\rho_t = (\rho_{t,1}, ..., \rho_{t,n})$ such that $0 \leq \rho_{t,i} \leq 1$, and $(R_t)_{t\geq 1}$ a sequence of $\mathcal{F}_t$-measurable random vectors $R_t = (R_{t,1}, ..., R_{t,n})$ satisfying

$$R_t \geq 0, \ \sum_{i=1}^{n} R_{t,i} = 1.$$

Recall that Λ^* was defined as the solution to equation (13). To prove that this solution exists and is unique let us define for each $t \geq 0$ the linear operator A_{t+1}:

$$(A_{t+1}x)_i = \rho_{t+1,i}x_i + (\sum_{m=1}^{n} x_m - \sum_{m=1}^{n} \rho_{t+1,m}x_m)R_{t+1,i}$$

$$= \rho_{t+1,i}x_i + \sum_{m=1}^{n}(1 - \rho_{t+1,m})x_m R_{t+1,i}.$$

This operator transforms Δ into itself, and for $x \in \Delta$ we have

$$(A_{t+1}x)_i = \rho_{t+1,i}x_{t+1,i} + (1 - \sum_{m=1}^{n} \rho_{t+1,m}x_{t+1,m})R_{t+1,i}.$$

Consequently, equation (13) can be written in the form (67) (with obvious changes in notation).

Let us introduce the following condition.

($\mathcal{R}$) There exist constants $\gamma > 0$ and $l \geq 0$ such that for each t and i, we have

$$\max_{1 \leq m \leq l} R_{t+m,i} \geq \gamma. \tag{73}$$

Theorem B.2. *Suppose that condition ($\mathcal{R}$) holds and there exists a constant $\theta > 0$ such that $\min\{\rho_{t,i},\ 1 - \rho_{t+1,i}\} \geq \theta$ for all t and i. Then a solution $(x_t^*)_{t\geq 0}$ to equation (67) exists, is unique up to stochastic equivalence, and satisfies (68) for some $\delta > 0$.*

Proof. Take any $x \in \Delta$ and define recursively $x_{t+1} = A_{t+1}x$, and $x_{t+m+1} = A_{t+m+1}x_{t+m}$. Then we have

$$(A_{t+1}x)_i = \rho_{t+1,i}x_i + \sum_{m=1}^{n}(1 - \rho_{t+1,m})x_m R_{t+1,i} \geq \theta(x_i + R_{t+1,i}),$$

which yields

$$x_{t+l} \geq \theta^l x_{t,i} + \theta^l R_{t+1,i} + \theta^{l-1}R_{t+2,i} + \ldots + \theta R_{t+l,i} \geq \theta^l \max_{1\leq m\leq l} R_{t+m,i} \geq \theta^l\gamma.$$

Thus condition ($\mathcal{A}$) holds with $\delta := \theta^l\gamma$, and so Theorem B.2 follows from Theorem B.1.

CHAPTER 3. AN EVOLUTIONARY FINANCE MODEL WITH ENDOGENOUS DIVIDENDS

In models considered in the current literature on Evolutionary Finance, asset payoffs or dividends are given exogenously and do not depend on investment strategies of the market players. In reality, however, such a dependence is more a rule than an exception. This chapter develops a model with endogenous asset payoffs determined by the fraction of total market wealth invested in each particular asset. The main results of this chapter are published in Amir et al. [8].

3.1. The model: Characteristic features and motivation

The present chapter draws on previous work by Evstigneev et al. [54], where a prototype of the model studied here was developed and some versions of the results of the present chapter were obtained. The main novelty of the modeling framework considered here lies in the fact that the dividends paid by the assets depend not only on exogenous random factors but also on the fraction of total market wealth invested in each particular asset. This is an important extension of EF models since in reality dividends are produced by firms that use capital as one of their inputs. The more wealth is invested in the outstanding stocks of a company, the easier it is to raise new capital and thus to produce more dividends.

This claim follows from a long tradition of capital-based asset pricing models. First Tobin [169] claimed and Tobin and Brainard [170] gave evidence that firms increase their capital stock if their market value increases above the value of their capital in place, i.e. above their book value. This is the famous q-theory of investments according to which the ratio of book to market is essential for investments. Li et al. [112] estimate the production function that is implicit in q-theory as a concave power function determining profits from the amount of capital employed by the firm.

Finally, as Lintner [114] first showed, dividends are a fixed proportion of profits, so that our assumption that dividends depend on the market capitalization of the firm has a good foundation in finance. Moreover, Fama and French [60] demonstrated using market data that firms with larger capitalization and profitability are more likely to pay dividends. Namely, they examined the factors affecting dividends using historical data on dividend payers during 1929–1999 period with a special attention to the period after 1972, when the data include NASDAQ, NYSE and AMEX firms. Fama and French concluded, not surprisingly, that dividend payers tend to be several times larger then non-payers, while non-payers are typically relatively small with broad growth opportunities. Other studies focusing on firms' characteristics and patterns in its life cycle depending on whether a firm pays dividends or not include DeAngelo, DeAngelo and Stulz [45], Kuo, Phillip and Zhang [105], and Li and Lie [111]. These papers support observations in [60] pointing out positive correlation between a firm's market capitalization and dividends.

Fig. 1 below shows market capitalization and dividend data of three firms (General Motors, Exxon and Procter&Gamble) that have dividend payouts in each year of the sample 1981–2019. The data are available in the online appendix to the paper Amir et al. [8].[4] We fit a dividend production function of the form $(c_1 b)^{c_2}$, where b is the firm's capital, and c_1, c_2 are firm-specific and estimated. We find that the average relation between the market capitalization of dividend-paying firms and their total cash dividends is given by such a concave function. Of course these dividend functions differ across firms. A limitation of our current model is that it does not capture firms that do not pay dividends or make any other disbursements to shareholders. The solid lines on Fig.1 are the result of a linear regression of logarithms of both quantities and estimated separately for each firm. To compare the shape of the dividend production function across companies, each firm's market capitalization and dividend payout is divided by the firm's maximum market capitalization in the sample. General Motors is shown in green, Exxon is in red, and Procter & Gamble is in blue. The fits, measured as R^2, are 0.73, 0.83, and 0.92, respectively.

[4] https://www.pnas.org/content/suppl/2021/06/24/2016514118.DCSupplemental

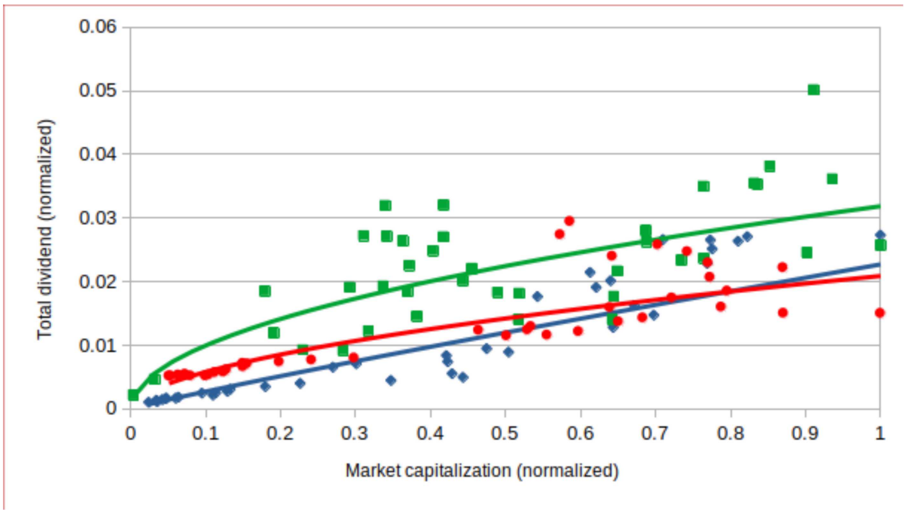

Fig. 1

To the best of our knowledge, we present the first rigorous EF stock market model with endogenous dividends[5]. On the other hand, this important extension of the EF model comes at a cost. For this chapter we limit the attention to so-called fixed-mix strategies, which hold the investment proportions constant over time. We note that the class of fixed-mix, or constant proportions, strategies we consider in this work is quite common in financial theory and practice; see e.g. Perold and Sharpe [136], Mulvey and Ziemba [130], and Browne [36].[6] Moreover, recent empirical evidence by DeMiguel et al. [46] has shown that even the simplest fix-mix strategy that invests the same fraction in all assets is at least as good as sophisticated mean-variance optimization strategies. Thus from the practical and from the theoretical standpoints, this class of strategies provides a convenient laboratory for the analysis of questions we are interested in. It makes it possible to formalize in a clear and compact way the concept of a *type* ("genetic code") of an investment strategy, which determines the evolutionary performance of its portfolio rule in the long run. From the practical standpoint, fixed-mix strategies are of importance since under

[5] A similar feedback effect has been studied by Cherkashin et al. [39] in a much simpler setting. They analyze a model with short-lived assets – one for each state of the world - in which the probability of occurrence of a state of the world depends on the amount invested in the asset paying off in that state.

[6] In fact, such strategies are routinely solicited and used by pension and investment funds, such as the TIIA-CREF and Vanguard.

certain general conditions they might lead to endogenous growth of wealth—volatility-induced growth, see Dempster et al. [47]. Finally, it should be noted that in models with i.i.d. random factors, fixed-mix strategies typically outperform all the others (see [55]), and we conjecture that this is the case for the model at hand, though a proof of this conjecture is not available at this point.[7]

The strategies determine the *ecology* of the market and its random dynamics over time. In the evolutionary perspective, the outcome of survival or extinction of investment strategies is governed by the long-run behaviour of the relative wealth of the strategies, which in turn depends on the combination of the strategies in the ecology. A strategy is said to *survive* if it generates with probability one a strictly positive share of market wealth, bounded away from zero, over an infinite time horizon, irrespective of the set of investment strategies in the ecology. It is said to become *extinct* if the share of market wealth corresponding to it tends to zero.

An investment strategy, λ^* is called *evolutionarily stable* if the following condition holds. Suppose the ecology consists of $N-1$ strategies $2, 3, ..., N$ (*non-mutants*), and a new strategy 1 (*mutant*) enters the existing ecology, and moreover, the initial share of wealth of this new strategy is small enough. Then the new strategy 1 will be driven out of the market by the other strategies in the long run: its market share will tend to zero with probability one as time goes to infinity. This definition combines ideas from two fundamental solution concepts of Evolutionary Game Theory proposed by Maynard Smith and Price [128] and Schaffer [146]. We provide an effective construction of the evolutionarily stable strategy λ^* and trace its links to the famous Kelly portfolio rule of *betting your beliefs*, see Kelly [100], Breiman [34], Thorp [168], Algoet and Cover [2], and Hakansson and Ziemba [79].

Our main result – Theorem 1 stated in the next section – demonstrates the existence and uniqueness of an evolutionary stable strategy (ESS) for our model. This result makes an important contribution to the asset pricing as well as to the portfolio theory aspect of evolutionary finance. Our result recommends investors to structure their portfolio based on fundamentals such as dividends. Moreover, the portfolio should be completely diversified and needs to be rebalanced over time, i.e. the investment proportions need to be restored after deviations resulting

[7]Numerical simulations of the model are described in section 3.7.

from price changes. If these rules are followed by all investors then any other investment strategy will lose wealth relative to this fundamental strategy. And if the market were governed by another strategy then this strategy could not survive since there exist better strategies that will gain against the incumbent strategy even when initially the entrant strategy has little wealth. Thus, in order to survive, it is necessary to follow the strategy identified in Theorem 1.

Theorem 1 gives support to the discounted cash flow rule, which is the classical asset pricing rule in traditional finance models with utility maximization given rational expectations. The price of any equity should be equal to the discounted sum of its future dividends[8]. However, there are important differences. First, Theorem 1 shows that in order to survive one needs to discount the future *relative* dividends. Second, observing these prices as the market outcome is more likely since this is the unique ESS; but this is not guaranteed because global stability is unresolved. Finally, note that without rational expectations one would have to learn the process determining future dividends. A natural approach would be to estimate it from the history of dividends one has observed. As our model shows, this might however be misleading since actual dividends depend on the wealth invested in the assets. By the interaction of the heterogeneous strategies in the market we would expect to see quite complicated trajectories of realized dividends. Nevertheless, the ESS does not depend on the ecology of the market, neither when strategies are still competing with each other nor when the ESS is established.

The intuition for our main result, the identification of an evolutionarily stable investment strategy and its characterization, is as follows. As the capital invested in a particular investment strategy increases, the assets that are overweighted (relative to the market portfolio) become more expensive, lowering their returns. Likewise, assets that are underweighted become cheaper and see their returns increase. Both forces are to the disadvantage of the investment strategy at hand (and to the benefit of other investment strategies that have 'opposing' weights). An evolutionarily stable investment strategy must therefore, with increasing capital, move prices

[8] The traditional argument goes as follows: The price of equity today should be equal to the discounted payoffs the equity holding entitles to next period, i.e. equal to the resale value and the dividends being paid. Iterating this argument forward, at any point in time the price is then equal to the discounted sum of all future dividends.

into a direction that does not offer such an advantage to other strategies. As it turns out, a fundamental value in relative terms provides these conditions. Thinking in terms of growth rates in random dynamical systems, an evolutionarily stable investment strategy must imply asset returns such that no other investment strategy can have a positive growth rate.

The structure of the remainder of the chapter is as follows. Section 3.2 describes the model. In Section 3.3, we formulate and discuss the main results. Section 3.4 contains some auxiliary propositions. Section 3.5 proves the main results. Section 3.6 examines an important example: the linear model.

3.2. The model

We consider a market where $K \geq 2$ *assets* are traded at moments of time $t = 0, 1, ...$ The total volume of each asset $k = 1, ..., K$ is constant (independent of time) and is denoted by V_k. There are $N \geq 2$ *investors* (*traders*) acting in the market. The market is influenced by random factors modeled in terms of a sequence of independent identically distributed elements $s_1, s_2, ...$ in a measurable space S. The random element s_t is interpreted as the "state of the world" at time/date t. The wealth of investor $i = 1, 2, ..., N$ at date $t \geq 1$ is denoted by $w_t^i = w_t^i(s^t)$, where $s^t := (s_1, ..., s_t)$ stands for the history of states of the world up to date t. Initial endowments $w_0^i > 0$ of all the investors at date 0 are given. An *investment strategy* (*portfolio rule*) of investor $i = 1, ..., N$ is represented by a vector $\lambda^i = (\lambda_1^i, ..., \lambda_K^i)$ in the unit simplex

$$\Delta^K := \{(a_1, ..., a_K) \in \mathbb{R}_+^K : a_1 + ... + a_K = 1\}.$$

An investor i at each time t allocates her wealth w_t^i across assets $k = 1, ..., K$ in constant (independent of time and random factors) proportions λ_k^i.

Given the set of investment strategies λ^i, $i = 1, ..., N$, the total amount allocated by all the investors $i = 1, ..., N$ for purchasing asset k at time t is expressed as

$$\langle \lambda_k, w_t \rangle := \sum_{i=1}^{N} \lambda_k^i w_t^i, \;\; \lambda_k := (\lambda_k^1, ..., \lambda_k^N), \;\; w_t := (w_t^1, ..., w_t^N), \;\; t = 0, 1, ... \tag{74}$$

At each time $t = 1, 2, ...$ assets $k = 1, ..., K$ pay dividends

$$D_{t,k} = D_k(s_t, W_{t-1,k}) \geq 0 \tag{75}$$

depending on the fraction

$$W_{t-1,k} := \frac{\langle \lambda_k, w_{t-1} \rangle}{\sum_{j=1}^K \langle \lambda_j, w_{t-1} \rangle} \tag{76}$$

of total market wealth

$$W_{t-1} := \sum_{j=1}^K \langle \lambda_j, w_{t-1} \rangle = \sum_{i=1}^I w_{t-1}^i \tag{77}$$

allocated to asset k. The functions $D_{t,k}(s, b)$, $b \in [0, 1]$, are assumed to be jointly measurable with respect to their arguments and satisfy

$$\sum_{k=1}^K D_{t,k} > 0. \tag{78}$$

We denote by $p_t = p_t(s^t) \in \mathbb{R}_+^K$ the vector of market prices of the assets. For each $k = 1, ..., K$, the coordinate $p_{t,k}$ of the vector $p_t = (p_{t,1}, ..., p_{t,K})$ stands for the price of one unit of asset k at date t. Below we describe how these prices are formed in equilibrium over each time period. A *portfolio* of investor i at date $t = 0, 1, ...$ is specified by a vector $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i) \in \mathbb{R}_+^K$ where $x_{t,k}^i$ is the amount (the number of units) of asset k in the portfolio x_t^i. The scalar product $\langle p_t, x_t^i \rangle = \sum_{k=1}^K p_{t,k} x_{t,k}^i$ expresses the value of the investor i's portfolio x_t^i at date t in terms of the prices $p_{t,k}$. The portfolio vector x_t^i depends on the history s^t of states of the world: $x_t^i = x_t^i(s^t)$. This vector function of s^t, as well as all the other functions of s^t we deal with, is measurable. To alleviate notation, we will often omit "s^t" in what follows.

At date $t = 0$ the investors' budgets are given by their (non-random) *initial endowments* $w_0^i > 0$. Investor i's budget/wealth at date $t \geq 1$ is

$$w_t^i = \langle D_t + p_t, x_{t-1}^i \rangle = \sum_{k=1}^K (D_{t,k} + p_{t,k}) x_{t-1,k}^i, \tag{79}$$

where

$$D_t := (D_{t,1}, ..., D_{t,K}), \ D_{t,k} = D_k(s_t, W_{t-1,k}), \ k = 1, ..., K. \tag{80}$$

The budget consists of two components: the dividends $\langle D_t, x_{t-1}^i \rangle$ paid by the yesterday's portfolio x_{t-1}^i and the market value $\langle p_t, x_{t-1}^i \rangle$ of x_{t-1}^i expressed in terms of the today's prices p_t. If investor i allocates the fraction λ_k^i of wealth w_k^i to asset k, then the number of units of asset k that can be purchased for this amount is

$$x_{t,k}^i = \rho \frac{\lambda_k^i w_t^i}{p_{t,k}}, \tag{81}$$

where $\rho \in (0,1)$ is the *transaction cost rate.* Thus, by employing the portfolio rule $\lambda^i = (\lambda_1^i, ..., \lambda_K^i)$, trader i constructs a portfolio whose positions are specified by (81).

Suppose that each investor i has selected some strategy $\lambda^i = (\lambda_1^i, ..., \lambda_K^i) \in \Delta^K$. Assume that the market is always in equilibrium: for all $t = 0, 1, ...$and $k = 1, ..., K$, total asset supply is equal to total asset demand

$$V_k = \sum_{i=1}^{N} x_{t,k}^i, \tag{82}$$

i.e.

$$V_k = \rho \sum_{i=1}^{N} \frac{\lambda_k^i w_t^i}{p_{t,k}}, \tag{83}$$

(see (81)). Then we get

$$p_{t,k} = \frac{\rho}{V_k} \sum_{i=1}^{N} \lambda_k^i w_t^i. \tag{84}$$

By combining (84) and (79), we obtain a system of equations that determines the equilibrium (market clearing) prices

$$p_{t,k} = \frac{\rho}{V_k} \sum_{i=1}^{N} \lambda_k^i \langle D_t + p_t, x_{t-1}^i \rangle, \ k = 1, ..., K. \tag{85}$$

It can be shown that a non-negative vector p_t satisfying these equations exists and is unique (for any s^t, $D_t \geq 0$ and any feasible x_{t-1}^i and λ_t^i) — see Proposition 1 in Section 4.

Given a strategy profile $(\lambda^1, ..., \lambda^N)$ of the investors and their initial endowments $w_0^1, ..., w_0^N$, we can, by using equations (79)–(85), generate recursively, the path of the system specified by the sequences of variables

$$w_t^i, \ i = 1, ..., N, \ t = 0, 1, ..., \tag{86}$$

(the investors' budgets)

$$W_{t,k}, \ k = 1, 2, ..., K, \ t = 0, 1, ..., \tag{87}$$

(the fractions of wealth allocated to each of the assets k)

$$p_t = (p_{t,1}, ..., p_{t,K}),\ t = 0, 1, ..., \tag{88}$$

(the vectors of equilibrium asset prices) and

$$x_t^i = (x_{t,1}^i, ..., x_{t,K}^i),\ i = 1, 2, ..., N,\ t = 0, 1, ... \tag{89}$$

(the investors' portfolios). The recursive procedure is as follows. For $t = 0$, the budgets $w_0^i > 0$ are given as initial endowments; $W_{0,k}$ are computed by the formula $W_{0,k} = \langle \lambda_k, w_0 \rangle / \sum_{j=1}^K \langle \lambda_j, w_0 \rangle$ (see (76)); the prices $p_{0,k}$ and the portfolio positions $x_{0,k}^i$ are obtained from equations (84) and (81):

$$p_{0,k} = \frac{\rho}{V_k} \sum_{i=1}^N \lambda_k^i w_0^i,\ x_{0,k}^i = \rho \frac{\lambda_k^i w_0^i}{p_{0,k}}. \tag{90}$$

Suppose the variables are defined up to some $t-1$. Then we define w_t^i by (79) and (80), and $W_{t,k}$ by (76). The prices p_t are determined by the system of equations (85), and the investors' portfolios by formula (81).

The above description of asset market dynamics requires clarification. Equations (83) and (81) make sense only if $p_{t,k} > 0$, or equivalently, if the aggregate demand for each asset (under the equilibrium prices) is strictly positive. Those strategy profiles which guarantee that the recursive procedure described above leads at each step to strictly positive equilibrium prices will be called *admissible*. In what follows, we will deal only with such strategy profiles. The hypothesis of admissibility guarantees that the random dynamical system under consideration is well-defined. Under this hypothesis, we obtain by induction that on the equilibrium path all the portfolios $x_t^i = (x_{t,1}^i, ..., x_{t,K}^i)$ are non-zero and the wealth $w_t^i := \langle D_t + p_t, x_{t-1}^i \rangle$ of each investor is strictly positive.

We give a simple sufficient condition for a strategy profile to be admissible. This condition will hold for all the strategy profiles we shall deal with in the present chapter, and in this sense it does not restrict generality. Suppose that some trader, say trader 1, uses a portfolio rule that always prescribes to invest into all the assets in strictly positive proportions λ_k^1. Then a strategy profile containing this portfolio rule is admissible. Indeed, for $t = 0$, we get from (90) that $p_{0,k} > 0$ and $x^1_{0,k} > 0$ for all k. Assuming that $x^1_{t-1} > 0$ and arguing by induction, we obtain $\langle D_t + p_t, x^1_{t-1}\rangle \geq \langle D_t, x^1_{t-1}\rangle > 0$ in view of (78), which in turn yields $p_t > 0$ and $x^1_t > 0$ by virtue of (85) and (81), as long as $\lambda^1_k > 0$.

3.3. The main results

Let $(\lambda^1, ..., \lambda^N)$ be an admissible strategy profile of the investors. Consider the path (86)–(89) of the asset market generated by this strategy profile and the given initial endowments $w^i_0 > 0,\ i = 1, ..., N$. We are primarily interested in the long-run behaviour of the *relative wealth* or the *market shares* $r^i_t := w^i_t/W_t$ of the traders, where $W_t = \sum_{i=1}^N w^i_t$ is the total market wealth. The main concept we analyze in this chapter is that of an *evolutionary stable strategy.*

Definition. A portfolio rule λ^* is called *evolutionary stable* if it possesses the following property. Suppose all the investors except one, say $i = 1$, use the strategy λ^*, while investor 1 uses some other strategy λ. Furthermore, suppose that the initial market share r^1_0 of investor 1 is small enough: $r^1_0 < \delta$, where $\delta > 0$ is some random variable. Then the market share r^1_t of trader 1 will tend to 0 almost surely, i.e. trader 1 will be driven out of the market by the other traders (using λ^*) with probability one.

