# On some Topics on the Irregularity of Distribution pertaining to Combinatorics and Geometric Discrepancy

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A thesis submitted to the
***University of Manchester***
for the degree of
***Doctor of Philosophy***
in the
***Faculty of Science and Engineering.***

*School of Natural Sciences*
*Department of Mathematics*
2022

The University of Manchester

Πειρώ τω μεν σώματι είναι
φιλόπονος, τη δε ψυχή φιλόσοφος.

Ισοκράτης, 436-338 π.Χ.

# Contents

Word Count: 69.337

# Abstract

This thesis deals with two topics: the Danzer problem in geometric discrepancy and the Sárközy-Fürstenberg theorem in combinatorics.

The first part is devoted to the Danzer problem. A suitable weakening of its statement leads one to a problem of visibility in so-called dense forests. These are discrete point sets in the Euclidean space getting uniformly close to long enough line segments. This motivates the investigation of visibility concepts emerging from discrete geometry as well as the study of the distribution of sequences in the Euclidean space, the torus and the sphere. The following types of results are established: (1) the best known visibility bounds for dense forests are improved in any dimension, (2) geometrical and visibility concepts concerning planar spiral point sets are generalised to higher dimensions and (3) density properties of oscillating sequences in the real line are established.

The second part concerns the Sárközy-Fürstenberg theorem. A multivariate version of the theorem is proved thanks to methods from Fourier analysis and with the help of a density increment argument.

# Declaration

No portion of the work referred to in the thesis has been submitted in support of an application for another degree or qualification of this or any other university or other institute of learning.

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# Acknowledgments

The submission of this thesis marks the end of a decision made a long time ago, namely, that of studying mathematics at a high level. I would like to devote this thesis to my parents Ilias and Ioanna as a minor act of gratitude for supporting me so far and to my brothers Georgios and Alexios without whom growing up would have been easier but not fulfilling. I want to thank also my two beloved grandmas, Eutuxia and Christina, who I would have hoped to be able to visit more often over the last four years. Many thanks to my uncles George & Ilias, my aunts Vaso & Margaret and my whole extended family for their care and love.

As for those immediately involved in this research, I would like to thank my supervisor during the first year of my studies, Dr. Sean Prendiville, for giving me the chance to be part of an institute of the caliber of the University of Manchester. Deep is my appreciation for my second, in chronological terms, supervisor: Dr. Faustin Adiceam. Faustin was an excellent supervisor and I want to express my gratefulness towards him on three levels, starting from what I consider less important and moving to the more significant ones:

1. For the *mathematical knowledge* he shared with me and for teaching me the way to work. The completion of this thesis is based on his exemplary mentorship.

2. For triggering my interests in *new directions.* His vast knowledge in many fields was the starting point for many long conversations, the memory of which I will keep for many years.

3. For being one of the few people I could talk when I was feeling all alone.

It is not clear if it was the fate, providence or another kind of power that made this collaboration happen. However, I am sure that the whole world conspired to enable me to spend Christmas 2020 and Easter 2021 with Faustin. I will remember these two occasions forever, not only because it was the first time celebrating them without my family, but also because I was surprised by Faustin's cooking skills.

I want to thank also a number of friends who may not have been directly involved in this PhD but who stood by my side for many years. My childhood friend Ioannis Koukouras, my team-mate Spiros Verikios, my mathematician friends Konstantinos Frementitis, Georgios Baziotis, Petros Chatzopoulos, Georgios Kalantzis, Michaela Paschalidou and my friends Nikolaos Gartaganis and Vasilis Lolos. There are also many other friends and familiar faces to thank. Even though it is impossible to name all of them, I would like them to know that they have my gratitude.

Lastly, I want to thank Vasiliki Torega, Ricardo Jesús Palomino Piepenborn, Borys Kuca, Marco Monti and Anja Meyer for filling up Manchester with beautiful memories. Special thanks to Charitini Tsiampoula.

# Publications

Part of the content of this thesis has been submitted for publication in the form of four papers:

- Faustin Adiceam, Ioannis Tsokanos, *Higher Dimensional Spiral Delone Sets*, to appear in *Funct. Approx. Comment. Math.*
- Faustin Adiceam, Ioannis Tsokanos, *Visibility Properties of Spiral Sets*, Submitted, to appear in *Mosc. J. Comb. Number Theory.*
- Ioannis Tsokanos, *Danzer's Problem, Effective Constructions of Dense Forests and Digital Sequences*, to appear in *Mathematika.*
- Ioannis Tsokanos, *Density of Oscillating Sequences in the Real Line*, to appear in *Unif. Distrib. Theory.*

The thesis takes into account the revisions of these papers following referees reports.

# Basic Notation

1. The following notations related to sets and set-theoretical notions will be used throughout the thesis.

   - $\mathbb{N} = \{1, 2, ...\}$: the set of positive natural numbers.
   - $\mathbb{N}_0 = \{0, 1, , 2...\}$: the set of non-negative integers.
   - $\mathbb{Z} = \{0, \pm 1, \pm 2, ...\}$: the set of integers.
   - $\mathbb{Q}$: the set of rational numbers.
   - $\mathbb{R}, \mathbb{R}^+$ and $\mathbb{R}^-$: the sets of reals, of positive and of negative numbers, respectively. Similarly, $\mathbb{R}_0^+$ stands for the non-negative real numbers and $\mathbb{R}_0^-$ stands for the non-positive reals.
   - $\mathbb{C}$: the set of complex numbers.
   - $\mathbb{F}_q, \mathbb{F}_q^N$: the finite field with $q$ elements and the $N$-dimensional vector space over the field $\mathbb{F}_q$, respectively, where $q$ is a power of a prime number.
   - $\mathbb{T} = \mathbb{R}/\mathbb{Z}$: the unit torus.
   - $\mathbb{S}^d$: the $d$-dimensional unit sphere in $\mathbb{R}^{d+1}$ with respect to the Euclidean norm.
   - $[\![a, b]\!] = [a, b] \cap \mathbb{Z}$ the integer interval between the reals $a \leq b$. If $a = 1$ one may write $[\![b]\!] = [\![1, b]\!]$.
   - $\#A$ stands for the cardinality of a finite set $A$.
   - Given two sets $A \subseteq B$, the function $\chi_A : B \mapsto \{0, 1\}$ denotes the charecteristic function of the set $A$.

2. Given a vector $\boldsymbol{x} \in \mathbb{R}^d$, $||\boldsymbol{x}||_2$ denotes its *Euclidean norm* and $||\boldsymbol{x}||_\infty$ its *supremum norm*. The dimension $d$ of the Euclidean space will be clear from the context. Given $\boldsymbol{x} \in \mathbb{R}^d$ and $r \geq 0$,

$$B_2(\boldsymbol{x}, r) \quad \text{and} \quad B_\infty(\boldsymbol{x}, r)$$

stand for the balls with radius $r$ centred at the point $\boldsymbol{x}$ with respect to the Euclidean and sup norms, respectively.

3. The sup-norm in the $d$-dimensional torus $\mathbb{T}^d$ is defined as

$$||\boldsymbol{x}|| \quad = \quad \min_{\boldsymbol{n} \in \mathbb{Z}^d} \{||\boldsymbol{x} - \boldsymbol{n}||_\infty\};$$

that is, the distance between $\boldsymbol{x}$ and its nearest point with integer coordinates.

4. Given two non-empty subsets $A, B \subseteq \mathbb{R}^d$, the quantity

$$\operatorname{dist}(A, B) = \inf\{||\boldsymbol{a} - \boldsymbol{b}||_2 : \boldsymbol{a} \in A, \boldsymbol{b} \in B\} \tag{1}$$

stands for the distance between the two sets. If one of the sets contains only one element, say $A = \{a\}$, then one may write $\operatorname{dist}(a, B)$.

5. Given a real number $x \in \mathbb{R}$, denote by $\lfloor x \rfloor = \max\{n \in \mathbb{Z} : n \leq x\}$ the integer part of $x$ and by $\{x\} = x - \lfloor x \rfloor$ its fractional part. Similarly, $\{x\}_2$ stands for the signed fractional part of $x$; that is, the unique real number in $\left[-\frac{1}{2}, \frac{1}{2}\right)$ such that $x - \{x\}_2 \in \mathbb{Z}$.

6. The Lebesgue measure in $\mathbb{R}^d$ is denoted by $\lambda_d(\cdot)$. If the ambient dimension is clear from the context, then the index $d$ may be omitted.

7. The complex exponential is denoted by $e(x) = e^{2\pi i \cdot x}$, where $x \in \mathbb{C}$.

8. Given a finite set $X$ of complex numbers, the average of the set $X$ is denoted by

$$\mathop{\mathbb{E}}_{x \in X} x \quad = \quad \frac{1}{\# X} \cdot \sum_{x \in X} x.$$

9. Throughout the thesis, we use Vinogradov's asymptotic notation: if, given two real functions $f, g : \mathbb{R}^+ \to \mathbb{R}^+$, there exists a positive constant $C \; > 0$ such that for every $x \in \mathbb{R}^+$, it holds that $f(x) \leq C \cdot g(x)$, then we write $f(x) \ll g(x)$. Equivalently, we may use Landau's Big-$O$ notation and write $f(x) = O\left(g(x)\right)$. Similarly, we write $f(x) = \Omega\left(g(x)\right)$ if $f(x) \gg g(x)$. The constant $C$ is referred to as the implicit constant. If the implicit constant depends on some parameter, say $t$, then we index the notation as $f(x) \ll_t g(x)$ (equivalently, as $f(x) = O_t\left(g(x)\right)$. If for two functions $f$ and $g$, it holds that $f(x) \ll g(x)$ and $g(x) \ll f(x)$ for all admissible values of $x$, then we write $f(x) \asymp g(x)$. We make also use of Landau's little-$o$ notation: if $f(x)/g(x) \to 0$ as $x \to +\infty$ (respectively, as $x \to 0$), then we may write $f(x) = o\left(g(x)\right)$ as $x \to +\infty$ (respectively, as $x \to 0$).

# Chapter 1

# Introduction

This thesis is divided into two parts. The first and longest one surrounds the problem of Danzer, which is a problem in convex geometry posed in the 1960's by the German mathematician Ludwig Danzer. Specifically, Danzer asked if there exists a point set $\mathfrak{D} \subseteq \mathbb{R}^d$ which intersects every convex set with volume one and which, moreover, satisfies the condition $\#\left(\mathfrak{D} \cap B_2\left(\mathbf{0}, T\right)\right) = O\left(T^d\right)$ for any $T > 0$.

The second part of the thesis concerns the Sárközy-Fürstenberg theorem in additive combinatorics. Sárközy-Fürstenberg proved independently in 1978 that if $A_N$ is a subset of the integer interval $[\![1, N]\!]$ such that no two elements of $A_N$ differ by a perfect square, then the ratio $\left|A_N\right|/N$ tends to 0 when $N \to +\infty$, where $\#A_N$ denotes the cardinality of $A$.

Progress in the direction of the Danzer problem is outlined in Chapters 2, 3 and in Appendix A. Considerations closely related to Danzer's problem give rise to problems in discrete geometry concerning the distribution properties of *spiral* point sets in $\mathbb{R}^{d+1}$; that is, of point sets of the form $\left\{ \sqrt[d+1]{k} \cdot \boldsymbol{u}_k \right\}_{k \in \mathbb{N}}$, where $\left(\boldsymbol{u}_k\right)_{k \in \mathbb{N}}$ is a sequence in $\mathbb{S}^d$. The study of these sets is carried out in Chapters 4 and 5. In turn, the latter investigations lead one to a problem concerning the density in the real line of oscillating sequences of the form $\left(k^\beta \cdot \sin\left(2\pi \cdot k\alpha\right)\right)_{k \in \mathbb{N}}$, with $\beta > 0$ and $\alpha \in \mathbb{R}$. This is the topic of Chapter 6. A generalisation of the Sárközy-Fürstenberg theorem for equations of the form

$$a_1x_1 + ... + a_sx_s \quad = \quad \sum_{1\leq k\leq l\leq t} c_{k,l}y_ky_l \quad \text{with } a_1 + ... + a_s = 0,$$

where $a_j, c_{k,l} \in \mathbb{N}$ for any $i \in \{1, ..., s\}$, $k, l \in \{1, ..., t\}$, is proved in Chapter 7. Appendix B deals with the proof of a distribution result concerning the multiples of vectors in the $d$-dimensional torus which will be exploited in Chapters 4 and 5.

## 1.1 Introduction to the Danzer Problem

Throughout this section, the integer $d \geq 2$ is fixed and is related to the dimension of the Euclidean space (either $\mathbb{R}^d$ or $\mathbb{R}^{d+1}$).

A **convex body** in $\mathbb{R}^d$ is a convex set with non-empty interior. A function $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is said to be a growth rate bound for a discrete subset $\mathfrak{Y} \subseteq \mathbb{R}^d$ if

$$\# \left(B_2\left(\mathbf{0}, T\right) \cap \mathfrak{Y}\right) \quad = \quad O\left(g(T)\right). \tag{1.1}$$

Moreover, a subset $\mathfrak{Y} \subseteq \mathbb{R}^d$ has **finite density** if it admits $O\left(T^d\right)$ as a growth rate bound; that is, if

$$\limsup_{T\to+\infty} \frac{\# \left(\mathfrak{Y} \cap B_d\left(\mathbf{0}, T\right)\right)}{T^d} \quad < \quad +\infty. \tag{1.2}$$

In 1965, *Danzer* asked the following problem.

**Problem 1.1.1 (The Danzer Problem)** *Let $d \geq 2$. Does there exists a set with finite density in $\mathbb{R}^d$ which intersects every convex body of volume 1?*

Fifty seven years on, Danzer's question still remains unanswered. Although this introduction provides a reasonably complete overview of the literature related to Danzer's problem, where all the references needed later in this thesis are provided, the reader is also referred to the survey by Adiceam [1] for a thorough account on this topic.

In view of this problem, given a positive real number $c > 0$, a subset $\mathfrak{D} \subseteq \mathbb{R}^d$ is called a **Danzer set** if $\mathfrak{D}$ intersects every convex body of volume $c$. In this case, $\mathfrak{D}$ is said to have the Danzer property. It is clear from the statement of Danzer's problem (equivalently, from the definition of a Danzer set) that, without loss of generality, the volume can be taken to be any fixed positive constant. Indeed, assume that a set $\mathfrak{D} \subseteq \mathbb{R}^d$ intersects every convex body of volume $c > 0$. Then, the dilated set

$$\frac{1}{\sqrt[d]{c}} \cdot \mathfrak{D} = \left\{ \frac{1}{\sqrt[d]{c}} \cdot \boldsymbol{x} : \boldsymbol{x} \in \mathfrak{D} \right\} \tag{1.3}$$

intersects every convex body of volume 1. This remark will be used implicitly throughout this exposition.

Although Danzer's problem is easy to comprehend, it would be easier to study if one could reduce the family of convex bodies to a smaller family of sets. This is achieved thanks to John's theorem on convex bodies [49], which shows that Danzer's problem can be reduced to the consideration of ellipsoids alone.

To state John's theorem, recall that an **affine map** $\Phi : \mathbb{R}^d \to \mathbb{R}^d$ is a map of the form

$$\Phi(\boldsymbol{x}) = A\boldsymbol{x} + \boldsymbol{b}$$

where $A : \mathbb{R}^d \to \mathbb{R}^d$ is linear and $\boldsymbol{b} \in \mathbb{R}^d$ is a constant vector. It is *nonsingular* if $\det A \neq 0$. An **ellipsoid** in $\mathbb{R}^d$ is the image of the closed unit ball $B_d(\mathbf{0}, 1)$ of $\mathbb{R}^d$ under a nonsingular affine map. Thus, an ellipsoid is a set $E \subseteq \mathbb{R}^d$ of the form

$$E = \left\{ \boldsymbol{c} + \boldsymbol{x} : \sum_{i=1}^{d} \frac{\langle \boldsymbol{x}, \boldsymbol{e_i} \rangle^2}{a_i^2} \leq 1 \right\} \tag{1.4}$$

where $(\boldsymbol{e_i})_1^n$ is an orthonormal basis of $\mathbb{R}^d$ and $(a_i)_1^n$ are positive numbers. The point $\boldsymbol{c}$ is called the center of the ellipsoid.

**Theorem 1.1.2 (John's Theorem on Convex Bodies [49])** *Let $\mathcal{K} \subseteq \mathbb{R}^d$ be a convex body. Then, there is a unique ellipsoid $E$ of maximal volume (called John's ellipsoid) such that if $\mathbf{c}$ is the center of $E$, then the inclusions*

$$E \subseteq \mathcal{K} \subseteq \boldsymbol{c} + d(E - \boldsymbol{c})$$

*hold. Here* $\boldsymbol{c} + d(E - \boldsymbol{c})$ *is the set of points* $\{\boldsymbol{c} + d(\boldsymbol{x} - \boldsymbol{c}) : \boldsymbol{x} \in E\}$*, namely, the dilation of* $E$ *by a factor* $d$ *with center* $\boldsymbol{c}$*.*

It is of central importance to note that one can replace the ellipsoids in the statement of John's theorem with boxes (recall that a **box** is a *parallelotope* whose adjacent faces are orthogonal to each other). Indeed, from the statement of Theorem 1.1.2, one obtains that

$$\frac{1}{d} \leq \frac{\lambda_d(E)}{\lambda_d(\mathcal{K})} \leq 1 \quad \text{and} \quad 1 \leq \frac{\lambda_d\left(c + d(E - c)\right)}{\lambda_d(\mathcal{K})} \leq d,$$

where $E \subseteq \mathbb{R}^d$ is an ellipsoid as described in (1.4). The volume of $E$ is equal to $\lambda_d(E) = u_d \cdot \prod a_i$, where $u_d$ is the volume of the ball $B_d(\mathbf{0}, 1)$. The box

$$I = \left\{c + x_1\boldsymbol{e_1} + ... + x_d\boldsymbol{e_d} \in \mathbb{R}^d : x_i \in \left[-\frac{a_i}{\sqrt{d}}, \frac{a_i}{\sqrt{d}}\right]\right\}$$

is contained in $E$ and moreover one has that $\lambda_d(I) = \frac{2^d \prod a_i}{d^{\frac{d}{2}}}$; that is,

$$\frac{\lambda_d(E)}{\lambda_d(I)} \quad = \quad u_d \cdot \frac{d^{\frac{d}{2}}}{2^d}.$$

Similarly, the box

$$J \quad = \quad \left\{c + x_1\boldsymbol{e_1} + ... + x_d\boldsymbol{e_d} \in \mathbb{R}^d : x_i \in [-a_i, a_i]\right\}$$

is such that $\boldsymbol{c} + d(E - \boldsymbol{c}) \subseteq J$ and $\lambda_d(J) = 2^d \prod a_i$. Therefore,

$$\frac{\lambda_d(E)}{\lambda_d(J)} \quad = \quad \frac{u_d}{2^d}.$$

Thus, the following corollary has just been established (see also Figure 1.1).

**Corollary 1.1.3 (Corollary to John's Theorem)** *For every convex body* $\mathcal{K} \subseteq \mathbb{R}^d$ *there exist boxes* $\mathcal{I}, \mathcal{J} \subseteq \mathbb{R}^d$ *and constants* $C_d, c_d > 0$ *such that*

$$\mathcal{I} \quad \subseteq \quad \mathcal{K} \quad \subseteq \quad \mathcal{J}$$

*and*

$$c_d \quad \leq \quad \frac{Vol(\mathcal{I})}{Vol(\mathcal{K})} \leq 1 \quad and \quad 1 \quad \leq \quad \frac{Vol(\mathcal{J})}{Vol(\mathcal{K})} \quad \leq \quad C_d.$$

*A suitable choice for the constants* $C_d, c_d > 0$ *is*

$$C_d \quad = \quad \frac{d \cdot 2^d}{u_d} \quad and \quad c_d \quad = \quad \frac{2^d}{d^{\frac{d+2}{2}} \cdot u_d}.$$

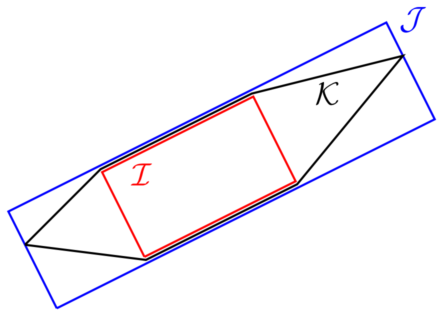



*Figure 1.1.* John's Theorem. A convex body $\mathcal{K}$ with an inner parallelotope $\mathcal{I}$ and an outer parallelotope $\mathcal{J}$ .

As a consequence of Corollary 1.1.3, one can replace, without loss of generality, the family of convex bodies in the statement of Danzer's problem (respectively, in the definition of a Danzer set) with the smaller family of boxes in $\mathbb{R}^d$.

**Corollary 1.1.4** *A subset* $\mathfrak{D} \subseteq \mathbb{R}^d$ *is a Danzer set if, and only if, there exists a constant* $c > 0$ *such that* $\mathfrak{D}$ *intersects every box with volume c.*

A strengthened version of Danzer's problem due to Gowers [1, Section 2.2] asks whether there exists a Danzer set $\mathfrak{D}$ for which there is a finite bound $M$ on the number of points of intersection between $\mathfrak{D}$ and any convex body of unit volume. Gower's version has been disproved by Solan, Solomon and Weiss [73].

There have been four main approaches to tackle Danzer's problem, namely:

- Relaxing the density constraint,
- Relaxing the volume constraint,
- Studying families of discrete sets enjoying extra structure to see if they satisfy the Danzer property,

- Considering a class of boxes smaller than the set of all possible boxes in the statement of Danzer's problem.

The state of the art regarding each of these approaches is given in the upcoming Subsections 1.1.1, 1.1.2, 1.1.3 and 1.1.4. Subsections 1.1.5-1.1.9 will give the background material needed for the study of Danzer's problem undertaken in this thesis.

### 1.1.1 Relaxing the Density Constraint

In this approach, one allows Danzer sets in $\mathbb{R}^d$ with growth rate bound larger than the optimal bound $O\left(T^d\right)$ but as close to it as possible; that is, one allows Danzer sets which do not have a finite density. The first result in this direction was established by Bambah and Woods in [15].

**Theorem 1.1.5** *[15, Theorem 2] There exists a Danzer set $\mathfrak{D} \subseteq \mathbb{R}^d$ with growth rate bound $g(T)$, where*

$$g(T) \quad = \quad O\left(T^d \cdot \left(\log(T)^{d-1}\right)\right).$$

In [74], Solomon and Weiss provide a probabilistic construction to improve on the density bound stated in Theorem 1.1.5. Their result currently stands as the best known bound.

**Theorem 1.1.6** *[74, Theorem 1.6] There exists a Danzer set in $\mathbb{R}^d$ with growth rate bound $O\left(T^d \cdot \log(T)\right)$.*

It is asked in [1, Problem 8] if one can provide a deterministic example of a Danzer set satisfying the growth rate bounds provided by Theorem 1.1.6. In Chapter 2, an affirmative answer is given when one considers a weakening of this problem; namely, when one replaces the family of all boxes of volume 1 in $\mathbb{R}^d$, denoted it by

$$\mathcal{B}_d \quad = \quad \left\{\text{Boxes in } \mathbb{R}^d \text{ with volume } 1\right\}, \tag{1.5}$$

with the smaller family

$$\mathcal{B}'_d \quad = \quad \left\{ \begin{array}{r} B \subseteq \mathbb{R}^d : B \text{ is a box which has } d-1 \text{ sides of length } \epsilon \in (0,1) \\ \text{and the remaining side with length } \epsilon^{-(d-1)} \end{array} \right\}. \tag{1.6}$$

When $d = 2$, this is not a weakening since $\mathcal{B}_2 = \mathcal{B}'_2$.

### 1.1.2 Relaxing the Volume Constraint

This approach is recent and is achieved in two steps. First, replace in the statement of Danzer's problem the family $\mathcal{B}_d$ defined in (1.5), with the smaller family of boxes $\mathcal{B}'{}_d$, defined in (1.6). In this setup, the relaxation of the volume constraint consists in letting the longest side of the box to have length $V(\epsilon) \gg \epsilon^{-(d-1)}$, where $V : (0,1) \to \mathbb{R}^+$ is a decreasing function such that $V(\epsilon) \to +\infty$ when $\epsilon \to 0^+$. Each such box can then be seen as the $\epsilon$-neighbourhood of the line segment connecting the centres of its two opposite faces with sidelengths $\epsilon$, upon ignoring a small neighbourhood of the edges of the line segment (see Figure 1.1.2). Ignoring the (lower order) volumes of those neighbourhoods is with no loss of generality: all the statements are up to multiplicative constants as one can work, for instance, with line segments with double length. This approach leads one to the problem of a dense forest.