The above definition of an evolutionary stable strategy combines two fundamental concepts of Evolutionary Game Theory: the classical definition of an evolutionary stable strategy (ESS) for continuous populations by Maynard Smith and Price [128] and its version for discrete populations proposed by Schaffer [146]. The analogy with the former lies in the fact that the initial relative wealth of the "mutant" (λ-investor) is assumed to be small enough; under this assumption, the λ-investor cannot survive in competition with "non-mutants" (λ^*-investors). A parallel with the latter is in the assumption that there is only one mutant type represented by the λ-investor 1; all the others, $2, 3, ..., N$, are non-mutants. Relative wealth is the counterpart of the relative mass of a continuous population of mutants or non-mutants in the biological context. A fundamental distinction between the notion introduced and the classical ones is that in the present EF setting we are dealing with properties holding *with probability one*, while the classical biological notions of evolutionary stability are concerned with frequencies, probability distributions and properties holding on average.

To formulate the main result of this work (Theorem 1 below) we introduce some assumptions and notation. Put

$$g_k(s,b) = V_k D_k(s,b).$$

The function $g_k(s,b)$ represents the total amount of dividends paid by all the assets k available in the market.

(G1) For each s and k the functions $g_k(s,b)$ ($b \in [0,1]$) are strictly positive, differentiable, strictly monotone increasing and concave in b.

(G2) For any $\lambda = (\lambda_1, ..., \lambda_K) \in \Delta^K$, the functions $g_k(s,\lambda_k)$ are linearly independent, i.e., if for some constants $a_1, ..., a_K$ the equality $\sum a_k g_k(s,\lambda_k) = 0$ holds for all s, then $a_1 = ... = a_K = 0$.

(G3) There exist constants $g'_{\max} > 0$ and $g_{\min} > 0$ such that

$$g'_{\max} < g_{\min} \tag{91}$$

and for all s, b and k we have

$$g_k(s,b) \geq g_{\min}, \; g'_k(s,b) \leq g'_{\max},$$

where $g'_k(s,b)$ stands for the derivative of the function $g_k(s,b)$ with respect to b.

Assumption (G1) contains standard regularity conditions on the functions $g_k(s,b)$ which are typical assumptions on a production function. Property (G2) means the absence of redundant assets: one cannot construct a "synthetic asset", a portfolio with fixed weights consisting of assets $j \neq k$, that yields the same dividends as any given asset k. Condition (G3) says that although the growth of the total investment in an asset k leads to the growth the dividend paid by this asset, this growth is moderate: its rate $g'_k(s,b) = V_k D'_k(s,b)$ cannot exceed the constant specified in (91). Such an assumption is natural when, in addition to capital, a second production factor (e.g., labor) is essential.

Theorem 1. *There exists a unique solution $\lambda^* = (\lambda^*_1, ..., \lambda^*_K) \in \Delta^K$ to the system of equations*

$$E\frac{g_k(s,\lambda^*_k)}{\sum_{m=1}^K g_m(s,\lambda^*_m)} = \lambda^*_k, \; k = 1,2,...,K. \tag{92}$$

We have $\lambda_k^* > 0$, $k = 1, ..., K$. The portfolio rule represented by the vector λ^* is evolutionary stable.

In (92) s is a random element in the space S having the same distribution as s_t $(t = 1, 2, ...)$. The symbol E stands for the expectation with respect to this distribution. The meaning of equation (92) is as follows. It says that the *relative dividends*

$$R_k^*(s) := \frac{g_k(s, \lambda_k^*)}{\sum_{m=1}^K g_m(s, \lambda_m^*)} = \frac{V_k D_k(s, \lambda_k^*)}{\sum_{m=1}^K V_m D_m(s, \lambda_m^*)}, \ k = 1, ..., K,$$

corresponding to the allocation of wealth across assets in the proportions $\lambda_1^*, ..., \lambda_K^*$ prescribed by the evolutionary stable portfolio rule λ^* coincide on average with these proportions.

If the functions $g_k(s, b)$ do not depend on b, equations (92) boil down to

$$\lambda_k^* = ER_k^*, \ k = 1, 2, ..., K.$$

In this case λ^* reduces to the prescription to invest in accordance with the expected relative dividends. This is the classical *Kelly portfolio rule* —"betting your beliefs" (Kelly [100])[9]. In EF models with exogenous dividends, stronger (global) versions of results of this kind were obtained in Evstigneev et al. [55] and Amir et al. [5].

3.4. Dynamics of the asset market

In this section we prove some auxiliary propositions needed for the proof of Theorem 1. They are concerned with the structure of the random dynamical system under consideration. The first proposition establishes the existence and uniqueness of an equilibrium price vector at each date $t \geq 0$.

[9] In the classical capital growth theory with exogenous asset returns (Kelly [100], Breiman [34], Algoet and Cover [2], and Hakansson and Ziemba [79]), the portfolio rule of "betting your beliefs" is obtained as a result of the maximization of the expected logarithm of the portfolio return. In our game-theoretic setting, where the performance of a strategy depends not only on the strategy itself but on the whole strategy profile, the evolutionary stable portfolio rule cannot be obtained as a solution to a single-agent optimization problem with a logarithmic, or any other, objective function.

Proposition 1. *Let $x_{t-1} = (x^1_{t-1}, ..., x^N_{t-1})$ be a set of vectors $x^i_{t-1} \in \mathbb{R}^K_+$ satisfying (82). Then for any s^t there exists a unique solution $p_t \in \mathbb{R}^K_+$ to equations (85). This solution is measurable with respect to all the parameters involved in (85).*

Proof. Fix some t and s^t and consider the operator transforming a vector $p = (p_1, ..., p_K) \in \mathbb{R}^K_+$ into the vector $q = (q_1, ..., q_K) \in \mathbb{R}^K_+$ with coordinates

$$q_k = \rho V_k^{-1} \sum_{i=1}^{N} \lambda^i_k \langle D_t + p, x^i_{t-1} \rangle.$$

This operator is contracting in the norm $||p||_V := \sum_k |p_k| V_k$. Indeed, we have

$$||q - q'||_V = \sum_{k=1}^{K} |q_k - q'_k| V_k \le \rho \sum_{k=1}^{K} \sum_{i=1}^{N} \lambda^i_k |\langle p - p', x^i_{t-1} \rangle|$$

$$= \rho \sum_{i=1}^{N} \sum_{k=1}^{K} \lambda^i_k |\langle p - p', x^i_{t-1} \rangle| = \rho \sum_{i=1}^{N} |\langle p - p', x^i_{t-1} \rangle|$$

$$= \rho \sum_{i=1}^{N} \sum_{m=1}^{K} |p_m - p'_m| x^i_{t-1,m} = \rho \sum_{m=1}^{K} \sum_{i=1}^{N} |p_m - p'_m| x^i_{t-1,m}$$

$$= \rho \sum_{m=1}^{K} |p_m - p'_m| V_m = \rho ||p - p'||_V.$$

By using the contraction principle, we obtain the existence, uniqueness and measurability of the solution to (85). □

In the next proposition, we derive a system of equations governing the dynamics of the market shares of the investors given their admissible strategy profile $(\lambda^1, ..., \lambda^N)$. Consider the path (86)–(89) of the random dynamical system generated by the strategy profile $(\lambda^1, ..., \lambda^N)$ and the sequence of vectors $r_t = (r^1_t, ..., r^N_t)$, where r^i_t is the investor i's market share at date t.

Denote by $R_{t,k}$ the *relative dividends*

$$R_{t,k} := \frac{D_{t,k} V_k}{\sum_{m=1}^{K} D_{t,m} V_m}, \ t = 1, 2, ..., \ k = 1, ..., K. \tag{93}$$

Note that $R_{t,k} > 0$ and $\sum R_{t,k} = 1$.

Proposition 2. *The following equations hold:*

$$r^i_{t+1} = \sum_{k=1}^{K} [\rho \langle \lambda_k, r_{t+1} \rangle + (1 - \rho) R_{t+1,k}] \frac{\lambda^i_k r^i_t}{\langle \lambda_k, r_t \rangle}, \tag{94}$$

$i = 1, ..., N$, $t \geq 0$.

Proof. From (85) and (81) we get

$$p_{t,k} = \rho V_k^{-1} \sum_{i=1}^{N} \lambda_k^i \langle p_t + D_t, x_{t-1}^i \rangle = \rho V_k^{-1} \sum_{i=1}^{N} \lambda_k^i w_t^i = \rho V_k^{-1} \langle \lambda_k, w_t \rangle,$$

$$x_{t,k}^i = \frac{V_k \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle},$$

where $t \geq 1$, $w_t := (w_t^1, ..., w_t^N)$ and $\lambda_k := (\lambda_k^1, ..., \lambda_k^N)$. The analogous formulas for $t = 0$,

$$p_{0,k} = \rho V_k^{-1} \langle \lambda_k, w_0 \rangle, \; x_{0,k}^i = \frac{V_k \lambda_k^i w_0^i}{\langle \lambda_k, w_0 \rangle},$$

follow from (90). Consequently, we have

$$w_{t+1}^i = \sum_{k=1}^{K} (p_{t+1,k} + D_{t+1,k}) x_{t,k}^i = \sum_{k=1}^{K} (\rho \frac{\langle \lambda_k, w_{t+1} \rangle}{V_k} + D_{t+1,k}) \frac{V_k \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}$$

$$= \sum_{k=1}^{K} (\rho \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_k) \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle}, \; t \geq 0. \tag{95}$$

By summing up these equations over $i = 1, ..., N$, we obtain

$$W_{t+1} = \sum_{k=1}^{K} (\rho \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_k) \frac{\sum_{i=1}^{N} \lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} =$$

$$\sum_{k=1}^{K} (\rho \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_k) = \rho W_{t+1} + \sum_{k=1}^{K} D_{t+1,k} V_k.$$

This implies:

$$W_{t+1} = \frac{1}{1-\rho} \sum_{m=1}^{K} D_{t+1,m} V_m. \tag{96}$$

From (95) we find

$$w_{t+1}^i = \sum_{k=1}^{K} (\rho \langle \lambda_k, w_{t+1} \rangle + D_{t+1,k} V_k) \frac{\lambda_k^i w_t^i}{\langle \lambda_k, w_t \rangle} \; t \geq 0.$$

Dividing both sides of this equation by W_{t+1} and using (96), we get

$$r_{t+1}^i = \sum_{k=1}^{K} [\rho \langle \lambda_k, r_{t+1} \rangle + (1-\rho) \frac{D_{t+1,k} V_k}{\sum_{m=1}^{K} D_{t+1,m} V_m}] \frac{\lambda_k^i w_t^i / W_t}{\langle \lambda_k, w_t \rangle / W_t},$$

which yields (94) by virtue of (93). □

Let us observe that it is sufficient to prove Theorem 1 when $N = 2$, i.e., the general model can be reduced to the case of two investors. Indeed, suppose that investor 1 uses some strategy λ and all the others follow the strategy λ^*. By setting

$$r_t^* := \sum_{i=2}^{N} r_t^i,$$

we get $r_t^1 = 1 - r_t^*$, and from equations (94) we obtain

$$r_{t+1}^1 = \sum_{k=1}^{K} [\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_{t+1,k}] \frac{\lambda_k^1 r_t^1}{\lambda_k^1 r_t^1 + \lambda_k^* r_t^*},$$

$$r_{t+1}^* = \sum_{k=1}^{K} [\rho\langle\lambda_k, r_{t+1}\rangle + (1-\rho)R_{t+1,k}] \frac{\lambda_k^* r_t^*}{\lambda_k^1 r_t^1 + \lambda_k^* r_t^*}.$$

Thus the dynamics of the market share r_t^1 of the λ-investor is the same as if she faces as the rival the λ^*-investor with the market share equal to $r_t^1 = 1 - r_t^* = 1 - \sum_{i=2}^{N} r_t^i$. Our goal is to show that $r_t^1 \to 0$ with probability one as long as r_0^1 is small enough.

The model with two investors will be examined by using Proposition 3 below. Assume that investors $i = 1, 2$ use strategies $\lambda^i = (\lambda_1^i, ..., \lambda_K^i), i = 1, 2$, and denote by x_t the ratio $r_t^1 / r_t^2 = w_t^1 / w_t^2$ of their market shares.

Proposition 3. *The process x_t is governed by the random dynamical system*

$$x_{t+1} = H(s_{t+1}, x_t),\ t = 0, 1, ...,$$

where

$$H(s_{t+1}, x) = x \frac{\sum_{k=1}^{K} [\rho\lambda_k^2 + (1-\rho)R_k(s_{t+1}, x)] \dfrac{\lambda_k^1}{\lambda_k^1 x + \lambda_k^2}}{\sum_{k=1}^{K} [\rho\lambda_k^1 + (1-\rho)R_k(s_{t+1}, x)] \dfrac{\lambda_k^2}{\lambda_k^1 x + \lambda_k^2}} \tag{97}$$

and

$$R_k(s_{t+1}, x) = \frac{g_k\left(s_{t+1}, \dfrac{\lambda_k^1 x + \lambda_k^2}{\sum_{j=1}^{K} (\lambda_j^1 x + \lambda_j^2)}\right)}{\sum_{m=1}^{K} g_m\left(s_{t+1}, \dfrac{\lambda_m^1 x + \lambda_m^2}{\sum_{j=1}^{K} (\lambda_j^1 x + \lambda_j^2)}\right)}. \tag{98}$$

Proof. By using (94) with $N = 2$, we get

$$r^i_{t+1} = \sum_{k=1}^{K} [\rho(\lambda^i_k r^i_{t+1} + \lambda^j_k (1 - r^i_{t+1})) + (1-\rho) R_{t+1,k}] \frac{\lambda^i_k r^i_t}{\lambda^i_k r^i_t + \lambda^j_k r^j_t},$$

where $i, j \in \{1, 2\}$ and $i \neq j$. Setting $C^{ij}_{t,k} := \lambda^i_k r^i_t / (\lambda^i_k r^i_t + \lambda^j_k r^j_t)$, we obtain

$$r^i_{t+1}[1 + \rho \sum_{k=1}^{K} (\lambda^j_k - \lambda^i_k) C^{ij}_k] = \sum_{k=1}^{K} [\rho \lambda^j_k + (1-\rho) R_{t+1,k}] C^{ij}_{t,k}.$$

Thus

$$\frac{r^i_{t+1}}{r^j_{t+1}} = \frac{A^{ij}_{t+1}/B^{ij}_{t+1}}{A^{ji}_{t+1}/B^{ji}_{t+1}},$$

where

$$A^{ij}_{t+1} := \sum_{k=1}^{K} [\rho \lambda^j_k + (1-\rho) R_{t+1,k}] C^{ij}_{t,k},$$

$$B^{ij}_{t+1} := 1 + \rho \sum_{k=1}^{K} (\lambda^j_k - \lambda^i_k) C^{ij}_{t,k}.$$

Observe that $B^{ji}_{t+1} = B^{ij}_{t+1}$. Indeed,

$$B^{ij}_{t+1} - B^{ji}_{t+1} = \rho \sum_{k=1}^{K} [(\lambda^j_k - \lambda^i_k) C^{ij}_{t,k} - (\lambda^i_k - \lambda^j_k) C^{ji}_{t,k}] =$$

$$\rho \sum_{k=1}^{K} (\lambda^j_k - \lambda^i_k) = 0$$

because $C^{ij}_{t,k} + C^{ji}_{t,k} = 1$. Consequently,

$$\frac{r^1_{t+1}}{r^2_{t+1}} = \frac{A^{12}_{t+1}}{A^{21}_{t+1}} = \frac{r^1_t}{r^2_t} \frac{\sum_{k=1}^{K} [\rho \lambda^2_k + (1-\rho) R_{t+1,k}] \dfrac{\lambda^1_k}{\lambda^1_k r^1_t / r^2_t + \lambda^2_k}}{\sum_{k=1}^{K} [\rho \lambda^1_k + (1-\rho) R_{t+1,k}] \dfrac{\lambda^2_k}{\lambda^1_k r^1_t / r^2_t + \lambda^2_k}},$$

which proves (97).

Finally, we get

$$R_{t+1,k} = \frac{D_{t+1,k} V_k}{\sum_{m=1}^{K} D_{t+1,m} V_m} = \frac{g_k(s_{t+1}, W_{t,k})}{\sum_{m=1}^{K} g_m(s_{t+1}, W_{t,m})}, \tag{99}$$

where (see (76))

$$W_{t,k} = \frac{\langle \lambda_k, w_t \rangle}{\sum_{j=1}^{K} \langle \lambda_j, w_t \rangle} = \frac{\lambda^1_k w^1_t + \lambda^2_k w^2_t}{\sum_{j=1}^{K} (\lambda^1_j w^1_t + \lambda^2_j w^2_t)} = \frac{\lambda^1_k x_t + \lambda^2_k}{\sum_{j=1}^{K} (\lambda^1_j x_t + \lambda^2_j)},$$

which yields (98). □

3.5. Proofs of the main results

In this section we give a proof of Theorem 1. We begin with proving the existence, uniqueness and strict positivity of the solution $y = (y_1, ..., y_K)$ to the system of equations

$$E\frac{g_k(s, y_k)}{\sum_{m=1}^{K} g_m(s, y_m)} = y_k,\ k = 1, 2, ..., K. \tag{100}$$

In fact, existence is straightforward from Brouwer's fixed-point theorem, and strict positivity follows from the strict positivity of g_k, so that we only need to establish uniqueness (however, the argument below yields existence as well).

We will fix s and omit it in the notation. Consider the mapping $F(y) = F(s, y)$ assigning to a vector $y = (y_1, ..., y_K) \in \mathbb{R}_+^K$ the vector

$$F(y) := (F_1(y), ..., F_K(y)) := |y|\, G(y),$$

where

$$G(y) := (G_1(y), ..., G_K(y)),\quad G_k(y) := \frac{g_k(y_k\, |y|^{-1})}{\sum_{m=1}^{K} g_m(y_m\, |y|^{-1})}$$

and $|y| = \sum_k y_k$. Observe that $G(y)$ is homogeneous of degree 0, $F(y)$ is homogeneous of degree 1, and if $|y| = 1$, then we have

$$F_k(y) = \frac{g_k(y_k)}{\sum_{m=1}^{K} g_m(y_m)}.$$

Thus the mapping $F(y)$ is a homogeneous of degree 1 extension (to the whole non-negative cone $\mathbb{R}_+^K$) of the mapping of Δ^K into itself appearing under the sign of expectation in (100).

Suppose we have found $y \in R_+^K$, $y \neq 0$, such that

$$EF(y) = \lambda y. \tag{101}$$

Then $\lambda = 1$, and $z := y\, |y|^{-1}$ is a solution to (100). Indeed, if $EF(y) = |y|\, EG(y) = \lambda y$, then

$$EG(z) = EG(y) = \lambda y\, |y|^{-1} = \lambda z,$$

where $\lambda = 1$ because $|z| = 1$ and $|EG(z)| = E\,|G(z)| = 1$.

The existence and uniqueness of a solution to (101) follows from a nonlinear version of the Perron-Frobenius theorem, holding under the assumption of strict monotonicity of the mapping

$EF(y)$ (Kohlberg [102]), which follows from the strict monotonicity of the mapping $F(y) = F(s,y)$ holding for each s:

$$\frac{\partial F_k(y)}{\partial y_j} > 0 \tag{102}$$

for each k,j and $y \in \mathbb{R}_+^K$. To prove (102) put

$$D := \sum_{m=1}^{K} g_m(y_m\,|y|^{-1}).$$

We have

$$\frac{\partial F_k(y)}{\partial y_j} = \frac{C}{D^2},$$

where

$$C = D\frac{\partial}{\partial y_j}[|y| \cdot g_k(y_k\,|y|^{-1})] - |y| \cdot g_k(y_k\,|y|^{-1}) \cdot \frac{\partial D}{\partial y_j} =$$

$$D\{g_k(y_k\,|y|^{-1}) + |y|\,\frac{\partial}{\partial y_j}[g_k(y_k\,|y|^{-1})]\} - |y|\,g_k(y_k\,|y|^{-1})\sum_{m=1}^{K}\frac{\partial}{\partial y_j}[g_m(y_m\,|y|^{-1})]$$

$$\frac{\partial}{\partial y_j}[g_k(y_k\,|y|^{-1})] = g_k'(y_k\,|y|^{-1}) \cdot \frac{|y|\delta_{j,k} - y_k}{|y|^2}, \tag{103}$$

where

$$\delta_{j,k} = 1 \text{ if } j = k \text{ and } \delta_{j,k} = 0 \text{ if } j \neq k.$$

Since the function $F_k(y)$ is homogeneous of degree 1, its partial derivatives are homogeneous of degree 0, and so we can assume without loss of generality that $|y| = 1$. In the chain of relations below, we use the inequalities $g_k(y_k) - g_k'(y_k)y_k \geq 0$, following from the concavity of $g_k(b)$. We have from (103)

$$D = \sum_{m=1}^{K} g_m(y_m),$$

$$C = [\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) + g_k'(y_k)(\delta_{j,k} - y_k)] - g_k(y_k)\sum_{m=1}^{K} g_m'(y_m)(\delta_{j,m} - y_m)] \geq$$

$$[\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) - g_k'(y_k)y_k)] + g_k(y_k)[\sum_{m=1}^{K} g_m'(y_m)y_m] - g_k(y_k)g_j'(y_j) \geq \tag{104}$$

$$g_k(y_k)[g_k(y_k) - g_k'(y_k)y_k)] + g_k(y_k)[\sum_{m=1}^{K} g_m'(y_m)y_m] - g_k(y_k)g_j'(y_j).$$

Finally, we get

$$[g_k(y_k) - g_k'(y_k)y_k)] + [\sum_{m=1}^{K} g_m'(y_m)y_m] - g_j'(y_j) \geq$$

$$g_k(y_k) - g_k'(y_k)y_k + g_k'(y_k)y_k - g_j'(y_j) =$$

$$g_k(y_k) - g_j'(y_j) > 0,$$

where the last inequality follows from the assumption (G3).

To complete the proof of Theorem 1 it remains to prove the property of evolutionary stability of the portfolio rule λ^*. By virtue of Evstigneev et al. [58, Theorem 1 and Section 6], it is sufficient to show that if $\lambda \neq \lambda^*$, then

$$E \ln H'(s,0) < 0, \tag{105}$$

where $H(s,0)$ is defined by (97) and (98) with $\lambda^1 = \lambda$ and $\lambda^2 = \lambda^*$. We have

$$R_k(s,0) = \frac{g_k\left(s, \lambda_k^*\right)}{\sum_{m=1}^{K} g_m\left(s, \lambda_m^*\right)}$$

and so

$$H'(s,0) = \sum_{k=1}^{K} \mu_k^*(s) \frac{\lambda_k}{\lambda_k^*}, \tag{106}$$

where

$$\mu_k^*(s) := \rho\lambda_k^* + (1-\rho)\frac{g_k(s, \lambda_k^*)}{\sum_{m=1}^{K} g_m(s, \lambda_m^*)}, \ \ k = 1, ..., K. \tag{107}$$

Observe that

$$E\mu_k^*(s) = \rho\lambda_k^* + (1-\rho)\lambda_k^* = \lambda_k^* \tag{108}$$

by virtue of (92).

Let us prove that the functions $\mu_k^*(s)$ are linearly independent (this will be needed below). Suppose $\sum_k a_k\mu_k^*(s) = 0$ for all s. Let us show that $a_1 = ... = a_K = 0$. Put

$$\gamma_k^*(s) := \frac{g_k(s, \lambda_k^*)}{\sum_{m=1}^{K} g_m(s, \lambda_m^*)}. \tag{109}$$

The equality

$$\sum_{k=1}^{K} a_k \mu_k^*(s) = \sum_{k=1}^{K} a_k[\rho\lambda_k^* + (1-\rho)\gamma_k^*(s)] = 0$$

implies

$$[\sum_{k=1}^{K} \gamma_m^*(s)] \sum_{m=1}^{K} a_k \rho\lambda_k^* + (1-\rho) \sum_{k=1}^{K} a_k \gamma_k^*(s) = 0$$

because $\sum_{m=1}^{K} \gamma_m^*(s) = 1$. This can be written (replacing k by m in the second sum):

$$[\sum_{k=1}^{K} \gamma_m^*(s)] \sum_{m=1}^{K} a_k \rho\lambda_k^* + (1-\rho) \sum_{m=1}^{K} a_m \gamma_m^*(s) = 0,$$

or equivalently,

$$\sum_{m=1}^{K} \gamma_m^*(s)[\rho \sum_{k=1}^{K} a_k \lambda_k^* + (1-\rho) a_m^*] = 0.$$

The functions $\gamma_m^*(s)$, $m = 1, 2, ..., K$, are linearly independent because the functions $g_k(s, \lambda_k^*)$ are linearly independent (see (109) and assumption (G2)). Therefore

$$\rho \sum_{k=1}^{K} a_k \lambda_k^* + (1-\rho) a_m = 0, \ m = 1, 2, ..., K,$$

and so for all m we have $a_m = A$, where

$$A := -\rho(1-\rho)^{-1} \sum_{k=1}^{K} a_k \lambda_k^*.$$

Consequently,

$$A := -\rho(1-\rho)^{-1} \sum_{k=1}^{K} A\lambda_k^* = -\rho(1-\rho)^{-1} A,$$

which can be true only if $A = 0$, from which we conclude that $a_m = 0$ for all m.

To prove (105) we use Jensen's inequality for the logarithmic function and write

$$E \ln H'(s,0) = E \ln \sum_{k=1}^{K} \mu_k^* \frac{\lambda_k}{\lambda_k^*} < \ln E \sum_{k=1}^{K} \mu_k^* \frac{\lambda_k}{\lambda_k^*} = \ln \sum_{k=1}^{K} (E\mu_k^*) \frac{\lambda_k}{\lambda_k^*} = \ln \sum_{k=1}^{K} \lambda_k = 0, \qquad (110)$$

where the last but one inequality follows from (108). To complete the proof it remains to justify the strict inequality in the above chain of relations. Assume the contrary: we have equality "=", rather than inequality "<", in (110). This can happen only if the random variable

$\sum_{k=1}^{K} \mu_k^*(s)(\lambda_k/\lambda_k^*)$ is in fact constant, i.e., coincides (a.s.) with its expectation $\sum_{k=1}^{K} E\mu_k^*(s)(\lambda_k/\lambda_k^*)$. But this expectation is equal to 1 (see (108)). Thus $\sum_{k=1}^{K} \mu_k^*(s)\lambda_k/\lambda_k^* = 1$ (a.s.) or equivalently, $\sum_{k=1}^{K} \mu_k^*(s)[(\lambda_k/\lambda_k^*)-1] = 0$ (a.s.), which implies that $(\lambda_k/\lambda_k^*)-1 = 0$ (a.s.) for all $k = 1, 2, ..., K$ because, as we have proved above, the functions $\mu_k^*(s)$ are linearly independent. Consequently, $\lambda = \lambda^*$, which is a contradiction.

3.6. Linear model

In this section we examine in detail an important special case of the model in which the dividend payoff functions $g_k(s, b)$ are linear:

$$g_k(s, b) = c_k(s) + a_k(s)b, \;\; c_k(s), a_k(s) > 0. \tag{111}$$

The general results obtained above are based on certain simple sufficient conditions for the model to be workable and estimates that most probably can be improved. In the present, linear context we succeed in finding necessary and sufficient conditions and obtain exact estimates. In what follows, the parameter s will be fixed and omitted in the notation.

In the previous section, it was shown that the strict monotonicity of the operator $F(y)$ (guaranteeing the existence and uniqueness of λ^*) is equivalent to the strict positivity of the following expression

$$I_{k,j}(y) := [\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) + g_k'(y_k)(\delta_{j,k} - y_k)] - g_k(y_k) \sum_{m=1}^{K} g_m'(y_m)(\delta_{j,m} - y_m)]$$

for all $y = (y_1, ..., y_K) \in \Delta^K$ and all k and j. We wish to characterize the set of pairs of vectors $a = (a_1, ..., a_K) > 0$ and $c = (c_1, ..., c_K) > 0$ under which $I_{k,j}(y) > 0$ for any y, k and j. Let us call such pairs of vectors *feasible.*

To formulate the result we introduce some notation. Let us assume (without loss of generality) that $a_1 \geq a_2 \geq ... \geq a_K$. Then

$$a_{\max}^k := \max_{j \neq k} a_j = \begin{cases} a_1, & \text{if } k > 1, \\ a_2, & \text{if } k = 1, \end{cases} \quad , \quad a_{\min}^k := \min_{j \neq k} a_j = \begin{cases} a_K, & \text{if } k < K, \\ a_{K-1}, & \text{if } k = K. \end{cases} \tag{112}$$

For those $k = 1, 2, ..., K-1$ for which $a_k > a_{\min}^k$, we define

$$\gamma_k^*(a, c) := \frac{a_k \left(a_{\max}^k - a_{\min}^k\right) - 2c_k(a_k - a_{\min}^k)}{2a_k(a_k - a_{\min}^k)} \tag{113}$$

and

$$A_k(a, c) := c_k \left(C + 2a_{\min}^k - a_{\max}^k\right) - (\gamma_k^*)^2 a_k(a_k - a_{\min}^k), \tag{114}$$

where

$$C := \sum_{m=1}^{K} c_m.$$

For each $k = 1, ..., K$ we denote by $B_k(a, c)$ the minimum of the two numbers

$$c_k \left(C + 2a_{\min}^k - a_{\max}^k\right),$$

$$c_k \left(C + 2a_k - a_{\max}^k\right) + a_k(a_k - a_{\max}^k). \tag{115}$$

Finally, we put

$$D_k(a, c) := \begin{cases} A_k(a, c), & \text{if } a_k > a_{\min}^k \text{ and } \gamma_k^*(a, c) \in (0, 1), \\ B_k(a, c), & \text{otherwise}, \end{cases}$$

for $k = 1, ..., K-1$ and $D_K(a, c) := B_K(a, c)$.

Proposition 4. *A pair of vectors* $a = (a_1, ..., a_K) > 0$ and $c = (c_1, ..., c_K) > 0$ *is feasible if and only if* $D_k(a, c) > 0$ *for all* $k = 1, ..., K$.

Proof. 1st step. We begin with the following remark. If $k = j$, then

$$[\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) + g_k'(y_k)(\delta_{j,k} - y_k)] - g_k(y_k) \sum_{m=1}^{K} g_m'(y_m)(\delta_{j,m} - y_m)]$$

$$= [\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) - g_k'(y_k)y_k + g_k'(y_k)] + g_k(y_k) \sum_{m=1}^{K} g_m'(y_m)y_m - g_k(y_k)g_k'(y_k)$$

$$> [\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) - g_k'(y_k)y_k + g_k'(y_k)] - g_k(y_k)g_k'(y_k)$$

$$\geq g_k(y_k)[g_k(y_k) - g_k'(y_k)y_k + g_k'(y_k)] - g_k(y_k)g_k'(y_k)$$

$$\geq g_k(y_k)g_k'(y_k) - g_k(y_k)g_k'(y_k) = 0.$$

Therefore if $j = k$, then $I_{k,j}(y) > 0$ always, and we can exclude this case from consideration. Clearly, if $k \neq j$, then

$$I_{k,j}(y) = [\sum_{m=1}^{K} g_m(y_m)][g_k(y_k) - g_k'(y_k)y_k] + g_k(y_k)[\sum_{m=1}^{K} g_m'(y_m)y_m] - g_k(y_k)g_j'(y_j).$$

Thus our goal is to characterize pairs vectors $a = (a_1, ..., a_K) > 0$ and $c = (c_1, ..., c_K) > 0$ such that this expression is strictly positive for all $y = (y_1, ..., y_K) \in \Delta^K$ and all $k \neq j$.

2nd step. In the model at hand (see (111)), we have

$$g_k'(b) = a_k, \;\; g_k'(b)b = a_k b, \;\; g_k(b) - g_k'(b)b = c_k + a_k b - a_k b = c_k,$$

and so

$$I_{k,j}(y) = \sum_{m=1}^{K}(c_m + a_m y_m)c_k + (c_k + a_k y_k)(\sum_{m=1}^{K} a_m y_m) - (c_k + a_k y_k)a_j.$$

Clearly, $I_{k,j}(y) > 0$ for all y and $k \neq j$ if and only if

$$I_k(y) := \sum_{m=1}^{K}(c_m + a_m y_m)c_k + (c_k + a_k y_k)(\sum_{m=1}^{K} a_m y_m) - (c_k + a_k y_k)\max_{j \neq k} a_j > 0 \tag{116}$$

for all y and k.

Fix $k = 1, ..., K$ and some $\gamma \in [0, 1]$, and put

$$\Delta^K(\gamma) := \{y = (y_1, ..., y_K) \in \Delta^K : \; y_k = \gamma\}.$$

For $y \in \Delta^K(\gamma)$, we have (see (116))

$$I_k(y) = \sum_{m=1}^{K}(c_m + a_m y_m)c_k + (c_k + a_k y_k)(\sum_{m=1}^{K} a_m y_m) - (c_k + a_k y_k)a^k_{\max}$$

$$= c_k C + c_k(a_k\gamma + \sum_{m \neq k}^{K} a_m y_m) + (c_k + a_k\gamma)(a_k\gamma + \sum_{m \neq k}^{K} a_m y_m) - (c_k + a_k\gamma)a^k_{\max}. \tag{117}$$

Thus a pair of vectors $(a_1, ..., a_K) > 0$ and $(c_1, ..., c_K) > 0$ is feasible if and only if

$$J_k(\gamma) := \min_{y \in \Delta^K(\gamma)} I_k(y) > 0 \text{ for all } k \text{ and all } \gamma \in [0, 1].$$

3rd step. Let us fix k and γ and regard the expression in (117) as a linear function of a $(K-1)$-dimensional vector $(y_1, ..., y_{k-1}, y_{k+1}, ..., y_K)$ belonging to the set

$$\{(y_1, ..., y_{k-1}, y_{k+1}, ..., y_K) \geq 0 : \sum_{m \neq k} y_m = 1 - \gamma\}.$$

This set is a convex polyhedron (simplex), and therefore any linear function attains its minimum on it at one of its vertices—that one for which the value of the linear function is the smallest. Therefore we have

$$J_k(\gamma) = c_k C + c_k[a_k \gamma + a^k_{\min}(1-\gamma)]$$

$$+(c_k + a_k\gamma)[a_k\gamma + a^k_{\min}(1-\gamma)] - (c_k + a_k\gamma)a^k_{\max} = r_k + p_k\gamma + q_k\gamma^2, \quad (118)$$

where

$$r_k = c_k C + c_k a^k_{\min} + c_k(a^k_{\min} - a^k_{\max}) = c_k(C + 2a^k_{\min} - a^k_{\max}),$$

$$p_k = 2c_k(a_k - a^k_{\min}) - a_k(a^k_{\max} - a^k_{\min}), \; q_k = a_k(a_k - a^k_{\min}).$$

Now our goal is to describe the set of pairs of vectors $a = (a_1, ..., a_K) > 0$ and $c = (c_1, ..., c_K) > 0$ such that

$$J^{\min}_k := \min_{\gamma \in [0,1]} J_k(\gamma) > 0 \;, k = 1, 2, ..., K. \quad (119)$$

We will show that (119) is fulfilled if and only if the inequalities $D_k(a, c) > 0$ hold for all $k = 1, ..., K$.

4th step. In view of (112), we have

$$q_k = a_k(a_k - a^k_{\min}) = \begin{cases} a_k(a_k - a_K) \geq 0, & \text{if } k < K, \\ a_K(a_K - a_{K-1}) \leq 0, & \text{if } k = K, \end{cases}$$

and so the function $J_k(\gamma)$ is convex quadratic or linear for $k < K$ and concave quadratic or linear for $k = K$. Fix some $k < K$ and consider two cases: (i) $a_k - a^k_{\min} > 0$ and $\gamma^*_k \in (0, 1)$ (see (113)); (ii) either $a_k - a^k_{\min} = 0$ or $a_k - a^k_{\min} > 0$ and $\gamma^*_k \notin (0, 1)$.

In the first case, the strictly convex quadratic function (118) attains its minimum on $[0, 1]$ at the point

$$-\frac{p_k}{2q_k} = \frac{a_k\left(a^k_{\max} - a^k_{\min}\right) - 2c_k(a_k - a^k_{\min})}{2a_k(a_k - a^k_{\min})} = \gamma^*_k, \quad (120)$$

(see (113)). This minimum is equal to

$$J_k(\gamma_k^*) = r_k + \gamma_k^*(p_k + q_k\gamma_k^*) = r_k + \gamma_k^*\frac{p_k}{2} = r_k - (\gamma_k^*)^2\, q_k$$

$$= c_k\left(C + 2a_{\min}^k - a_{\max}^k\right) - (\gamma_k^*)^2\, a_k(a_k - a_{\min}^k) = A_k$$

(see (114)).

In the second case, the function $J_k(\gamma)$ is linear or convex quadratic with the stationary point $\gamma_k^* \notin (0,1)$ (see (120)). Consequently, the minimum of this function on $[0,1]$ is attained either at 0 or at 1, i.e.,

$$J_k^{\min} = \min\{J_k(0), J_k(1)\}, \tag{121}$$

where

$$J_k(0) = c_k\left(C + 2a_{\min}^k - a_{\max}^k\right), \tag{122}$$

$$J_k(1) = c_k\left(C + 2a_{\min}^k - a_{\max}^k\right) + 2c_k(a_k - a_{\min}^k) - a_k\left(a_{\max}^k - a_{\min}^k\right) + a_k(a_k - a_{\min}^k)$$

$$= c_k\left(C + 2a_k - a_{\max}^k\right) + a_k(a_k - a_{\max}^k), \tag{123}$$

which shows that $J_k^{\min} = B_k$ (see (115)).

It remains to consider the case $k = K$. As we have noticed above, the function $J_K(\gamma)$ is concave quadratic or linear. Therefore the arguments conducted in case (ii) and the relations in (121)-(123) apply to the case $k = K$, which proves that $J_K^{\min} = B_K$.

□

We provide a simple *sufficient* condition (124) under which all the (necessary and sufficient) conditions listed in Proposition 4 hold. Define:

$$a_{\min} = \min_{1\le m\le K} a_m,\; a_{\max} = \max_{1\le m\le K} a_m,\; c_{\min} = \min_{1\le m\le K} c_m$$

and recall that $C = \sum_{m=1}^K c_m$.

Proposition 5. *Suppose that*

$$c_{\min}\left(C + 2a_{\min} - a_{\max}\right) > a_{\max}(a_{\max} - a_{\min}). \tag{124}$$

Then $D_k(a,c) > 0$ for all $k = 1, ..., K$.

Proof. We show that for all $k = 1, ..., K$ condition (124) ensures that $B_k > 0$ and $A_k > 0$ as long as $\gamma_k^* \in (0, 1)$, which implies that $D_k > 0$ for all k. Indeed, we have

$$c_k \left(C + 2a_{\min}^k - a_{\max}^k\right) \geq c_{\min} \left(C + 2a_{\min} - a_{\max}\right)$$

$$> a_{\max}(a_{\max} - a_{\min}) \geq (\gamma_k^*)^2 \, a_k(a_k - a_{\min}^k),$$

$$c_k \left(C + 2a_{\min}^k - a_{\max}^k\right) \geq c_k \left(C + 2a_{\min} - a_{\max}\right) > a_{\max}(a_{\max} - a_{\min}) \geq 0$$

and

$$c_k \left(C + 2a_k - a_{\max}^k\right) + a_k(a_k - a_{\max}^k) \geq c_{\min} \left(C + 2a_{\min} - a_{\max}\right) + a_k(a_{\min} - a_{\max})$$

$$\geq c_{\min} \left(C + 2a_{\min} - a_{\max}\right) + a_{\max}(a_{\min} - a_{\max}) > 0,$$

which completes the proof.

3.7. A numerical example with time-dependent strategies

A numerical example with time-dependent investment strategies is provided to illustrate (a) the capability of λ^* and (b) the pitfall of not including such a fundamental strategy in agent-based models of financial markets. This example supports the hope that the results obtained in this chapter can be extended to classes of strategies more general than fixed-mix ones.

There are two assets in supply $V_k = k$ and dividends $d_k(s_t, W_{t-1,k}) = 1 + s_t W_{t-1,k}$, $k = 1, 2$. The total amount of dividends paid by the asset k in period t is $D_k(s_t, W_{t-1,k}) = V_k d_k(s_t, W_{t-1,k})$. The process s_t is i.i.d. and log-normal with parameters $(1, 1)$.

There are three investment strategies. First, the ESS $\lambda^1 = \lambda^* = (0.2, 0.8)$ which is fixed over time. Second, λ_t^2 is a history-dependent, trend chaser (momentum) strategy. Denote by $R_{t-1,k}$ asset k's realized return from period $t-2$ to $t-1$ and by $\overline{R}_{t-1}$ its average over $k = 1, 2$. Then $\lambda_{t,1}^2 := \arctan(R_{t-1,1} - \overline{R}_{t-1})/\pi + 0.5$ and $\lambda_{t,2}^2 := \arctan(R_{t-1,2} - \overline{R}_{t-1})/\pi + 0.5 = 1 - \lambda_{t,1}^2$. Since there are no previous returns in the initial period, the strategy is chosen randomly. Third, a noise trader strategy which varies from period to period. In each period this strategy is determined by randomly drawing $\lambda_{t,1}^3$ uniformly from $[0, 1]$ and setting $\lambda_{t,2}^3 = 1 - \lambda_{t,1}^3$.

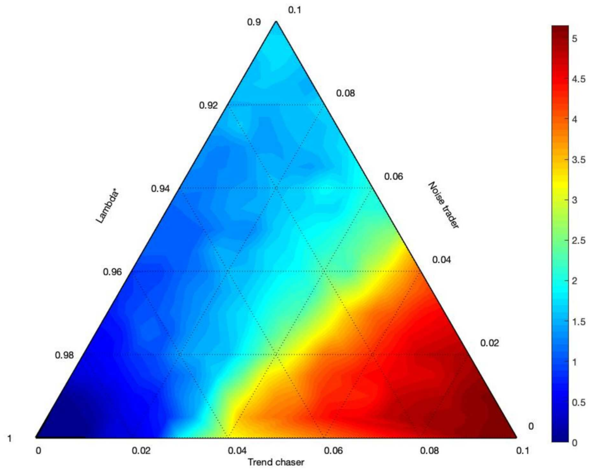

Fig. 2

The simulation is carried out as follows. Strategies start with different initials wealth shares, λ^*'s initial share is $r_0^1 \geq 0.9$. In each period of time, prices for both assets are computed by (84), as well as the ratio of prices $p_t := p_{t,1}/p_{t,2}$. Then we calculate $|p^* - p_t|$ where $p_k^* = \frac{\rho}{V_k}\lambda_k^*$ are the equilibrium prices (the prices that prevail when the ESS λ^* has all wealth, i.e. $r^1 = 1$). Portfolios are defined by (81). Additionally, we calculate fractions of total market wealth allocated to assets by (76). Next, we generate random state of the world s_t and calculate dividends $d_k(s_t, W_{t-1,k})$. Afterwards, we calculate the new wealth of all strategies by (79) as well as returns and relative dividends, which are needed for the trend chaser.

Fig. 3 below illustrates the number of model iterations to obtain prices close to those prevailing in the long-run equilibrium. We calculate the expected number of periods to obtain $|p^* - p_t| < 2.5\%$, where p is the relative price of the two assets (of course, full equilibrium is attained only asymptotically). Each point in the diagram corresponds to a distribution of wealth across the three strategies. First, we observe that the equilibrium is indeed stable. Even for large perturbations, prices revert to equilibrium. Second, it requires a large deviation from equilibrium to have long-lasting mispricing. For instance, when the noise trader acquires 10% of wealth, it takes 12 iterations of the wealth dynamic to return to equilibrium prices. For the trend chaser, 4% of total wealth has the same effect.

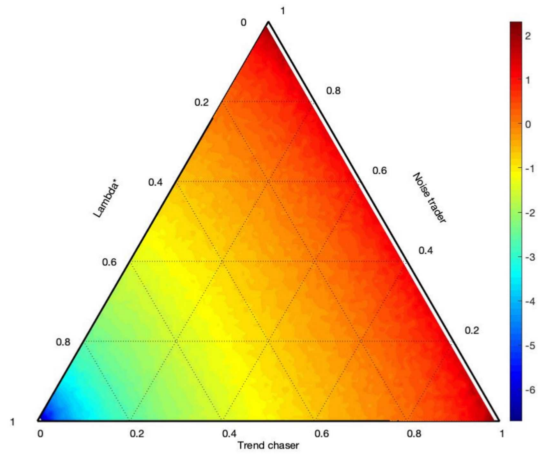

Fig. 3

One can also determine which strategy has the highest growth rate of wealth for each particular initial distribution of wealth. It turns out that λ^* has the highest expected growth for all initial distributions. As an implication it follows that λ^* is globally stable against the history-dependent momentum strategy and the noise trader. Fig. 3 shows that the wealth of λ^* grows the fastest, the poorer the strategy. The composition of the market ecology (moving parallel to the vertex of the noise trader) has no impact on λ^*'s growth rate. One needs to take into account that the wealth dynamics is driven solely by capital gains. There is no exogenous flow of capital from worse to better performing strategies. The result in Fig. 3 indicates that if such a flow were present, convergence to equilibrium prices would be even faster, enhancing the stability of the ESS.

3.8. Conclusion

Financial markets are modeled from a biological perspective where investment strategies and wealth take on the role of species and their fitness. The main innovation is that dividends paid by a firm's stocks are determined by the firm's market value and random events. This creates a feedback loop between the wealth distribution and the production of dividends that are paid to the investment strategies. We analyze the resulting stochastic dynamical system with the aim to determine whether there are investment strategies such that the wealth distribution is (locally) stable.

Our main result is the explicit description of a unique evolutionary stable investment strategy λ^* such that the state in which this strategy has all the wealth is locally stable. 'Invading' investment strategies will be driven out by the wealth dynamics. The remarkable property of λ^* is that it only depends on the dividend production function and not on the ecology of the financial market. It has several interesting properties such as full diversification, which is of importance in portfolio management, and the valuation of financial assets in terms of relative fundamentals, which matters in asset pricing.

Our numerical results suggest that λ^* might be globally stable. A drawback of the current model is the restriction to i.i.d. shocks, which allow us to work with fixed-mix investment strategies. Future work will aim at relaxing this assumption to an arbitrary time- and history-dependent setting, which has been done successfully for exogenous dividend processes.