**Definition 1.1.7 (Dense Forest)** *A set $\mathfrak{F} \subseteq \mathbb{R}^d$ is a dense forest if it has finite density and if it satisfies the following property: there exists a decreasing function $V : (0,1) \to \mathbb{R}^+$ tending to $+\infty$ as $\epsilon \to 0^+$ such that for any $\epsilon \in (0,1)$ and any line segment $L \subseteq \mathbb{R}^d$ with length $V(\epsilon)$, there is a point $\boldsymbol{x} = \boldsymbol{x}(L) \in \mathfrak{F}$ such that $dist(\boldsymbol{x}, L) \leq \epsilon$. The function $V$ is said to be a visibility function for the dense forest $\mathfrak{F}$.*

The problem of constructing dense forests is about the existence of a dense forest in $\mathbb{R}^d$ with visibility function

$$V(\epsilon) \quad = \quad O\left(\epsilon^{-(d-1)}\right), \tag{1.7}$$

As will be proved in Chapter 2, this bound is the best one can hope for since, given a dense forest $\mathfrak{F} \subseteq \mathbb{R}^d$, a visibility function $V$ in $\mathfrak{F}$ always satisfies the relation

$$V(\epsilon) \quad \gg \quad \epsilon^{-(d-1)}.$$

It is easy to see that a Danzer set of finite density is a dense forest with optimal visibility. This means that one can potentially disprove Danzer's problem by showing that there do not exist dense forests with optimal visibility. For more details regarding the connection between Danzer's problem and that of dense forests, see [1, Sections 3 & 4].

In the planar case, the two notions are equivalent; to see this, assume that $\mathcal{R} \subseteq \mathbb{R}^2$ is a rectangle of volume two (recall that it is sufficient to prove the Danzer condition for the family of rectangles with any fixed positive volume). In particular, assume that $\mathcal{R}$ has a short side with length $2\epsilon$ and a long side with length $\epsilon^{-1}$, for some $\epsilon \in (0,1)$. Let $L$ be the line segment which connects the middle points of the short edges of $\mathcal{R}$. Let $L_{\mathcal{R}}$ be the line segment which results from $L$ by removing from both its ends a line segment of length $\epsilon$ (see Figure 1.1.2). When $\epsilon$ is sufficiently small, say $\epsilon \leq 1/4$, the lengths $l_{\mathcal{R}}$ and $l$ of $L_{\mathcal{R}}$ and $L$, respectively, have the same order, in the sense that $l \ll l_{\mathcal{R}} \ll l$. This is true since the part which was removed from $L$ to obtain $L_{\mathcal{R}}$ has length $2\epsilon$. It is obvious that, given a point $\boldsymbol{x} \in \mathbb{R}^2$;

1. if the ball $B_2(\boldsymbol{x}, \epsilon)$ intersects $L_{\mathcal{R}}$, then $\boldsymbol{x}$ belongs to $\mathcal{R}$ and, conversely,

2. if $\boldsymbol{x}$ belongs in $\mathcal{R}$ then the ball $B_2(\boldsymbol{x}, \epsilon)$ intersects $L$.

These two conditions are of course equivalent to saying that the point $\boldsymbol{x}$ is $\epsilon$-close to the line segments $L_{\mathcal{R}}$ and $L$, respectively. Let $\mathfrak{D}$ be a subset of $\mathbb{R}^2$. Adjusting the set $\mathfrak{D}$ by rescaling it properly if necessary (see equation 1.3), it is clear that $\mathfrak{D}$ is a Danzer set which intersects every box of volume 2 if, and only if, it is $\epsilon$-close to any line segment $L'$ with length $V(\epsilon) = 2\epsilon^{-1} - 2\epsilon \leq 2\epsilon^{-1}$. Therefore, in the planar case, a set $\mathfrak{D}$ is a Danzer set if, and only if, it is a dense forest with optimal visibility. It should be noted that this equivalence is not true in higher dimensions [1, Section 3, p.12].

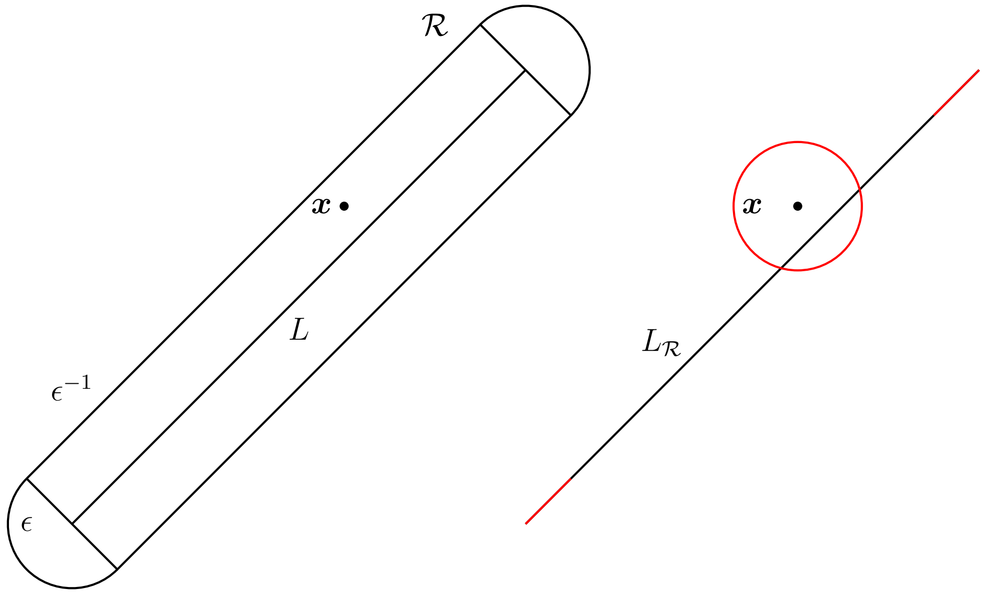



*Figure 1.1.2.* The equivalence between a Danzer set with finite density and a dense forest with optimal visibility in the plane. On the left-hand side, the point $\boldsymbol{x}$ lies in the rectangle $\mathcal{R}$ with sides $2\epsilon$ and $\epsilon^{-1}$; on the right-hand side, the ball with radius $\epsilon$ centred at the point $\boldsymbol{x}$ intersects the middle line segment $L$ of the rectangle. The rectangle $R$ with the half disks is the full $\epsilon$-neighbourhood of the line segment $L$.

Over the past ten years, many authors have taken an interest in the study of dense forests, mostly because of its connection with the Danzer problem. A strengthening of the dense forest problem asks for the set under consideration to enjoy the extra property of being a Delone set.

**Definition 1.1.8 ($s$-Uniform Discreteness, $r$-Relatively Dense, Delone Set)** *Let $r, s > 0$ be constants.*

- *A subset $A \subseteq \mathbb{R}^d$ is* $s$-uniformly discrete *if for every $\boldsymbol{x}, \boldsymbol{y} \in A$ with $\boldsymbol{x} \neq \boldsymbol{y}$, it holds that*

$$||\boldsymbol{x} - \boldsymbol{y}||_2 \geq s.$$

  *It is* uniformly discrete *if it is $s$-uniformly discrete for some $s > 0$.*

- *A subset* $A \subseteq \mathbb{R}^d$ *is* $r$-relatively dense *if for every* $\boldsymbol{x} \in \mathbb{R}^d$*, it holds that*

$$\# \left(B_2 \left(\boldsymbol{x}, r\right) \cap A\right) \geq 1.$$

  *It is* relatively *dense if it is* $r$*-relatively dense for some* $r > 0$*.*

- *A subset* $A \subseteq \mathbb{R}^d$ *is a* Delone *set if it is both uniformly discrete and relatively dense. In this case, the set* $A$ *is said to have the* Delone property*.*

This strengthening is related to a version of Danzer's problem, known also as *Conway's dead fly problem* [1, Section 2.2], which asks whether there exists a Danzer set which enjoys the extra property of being Delone. This question was also posed independently by Boshernitzan [1, Section 2.2].

In 2014, Solomon and Weis [74], using homogeneous dynamics and ergodic theory, proved the existence of a Delone dense forest in $\mathbb{R}^d$, $d \geq 2$. Their methods, however, did not provide a bound for the visibility in the corresponding forest.

It is clear that any Delone set has finite density and that, in general, finite density does not imply the Delone property. The best result in the direction of constructing Delone dense forests is due to Alon [13]. By utilising Lovasz's Local Lemma from probability theory, he provides a planar construction of a Delone dense forest with almost optimal visibility (in the sense stated in the following theorem). However, because of the nature of the argument, Alon's construction is not deterministic.

**Theorem 1.1.9 (Alon's Delone Dense Forest)** *There is a constant* $C > 0$ *such that there exists a Delone planar dense forest with visibility function*

$$V(\epsilon) \quad = \quad O\left(\epsilon^{-1} \cdot 2^{C \cdot \sqrt{\log(1/\epsilon)}}\right). \tag{1.8}$$

Given a function $V : (0, 1) \mapsto \mathbb{R}^+$ and $d \geq 2$, in Chapter 2 is proved a sufficient condition for the existence of a dense forest $\mathfrak{F}$ in $\mathbb{R}^d$ with visibility $V$. Furthermore, the related construction is *fully deterministic.* As a consequence, we will obtain a (deterministic) dense forest with almost optimal visibility $O_\eta\left(\epsilon^{-(d-1)} \cdot \ln\left(1/\epsilon\right) \cdot \ln\ln\left(1/\epsilon\right)^{1+\eta}\right)$, where $\eta > 0$ can be chosen arbitrary small. This bound should be compared with (1.8) when $d = 2$.

### 1.1.3 Studying Families of Discrete Sets enjoying extra Structure, to see if they satisfy the Danzer property

The main idea in this approach is to try and understand further the structure of a Danzer set and/or of a dense forest by finding concrete and, if possible, *deterministic* examples; that is, to show that classes of discrete points sets enjoying some additional structure do (not) satisfy the Danzer property or the property of being a dense forest despite having a finite density. For instance, in [15], Bambah and Woods prove that the union of a finite number of grids cannot be a Danzer set (recall that a **grid** is a translated lattice).

Over the last years, there has been a lot of progress in the problem of dense forests, mostly thanks to a construction due to Yuval Peres [24, Lemma 2.4] described in the following definition. In fact, Peres' construction corresponds to the planar case of the definition while the higher dimensional analogue is due to Adiceam, Solomon and Weiss [4, p.16, Equation (5.5)].

**Definition 1.1.10 (Peres-Type Forest)** *Let $R : \mathbb{R}^{d+1} \to \mathbb{R}^{d+1}$ be the linear transformation which acts by permuting the coordinates as follows:*

$$R\left(x_1, x_2, ..., x_d, x_{d+1}\right)^T = \left(x_2, ..., x_d, x_{d+1}, x_1\right)^T. \tag{1.9}$$

*Given $s \geq 2$ and $\boldsymbol{\Theta}_{s,d} = (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_s)$ an $s$-tuple of $d$-dimensional vectors, define the Peres' forest generated by the tuple $\boldsymbol{\Theta}_{s,d}$ to be the set*

$$\mathfrak{F}\left(\boldsymbol{\Theta}_{s,d}\right) \quad = \quad \bigcup_{l=1}^{d+1} \mathfrak{F}_l\left(\boldsymbol{\Theta}_{s,d}\right). \tag{1.10}$$

*Here,*

$$\mathfrak{F}_1\left(\boldsymbol{\Theta}_{s,d}\right) = \bigcup_{i=1}^{s} \begin{pmatrix} 1 & \mathbf{0}^T \\ \boldsymbol{\theta}_i & I_d \end{pmatrix} \cdot \mathbb{Z}^{d+1}$$

*and*

$$\mathfrak{F}_l\left(\boldsymbol{\Theta}_{s,d}\right) = R^{l-1}\left(\mathfrak{F}_1\left(\boldsymbol{\Theta}_{s,d}\right)\right) \quad \text{for every } l \in \{2, 3, ..., d+1\},$$

*where $I_d$ stands for the $d \times d$ identity matrix. The set $\mathfrak{F}\left(\boldsymbol{\Theta}_{s,d}\right)$ is called Peres'* *forest generated from the s-tuple* $\mathbf{\Theta}_{s,d}$*.*

*When* $d = 1$*, given a sequence* $\boldsymbol{a} = (a_k)_{k\in\mathbb{N}}$ *in the unit torus* $\mathbb{T}$*, define the planar set*

$$\mathfrak{F}(\boldsymbol{a}) \quad = \quad \mathfrak{F}_1(\boldsymbol{a}) \cup \mathfrak{F}_2(\boldsymbol{a}), \tag{1.11}$$

*where*

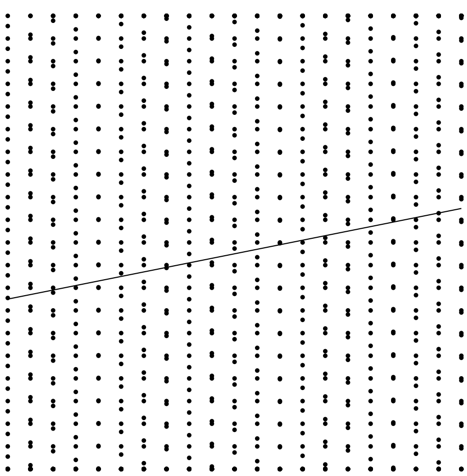
$$\mathfrak{F}_1(\boldsymbol{a}) = \left\{ \left(k, a_{|k|} + l\right) : k \in \mathbb{Z}\setminus\{0\}, l \in \mathbb{Z} \right\} \quad \text{and} \quad \mathfrak{F}_2(\boldsymbol{a}) = R_{\frac{\pi}{2}}(\mathfrak{F}_1).$$

*Here,* $R_{\frac{\pi}{2}}(\cdot)$ *is the* $\frac{\pi}{2}$*-radian rotation around the origin* $(0,0)$*. The set* $\mathfrak{F}(\boldsymbol{a})$ *is called Peres' forest generated from the sequence* $\boldsymbol{a}$*.*

When the toral sequence $\boldsymbol{a}$ is defined as the multiples of a real number $\theta$; that is, when $\boldsymbol{a} = (k\cdot\theta)_{k\in\mathbb{N}}$ modulo 1, the forest defined in (1.11) is the same as the one defined in (1.10) with $\mathbf{\Theta}_{1,1} = \theta$. Peres specialises the construction in (1.11) to the sequence $\boldsymbol{a} = (a_k)_k$ with

$$a_k = \begin{cases} 0 & \text{if } k \in 2\mathbb{N}-1 \\ \frac{k}{2}\cdot\phi & \text{if } k \in 2\mathbb{N} \end{cases}, \tag{1.12}$$

where $\phi$ is the golden ratio. Furthermore, Peres [24, Lemma 2.4] proves that $\mathfrak{F}(\boldsymbol{a})$ has visibility $O(\epsilon^{-4})$.



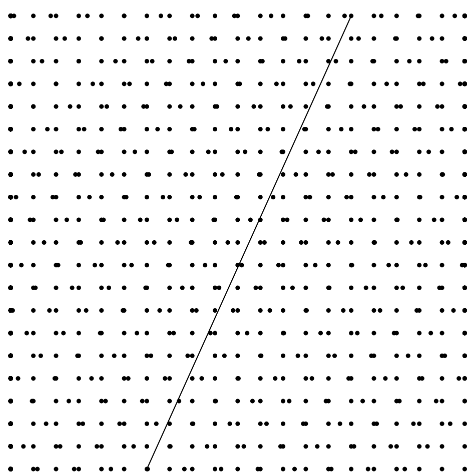


*Figure 1.1.3(a).* An illustration of Peres' original forest. The sets $\mathfrak{F}_1$ (left figure) and $\mathfrak{F}_2$ (right figure) defined in (1.11) are generated from the sequence $\boldsymbol{a}$ define in (1.12) for $n \in [\![-10, 10]\!]$ . The forest $\mathfrak{F}_1$ is dense when considering only (nearly horizontal) line segments with slope $|\xi| \leq 1$. Similarly, the forest $\mathfrak{F}_2$ is dense when considering only (nearly vertical) line segments with slope $|\xi| \geq 1$.

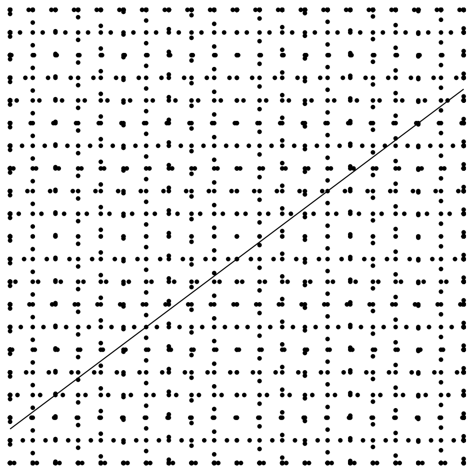


*Figure 1.1.3(b).* Peres' dense forest $\mathfrak{F}$ is defined as the union of the forests $\mathfrak{F}_1$ and $\mathfrak{F}_2$.

In [2, 4], Adiceam, Solomon and Weiss refine construction (1.11) and generalise it to higher dimensions to arrive at the definition of the set in (1.10). In particular, in [4], the authors exploit these higher dimensional constructions to show the existence of dense forests in $\mathbb{R}^d$ with (almost optimal) visibility $O\left(\epsilon^{-(d-1)-\eta}\right)$ for every $d \geq 2$, where $\eta > 0$ can be chosen arbitrarily small. However, these constructions are almost-deterministic in the sense that they still depend on the probabilistic choice of a set of parameters:

**Theorem 1.1.11** *[4, Theorem 1.4] For each $d \geq 2$, each $s \geq d$ and each $\eta > 0$, for almost every choice, with respect to the Lebesgue measure, of an $s$-tuple of $(d-1)$-dimensional vectors $\boldsymbol{\Theta}_{s,d-1} = (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_s)$, the point set $\mathfrak{F}(\boldsymbol{\Theta}_{s,d-1})$ defined in equation (1.10) is a dense forest in $\mathbb{R}^d$ with visibility satisfying the bound*

$$V(\epsilon) \quad = \quad O_\eta\left(\epsilon^{-(d-1+\alpha_d(s))+\eta}\right),$$

*where*

$$\alpha_d(s) \quad = \quad \frac{d(d-1)^2}{s-d+1} \quad \underset{s\to+\infty}{\longrightarrow} \quad 0.$$

Given a sequence $\boldsymbol{a}$ in $\mathbb{T}$, it is easy to check that the forest $\mathfrak{F}(\boldsymbol{a})$ has finite density. The construction of dense forests of the form (1.11) is of independent interest, as the visibility properties of the point set $\mathfrak{F}(\boldsymbol{a})$ depend on the properties of the sequence $\boldsymbol{a}$. This remains true for the corresponding constructions in higher dimensions as well. This enables the use of tools from the theory of distribution of sequences modulo 1 and from Diophantine approximation to tackle the dense forest problem.

**Remark 1.1.12** *Given a toral sequence $\boldsymbol{a}$, let $\mathfrak{F}(\boldsymbol{a})$ be the planar Peres-type forest generated from $\boldsymbol{a}$ as defined in* (1.11)*. If the point set $\mathfrak{F}(\boldsymbol{a})$ is a dense forest, then the terms of the sequence $\boldsymbol{a}$ are dense in $\mathbb{T}$. To see this, it is enough to consider the horizontal half-lines of the form $L_c = \{(x, y) : x > 0, y = c\}$, with $c \in \mathbb{R}$. Fix $c \in \mathbb{R}$ and $\epsilon > 0$. Since the forest $\mathfrak{F}(\boldsymbol{a})$ is assumed to be dense, it holds that $dist(L_c, \mathfrak{F}(\boldsymbol{a})) \leq \epsilon$. This implies the existence of natural numbers $k, l \in \mathbb{N}$ such that*

$$||c - a_k|| \quad = \quad ||(k, c) - (k, a_k + l)||_2 \quad = \quad dist(L_c, \mathfrak{F}(\boldsymbol{a})) \quad \leq \quad \epsilon.$$

*The choice of $c \in \mathbb{R}$ and $\epsilon > 0$ is arbitrary, therefore, the density of the terms of $\boldsymbol{a}$ follows. As will be detailed in Chapter 3, the forest $\mathfrak{F}(\boldsymbol{a})$ is dense with visibility $O(V)$, for some function $V : (0, 1) \mapsto \mathbb{R}^+$, if and only if for every $\epsilon > 0$, $m \in \mathbb{N}_0$ and $\xi, \gamma \in \mathbb{T}$, there exists $k \in [\![m + 1, m + V(\epsilon)]\!]$ such that $||a_{k+m} - k\xi - \gamma|| \leq \epsilon$. This makes it clear that the density properties of the sequence of the multiples of real numbers lies at the heart of the study of visibility bounds in Peres-type forests. This approach is adopted in Chapter 3 and Appendix A and will be justified more rigorously therein. The material needed for these investigations is developed in Section 1.1.7 while results concerning the distribution of sequences modulo one are given in Section 1.1.6.*

The main result of Chapter 3 concerns the (deterministic) construction of a planar Peres-type forest with the best known visibility bound, namely, $O_\eta(\epsilon^{-2-\eta})$

for any $\eta > 0$. This is achieved by constructing a deterministic digital sequence in $\mathbb{T}$ satisfying strong distribution properties. This should be compared with Peres' aforementioned bound $O\left(\epsilon^{-4}\right)$ improved by Adiceam, Solomon and Weiss to $O\left(\epsilon^{-3}\right)$.

Another structure which is considered in this thesis as a candidate for satisfying the dense forest property is the class of *spiral* point sets. A planar spiral is defined as the point set of the form

$$\mathfrak{S}\left(\boldsymbol{u}\right) \quad = \quad \left\{\sqrt{k}\cdot e\left(u_k\right)\right\}_{k\in\mathbb{N}}, \tag{1.13}$$

where $\boldsymbol{u} = \left(u_k\right)_{k\in\mathbb{N}}$ is a sequence in the unit torus $\mathbb{T}$. When $\boldsymbol{u} = \left(k\cdot\alpha\right)_{k\in\mathbb{N}}$ for some $\alpha \in \mathbb{R}$, Akiyama [9] proved that the spiral $\mathfrak{S}\left(\boldsymbol{u}\right)$ is Delone if, and only if, the real number $\alpha$ is badly approximable. Marklof [59] took a more general study of spiral point sets and established necessary and sufficient conditions on an arbitrary toral sequence $\boldsymbol{u}$ for the spiral $\mathfrak{S}(\boldsymbol{u})$ to be Delone.

In his work, Akiyama [9] raised the question as to whether this theory can be generalised to higher dimensions and, in particular, if examples of higher dimensional spiral Delone sets can be obtained. Analogous to the definition of a planar spiral (1.13), define a spiral set in $\mathbb{R}^{d+1}$ as a set of points of the form $\left\{\sqrt[d+1]{k}\cdot\boldsymbol{u}_k\right\}_{k\in\mathbb{N}}$, where $\boldsymbol{U} = \left(\boldsymbol{u}_k\right)_{k\in\mathbb{N}}$ is a sequence in the $d$-dimensional sphere $\mathbb{S}^d$. Akiyama's question is answered in Chapter 4 by both generalising the results of Marklof [59] and providing explicit examples of higher dimensional spiral Delone sets. Visibility properties of spirals in any dimension and their connection with dense forests are considered and studied in Chapter 5.

### 1.1.4 Considering a class of boxes smaller than the set of all possible boxes in the statement of Danzer's problem

In Section 1.1.1 it was established that the Danzer problem amounts to the existence of a Danzer set with finite density which intersects every element of the family of boxes $\mathcal{B}_d$, as defined in (1.5). In this approach, one asks for this intersection property to hold for a subfamily of $\mathcal{B}_d$. A natural choice is the subfamily

$\mathcal{AB}_d$ of **aligned boxes** (of volume 1); that is, of boxes with edges parallel to the canonical basis of $\mathbb{R}^d$. A set $\mathfrak{D} \subseteq \mathbb{R}^d$ is an **aligned-Danzer set** if $\mathfrak{D}$ intersects any aligned box of a given positive volume. This problem has been completely settled by Simmons and Solomon [72]. To state their result, recall the definition of an *admissible lattice* in $\mathbb{R}^d$.

**Definition 1.1.13 (Admissible Lattice)** *A lattice $\Lambda \subseteq \mathbb{R}^d$ is admissible if*

$$\inf_{(l_1,\ldots,l_d)\in\Lambda\setminus\{0\}} |l_1 \cdots l_d| \quad > \quad 0.$$

For more details in lattice theory, the reader is referred to Section 1.1.9.

**Theorem 1.1.14 (Aligned-Danzer Sets)** *[72, Theorems 1.1 & 1.3] For every $d \geq 2$, there exists an admissible lattice in $\mathbb{R}^d$, and every admissible lattice is an aligned-Danzer set with finite density.*

*Moreover, the set*

$$\mathfrak{D} \quad = \quad \left\{ \left( \pm \sum_{n\in\mathbb{N}} a_n 2^n, \pm \sum_{n\in\mathbb{N}} a_n 2^{-n} \right) \in \mathbb{R}^2 : \quad (a_n)_{n\in\mathbb{N}} \in \{0,1\}^{\mathbb{Z}}_{Fin} \right\} \tag{1.14}$$

*is an aligned-Danzer set in $\mathbb{R}^2$ with finite density. Here, the set $\{0,1\}^{\mathbb{Z}}_{Fin}$ is the subset of $\{0,1\}^{\mathbb{Z}}$ consisting of those bi-infinite sequences that contain only finitely many 1's.*

The set $\mathfrak{D}$ defined in (1.14) is a 2-dimensional variant of the well-known van der Corput sequence [36, 43].