CHAPTER 4. UNBEATABLE STRATEGIES

This chapter analyzes the notion of an unbeatable strategy as a game solution concept. A general framework (game with relative preferences) suitable for the analysis of this concept is proposed. Basic facts regarding unbeatable strategies are presented and a number of examples and applications considered. The study is motivated by the applications to Evolutionary Finance. In the context of EF models, the concepts of a survival strategy and an unbeatable strategy are equivalent (see Sect. 2.4). The results of this chapter are presented in the preprint Amir el al. [7].

4.1. Unbeatable strategy as a game solution concept

Nowadays Nash equilibrium is the most common solution concept in game theory. However, a century ago, when the discipline was in its infancy, the term "solving a game" was understood quite differently. The focus was not on finding a strategy profile that would equilibrate conflicting interests of the players. The main goal was to find, if possible, an individual strategy enabling the player to win (or at least not to lose) the game, or in other words, to construct an *unbeatable strategy.* This question was considered in the paper by Bouton [33], apparently the earliest mathematical paper in the field. Borel [26] formulated in his study the general problem "to investigate whether it is possible to determine a method of play better than all others; i.e., one that gives the player who adopts it a superiority over every player who does not adopt it". When developing this idea, Borel introduced a famous class of games, that later received the name "Colonel Blotto games" (Borel [26]).

It should be noted that the problem of constructing explicit unbeatable strategies turned out to be unsolvable for the vast majority of mind games of common interest (such as chess). What mathematicians could achieve, at most, was to prove that games in certain classes were determinate. A game is called *determinate* if at least one of the players has an unbeatable strategy.

Problems related to the determinacy of chess were considered in the paper by Zermelo [180]. Although this result is traditionally referred to as Zermelo's theorem, the determinacy of chess was apparently established for the first time by Kalmár [97]. For a discussion of the history of this question see Schwalbe and Walker [152].

A deep mathematical analysis of the determinacy of infinite win-or-lose games of complete information was initiated by Gale and Stewart [69]. This line of studies has led to remarkable achievements in set theory and topology. The highlight in the field was "Martin's determinacy axiom" and a proof of its independence of the Zermelo–Fraenkel axioms of set theory (Martin [127]). For comprehensive surveys of research in this area see Telgársky [166] and Kehris [99] ; for reviews of topics related to unbeatable strategies in combinatorial game theory see Berlekamp et al. [20].

However, the above achievements had for the most part purely theoretical value, having nothing to do with real-life applications. They dealt with elegant games created in the minds of mathematicians. A classical example is the Bouton's [33] game "Nim", a theory of which was developed in his pioneering paper. Therefore new, applications-oriented solution concepts based on the idea of minimax (von Neumann [176]) and equilibrium (Nash [131]) came to the fore. These new concepts promptly emerged as central in game-theoretic analysis related to operations research and economics.

For symmetric games, unbeatable strategies, if they exist, are the same for both players and form a saddle point for the associated zero-sum game (see Section 3 below) whose payoffs are defined as differences between the payoffs of the players in the original game. This provided an ex post justification for the shift that occurred much earlier from the notion of unbeatable strategies to the concept of saddle points, with the former re-emerging only decades after the seminal von Neumann paper [176]. In the 1950s, when game theory started developing primarily as a mathematical framework for economic modeling, non-zero sum N-player games provided the main paradigm, and the notion of Nash equilibrium became the fundamental solution concept for the study of strategic behaviour.

The concept of an unbeatable strategy as such emerged again in theoretical biology and served as a starting point for the development of evolutionary game theory (EGT). Hamilton

[80] used this notion, and the term “unbeatable strategy”—without a rigorous formalization—in his paper on the analysis of sex ratios in populations of some species, which turned out to be extremely influential. Maynard Smith and Price [128] formalized Hamilton’s idea, but at the same time somewhat changed its content. The notion they introduced, usually referred to as an evolutionary stable strategy (ESS), should be called, more precisely, a *conditionally unbeatable* strategy. It it is indeed unbeatable, but only if the rival is "weak enough." In the context of evolutionary biology, ESS is a strategy that cannot be beaten if the fraction of the rivals (mutants) in the population is sufficiently small. This definition requires the population to be infinite, since one has to speak of its arbitrarily small fractions. Versions of ESS applicable to finite populations were suggested by Schaffer ([146, 147]). Schaffer’s notions of ESS—there are two of these—are also in a sense conditionally unbeatable strategies. The first requires the population to contain only *one* mutant; the second assumes that there are several *identical* mutants.

It is not surprising that an unbeatable strategy, rather than a Nash equilibrium, turned out to be a key idea that fitted ideally the purposes of evolutionary modeling in biology. Nash equilibrium presumes full rationality of players, understood in terms of utility maximization, and their ability to coordinate their actions (or the presence of Harsanyi’s "mediator") to establish an equilibrium, especially if it is non-unique. In a biological context such possibilities are absent, and moreover the role of individual utilities, always having a subjective nature, is played in EGT by a *fitness function*, an objective characteristic reflecting the survival rate in the natural selection process.

It is standard to present EGT models in conventional game-theoretic terms, with utilities/payoffs and Nash equilibrium, but this is just a matter of convenience, that makes it possible to employ the terminology and the results of conventional game theory. Moreover, EGT models are nearly exclusively symmetric, and as has been said above, the analysis of unbeatable strategies in the symmetric case boils down to the consideration of symmetric Nash equilibria (possessing some additional properties). At the same time, this kind of exposition, although convenient in some respects, might be misleading in others. In EGT, in contrast with conventional game theory, players do not select their strategies. Strategies are nothing but "genetic codes" of the players they have no influence on, while payoffs or utilities are not their individual characteristics (which are typically unobservable), but as has been noted, represent their fitness functions amenable to observations and statistical estimates.

The notion of ESS proposed by Maynard Smith and Price [128] reigned in Evolutionary Game Theory for many years. An unconditional variant of ESS—fully corresponding to Hamilton's idea of an unbeatable strategy[10]—was first revived in the context of economic applications of EGT in a remarkable paper by Kojima [103], three decades after Maynard Smith and Price and four decades after Hamilton. Kojima's study was motivated by economic applications, where the assumption of smallness of the population of "mutants" is obviously not realistic: a new technology or a new product may be thrown into the economy in any quantities.

Several years after Kojima's work, it was discovered (Amir et al. [5], [6]) that the concept of an unbeatable strategy represents a very convenient and efficient tool in the analysis of financial market models combining evolutionary and behavioural principles—see the surveys in Evstigneev et al. [57] and Holtfort [94]. This circumstance motivated us to undertake a systematic study of unbeatable strategies, which is conducted in the present chapter.

The next section introduces a general framework (game with relative preferences) in which we examine unbeatable strategies. Section 3 analyzes in detail the classical case of a game with two players and cardinal preferences. In the remainder of the chapter, various examples and applications are discussed. Some of these examples and applications are entirely new, some other are essentially known, and the novelty consists in their presentation from the perspective of unbeatable strategies.

[10] Hamilton [80] did not give a rigorous general definition of an unbeatable strategy, using this notion in the specific context of that particular paper. In later papers (Hamilton and May, [81]; Comins et al., [40]), he used the notion of an ESS and emphasized its "combination of simplicity and generality". However, in Hamilton [82], three decades later, he stated that in [80] he had in mind indeed a genuine notion of an unbeatable strategy, without the additional assumption of a small fraction of mutants in the population. For a discussion of the history of these ideas see Sigmund [161].

4.2. Game with relative preferences

Game description. This section develops a general framework—*game with relative preferences*—serving as the basis for the analysis of unbeatable strategies. This framework was suggested in Amir et al. [5] in connection with a study on Evolutionary Finance. There are N players $i = 1, ..., N$ choosing their *strategies* x^i from some given sets X^i. A set $Z \subseteq X^1 \times ... \times X^N$ of *admissible strategy profiles* is given. For each i there is a mapping $w^i : Z \to W^i$ from Z into the set W^i of *outcomes* of the game for player i. If players $i = 1, 2, ..., N$ select strategies $x^1, ..., x^N$ such that $(x^1, ..., x^N) \in Z$, then the outcome of the game for player i is $w^i(x^1, ..., x^N) \in W^i$.

We would like to define the notion of an unbeatable strategy of some player i. To this end we assume that for any pair of outcomes $w^i \in W^i$, $w^j \in W^j$ $(j \neq i)$ a preference relation

$$w^i \succeq_{ij} w^j, \; w^i \in W^i, \; w^j \in W^j$$

is given. This preference relation is used to compare the game outcomes w^i and w^j of players i and j by estimating their relative performance. We do not impose any assumptions on the preference relation $\succeq_{ij}$. It is defined in terms of an arbitrary non-empty set $\{(w^i, w^j) \in W^i \times W^j : w^i \succeq_{ij} w^j\}$.

Definition 1. A strategy x^* of player i is called *unbeatable* if for any admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ with $x^i = x^*$, we have

$$w^i(x^1, x^2, ..., x^N) \succeq_{ij} w^j(x^1, x^2, ..., x^N) \text{ for all } j \neq i. \tag{125}$$

According to this definition, player i adopting the strategy x^* cannot be outperformed by any other player $j \neq i$ irrespective of what strategies player i's rivals $j \neq i$ use.

Reduction to the case of two players. Suppose we are interested in unbeatable strategies for player i in the above game G. Then, as is easily seen, we can reduce the general game G with N players to a game $\hat{G}$ with two players, one of whom is player i, while the other is the team $\{j : j \neq i\}$ of i's rivals, which will be referred to as "player $-i$". Define the strategy set X^{-i} of player $-i$ in the new game $\hat{G}$ as

$$X^{-i} = \prod_{j \neq i} X^j$$

and denote by

$$W^{-i} = \prod_{j \neq i} W^j$$

the set of game outcomes for this player. We say that a strategy $x^i \in X^i$ of player i and a strategy $x^{-i} = (x_j)_{j \neq i} \in X^{-i}$ of player $-i$ form an admissible strategy profile in the game $\hat{G}$ and write $(x^i, x^{-i}) \in \hat{Z}$ if

$$(x^i, x^{-i}) = (x^1, ..., x^N) \in Z.$$

To each $(x^i, x^{-i}) \in \hat{Z}$ there correspond the game outcomes for players i and $-i$:

$$w^i(x^i, x^{-i}) \in W^i, \quad w^{-i}(x^i, x^{-i}) = (w^j(x^i, x^{-i}))_{j \neq i} \in W^{-i}.$$

The preference relations between the game outcomes $w^i \in W^i$ and $w^{-i} \in W^{-i}$ for players i and $-i$ are defined as follows. If $w^{-i} = (w^j)_{j \neq i}$, where $w^j \in W^j$, then, by definition, the relation $w^i \succcurlyeq_{i,-i} w^{-i}$ holds when $w^i \succcurlyeq_{ij} w^j$ for all $j \neq i$ and the relation $w^{-i} \succcurlyeq_{-i,i} w^i$ holds when $w^j \succcurlyeq_{ji} w^i$ for some $j \neq i$. The former means that player i cannot be outperformed by any member j of the team $-i$ and the latter means that at least one of the members j of the team $-i$ cannot be outperformed by i.

Clearly x^* is an unbeatable strategy of player i in the original game G if and only if x^* is an unbeatable strategy of player i in the two player game $\hat{G}$.

Cardinal preferences (numerical measures of performance). Suppose that for each player i, a function $F_i(x^1, ..., x^N)$ of an admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ is given. One can interpret the number $F_i(x^1, ..., x^N)$ as the "score" or payoff which player j gets if the strategy profile of the all players is $(x^1, ..., x^N)$. This number characterizes the outcome of the game for player i. The sets of outcomes W^i for all the players i are the same and coincide with the real line R. The preference relations between the game outcomes are defined as usual non-strict inequalities between real numbers: $w^i \succcurlyeq_{ij} w^j$ if and only if $w^i \geq w^j$. In this setting, a strategy x^* of player i is unbeatable if for any admissible strategy profile $(x^1, x^2, ..., x^N) \in Z$ with $x^i = x^*$, we have

$$F_i(x^1, x^2, ..., x^N) \geq F_j(x^1, x^2, ..., x^N) \text{ for all } j \neq i,$$

or equivalently,

$$F_i(x^1, x^2, ..., x^N) \geq \max_{j \neq i} F_j(x^1, x^2, ..., x^N).$$

This shows that the analysis of unbeatable strategies of player i in the N-player game at hand reduces to the analysis of such strategies in the two player game between i and the team $\{j : j \neq i\}$ of i's rivals , where

$$F_{-i}(x^1, x^2, ..., x^N) = \max_{j \neq i} F_j(x^1, x^2, ..., x^N)$$

(cf. the previous subsection).

Symmetric N-player games. Let us say that the game introduced in the previous subsection is *symmetric* if $X^1 = X^2 = ... = X^N = X$ and for every permutation $\pi(i)$ of the numbers $1, ..., N$ we have $(x^1, x^2, ..., x^N) \in Z$ if and only if $(x^{\pi(1)}, x^{\pi(2)}, ..., x^{\pi(N)}) \in Z$ and

$$F_i(x^1, ..., x^i, ..., x^N) = F_{\pi(i)}(x^{\pi(1)}, ..., x^{\pi(i)}, ..., x^{\pi(N)}), \tag{126}$$

i.e. both the class of admissible profiles Z and the payoff functions $F_i(x^1, ..., x^i, ..., x^N)$ are permutation-invariant. Clearly it follows from (126) that

$$F_i(x^1, ..., x^i, ..., x^j, ..., x^N) = F_j(x^1, ..., x^j, ..., x^i, ..., x^N). \tag{127}$$

In the general case, if we wish to verify that x^* is an unbeatable strategy of some player, say, player 1, then we need to check the validity of $N-1$ inequalities

$$F_1(x^*, x^2, ..., x^N) \geq F_j(x^*, x^2, ..., x^N) \text{ for all } j = 2, ..., N \text{ and } x^2, ..., x^N. \tag{128}$$

However, if the game is symmetric, it is sufficient to verify only one of these inequalities, for some particular j, say $j = 2$:

$$F_1(x^*, x^2, ..., x^N) \geq F_2(x^*, x^2, ..., x^N). \tag{129}$$

Indeed assume (129) holds, consider any $j = 3, ..., N$, and observe that inequality (128) is equivalent to (129) because

$$F_j(x^*, x^2, ..., x^j, ..., x^N) = F_2(x^*, x^j, ..., x^2, ..., x^N)$$

by virtue of (127).

Schaffer's ESS. We define within the present framework the important notions of evolutionary stable strategies (ESS) for finite populations introduced in the seminal works of Schaffer [146, 147]. Consider the symmetric model with cardinal preferences described in the previous subsection. Let us define the class Z of admissible strategy profiles as follows. Let $(x^1, x^2, ..., x^N) \in Z$ if there exists $i = 1, 2, ..., N$ such that all the strategies x_j for j distinct from i coincide:

$$x_j = x_{j'} \text{ for all } j, j' \neq i.$$

Thus all the admissible strategy profiles are of the following form

$$z_i = (y, ..., y, x, y, ...y), \;\; x, y \in X, \;\; i = 1, 2..., N,$$

where "x" stands at the ith place. By symmetry, for such a strategy profile we have

$$F_j(y, ..., y, x, y, ...y) = F_{j'}(y, ..., y, x, y, ...y) \text{ for all } j, j' \neq i.$$

Definition 2. A strategy x^* is called a *Schaffer's ESS of the first kind* if

$$F_j(x^*, ..., x^*, x, x^*, ..., x^*) \geq F_i(x^*, ..., x^*, x, x^*, ..., x^*) \text{ for all } x \in X \text{ and } j \neq i. \tag{130}$$

It is termed a *Schaffer's ESS of the second kind* if

$$F_i(x, ..., x, x^*, x, ...x) \geq F_j(x, ..., x, x^*, x, ...x) \text{ for all } x \in X \text{ and } j \neq i. \tag{131}$$

It is said in (130) that a group of $N-1$ identical non-mutants x^* cannot be outperformed by a mutant x. According to (131), a non-mutant x^* cannot be beaten by a pool of $N-1$ identical mutants x.

Remark 1. As we have seen, the analysis of unbeatable strategies in an N-person game G with relative preferences can be reduced, at least formally, to the analysis of such strategies in a certain two-person game $\hat{G}$. However, one may doubt that this reduction can actually be of help in constructing unbeatable strategies. Since in the game $\hat{G}$ player i faces a coalition $\{j : j \neq i\}$ of rivals, one may expect this coalition to have substantially more "power" to beat player i than any of i's individual rivals. Quite often this consideration happens to be true, and in many classical N-person games unbeatable strategies exist only for $N = 2$. Nevertheless, there are important cases when this idea works. Suppose we have a family of games $G(N)$, $N = 1, 2, ...$, with the same structure, so that the only distinction between them consists in the number of players. Assume that for the N-person game $G(N)$ we can represent a group of $N-1$ players by one "aggregate player" and obtain thereby a two-player game $G(2)$ from the family under consideration. This turns out to be the case in most of the known evolutionary models of financial markets, where a pool of market players/investors can indeed be represented by an "aggregate investor" (such as a *mutual fund*). This approach has been successfully applied to the construction of unbeatable strategies in models developed in Evolutionary Finance (Amir at al. [5], [6], [8]), which served as basic sources of motivation for the present work. The papers mentioned contain several models that can be included into the general framework of a game with relative preferences considered in this section.

4.3. Two players, cardinal preferences

Two-player game with cardinal preferences. In this section, we examine in detail unbeatable strategies in the classical framework of static two-player games with cardinal preferences specified by payoff functions. Consider a game G with strategy sets A, B and payoff functions $u(a,b)$, $v(a,b)$ of players 1 and 2 which are interpreted as their *measures of performance* ("scores"). The goal of a player is to construct a strategy that cannot be outperformed in terms of higher payoffs by the rival, whatever the rival's strategy might be.

The general definition of an unbeatable strategy (see Section 2) takes on in this context the following form:

Definition 3. A strategy $a^* \in A$ of player 1 is said to be *unbeatable* if

$$u(a^*, b) \geq v(a^*, b) \tag{132}$$

for any strategy $b \in B$ of player 2. Analogously, a strategy $b^* \in B$ of player 2 is called *unbeatable* if

$$v(a, b^*) \geq u(a, b^*) \tag{133}$$

for any strategy $a \in A$ of player 1. The game is said to be *determinate* if at least one of the players has an unbeatable strategy.

According to (132), player 1 using the strategy a^*cannot be outperformed by player 2, irrespective of the strategy b of player 2. Condition (133) expresses the analogous property of the strategy b^* of player 2.

The associated zero-sum game. To analyze the concept of an unbeatable strategy we will associate with the original game G defined in terms of the strategy sets A, B and payoff functions $u(a,b)$, $v(a,b)$ a zero-sum game G^0 in which the strategy sets of players 1 and 2 are the same as above, A and B, while the payoff functions of players 1 and 2 are given by

$$f(a,b) = u(a,b) - v(a,b) \text{ and } g(a,b) = -f(a,b).$$

The game G^0 will be called *the zero-sum game associated with the game* G.

Remark 2. If the original game is zero-sum, then $v(a,b) = -u(a,b)$, and so

$$f(a,b) = u(a,b) - v(a,b) = 2u(a,b),$$

which means that the associated zero-sum game G^0 is isomorphic to the original one.

Remark 3. If the original game is symmetric, i.e. $A = B$ and $v(a,b) = u(b,a)$, then

$$f(a,b) = u(a,b) - v(a,b) = v(b,a) - u(b,a) = -f(b,a),$$

and consequently, the payoff function $f(a,b)$ in the associated zero-sum game G^0 is skew-symmetric:

$$f(a,b) = -f(b,a).$$

Thus $f(a,b) = g(b,a)$, which means that the game G^0 is symmetric.

The associated game and unbeatable strategies. We reformulate the definition of an unbeatable strategy in the game G in terms of the zero-sum game G^0. A strategy a^* of player 1 is unbeatable if and only if $f(a^*,b) \geq 0$ for every strategy b of player 2, or equivalently,

$$\inf_{b \in B} f(a^*,b) \geq 0.$$

A strategy b^* of player 2 is unbeatable if and only if $f(a, b^*) \leq 0$ for every strategy a of player 1, or equivalently,

$$\sup_{a \in A} f(a, b^*) \leq 0.$$

Clearly, an unbeatable strategy of player 1 exists if and only if either

$$\underline{f} := \sup_{a \in A} \inf_{b \in B} \ f(a, b) > 0$$

or

$$\max_{a \in A} \inf_{b \in B} \ f(a, b) \geq 0.$$

Player 2 possesses an unbeatable strategy if and only if either

$$\overline{f} := \inf_{b \in B} \sup_{a \in A} f(a, b) < 0$$

or

$$\min_{b \in B} \sup_{a \in A} f(a, b) \leq 0.$$

Here and in what follows, we write "max" in place of "sup" and "min" in place of "inf" if the corresponding extremum is attained.

The numbers $\overline{f}$ and $\underline{f}$ are called the *upper* and the *lower values* of the zero-sum game G^0, respectively. The former is always not less than the latter.

Existence of unbeatable strategies. A function $F(x)$ defined on a topological space X is called *upper semicontinuous* if for every real number r the upper-level set of this function $\{x : F(x) \geq r\}$ is closed. This function is termed *lower semicontinuous* if every lower-level set $\{x : F(x) \leq r\}$ of this function is closed. An upper semicontinuous function attains its maximum and a lower semicontinuous function attains its minimum on a compact set.

Consider the following conditions.

(**A**) The strategy set A is a compact topological space, for each b the function $f(a, b)$ is upper semicontinuous with respect to a, and $\underline{f} \geq 0$.

(**B**) The strategy set B is a compact topological space, for each a the function $f(a, b)$ is lower semicontinuous with respect to b, and $\overline{f} \leq 0$.

Define

$$\underline{f}(a) = \inf_{b \in B} f(a,b), \quad \overline{f}(b) = \sup_{a \in A} f(a,b).$$

The following result provides simple and general sufficient conditions for the existence of unbeatable strategies.

Theorem 1. *If assumption (**A**) holds, then the function $\underline{f}(a)$ attains its maximum on A, and any element a^* of the set A maximizing $\underline{f}(a)$ is an unbeatable strategy of player 1. If condition (**B**) is fulfilled, then the function $\overline{f}(b)$ attains its minimum on B, and any element b^* of the set B minimizing $\overline{f}(b)$ is an unbeatable strategy of player 2. If one of assumptions (**A**) and (**B**) holds, then the game is determinate.*

Proof. Suppose (**A**) is satisfied. Then the function $\underline{f}(a)$ is upper semicontinuous in a because for each r the intersection of the closed sets

$$\{a : \underline{f}(a) \geq r\} = \bigcap_{b \in B} \{a : f(a,b) \geq r\}$$

is closed. Therefore $\underline{f}(a)$ attains its maximum on the compact set A. If $a^* \in A$ is a point where this maximum is attained, then

$$\underline{f}(a^*) = \max_{a \in A} \underline{f}(a) = \max_{a \in A} \inf_{b \in B} f(a,b) = \underline{f} \geq 0$$

by virtue of assumption (**A**), consequently,

$$\underline{f}(a^*) = \inf_{b \in B} f(a^*, b) \geq 0,$$

which means that a^* is an unbeatable strategy of player 1.

Let (**B**) hold. In this case the function $\overline{f}(b)$ is lower semicontinuous in b since for every r the set

$$\{b : \overline{f}(b) \leq r\} = \bigcap_{a \in A} \{b : f(a,b) \leq r\}.$$

is closed. Consequently, $\overline{f}(b)$ attains its minimum on the compact set B. If b^* minimizes $\overline{f}(b)$, then

$$\overline{f}(b^*) = \min_{b \in B} \overline{f}(b) = \min_{b \in B} \sup_{a \in A} f(a,b) = \overline{f} \leq 0.$$

according to condition (**B**). Thus

$$\overline{f}(b^*) = \sup_{a \in A} f(a, b^*) = \overline{f} \leq 0,$$

and so b^* is an unbeatable strategy of player 2. □

Remark 4. The above result suggests that the analysis of unbeatable strategies in static two-player games (with standard numerical preferences) is a "much easier" task than, say, the analysis of Nash equilibrium strategies. First of all, a general game G reduces to a zero-sum one, the associated game G^0. If G^0 satisfies some very general assumptions of semicontinuity and compactness, then to prove the existence of an unbeatable strategy for one player or another we have only to find the upper value $\overline{f}$ or the lower value $\underline{f}$ of the game G^0 and simply check its sign. If $\underline{f} \geq 0$ (resp. $\overline{f} \leq 0$), then player 1 (resp. player 2) has an unbeatable strategy. Moreover, results of the above type provide effective constructions of unbeatable strategies based on minimization and maximization procedures. They do not rely upon pure existence theorems, such as Brouwer's or Kakutani's ones. Note that in Nash equilibrium analysis, such constructions are possible only for potential games.

Unbeatable strategies and saddle points. Let us examine relations between unbeatable strategies in the original game G and saddle points (Nash equilibria) in the associated zero-sum game G^0. A pair $(\bar{a}, \bar{b}) \in A \times B$ is called a *saddle point* of the function $f(a, b)$, or of the zero-sum game G^0, if

$$f(a, \bar{b}) \leq f(\bar{a}, \bar{b}) \leq f(\bar{a}, b) \text{ for all } a \text{ and } b. \tag{134}$$

Proposition 1. *If the associated zero-sum game G^0 has a saddle point $(\bar{a}, \bar{b})$, then the original game G is determinate. Specifically, if $f(\bar{a}, \bar{b}) \geq 0$, then $\bar{a}$ is an unbeatable strategy of player 1. If $f(\bar{a}, \bar{b}) \leq 0$, then $\bar{b}$ is an unbeatable strategy of Player 2. If $f(\bar{a}, \bar{b}) = 0$, then $\bar{a}$ and $\bar{b}$ are unbeatable strategies of Players 1 and 2, respectively.*

Proof. If $f(\bar{a}, \bar{b}) \geq 0$, then $\bar{a}$ is an unbeatable strategy of player 1 because by virtue of (134),

$$f(\bar{a}, b) \geq f(\bar{a}, \bar{b}) \geq 0$$

for all b. If $f(\bar{a}, \bar{b}) \leq 0$, then

$$f(a, \bar{b}) \leq f(\bar{a}, \bar{b}) \leq 0$$

for all a, and so $\bar{b}$ is an unbeatable strategy of player 2. Consequently, if $f(\bar{a}, \bar{b}) = 0$, then $\bar{a}$ and $\bar{b}$ are unbeatable strategies of players 1 and 2, respectively. □

Remark 5. By virtue of Proposition 1, the existence of a saddle point in the associated zero-sum game G^0 is a sufficient condition for the determinacy of the original game G. But this condition is by no means necessary—see, e.g., Example 1 below. Moreover, in Section 4 we will show that determinacy occurs in a sense "substantially more often" than a saddle point. As Theorem 1 demonstrates, the assumptions guaranteeing the existence of unbeatable strategies are quite general and make it possible to construct such strategies effectively. Existence theorems for saddle points, or *minimax theorems*, are mathematically deeper and require stronger assumptions. Although such results are typically non-constructive, they might be useful when one needs to establish only the fact of existence of a saddle point in the associated zero-sum game G^0—and consequently, the determinacy of G—without finding unbeatable strategies. The literature contains a whole variety of minimax theorems, most of which pertain to zero-sum games with strategy sets in linear spaces and require assumptions of convexity— see, e.g., Willem [178]. Results on the existence of saddle points that do not assume convexity and use other conditions (submodularity, increasing differences, finite actions, discrete quasiconcavity, existence of potentials, etc.) are contained in the paper by Duersch, Oechssler and Schipper [49], where special attention is paid to symmetric relative payoff games.

Minimax theorems and unbeatable strategies. As an important example of a minimax theorem that can be employed for proving the existence of saddle points in zero-sum games and the existence of unbeatable strategies, we will consider the classical Sion's theorem, one of the most powerful and widely used results of this kind. A function $F(x)$ defined on a linear space X is termed *quasi-concave* if for every real number r the upper-level set of this function $\{x : F(x) \geq r\}$ is convex. This function is called *quasi-convex* if every lower-level set $\{x : F(x) \leq r\}$ of this function is convex.

Let us introduce the following conditions.

(**S1**) A and B are convex sets in linear topological spaces, the function $f(a, b)$ is quasi-concave in a and quasi-convex in b. The set A is compact, and the function $f(a, b)$ is upper semicontinuous in a.

(**S2**) The set B is compact and the function $f(a,b)$ is lower semicontinuous in b.

Theorem 2 (Sion [163]). *(i) Under assumption (**S1**), we have*

$$\max_{a\in A}\inf_{b\in B}\ f(a,b) = \inf_{b\in B}\max_{a\in A}\ f(a,b). \tag{135}$$

*(ii) If additionally assumption (**S2**) holds, then the zero-sum game at hand has a saddle point.*

Clearly assertion (ii) in the above theorem is a direct consequence of (i).

Counter-strategies. Let us show how Theorem 2 can be used for proving the existence of unbeatable strategies. Consider the following hypothesis.

(**P**) For each strategy b of player 2, there exists a strategy $a^*(b)$ of player 1 satisfying

$$u(a^*(b),b) \geq v(a^*(b),b).$$

This hypothesis means that player 1 can respond to every strategy b of player 2 with a *counter-strategy* $a^*(b)$ that does not permit the latter to beat the former. Clearly condition (**P**) is necessary for the existence of an unbeatable strategy for player 1. Indeed, an unbeatable strategy is nothing but a counter-strategy $a^* = a^*(b)$ that *does not depend on* b. We will show that under assumption (**S1**), condition (**P**) is not only necessary, but also sufficient for the existence of an unbeatable strategy of player 1.

Proposition 2. *Under assumptions (**S1**) and (**P**), player 1 possesses an unbeatable strategy.*

Proof. From (**P**) we get $f(a^*(b),b) \geq 0$. Consequently, $\max_{a\in A}\ f(a,b) \geq 0$ for each b, and therefore $\inf_{b\in B}\max_{a\in A}\ f(a,b) \geq 0$. By using (135), we obtain $\max_{a\in A}\inf_{b\in B}\ f(a,b) = \inf_{b\in B}\max_{a\in A}\ f(a,b) \geq 0$. Let a^* be the point in A where the maximum on the left-hand side of this equality is attained. Then $\inf_{b\in B}\ f(a^*,b) \geq 0$, i.e., a^* is an unbeatable strategy of Player 1. □

When does determinacy imply a saddle point? As Proposition 1 shows, the existence of a saddle point in the zero-sum game G^0 implies the determinacy of the original game G. The proposition below lists some cases when the converse is true.

Proposition 3. *(a) If both players have unbeatable strategies, a^* and b^*, then (a^*,b^*) is a saddle point in the game G^0, and $f(a^*,b^*) = 0$. (b) If the game G is symmetric, then any unbeatable strategy a^* of one of the players is an unbeatable strategy of the other, the pair (a^*,a^*)*

is a saddle point in the game G^0, *and* $f(a^*, a^*) = 0$. *Thus a symmetric game* G *is determinate if and only if the zero-sum game* G^0 *possesses a symmetric saddle point.*

Proof. To prove (a) we observe that $f(a, b^*) \leq 0 \leq f(a^*, b)$ for all a and b, which implies that $f(a^*, b^*) = 0$ and shows that (a^*, b^*) is a saddle point in the game G^0. Assertion (b) is a consequence of (a). □

Finite games. Suppose that the strategy sets A and B are finite: $A = \{a_1, ..., a_k\}$ and $B = \{b_1, ..., b_l\}$. Put $u_{ij} = u(a_i, b_j)$ and $v_{ij} = v(a_i, b_j)$. Consider the payoff matrix (f_{ij}) of the associated zero-sum game G^0:

$$f_{ij} = f(a_i, a_j) = u_{ij} - v_{ij}.$$

According to the above definition, a strategy a_n of player 1 is unbeatable if in the nth row in the payoff matrix of the original game G, the payoffs of the first player are not less than the payoffs of the second player:

$$u_{nj} \geq v_{nj} \text{ for all } j.$$

A strategy b_m of player 2 is unbeatable if in the mth column of the payoff matrix, the payoffs of the second player are not less than the payoffs of the first player:

$$v_{im} \geq u_{im} \text{ for all } i.$$

Consequently, a strategy a_n of player 1 is unbeatable if all the elements f_{nj} in the nth row of the matrix (f_{ij}) are non-negative:

$$f_{nj} \geq 0 \text{ for all } j.$$

A strategy b_m of player 2 is unbeatable if all the elements f_{im} in the mth column of this matrix are non-positive:

$$f_{im} \leq 0 \text{ for all } i.$$

Thus we obtain the following result regarding unbeatable strategies in the original game G formulated in terms of the payoff matrix (f_{ij}) of the associated zero-sum game G^0.

Proposition 4. *Player 1 possesses an unbeatable strategy if and only if the matrix* (f_{ij}) *has a non-negative row. Player 2 possesses an unbeatable strategy if and only if the matrix* (f_{ij})

has a non-positive column. The game is determinate if and only if the matrix (f_{ij}) has either a non-negative row or a non-positive column.

Example 1. Consider the following game G with the payoffs $u_{ij} = u(a_i, b_j)$, $v_{ij} = v(a_i, b_j)$ of players 1 and 2 and the associated zero-sum game G^0 with the payoffs $f_{ij} = u_{ij} - v_{ij}$ of player 1:

G	b_1	b_2	b_3
a_1	$-3, -1$	$3, 1$	$2, 0$
a_2	$0, 1$	$4, 6$	$5, 4$
a_3	$1, 2$	$2, 4$	$3, 3$

G^0		b_1	b_2	b_3	$\min_j f_{ij}$
	a_1	-2	2	2	-2
	a_2	-1	-2	1	-2
	a_3	-1	-2	0	-2
$\max_i f_{ij}$		-1	2	2	

We have

$$\min_j \max_i f_{ij} = -1, \ \max_i \min_j f_{ij} = -2.$$

• The matrix (f_{ij}) has a non-positive column, and so player 2 has an unbeatable strategy.

• There are no non-negative rows in the matrix (f_{ij}), and so player 1 does not have unbeatable strategies.

• The game is determinate but the associated zero sum game does not have a saddle point because $\max \min f_{ij} \neq \min \max f_{ij}$.

Pyrrhic victory. Let us look at the game considered in the above example. Clearly the strategy b_1 of player 2 is unbeatable: it yields payoff greater than the payoff of player 1, irrespective of his strategy. Thus b_1 is good in terms of the *relative* payoffs. However, in terms of the *absolute* payoffs, b_1 is the worst (strictly dominated by any other!) strategy of player 2. This seeming paradox demonstrates that the rationality in terms of a relative criterion may be wildly inconsistent with the rationality in terms of the absolute one. The strategy b_2 allows player 2 to gain the victory over player 1, but this is a Pyrrhic victory—a victory that is so devastating for the victor that it is tantamount to defeat. It is achieved at the expense of a dramatic reduction in player 2's payoff, which is less, however, than the reduction in the payoff of player 1.

4.4. Random game: Determinacy vs. saddle point

Finite game with random payoffs. As we have seen above, the existence of a saddle point in the associated zero-sum game G^0 is sufficient but not necessary for the determinacy of the original game G. Let us show that in a natural probabilistic sense the determinacy of G is a much more frequent event than a saddle point in G^0. Consider a zero-sum game G_n with a finite set of strategies $A = \{a_1, a_2, ..., a_n\}$, the same for both players, and payoffs

$$u_{ij} = u(a_i, a_j),\ v_{ij} = -u(a_i, a_j)$$

of the first and the second players, respectively. The payoff matrix (f_{ij}) of the associated zero-sum game G^0 is defined by

$$f_{ij} = f(a_i, a_j) = u_{ij} - v_{ij} = 2u_{ij}.$$

Suppose that u_{ij}, and therefore f_{ij}, are independent identically distributed random variables with a continuous distribution. Denote by p_1 the probability that $f_{ij} \geq 0$ and by p_2 the probability that $f_{ij} \leq 0$, in symbols,

$$p_1 = P\{f_{ij} \geq 0\},\ p_2 = P\{f_{ij} \leq 0\}.$$

We will assume that both numbers p_1 and p_2 are strictly positive.

Probabilities of determinacy and saddle point. Denote by Δ_n^i $(i = 1, 2)$ the probability that player i in the game G_n has an unbeatable strategy, by Δ_n the probability that this game is determinate, and by Σ_n the probability that it has a saddle point.

Proposition 5. *We have*

$$\Sigma_n = \frac{(n!)^2}{(2n-1)!}, \tag{136}$$

$$\Delta_n^i = 1 - (1 - p_i^n)^n,\ i = 1, 2, \tag{137}$$

$$\Delta_n = \Delta_n^1 + \Delta_n^2. \tag{138}$$

Proof. Formula (136) is obtained in Goldman [73].

As we have shown in the previous section (see Proposition 2), Δ_n^1 coincides with the probability that the random matrix (f_{ij}) has a non-negative row. The probability that some particular row is non-negative is equal to p_1^n, and so the probability that at least one row is non-negative is $1-(1-p_1^n)^n$. This yields (137) for $i=1$.

By virtue of Proposition 2, the number Δ_n^2 represents the probability that the random matrix (f_{ij}) possesses a non-positive column. It is equal to $1-(1-p_2^n)^n$, which implies (137) for $i=2$.

If the matrix f_{ij} has both a non-negative row and a non-positive column, it must have a zero element. Since the distribution of f_{ij} is continuous, the probability of this event is equal to zero. Consequently, $\Delta_n = \Delta_n^1 + \Delta_n^2$. □

The ratio Σ_n/Δ_n. Clearly both the probability Σ_n of a saddle point and the probability Δ_n of determinacy tend to zero as $n \to \infty$, but the former tends to zero faster than the latter.

Proposition 6. *The ratio Σ_n/Δ_n tends to zero at an exponential rate.*

Proof. By using the Stirling formula $n! \sim \sqrt{2\pi n}n^n e^{-n}$, we get

$$\Sigma_n = \frac{(n!)^2}{(2n-1)!} \sim \frac{2\pi n \cdot n^{2n}e^{-2n}}{\sqrt{2\pi(2n-1)}(2n-1)^{2n-1}e^{-(2n-1)}} =$$

$$= \frac{\sqrt{2\pi}e^{-1}\cdot n \cdot (2n-1)\cdot n^{2n}}{\sqrt{2n-1}(2n-1)^{2n}} = \sqrt{2\pi}e^{-1}n\sqrt{2n-1}\left(\frac{n}{2n-1}\right)^{2n}, \tag{139}$$

where

$$\left(\frac{n}{2n-1}\right)^2 = \left(\frac{1}{2-1/n}\right)^2 \le 1/3$$

for all sufficiently large n. Consequently, for some constant C and all n large enough, we have

$$\Sigma_n \le Cn^2 3^{-n}. \tag{140}$$

Let us show that

$$g_n := 1-(1-p^n)^n \sim np^n \tag{141}$$

for any $0<p<1$. We can represent $\exp x$ and $\ln(1+x)$ as

$$\exp x = 1 + x(1+\alpha(x)),\ \ln(1+x) = x(1+\beta(x)),$$

with $\alpha(x)\to 0$, $\beta(x)\to 0$ as $x\to 0$. Therefore

$$g_n = 1-\exp[n\ln(1-p^n)] = 1-\exp[n\ln(1-p^n)] = 1-\exp r_n$$

where $r_n := -np^n(1+\beta_n) \to 0$ and $\beta_n := \beta(-p^n) \to 0$. By setting $\alpha_n := \alpha(r_n)$, we obtain $g_n = 1 - \exp r_n = -r_n(1+\alpha_n)$, which yields

$$\frac{g_n}{np^n} = \frac{np^n(1+\beta_n)(1+\alpha_n)}{np^n} \to 1.$$

Since $p_1 + p_2 = 1$, one of the numbers p_1, p_2 is not less than 1/2, and so

$$\Delta_n = \Delta_n^1 + \Delta_n^2 \geq 1 - [1-(1/2)^n]^n \geq cn2^{-n} \tag{142}$$

for some constant c and all sufficiently large n. From (140) and (142) we obtain that

$$\Sigma_n/\Delta_n \leq Cn^2 3^{-n}/cn2^{-n} \leq (C/c)n(2/3)^n$$

for all n large enough. □

4.5. Unbeatable strategies in evolutionary game theory

Population model (see, e.g., Weibull [177]). Our next goal is to demonstrate applications of unbeatable strategies in evolutionary game theory[11]. Members of a population of organisms (e.g. animals, human beings, plants, etc.) interact pairwise. Each organism can be of a certain type x. The set of possible types is X. There is a function $u(x, y)$, $x, y \in X$ (*fitness function*) that characterizes the ability of organisms to survive. If an organism is of a type x and faces the probability distribution β of types y in the population, then its ability to survive is characterized by the expectation of $u(x, y)$ with respect to β. In evolutionary biology, elements x in X might represent *genotypes* of species and $u(x, y)$ the (average) number of surviving offspring. In evolutionary economics, such models serve to describe interactions in large populations of economic agents. Types x can represent various characteristics of economic agents and/or patterns of their behaviour.

Symmetric game. With the given model, we associate a symmetric two-player game in which the payoff functions of the players are $u(x, y)$ and $v(x, y) = u(y, x)$, and their common

[11] The main source for this section is the paper by Kojima [103]. However, to simplify the exposition we focus primarily on models based on symmetric two-player two-strategy games. This makes it possible to elucidate key concepts in an elementary but sufficiently rich setting.

strategy set is X. In this context, we will use the terms "types" and "strategies" interchangeably. The values of the fitness function $u(x,y)$ will be interpreted as "payoffs".

Let us say that a strategy x^* is *(strictly) unbeatable* if

$$(1-\varepsilon)u(x^*,x^*)+\varepsilon u(x^*,x) > \ (1-\varepsilon)u(x,x^*)+\varepsilon u(x,x) \tag{143}$$

for all $x \neq x^*$ and all $0<\varepsilon<1$. We will omit "strictly" in what follows. For an unbeatable strategy x^*, inequality (143) must hold for all $x \neq x^*$ and all $0<\varepsilon<1$. For an *evolutionary stable strategy (ESS)* x^*, according to its definition, it must hold only for $\varepsilon>0$ small enough, which means that "non-mutants" x^* outperform "mutants" x only if the fraction of the mutants is small enough. The definition of an unbeatable strategy requires that this should be true always, not only when the fraction of mutants is sufficiently small.

Proposition 7. *A strategy x^* is unbeatable if and only if for each $x \neq x^*$ at least one of the following conditions is fulfilled:*

$$u(x,x^*)<u(x^*,x^*) \textit{ and } u(x,x)\leq u(x^*,x), \tag{144}$$

$$u(x,x^*)\leq u(x^*,x^*) \textit{ and } u(x,x)<u(x^*,x). \tag{145}$$

Proof. Suppose (144) holds. Multiply the first inequality in (144) by $1-\varepsilon$, the second by ε, and add up. This will yield (143). The same argument shows that (145) implies (143). Conversely, observe that inequality (143) holds for each $0<\varepsilon<1$ if and only if it holds as a non-strict inequality both for $\varepsilon=0$ and $\varepsilon=1$ and as a strict inequality in at least one of the two cases: $\varepsilon=0$ and $\varepsilon=1$ The former case corresponds to (144) and the latter to (145). □

Remark 6. We compare (144) and (145) with the conditions characterizing ESS: for each $x \neq x^*$, we have either

$$u(x,x^*)<u(x^*,x^*) \tag{146}$$

(x^* is a strict Nash equilibrium) or

$$u(x,x^*)=u(x^*,x^*) \text{ and } u(x,x)<u(x^*,x). \tag{147}$$

Note that the assertion that at least one of the conditions (144) and (145) holds is equivalent to the assertion that for each $x \neq x^*$, one (and only one) of the following two requirements is fulfilled:

(I) $u(x, x^*) < u(x^*, x^*)$ and $u(x, x) \leq u(x^*, x)$;

(II) $u(x, x^*) = u(x^*, x^*)$ and $u(x, x) < u(x^*, x)$.

Indeed, the inequality $u(x, x^*) \leq u(x^*, x^*)$ involved in (145) can hold either as a strict inequality, and then we have (I), or as equality, which leads to (II). Note that condition (II) coincides with property (147) in the definition of ESS, but condition (I) contains together with the strict equilibrium property $u(x, x^*) < u(x^*, x^*)$ stated in (146) the additional requirement $u(x, x) \leq u(x^*, x)$. This shows, in particular, that a strict symmetric Nash equilibrium is always an ESS, but it is not necessarily an unbeatable strategy.

Mixed strategies in two-player two-strategy games. Let us consider the concept of an unbeatable strategy in the case where X is the set of mixed strategies in a symmetric two-player game with two strategies a_1, a_2 and the payoffs $u_{ij} = u(a_i, a_j)$ of the first player:

	a_1	a_2
a_1	$u_{11} = u(a_1, a_1)$	$u_{12} = u(a_1, a_2)$
a_2	$u_{21} = u(a_2, a_1)$	$u_{22} = u(a_2, a_2)$

. (148)

Note that unbeatable strategies, as well as ESS, are defined in terms of the differences $u(\alpha, \beta) - u(\beta, \beta)$, where $\alpha = (p, 1-p)$ and $\beta = (q, 1-q)$ are mixed strategies. It follows from this that unbeatable strategies and ESS are the same for the original game and the following *simple* game:

	a_1	a_2
a_1	$u_1 = u_{11} - u_{21}$	0
a_2	0	$u_2 = u_{22} - u_{12}$

(149)

"Simple", by definition, means that non-diagonal payoffs are equal to zero. In the analysis of such games, we will assume (to exclude degenerate cases) that $u_1 \neq 0$ and $u_2 \neq 0$. The simple game (149) will be called the *reduced version* of the original one.

Our goal is to characterize those mixed strategies $\beta = (q, 1-q)$ which are unbeatable, i.e. satisfy for all $\alpha \neq \beta$ conditions (**I**) or (**II**):

(**I**) $u(\alpha,\beta) < u(\beta,\beta)$ and $u(\alpha,\alpha) \le u(\beta,\alpha)$,

(**II**) $u(\alpha,\beta) = u(\beta,\beta)$ and $u(\alpha,\alpha) < u(\beta,\alpha)$.

ESS in simple games. As is known (see, e.g., Weibull [177]), the structure of ESS in the game at hand is as follows:

		ESS	$\beta = (q, 1-q)$
Case 1	$u_1 < 0, u_2 < 0$	one	$q = q^*$, $q^* = \dfrac{u_2}{u_1+u_2}$, $1 - q^* = \dfrac{u_1}{u_1+u_2}$,
Case 2	$u_1 > 0, u_2 > 0$	two	$q = 0,\ 1,$
Case 3	$u_1 < 0, u_2 > 0$	one	$q = 0,$
Case 4	$u_1 > 0, u_2 < 0$	one	$q = 1.$

In cases 2, 3 and 4, ESS are strict Nash equilibria: $u(\alpha,\beta) < u(\beta,\beta)$ for all $\alpha \neq \beta$.