### 1.1.5 Visibility Problems

The problem of dense forests (see Definition 1.1.7 and the discussion around it) is an example of a *visibility problem.* To see more precisely what such problems are about we introduce some definitions.

A set $\mathfrak{Y} \subseteq \mathbb{R}^d$ is **discrete** if for every $\boldsymbol{x} \in \mathbb{R}^d$ and $r > 0$, the intersection $\mathfrak{Y} \cap B_2(\boldsymbol{x}, r)$ is finite. A **ray** emanating from a point $\boldsymbol{x} \in \mathbb{R}^d$ in direction $\boldsymbol{v} \in \mathbb{S}^{d-1}$ is the half-line

$$L(\boldsymbol{x}, \boldsymbol{u}) \quad = \quad \{\boldsymbol{x} + t\boldsymbol{v}\}_{t\geq 0}\,. \tag{1.15}$$

The set $\mathfrak{Y}$ blocks a ray $L$ if its distance from $L$ is zero; that is, if $\operatorname{dist}(L, \mathfrak{Y}) = 0$. The statement of a visibility problem (in *discrete geometry*) is usually of the form: given a family of rays $\mathfrak{R}$ in $\mathbb{R}^d$ and a discrete subset (of obstacles) $\mathfrak{Y} \subseteq \mathbb{R}^d$, does $\mathfrak{Y}$ block every ray belonging to $\mathfrak{R}$? This is formalised in the following definition.

**Definition 1.1.15 (Visible Points)** *Let $\mathfrak{Y} \subseteq \mathbb{R}^{d+1}$ be non-empty. Fix a direction $\boldsymbol{v} \subseteq \mathbb{S}^d$.*

*The set of* visible points *of $\mathfrak{Y}$ in direction $\boldsymbol{v}$ is defined as*

$$Vis\left(\mathfrak{Y}, \boldsymbol{v}\right) \quad = \quad \left\{\boldsymbol{x} \in \mathbb{R}^{d+1} : dist_2\left(L\left(\boldsymbol{x}, \boldsymbol{v}\right), \mathfrak{Y}\right) \ > 0\right\}.$$

*The set of* visible points *of $\mathfrak{Y}$ is defined as*

$$Vis\left(\mathfrak{Y}\right) \quad = \quad \left\{\boldsymbol{x} \in \mathbb{R}^{d+1} : \exists \boldsymbol{v} \in \mathbb{S}^d, \boldsymbol{x} \in Vis\left(\mathfrak{Y}, \boldsymbol{v}\right)\right\}.$$

*The set of* hidden points *of $\mathfrak{Y}$ is the complement set $\mathbb{R}^{d+1} \backslash Vis(\mathfrak{Y})$.*

It is clear that Definition 1.1.7 of a dense forest is a quantitative version of that of hidden points. The condition regarding discreteness is naturally imposed on the set of obstacles $\mathfrak{Y}$ for the visibility conditions under consideration not to be trivially met. For instance, if one were imposing a weaker restriction, say that $\mathfrak{Y} \subseteq \mathbb{R}^d$ is just countable, then the set $\mathfrak{Y} = \mathbb{Q}^d$ blocks any ray.

Visibility concepts such as the notion of visible points are employed to characterise density properties of point sets (of obstacles). Such formalisations can be tracked back at least to Pólya's classical orchard's problem [67, Problem 239]: "how thick must be the trunks of the trees (viewed as disks in $\mathbb{R}^2$) in a circular orchard grow if they are to block completely the view from the center?" . Assume that the observer stands at the origin, that the centres of the trees are located at the nonzero points of the lattice $\mathbb{Z}^2$ and that they have Euclidean norms at most $Q \ > 0$. Pólya, based on Minkowski's Convex Body Theorem, showed that it is enough for the trees to have radius $1/Q$ to block any ray emanating from the origin. We recall here the statement of Minkowski's Convex Body Theorem given its importance for later discussions.

**Theorem 1.1.16 (Minkowski's Convex Body Theorem)** *[29, p.68-73, Chapter III.2] Let $\mathcal{K}$ be a convex body in $\mathbb{R}^d$ symmetric with respect to the origin. If $\lambda_d(\mathcal{K}) > 2^d$, then $\mathcal{K}$ contains a non-zero integer point.*

Pólya's problem was also solved by Allen [11] with slightly sharper estimates. It motivates the following definition.

**Definition 1.1.17 (Orchard)** *A subset $\mathcal{O} \subseteq \mathbb{R}^d \setminus \{\mathbf{0}\}$ is an* orchard *if it has finite density and if there exists a function $V : \epsilon \in (0,1) \to V(\epsilon) \in \mathbb{R}^+$, increasing as $\epsilon \to 0^+$, such that the following holds: for every $\epsilon > 0$ and every direction $\boldsymbol{v} \in \mathbb{S}^{d-1}$, there exists a point $\boldsymbol{o} \in \mathcal{O}$ and a real number $0 < t \leq V(\epsilon)$ such that $||\boldsymbol{o} - t\boldsymbol{v}||_2 \leq \epsilon$.*

An orchard can be viewed as a weaker version of a dense forest where the position of the viewer is fixed at the origin. Here, by **viewer** we mean the point $\boldsymbol{x}$ from which a ray is emanating. Any visibility function $V$ of an orchard satisfies the same lower bound (1.7) satisfied by a visibility function of a dense forest. This will be justified in detail in Chapter 5.

### 1.1.6 The Distribution of Sequences modulo One

Throughout this section, $\boldsymbol{a} = (a_k)_{k\in\mathbb{N}}$ stands for a sequence in $\mathbb{T}$. The qualitative and quantitative features in the distribution of toral sequences is the object of study of the theory of distribution of sequences modulo one. The goal of this section is to develop the basic notions and results of this theory which will be used later in Chapters 3, 4, 5, 6 and Appendices A, B. Recall that given a vector $\boldsymbol{x} \in \mathbb{R}^d$ (resp. a real number $x \in \mathbb{R}$), $||\boldsymbol{x}||$ (resp. $||x||$) stands for the distance of $\boldsymbol{x}$ (resp. of $x$) from a nearest integer point.

Given a finite subset $A \subseteq \mathbb{T}$, a way to quantify how well the points of $A$ are spread in $\mathbb{T}$ is by determining how *dense* they are.

**Definition 1.1.18** *($\epsilon$-Dense Set) A set $A \subseteq \mathbb{T}$ (resp. $\mathbb{T}^d$) is $\epsilon$-dense if for every $\gamma \in \mathbb{T}$ (resp. for every $\boldsymbol{\gamma} \in \mathbb{T}^d$) there exists $x \in A$ (resp. $\boldsymbol{x} \in A$) such that $||x - \gamma|| \leq \epsilon$ (resp. such that $||\boldsymbol{x} - \boldsymbol{\gamma}|| \leq \epsilon$).*

Dispersion is a measure of density of the terms of a given sequence $\boldsymbol{a}$ and is of a metric nature. It will play a central role in our investigations.

**Definition 1.1.19 (Dispersion)** *Let $\boldsymbol{a}$ be a sequence in the unit torus. The dispersion of the first $N \in \mathbb{N}$ terms of the sequence $\boldsymbol{a}$ is the quantity*

$$\delta_{\boldsymbol{a}}(N) \quad = \quad \sup_{\gamma \in \mathbb{T}} \quad \min_{j \in [\![N]\!]} \quad ||\gamma - a_j|| \,. \tag{1.16}$$

Given a sequence $\boldsymbol{a}$ in $\mathbb{T}$, it holds that $\delta_{\boldsymbol{a}}(N) \le \epsilon$, for some $\epsilon > 0$, if, and only if, the set $A_N = \{a_n : n \in [\![N]\!]\}$ is $\epsilon$-dense in $\mathbb{T}$. Note that for any $\epsilon \in (0,1)$,

$$\delta_{\boldsymbol{a}}\left(\epsilon^{-1}\right) \quad \ge \quad \frac{\epsilon}{2} \tag{1.17}$$

since a set of $N$ points in $\mathbb{T}$ is never more than $\eta$-dense when $\eta < 1/2$.

Another notion to measure the irregularity in the distribution of a given sequence is the discrepancy. It is of a measure-theoretical nature.

**Definition 1.1.20 (Discrepancy - Equidistribution)** *The* discrepancy *of $N$ points $x_1, x_2, ..., x_N \in \mathbb{T}$ is the quantity*

$$d(x_1, x_2, ..., x_n) \quad = \quad \sup_{I \subseteq \mathbb{T}} \left| \frac{1}{N} \sum_{k=1}^{N} \chi_I(x_k) - \lambda(I) \right|, \tag{1.18}$$

*where the supremum is over all intervals $I \subseteq \mathbb{T}$, $\chi_I$ is the characteristic function of the interval $I$ and $\lambda$ stands for the Lebesgue measure. For an infinite sequence $\boldsymbol{a}$ in the unit torus, the discrepancy $d_{\boldsymbol{a}}(N)$ is the* discrepancy *of the first $N$ terms of the sequence $\boldsymbol{a}$.*

*A sequence $\boldsymbol{a}$ in the unit torus is* equidistributed *(or* uniformly distributed*) modulo* 1 *if for every interval $I \subseteq \mathbb{T}$, it holds that*

$$\lim_{N \to +\infty} \frac{1}{N} \sum_{k=1}^{N} \chi_I(a_k) \quad = \quad \lambda(I). \tag{1.19}$$

It is well known that a toral sequence $\boldsymbol{a}$ is equidistributed modulo 1 if, and only if, $d_{\boldsymbol{a}}(N) \to 0$ as $N \to +\infty$ [55, p.89, Theorem 1.1]. Given a sequence $\boldsymbol{a}$ in $\mathbb{T}$, $N \in \mathbb{N}$ and $\epsilon \in (0,1)$ such that $d_{\boldsymbol{a}}(N) \le \epsilon$, it follows immediately from the definition of discrepancy that every interval $I \subseteq \mathbb{T}$ with length larger than $\epsilon$ contains at least one of the first $N$ terms of $\boldsymbol{a}$. Therefore, the following inequality between the discrepancy and dispersion always holds:

$$\delta_{\boldsymbol{a}}(N) \quad \leq \quad \frac{d_{\boldsymbol{a}}(N)}{2}. \tag{1.20}$$

The main advantage of discrepancy compared to dispersion is that there exist known analytic tools to estimate it. The following result, due to Weyl, provides a characterisation of an equidistributed sequence modulo 1 in terms of an exponential sum.

**Theorem 1.1.21 (Weyl's Criterion)** *[55, p.7, Theorem 2.1] Let $\boldsymbol{a} = (a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{T}$. The following claims are equivalent.*

1. *The sequence $\boldsymbol{a}$ is equidistributed; that is, $d_{\boldsymbol{a}}(N) \to 0$ as $N \to +\infty$.*
2. *For every $h \in \mathbb{Z}\setminus\{0\}$ it holds that*

$$\lim_{N\to+\infty} \frac{1}{N}\sum_{k=1}^{N} e\left(h \cdot a_k\right) \quad = \quad 0.$$

Weyl's criterion says that a sequence $\boldsymbol{a}$ is equidistributed modulo one if, and only if, when one embeds the sequence $(h \cdot a_n)_{n\in\mathbb{N}}$ in the complex unit circle, the average of the terms tends to zero for any dilation factor $h \in \mathbb{N}$. Notice that the convergence to zero guarantees that the placement of the points is not biased in any particular direction. The reason for the presence of the dilation factors $h \in \mathbb{N}$ is to avoid rationality obstructions. For instance, if one chooses the sequence $1/2, 0, 1/2, 0, \ldots$, then the average of the points on the complex unit circle converges to zero for $h = 1$ while, for $h = 2$, the average of the points converges to one.

The following two results make it clear that the exponential sums appearing in the statement of Theorem 1.1.21 are closely related to its discrepancy. First, Koksma's inequality yields a lower bound for the discrepancy of a sequence in terms of these sums:

**Theorem 1.1.22 (Koksma's Inequality )** *[55, p.143, Corollary 5.1] Let $x_1, \ldots, x_n$ be $N$ points in the unit torus with discrepancy $d(x_1, \ldots, x_N)$. For any $h \in \mathbb{Z}\setminus\{0\}$,* *it holds that*

$$\left|\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i\cdot hx_k}\right| \quad \leq \quad 4\cdot h\cdot d\left(x_1,...,x_n\right).$$

Second, the Erdös-Turán inequality yields an upper bound for the discrepancy of a sequence.

**Theorem 1.1.23 (Erdös-Turán Inequality)** *[55, p.112, Theorem 2.5] Let $x_1,...,x_n$ be $N$ points in the unit torus with discrepancy $d\left(x_1,...,x_N\right)$. For any $H\in\mathbb{N}$ it holds that*

$$d\left(x_1,...,x_N\right) \quad \leq \quad \frac{6}{H+1}+\frac{4}{\pi}\sum_{h=1}^{H}\left(\frac{1}{h}-\frac{1}{H+1}\right)\cdot\left|\frac{1}{N}\sum_{k=1}^{N} e^{2\pi i\cdot hx_k}\right|.$$

The Erdös-Turán inequality is usually sharp up to a logarithmic factor, in the sense that in most cases the estimation of the discrepancy fails only up to a logarithmic factor.

A most common example of an equidistributed sequence is a *rotation* by an irrational number $\alpha$; that is, the sequence $\boldsymbol{\alpha}$ given by the multiples of $\alpha$ modulo 1:

$$\boldsymbol{\alpha} \quad = \quad (k\alpha)_{k\in\mathbb{N}}. \tag{1.21}$$

The equidistribution of $\boldsymbol{\alpha}$ can be proved as an immediate application of Weyl's criterion [28, p.2, Theorem 3]. Moreover, it is not hard to check that the sequence of multiples of a rational number is not dense in $\mathbb{T}$ and thus cannot be equidistributed. This result can be extended to polynomial sequences. The proof is based on the following theorem due to J. G. van der Corput which provides a strong way to prove that a given sequence is equidistributed.

**Theorem 1.1.24 (van der Corput's Difference Theorem)** *[55, p.26, Theorem 3.1] Let $\boldsymbol{a}$ be a sequence in $\mathbb{T}$. If, for every positive integer $h\in\mathbb{N}$, the sequence $\left(a_{k+h}-a_k\right)_{k\in\mathbb{N}}$ is equidistributed modulo 1, then the sequence $\left(a_k\right)_{k\in\mathbb{N}}$ is equidistributed modulo 1.*

Weyl, by using the equidistribution of the sequence (1.21) to initialise the induction and by exploiting Theorem 1.1.24, proved the following general result concerning the equidistribution of polynomial sequences.

**Theorem 1.1.25** *[55, p.27, Theorem 3.2] Let* $P(x) = a_n x^n + ...a_1 x + a_0$ *be a polynomial with real coefficients and degree* $\deg(P) \geq 1$. *The sequence* $(P(k))_{k\in\mathbb{N}}$ *is equidistributed mod* 1 *if, and only if, at least one of the coefficients* $a_1, ..., a_n$ *is irrational.*

The *well-distribution* and the *spectrum* of sequence are two additional notions which describe qualitative features of the distribution of a sequence. Given a toral sequence $\boldsymbol{a}$, $m \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$, denote by

$$d_{\boldsymbol{a}}(N, m, \xi) \quad = \quad \sup_{I\subseteq\mathbb{T}} \left| \frac{1}{N} \sum_{k=1}^{N} \chi_I(a_{m+k} - k\xi) - \lambda(I) \right|, \tag{1.22}$$

where the supremum is over intervals $I$ in $\mathbb{T}$, the discrepancy of the $N$ first terms of the sequence $(a_{m+k} - k\xi)_{k\in\mathbb{N}}$.

**Definition 1.1.26 (Well-Distribution - Spectrum)** *The toral sequence* $\boldsymbol{a}$ *is* well-distributed *in* $\mathbb{T}$ *if it holds that*

$$\sup_{m\in\mathbb{N}_0} d_{\boldsymbol{a}}(N, m, 0) \to 0 \quad \textit{as } N \to +\infty; \tag{1.23}$$

*that is, if* $\boldsymbol{a}$ *is equidistributed uniformly in the starting index.*

*The* spectrum *of the sequence* $\boldsymbol{a}$ *is the set*

$$\mathcal{SP}(\boldsymbol{a}) = \left\{ \xi \in \mathbb{T} : (a_k - k\xi)_{k\in\mathbb{N}} \ \textit{ is not equidistributed} \right\}; \tag{1.24}$$

*that is,* $\xi \notin \mathcal{SP}(\boldsymbol{a})$ *if, and only if,*

$$d_{\boldsymbol{a}}(N, 0, \xi) \to 0 \quad \textit{as } N \to +\infty. \tag{1.25}$$

*The sequence* $\boldsymbol{a}$ *has an* empty spectrum *if equation* (1.25) *holds for every* $\xi \in \mathbb{T}$.

**Remark 1.1.27** *Almost all sequences in* $\mathbb{T}$ *have empty spectrum with respect to the Haar measure on the infinite torus* $\mathbb{T}^{\mathbb{N}}$ *[37, p.177]. Conversely, almost no sequence is well distributed with respect to the same measure on* $\mathbb{T}^{\mathbb{N}}$ *[55, p.201, Theorem 3.8].*

In the context of this thesis, we will be interested in toral sequences which satisfy both of these two antagonist properties. Indeed, it will be proved in detail in Chapter 3 and Appendix A that the Peres-type forest $\mathfrak{F}(\boldsymbol{a})$ defined in (1.11) is a dense forest if, and only if, the sequence $\boldsymbol{a}$ is well-distributed and has an empty spectrum.

There are a lot of examples of well-distributed sequences in the literature. The most common of them is the sequence of rotations of an irrational number $\alpha$ as defined in (1.21). This follows from a straightforward application of Weyl's criterion (Theorem 1.1.21).

In [50], Keogh, Lawton and Petersen provide a way to construct a well-distributed sequence of the form

$$\left(\left(\prod_{i=1}^{k} n_i\right) \cdot \theta\right)_{k \in \mathbb{N}},$$

where $\theta$ is an irrational number and $(n_k)_{k \in \mathbb{N}}$ is an increasing sequence of natural numbers [50, Theorem 4]. Also, in the same paper is proved that, given positive integers $p, q$, the sequence

$$\left(\left(\frac{p}{q}\right)^{k} \cdot \theta\right)_{k \in \mathbb{N}}$$

is not well distributed for any $\theta$ [50, Theorem 5].

In [39], Drmota studies the property of well-distribution in the context of *strongly $q$-additive* functions. To define this concept, given natural numbers $n \in \mathbb{N}$ and $q \geq 2$, let

$$n \quad = \quad \sum_{k=1}^{+\infty} d_k\left(q, n\right) \cdot q^k, \quad \text{with} \quad 0 \leq d_k(q, n) \leq q - 1, \qquad (1.26)$$

be the representation of $n$ in base $q$.

**Definition 1.1.28 ((Strongly) $q$-Additive Functions)** *Let $g : \mathbb{N}_0 \to \mathbb{N}_0$ be an arithmetic function such that $g(0) = 0$. The function $g$ is $q$-*additive *if*

$$g(n) \quad = \quad \sum_{k=0}^{+\infty} g\left(d_k(q, n) \cdot q^k\right),$$

*where* $d_k(q,n)$ *is as in* (1.26)*. It is* strongly $q$-additive *if*

$$g(n) \quad = \quad \sum_{k=0}^{+\infty} g\left(d_k(q,n)\right).$$

The most common example of a strongly $q$-additive function is the sum-of-digits function given by the formula

$$s(q,n) = \sum_{k=0}^{+\infty} d_k(q,n).$$

Also, the identity function

$$id(n) \quad = \quad n \quad = \quad \sum_{k=0}^{+\infty} d_k(q,n) \cdot q^k$$

is an example of a $q$-additive function which is not strongly $q$-additive.

Given $g$ a non-negative strongly $q$-additive function such that $\gcd\{0 < j < q : g(j) > 0\} = 1$, Drmota [39, Theorem 1.2] shows that if a sequence $(x_k)_{k\in\mathbb{N}}$ is well-distributed, then so is the sequence $\left(x_{g(k)}\right)_{k\in\mathbb{N}}$.

It is not hard to list explicit examples of sequences in $\mathbb{T}$ with empty spectrum. For instance, the sequence $(P(k))_{k\in\mathbb{N}}$, where $P(x) = a_n x^n + ... + a_1 x + a_0$, is a polynomial with degree $\deg(P) \geq 2$ and with at least one of the coefficients $a_2, .., a_n$ being irrational, has empty spectrum. This follows from an easy application of Theorem 1.1.25.

Another example of a sequence with empty spectrum is obtained from $b$-normal numbers. Specifically, if $b \geq 2$ and $x$ is a $b$-normal number, then the sequence

$$\boldsymbol{x} = \left(x \cdot b^k\right)_{k\in\mathbb{N}} \tag{1.27}$$

has empty spectrum. We recall here a few related definitions.

**Definition 1.1.29** *Let* $b \in \mathbb{N}$ *be a natural number with* $b \geq 2$ *and let* $x \in \mathbb{R}$ *be a real number. Denote by*

$$x \quad = \quad \lfloor x \rfloor + \sum_{k=1}^{+\infty} \frac{x_k}{b^k}$$

*the expansion of $x$ in base $b$, where $x_k$ are integers such that $0 \leq x_k < b$.*

*The number $x$ is called simply $b$-normal if for every $y \in \{0, 1, ..., b-1\}$, it holds that*

$$\lim_{N \to +\infty} \frac{\#\{x_k = y : k \in [\![N]\!]\}}{N} = \frac{1}{b}.$$

*The number $x$ is called $b$-normal if for every $k \in \mathbb{N}$ and every choice $y_1, y_2, ..., y_m \in [\![0, b-1]\!]$, it holds that*

$$\lim_{N \to +\infty} \frac{\#\{x_{k+1} = y_1, ..., x_{k+m} = y_m : k \in [\![0, N-1]\!]\}}{N} = \frac{1}{b^m}.$$

The claim concerning the spectrum of the sequence $\boldsymbol{x}$, as defined in (1.27), follows upon applying Weyl's criterion (Theorem 1.1.21) in combination with Theorem 1.1.24. For the application of Weyl's criterion one needs [55, p.70, Theorem 8.1], which states that the number $x$ is $b$-normal if, and only if, the sequence $\boldsymbol{x}$ is equidistributed modulo 1.

In [40, p.101, Theorem 1.108], Drmota and Tichy show that, given a strongly $q$-additive function $g$ and an irrational $x$, the sequence $(x \cdot g(k))_{k \in \mathbb{N}}$ has empty spectrum.

In [37, Corollary 2], Dabousi and France prove that a sequence $(x_k)_{k \in \mathbb{N}}$ has empty spectrum if, and only if, all the subsequences $\boldsymbol{M} = (x_{m_k})_{k \in \mathbb{N}}$ are equidistributed modulo 1, where $(m_k)_{k \in \mathbb{N}}$ is a non-decreasing sequence of natural numbers whose characteristic function defined as $x_{\boldsymbol{M}} : n \in \mathbb{N} \to \#\{k : m_k = n\}$ is almost periodic in the following sense:

**Definition 1.1.30 (Trigonometric Polynomials - Almost Periodic Functions)** *Let $\Omega$ be the complex vector space of arithmetic functions $f$ such that*

$$||f||_\Omega := \limsup_{N \to +\infty} \frac{1}{N} \cdot \sum_{k \leq N} |f(k)| < +\infty.$$

*Let $\mathfrak{M}$ be the quotient space $\Omega / (\mathbf{0})$, where $(\mathbf{0})$ is the set of null functions, i.e. the set of functions $f$ such that $||f||_\Omega = 0$. The space $\mathfrak{M}$ is known as the Marcinkiewicz space; it is a Banach space for the norm induced by the semi-norm $||\cdot||_\Omega$.*

*A* trigonometric polynomial $P : \mathbb{R} \to \mathbb{C}$ *is a finite linear combination of imaginary exponentials:*

$$P(x) = \sum_k c_k \cdot e\left(-a_k x\right), \quad \textit{with } c_k \in \mathbb{C}, \quad a_k \in \mathbb{T},$$

*where the summation extends over a finite range.*

*Let $A \subseteq \mathbb{T}$ be a subset of the unit torus and let $\mathcal{V}(A) \subseteq \mathfrak{M}$ be the subspace of all those trigonometric polynomials with their exponents $a_k$ taken from $A$. The vector space $\mathcal{B}(A)$ of* almost-periodic functions *with exponents in $A$ is defined as the closure of $\mathcal{V}(A)$ with respect to the norm $||\cdot||_\Omega$.*

In the case where one wants to prove that a sequence $\boldsymbol{a}$ in $\mathbb{T}$ does not have an empty spectrum, it is often easier to work with the *Fourier-Bohr* spectrum of $\boldsymbol{a}$ defined as follows:

**Definition 1.1.31 (Fourier-Bohr Spectrum)** *Let $\boldsymbol{a} = (a_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{T}$. The Fourier-Bohr spectrum of $\boldsymbol{a}$ is the set*

$$\mathcal{SP}_{FB}\left(\boldsymbol{a}\right) = \left\{\xi \in \mathbb{T} : \limsup_{N\to+\infty} \frac{1}{N} \cdot \left|\sum_{k=1}^{N} e\left(a_k - k\xi\right)\right| > 0\right\}. \tag{1.28}$$

Given a sequence $\boldsymbol{a}$ in $\mathbb{T}$, an immediate application of Weyl's criterion yields that if $\xi \in \mathcal{SP}_{FB}(\boldsymbol{a})$, then the sequence $(a_k - k\xi)_{k\in\mathbb{N}}$ is not equidistributed. Therefore, one has that

$$\mathcal{SP}_{FB}(\boldsymbol{a}) \quad \subseteq \quad \mathcal{SP}(\boldsymbol{a}). \tag{1.29}$$

A class of sequences which have an empty Fourier-Bohr spectrum is the family of *pseudorandom* sequences [75, p.3, Lemma 2.3] defined below.