Evolutionary stable and unbeatable strategies. Let us find out which of the ESS described above are unbeatable.

Case 1. Let us show that $\beta^* = (q^*, 1-q^*)$ is an unbeatable strategy. We have

$$u(\alpha,\beta^*) = u(\beta^*,\alpha) = p \cdot q^* u_1 + (1-p)\cdot(1-q^*)u_2 = \frac{u_1u_2}{u_1+u_2} = u(\beta^*,\beta^*)$$

for each $\alpha = (p, 1-p)$ because

$$q^*u_1 = (1-q^*)u_2 = \frac{u_1u_2}{u_1+u_2}.$$

Thus $u(\alpha,\beta^*) = u(\beta^*,\beta^*)$, and so we need to verify the inequality in (**II**): $u(\alpha,\alpha) < u(\beta^*,\alpha)$ for all $\alpha \neq \beta^*$. Since

$$u(\alpha,\alpha) = p^2u_1 + (1-p)^2u_2,$$

this inequality can be written:

$$p^2u_1 + (1-p)^2u_2 < u_1u_2/(u_1+u_2),\ p \neq q^*.$$

But this is indeed true: the concave quadratic function $\psi(p) = p^2u_1 + (1-p)^2u_2$ ($u_1 < 0$, $u_2 < 0$) attains its maximum $u_1u_2/(u_1+u_2)$ at the unique point $p = q^* = \dfrac{u_2}{u_1+u_2}$, where its derivative is equal to zero.

In all the other cases (2, 3 and 4) ESS are strict, and therefore we have to check the second inequality in (**I**), which can be written as

$$u(\alpha,\alpha) = p^2u_1 + (1-p)^2u_2 \le pqu_1 + (1-p)(1-q)u_2 = u(\beta,\alpha). \quad (150)$$

Case 2: neither $q = 0$, nor $q = 1$ are unbeatable. Indeed, if $q = 0$, then (150) becomes $p^2u_1 + (1-p)^2u_2 \le (1-p)u_2$, which is not true for $p = 1$. If $q = 1$, then (150) yields $p^2u_1 + (1-p)^2u_2 \le pu_1$, which is wrong for $p = 0$.

Case 3: $q = 0$ is unbeatable because $p^2u_1 + (1-p)^2u_2 \le (1-p)^2u_2 \le (1-p)u_2$.

Case 4: $q = 1$ is unbeatable because $p^2u_1 + (1-p)^2u_2 \le p^2u_1 \le pu_1$.

We summarize the results obtained in the following table:

		unbeatable strategies	$(q, 1-q)$
Case 1	$u_1 < 0, u_2 < 0$	one	$q = q^*$, $q^* = \dfrac{u_2}{u_1 + u_2}$,
Case 2	$u_1 > 0, u_2 > 0$	no unbeatable strategies	
Case 3	$u_1 < 0, u_2 > 0$	one	$q = 0$,
Case 4	$u_1 > 0, u_2 < 0$	one	$q = 1$.

Are unbeatable strategies that rare? There is a widespread view that unbeatable strategies are rare compared with ESS. However, the model at hand does not confirm this view. Only in one of four cases (case 2) it happens that an ESS fails to be unbeatable.

Selection model. Our next goal is to examine unbeatable strategies in a different model. The previous one considered mutations, this one focuses on selection. Although they are different, there are deep connections between them. Consider the symmetric two-strategy game (148). There is a large population of players. Some of them select the strategy a_1 and the others a_2. The fraction of a_1 players is $x \in (0,1)$, and the fraction of a_2 players is $1-x$. The average payoff of those playing a_1 is $U_1 = xu_{11} + (1-x)u_{12}$ because an a_1 player encounters another a_1 player in the population with probability x and an a_2 player with probability $1-x$. For similar reasons the average payoff of those playing a_2 is $U_2 = xu_{21} + (1-x)u_{22}$. The average payoff across the population can be expressed as $U = xU_1 + (1-x)U_2$.

Replicator Dynamics. The proportion of a_1 players goes up if on average they are doing better than the overall average, and down otherwise. This principle is expressed by the *replicator dynamics (RD)* equation (Taylor and Jonker [165]):

$$\frac{x'}{x} = U_1 - U. \tag{151}$$

Here $x' = x'(t)$ is the derivative of the function $x(t)$ with respect to time t, and x'/x is the growth rate of $x(t)$.

By using the definitions of U_1, U_2 and U we transform (151) to

$$x' = x(1-x)[x(u_{11} - u_{21}) - (1-x)(u_{22} - u_{12})],$$

which can also be written as

$$x' = f(x), \tag{152}$$

where

$$f(x) := x(1-x)[xu_1 - (1-x)u_2], \; u_1 = u_{11} - u_{21}, \; u_2 = u_{22} - u_{12}.$$

Note that the function $f(x)$ is the same for the original game and its reduced version (see (149)). Thus we can focus on the differential equation (152) for the simple game (149). As before it will be assumed that $u_1 u_2 \neq 0$. Note that the equation $f(x) = 0$, has always the roots $x = 0$ and $x = 1$. This equation has a root x in the interval $0 < x < 1$ if and only if u_1 and u_2 have the same signs, and then this root is given by

$$x = q^* = \frac{u_2}{u_1 + u_2}$$

(compare with the ESS in the previous model!).

Definition 4. An *evolutionary stable steady state (ESSS)* is defined as an asymptotically stable steady state[12] of the dynamical system (152).

Replicator dynamics and ESS. We wish to analyze the asymptotic behaviour of paths of the dynamical system described by the differential equation (152).

[12] A steady state $\bar{x}$ of a dynamical system is called *asymptotically stable* if trajectories of the system starting from points x sufficiently close to $\bar{x}$ converge to $\bar{x}$.

Case 1: $u_1 < 0, u_2 < 0$. Then $0 < q^* < 1$, and for $0 < x < 1$, we have

$$xu_1 - (1-x)u_2 > 0 \text{ if and only if } x < q^* = \frac{u_2}{u_1 + u_2}.$$

Consequently, $f(x) > 0$ when $0 < x < q^*$ and $f(x) < 0$ when $q^* < x < 1$. Thus we have *convergence to* q^* *from every starting point* $0 < x < 1$*:*

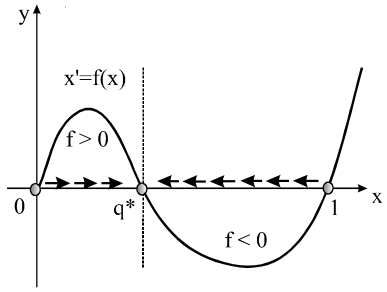

Fig.1

Case 2: $u_1 > 0, u_2 > 0$. Then $0 < q^* < 1$, and for $0 < x < 1$, we have

$$xu_1 - (1-x)u_2 > 0 \text{ if and only if } x > q^* = \frac{u_2}{u_1 + u_2}.$$

Thus $f(x) > 0$ when $q^* < x < 1$ and $f(x) < 0$ when $0 < x < q^*$. Consequently, *paths of the RD system converge to* 0 *if they start from any* $0 < x < q^*$ *and to* 1 *if they start from any* $q^* < x < 1$:

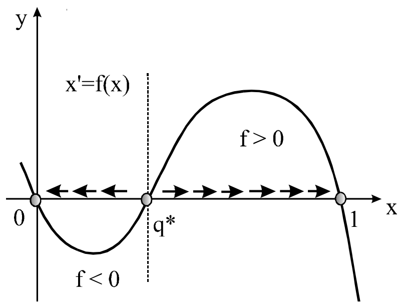

Fig. 2

Case 3: $u_1 < 0$, $u_2 > 0$. The dynamics of the RD process is shown in the following diagram:

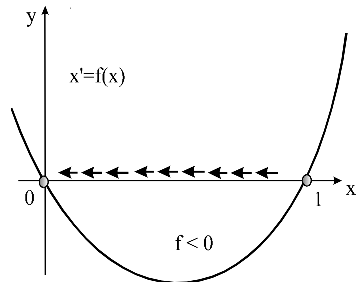

Fig. 3

We have $f(x) < 0$ for $0 < x < 1$, therefore the *RD process starting from every initial state* $0 < x < 1$ *converges to* 0.

Case 4: $u_1 > 0$ and $u_2 < 0$. The function $f(x)$ is strictly positive for $0 < x < 1$, and therefore *the RD process starting from every point* $0 < x < 1$ *converges to* 1.

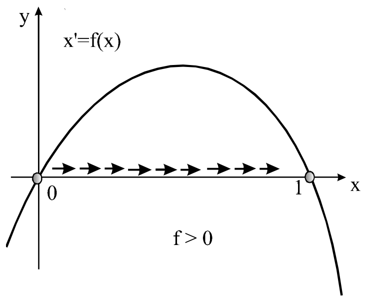

Fig. 4

In all the cases considered, *the evolutionary stable steady states of the replicator dynamics process coincide with the evolutionary stable strategies of the underlying game!*

Replicator dynamics and unbeatable strategies. We will show that for the replicator dynamics process those ESSS which correspond to unbeatable strategies form globally evolutionary stable steady states.

Definition 5. A *globally evolutionary stable steady state (GESSS)* is a globally asymptotically stable steady state[13] of the dynamical system $x' = f(x)$.

[13]A steady state $\bar{x}$ of a dynamical system is called *globally asymptotically stable* if trajectories of the system

Case 1: $u_1 < 0$, $u_2 < 0$. As it was shown, $q^* = u_1/(u_1+u_2)$ corresponds to the unique ESS, which is an unbeatable strategy. We can see from Fig. 1 that the RD process converges to q^* starting from *any* initial point x in the interval $0 < x < 1$, and not only from points sufficiently close to q^*. Consequently, q^* is GESSS.

Case 2: $u_1 > 0, u_2 > 0$. The dynamics of the RD process is illustrated in Fig. 2. There are two ESS: $q = 0$ and $q = 1$, but *none* of them represents a GESSS. Convergence to 1 takes place only for trajectories of the RD process starting from initial states $q^* < x < 1$ and convergence to 0 takes place only for trajectories starting from $0 < x < q^*$ (recall that $q^* = u_2/(u_1 + u_2)$).

Case 3: $u_1 < 0$, $u_2 > 0$. According to Fig. 3, the ESS $q = 0$ is globally evolutionary stable since the RD process starting from *every* initial state $0 < x < 1$ converges to 0.

Case 4: $u_1 > 0$ and $u_2 < 0$. The function $f(x)$ is strictly positive for $0 < x < 1$, and therefore the RD process starting from *every* point $0 < x < 1$ converges to 1 (see Fig. 4).

In all the cases considered, *the globally evolutionary stable steady states of the replicator dynamics process correspond to the unbeatable strategies of the underlying game!*

Convergence of the RD process: justification. The above conclusions on the asymptotic behaviour of the RD process were made based on the analysis of diagrams. Here, we provide some additional comments to justify these results.

Proposition 8. *Let $f(x)$ be a differentiable function on a segment $[a, b]$. (i) If $f(x) > 0$ for $x \in (a, b)$ and $f(b) = 0$, then the solution $x(t)$, $t \geq 0$, to the differential equation*

$$x'(t) = f(x(t)),\ x(0) = x_0 \tag{153}$$

is a strictly increasing function taking on its values in (x_0, b) and converging to b as $t \to \infty$. (ii) If $f(x) < 0$ for $x \in (a, b)$ and $f(a) = 0$, then the solution $x(t)$, $t \geq 0$, to with values in (a, b) and converging to a as $t \to \infty$.

Proof. Let us prove the first assertion, the second is proved analogously. Define

$$F(x) = \int_{x_0}^{x} \frac{dy}{f(y)},\ x_0 \leq x < b.$$

starting from *any* initial point in the domain of the system (and not only from points in some sufficiently small neighborhood of $\bar{x}$) converge to $\bar{x}$.

The function $F(x)$ is differentiable, strictly increases on $[x_0, b)$ and satisfies $F(x_0) = 0$. Since $f(y)$ is differentiable at b, we have $f(y) \leq C(b - y)$ for some constant C and all $y \in [x_0, b)$ sufficiently close to b. Therefore $F(x) \uparrow \infty$ as $x \uparrow b$, and so the equation

$$F(x(t)) = t, \; x(0) = x_0, \tag{154}$$

has a unique solution (the inverse function to $F(x)$) converging to b as $t \to \infty$. Clearly a function $x(t)$ is a solution to (153) if and only if it is a solution to (153). □

4.6. Unbeatable strategies in an asymmetric homogeneous-good Cournot duopoly

Asymmetric duopoly with capacity constraints. In this section we examine unbeatable strategies in an asymmetric Cournot duopoly model with a homogeneous good. The production units in this model feature different production costs and different capacity constraints. We conduct a comparative analysis of the Cournot-Nash equilibrium strategies and unbeatable strategies in this framework. In symmetric settings, questions of this kind were considered in the seminal paper by Schaffer [147] and related works: Vega-Redondo [175], Rhode and Stegeman [141], Possajennikov [137][138], Hehenkamp et al. [85], Cressman and Hofbauer [44], Schipper [148], Hehenkamp et al. [86], and Duersch et al. [50, 51]. Asymmetry of the model adds interesting new aspects to the study and leads to results which to the best of our knowledge have no direct analogues in the existing literature.

Model description. A firm owns two production units/plants $i = 1, 2$ producing quantities $q_1 \geq 0$, $q_2 \geq 0$ of a homogeneous good, the inverse demand for which is $1 - q_1 - q_2$. The plant i's production cost is $c_i \in (0, 1)$ and its capacity (the maximum quantity it can produce) is $Q_i > 0$. The plants are run by two managers who select the quantities $q_i \in [0, Q_i]$, $i = 1, 2$, of the good to be produced representing strategies in the game at hand. If strategies q_1 and q_2 are chosen, then the profits of the plants are

$$\pi_i(q_1, q_2) = q_i(1 - q_1 - q_2 - c_i), \; i = 1, 2.$$

The goal of the firm, serving the whole market, is to maximize profits. To achieve this goal it contemplates an incentive scheme for the managers. A standard possibility would be to share with them a certain fraction of profits. This would lead to the conventional Cournot-Nash equilibrium outcome. However, the "parsimonious" firm, rather than allocating to the managers some fixed percentage of profits, sets up a contest. The principle of this contest is that the managers are rewarded for getting a *higher profit* than their rivals. What matters is not the absolute value π_i of the profit that plant i obtains, but the difference $\pi_i\,(q_1, q_2) - \pi_j(q_1, q_2)$ between the profits of plants i and j. The contest represents a game with relative preferences in which the players/managers strive to employ unbeatable strategies. In this section we examine this contest and compare its outcome with the conventional Cournot-Nash equilibrium outcome in the original game.

Contest setup. The firm establishes a *bonus fund* B some part of which is allocated to manager 1 and the rest to manager 2 depending on their relative performance. Specifically, there is a constant $\theta > 0$ such that player 1 gets

$$g_1(q_1, q_2) = B/2 + \theta f(q_1, q_2)$$

where

$$f(q_1, q_2) = \pi_1(q_1, q_2) - \pi_2(q_1, q_2),\ \ q_i \in [0, Q_i],\ \ i = 1, 2, \tag{155}$$

and player 2 receives

$$g_2(q_1, q_2) = B/2 - \theta f(q_1, q_2).$$

If the profits obtained by both plants are equal, both managers receive $B/2$, half the amount contained in the bonus fund.

This is a constant-sum game, in which the payoffs of the players sum up to B:

$$g_1(q_1, q_2) + g_2(q_1, q_2) = B.$$

Clearly this game is isomorphic to the zero-sum game with the payoffs $f(q_1, q_2)$ and $-f(q_1, q_2)$. Player 1 will maximize the payoff function $g_1(q_1, q_2)$, or equivalently, maximize $f(q_1, q_2)$. Player 2 will maximize $g_2(q_1, q_2)$, or equivalently, minimize $f(q_1, q_2)$. Thus the solution to the contest game will be a saddle point $(\bar{q}_1, \bar{q}_2)$ of the zero-sum game with the payoffs $f(q_1, q_2)$ and $-f(q_1, q_2)$. The strategy $\bar{q}_1$ will be unbeatable for player 1 if $f(\bar{q}_1, \bar{q}_2) \geq 0$, and the strategy $\bar{q}_1$ will be unbeatable for player 2 if $f(\bar{q}_1, \bar{q}_2) \leq 0$ (see Section 3).

Saddle point in the associated zero-sum game. Put

$$\gamma_i = 1 - c_i \in (0, 1], \; i = 1, 2.$$

Then the profit of production unit 1 can be expressed as

$$\pi_i(q_1, q_2) = q_i(\gamma_i - q_1 - q_2). \tag{156}$$

Define

$$\bar{q}_1 = \min\{Q_1, \frac{\gamma_1}{2}\}, \; \bar{q}_2 = \min\{Q_2, \frac{\gamma_2}{2}\}. \tag{157}$$

Proposition 9. *The pair $(\bar{q}_1, \bar{q}_2)$ is the unique saddle point of the function $f(q_1, q_2)$, i.e.*

$$f(q_1, \bar{q}_2) \leq f(\bar{q}_1, \bar{q}_2) \leq f(\bar{q}_1, q_2) \text{ for all } q_i \in [0, Q_i], \; i = 1, 2; \tag{158}$$

$\bar{q}_1$ is an unbeatable strategy of player 1 if $f(\bar{q}_1, \bar{q}_2) \geq 0$; $\bar{q}_2$ is an unbeatable strategy of player 2 if $f(\bar{q}_1, \bar{q}_2) \leq 0$.

Proof. We have

$$\begin{aligned} f(q_1, q_2) &= q_1(\gamma_1 - q_1 - q_2) - q_2(\gamma_2 - q_1 - q_2) \\ &= q_2^2 - \gamma_2 q_2 + q_1\gamma_1 - q_1^2. \end{aligned} \tag{159}$$

For each q_1, this function attains its unique minimum with respect to q_2 at $\bar{q}_2 = \min\{Q_2, \gamma_2/2\}$, and for each q_2 it attains its unique maximum with respect to q_1 at $\bar{q}_1 = \min\{Q_1, \gamma_1/2\}$, which proves that $(\bar{q}_1, \bar{q}_2)$ is the unique saddle point of $f(q_1, q_2)$. If $f(\bar{q}_1, \bar{q}_2) \geq 0$, then from the second inequality in (158) we obtain

$$0 \leq f(\bar{q}_1, q_2) = \pi_1(\bar{q}_1, q_2) - \pi_2(\bar{q}_1, q_2),$$

which means that $\bar{q}_1$ is an unbeatable strategy of player 1. If $f(\bar{q}_1, q_2) \leq 0$, then the first inequality in (158) implies

$$0 \geq f(q_1, \bar{q}_2) = \pi_1(q_1, \bar{q}_2) - \pi_2(q_1, \bar{q}_2),$$

consequently, $\bar{q}_2$ is an unbeatable strategy of player 2. □

The value of the associated zero-sum game. By virtue of (158) and (155) we get

$$f(\bar{q}_1, \bar{q}_2) = \begin{cases} -\gamma_2^2/4 + \gamma_1^2/4 & \text{if } Q_1 \geq \gamma_1/2, Q_2 \geq \gamma_2/2, \\ -\gamma_2^2/4 + Q_1(\gamma_1 - Q_1) & \text{if } \gamma_2/2 \leq Q_2, \gamma_1/2 \geq Q_1, \\ Q_2(Q_2 - \gamma_2) + \gamma_1^2/4 & \text{if } \gamma_2/2 \geq Q_2,\ \gamma_1/2 \leq Q_1, \\ Q_2(Q_2 - \gamma_2) + Q_1(\gamma_1 - Q_1) & \text{if } Q_1 \leq \gamma_1/2, Q_2 \leq \gamma_2/2. \end{cases} \tag{160}$$

As we have seen above, the sign of this value determines who of the players has an unbeatable strategy.

Capacity constraints: an assumption. If there are no capacity constraints, or they are not binding, then as is easily seen, that plant which has the lower production cost can beat the rival. However, this might not necessarily be the case if this plant does not have a sufficient capacity to fully realize its potential of producing the good at the lower cost. The rival might have a greater capacity so that by producing more at a greater cost it may achieve a higher profit. To focus on the essence of the model, we will exclude the cases where the capacity constraints are "too binding": $Q_1 \leq \gamma_1/2$, $Q_2 \leq \gamma_2/2$ or "not binding enough": $Q_1 \geq \gamma_1/2, Q_2 \geq \gamma_2/2$. We will concentrate on the cases $Q_1 \leq \gamma_1/2$, $Q_2 \geq \gamma_2/2$ and $Q_1 \geq \gamma_1/2$, $Q_2 \leq \gamma_2/2$. Each of this cases can be reduced to the other by changing the notation, and therefore it can be assumed without loss of generality that the following condition is satisfied:

(**A1**) The following inequalities hold: $Q_1 \geq \gamma_1/2$, $Q_2 \leq \gamma_2/2$.

Under this assumption, we have

$$\bar{q}_1 = \min\{Q_1, \gamma_1/2\} = \gamma_1/2,\ \bar{q}_2 = \min\{Q_2, \gamma_2/2\} = Q_2 \tag{161}$$

(see (157)).

When is the strategy $\bar{q}_1$ unbeatable? By virtue of (160) and (161), $\bar{q}_1$ is an unbeatable strategy of player 1 when

$$f(\bar{q}_1, \bar{q}_2) = Q_2(Q_2 - \gamma_2) + \gamma_1^2/4 \geq 0. \tag{162}$$

If $\gamma_1 \geq \gamma_2$, then inequality (162) holds always (as long as (**A1**) is satisfied) because then $Q_2(Q_2 - \gamma_2) \geq -\gamma_2^2/4 \geq -\gamma_1^2/4$. Let us assume that the opposite inequality holds:

(A2) We have $\gamma_1 \leq \gamma_2$.

Recall that $\gamma_i = 1 - c_i$. Thus condition (**A2**) means that plant 1 has the production cost c_1 greater than the production cost c_2 of plant 2. As the following proposition shows, the strategy $\bar{q}_1$ of player 1 happens to be unbeatable if the capacity of production unit 2 is below a certain threshold.

Proposition 10. *Under assumptions (**A1**)and (**A2**), the strategy* $\bar{q}_1 = \gamma_1/2$ *of player* 1 *is unbeatable if and only if*

$$Q_2 \leq \frac{1}{2}\left(\gamma_2 - \sqrt{\gamma_2^2 - \gamma_1^2}\right) \quad [\leq \gamma_2/2]. \tag{163}$$

Proof. The value of the function $\phi(Q_2) = Q_2(Q_2 - \gamma_2) + \gamma_1^2/4$ at the point $0 \leq Q_2 \leq \gamma_2/2$ is non-negative if and only if Q_2 does not exceed the smaller root of the quadratic equation $Q_2(Q_2 - \gamma_2) + \gamma_1^2/4 = 0$, which is equal to $\left(\gamma_2 - \sqrt{\gamma_2^2 - \gamma_1^2}\right)/2$. □

Note that the greater is the asymmetry in costs (i.e. the greater is the difference between $\gamma_1 = 1 - c_1$ and $\gamma_2 = 1 - c_2$) the lower is the threshold for Q_2 in (163) guaranteeing that the strategy $\bar{q}_1 = \gamma_1/2$ is unbeatable.

Total output, price and profits. Suppose player 1 follows the unbeatable strategy $\bar{q}_1 = \gamma_1/2$ and player 2 employs the strategy $\bar{q}_2 = Q_2$. Then the total production output and the price of the good can be expressed as follows:

$$\bar{Q} = \bar{q}_1 + \bar{q}_2 = \frac{\gamma_1}{2} + Q_2 = \frac{1 - c_1}{2} + Q_2, \tag{164}$$

$$\bar{p} = 1 - \bar{Q} = 1 - \frac{\gamma_1}{2} - Q_2 = \frac{1 + c_1}{2} - Q_2. \tag{165}$$

Since $\pi_i(\bar{q}_1, \bar{q}_2) = \bar{q}_i(\gamma_i - \bar{Q})$, we get the following formulas for the profits $\bar{\pi}_i := \pi_i(\bar{q}_1, \bar{q}_2)$ of plants $i = 1, 2$:

$$\bar{\pi}_1 = \bar{q}_1(\gamma_1 - \bar{Q}) = \frac{\gamma_1}{2}(\frac{\gamma_1}{2} - Q_2) = \frac{1 - c_1}{2}(\frac{1 - c_1}{2} - Q_2), \tag{166}$$

$$\bar{\pi}_2 = \bar{q}_2(\gamma_2 - \bar{Q}) = Q_2(\gamma_2 - \gamma_1/2 - Q_2) = Q_2(\frac{1 + c_1 - 2c_2}{2} - Q_2). \tag{167}$$

Schaffer's paradox (Schaffer [147]). Consider the symmetric case. Suppose the production units have the same capacities $Q_1 = Q_2 = Q$ and the same production costs: $c_1 = c_2 = c$, so that $\gamma_i = (1 - c_i) = \gamma$, where $\gamma = 1 - c$. Then in view of (**A2**) we have $Q \geq \gamma/2$, $Q \leq \gamma/2$, which yields $Q = \gamma/2$. Furthermore, by virtue of (164) $\bar{Q} = \gamma = 1 - c$, and so the price $\bar{p} = 1 - \bar{Q}$ coincides with the production cost c. This implies that the profits $\pi_i(\bar{q}_1, \bar{q}_2)$ of both plants are equal to zero—an outcome disastrous for the profit maximizing firm!

Nash equilibrium. To find a Nash equilibrium in the Cournot game at hand observe that for each q_2 the function

$$\pi_1(q_1, q_2) = q_1(1 - q_1 - q_2 - c_1) = -q_1^2 + q_1(\gamma_1 - q_2)$$

attains its maximum with respect to q_1 on $[0, Q_1]$ at $q_1 = \min\{(\gamma_1 - q_2)/2, Q_1\}$. Analogously for each q_1 the function of q_2

$$\pi_2(q_1, q_2) = q_2(1 - q_1 - q_2 - c_2) = -q_2^2 + q_2(\gamma_2 - q_1)$$

reaches its maximum on $[0, Q_2]$ when $q_2 = \min\{(\gamma_2 - q_1)/2, Q_2\}$. A Nash equilibrium (q_1^*, q_2^*) is a solution to the system of two equations

$$q_1^* = \min\{\frac{\gamma_1 - q_2^*}{2}, Q_1\}, \ q_2^* = \min\{\frac{\gamma_2 - q_1^*}{2}, Q_2\}. \tag{168}$$

Sufficiency of capacities. We introduce an assumption from which it will follow that the capacities Q_1 and Q_2 are not "too binding": they make it possible to produce those quantities of the good which correspond to the Nash equilibrium strategies in the Cournot game without capacity constraints.

(**A3**) The numbers $\gamma_i = 1 - c_i$ and Q_i satisfy the following inequalities:

$$0 \leq \frac{2\gamma_1 - \gamma_2}{3} \leq Q_1, \ 0 \leq \frac{2\gamma_2 - \gamma_1}{3} \leq Q_2. \tag{169}$$

Note that the inequality $2\gamma_2 - \gamma_1 \geq 0$ follows from (**A2**), and so it does not impose any new constraints. However, the inequality $2\gamma_1 - \gamma_2 \geq 0$ does. It says that the asymmetry in the model is not "too big". Although the number $\gamma_1 = 1 - c_1$ is smaller than $\gamma_2 = 1 - c_2$, it should not be smaller by more than two times.

Proposition 11. *Under (**A3**), the Nash equilibrium (q_1^*, q_2^*) is given by*

$$q_1^* = \frac{2\gamma_1 - \gamma_2}{3}, \ q_2^* = \frac{2\gamma_2 - \gamma_1}{3}. \tag{170}$$

Proof. To verify the first equation in (168) we write

$$\min\{\frac{\gamma_1 - q_2^*}{2}, Q_1\} = \min\{\frac{3\gamma_1 - 2\gamma_2 + \gamma_1}{6}, Q_1\} = \min\{\frac{2\gamma_1 - \gamma_2}{3}, Q_1\} = q_1^*.$$

The second equation in (168) is proved analogously. □

The assumptions are consistent. We introduced a number of assumptions on the data of the model that are needed for the comparative analysis of Nash equilibrium and unbeatable strategies. Are these assumptions consistent, i.e. do not some of them contradict the others? How to find a simple condition under which all of them are satisfied? The following proposition gives answers to these questions. Put

$$\bar{b} = \frac{3\sqrt{3} - 1}{4} \ (\approx 1.049),$$

$$u(\gamma_1, \gamma_2) = \frac{2\gamma_2 - \gamma_1}{3}, \ v(\gamma_1, \gamma_2) = \frac{\gamma_2}{2} - \frac{1}{2}\sqrt{\gamma_2^2 - \gamma_1^2}. \tag{171}$$

Proposition 12. *If*

$$0 < \gamma_1 \leq \gamma_2 \leq \bar{b}\gamma_1, \tag{172}$$

then

$$u(\gamma_1, \gamma_2) \leq \ v(\gamma_1, \gamma_2), \tag{173}$$

and if

$$Q_1 \geq \gamma_1/2, \ u(\gamma_1, \gamma_2) \leq Q_2 \leq v(\gamma_1, \gamma_2). \tag{174}$$

*then conditions (**A1**) – (**A3**) and (163) are satisfied.*

Typically, asymmetric Cournot duopoly models are workable if their asymmetry is in a sense not too big. The most standard assumption expressing this idea is the inequality $\gamma_1 \leq 2\gamma_2$. (As we already noticed, it follows from (**A3**)). In the present context we need a stronger condition (172), meaning that the asymmetry of the model is small enough.

Proposition 12 can be used as follows. Inequality (173) holding under assumption (b) implies that the segment $[u(\gamma_1, \gamma_2), v(\gamma_1, \gamma_2)]$ is not empty. Using this, we can select any Q_2 in this segment and any $Q_1 \geq \gamma_1/2$. Then according to Proposition 12 all the assumptions imposed above and hence all the assertions proved will be valid.

Proof of Proposition 12. We have

$$v(\gamma_1, \gamma_2) - u(\gamma_1, \gamma_2) = \frac{\gamma_2}{2} - \frac{1}{2}\sqrt{\gamma_2^2 - \gamma_1^2} - \frac{2\gamma_2 - \gamma_1}{3}$$
$$= -\frac{1}{2}\sqrt{\gamma_2^2 - \gamma_1^2} - \frac{\gamma_2 - 2\gamma_1}{6}.$$

This expression is non-negative if and only if

$$8\gamma_2^2 - 13\gamma_1^2 + 4\gamma_1\gamma_2 \leq 0.$$

or equivalently,

$$8(\gamma_2/\gamma_1)^2 - 13 + 4(\gamma_2/\gamma_1) \leq 0,$$

which holds if and only if $\gamma_2/\gamma_1 \leq \bar{b}$.

To complete the proof we observe that (**A1**) holds because $Q_1 \geq \gamma_1/2$, as assumed in (174), and $Q_2 \leq \gamma_2/2$ by virtue of the second inequality in (174) and the definition of $v(\gamma_1, \gamma_2)$ in (171). Condition (**A2**) follows from (172). To check (**A3**) we observe the following. The first inequality in (169) holds because $\gamma_2 \leq \bar{b}\gamma_1 < 2\gamma_1$ by virtue of (172). The second is true since $\gamma_2 \geq \gamma_1$, and so $(2\gamma_1 - \gamma_2)/3 \leq \gamma_1/3 < \gamma_1/2 \leq Q_1$ (see (174)). The third is fulfilled by virtue of (**A2**) and the fourth follows from the second inequality in (174). Finally, (163) coincides with the third inequality in (174). □

Nash equilibrium outcome. Suppose that the production units 1 and 2 select Nash equilibrium strategies (170). Then the total production output Q^*, the market price p^* of the good, and the profits π_1^*, π_2^* of plants 1 and 2 will be as follows:

$$Q^* = \frac{2\gamma_1 - \gamma_2}{3} + \frac{2\gamma_2 - \gamma_1}{3} = \frac{\gamma_1 + \gamma_2}{3},$$
$$p^* = 1 - Q^* = 1 - \frac{\gamma_1 + \gamma_2}{3},$$
$$\pi_1^* = q_1^*(\gamma_1 - Q^*) = \frac{2\gamma_1 - \gamma_2}{3}(\gamma_1 - \frac{\gamma_1 + \gamma_2}{3}) = \frac{(2\gamma_1 - \gamma_2)^2}{9},$$
$$\pi_2^* = q_2^*(\gamma_2 - Q^*) = \frac{2\gamma_2 - \gamma_1}{3}(\gamma_2 - \frac{\gamma_1 + \gamma_2}{3}) = \frac{(2\gamma_2 - \gamma_1)^2}{9}.$$

Contest vs. Nash equilibrium. Define

$$\mu = \frac{\gamma_1 + \gamma_2}{2} = 1 - \frac{c_1 + c_2}{2} \in (0, 1),$$

$$\sigma = \frac{\gamma_2 - \gamma_1}{2} = \frac{c_1 - c_2}{2} \in [0, \mu).$$

Here $1 - \mu$ is the average production cost. The number σ may be regarded as a "measure of asymmetry" of the model: 2σ is equal to the difference between the production costs. In further analysis, it will be convenient to change the variables γ_1 and γ_2 to μ and σ. Clearly,

$$\gamma_1 = \mu - \sigma, \ \gamma_2 = \mu + \sigma. \tag{175}$$

We will show that the contest under consideration leads, compared with the Nash equilibrium, to a greater output, smaller price, and lower profits for both production units. It is important to note that the differences between the corresponding variables in the two settings grow as the index $\sigma = (\gamma_2 - \gamma_1)/2$ of the model asymmetry grows. Thus, although Schaffer's paradox can essentially be observed in the present context as well, the asymmetry of the model, surprisingly or not, makes it "milder".

Proposition 13. *The following inequalities hold:*

$$\bar{Q} \geq Q^* + \frac{\mu + 3\sigma}{6}, \tag{176}$$

$$\bar{p} \leq p^* - \frac{\mu + 3\sigma}{6}, \tag{177}$$

$$\bar{\pi}_1 \leq \pi_1^* - \frac{(\mu + 3\sigma)^2}{36}, \tag{178}$$

$$\bar{\pi}_2 \leq \pi_2^* - \frac{7(\mu + 3\sigma)^2}{144}. \tag{179}$$

Proof. By using (164), the first inequality in (174) and (175), we write

$$\bar{Q} = \frac{\gamma_1}{2} + Q_2 \geq \frac{\gamma_1}{2} + \frac{2\gamma_2 - \gamma_1}{3} = \frac{3\gamma_1 + 4\gamma_2 - 2\gamma_1}{6} =$$

$$= \frac{\gamma_1 + \gamma_2}{3} + \frac{2\gamma_2 - \gamma_1}{6} = Q^* + \frac{\mu + 3\sigma}{6},$$

which implies (176) and (177). Further, in view of (164), (166), the first inequality in (174) and (175), we have

$$\bar{\pi}_1 = \bar{q}_1(\gamma_1 - \bar{Q}) = \frac{\gamma_1}{2}(\frac{\gamma_1}{2} - Q_2) \leq \frac{\gamma_1}{2}(\frac{\gamma_1}{2} - \frac{2\gamma_2 - \gamma_1}{3}) = \frac{\gamma_1(5\gamma_1 - 4\gamma_2)}{12},$$

and consequently,

$$\pi_1^* - \bar{\pi}_1 \geq \frac{(2\gamma_1 - \gamma_2)^2}{9} - \frac{\gamma_1(5\gamma_1 - 4\gamma_2)}{12}.$$

$$= \frac{[2(\mu - \sigma) - (\mu + \sigma)]^2}{9} - \frac{(\mu - \sigma)[5(\mu - \sigma) - 4(\mu + \sigma)]}{12} = \frac{(\mu + 3\sigma)^2}{36},$$

and thus we obtain (178). Finally, we get

$$\bar{\pi}_2 = \bar{q}_2(\gamma_2 - \bar{Q}) = Q_2(\gamma_2 - \gamma_1/2 - Q_2) \leq \max_{Q_2} Q_2(\gamma_2 - \gamma_1/2 - Q_2)$$

$$= \frac{(\gamma_2 - \gamma_1/2)^2}{4} = \frac{(\mu + 3\sigma)^2}{16} = \frac{(\mu + 3\sigma)^2}{9} - \frac{7(\mu + 3\sigma)^2}{144} = \pi_2^* - \frac{7(\mu + 3\sigma)^2}{144},$$

which yields (179). □

From symmetric to asymmetric case. We provide some estimates for the variables (164) - (167) showing how they may change when the degree of asymmetry of the model changes.

Fix some $\mu \in (0, 1)$ and consider the following functions of $\sigma \in [0, \mu]$:

$$\Phi(\sigma) = \mu - \sqrt{\mu\sigma},\ \Pi_1(\sigma) = \frac{\mu - \sigma}{2}(\sqrt{\mu\sigma} - \sigma),\ \Pi_2(\sigma) = \frac{\mu + 3\sigma}{3}(\sigma + \sqrt{\mu\sigma}).$$

Proposition 14. *The following inequalities hold:*

$$\bar{Q} \leq \Phi(\sigma),\ \bar{p} \geq 1 - \Phi(\sigma),\ \bar{\pi}_1 \geq \Pi_1(\sigma),\ \bar{\pi}_2 \geq \Pi_2(\sigma).$$

For $0 < \sigma < \mu$ *we have* $\Phi(\sigma) > 0$, $\Pi_1(\sigma) > 0$, $\Pi_2(\sigma) > 0$, $\Phi'(\sigma) < 0$, $\Pi_2'(\sigma) > 0$, *and* $\Pi_1'(\sigma) > 0$ *if* σ *is small enough.*

This proposition shows that when the asymmetry of the model (measured in terms of σ) increases, the outcome of the asymmetric contest becomes more and more distinct from the outcome of its symmetric counterpart. The total output $\bar{Q}$ exhibits a tendency to decrease as it is bounded above by a decreasing function of σ. The price $\bar{p}$ and the profits $\bar{\pi}_1$ and $\bar{\pi}_2$ are bounded below by strictly positive increasing functions of σ, and so they tend to grow when σ grows.

Proof of Proposition 14. Using the fact that

$$\frac{1}{2}\sqrt{\gamma_2^2 - \gamma_1^2} = \sqrt{\frac{(\gamma_2 + \gamma_1)}{2}\frac{(\gamma_2 - \gamma_1)}{2}} = \sqrt{\mu\sigma},$$

we can write inequality (163) as

$$Q_2 \leq \frac{\gamma_2}{2} - \sqrt{\mu\sigma}. \tag{180}$$

By employing (180) and (175), we get

$$\bar{Q} = \frac{\gamma_1}{2} + Q_2 \leq \frac{\gamma_1}{2} + \frac{\gamma_2}{2} - \sqrt{\mu\sigma} = \mu - \sqrt{\mu\sigma} = \Phi(\sigma),$$

$$\bar{p} = 1 - \bar{Q} \geq 1 - \mu + \sqrt{\mu\sigma} = 1 - \Phi(\sigma),$$

$$\bar{\pi}_1 = \bar{q}_1(\gamma_1 - \bar{Q}) = \frac{\gamma_1}{2}(\frac{\gamma_1}{2} - Q_2) \geq \frac{\gamma_1}{2}(\frac{\gamma_1}{2} - \frac{\gamma_2}{2} + \sqrt{\mu\sigma})$$

$$= \frac{\mu - \sigma}{2}(\sqrt{\mu\sigma} - \sigma) = \Pi_1(\sigma),$$

$$\bar{\pi}_2 = \bar{q}_2(\gamma_2 - \bar{Q}) \geq Q_2(\gamma_2 - \mu + \sqrt{\mu\sigma}) = Q_2(\mu + \sigma - \mu + \sqrt{\mu\sigma})$$

$$= Q_2(\sigma + \sqrt{\mu\sigma}) \geq \frac{2\gamma_2 - \gamma_1}{3}(\sigma + \sqrt{\mu\sigma}) = \frac{\mu + 3\sigma}{3}(\sigma + \sqrt{\mu\sigma}) = \Pi_2(\sigma),$$

where the last inequality in the above chain of relations follows from the first inequality in (174). Strict positivity of the functions in question follows from the inequalities

$$\sigma < \sqrt{\mu\sigma} < \mu$$

(holding when $\sigma < \mu$). It is clear that $\Phi'(\sigma) < 0$ and $\Pi_2'(\sigma) > 0$. Finally, we can see from the equation

$$\Pi_1'(\sigma) = \frac{\mu - \sigma}{2}(\frac{\mu}{2\sqrt{\mu\sigma}} - 1) - \frac{1}{2}(\sqrt{\mu\sigma} - \sigma)$$

that $\Pi_1'(\sigma) > 0$ for sufficiently small $\sigma > 0$ because the above expression tends to infinity as σ tends to zero. □

4.7. Unbeatable strategies in an asymmetric Cournot duopoly with differentiated goods

Asymmetric duopoly with differentiated goods. In this section we examine unbeatable strategies in an asymmetric Cournot duopoly model with differentiated goods. We conduct a comparative analysis of the Cournot-Nash equilibrium strategies and unbeatable strategies in this framework. The duopoly model we deal with is described as follows. A firm owns two production units/plants $i = 1, 2$ producing goods $i = 1, 2$ in quantities $q_i \geq 0$. The inverse demand for good i is $1 - a_i q_i - b q_j$ ($j \neq i$), where $a_i \geq 1$ is the inverse elasticity of demand for good i and $0 \leq b \leq 1$ is the product substitutability coefficient. The plants are run by two managers who select the quantities $q_i \geq 0$ of goods $i = 1, 2$ to be produced (*strategies* of players $i = 1, 2$). If strategies q_1 and q_2 are chosen, then the profits of the plants are

$$\pi_i(q_1, q_2) = q_i(1 - a_i q_i - b q_j - c_i), \; j \neq i, \tag{181}$$

where $c_i \in (0, 1)$ is the plant i's production cost.

Cournot contest. We will examine the contest analogous to that described in the previous section. It will be based on the zero-sum game associated with the Cournot duopoly model under consideration. The payoff function of the first player in this game is the difference of profits

$$f(q_1, q_2) = \pi_1(q_1, q_2) - \pi_2(q_1, q_2). \tag{182}$$

The solution to the contest problem is the saddle point in this game. To examine this problem it will be convenient to introduce the notation:

$$\gamma_i := 1 - c_i \in (0, 1], \; \delta_i := \frac{\gamma_i}{a_i}, \; i = 1, 2. \tag{183}$$

The profit of production unit i can be expressed via γ_i as follows:

$$\pi_i(q_1, q_2) = q_i(\gamma_i - a_i q_i - b q_j), \; j \neq i. \tag{184}$$

For a comprehensive review of the Cournot duopoly model with differentiated goods see Singh and Vives [162]. Evolutionary aspects of symmetric contests based on this model are considered in Rhode and Stegeman [141]. For related work see the references provided at the beginning of the previous section. The results on asymmetric contests obtained in the present chapter seem to be entirely new.

The next proposition describes the solution to the contest problem. Define

$$\bar{q}_1 = \frac{\delta_1}{2}, \; \bar{q}_2 = \frac{\delta_2}{2}. \tag{185}$$

Proposition 15. *(i) The pair* $(\bar{q}_1, \bar{q}_2)$ *is the unique saddle point of the function* $f(q_1, q_2)$, *i.e.*

$$f(q_1, \bar{q}_2) \le f(\bar{q}_1, \bar{q}_2) \le f(\bar{q}_1, q_2) \text{ for all } q_i \ge 0,\ i = 1, 2. \tag{186}$$

(ii) We have

$$f(\bar{q}_1, \bar{q}_2) = \frac{\gamma_1^2}{4a_1} - \frac{\gamma_2^2}{4a_2} = \frac{a_1\delta_1^2}{4} - \frac{a_2\delta_2^2}{4}. \tag{187}$$

(iii) If $f(\bar{q}_1, \bar{q}_2) \ge 0$, *then* $\bar{q}_1 = \delta_1/2$ *is an unbeatable strategy of player* 1. *(iv) If* $f(\bar{q}_1, \bar{q}_2) \le 0$, *then* $\bar{q}_2 = \delta_2/2$ *is an unbeatable strategy of player* 2.

Proof. By virtue of (182) and (184),

$$f(q_1, q_2) = q_1(\gamma_1 - a_1 q_1 - b q_2) - q_2(\gamma_2 - a_2 q_2 - b q_1)$$

$$= \gamma_1 q_1 - a_1 q_1^2 - q_2\gamma_2 + a_2 q_2^2. \tag{188}$$

The function $\gamma x - a x^2$ attains its maximum $\gamma^2/4a$ at $x = \gamma/2a$, and the function $ax^2 - \gamma x$ attains its minimum $-\gamma^2/4a$ at $x = \gamma/2a$. This proves (i) and (ii). Assertions (iii) and (iv) are consequences of (186). □

In what follows we will assume that

$$\delta_1 \ge \delta_2, \tag{189}$$

or equivalently, that $\bar{q}_1 \ge \bar{q}_2$.

Contest outcome. As a result of the contest, plants $i = 1, 2$ will produce quantities $\bar{q}_i = \delta_i/2$ of goods $i = 1, 2$ and get profits

$$\bar{\pi}_1 = \frac{\delta_1}{2}(\gamma_1 - a_1\frac{\delta_1}{2} - b\frac{\delta_2}{2}) = \frac{\delta_1}{2}(a_1\delta_1 - a_1\frac{\delta_1}{2} - b\frac{\delta_2}{2}) = \frac{\delta_1}{4}(a_1\delta_1 - b\delta_2), \tag{190}$$

$$\bar{\pi}_2 = \frac{\delta_2}{2}(\gamma_2 - a_2\frac{\delta_2}{2} - b\frac{\delta_1}{2}) = \frac{\delta_2}{2}(a_2\delta_2 - a_2\frac{\delta_2}{2} - b\frac{\delta_1}{2}) = \frac{\delta_2}{4}(a_2\delta_2 - b\delta_1). \tag{191}$$

(see the definition of δ_i in (183)). The total profit $\bar{\pi} = \bar{\pi}_1 + \bar{\pi}_2$ received by the firm from production units $i = 1, 2$ is expressed as

$$\bar{\pi} = \frac{\delta_1}{4}(a_1\delta_1 - b\delta_2) + \frac{\delta_2}{4}(a_2\delta_2 - b\delta_1) = \frac{a_1\delta_1^2 - 2b\delta_1\delta_2 + a_2\delta_2^2}{4} \tag{192}$$

We can see that $\bar{\pi} \geq 0$ because

$$\bar{\pi} \geq \frac{\delta_1^2 - 2\delta_1\delta_2 + \delta_2^2}{4} = \frac{(\delta_1 - \delta_2)^2}{4}, \tag{193}$$

where the inequality is valid since $b \leq 1$ and $a_i \geq 1$.