**Definition 1.1.32 (Pseudorandom Arithmetic Functions)** *Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function. The function $g$ is called pseudorandom if the limit*

$$\gamma_r = \lim_{N\to+\infty} \frac{1}{N} \cdot \sum_{0\leq k < N} g(k+r) \cdot \bar{g}(k)$$

*exists for all* $r \in \mathbb{N}_0$ *and*

$$\lim_{R\to+\infty} \frac{1}{R} \cdot \sum_{0\leq r<R} |\gamma_r|^2 = 0.$$

For more information regarding dispersion, discrepancy, spectrum, well-distribution and the other concepts mentioned in this section, we refer the reader to the books [28, Chapter 1 & 4], [40, Chapters 1 & 2], [55, Chapters 1 & 2], and the papers [26, 33], [34, 50, 56, 62], [35, 37, 75] and [39, 42]. The reader is also referred to the books [46, Chapters 2 & 3], [48, Chapter 8], [64, Chapter 3] and [77, Chapter I.6] for related considerations in exponential sums.

### 1.1.7 Diophantine Approximation and Continued Fractions

The goal of this section is to develop the basic notions and results from the theory of Diophantine approximation which will be extensively used for the analysis undertaken in Chapter 3 and Appendices A and B. Recall that $||x||$ denotes the distance of the real number $x$ from its nearest integer.

In Section 1.1.6 was mentioned that the sequence (1.21) of rotations of $\alpha$ is equidistributed modulo 1 if, and only if, $\alpha$ is irrational. This discrimination between irrational and rational numbers is only a first step towards a fine classification of the distributional properties of the rotations of real numbers. To understand the more refined properties of the distribution of $(k\alpha)_{k\in\mathbb{N}}$, one needs some of the tools from Diophantine approximation developed in this section.

The theory of *Diophantine approximation* is a field of mathematics which deals with the approximation of real numbers by rationals. It is well-known that the rational numbers are dense in $\mathbb{R}$. This is to say that given a real $\alpha$ and a positive $\epsilon$, there exists a rational $p/q$ such that

$$\left|\alpha - \frac{p}{q}\right| \quad \leq \quad \epsilon. \tag{1.30}$$

Thinking of the denominator $q$ as the *complexity* of the irreducible fraction $p/q$, where the larger the denominator, the larger the complexity, one may ask if the choice of $q$ could be efficient compared with the choice of $\epsilon$. For instance, one can trivially improve on inequality (1.30) by choosing $q \geq \epsilon^{-1}/2$: indeed, there always exists $p \in \mathbb{Z}$ such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q}. \tag{1.31}$$

A fundamental result in the theory of Diophantine approximation, due to Peter Gustav Lejeune Dirichlet, is an immediate application of the *pigeonhole principle* and improves non-trivially on inequality (1.31).

**Theorem 1.1.33 (Dirichlet's Theorem on Diophantine Approximation)** *[27, p.2, Theorem 1.1] Let $\alpha$ and $Q$ be real numbers with $Q \geq 1$. There exists a rational number $p/q$ with $\gcd(p, q) = 1$ and $1 \leq q \leq Q$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{qQ}. \tag{1.32}$$

*Furthermore, if $\alpha$ is irrational, then there exist infinitely many rational numbers $p/q$ such that*

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}, \tag{1.33}$$

*and if $\alpha = a/b$ is rational, then for any rational $p/q \neq a/b$ with $q > 0$, it holds that*

$$\left|\alpha - \frac{p}{q}\right| \geq \frac{1}{|b|q}.$$

Inequality (1.33) holds for every irrational number $\alpha$. More generally, given a function $\psi : \mathbb{N} \to \mathbb{R}^+$ called an **approximation function**, the set of $\psi$-approximable numbers is defined as

$$W(\psi) = \{\alpha \in \mathbb{R} : \quad ||q\alpha|| < \psi(q) \text{ for infinitely many } q \in \mathbb{N}\}. \tag{1.34}$$

Khinchin's theorem in metric Diophantine approximation provides an elegant criterion for the size of the set $W(\psi)$ expressed in terms of the Lebesgue measure. Here, we present an improved version due Beresnevich and Velani [23].

**Theorem 1.1.34 (Khinchin's Theorem)** *Let* $\psi : \mathbb{N} \to \mathbb{R}^+$ *be a monotonic function. Then,*

$$\lambda\left(W(\psi) \cap [0,1]\right) \quad = \quad \begin{cases} 0 & \textit{if } \sum_{q=1}^{+\infty} \psi(q) \; < +\infty, \\ 1 & \textit{if } \sum_{q=1}^{+\infty} \psi(q) = +\infty. \end{cases} \tag{1.35}$$

Khinchin's original statement asks for the stronger assumption that the function $q \mapsto q \cdot \psi(q)$ should be non-increasing [51, p.69, Theorem 32]. Duffin and Shaeffer [41] constructed a non-monotonic approximation function $\theta : \mathbb{N} \mapsto \mathbb{R}^+$ such that the sum $\sum_q \theta(q)$ diverges but $\lambda\left(W(\theta) \cap [0,1]\right) = 0$. In the same paper [41], they conjecture a version of Khinchin's theorem where the monotonic condition on the approximation function $\psi$ is not necessary. The *Duffin-Schaeffer conjecture*, which stood for 79 years as a key open problem in number theory, was proved in 2020 by *Koukoulopoulos* and *Maynard* [53]. The conjecture (now a theorem) is given in the following statement:

**Theorem 1.1.35 (Koukoulopoulos & Maynard)** *Let* $\psi : \mathbb{N} \mapsto \mathbb{R}^+$ *be a real valued function. Then for almost all* $\alpha \in \mathbb{R}$ *(with respect to the Lebesgue measure), the inequality*

$$\left|\alpha - \frac{p}{q}\right| \quad \leq \quad \frac{\psi(q)}{q}$$

*has infinitely many solutions in coprime integers* $p, q$ *with* $q \; > 0$ *if and only if*

$$\sum_{q=1}^{+\infty} \phi(q) \cdot \frac{\psi(q)}{q} \quad = \quad +\infty,$$

*where* $\phi$ *is Euler's totient function; that is,*

$$\phi(q) := \quad \#\left\{q' \in \mathbb{N} : \quad 1 \leq q' \leq q, \quad \gcd\left(q', q\right) = 1\right\}.$$

Assume that, given a real number $\alpha$, its distance $|\alpha - p/q|$ from any rational $p/q$ satisfies a lower bound of the form $\geq c/q^2$ for some constant $c \; > 0$ depending on $\alpha$; thas is, a lower bound of the same magnitude as the upper bound in inequality (1.33). In this case, the number $\alpha$ is called *badly approximable* (the name is justified by the fact that inequality (1.33) cannot be improved except by a multiplicative constant).

**Definition 1.1.36 (Badly Approximable Numbers)** *[27, p.11, Definition 1.3]*
*An irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ is called badly-approximable if there exists a constant $c(\alpha) > 0$ such that*

$$\left|\alpha - \frac{p}{q}\right| \quad \geq \quad \frac{c(\alpha)}{q^2} \quad \text{for every rational } \frac{p}{q}. \tag{1.36}$$

*The set of badly-approximable numbers is defined as*

$$Bad := \quad \left\{\alpha \in \mathbb{R}\backslash\mathbb{Q} : \inf_{q\in\mathbb{N}} q\,||q\alpha|| \ > 0\right\} \quad = \quad \left\{\alpha \in \mathbb{R}\backslash\mathbb{Q} : c(\alpha) := \liminf_{q\to+\infty} q\,||q\alpha|| \ > 0\right\}. \tag{1.37}$$

Applying Theorem 1.1.34 to every function $\psi_n(q) = 1/(nq)$ with $n \in \mathbb{N}$ yields that the set *Bad* has zero Lebesgue measure. However, the set of badly-approximable numbers is much more than non-empty: it is well known that the golden ratio $\phi$ belongs to it [51, p.33, Discussion and p.34, Theorem 21 ].

Given $u \geq 2$, it is not hard to construct examples of real numbers $\alpha$ satisfying

$$\left|\alpha - \frac{p}{q}\right| \quad \leq \quad \frac{1}{q^u} \tag{1.38}$$

for infinitely many pairs of integers $(p, q)$ [51, p.35, Theorem 22]. For instance, for any choice of $u \geq 2$, the number

$$\mathtt{L} \quad = \quad \sum_{n\in\mathbb{N}} \frac{1}{10^{n!}} \tag{1.39}$$

satisfies inequality (1.38) for an infinite number of rationals $p/q$. One could "measure" how irrational a real number $\alpha$ is, in the sense given by the following definition:

**Definition 1.1.37 (Irrationality Measure)** *The* irrationality measure *of* $\alpha \in$

$\mathbb{R}$ *is defined to be the quantity*

$$\mu(\alpha) \quad = \quad \sup\left\{ \begin{array}{r} u > 0 : \left|\alpha - \dfrac{p}{q}\right| \leq \dfrac{1}{q^u} \quad \textit{holds for infinitely many rationals } \dfrac{p}{q} \\ \textit{with } \gcd(p,q) = 1 \end{array} \right\}. \tag{1.40}$$

It can readily be checked that every rational number $r \in \mathbb{Q}$ has irrationality measure $\mu(r) = 1$ while, from Dirichlet's theorem in Diophantine approximation, for every irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, it holds that $\mu(\alpha) \geq 2$. Actually, almost every real number $\alpha$ (with respect to the Lebesgue measure) has irrationality measure $\mu(\alpha) = 2$. The result follows upon applying Kinchin's theorem (Theorem 1.1.34) to the approximation functions $\psi_0(q) = (q\log(q))^{-1}$ and $\psi_u(q) = q^{-u}$ with $u > 1$: indeed, the theorem yieds that $\lambda\left(W\left(\psi_0\right) \cap [0,1]\right) = 1$ and $\lambda\left(W\left(\psi_u\right) \cap [0,1]\right) = 0$, which easily implies the claims. Since the set of badly approximable numbers is non-empty, one has that $W\left(\psi_0\right) \cap [0,1] \neq [0,1]$.

An *irrational* number $\alpha$ such that $\mu(\alpha) = +\infty$ is called a **Liouville number**. A result of historical importance in the theory of Diophantine approximation is the Liouville Theorem which led to the first discovery of *transcendental numbers*. Before stating it, recall that a real number is **algebraic of degree** $n$ if it is the root of an irreducible polynomial of degree $n$ with integer coefficients. A **transcendental number** is a real number which is not algebraic.

**Theorem 1.1.38 (Liouville's Theorem)** *[27, p.3, Theorem 1.2] Let $\alpha$ be a real root of an irreducible polynomial $P(x)$ of degree $n \geq 2$. There exists a positive constant $c(\alpha)$ such that*

$$\left|\alpha - \frac{p}{q}\right| \quad \geq \quad \frac{c(\alpha)}{q^n}$$

*for all rational numbers $p/q$. A suitable choice for $c(\alpha)$ is*

$$c(\alpha) := \quad \frac{1}{1 + \max_{|t-\alpha|\leq 1} |P'(t)|}.$$

Theorem 1.1.38 implies immediately that every Liouville number is transcendental. An example of such a number is the number $\mathtt{L}$ defined in (1.39). The original proof of Theorem 1.1.38 is given in [58]. The main idea, however, already appeared in Liouville's note [57].

A basic tool in the theory of Diophantine approximation is the *continued fraction expansion* of real numbers. Continued fractions and the theory surrounding them will be exploited throughout the thesis. Given a real number $\alpha$, this theory provides a natural answer to the question of determining the best approximants to $\alpha$. This may be rigorously rephrased as: what are those $q \in \mathbb{N}$ and $p \in \mathbb{Z}$ such that

$$|q\alpha - p| < |k\alpha - l| \quad \text{for every} \quad k \in [\![q-1]\!] \quad \text{and every} \quad l \in \mathbb{Z}? \tag{1.41}$$

Given $a_0 \in \mathbb{Z}$ and $a_i \in \mathbb{N}$ for every $i \geq 1$, a finite continued fraction denotes any expression of the form

$$[a_0; a_1, a_2, ..., a_n] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}. \tag{1.42}$$

More generally, we call any expression of the above form or of the form

$$[a_0; a_1, a_2, ...] = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\cdots}}} = \lim_{n \to +\infty} [a_0; a_1, a_2, ..., a_n] \tag{1.43}$$

(when the limit exists) a *continued fraction.* At this point it is necessary to justify two things: first, that every real number has an (infinite or finite) continued fraction expansion and, second, that the infinite continued fraction expansion of the form (1.43) always converges. At the heart of both of these claims lies the Euclidean algorithm hereafter described.

The continued fraction expansion of a real $\alpha$ is defined inductively by iterating the following algorithm [51, Chapters 1&2]: set $\eta_{-1} = 1$, $a_0 = \lfloor \alpha \rfloor$, $\eta_0 = \alpha - a_0$ and, for every $n \in \mathbb{N}$, define

$$a_n = \left\lfloor \frac{\eta_{n-2}}{\eta_{n-1}} \right\rfloor \quad \text{and} \quad \eta_n = \eta_{n-2} - a_n \cdot \eta_{n-1}. \tag{1.44}$$

The process stops if at the $n$-th iteration one has as an output $\eta_n = 0$. The connection between this algorithm and the continued fraction expansion of $\alpha$ is made clear through this relation which can be proved inductively:

$$\alpha = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n + \frac{\eta_n}{\eta_{n-1}}}}}}.$$

If the algorithm stops after finitely many steps; that is, if one has as an output $\eta_n = 0$ for some $n \in \mathbb{N}$, then the result is a finite continued fraction expansion and, therefore, $\alpha$ is rational. Furthermore, a real number $\alpha$ is rational if, and only if, it has a finite continued fraction expansion [27, p.1, Lemma 1.1]. More precisely, every rational number $r \in \mathbb{Q}\backslash\mathbb{Z}$ has exactly two (finite) continued fraction expansions of the form

$$r = [a_0; a_1, ..., a_n] \quad \text{and} \quad [a_0; a_1, ..., a_n - 1, 1] \quad \text{with } a_n \geq 2$$

(similarly, every integer $r$ can be written as $r = [r]$ and $r = [r-1; 1]$).

**Definition 1.1.39 (Convergents and Partial Quotients of $\boldsymbol{\alpha}$)** *Let* $\alpha = [a_0; a_1, a_2, ...]$ *be an irrational number where the sequence* $(a_0)_{n=0}^{+\infty}$ *is defined in* (1.44)*. Given* $n \in \mathbb{N}_0$*, the rational number*

$$\frac{p_n}{q_n} = [a_0; a_1, ..., a_n] \tag{1.45}$$

*is called the n-th* convergent *of* $\alpha$ *and the integer* $a_n$ *is the n-th* partial quotient *of $\alpha$.*

*If $\alpha = [a_0; a_1, ..., a_N]$ is a continued fraction expansion of a rational number with $a_N \neq 1$, then, for every $0 \leq n \leq N$, the $n$-th convergent and the $n$-th partial quotient of $\alpha$ are defined in the same way as in the irrational case.*

The convergents of a real number $\alpha$ can by computed from its continued fraction expansion and vice versa. By induction, one can prove the following basic results.

**Theorem 1.1.40** *[27, p.8-9, Theorems 1.3, 1.4 and Lemma 1.3] Let $\alpha = [a_0; a_1, a_2, ...]$ be a real number. The following statements hold.*

1. *For every $n \in \mathbb{N}$, the numerator and denominator in (1.45) are given by*

$$\begin{aligned} p_n &= a_n p_{n-1} + p_{n-2} \quad \textit{with} \quad p_0 = a_0 \quad \textit{and} \quad p_{-1} = 1 \\ q_n &= a_n q_{n-1} + q_{n-2} \quad \textit{with} \quad q_0 = 1 \quad \textit{and} \quad q_{-1} = 0. \end{aligned} \tag{1.46}$$

   *Moreover, for every $n \geq 0$ (resp. $n \geq 1$) it holds that*

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n \quad \Big(\textit{resp.} \quad q_n p_{n-2} - p_n q_{n-2} = (-1)^{n-1} a_n\Big). \tag{1.47}$$

   *In particular, $gcd\,(p_n, q_n) = 1$.*

2. *The subsequence $(p_{2n}/q_{2n})_{n\in\mathbb{N}}$ of convergents of even order is strictly increasing and the subsequence $(p_{2n-1}/q_{2n-1})_{n\in\mathbb{N}}$ of convergents of odd order is strictly decreasing.*

The first part of Theorem 1.1.40 yields that for every $n \in \mathbb{N}$

$$\left|\frac{p_n}{q_n} - \frac{p_{n+1}}{q_{n+1}}\right| \quad \leq \quad \frac{1}{q_n q_{n+1}}.$$

In turn, from the second part, one concludes that the sequence of convergents $(p_n/q_n)_{n\in\mathbb{N}}$ converges to $\alpha$ and, furthermore, that for every $n \in \mathbb{N}$

$$\left|\alpha - \frac{p_n}{q_n}\right| \quad < \quad \frac{1}{q_n q_{n+1}}. \tag{1.48}$$

Finally, from equation (1.45) one concludes that every real number has a continued fraction expansion. In the case of irrational numbers, the expansion is unique [27, Theorem 1.5]. Upon noticing that $q_n q_{n+1} \geq a_{n+1} q_n^2$, inequality (1.48) implies that the convergents of $\alpha$ satisfy inequality (1.33). Moreover, in view of the question of determining the best approximants[1] (defined in (1.41)), an elementary argument [51, p.24, Theorem 16] yields that, for every natural $1 \leq k < q_n$ and every integer $l \in \mathbb{Z}$,

$$|q_n \alpha - p_n| \quad < \quad |k\alpha - l| \,. \tag{1.50}$$

With the trivial exception of numbers $\alpha$ of the form $\alpha = a_0 + (1/2)$, $a_0 \in \mathbb{Z}$, the converse is also true: every best approximant of $\alpha$ defined as in inequality (1.41) is also a convergent of $\alpha$ [51, p.26, Theorem 17].

The continued fraction expansion contains the Diophantine properties of numbers. For instance, the following result provides an elegant characterization of bad-approximability in terms of partial quotients.

**Theorem 1.1.41** *[51, p.36, Theorem 23] Let $\alpha = [a_0; a_1, a_2, ...]$ be an irrational number. Then, $\alpha$ is a badly-approximable number if, and only if, its partial quotients are bounded, that is,*

$$(\alpha \in Bad) \quad \Leftrightarrow \quad (\exists M = M(\alpha): \quad \forall i \in \mathbb{N}, \quad a_i \leq M) \,.$$

For instance, the real number $[1; 1, 1, ...]$ is a badly approximable number according to Theorem 1.1.41. This is the continued fraction expansion of the golden ratio $\phi = \left(\left(1+\sqrt{5}\right)/2\right)$ [51, p.33].

[1] One could define the best approximants of a real number $\alpha$ as those rationals $p/q$, $\gcd(p,q) = 1$, such that

$$\left|\alpha - \frac{p}{q}\right| \quad < \quad \left|\alpha - \frac{l}{k}\right| \quad \text{for every} \quad k \in [\![q-1]\!] \quad \text{and every} \quad l \in \mathbb{Z}. \tag{1.49}$$

Here, two things should be mentioned. Firstly, every best approximant in the sense of inequality (1.41) is also a best approximant in the sense of (1.49) [51, p.24, Discussion]. Secondly, the best approximants defined in (1.49) are either the convergents $p_n/q_n$ of $\alpha$ defined in (1.46) or a fraction of the form

$$\frac{p_{n-1} + m p_n}{q_{n-1} + m q_n} \quad \text{with} \quad 1 \leq m < a_{n+1} \quad \text{and} \quad n \in \mathbb{N}$$

[51, p.22, Theorem 15]. The above fractions are usually called **semiconvergents**, **secondary convergents** or **intermediate fractions**.

The continued fraction expansion (1.43) can be defined more generally for $a_0 \in \mathbb{R}$, $a_i \in \mathbb{R}^+, i \geq 1$. The corresponding expansion converges if, and only if, the series $\sum_{i=1}^{+\infty} a_i$ diverges [51, p.10, Theorem 10]. When the continued fraction expansion is well-defined, relations (1.46), (1.47) and (1.48) hold true in this set-up as well [51, Chapter 1] .

Complementary to the notion of a continued fraction is that of an Ostrowski expansion [22, Section 3]. It contains the information of the approximation of a real number $\rho$ by multiples of an irrational $\alpha$ modulo one.

**Definition 1.1.42 (Ostrowski Numeration System and Ostrowski Expansion)** *Given an irrational* $\alpha = [a_0; a_1, a_2, ...]$ *and the sequence* $(q_n)_{n=0}^{+\infty}$ *of the denominators of the convergents of* $\alpha$ *defined in* (1.45)*, the* Ostrowski numeration system *in base* $\alpha$ *has as its scale of numeration the sequence* $(q_n)_{n=0}^{+\infty}$ *and the Ostrowski expansion of a non-negative integer* $k \in \mathbb{N}_0$ *is the unique choice of integers* $\{e_n(k)\}_{n\in\mathbb{N}_0}$ *such that*

$$k = \sum_{n=0}^{+\infty} e_n(k) \cdot q_n, \tag{1.51}$$

*where*

$$\sum_{n=0}^{N} e_n(k) \cdot q_n < q_{N+1} \qquad \textit{for all} \quad N \geq 0.$$

*Moreover, this unique expansion has the property that* $e_0(k) \in [\![0, a_1 - 1]\!]$ *and* $e_n(k) \in [\![0, a_{n+1}]\!]$ *for every* $n \geq 1$,

$$\textit{with} \quad e_n(\rho) = 0 \quad \textit{whenever} \quad e_{n+1}(\rho) = a_{n+2} \quad \textit{for} \quad n \geq 0.$$

*Similarly, given a real number* $\rho \in \mathbb{R}$*, the* Ostrowski expansion *of* $\rho$ *in base* $\alpha$ *is the unique choice of natural numbers* $\{e_n(\rho)\}_{n\in\mathbb{N}_0}$ *and of an integer* $\rho_0$ *such that*

$$\rho = \rho_0 + e_0(\rho) \cdot \{\alpha\} + \sum_{n=1}^{+\infty} e_n(\rho) \cdot \{q_n\alpha\}_2, \tag{1.52}$$

*where* $\rho - \rho_0 \in [-\alpha, 1-\alpha)$, $\quad e_0(\rho) \in [\![0, a_1 - 1]\!]$ *and* $e_n(\rho) \in [\![0, a_{n+1}]\!]$ *for every* $n \geq 1$,

$$\textit{with} \quad e_n(\rho) = 0 \quad \textit{whenever} \quad e_{n+1}(\rho) = a_{n+2} \quad \textit{for} \quad n \geq 0.$$

*Here, $\{x\}_2$ denotes the signed fractional part of $x \in \mathbb{R}$; that is, the unique real number in $\left[-\frac{1}{2}, \frac{1}{2}\right)$ such that $x - \{x\}_2 \in \mathbb{Z}$.*

Definition 1.1.42 is well-defined [22, Lemmas 3.1 & 3.2]. The Ostrowski expansion is one of the basic tools in inhomogeneous Diophantine approximation which, given real numbers $\theta, s \in \mathbb{R}$ and an approximation function $\psi : \mathbb{N} \mapsto \mathbb{R}^+$, enables one to deal with the problem of finding the (number of) solutions to the inequality

$$||k\theta - s|| \leq \psi(k).$$

For instance, Beresnevich, Haynes and Velani [22] use the Ostrowski expansion to estimate sums of reciprocals of the form

$$S_N(\alpha, \gamma) := \sum_{k=1}^{N} \frac{1}{k \cdot ||k\alpha - \gamma||} \quad \text{and} \quad R_N(\alpha, \gamma) := \sum_{k=1}^{N} \frac{1}{||k\alpha - \gamma||}$$

as well as to count solutions of inequalities of the form $||k\alpha - \gamma|| \leq \epsilon$ with $k \in [\![N]\!]$.