Note that the strategies $\bar{q}_i = \gamma_i/a_i$ representing the ratios of the numbers $\gamma_i = 1 - c_i$ (where c_i are production costs) and the inverse elasticities of demand for goods $i = 1, 2$ do not depend on the product substitutability coefficient $0 \leq b \leq 1$. However, the profits $\bar{\pi}_1$, $\bar{\pi}_2$, and $\bar{\pi}$ do. They are decreasing functions of b. They attain their maximum values when $b = 0$, when each plant i is a monopolist in producing good i, and their minimum values when $b = 1$. In the latter case, goods $i = 1, 2$ are indistinguishable from each other, and the duopoly model essentially reduces to its version with a homogeneous good.

Symmetric contest outcome. Suppose the production units have the same production costs $c_1 = c_2 = c$ and the goods produced feature the same inverse elasticities $a_1 = a_2 = a$, so that $\gamma_i = (1 - c_i) = \gamma$ and $\delta_i = \gamma_i/a_i = \gamma/a = \delta$. Then we have:

$$\bar{q}_1 = \bar{q}_2 = \frac{\delta}{2} = \frac{\gamma}{2a}, \tag{194}$$

$$\bar{\pi}_1 = \bar{\pi}_2 = \frac{\delta^2}{4}(a - b) = \frac{\gamma^2}{4a^2}(a - b), \tag{195}$$

$$\bar{\pi} = \bar{\pi}_1 + \bar{\pi}_2 = \frac{\delta^2}{2}(a - b) = \frac{\gamma^2}{2a^2}(a - b). \tag{196}$$

If $a = b = 1$, the model becomes isomorphic to the classical homogeneous-good Cournot duopoly, and we have $\bar{\pi}_1 = \bar{\pi}_2 = 0$ (Schaffer's [147] paradox).

Degree of asymmetry. The above formulas show that key roles in the contest model at hand are played by the ratios $\delta_i = \gamma_i/a_i$. By using these ratios we define

$$\mu = \frac{\delta_1 + \delta_2}{2}, \ \sigma = \frac{\delta_1 - \delta_2}{2}.$$

By virtue of (189) the number σ is non-negative. This number turns out to be a natural measure of "asymmetry of the model" (cf. the previous section). The variables δ_i are expressed trough μ and σ as

$$\delta_1 = \mu + \sigma, \ \delta_2 = \mu - \sigma. \tag{197}$$

Expressing model variables via μ and σ. From (185) we get:

$$\bar{q}_1 = \frac{\mu+\sigma}{2}, \ \bar{q}_2 = \frac{\mu-\sigma}{2}. \tag{198}$$

The following expressions for the profits of the first and the second player follow from (190) and (191):

$$\begin{aligned}\bar{\pi}_1 &= \frac{\delta_1}{4}(a_1\delta_1 - b\delta_2) = \frac{\mu+\sigma}{4}[a_1(\mu+\sigma) - b(\mu-\sigma)] \\ &= \frac{\mu+\sigma}{4}[(a_1-b)\mu + (a_1+b)\sigma], \end{aligned}\tag{199}$$

$$\begin{aligned}\bar{\pi}_2 &= \frac{\delta_2}{4}(a_2\delta_2 - b\delta_1) = \frac{\mu-\sigma}{4}[a_2(\mu-\sigma) - b(\mu+\sigma)] \\ &= \frac{\mu-\sigma}{4}[(a_2-b)\mu - (a_2+b)\sigma)]. \end{aligned}\tag{200}$$

By using (187), we write

$$\bar{\pi}_1 - \bar{\pi}_2 = \frac{a_1\delta_1^2}{4} - \frac{a_2\delta_2^2}{4} = \frac{1}{4}[a_1(\mu+\sigma)^2 - a_2(\mu-\sigma)^2]. \tag{201}$$

Finally, by employing the expression (192) for the total profit $\bar{\pi}$ generated by plants 1 and 2, we find:

$$\begin{aligned}\bar{\pi} &= \frac{a_1\delta_1^2 - 2b\delta_1\delta_2 + a_2\delta_2^2}{4} \\ &= \frac{a_1(\mu+\sigma)^2 - 2b(\mu+\sigma)(\mu-\sigma) + a_2(\mu-\sigma)^2}{4}. \end{aligned}\tag{202}$$

Growing asymmetry. What happens with the variables (198) - (202) when the degree σ of asymmetry of the model grows? This question is examined in the next proposition, where we regard the variables under consideration as functions of σ. Some of these functions increase, and the others decrease.

Proposition 16. *(i) The function $\bar{q}_1(\sigma)$ is increasing, while $\bar{q}_2(\sigma)$ is decreasing. (ii) The function $\bar{\pi}_1(\sigma)$ is increasing. (iii) If σ is small enough, specifically if $\sigma \le \mu/2$, then $\partial\bar{\pi}_2/\partial\sigma \le 0$. (iv) The difference $\bar{\pi}_1 - \bar{\pi}_2$ is increasing. (v) If $a_1 \ge a_2$, then $\partial\bar{\pi}/\partial\sigma \ge 0$. If $a_1 < a_2$ and σ is small enough, more precisely, if*

$$\sigma < \frac{(a_2-a_1)\mu}{a_1+a_2+2b}, \tag{203}$$

then $\partial\bar{\pi}/\partial\sigma < 0$.

Proof. Assertions (i), (ii) and (iv) are clear from (198), (199) and (201). To prove (iii) we compute the derivative

$$\frac{\partial\bar{\pi}_2}{\partial\sigma} = -\frac{1}{4}[(a_2 - b)\mu - (a_2 + b)\sigma] - \frac{\mu - \sigma}{4}(a_2 + b) =$$
$$\frac{1}{4}[-(a_2 - b)\mu + (a_2 + b)\sigma - \mu(a_2 + b) + \sigma(a_2 + b)] =$$
$$\frac{1}{4}[2(a_2 + b)\sigma - 2a_2\mu] = \frac{1}{2}[(a_2 + b)\sigma - a_2\mu].$$

The fact that it is non-positive follows from the relations

$$\frac{(a_2 + b)\sigma}{a_2} \leq \frac{(a_2 + 1)\sigma}{a_2} = \sigma + \frac{\sigma}{a_2} \leq 2\sigma \leq \mu$$

(we used here the inequalities $b \leq 1$ and $a_i \geq 1$).

To verify (v) we write

$$\bar{\pi} = \frac{a_1(\mu^2 + 2\mu\sigma + \sigma^2) - 2b(\mu^2 - \sigma^2) + a_2(\mu^2 - 2\mu\sigma + \sigma^2)}{4}$$

and observe that the sign of the derivative $\partial\bar{\pi}/\partial\sigma$ is the same as the sign of the derivative of the function

$$a_1(2\mu\sigma + \sigma^2) + 2b\sigma^2 + a_2(-2\mu\sigma + \sigma^2)$$

which is

$$2(a_1 - a_2)\mu + 2(a_1 + a_2 + 2b)\sigma.$$

This expression is positive if $a_1 \geq a_2$. It is negative if $a_1 < a_2$ and $(a_1 + a_2 + 2b)\sigma < (a_2 - a_1)\mu$, which is equivalent to (203). □

Nash equilibrium. Assume that

$$\delta_1 \geq \delta_2/2, \ \delta_2 \geq \delta_1/2. \quad (204)$$

(the asymmetry of the model is "not too big"). Observe that the first inequality follows from

(189). Let us find the Nash equilibrium (q_1^*, q_2^*) in the game at hand. Put

$$\Delta = 1 - \frac{b^2}{4a_1a_2}.$$

Proposition 17. *The Nash equilibrium in the Cournot game under consideration is given by*

$$q_1^* = \frac{1}{\Delta}(\frac{\delta_1}{2} - \frac{b\delta_2}{4a_1}), \tag{205}$$

$$q_2^* = \frac{1}{\Delta}(\frac{\delta_2}{2} - \frac{b\delta_1}{4a_2}). \tag{206}$$

We have $q_i^* \geq 0$, $i = 1, 2$.

Proof. In the equilibrium, the strategy q_i^* should maximize

$$\pi_i(q_1, q_2) = q_i(\gamma_i - a_iq_i - bq_j^*).$$

Therefore the equations

$$\gamma_1 - 2a_1q_1^* - bq_2^* = 0,\ \gamma_2 - 2a_2q_2^* - bq_1^* = 0,$$

which are equivalent to the following ones

$$q_1^* = \frac{\delta_1}{2} - \frac{b}{2a_1}q_2^*,\ q_2^* = \frac{\delta_2}{2} - \frac{b}{2a_2}q_1^*, \tag{207}$$

have to be satisfied. The fact that these equations indeed hold can be verified directly by substituting (205) and (206) into (207).

To check the first equation in (207) we have to show that

$$\frac{1}{\Delta}(\frac{\delta_1}{2} - \frac{b}{2a_1}\frac{\delta_2}{2}) = \frac{\delta_1}{2} - \frac{b}{2a_1}\frac{1}{\Delta}(\frac{\delta_2}{2} - \frac{b}{2a_2}\frac{\delta_1}{2}).$$

By transforming this equation, we write

$$\frac{\delta_1}{2} - \frac{b}{2a_1}\frac{\delta_2}{2} = \frac{\delta_1}{2} - \frac{\delta_1}{2}\frac{b^2}{4a_1a_2} - \frac{b}{2a_1}(\frac{\delta_2}{2} - \frac{b}{2a_2}\frac{\delta_1}{2}),$$

$$2\delta_1 - \frac{b}{a_1}\delta_2 = 2\delta_1 - \frac{\delta_1}{2}\frac{b^2}{a_1 a_2} - \frac{b}{a_1}(\delta_2 - \frac{b}{a_2}\delta_1),$$

$$-\frac{b}{a_1}\delta_2 = -\frac{\delta_1 b^2}{2a_1 a_2} - \frac{b}{a_1}(\delta_2 - \frac{b}{a_2}\delta_1),$$

and arrive at the identity

$$-\frac{b}{a_1}\delta_2 = -\frac{\delta_1 b^2}{a_1 a_2} - \frac{b}{a_1}\delta_2 + \frac{b^2\delta_1}{a_1 a_2}.$$

The second equation in (207) can be obtained from the first by swapping the subscripts 1 and 2.

By virtue of (204), we have

$$\delta_1 \geq \frac{\delta_2}{2} \geq \frac{b\delta_2}{2a_1},\ \delta_2 \geq \frac{\delta_1}{2} \geq \frac{b\delta_1}{2a_2},$$

which implies that $q_i^* \geq 0$ (see (205) and (206)). □

In the equilibrium, production units 1 and 2 get profits

$$\pi_1^* = q_1^*(\gamma_1 - a_1 q_1^* - b q_2^*),\ \pi_2^* = q_2^*(\gamma_2 - a_2 q_2^* - b q_2^*), \tag{208}$$

so that the total profit is

$$\pi^* = q_1^*(\gamma_1 - a_1 q_1^* - b q_2^*) + q_2^*(\gamma_2 - a_2 q_2^* - b q_1^*). \tag{209}$$

Symmetric case. Assume that $c_1 = c_2 = c$ and $a_1 = a_2 = a$, so that $\gamma_i = 1 - c_i = 1 - c = \gamma$ and $\delta_i = \gamma_i/a_i = \gamma/a = \delta$.

Proposition 18. *In the symmetric case we have*

$$q_1^* = q_2^* = \frac{\gamma}{2a + b}, \tag{210}$$

$$\pi_1^* = \pi_2^* = \frac{\gamma^2 a}{(2a+b)^2}, \tag{211}$$

$$\pi^* = \pi_1^* + \pi_2^* = \frac{2\gamma^2 a}{(2a+b)^2}. \tag{212}$$

Proof. To check (210) we use (205) and write

$$q_i^* = \frac{\delta}{2}(1 - \frac{b}{2a})(1 - \frac{b^2}{4a^2})^{-1} = \frac{\gamma}{2a(1 + b/2a)} = \frac{\gamma}{2a+b}.$$

Formulas (211) and (212) follow from (208), (209) and the chain of relations

$$\pi_i^* = \frac{\gamma}{2a+b}(\gamma - a\frac{\gamma}{2a+b} - b\frac{\gamma}{2a+b}) = \frac{\gamma^2}{(2a+b)^2}(2a+b-a-b) = \frac{\gamma^2 a}{(2a+b)^2}.$$

□

Equilibrium vs. contest in the symmetric case. Let us compare the Nash equilibrium outcome (210) - (212) and the outcome of the contest (194) - (196) in the symmetric model. We will regard the variables

$$\bar{q} := \bar{q}_i,\ q^* := q_i^*,\ \bar{\pi} := \bar{\pi}_1 + \bar{\pi}_2,\ \pi^* := \pi_1^* + \pi_2^*$$

as functions of the substitutability coefficient $b \in [0, 1]$. Define

$$\phi(b) = \frac{\bar{q}(b)}{q^*(b)},\ \psi(b) = \frac{\bar{\pi}(b)}{\pi^*(b)}. \tag{213}$$

Proposition 19. *The following relations hold:*

$$\phi(b) = \frac{2a+b}{2a},\ \phi(b) \geq 1,\ \phi'(b) > 0, \tag{214}$$

$$\psi(b) = \frac{(a-b)(2a+b)^2}{4a^3},\ \psi(b) \leq 1,\ \psi'(b) < 0. \tag{215}$$

It follows from this proposition that *the contest production output* $\bar{q}(b)$ *is greater than the Nash equilibrium production output* $q^*(b)$, and the ratio $\bar{q}(b)/q^*(b)$ is an increasing function of the substitutability coefficient b. At the same time, *the contest profit* $\bar{\pi}(b)$ *is less than the equilibrium profit* $\pi^*(b)$, and the ratio $\bar{\pi}(b)/\pi^*(b)$ decreases as b increases.

Proof of Proposition 19. The relations in (214) are immediate from (194) and (210). To check (214) we write

$$\psi(b) = \frac{\gamma^2(a-b)}{2a^2}\frac{(2a+b)^2}{2\gamma^2 a} = \frac{(a-b)(2a+b)^2}{4a^3}.$$

The sign of $\psi'(b)$ is the same as the the sign of the derivative of the function $(a-b)(2a+b)^2$, which is

$$-(2a+b)^2 + 2(2a+b)(a-b) = -6ab - 3b^2 < 0.$$

The inequality $\psi(b) \leq 1$ holds because $\psi(0) = 1$ and $\psi'(b) < 0$.

$\square$

From the symmetric to the asymmetric case. We extend the above results to the general, asymmetric model.

Proposition 20. *Equilibrium quantities are not greater than the quantities resulting from the contest:*

$$q_1^* \leq \bar{q}_1, \;\; q_2^* \leq \bar{q}_2.$$

Proof. This follows from (207) because $\bar{q}_i = \delta_i/2$. $\square$

Proposition 21. *The total equilibrium profit (209) is not less than the total profit (192) obtained by the contest participants:*

$$\pi^* \geq \bar{\pi}.$$

Proof. From (209) we get $\pi^* = A - B$, where

$$A = q_1^* \gamma_1 + q_2^* \gamma_2, \tag{216}$$

$$B = a_1 \left(q_1^*\right)^2 + a_2 \left(q_2^*\right)^2 + 2b q_1^* q_2^*. \tag{217}$$

By substituting (205) and (206) into (216), we obtain

$$A = \frac{1}{\Delta}\left(\frac{\gamma_1\delta_1}{2} - \frac{\gamma_1 b\delta_2}{4a_1} + \frac{\gamma_2\delta_2}{2} - \frac{\gamma_2 b\delta_1}{4a_2}\right) = \frac{1}{\Delta}\left(\frac{a_1\delta_1^2}{2} + \frac{a_2\delta_2^2}{2} - \frac{b\delta_1\delta_2}{2}\right)$$

(recall that $\gamma_i = \delta_i a_i$). By using (205), (206) and applying elementary algebraic transformations, we arrive at the following identity[14]:

$$B = \frac{a_1\delta_1^2 + a_2\delta_2^2}{4\Delta^2} - \frac{1-\Delta}{\Delta^2}\left(\frac{3\delta_1^2 a_1 - 2b\delta_2\delta_1 + 3\delta_2^2 a_2}{4}\right). \tag{218}$$

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[14] Equality (218) is verified as follows:

$$B = \frac{a_1}{\Delta^2}\left(\frac{\delta_1}{2} - \frac{b\delta_2}{4a_1}\right)^2 + \frac{a_2}{\Delta^2}\left(\frac{\delta_2}{2} - \frac{b\delta_1}{4a_2}\right)^2 + \frac{2b}{\Delta^2}\left(\frac{\delta_1}{2} - \frac{b\delta_2}{4a_1}\right)\left(\frac{\delta_2}{2} - \frac{b\delta_1}{4a_2}\right) = \frac{a_1}{\Delta^2}\left(\frac{\delta_1^2}{4} + \frac{b^2\delta_2^2}{16a_1^2} - \frac{b\delta_2\delta_1}{4a_1}\right)$$
$$+\frac{a_2}{\Delta^2}\left(\frac{\delta_2^2}{4} + \frac{b^2\delta_1^2}{16a_2^2} - \frac{b\delta_2\delta_1}{4a_2}\right) + \frac{2b}{\Delta^2}\left(\frac{\delta_1\delta_2}{4} - \frac{b\delta_1^2}{8a_2} - \frac{b\delta_2^2}{8a_1} + \frac{b^2\delta_2\delta_1}{16a_2a_1}\right) = \left(\frac{a_1\delta_1^2}{4} + \frac{b^2\delta_2^2}{16a_1} - \frac{b\delta_2\delta_1}{4} + \frac{a_2\delta_2^2}{4}\right)$$
$$+\frac{1}{\Delta^2}\left(\frac{b^2\delta_1^2}{16a_2} - \frac{b\delta_2\delta_1}{4} + \frac{b\delta_1\delta_2}{2} - \frac{b^2\delta_1^2}{4a_2} - \frac{b^2\delta_2^2}{4a_1} + \frac{b^3\delta_2\delta_1}{8a_2a_1}\right) = \frac{1}{\Delta^2}\left(\frac{a_1\delta_1^2}{4} + \frac{a_2\delta_2^2}{4} - \frac{3b^2\delta_1^2}{16a_2} - \frac{3b^2\delta_2^2}{16a_1} + \frac{b^3\delta_2\delta_1}{8a_2a_1}\right)$$
$$= \frac{a_1\delta_1^2 + a_2\delta_2^2}{4\Delta^2} - \frac{b^2}{4\Delta^2 a_1 a_2}\left(\frac{3\delta_1^2 a_1 - 2b\delta_2\delta_1 + 3\delta_2^2 a_2}{4}\right) = \frac{a_1\delta_1^2 + a_2\delta_2^2}{4\Delta^2} - \frac{1-\Delta}{\Delta^2}\left(\frac{3\delta_1^2 a_1 - 2b\delta_2\delta_1 + 3\delta_2^2 a_2}{4}\right).$$

Consequently,

$$\pi^* = A - B = \frac{a_1\delta_1^2 + a_2\delta_2^2 - b\delta_1\delta_2}{2\Delta} + (1-\Delta)\left(\frac{3\delta_1^2 a_1 - 2b\delta_2\delta_1 + 3\delta_2^2 a_2}{4\Delta^2}\right) - \frac{a_1\delta_1^2 + a_2\delta_2^2}{4\Delta^2} =$$

$$\frac{2\Delta a_1\delta_1^2 + 2\Delta a_2\delta_2^2 - 2\Delta b\delta_1\delta_2 + (1-\Delta)\left(3\delta_1^2 a_1 - 2b\delta_2\delta_1 + 3\delta_2^2 a_2\right) - a_1\delta_1^2 - a_2\delta_2^2}{4\Delta^2} =$$

$$\frac{(2-\Delta)\, a_1\delta_1^2 + (2-\Delta)\, a_2\delta_2^2 - 2b\delta_2\delta_1}{4\Delta^2} \geq \frac{a_1\delta_1^2 - 2b\delta_1\delta_2 + a_2\delta_2^2}{4} = \bar{\pi},$$

where the last inequality holds because $\Delta < 1$. □

Classical model: $a = 1$**.** In this model the inverse demand is $1 - q_1 - bq_2$. The formulas (194), (210), (196) and (212) reduce to the following ones (recall that $c = 1 - \gamma$):

$$\bar{q} = \frac{1-c}{2},\ q^* = \frac{1-c}{2+b},$$

$$\bar{\pi} = \frac{(1-c)^2(1-b)}{2},\ \pi^* = \frac{2(1-c)^2}{(2+b)^2}.$$

The ratios (213) can be expressed as follows (see (214) and (215)):

$$\phi(b) = \frac{\bar{q}(b)}{q^*(b)} = \frac{2+b}{2},$$

$$\psi(b) = \frac{\bar{\pi}(b)}{\pi^*(b)} = \frac{(1-b)(2+b)^2}{4}.$$

The former function is increasing and ranges from $\phi(0) = 1$ to $\phi(1) = 3/2$. The latter decreases from $\psi(0) = 1$ to $\psi(1) = 0$. Note that both ratios do not depend on the production cost c!

Extreme cases. If $b = 0$, then goods 1 and 2 are totally different, and plants 1 and 2 producing them represent monopolies in their respective market segments. The quantities produced and the profits obtained are the same both for the contest and for the Nash equilibrium:

$$\bar{q} = q^* = \frac{1-c}{2},\ \bar{\pi} = \pi^* = \frac{(1-c)^2}{2}.$$

In the case when $b = 1$ we have a homogeneous-good model, where the results of the former and the latter are substantially distinct:

$$\bar{q} = \frac{1-c}{2} > \ q^* = \frac{1-c}{3},$$

$$\bar{\pi} = 0 < \ \pi^* = \frac{2(1-c)^2}{9}.$$

The fact that $\bar{\pi} = 0$ means *perfect competition* in a market with only two participants. This situation is akin to that observed in the classical Bertrand duopoly model (see Amir and Evstigneev [4] for its most recent treatment). When passing from contest to competition, the profit increases from 0 to $2(1-c)^2/9$, but the production output drops from $(1-c)/2$ to $(1-c)/3$.

4.8. Concluding remarks

In this chapter we developed a general framework—game with relative preferences—suitable for the analysis of the concept of an unbeatable strategy and considered various applications of this concept. The framework proposed extends its earlier version considered in the paper by Amir et al. [6], Section 6. In that paper it was observed that the classes of survival and unbeatable strategies in Evolutionary Finance models coincide and that unbeatable portfolio rules, rather than Nash equilibrium ones, represent the essence of EF models.

By defining proper relative preferences in the general setting described in Section 4.2, we can analyze within this setting:

(a) survival/unbeatable strategies in the EF model with long-lived dividend-paying assets—Evstigneev et al. [53] and Chapter 2 of this thesis;

(b) unbeatable strategies vs. Maynard Smith and Price's [128] ESS in the classical evolutionary game theory—Section 4.5;

(c) Shaffer's [146, 147] ESS for finite populations—Section 4.5, Definition 2;

(d) the notion of an evolutionary stable strategy in the EF model with endogenous dividends combining the concepts of Schaffer's and Maynard Smith and Price's ESS—Amir et al. [8], p. 5, and Section 3.3 of the dissertation;

(e) solutions to contest problems represented by unbeatable strategies vs. Nash equilibria in asymmetric Cournot duopolies—Sect. 4.6 and 4.7.

Comparative analysis of unbeatable strategies and Nash equilibria in various classical game-theoretic models may constitute an interesting direction of further research.

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