The concept of bad approximability can be generalised to higher dimensions. Indeed, given $Q \in \mathbb{N}$, by partioning the cube $[0,1]^d$ into smaller cubes with side-length $1/\left\lfloor \sqrt[d]{Q} \right\rfloor$, one can prove from Dirichlet's pigeonhole principle the following higher-dimensional analogue of Theorem 1.1.33.

**Theorem 1.1.43 (Dirichlet's Theorem in Higher Dimensions)** *[70, p.27, Chapter 2, Theorem 1A] There exists an absolute constant $C_d > 0$ such that, for any $\boldsymbol{a} \in \mathbb{T}^d$ and any $Q \in \mathbb{N}$, there is a natural number $1 \leq q \leq Q$ such that*

$$||q\boldsymbol{a}|| \quad \leq \quad \frac{C_d}{Q^{\frac{1}{d}}}. \tag{1.53}$$

In the same way as for badly approximable numbers, a badly approximable vector is defined as a vector for which inequality (1.58) cannot be improved but up to a constant:

**Definition 1.1.44 (Badly Approximable Vectors)** *A real vector $\boldsymbol{a} = (a_1, a_2, ..., a_d) \in$*

$\mathbb{R}^d$ *is called badly approximable if*

$$c_S(\boldsymbol{a}) := \quad \inf_{q\in\mathbb{N}} q^{\frac{1}{d}} \cdot \max_{1\leq j\leq d} ||qa_j|| \quad > \quad 0. \tag{1.54}$$

The left-hand side quantity in (1.54) expresses the *simultaneous* smallness of the multiples of the coordinates of $\boldsymbol{a}$. The famous Perron-Khintchine Transference Theorem, stated below, shows that inequality (1.54) is equivalent to a similar inequality for the linear form

$$\mathfrak{L}_{\boldsymbol{a}}(m_1, ..., m_d) := \quad m_1a_1 + ... + m_da_d \qquad \text{with} \qquad (m_1, ..., m_d) \in \mathbb{Z}^d. \tag{1.55}$$

**Theorem 1.1.45 (The Perron-Khintchin Transference Theorem)** *[70, Chapter IV, Theorem 5B] Let* $\boldsymbol{a} = (a_1, ..., a_d)$ *be a vector in* $\mathbb{T}^d$*. Then, inequality* (1.54) *holds if and only if*

$$c_L(\boldsymbol{a}) := \quad \inf_{\boldsymbol{m}\in\mathbb{Z}^d\setminus\{\boldsymbol{0}\}} \left(\max_{1\leq j\leq d} |m_j|\right)^d \cdot ||\mathfrak{L}_{\boldsymbol{a}}(\boldsymbol{m})|| \quad > \quad 0, \tag{1.56}$$

*where the linear form* $\mathfrak{L}_{\boldsymbol{a}}$ *is defined in* (1.55)*.*

Although there does not exist a known algorithm to compute them in higher dimensions, the **best approximation vectors for the simultaneous approximation** of a vector $\boldsymbol{a} \in \mathbb{R}^d$ (resp.$\boldsymbol{a} \in \mathbb{T}^d$), denoted by

$$\boldsymbol{b}_n := \left(\frac{p_1^{(n)}}{\mathsf{q}_n}, \frac{p_2^{(n)}}{\mathsf{q}_n}, ..., \frac{p_d^{(n)}}{\mathsf{q}_n}\right) \tag{1.57}$$

with

$$p_1^{(n)}, ..., p_d^{(n)} \in \mathbb{Z}, \mathsf{q}_n \in \mathbb{N} \quad \text{and} \quad \gcd\left(\mathsf{q}_n, p_1^{(n)}, ..., p_d^{(n)}\right) = 1,$$

can be defined in such a way that a higher-dimensional analogue of the property (1.50) holds. More precisely, they are defined as follows:

- the sequence $(\mathsf{q}_n)_{n\in\mathbb{N}}$ is increasing, that is,

$$\mathsf{q}_1 \quad < \quad \mathsf{q}_2 \quad < \quad ... \quad < \quad \mathsf{q}_n \quad < \quad ... \quad .$$

- the rational vector $\boldsymbol{p}_n/\mathsf{q}_n$, where $\boldsymbol{p}_n = \left(p_1^{(n)}, ..., p_d^{(n)}\right)$, is the best approximation to the vector $\boldsymbol{a}$ in the sense that for every $q < \mathsf{q}_n$

$$\xi_n := \max_{1\leq j\leq d} ||\mathsf{q}_n a_j|| = \max_{1\leq j\leq d} \left|\mathsf{q}_n a_j - p_j^{(n)}\right| < \max_{1\leq j\leq d} ||q a_j|| .$$

- the sequence of simultaneous approximations $(\xi_n)_{n\in\mathbb{N}}$ is decreasing, that is,

$$\xi_1 > \xi_2 > ... > \xi_n > ... .$$

There is no known analogue of the continued fraction algorithm in higher dimensions preserving all properties of the convergents. From Theorem 1.1.43 and the definition of the sequence $(\mathsf{q}_n)_{n\in\mathbb{N}}$, one has that

$$||\mathsf{q}_n \boldsymbol{a}|| \leq \frac{C_d}{\mathsf{q}_n^{1/d}}. \tag{1.58}$$

The **best approximation vectors for the dual approximation** of the vector $\boldsymbol{a} \in \mathbb{R}^d$ (resp. $\boldsymbol{a} \in \mathbb{T}^d$), denote them

$$\boldsymbol{m}_n := \left(m_1^{(n)}, ..., m_d^{(n)}\right), \quad n \in \mathbb{N}, \tag{1.59}$$

are defined as those vectors satisfying the following properties:

- the sequence $M_n = \max_{1\leq j\leq d} \left|m_j^{(n)}\right|$, $n \in \mathbb{N}$ is increasing; that is,

$$M_1 < M_2 < .... < M_n < ... < . \tag{1.60}$$

- the values of the linear form

$$L_n := \mathfrak{L}(\boldsymbol{m}_n)$$

satisfy for all $\boldsymbol{m} \in \mathbb{Z}^d \setminus \{\boldsymbol{0}\}$ with $||\boldsymbol{m}||_\infty < M_n$ the inequalities

$$L_n < \mathfrak{L}(\boldsymbol{m})$$

and

$$L_1 \quad > \quad L_2 \quad > \quad ... \quad > \quad L_n \quad > \quad ... \quad . \tag{1.61}$$

From Theorem 1.1.45 and the definition of the quantities $c_L(\boldsymbol{a})$ (defined in (1.56)), $M_n$ and $L_n$, one has that

$$L_n \quad \geq \quad \frac{c_L(\boldsymbol{a})}{M_n^d}.$$

The following theorem due to Akhunzhanov and Moshchevitin characterises the property of a vector being badly approximable in terms of the above defined sequences of simultaneous approximation $(\xi_n)_{n\in\mathbb{N}}$ and of the linear approximation $(L_n)_{n\in\mathbb{N}}$.

**Theorem 1.1.46** *[8, p.3, Theorem 1 and Remark 1] Assume that the real numbers* $a_1, a_2, ..., a_d, 1$ *are linearly independent over* $\mathbb{Q}$*. Then, the following statements are equivalent:*

1. $\boldsymbol{a} = (a_1, ..., a_d)$ *is badly approximable;*

2. $\sup_{n\in\mathbb{N}} \quad \mathsf{q}_{n+1}/\mathsf{q}_n \quad < \quad +\infty$*;*

3. $\inf_{n\in\mathbb{N}} \quad L_{n+1}/L_n \quad > \quad 0$*,*

*where the sequences* $(\mathsf{q}_n)_{n\in\mathbb{N}}$ *and* $(L_n)_{n\in\mathbb{N}}$ *are defined in* (1.57) *and* (1.61)*, respectively. Moreover, it then holds that*

$$\inf_{n\in\mathbb{N}} \frac{\xi_{n+1}}{\xi_n} \quad > \quad 0 \quad \textit{and} \quad \sup_{n\in\mathbb{N}} \frac{M_{n+1}}{M_n} \quad < \quad +\infty.$$

For more details on (homogeneous and inhomogeneous) Diophantine approximation, the reader is referred to the books [27, Chapters 1 & 2], [28, Chapters 7 & 9], [51, Chapters 1, 2 & 3], [70, Chapters 1 & 2], [77, Chapter I.7] and the papers [22, 23, 38, 45, 54, 71]. Methods and algorithms for the estimation of the irrationality measure of a real number are given in [83] and in the references therein. More details about best approximation vectors can be found in [8, 30, 66].

### Uniformly-Diophantine Type Vectors

In Chapter 3 we exploit the theory of uniformly-Diophantine type vectors to show the existence of a class of sequences $\boldsymbol{a}$ in $\mathbb{T}$ which generate Peres-type forests with almost optimal visibility $O\left(\epsilon^{-1-\eta}\right)$. This theory was developed by Adiceam, Solomon and Weiss [4] to be combined with the construction of a higher dimensional Peres-type forest as in (1.10). It resulted in Theorem 1.1.11, which guarantees the existence of dense forests (Definition 1.1.7) in $\mathbb{R}^d$ with visibility bounds close to the optimal $O\left(\epsilon^{-(d-1)}\right)$.

The following is a preliminary definition to introduce the notion of Uniformly Diophantine-Type vectors.

**Definition 1.1.47 (Diophantine Vectors of type $\boldsymbol{\tau}$)** *A vector $\boldsymbol{\theta} \in \mathbb{R}^d$ is called Diophantine of type $\tau > 0$, if there exists a constant $c = c(\boldsymbol{\theta}) > 0$ such that for every $\boldsymbol{u} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$, it holds that*

$$||\boldsymbol{\theta} \cdot \boldsymbol{u}|| \geq \frac{c}{||\boldsymbol{u}||_\infty^\tau}.$$

From Perron-Khintchin Transference Theorem (Theorem 1.1.45), it follows that for a Diophantine vector of type $\tau$ in $\mathbb{R}^d$ it always holds that $\tau \geq d$.

**Definition 1.1.48 (Uniformly Diophantine Vectors of Type $\boldsymbol{\Phi}$)**

*[4, p.16, Definition 5.1] Let $\Phi$ be a non-increasing function tending to zero at infinity. An $s$-tuple of $d$-dimensional vectors $\boldsymbol{\Theta}_{s,d} = (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_s)$ is a uniformly Diophantine vector of type $\Phi$ if for any $T \geq 1$ and any $\boldsymbol{\xi} \in \mathbb{R}^d$, there exists $j \in \{1, 2, ..., s\}$ such that for all $\boldsymbol{u} \in \mathbb{Z}^d \setminus \{\mathbf{0}\}$ with $||\boldsymbol{u}||_\infty \leq T$, it holds that*

$$||\boldsymbol{u} \cdot (\boldsymbol{\xi} - \boldsymbol{\theta}_j)|| \geq \Phi(T).$$

*The set of $\boldsymbol{\Theta}_{s,d}$ which are uniformly Diophantine vectors of type $\Phi$ will be denoted by $UDT_s^d(\Phi)$. One has that $\boldsymbol{\Theta}_{s,d} \in UDT_s^d(\Phi)$ if, and only if,*

$$\inf_{T\geq 1} \sup_{\boldsymbol{\xi}\in\mathbb{R}^d} \max_{1\leq j\leq s} \min_{\substack{1\leq ||\boldsymbol{u}||_\infty \leq T \\ \boldsymbol{u}\in\mathbb{Z}^d}} \Phi(T)^{-1} \cdot ||\boldsymbol{u} \cdot (\boldsymbol{\xi} - \boldsymbol{\theta}_j)|| \quad \geq \quad 1. \tag{1.62}$$

*Also, given $\tau > 0$, define the set of uniformly Diophantine vectors of type $\tau$ as*

$$UDT_s^d(\tau) = \bigcup_{c > 0} UDT_s^d\left(x \to c \cdot x^{-\tau}\right). \tag{1.63}$$

Given a uniformly-Diophantine vector of type $\Phi$, the discussion following Definition 1.1.47 yields that the function $\Phi$ has to satisfy the bound

$$\Phi(T) = O\left(T^{-d}\right). \tag{1.64}$$

The existence of uniformly Diophantine-type vectors is implied by the following metric result.

**Theorem 1.1.49** *[4, p.17, Theorem 5.2] Assume that $s \geq d+1$. Let $\Phi$ be a non-increasing function tending to zero at infinity such that*

$$\liminf_{T \to +\infty} \frac{\Phi(2T)}{\Phi(T)} > 0$$

*and*

$$\sum_{i=1}^{+\infty} 2^{id(s+1)} \cdot \Phi\left(2^i\right)^{s-d} < +\infty.$$

*Then, with respect to the $d \times s$-dimensional Lebesgue measure, for almost all $\boldsymbol{\Theta}_{s,d} \in \mathbb{R}^{d\times s}$ there is a $c = c(\boldsymbol{\Theta}_{s,d}) > 0$ such that $\boldsymbol{\Theta}_{s,d} \in UDT_s^d(c\Phi)$.*

For instance, applying Theorem 1.1.49 to the function

$$\Phi(T) = T^{-\left(\frac{d(s+1)}{s-d}+\eta\right)} \quad \text{for } \eta > 0, \tag{1.65}$$

where $s \geq d+1$, yields that almost every $s$-tuple $\boldsymbol{\Theta}_{s,d}$ of $d$-dimensional vectors (with respect to the Lebesgue measure) belongs to $UDT_s^d(c\Phi)$ for some $c = c(\boldsymbol{\Theta}_{s,d})$.

Given a uniformly Diophantine vector of type $\Phi$, say $\boldsymbol{\Theta}_{s,d}$, the visibility bound of the forest $\mathfrak{F}(\boldsymbol{\Theta}_{s,d})$ defined in equation (1.10) is given by the following theorem.

**Theorem 1.1.50** *[4, p.17, Theorem 5.2] Assume that $\boldsymbol{\Theta}_{s,d} \in UDT_s^d(\Phi)$. Then,* *the set $\mathfrak{F}\left(\mathbf{\Theta}_{s,d}\right)$ constructed in* (1.10) *is a dense forest in* $\mathbb{R}^{d+1}$ *with visibility function satisfying*

$$V(\epsilon) \quad = \quad O\left(\left(\epsilon^{d-1}\cdot\Phi\left(d\epsilon^{-1}\right)^{-1}\right)^{d}\right). \tag{1.66}$$

The visibility bound (1.66) approaches the optimal $O\left(\epsilon^{-d}\right)$ as the bound on the uniformly Diophantine type $\Phi$ comes closer to the upper bound (1.64). Given a vector $\boldsymbol{v}\in\mathbb{T}^{d}$, let

$$\boldsymbol{V}=(k\cdot\boldsymbol{v})_{k\in\mathbb{N}} \tag{1.67}$$

be the sequence of multiples of $\boldsymbol{v}$. The proof of Theorem 1.1.50 is based on the following proposition which provides an effective way to distinguish for which vectors $\boldsymbol{v}$ in $\mathbb{T}^{d}$ the sequence $\boldsymbol{V}$ is $\epsilon$-dense in $\mathbb{T}^{d}$ (see Definition 1.1.18), for a given $\epsilon\in(0,1)$. Specifically, the proposition claims quantitatively that the obstruction to a good distribution modulo 1 of the sequence (1.67) is the existence of good rational approximations to the vector $\boldsymbol{v}$. The underlying idea will also be considered in Appendix B where it is proved that the multiples of badly approximable vectors satisfy some optimal dispersion properties.

**Proposition 1.1.51** *[4, p.19, Proposition 5.6] Let $\epsilon\in(0,1)$ be a positive number. Assume that*

$$N\geq 2^{d}\epsilon^{-d}.$$

*Then, the inclusion*

$$C_{d}\left(\epsilon,N\right)\quad\subseteq\quad S_{n}\left(\epsilon,N\right)$$

*holds, where*

$$C_{d}(\epsilon,N)=\left\{\boldsymbol{\xi}\in\mathbb{T}^{d}:\quad \textit{the sequence }\ (i\cdot\boldsymbol{\xi})_{0\leq i\leq N}\ \textit{ is not }\epsilon-\textit{dense in }\mathbb{T}^{d}\right\} \tag{1.68}$$

*and*

$$S_{d}(\epsilon,N)=\left\{\boldsymbol{\xi}\in\mathbb{T}^{d}:\quad \exists\boldsymbol{u}\in\mathbb{Z}^{d}\setminus\{\mathbf{0}\},\quad ||\boldsymbol{u}||_{\infty}\leq c_{d},\quad ||\boldsymbol{u}\cdot\boldsymbol{\xi}||\leq c_{d}'\cdot\frac{\epsilon^{d-1}}{N^{\frac{1}{d}}}\right\}, \tag{1.69}$$

*with*

$$c_{d}=d\quad\textit{and}\quad c_{d}'=d^{\frac{3}{2}}. \tag{1.70}$$

The following lemma, which is derived from Proposition 1.1.51, yields the proof of Theorem 1.1.50. It states that if $\mathbf{\Theta}_{s,d}$ is a uniformly-Diophantine vector of type $\Phi$, then the multiples of the vectors of $\mathbf{\Theta}_{s,d} = (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_s)$, when considered simultaneously, enjoy strong dispersion properties (recall the Definition (1.1.19) of the dispersion of a sequence).

**Lemma 1.1.52** *Let $\Phi$ be a non-increasing function tending to zero at infinity and assume that $\mathbf{\Theta}_{s,d} = (\boldsymbol{\theta}_1, ..., \boldsymbol{\theta}_s) \in UDT_s^d(\Phi)$. Then, for any*

$$N \quad \geq \quad \left(c_d' \cdot \epsilon^{d-1} \cdot \Phi\left(c_d \epsilon^{-1}\right)^{-1}\right)^d$$

*and any $\boldsymbol{\xi} \in \mathbb{R}^d$, there exists $i \in \{1, 2, ..., s\}$ such that the sequence*

$$(k \cdot (\boldsymbol{\theta}_i - \boldsymbol{\xi}))_{k=1}^N$$

*is $\epsilon$-dense in $\mathbb{T}^d$, where the constants $c_d, c_d'$ are as in* (1.70).

As illustrated in the statement of Theorem 1.1.11, applying Theorems 1.1.49 and 1.1.50 to the function (1.65) yields the existence of dense forests with very good visibility bounds in any dimension $d \geq 2$.

**Oscillating Sequences**

One can apply tools from Diophantine approximation such as the continued fraction expansion and the Ostrowski expansion to study the density in the real line of **oscillating sequences**; these are sequences of the form

$$(g(k) \cdot F(k\alpha))_{k\in\mathbb{N}}\,, \tag{1.71}$$

where $g$ is a positive increasing function and $F$ a real continuous 1-periodic function.

Relevant problems concerning the density properties of oscillating sequences have been studied extensively by Berend, Kolesnik and Boshernitzan [19, 20, 21] who established differential properties on the function $F$ ensuring that the sequence (1.71) is dense modulo 1. A similar question posed in [60] asks whether the sequence $(k \cdot \sin(k))_{k\in\mathbb{N}}$ is dense in $\mathbb{R}$. Given the function $g$, elementary considerations show that the density properties of the oscillating sequence depend, on the one hand on the choice of the real $\alpha$ and, on the other, on the local behavior of $F$ around its roots. This is no less than a problem concerning the distribution of the sequence $(k\alpha)_{k\in\mathbb{N}}$ and the quality of the approximation of the roots of $F$ by multiples of $\alpha$. The study of the densitiy properties in the real line of such oscillating sequences is the topic of Chapter 6.

### 1.1.8 Combinatorial Geometry and Range Spaces

In this section, we develop those notions from *combinatorial geometry* needed for the study of dense forests undertaken in Chapter 2. Combinatorial geometry is a branch of mathematics which applies ideas from the *probabilistic method*[2] and combinatorics to the study of some geometrical problems of a combinatorial nature. For instance, suppose that one chooses randomly $n$ points $P_1, P_2, ..., P_n$ on the unit circle in $\mathbb{R}^2$, according to the uniform distribution. What is the probability that the convex hull of these points contains the origin? To compute this probability[3], choose at first $n$ pairs of antipodal points $Q_1 = -Q_{n+1}$, $Q_2 = -Q_{n+2}$, ..., $Q_n = -Q_{2n}$. Choose $P_i$ to be either $Q_i$ or $Q_{n+i} = -Q_i$, where each choice is equally likely. This corresponds to a random choice of the point $P_i$. The probability that the origin does not belong to the convex hull of the points $\{P_j\}_{j=1}^{n}$, given the (distinct) points $\{Q_j\}_{j=1}^{2n}$, is precisely $x/2^n$, where $x$ is the number of subsets of $\{Q_j\}_{j=1}^{2n}$ of size $n$ contained in an open half-plane determined by a line through the origin, which does not pass through any of the points $\{Q_j\}_{j=1}^{2n}$. It is easy to see that $x = 2n$. Therefore, the probability that the origin is in the convex hull of $n$ randomly chosen points on the unit circle is precisely $1 - (2n/2^n)$.

[2]The basic probabilistic method can be described as follows: in order to prove the existence of a combinatorial structure with certain properties, one constructs an appropriate probability space and shows that a randomly chosen element in this space has the desired properties with positive probability.

[3]Here, we reproduce the answer as given in [14, Chapter 13, Introduction].

One of the most studied concepts in combinatorial geometry is that of a *Range Space*.

**Definition 1.1.53 (Range Space, $\epsilon$-Samples and $\epsilon$-Nets)** *[14, p.221-222] A* range space *is a pair $(\mathcal{S}, \mathcal{R})$, where $\mathcal{S}$ is a (finite or infinite) set and $\mathcal{R}$ is a (finite or infinite) family of subsets of $\mathcal{S}$. The members of $\mathcal{S}$ are* points *and those of $\mathcal{R}$ are* ranges*. The range space $(\mathcal{S}, \mathcal{R}')$ is a* subrange space *of $(\mathcal{S}, \mathcal{R})$ if $\mathcal{R}' \subseteq \mathcal{R}$.*

*Let $A$ be a finite subset of $\mathcal{S}$. Given $\epsilon \in (0,1)$, a subset $B \subseteq A$ is an* $\epsilon$-sample *of $A$ if for any range $R \in \mathcal{R}$, it holds that*

$$\left| \frac{\#(A \cap R)}{\#A} - \frac{\#(B \cap R)}{\#B} \right| \leq \epsilon.$$

*Similarly, a subset $\mathcal{N}_\epsilon \subseteq A$ is an* $\epsilon$-net *of $A$ if for any range $R \in \mathcal{R}$ satisfying $\#(R \cap A) \geq \epsilon \cdot \#A$, it holds that*

$$\mathcal{N}_\epsilon \cap R \neq \emptyset.$$

*If the set $\mathcal{S}$ is infinite and equipped with a probability measure $\mu$, then a subset $\mathcal{N}_\epsilon \subseteq \mathcal{S}$ is an* $\epsilon$-net *if for every range $R \in \mathcal{R}$ such that $\mu(R) \geq \epsilon$, it holds that $\mathcal{N}_\epsilon \cap R \neq \emptyset$.*

Many problems, especially of a geometric nature, can be restated in the language of range spaces. It is then useful to interpret, in a range space $(\mathcal{S}, \mathcal{R})$, $\mathcal{S}$ as being a geometrical space and $\mathcal{R}$ as being a family of geometrical shapes of interest. Both notions of $\epsilon$-samples and $\epsilon$-nets define subsets of $A \subseteq \mathcal{S}$ that represent approximately some of the behavior of $A$ with respect to the ranges. For instance, with these notions, Danzer's problem is rephrased as follows: does there exist a set of finite density in $\mathbb{R}^d$ which intersects all the ranges of the space $\mathbb{R}^d$, where the range space is the set of all boxes with volume 1?

Denote by

$$\mathcal{I} = \left[-\frac{1}{2}, \frac{1}{2}\right]$$

the closed interval of length one, centred at the origin. Given a natural number $d \geq 2$, let $\mathcal{B}$ be the family of all boxes in $\mathcal{I}^d$. The existence of $\epsilon$-nets with growth rate $O(\epsilon^{-1})$ in $\left(\mathcal{I}^d, \mathcal{B}\right)$ is known as the *Danzer-Rogers* problem and is the combinatorial analogue of Danzer's problem. Indeed, Solomon and Weiss [74] prove the following result which stresses the equivalence between these two problems.

**Theorem 1.1.54** *[74, Theorem 1.4] For a fixed $d \geq 2$, given a function $g$ of polynomial growth*[4]*, the following are equivalent:*

1. *There exists a Danzer set $\mathfrak{D} \subseteq \mathbb{R}^d$ of growth rate bound $g(T)$.*
2. *For every $\epsilon > 0$ there exists an $\epsilon$-net $\mathcal{N}_\epsilon \subseteq \mathcal{I}^d$ in the range space $\left(\mathcal{I}^d, \mathcal{B}\right)$ such that $\#\mathcal{N}_\epsilon = O\left(g\left(\epsilon^{-\frac{1}{d}}\right)\right)$.*

Furthermore, Solomon and Weiss utilise a probabilistic argument due to Haussler and Welzl [47] to show the existence of $\epsilon$-nets in $\left(\mathcal{I}^d, \mathcal{B}\right)$ which fail to have optimal growth rate only up to a logarithmic factor. Their result (stated below), when combined with Theorem 1.1.54, yields Theorem 1.1.6, p.21.

**Theorem 1.1.55** *[74, Theorem 1.6] For any $\epsilon > 0$ there exists an $\epsilon$-net $\mathcal{N}_\epsilon \in \mathcal{I}^d$ with growth rate $\#\mathcal{N}_\epsilon = O\left(\epsilon \cdot \ln\left(\epsilon^{-1}\right)\right)$. Equivalently, there exists a Danzer set $\mathfrak{D} \subseteq \mathbb{R}^d$ with growth rate bound $g(T) = O\left(T^d \cdot \log T\right)$.*

The proof of Theorem 1.1.55 uses the notion of the dimension of a range space, introduced in the following definition. To this end, given a range space $(\mathcal{S}, \mathcal{R})$ and a subset $A \subseteq \mathcal{S}$, define the **projection** of $\mathcal{R}$ onto $A$ to be the set

$$P_{\mathcal{R}}(A) = \{R \cap A : R \in \mathcal{R}\}; \tag{1.72}$$

that is, as the family of intersections of $A$ with all ranges.

**Definition 1.1.56 (Shattered Sets & Vapnik-Chervonenkis Dimension)** *[14, p.221] Let $(\mathcal{S}, \mathcal{R})$ be a range space.*

*A subset $A \subseteq \mathcal{S}$ is* shattered *if $P_{\mathcal{R}}(A) = 2^A$; that is, if the projection of $\mathcal{R}$ onto $A$ contains all subsets of $A$.*

[4] A function $g : \mathbb{R}^+ \to \mathbb{R}^+$ has polynomial growth rate if

$$\limsup_{x \to +\infty} \frac{\log g(x)}{\log x} < +\infty.$$

*The* Vapnik-Chervonenkis *dimension (or $VC$-dimension) of* $(\mathcal{S}, \mathcal{R})$*, denoted by* $VC(\mathcal{S}, \mathcal{R})$*, is the maximum cardinality of a shattered subset of* $\mathcal{S}$*. If there are arbitrarily large shattered subsets, then* $VC(\mathcal{S}, \mathcal{R}) = +\infty$*.*

In case there is no risk of confusion, one may omit the family of ranges $\mathcal{R}$ in the previous notation; that is, one may denote the $VC$-dimension of $(\mathcal{S}, \mathcal{R})$ by $VC(\mathcal{S})$.

The following combinatorial lemma was proved independently by Vapnik and Chervonenkis in [81] and by Sauer in [69]. Although it is elementary in nature, it provides strong upper bounds for the number of ranges in a finite range space with a given number of points and a given $VC$-dimension. Given integers $n \geq 0$ and $d \geq 0$, define the function $\Psi_d(n)$ by

$$\Psi_d(n) = \begin{cases} \sum_{i=0}^{d} \binom{n}{i} & \text{if } d < n, \\ 2^n & \text{if } d \geq n. \end{cases} \tag{1.73}$$

**Lemma 1.1.57** *Let* $(\mathcal{S}, \mathcal{R})$ *be a finite range space with* $\#\mathcal{S} = n$ *points and $VC$-dimension* $d$*. Then,* $\#\mathcal{R} \leq \Psi_d(n)$*.*

To sketch the proof of Lemma 1.1.57, assume that $(\mathcal{S}, \mathcal{R})$ is a finite range space with $VC$-dimension $d$. Let $s \in \mathcal{S}$ be a point of the space. Define the range spaces $(\mathcal{S} \backslash \{s\}, \mathcal{R} \backslash s)$ and $(\mathcal{S} \backslash \{s\}, \mathcal{R} - s)$ where

$$\mathcal{R} \backslash s = \{R \backslash \{s\} : R \in \mathcal{R}\} \quad \text{and} \quad \mathcal{R} - s = \{R \in \mathcal{R} : s \notin R, R \cup \{s\} \in \mathcal{R}\}.$$

Clearly the $VC$-dimension of $(\mathcal{S} \backslash \{s\}, \mathcal{R} \backslash s)$ is at most $d$ and the $VC$-dimension of $(\mathcal{S} \backslash \{s\}, \mathcal{R} - s)$ is at most $d - 1$. Observe that

$$\#\mathcal{R} = \#(\mathcal{R} \backslash s) + \#(\mathcal{R} - s)$$

and

$$\Psi_d(n) = \Psi_d(n-1) + \Psi_{d-1}(n-1) \quad \text{for all} \quad n, d \geq 1.$$

The lemma now follows from an easy induction on the number $n + d$ (where $n = \#\mathcal{S}$), upon noticing that the case $d = 0$ is trivially satisfied.

It is easy to check that the estimate given in Lemma 1.1.57 is sharp. Indeed, given a finite set $\mathcal{S}$, consider the set of ranges $\mathcal{R}$ which contains all the subsets of $\mathcal{S}$ with at most $d$ elements, then $VC(\mathcal{S}) = d$ and $\#\mathcal{R} = \Psi_d(n)$.

Note that, given a range space $(\mathcal{S}, \mathcal{R})$ of $VC$-dimension $d$ and $A \subseteq \mathcal{S}$, the $VC$-dimension of $(A, P_{\mathcal{R}}(A))$ is at most $d$. This leads one to the following corollary to Lemma 1.1.57:

**Corollary 1.1.58** *[47, Theorem 3.2] If $(\mathcal{S}, \mathcal{R})$ is a range space of $VC$-dimension $d$, then for every finite subset $A$ of $\mathcal{S}$, it holds that $\#P_{\mathcal{R}}(A) \leq \Psi_d(\#A)$.*

The space $\left(\mathbb{R}^d, \mathcal{H}\right)$, where $\mathcal{H}$ is the set of all open half-spaces in $\mathbb{R}^d$, is an example of a range space with finite $VC$-dimension. In particular, it holds that $VC\left(\mathbb{R}^d, \mathcal{H}\right) = d+1$. Indeed, on the one hand, one can prove that a set of $d+1$ points in *general position*[5] in $\mathbb{R}^d$ is shattered and, on the other, that every set of $d+2$ points in $\mathbb{R}^d$ is never shattered. The latter follows from Radon's Theorem on convex sets [68] which states that any set of $d+2$ points in $\mathbb{R}^d$ can be partitioned into two sets whose convex hulls intersect non trivially.

Let $\left(\mathbb{R}^d, \mathcal{R}'\right)$ with $\mathcal{R}' = \left\{\text{Boxes in } \mathbb{R}^d \text{ with volume } 1\right\}$ be the range space related to the Danzer problem. Denote by $\mathcal{H}_{2d} = \{H_1 \cap ... \cap H_{2d} : H_1, ..., H_{2d} \in \mathcal{H}\}$ the family of intersections of any $2d$ ranges from $\mathcal{H}$ defined as above. Since every box in $\mathbb{R}^d$ can be obtained as the intersection of $2d$ half-spaces of $\mathbb{R}^d$, $\left(\mathbb{R}^d, \mathcal{R}'\right)$ is a subrange space of $\left(\mathbb{R}^d, \mathcal{H}_{2d}\right)$. One can thus bound above the $VC$-dimension of $\left(\mathbb{R}^d, \mathcal{R}'\right)$ with that of $\left(\mathbb{R}^d, \mathcal{H}_{2d}\right)$. Since the space $\left(\mathbb{R}^d, \mathcal{H}_{2d}\right)$ is constructed from the space $\left(\mathbb{R}^d, \mathcal{H}\right)$, it is natural to ask if one can determine the dimension of the first space in terms of the dimension of the latter. The answer to this question, which lies at the heart of the proof of Theorem 1.1.55, is given by the following more general result.

**Corollary 1.1.59** *[14, Corollary 13.4.3] Let $(\mathcal{S}, \mathcal{R})$ be a range space of $VC$-dimension $d \geq 2$, and let $(\mathcal{S}, \mathcal{R}_h)$ be the range space where*

$$\mathcal{R}_h \quad = \quad \{(R_1 \cap ... \cap R_h) : R_1, ..., R_h \in \mathcal{R}\} \qquad \textit{for some } h \in \mathbb{N}. \tag{1.74}$$

[5]A set of points in the $d$-dimensional Euclidean space is in **general position** if no $k$ of them lie in a $(k-2)$-dimensional affine subspace for $k = 2, 3, ..., d+1$.

*Then,* $VC(\mathcal{S}, \mathcal{R}_h) \leq 2dh \cdot \log_2(dh)$.

Corollary 1.1.59 follows easily from Corollary 1.1.58. Indeed, given a range space $(\mathcal{S}, \mathcal{R})$ and $h \in \mathbb{N}$, assume that $A \subseteq \mathcal{S}$ is a shattered set of $(\mathcal{S}, \mathcal{R}_h)$ with cardinality $n \in \mathbb{N}$. From Corollary 1.1.58, one has that $\#P_{\mathcal{R}}(A) \leq \Psi_d(\#A) \leq n^d$, which yields in turn

$$\#P_{\mathcal{R}_h}(A) \quad \leq \quad \binom{\Psi_d(n)}{h} \quad \leq \quad n^{dh}.$$

Since $A$ is assumed to be shattered, one has $2^n \leq n^{dh}$, which yields that $n \leq 2dh \cdot \log_2(dh)$, whence the claim of Corollary 1.1.59.

For more details about the theory of range spaces, the reader is referred to the book by Alon and Spencer [14, Chapter 13], to the papers [12, 47] and to the references therein.

### 1.1.9 Lattices and Geometry of Numbers

The foundation of the geometry of numbers can be traced back at least to 1896 and to the fundamental monograph [63] by Hermann Minkowski. Minkowski's convex-body theorem (Theorem 1.1.16) states that, given a convex body $\mathcal{K}$ in $\mathbb{R}^d$ which is symmetric with respect to the origin, if $\lambda_d(\mathcal{K}) > 2^d$, then $\mathcal{K}$ contains a non-zero integer point. Geometry of numbers has a close relation with other fields of mathematics, especially Diophantine approximation.

One of the main objects of study in the geometry of numbers are lattices.

**Definition 1.1.60 (Lattice - Lattice Basis - Grid)** *A subset* $\Lambda$ *of* $\mathbb{R}^d$ *is a full-rank* lattice *if there are linearly independent vectors* $\boldsymbol{u}_1, ..., \boldsymbol{u}_d \in \mathbb{R}^d$ *such that*

$$\Lambda \quad = \quad span_{\mathbb{Z}}(\boldsymbol{u}_1, ..., \boldsymbol{u}_d) \quad = \quad \mathbb{Z}\boldsymbol{u}_1 \oplus ... \oplus \mathbb{Z}\boldsymbol{u}_d.$$

*The set of vectors* $\{\boldsymbol{u}_j\}_{k=1}^d$ *is a* basis of the lattice.

*A* grid *or translated lattice is a set of the form $\boldsymbol{x} + \Lambda$, where $\Lambda \subseteq \mathbb{R}^d$ is a lattice and $\boldsymbol{x} \in \mathbb{R}^d$.*

The set $\mathbb{Z}^d$ is the **canonical/standard lattice**. A lattice $\Lambda$ forms an abelian group under addition which can be characterised as follows:

**Theorem 1.1.61 (Characterisation of Lattices)** *[29, p.78, Theorem VI] A necessary and sufficient condition for a set of points $\Lambda \subseteq \mathbb{R}^d$ to be a (full-rank) lattice is that it should satisfy the following three properties:*

1. *If $\boldsymbol{a}, \boldsymbol{b} \in \Lambda$, then $\boldsymbol{a} \pm \boldsymbol{b} \in \Lambda$.*
2. *$\Lambda$ contains $d$ linearly independent points $\boldsymbol{a}_1, \boldsymbol{a}_2, ..., \boldsymbol{a}_d$.*
3. *There exists a constant $\eta > 0$ such that $\mathbf{0}$ is the only point of $\Lambda$ in the ball $B_2(\mathbf{0}, \eta)$, that is,*
$$\Lambda \cap B_2(\mathbf{0}, \eta) \quad = \quad \{\mathbf{0}\}.$$

To every lattice one can associate a dual lattice as follows:

**Definition 1.1.62 (Dual Lattice)** *Given a lattice $\Lambda \subseteq \mathbb{R}^d$, its dual lattice is defined as*
$$\Lambda^* := \quad \left\{\boldsymbol{x} \in \mathbb{R}^d : \quad \forall \boldsymbol{y} \in \Lambda, \quad \boldsymbol{x} \cdot \boldsymbol{y} \in \mathbb{Z}\right\}.$$

It is straightforward that a dual lattice is indeed a lattice.

The basis of a lattice $\Lambda$ is not uniquely determined. Indeed, let

$$M := \quad \begin{pmatrix} m_{1,1} & \cdots & m_{1,d} \\ \vdots & \ddots & \vdots \\ m_{d,1} & \cdots & m_{d,d} \end{pmatrix} \quad \in \quad SL_d(\mathbb{Z}) \tag{1.75}$$

be an integer matrix with $|\det(M)| = 1$. It is clear that, given a basis $\{\boldsymbol{u}_j\}_{j=1}^d$ of $\Lambda$, the vectors $\{\boldsymbol{v}_j\}_{j=1}^d$ with

$$\boldsymbol{v}_j := \quad \sum_{i=1}^d m_{j,i} \cdot \boldsymbol{u}_i \tag{1.76}$$

is also a basis of $\Lambda$. Furthermore, it can be easily checked that any basis $\{\boldsymbol{v}_j\}_{j=1}^d$ of a lattice $\Lambda$ is obtained from a given basis $\{\boldsymbol{u}_j\}_{j=1}^d$ in this way [29, p.10].

Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and let $\{\boldsymbol{u}_j\}_{j=1}^d$ be a basis for $\Lambda$. The **determinant of the lattice** $\Lambda$ is defined as

$$D(\Lambda) := \quad \left|\det(\boldsymbol{u}_1, \cdots, \boldsymbol{u}_d)\right|, \tag{1.77}$$

where $\det(\boldsymbol{u}_1, ..., \boldsymbol{u}_d)$ denotes the determinant of the $d \times d$-matrix whose $j$-th row is the vector $\boldsymbol{u}_j$. Since all the bases of $\Lambda$ are related through the relations (1.75) and (1.76), it is straightforward that the determinant of $\Lambda$ is independent of the particular choice of basis.

The following result relates the determinants of $\Lambda$ and $\Lambda^*$.

**Lemma 1.1.63** *[29, p.24, Lemma 5] Let $\Lambda \subseteq \mathbb{R}^d$ be a lattice and let $\Lambda^*$ be its dual lattice. Then, the dual lattice of $\Lambda^*$ is the lattice $\Lambda$; that is,*

$$(\Lambda^*)^* \quad = \quad \Lambda.$$

*Furthermore,*

$$D(\Lambda) \cdot D(\Lambda^*) \quad = \quad 1.$$

The following definition introduces the successive minima and the covering radius of a lattice $\Lambda$. These two quantitative parameters are closely related to the geometrical properties of $\Lambda$.

**Definition 1.1.64 (successive Minima and Covering Radius of $\Lambda$)** *Fix a norm $||\cdot||_*$ in $\mathbb{R}^d$ and let $\Lambda \subseteq \mathbb{R}^d$ be a lattice. The quantity $\lambda_1(\Lambda; ||\cdot||_*)$, called the* first successive minimum *of $\Lambda$ (with respect to the norm $||\cdot||_*$), equals the norm of the shortest nonzero element of $\Lambda$; that is,*

$$\lambda_1(\Lambda; ||\cdot||_*) := \quad \min\left\{||\boldsymbol{u}||_* : \boldsymbol{u} \in \Lambda, \boldsymbol{u} \neq \boldsymbol{0}\right\}.$$

*More generally, denote by*

$$\lambda_1(\Lambda; ||\cdot||_*) \quad \leq \quad \lambda_2(\Lambda; ||\cdot||_*) \quad \leq \quad ... \quad \leq \quad \lambda_d(\Lambda; ||\cdot||_*)$$

*the* successive minima *of* $\Lambda$ *defined for each* $1 \leq k \leq d$ *as*

$$\lambda_k\left(\Lambda; ||\cdot||_*\right) := \quad \min\left\{r \ > 0 : \quad \begin{array}{r}\textit{the ball } B_*\left(\mathbf{0}, r\right) \ \textit{contains } k\\ \textit{linearly independent vectors}\end{array}\right\},$$

*where* $B_*\left(\mathbf{0}, r\right)$ *stands for the ball of radius* $r$ *centered at the origin* $\mathbf{0}$ *with respect to the norm* $||\cdot||_*$.

*The* covering radius *of* $\Lambda$*, denoted by* $\mu\left(\Lambda; ||\cdot||_*\right)$*, equals the infimum of the radii* $r$ *such that*

$$\mathbb{R}^d \quad = \quad \bigcup_{\boldsymbol{\lambda}\in\Lambda} B_*\left(\boldsymbol{\lambda}, r\right).$$

*Equivalently,*

$$\mu\left(\Lambda; ||\cdot||_*\right) := \quad \sup_{\boldsymbol{x}\in\mathbb{R}^d} \quad \inf_{\boldsymbol{\lambda}\in\Lambda} ||\boldsymbol{x} - \boldsymbol{\lambda}||_*.$$

The successive minima and the covering radius depend on the chosen norm.

Given a lattice $\Lambda \subseteq \mathbb{R}^d$, the following well-known result due to Banaszczyk shows that the covering radius of $\Lambda$ and the first successive minimum of its dual lattice $\Lambda^*$ cannot be simultaneously too large.

**Theorem 1.1.65** *[17, Theorem 2.2] For any lattice* $\Lambda \subseteq \mathbb{R}^d$*,*

$$\frac{1}{2} \quad \leq \quad \mu\left(\Lambda; ||\cdot||_2\right) \cdot \lambda_1\left(\Lambda^*; ||\cdot||_2\right) \quad \leq \quad \left(\frac{1}{2\pi} + o(1)\right) \cdot d, \quad \textit{as } d \to +\infty, \tag{1.78}$$

*where* $||\cdot||_2$ *stands for the Euclidean norm .*

A slightly stronger version of Theorem 1.1.65 is proved in [7]. Therein, the right-hand side of inequality (1.78) is replaced with the sharper $\mu\left(\Lambda; ||\cdot||_2\right) \cdot \lambda_1\left(\Lambda^*; ||\cdot||_2\right) \leq (0.1275 + o(1)) \cdot d$.

Given an integer vector $\boldsymbol{p} = (p_1, ..., p_d) \in \mathbb{R}^d$ and a natural number $q \in \mathbb{N}$, denote by

$$\Lambda\left(\boldsymbol{p}, q\right) := \quad span_{\mathbb{Z}}\left\{\frac{\boldsymbol{p}}{q}, \boldsymbol{e}_1, ..., \boldsymbol{e}_d\right\} \quad \subseteq \quad \mathbb{R}^d \tag{1.79}$$

the lattice spanned by the vector $\boldsymbol{p}/q$ and the vectors $\boldsymbol{e}_1, ..., \boldsymbol{e}_d$ of the standard basis of $\mathbb{R}^d$. The vectors $\boldsymbol{p}/q, \boldsymbol{e}_1, ..., \boldsymbol{e}_d$ are not linearly independent (since there are $d+1$ of them). However, the point set $\Lambda(\boldsymbol{p}, q)$ is indeed a lattice. This follows from a straightforward application of Theorem 1.1.61.

In Appendix B, we study the dispersion of the sequence (1.67) of multiples of a badly approximable vector $\boldsymbol{v} \in \mathbb{R}^d$. A central tool in this investigation is the following result by Adiceam, Solomon and Weiss which relates the dispersion of the finite sequence $(k \cdot \boldsymbol{p}/q)_{k\in[\![q]\!]}$, where $\boldsymbol{p} \in \mathbb{Z}^d$ and $q \in \mathbb{N}$, to the first successive minimum of the lattice dual to $\Lambda(\boldsymbol{p}, q)$. From the way the lattice $\Lambda(\boldsymbol{p}, q)$ is defined in (1.79), one has that $\boldsymbol{x} \in \Lambda^*(\boldsymbol{p}, q)$ if and only if

$$\forall i \in \{1, ..., d\}, \quad \boldsymbol{x} \cdot \boldsymbol{e}_i \in \mathbb{Z} \quad \text{and} \quad \boldsymbol{x} \cdot \frac{\boldsymbol{p}}{q} \in \mathbb{Z}.$$

Therefore, one infers that

$$\Lambda^*(\boldsymbol{p}, q) \quad = \quad \left\{\boldsymbol{u} \in \mathbb{Z}^d : \quad \boldsymbol{p} \cdot \boldsymbol{u} \equiv 0 \pmod{q}\right\}. \tag{1.80}$$

**Lemma 1.1.66** *[4, p.21, Lemma 6.3] Assume that the Euclidean length of the shortest nonzero vector in $\Lambda^*(\boldsymbol{p}, q)$ satisfies*

$$\lambda_1(\Lambda^*(\boldsymbol{p}, q)) \quad > \quad d \cdot \epsilon^{-1}. \tag{1.81}$$

*Then, the sequence $(k \cdot (\boldsymbol{p}/q))_{1\leq k\leq q}$ is $(c_d \cdot \epsilon)$-dense in $\mathbb{T}^d$ for some constant $c_d > 0$ depending only on the choice of $d$. Moreover, for $d \in \mathbb{N}$ large enough one can choose $c_d = 1/2$.*

The proof of Lemma 1.1.66 is based on Theorem 1.1.65. Indeed, since the lattice $\Lambda(\boldsymbol{p}, q)$ contains $\mathbb{Z}^d$, the sequence $\left(k \cdot \frac{\boldsymbol{p}}{q}\right)_{k=1}^q$ will be $(c_d \cdot \epsilon)$-dense in $\mathbb{T}^d$ provided that the covering radius $\mu(\Lambda(\boldsymbol{p}, q); ||\cdot||_\infty)$ is at most $c_d \cdot \epsilon$. This follows immediately from Banaszczyk's bound (1.78), assumption (1.81) and the bound $||\boldsymbol{x}||_\infty \leq ||\boldsymbol{x}||_2$.

For more details on the theory of lattices, the reader is referred to the book by Cassels [29].

## 1.2 Introduction to the Sárközy-Fürstenberg Theorem

The Sárközy-Fürstenberg theorem was conjectured by László Lovász and proved independently in the late 1970's by András Sárközy [28, 29, 30] and Hillel Fürstenberg [9]. It states that if $A$ is a set of natural numbers with the property that no two numbers in it differ by a square number, then the **natural density** of $A$ is zero; that is, $\lim_{N\to+\infty} \#\left(A \cap [\![N]\!]\right)/N \to 0$. A subset $A \subseteq \mathbb{N}$ is **dense** if

$$\limsup_{N\to+\infty} \left\{ \frac{\#\left(A \cap [\![N]\!]\right)}{N} \right\} > 0.$$

**Theorem 1.2.1 (Sárközy-Fürstenberg, 1978)** *[9, 28] If $A_N \subseteq [\![N]\!]$ and $A_N$ lacks the configuration $x, x+y^2$, that is, there do not exist $x_1, x_2 \in A_N$ such that $x_1 - x_2 = y^2$ for some $y \in \mathbb{N}$, then $\#A_N = o(N)$.*

Actually, Sárközy [30] proved the stronger result that dense sets of integers contain two elements differing by a $k$th power. One can do even better and replace the $k$th-power configurations with the values of an arbitrary chosen integer polynomial with zero constant term.

**Theorem 1.2.2 (Sárközy's Theorem on Polynomials)** *[21, Theorem 1.1] If $P \in \mathbb{Z}[T]$ is a polynomial with $P(0) = 0$, there exists a constant $c_P > 0$ such that if $A_N \subseteq [\![N]\!]$ is a set with*

$$\#A_N \quad \gg_P \quad \frac{N}{(\log N)^{c_P}},$$

*then $A_N$ contains distinct elements $a, b$ such that*

$$a - b \quad = \quad P(y) \quad \text{for some } y \in \mathbb{Z}.$$

Sárközy in his proof uses ideas from Fourier analysis to show that if $A \subseteq [\![N]\!]$

is square-difference free[6], then

$$\#A \ll \frac{N}{(\log\log N)^c}$$

for a small, absolute constant $c > 0$. Note that the bound proved by Sárközy is weaker than the one provided by Theorem 1.2.2, which is due to Alex Rice [21]. Fürstenberg [9] proves Theorem 1.2.1 with techniques from the Ergodic Theory; however his methods provide no bound on the size of the square-difference free set $A$.

In the literature, one can find both lower and upper bounds for the size of square-free difference subsets of the integer interval $[\![N]\!]$. The best upper bound so far is due to Pintz, Steiger and Szemerédi [20]. Their results yields that if $A \subseteq [\![N]\!]$ is square-difference free, then

$$\#A \ll \frac{N}{(\log N)^{\frac{1}{12}\cdot\log\log\log\log N}}.$$

The best lower bound is given by Ruzsa [27] who proved, through an explicit construction, that

$$\#A_{max} \gg \frac{N}{N^{0.267}},$$

where $A_{max}$ is the largest square-difference free subset of $[\![N]\!]$.

The problems such as the one stated in Theorem 1.2.1 are called *density problems* for the reason that they ask a question of the form: how large must a substructure be to guarantee that a particular property holds? In our example, one measures largeness by the density of a given subset. The origin of this kind of problem can be traced back to the Ramsey theory of integers and in the somewhat easier considerations of *coloring problems*. Unlike in a density problem, in a coloring problem, one partitions a structure into a predetermined number of substructures and asks how big the structure must be for at least one substructure to satisfy a desired property. For instance, denote by $\mathtt{K}_n$ the complete graph on $n$ vertices;

[6] A subset $A \subseteq \mathbb{N}$ is square-difference free if there do not exist two elements of $A$ differing by a perfect square. In other words, for every $x_1, x_2 \in A$ and $y \in \mathbb{N}$, $x_1 - x_2 \neq y^2$.

that is, the graph which has an edge between any two distinct vertices. An **edge-coloring with $r$ colors** of the graph is a map $\mathfrak{C}$ from the set of edges $\mathtt{E}_n$ of $\mathtt{K}_n$ to a finite set with $r$-elements; that is,

$$\mathfrak{C}: \quad \mathtt{E}_n \quad \mapsto \quad \{c_1, ..., c_r\}\,.$$

A **monochromatic graph** is a graph with all its edges having the same colour. The following classical theorem of the British mathematician Frank P. Ramsey stands as a good illustration of a coloring problem:

**Theorem 1.2.3 (Ramsey's Theorem for Two Colors)** *[16, p.7, Theorem 1.15] Let $k, l \geq 2$ be two natural numbers. There exists a least positive integer $R = R(k,l)$ such that every edge-coloring with 2 colors $\{c_1, c_2\}$ of the complete graph $K_R$ admits a monochromatic subgraph $K_k$ of color $c_1$ or a monochromatic subgraph $K_l$ of color $c_2$.*

In the same vein, Bartel Leendert van der Waerden, a Dutch mathematician, proved in 1927 the following result concerned with arithmetic progressions in the set of natural numbers [37]. In the following statements, a **$k$-arithmetic progression** is a finite arithmetic progression with $k$ terms.

**Theorem 1.2.4 (Van der Waerden's Theorem)** *[16, p.23, Theorem 2.1] Let $k, r \geq 2$ be two integer numbers. There exists a least integer $W = W(k,r)$ such that for all $N \geq W$ and for every partition*

$$[\![N]\!] \quad = \quad \bigcup_{i=1}^{r} \mathtt{N}_i,$$

*there exists $j \in \{1, .., r\}$ such that the set $\mathtt{N}_j$ contains a $k$-arithmetic progression.*

The density version of van der Wearden's theorem was conjectured in 1936 by Erdós and Turán and proved by Endre Szemerédi in 1975.

**Theorem 1.2.5 (Szemerédi's Theorem)** *[33] If $A_N \subseteq [\![N]\!]$ lacks a configuration of points of the form*

$$a, \quad a+s, \quad ..., \quad a+(k-1)s$$

*for any choice of the natural numbers $a, s \in \mathbb{N}$; that is, if $A_N$ does not contain a $k$-arithmetic progression, then*

$$\frac{\#A}{N} \underset{N\to\infty}{\longrightarrow} 0.$$

Prior to Szemerédi's proof, the cases $k = 3$ and $k = 4$ had been proved by Roth [24] in 1953 and Szemerédi [32] in 1969, respectively. A non-linear generalisation of Szemerédi's theorem was established by Bergelson and Leibman [2] whose proof proceeds by adapting Fürstenberg's ergodic approach. It is therefore non effective.

**Theorem 1.2.6 (Polynomial Szemerédi Theorem)** *[2] Let $P_1, ..., P_k$ denote polynomials with integer coefficients and zero constant term. Any set $A \subseteq [\![N]\!]$ lacking the configuration*

$$x + P_1(y), \quad \dots \quad , x + P_k(y) \quad \textit{with } y \in \mathbb{Z}\setminus\{0\}$$

*is such that*

$$\frac{\#A}{N} \underset{N\to+\infty}{\longrightarrow} 0.$$

It must be noted that the statement of Theorem 1.2.6 applies to any polynomial configuration and thus implies non-quantitative versions of both Theorems 1.2.2 and 1.2.5.

Over the last decades, the Fourier analytic methods used in the proof of Sárközy's theorem have been refined and applied to configurations other than arithmetic progressions such as in Szemerédi's theorem, giving this way impetus to research surrounding the polynomial Szemerédi theorem and polynomial progressions in the primes. Configurations which have been studied in the literature are for instance polynomial images [17, 31], shifted primes $S = \{p - 1 : p \text{ prime}\}$ [18, 26, 30] and images of the primes under polynomials [23, 31].

In many of these Szemerédi-type density problems, the difference between the Abelian group $\mathbb{Z}/N\mathbb{Z}$ and $[\![N]\!]$ is purely technical. As noted by Ben Green in [10] (and by other authors before him), a great deal of these questions are expressed more naturally when they are posed in a general Abelian group. For instance, let

*$r_k(N)$ be the cardinality of the largest subset of $[\![N]\!]$ which is free of $k$-arithmetic progression.* Given a finite abelian group $G$, one can straight-forwardly define in a similar manner the quantity $r_k(G)$.

Many of these problems can be addressed more naturally in Abelian groups different than those in which they were originally asked. A model which has been used extensively in the literature is that of finite fields. Denote by $\mathbb{F}_p$ the finite field with $p$ elements, where $p$ is a fixed (small) prime number, and by $\mathbb{F}_p^N$ the $N$-dimensional vector space over the field $\mathbb{F}_p$. The reason for the choice of this model is that one can exploit ideas from linear algebra. For instance, subsets like subspaces are closed under addition, making it possible to run arguments locally and facilitating especially those relying on iteration.

Although the problem at hand is often easier when working in $\mathbb{F}_p^N$ (for some fixed small prime $p$), solving the problem over finite fields constitutes a significant step towards solving the problem over the integers. Since arguments in this model are usually more accessible because of its exact algebraic nature, this provides a good insight in their core idea which may be obscured by technical details when working directly with the integers.

Over the recent years, a lot of progress has been made to bound from above the cardinality of progression-free sets in finite fields. One of the breakthrough results in this direction is the paper by Croot, Lev and Pach [6] in which is developed a method to prove the following bound for $r_3\left(\mathbb{Z}_4^N\right)$.

**Theorem 1.2.7** *[6, Theorem 1] If $N \geq 1$ and $A \subseteq \mathbb{Z}_4^N$ contains no 3-arithmetic progressions, then*

$$\#A \quad \leq \quad 4^{\gamma n},$$

*where $\gamma = 0.927$.*

Here, we present this method, known as the *polynomial method*, in a form slightly more general than the one in [6]; more specifically, as it appears in the work by Ellenberg and Gijswijt [8]. Let $M_n$ be the set of monomials in $x_1, ..., x_n$ whose degree in each variable is at most $p-1$, and let $S_n$ be the $\mathbb{F}_p$-vector space they span. For any real number $d \in [0, 2n]$, let $M_n^d$ be the set of monomials in $M_n$ of degree at most $d$ and let $S_n^d$ be the subspace of $S_n$ they span. Write $m_d$ for the dimension of $S_n^d$.

**Theorem 1.2.8 (The Polynomial Method)** *[8, Proposition 2] Let $\mathbb{F}_p$ be a finite field and let $A$ be a subset of $\mathbb{F}_p^N$. Let $\alpha, \beta, \gamma$ be three elements of $\mathbb{F}_p$ which sum to* $0$.

*Suppose $P \in S_n^d$ satisfies $P(\alpha a + \beta b) = 0$ for every pair $a, b$ of distinct elements of $A$. Then, the number of $a \in A$ for which $P(-\gamma a) \neq 0$ is at most $2m_{\lfloor d/2 \rfloor}$.*

Ellenberg and Gijswijt [8] used the polynomial method to provide a far-reaching improvement on the known upper bounds of $r_3\left(\mathbb{F}_p^N\right)$, for every prime $p$:

**Theorem 1.2.9 (Ellenberg & Gijswijt)** *[8, Theorem 4 & Corollary 5]*
*Let $\alpha, \beta, \gamma$ be elements in $\mathbb{F}_p$ such that $\alpha + \beta + \gamma = 0$ and $\gamma \neq 0$, and let $A$ be a subset of $\mathbb{F}_p^n$ such that the equation*

$$\alpha a_1 + \beta a_2 + \gamma a_3 = 0$$

*has no solution $(a_1, a_2, a_3) \in A^3$ apart from those with $a_1 = a_2 = a_3$. As above, let $m_d$ be the number of monomials in $x_1, ..., x_n$ with total degree at most $d$ in which each variable appears with degree at most $p - 1$.*

*Then, $\#A \leq 3m_{\lfloor (p-1)n/3 \rfloor}$. In particular, let $A$ be a subset of $\mathbb{F}_3^n$ containing no* 3-*arithmetic progression. Then, $\#A = o(2.756^n)$.*

Estimating the size of the largest subset of $\mathbb{F}_3^N$ which is free of 3-arithmetic progressions is known as the *cap problem* and has a long history in number theory. Before the work of Ellenberg and Gijswijt, the best known bounds were

$$2.2^N \quad \ll \quad r_3\left(\mathbb{F}_3^N\right) \quad \ll_\epsilon \quad \frac{3^N}{N^{1+\epsilon}}$$

and were due to Edel [7] (for the lower bound) and Bateman and Katz [1] (for the upper bound). This should be compared with the current best bounds given by Theorem 1.2.9.

One can state all of the above-mentioned problems also in the model of *function fields*; that is, the vector space of polynomials over a finite field $\mathbb{F}_q$. As can be checked easily, the vector space $\mathbb{F}_q^N$ is isomorphic to the subspace of polynomials over $\mathbb{F}_q$ of degree at most $N-1$. The main difference between the space $\mathbb{F}_q^N$ and function fields is that in the latter one can define the notion of multiplication between two elements. In [11], Ben Green uses the polynomial method to prove the following analogue of the Polynomial Sárközy theorem (Theorem 1.2.2) over function fields. To state the result, denote by $\mathcal{P}_{q,N}$ the $N$-dimensional vector space over $\mathbb{F}_q$ (with $q$ being a prime power) consisting of all polynomials $c_{N-1}T^{N-1} + ... + c_1 T + c_0$ of degree less than $N$.

**Theorem 1.2.10 (Green)** *[11, Theorem 1.1] Let $k \geq 2$ be an integer and $q$ be a prime power. Then, there exists an explicit constant $c(k,q) > 0$ satisfying the following property: if $A \subseteq \mathcal{P}_{q,N}$ is a set with $\#A > 2q^{(1-c(k,q))N}$, then $A$ contains distinct polynomials $P(T)$, $Q(T)$ such that*

$$P(T) - Q(T) = b(T)^k$$

*for some $b \in \mathbb{F}_q[T]$.*

Another polynomial method-based result concerned with the existence of solutions to polynomial equations in dense subsets of function fields is due to Bienvenu.

**Theorem 1.2.11 (Bienvenu)** *[3] Let $r, k$ and $d$ be integers satisfying $k \geq 2r^2+1$. Suppose $(a_1, ..., a_k)$ are polynomials over $\mathbb{F}_p$ of degree at most $d$ such that $\sum_{j=1}^{k} a_j = 0$. Then, there exist constants $0 < c(r,p) < 1$ and $C = C(d,r,p) > 0$ such that any $A \subseteq \mathcal{P}_{p,N}$ satisfying the relation $\#A \geq kC \cdot p^{c(r,p)N}$ must contain a non-trivial solution to the equation*

$$\sum_{j=1}^{k} a_j f_j^r = 0.$$

In view of Sárközy's Theorem (Theorem 1.2.1), a complementary direction of study is considered in Chapter 7. Specifically, we prove a multivariable version of Theorem 1.2.1; namely, that dense sets $A \subseteq [\![N]\!]$ of integers enjoy non-trivial solutions for the equation

$$a_1 x_1 + ... + a_s x_s = \mathcal{Q}(y_1, ..., y_t) \quad \text{for some} \quad y_1, ..., y_t \in \mathbb{Z}, \tag{1.82}$$

where $a_1, ..., a_s$ are integers which sum to zero, $\{x_i\}_{i=1}^s$ is a subset of distinct elements of $A$ and $\mathcal{Q} \in \mathbb{Z}[y_1, ..., y_t]$ is a quadratic form in $t$ variables. In Section 1.2.1 below, we introduce the Fourier-analytic concepts needed for the proof of this result.

The reader interested in density problems or other aspects of additive combinatorics is referred to the book [35, Chapters 2, 10, 11 & 12], to the surveys [10, 38] and to the papers [5, 8, 11, 12, 13, 21, 22, 24, 25].

### 1.2.1 Fourier Analysis

In Chapter 7 we prove a generalised version of Sárközy's theorem (Theorem 1.2.1, p.70) in more variables (see Equation (1.82) and the discussion around it). The main tool used there is the discrete Fourier transform.

**Definition 1.2.12 (Fourier Transform)** *Given a finitely supported function $f : \mathbb{Z} \to \mathbb{C}$, the Fourier Transform of $f$ is the function $\hat{f} : \mathbb{T} \to \mathbb{C}$ defined by the formula*

$$\hat{f}(\alpha) \quad = \quad \sum_{n \in \mathbb{Z}} f(n) \cdot e(\alpha n).$$

The Fourier transform completely determines a finitely supported function $f : \mathbb{Z} \mapsto \mathbb{C}$, since by the *orthogonality relations*

$$\int_{\alpha \in \mathbb{T}} e(\alpha n) \, \mathrm{d}\alpha \quad = \quad \begin{cases} 0 & \text{if } n \in \mathbb{Z} \setminus \{0\} \\ 1 & \text{if } n = 0, \end{cases} \tag{1.83}$$

one has that

$$f(n) \quad = \quad \int_{\mathbb{T}} \hat{f}(\alpha) \, e(-\alpha n) \, \mathrm{d}\alpha.$$

This leads one to define the *inverse formula* of the Fourier transform; given an integrable function $F : \mathbb{T} \to \mathbb{C}$, define $\hat{F} : \mathbb{Z} \to \mathbb{C}$ by

$$\hat{F}(n) := \quad \int_{\mathbb{T}} F(\alpha) e(-\alpha n) \, \mathrm{d}\alpha. \tag{1.84}$$

**Lemma 1.2.13 (Orthogonality)** *Let $b, c_1, c_2, \cdots, c_s \in \mathbb{Z}$ and let $f_1, \cdots, f_s : \mathbb{Z} \to \mathbb{C}$ have finite support. Then*

$$\sum_{\substack{x_1,\cdots,x_s\in\mathbb{Z},\\ c_1x_1+\cdots+c_sx_s=b}} f_1(x_1)\cdots f_s(x_s) \quad = \quad \int_{\mathbb{T}} \hat{f}_1(c_1\alpha)\cdots\hat{f}_s(c_s\alpha)\, e(-b\alpha)\, d\alpha.$$

Given a finitely supported function $f : \mathbb{Z} \to \mathbb{C}$, Lemma 1.2.13 yields *Parseval's identity*:

$$\sum_{x\in\mathbb{Z}} |f(x)|^2 \quad = \quad \int_{\mathbb{T}} \left|\hat{f}(\alpha)\right|^2 \mathrm{d}\alpha. \tag{1.85}$$

Given two finite sets $A$ and $B$, define the density of $A$ over $B$ to be the quantity

$$\delta(A; B) \quad = \quad \mathop{\mathbb{E}}_{x\in B} \chi_A(x), \tag{1.86}$$

where $\chi_A$ is the characteristic function of the set $A$ and $\mathop{\mathbb{E}}_{x\in B}\chi_A(x)$ is the average of the numbers $\{\chi_A(x)\}_{x\in B}$ defined in the notation chapter, p.13.

**Definition 1.2.14 (Fourier Uniform Sets/Functions)** *Given $N \in \mathbb{N}$ and a subset $A \subseteq [\![N]\!]$, the set $A$ is $\epsilon$-Fourier Uniform if*

$$\left|\left|\hat{\chi}_A - \mathop{\mathbb{E}}_{n\in[\![N]\!]}\chi_A(n)\cdot\hat{\chi}_{[\![N]\!]}\right|\right|_\infty \quad \leq \quad \epsilon\cdot N.$$

*A function $f : [\![N]\!] \to \mathbb{C}$ is called $\epsilon$-Fourier uniform if*

$$\left|\left|\hat{f}\right|\right|_\infty \quad \leq \quad \epsilon\cdot N.$$

For more details on Fourier analysis and its connection with additive combinatorics, the reader is referred to the books [65, Chapters 2, 3, 5, 10], [84, Chapters 1 & 2], [35, Chapter 4].

## 1.3 Main Results and Structure of the Thesis

The thesis is structured as follows: Chapters 2 and 3 contain original results on the Danzer problem and the problem of dense forests, while Chapters 4, 5 & 6 and Appendices A & B deal with problems rising from related considerations. In Chapter 7 is proved a version of Sárközy's theorem for equations of the form (1.82). More specifically:

Chapter 2 This chapter contains two main results: (1) the construction of a dense forest with the best known visibility bound which, furthermore, enjoys the property of being deterministic; (2) the construction of a point set in $\mathbb{R}^d$ which shows that the growth rate bound obtained by Solomon and Weiss [74] in Theorem 1.1.6 can be achieved deterministically if one weakens the notion of a Danzer set in a suitable way. For the published version of these results, see [79].

Chapter 3 This chapter is concerned with the construction of planar Peres-type forests (see Definition 1.1.10). The main results are the probabilistic construction of Peres-type forests with (almost optimal) visibility $O_\eta\left(\epsilon^{-1-\eta}\right)$ and the fully deterministic construction of a Peres-type forest with visibility $O_\eta\left(\epsilon^{-2-\eta}\right)$, for any $\eta > 0$. Notice that the latter stands as the best known visibility bound of a deterministic Peres-forest in the literature. Both results are achieved by constructing sequences in the unit torus satisfying strong distribution properties. For the published version of these results, see [79].

Chapter 4 In this chapter we generalise to higher dimensions the work by Akiyama [9] and Marklof [59] concerned with planar Delone spiral sets. The main result of this chapter provides necessary and sufficient conditions on a spherical sequence for the spiral that it generates to be Delone. This allows for the construction of explicit examples of spiral Delone sets in $\mathbb{R}^{d+1}$ for all $d \geq 1$, which boils down to finding a sequence in $\mathbb{S}^d$ enjoying some optimal distribution properties. In turn, the constuction of such a sequence is achieved by *lifting* a suitable sequence from $\mathbb{T}^d$ to $\mathbb{S}^d$. For the published version of these results, see the joint work [5].

Chapter 5 In this chapter we extend the analysis undertaken in Chapter 4 in order to study visibility concepts in the context of spiral sets in $\mathbb{R}^{d+1}$. Necessary and sufficient conditions on the sequence $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ in $\mathbb{S}^d$ are proved for the spiral $\left\{ \sqrt[d+1]{k} \cdot \boldsymbol{u}_k \right\}_{k\in\mathbb{N}}$ (1) to be an orchard (cf. Definition 1.1.17); (2) to have an empty set of visible points (cf. Definition 1.1.15); (3) to be a dense forest (cf. Definition 1.1.7). As a consequence, the existence of a Delone spiral set which satisfies the extra properties of being an orchard with optimal visibility $O\left(\epsilon^{-d}\right)$ and having an empty set of visible points is established in any dimension $d+1$ ($d \geq 1$). The problem of determining the existence of a spiral set which, furthermore, is a dense forest remains unsettled. For the published version of these results, see the joint work [6].

Chapter 6 This chapter is concerned with the study of real-valued oscillating sequences as defined in (1.71). More precisely, when the oscillating function $F$ has finitely many roots in $[0, 1)$, necessary and also sufficient conditions for the oscillating sequences under consideration to be dense in $\mathbb{R}$ are provided. For the published version of these results, see [80].

Chapter 7 In this chapter is proved a multivariable version of Sárközy's Theorem 1.2.1. Specifically, it is proved that if a subset $A_N \subseteq [\![N]\!]$ does not contain solutions to the equation (1.82), then it holds that

$$\frac{\#A_N}{N} \quad = \quad o(1) \quad \text{as } N \to +\infty.$$

Appendix A This appendix complements the study undertaken in Chapter 3, develops the techniques used therein and provides further examples of sequences which both generate and do not generate Peres-type forests. In particular, it is proved that the Peres-type forest (1.11) generated from the sequence $(\alpha \cdot k^2)_{k\in\mathbb{N}}$, with $\alpha$ being a badly approximable number (Definition 1.1.36), has visibility $O\left(\epsilon^{-3}\right)$. Similarly, the Peres-type forest generated from the sequence $(x \cdot g(k))_{k\in\mathbb{N}}$, where $x$ is irrational and $g : \mathbb{N}_0 \mapsto \mathbb{N}_0$ is a $q$-additive function, is also a dense forest.

Appendix B In this appendix we prove some auxiliary results concerning the distribution of the sequence (1.67) of the multiples of a badly approximable vector $\boldsymbol{v} \in \mathbb{T}^d$.

Specifically, it is established that the sequence $\boldsymbol{V} = (k \cdot \boldsymbol{v})_{k \in \mathbb{N}}$ enjoys an optimal distribution property; namely, for every $m \in \mathbb{N}$, the consecutive terms $(k \cdot \boldsymbol{v})_{k=m+1}^{m+N}$ are $O\left(1/N^{\frac{1}{d}}\right)$-dense in $\mathbb{T}^d$ and the distance between any two of these terms is larger than $c \cdot N^{-\frac{1}{d}}$, for some constant $c = c(\boldsymbol{v}) > 0$. This result is used in Chapters 4 and 5.

# Chapter 2

# Dense and Optical Forests

## 2.1 Introduction

Let $d \geq 2$ be a natural number which, throughout the chapter, stands for a dimension. One can tackle the Danzer problem either by relaxing the density constraint (see Section 1.1.1), that is by allowing sets satisfying the Danzer property with growth rate bound larger than $O\left(T^d\right)$, or by studying the weaker concept of dense forests (Definition 1.1.7, p.8) obtained by a suitable relaxation of the volume constraint (see 1.1.2). Given a dense forest $\mathfrak{F} \subseteq \mathbb{R}^d$, a visibility function $V : (0, 1] \mapsto \mathbb{R}^+$ of $\mathfrak{F}$ satisfies the lower bound

$$V(\epsilon) \quad \gg \quad \epsilon^{-(d-1)}. \tag{2.1}$$

This bound will be proved in detail in Section 2.2.

In the definition of a dense forest in $\mathbb{R}^d$, one fixes the density of the point set and allows its visibility to grow to infinity faster than $O\left(\epsilon^{-(d-1)}\right)$ as $\epsilon \to 0^+$. In the definition of an optical forest introduced below for the first time, one fixes the visibility of the forest to be optimal and allows its growth rate bound to be "larger" than $O\left(T^d\right)$; that is, the forest has not necessarily finite density.

**Definition 2.1.1 (Optical Forest)** *Set $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $g(T) \gg T^d$. A set $\mathfrak{F} \subseteq \mathbb{R}^d$ is an* optical forest *with growth rate bound $g : \mathbb{R}^+ \to \mathbb{R}^+$ if its density has growth rate bound $g$ (as defined in* (1.1)*, p.17) and if for every $\epsilon \in (0, 1)$ and* *every line segment of length* $O\left(\epsilon^{-(d-1)}\right)$*, there is a point* $\boldsymbol{x} = \boldsymbol{x}(L) \in \mathfrak{F}$ *such that* $\mathit{dist}(\boldsymbol{x}, L) \leq \epsilon$*.*

An optical forest is a point set which intersects every box in $\mathbb{R}^d$, $(d-1)$ of the edges of which have equal length, say $\epsilon \in (0,1)$, and the remaining edge has length $C \cdot \epsilon^{-(d-1)}$ for some constant $C > 0$. Given $C > 0$, denote this family of boxes by $\mathcal{B}'_d(C)$. The problem of constructing optical forests is concerned with the existence of such a point set in $\mathbb{R}^d$ with growth rate bound as close as possible to the optimal bound $O\left(T^d\right)$. This problem is a weakening of Danzer's in two ways: (1) one allows the growth rate bound of the density of the point set to be larger than $O\left(T^d\right)$ and (2) one substitutes the family of boxes of a given volume $C > 0$ which a Danzer set must intersect with the smaller family $\mathcal{B}'_d(C)$. In particular, the concept of an optical forest is a weakening of that of a Danzer set in the sense that a Danzer set with growth rate bound $g : \mathbb{R}^+ \to \mathbb{R}^+$ (respectively, with finite density) is an optical forest with growth rate bound $g$ (respectively, with optimal growth rate bound $O\left(T^d\right)$). The two notions are equivalent only in dimension $d = 2$. This is similar to the connection which holds between Danzer sets and dense forests (see the discussion following Definition 1.1.7, p.8).

The goal of this chapter is to provide effective constructions of (1) dense forests with almost optimal visibility bounds (in a suitable sense) and of (2) optical forests with almost optimal growth rate bounds (in a suitable sense). As far as the construction of dense forests is concerned, the best known result is a purely probabilistic planar construction due to Alon [13] with visibility $V(\epsilon) = O\left(\epsilon^{-1} \cdot 2^{O\left(\sqrt{\ln(\epsilon^{-1})}\right)}\right)$ (see Theorem 1.1.9, p.25). Alon's forest enjoys the extra property of being a Delone set (Definition 1.1.8, p.10).

The main result of this chapter yields a completely effective construction of dense forests with almost optimal visibility in all dimensions $d \geq 2$.

**Theorem 2.1.2** *Let* $V : (0,1) \mapsto \mathbb{R}^+$ *be a decreasing function such that* $V(\epsilon) \longrightarrow +\infty$ *as* $\epsilon \to 0^+$*. Assume that there exists a decreasing sequence* $(e_j)_{j\geq 1}$

*in* $(0,1)$ *with* $e_j \underset{j\to+\infty}{\longrightarrow} 0$ *such that*

$$\sum_{j=1}^{+\infty} \frac{e_j^{-(d-1)}}{V(e_j)} < +\infty.$$

*Then, there exists a deterministic construction of a dense forest in* $\mathbb{R}^d$ *with visibility function* $W$ *such that* $W(\epsilon) = 2\sqrt{d}\cdot V(e_i)$, *where* $i = i(\epsilon)$ *is the unique index such that* $e_i \leq \epsilon < e_{i-1}$.

In [13], Alon claims that, by optimising his probabilistic construction, one could prove the existence of a planar dense forest with visibility bound $V(\epsilon) = O\left(\epsilon^{-1}\cdot\ln\left(\epsilon^{-1}\right)\cdot\ln\ln\left(\epsilon^{-1}\right)\right)$. However, the author could not verify this claim and further discussions with Prof. N. Alon confirm that its validity is doubtful.

By applying Theorem 2.1.2 to the function $V(\epsilon) = \epsilon^{-(d-1)}\cdot\ln\left(\epsilon^{-1}\right)\cdot\ln\ln\left(\epsilon^{-1}\right)^{1+\eta}$ for an arbitrary $\eta > 0$, one obtains the following corollary, which stands as the best known visibility bound for a dense forest in any dimension $d \geq 2$. Furthermore, the corresponding construction is deterministic.

**Corollary 2.1.3** *Given* $d \geq 2$, *for every* $\eta > 0$, *there exists a deterministic dense forest in* $\mathbb{R}^d$ *with visibility*

$$V(\epsilon) = O\left(\epsilon^{-(d-1)}\cdot\ln\left(\epsilon^{-1}\right)\cdot\ln\ln\left(\epsilon^{-1}\right)^{1+\eta}\right).$$

When relaxing the density constraints, the best known construction of a Danzer set is due to Solomon and Weiss [74] who prove the existence of a Danzer set in $\mathbb{R}^d$ with growth rate bound $O\left(T^d\cdot\ln(T)\right)$ for every $d \geq 2$. The second result of this chapter shows that the growth rate bound obtained by Solomon and Weiss can be achieved by a deterministic construction (in contrast with their probabilistic one) if one considers optical forests instead of Danzer sets.

To this end, the Definition 1.1.53 (p.36) of *range spaces* is used. In the context of this chapter, the set of points $\mathcal{S}$ of the range space $(\mathcal{S},\mathcal{R})$ will always be the $d$-dimensional box $\mathcal{I}^d$, where $\mathcal{I} = \left[-\frac{1}{2},\frac{1}{2}\right]$. The set $\mathcal{R}$ will be either the family of ranges $\mathcal{B}$ consisting of all boxes in $\mathcal{I}^d$ or the family $\mathcal{B}'$ of boxes in $\mathcal{I}^d$ with side lengths $s_1,\dots,s_d$ such that $s_1 = \dots = s_{d-1} \leq s_d$. Obviously, it holds that $\mathcal{B}' \subseteq \mathcal{B}$.

Given $\epsilon > 0$, recall that a subset $\mathcal{N}_\epsilon \subseteq \mathcal{I}^d$ is an $\epsilon$-net if $\mathcal{N}_\epsilon$ intersects non-trivially any box $B \in \mathcal{B}$ as soon as $\lambda_d(B) \geq \epsilon$, where $\lambda_d$ is the Lebesque measure in $\mathcal{I}^d$.

The following theorem shows that the claim in Theorem 1.1.54 (p.37) is true if one replaces, on the one hand the ranges $\mathcal{B}$ with $\mathcal{B}'$ and, on the other, Danzer sets with optical forests.

**Theorem 2.1.4** *Given $d \geq 2$ and an increasing function $g : \mathbb{R}^+ \mapsto \mathbb{R}^+$ satisfying*

$$g(x) \gg x^d \quad \text{and for every } x > 0, \quad 1 + c \leq \frac{g(2x)}{g(x)} \leq C \tag{2.2}$$

*for some positive constants $c, C > 0$, the following statements are equivalent:*

1. *There exists an optical forest $\mathfrak{F} \subseteq \mathbb{R}^d$ with growth rate bound $g(T)$.*
2. *For every $\epsilon > 0$ there exists $\mathcal{N}_\epsilon \subseteq \left[-\frac{1}{2}, \frac{1}{2}\right]^d$ such that $\#\mathcal{N}_\epsilon = O\left(g\left(\epsilon^{-\frac{1}{d}}\right)\right)$, and such that $\mathcal{N}_\epsilon$ intersects every box in $\mathcal{B}'$ of volume $\epsilon$. In other words, $\mathcal{N}_\epsilon$ is an $\epsilon$-net in the range space $\left(\mathcal{I}^d, \mathcal{B}'\right)$.*

In view of the results of Solomon and Weiss (see Theorem 1.1.55, p.37), it is asked in [1, Problem 8] if one can construct a deterministic $\epsilon$-net in $\left(\mathcal{I}^d, \mathcal{B}\right)$ with growth rate $O\left(\epsilon^{-1} \cdot \ln\left(\epsilon^{-1}\right)\right)$. Our result yields an affirmative answer if one replaces the range space $\left(\mathcal{I}^d, \mathcal{B}\right)$ with $\left(\mathcal{I}^d, \mathcal{B}'\right)$.

**Theorem 2.1.5** *Given $d \geq 2$ and the range space $\left(\mathcal{I}^d, \mathcal{B}'\right)$ defined above, for every $\epsilon > 0$, one can construct a deterministic $\epsilon$-net $\mathcal{N}_\epsilon$ with cardinality $\#\mathcal{N}_\epsilon = O\left(\epsilon^{-1} \cdot \ln\left(\epsilon^{-1}\right)\right)$. Equivalently, one can construct a deterministic optical forest in $\mathbb{R}^d$ with growth rate bound $O\left(T^d \cdot \ln(T)\right)$.*

Since a Danzer set is in particular an optical forest, the result of Solomon and Weiss [74, Theorem 1.6] yields an optical forest with a same growth rate bound as the one provided by Theorem 2.1.5. The main feature of Theorem 2.1.5 is that the construction is deterministic; however, for $d \geq 3$, it is not a Danzer set. This will be justified in detail after the proof of Theorem 2.1.5 in Section 2.2. Note that, in the case $d = 2$, it holds that $\mathcal{B} = \mathcal{B}'$ and thus an optical forest is also a Danzer set.

The idea underlying the proof of Theorem 2.1.5 is known but is reproduced here as the literature lacks any proper reference. Similar constructions are given in the work of Bambah and Woods [15] who proved the existence of a Danzer set in $\mathbb{R}^d$ with growth rate bound $O\left(T^d \cdot \log(T)^{d-1}\right)$.

The chapter is organised as follows. In Section 2.2 the proofs of Theorems 2.1.2, 2.1.5 and of Corollary 2.1.3 are given. Theorem 2.1.4 is proved in Section 2.3.

## 2.2 Proof of Theorems 2.1.2 and 2.1.5

Proof (Visibility Bound (2.1)) To prove inequality (2.1), first note that one can replace the Euclidean norm in the definition of the dense forest with the supremum norm. This change affects the visibility of a given forest only up to a constant. A similar remark can be made for the definition of a growth rate bound of a set, where one can replace the Euclidean ball of radius $T$ centered at the origin with the ball of radius $T$ with respect to the sup norm centered at the origin.

Now, assume that a given dense forest $\mathfrak{F}$ in $\mathbb{R}^d$ has visibility $V$. Fix $\epsilon > 0$ and set $C_\epsilon$ to be the hypercube centered at the origin $\mathbf{0}$ with sidelength $V(\epsilon)$; that is, $C_\epsilon = B_\infty(\mathbf{0}, V(\epsilon)/2)$. Decompose the hypercube $C_\epsilon$ into axes-parallel boxes which have $d-1$ sides of length $\epsilon$ and one side of length $V(\epsilon)$. From the definition of a dense forest, any such box contains at least one point from $\mathfrak{F}$. This yields that

$$\epsilon^{-(d-1)} \cdot V(\epsilon)^{d-1} \quad \leq \quad \#\left(B_\infty\left(\mathbf{0}, V(\epsilon)/2\right) \cap \mathfrak{F}\right) \quad \ll \quad V(\epsilon)^d,$$

where the left-hand side quantity stands for the number of boxes that the cube $C_\epsilon$ is decomposed into and where the middle quantity is, by definition, the total number of points of $\mathfrak{F}$ belonging to $C_\epsilon$. The right-hand side holds from the assumption that the forest $\mathfrak{F}$ has finite density. Therefore, one obtains that $V(\epsilon) \gg \epsilon^{-(d-1)}$. ■

Proof (Theorem 2.1.2) Fix a natural number $d \geq 2$ and a sequence $(e_j)_{j\geq 1}$

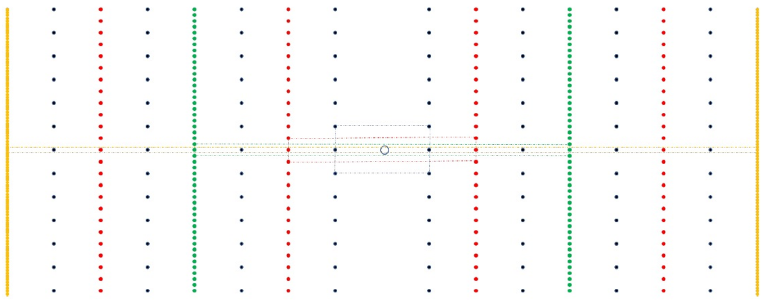


Figure 2.1: An illustration of the point sets $S_j, j \in \mathbb{N}$, defined in (2.3).

satisfying the assumptions of Theorem 2.1.2. For every $j \in \mathbb{N}$, define the sets

$$S_j = \left\{ \left( k \cdot V(e_j) \quad , \quad l_2 \cdot \frac{e_j}{\sqrt{d-1}} \quad , \quad \dots \quad , \quad l_d \cdot \frac{e_j}{\sqrt{d-1}} \right) : \begin{array}{l} k \in \mathbb{Z} \backslash \{\mathbf{0}\}, \\ l_2, \dots, l_d \in \mathbb{Z} \end{array} \right\} \tag{2.3}$$

(see Figure[1] 2.1 above). For every $l \in \{1, ..., d\}$, let $R_l : \mathbb{R}^d \mapsto \mathbb{R}^d$ be the map which permutes the first and the $l$-th coordinate of a point; that is,

$$R_l(x_1, ..., x_{l-1}, x_l, x_{l+1}, ..., x_d) = (x_l, ..., x_{l-1}, x_1, x_{l+1}, ..., x_d). \tag{2.4}$$

Define also the sets

$$\mathfrak{F}_j = \bigcup_{l \in \{1,...,d\}} R_l(S_j) \quad \text{and} \quad \mathfrak{F} = \bigcup_{j \in \mathbb{N}} \mathfrak{F}_j \tag{2.5}$$

We prove that the point set $\mathfrak{F}$ is a dense forest with visiblity $W : (0,1) \mapsto \mathbb{R}^+$, where $W(\epsilon) = 2\sqrt{d} \cdot V(e_i)$ and $i = i(\epsilon)$ is the unique index such that $e_i \leq \epsilon < e_{i-1}$. To this end, fix $\epsilon \in (0,1)$ and set $i = i(\epsilon)$. It is easy to check that every line segment $L$ of length $2\sqrt{d} \cdot V(e_i)$ is such that the distance $dist(L, \mathfrak{F}_i)$ from $L$ to the set $\mathfrak{F}_i$ is smaller than $e_i$. Indeed, if the line segment $L$ has length $2\sqrt{d} \cdot V(e_i)$, then $L$ contains at least one point which has at least one coordinate equal to $k \cdot V(e_i)$, for some $k \in \mathbb{Z}$. Thus, from the definition of the set $\mathfrak{F}_i$, one obtains that $\text{dist}(L, \mathfrak{F}_i) \leq e_i \leq \epsilon$. This implies the claim regarding the visiblity function of the forest $\mathfrak{F}$ in Theorem 2.1.2.

[1]The author would like to thank the anonymous referee of [79] for providing Figure 2.1.

As for the density of the forest $\mathfrak{F}$, it is enough to show that

$$\limsup_{T\geq 1}\left(\frac{\#\left(\mathfrak{F}\cap B_2(\mathbf{0},T)\right)}{T^d}\right) \quad < \quad +\infty. \tag{2.6}$$

Indeed, given $j\in\mathbb{N}$ and $T_j\geq V\left(e_j\right)$ with $kV\left(e_j\right)\leq T_j \ < (k+1)V\left(e_j\right)$ for some $k\in\mathbb{N}$, one has that

$$\begin{aligned}\frac{\#\left(\mathfrak{F}_j\cap B_2\left(\mathbf{0},T_j\right)\right)}{T_j^d} &\leq \frac{\#\left(\mathfrak{F}_j\cap B_\infty\left(\mathbf{0},(k+1)V\left(e_j\right)\right)\right)}{k^d\cdot V\left(e_j\right)^d}\\ &\leq \frac{d2^d\cdot(d-1)^{\frac{d-1}{2}}\cdot(k+1)^d}{k^d}\cdot\frac{V\left(e_j\right)^{d-1}\cdot e_j^{-(d-1)}}{V\left(e_j\right)^d} \\ &\leq C_d\cdot\frac{e_j^{-(d-1)}}{V\left(e_j\right)},\end{aligned} \tag{2.7}$$

where $C_d = 2^{d+1}\cdot d(d-1)^{\frac{d-1}{2}}$ and where the second inequality follows from the construction of $\mathfrak{F}_j$ as a union of $d$ rotations of the set $S_j$. Fix $T\geq 1$ and set $i_T=i(T)$ to be the unique index such that $V\left(e_{i_T}\right)\leq T \ < V\left(e_{i_T+1}\right)$. Notice that $\#\left(\mathfrak{F}_j\cap B_2(\mathbf{0},T)\right)=0$ for every $j \ > i_T$. Therefore, one has that

$$\frac{\#\left(\mathfrak{F}\cap B_2\left(\mathbf{0},T\right)\right)}{T^d} = \frac{\sum_{j=1}^{i_T}\#\left(\mathfrak{F}_j\cap B_2\left(\mathbf{0},T\right)\right)}{T^d} \underset{(2.7)}{\leq} C_d\cdot\sum_{j=1}^{+\infty}\frac{e_j^{-(d-1)}}{V\left(e_j\right)}\cdot \tag{2.8}$$

The right-hand side of inequality (2.8) converges by assumption. The choice of $T \ > 0$ was arbitrary, therefore, inequality (2.6) is proved and the forest $\mathfrak{F}$ has thus finite density. The proof is complete. ■

Proof (Corollary 2.1.3) Fix $\eta>0$. Applying Theorem 2.1.2 with $e_j=\frac{1}{2^j}$ and $V_0\left(\epsilon\right)=\epsilon^{-(d-1)}\cdot\ln\left(\epsilon^{-1}\right)\cdot\ln\ln\left(\epsilon^{-1}\right)^{1+\eta}$ yields the result. More explicitly, the deterministic construction of the corresponding dense forest is as follows: set

$$S_j=\left\{\left(k\cdot V_0\left(2^{-j}\right),\quad l_2\cdot\frac{2^{-j}}{\sqrt{d-1}},\quad ...,\quad l_d\cdot\frac{2^{-j}}{\sqrt{d-1}}\right):k\in\mathbb{Z}\backslash\left\{0\right\},l_2,...,l_d\in\mathbb{Z}\right\}$$

and $\mathfrak{F}=\bigcup_{j=1}^{+\infty}\bigcup_{l=1}^{d}R_l\left(S_j\right)$ with $R_l$ defined in (2.4). Given $\epsilon \ > 0$, let $i=i(\epsilon)$ be the unique index such that $e_i \leq \epsilon < e_{i-1}$. Then, it holds that

$$V(\epsilon) \quad = \quad O(V_0(e_i)) \quad = \quad \frac{V_0(e_i)}{V_0(e_{i-1})} \cdot O(V_0(\epsilon)) \quad = \quad O(V_0(\epsilon))$$

since, given $i \in \mathbb{N}$, one has that $\frac{V_0(e_i)}{V_0(e_{i-1})} \leq 2^{d+1}$. The proof of the corollary is complete. ■

Proof (Theorem 2.1.5) Fix a natural number $d \geq 2$. It is enough to prove the existence of an $\epsilon$-net $\mathcal{N}_\epsilon$ with $\#\mathcal{N}_\epsilon = \epsilon^{-1} \cdot \ln(\epsilon^{-1})$ in $\left(\mathcal{I}^d, \mathcal{B}'\right)$ for every $\epsilon \in \left\{2^{-(d-1)dn} : n \in \mathbb{N}\right\}$. For every $j \in \mathbb{N}$, define the sets

$$S_j = \left\{k \cdot 2^{(d-1)j}, \frac{l_2}{\sqrt{d-1} \cdot 2^{j+1}}, ..., \frac{l_d}{\sqrt{d-1} \cdot 2^{j+1}} : k \in \mathbb{Z} \backslash \{\mathbf{0}\}, l_2, ..., l_d \in \mathbb{Z}\right\}.$$

Given $l \in \{1, ..., d\}$, let $R_l : \mathbb{R}^d \to \mathbb{R}^d$ be the rotation defined in equation (2.4). Set also

$$\mathfrak{S}_j = \bigcup_{i=1}^{d} R_i(S_j) \quad \text{and} \quad \mathfrak{S} = \bigcup_{j \in \mathbb{N}} \mathfrak{S}_j. \tag{2.9}$$

The goal is to prove that every box in $\mathbb{R}^d$ with side lengths $s_1, ..., s_d$ such that

$$s_1 = ... = s_{d-1} \leq s_d$$

and with volume $2^d \cdot \sqrt{d}$ intersects $\mathfrak{S}$. To this end, fix such a box $B'$ in $\mathbb{R}^d$. Then, there exists $\epsilon > 0$ such that the sides of $B'$ have lengths $2^d \cdot \sqrt{d} \cdot \epsilon^{-(d-1)}, \epsilon, ..., \epsilon$. Define $j = j(\epsilon)$ to be the smallest natural number such that $2^{j-1} < \epsilon^{-1} \leq 2^j$. Then, the box $B'$ contains a box $B$ with sides of lengths $2 \cdot \sqrt{d} \cdot 2^{(d-1)j}, \frac{1}{2^j}, ..., \frac{1}{2^j}$. Define $L$ to be the line segment connecting the middle points of the two faces of $B$ which have sides of length $1/2^j$. Obviously, $L$ has length $2 \cdot \sqrt{d} \cdot 2^{j(d-1)}$. Therefore, $L$ contains at least one point $\boldsymbol{x} = \boldsymbol{x}(L)$ which has at least one coordinate equal to $k \cdot 2^{(d-1)j}$ for some $k = k(L) \in \mathbb{Z}$. By construction of the set $S_j$, there exists at least one point $\boldsymbol{y} \in S_j$ such that $||\boldsymbol{x} - \boldsymbol{y}||_2 \leq \frac{1}{2^{j+1}} \leq \epsilon$. Therefore, $\boldsymbol{y} \in B \subseteq B'$.

Thus, the claim is proved.

It is left to prove that the optical forest $\mathfrak{S}$ admits $O\left(T^d \cdot \ln(T)\right)$ as a growth rate bound. Applying inequality (2.7) to $e_j = 2^{-j}$ and to $V(\epsilon) = \epsilon^{-(d-1)}$ (that is, to $\mathfrak{F}_j = \mathfrak{S}_j$) yields that for every $T \geq 2^{(d-1)j}$,

$$\frac{\#\left(\mathfrak{S}_j \cap B_2\left(\mathbf{0}, T\right)\right)}{T^d} \quad \leq \quad C_d, \tag{2.10}$$

where $C_d = 2^{d+1} \cdot d(d-1)^{\frac{d-1}{2}}$. Fix $T \geq 1$ and set $i_T = i(T)$, which is the unique natural number such that $2^{i(d-1)} \leq T \ < 2^{(i+1)(d-1)}$. Notice that $\#\left(\mathfrak{S}_j \cap B_2\left(\mathbf{0}, T\right)\right) = 0$ for every $j \ > i_T$. Therefore, one has that

$$\frac{\#\left(\mathfrak{S} \cap B_2\left(\mathbf{0}, T\right)\right)}{T^d} \quad = \quad \frac{\sum_{j=1}^{i(T)} \#\left(\mathfrak{S}_j \cap B_2\left(\mathbf{0}, T\right)\right)}{T^d} \underset{(2.10)}{\leq} \frac{C_d}{d-1} \cdot \log_2(T) \cdot$$

Thus, this shows that $\#\left(\mathfrak{S} \cap B_2\left(\mathbf{0}, T\right)\right) = O\left(T^d \cdot \ln(T)\right)$.

As for the construction of an $\epsilon$-net $\mathcal{N}_\epsilon$ in $\left(\mathcal{I}^d, \mathcal{B}'\right)$ with growth rate $O\left(\epsilon^{-1} \cdot \ln\left(\epsilon^{-1}\right)\right)$, it is enough to construct it only in the case $\epsilon = \epsilon_n$, where $\epsilon_n = 2^{-(d-1)dn}$, $n \in \mathbb{N}$. To this end, fix $n \in \mathbb{N}$ and set $Q_n = \mathfrak{S} \cap \left[0, \tau_d(n)\right]^d$, where $\tau_d(n) = 2 \cdot d^{1/2d} \cdot 2^{(d-1)n}$. Furthermore, set

$$\mathcal{N}_{\epsilon_n} = \left\{ \left( \frac{x_1}{\tau_d(n)} - \frac{1}{2}, ..., \frac{x_d}{\tau_d(n)} - \frac{1}{2} \right) : (x_1, ..., x_d) \in Q_n \right\} \subseteq \mathcal{I}^d.$$

From the construction of the set $Q_n$, it follows easily that the set $\mathcal{N}_{\epsilon_n}$ intersects every box $\mathcal{B}'$ of volume larger than $\epsilon_n$ and, moreover, that $\#\mathcal{N}_{\epsilon_n} \ll 2^{(d-1)dn} \cdot \ln\left(2^{(d-1)dn}\right)$.

The proof is complete. ■

We conclude this section by showing that the optical forest defined in the proof of Theorem 2.1.5 is not a Danzer set for $d \geq 3$. Namely, fix $d \geq 3$ and let $\mathfrak{S} \subseteq \mathbb{R}^d$ be the optical forest defined in (2.9). The goal is to show that there are arbitrary large boxes in $\mathbb{R}^d$ which do not intersect $\mathfrak{S}$. To this end, for every $j \in \mathbb{N}$ set the box

$$B_j = \left\{ \begin{array}{ll} (x_1, x_2, ..., x_d) \in \mathbb{R}^d : & \dfrac{2^{-j-2}}{\sqrt{d-1}} \leq x_1 \leq \dfrac{2^{-j-2}}{\sqrt{d-1}} + \dfrac{2^{-j-3}}{\sqrt{d-1}} \\ & \text{and} \quad 0 \leq x_2, ..., x_d \leq 2^{(d-1)j-1} \end{array} \right\}.$$

It is readily checked that for any $j \in \mathbb{N}$ the box $B_j$ does not intersect the point set $\mathfrak{S}$. Moreover, for the volume of $B_j$ it holds that

$$Vol\left(B_j\right) = \frac{2^{-j-3}}{\sqrt{d-1}} \cdot \left(2^{(d-1)j-1}\right)^{d-1} = 2^{-(d+2)} \cdot \frac{2^{(d-1)^2 j - j}}{\sqrt{d-1}}.$$

Since it was assumed that $d \geq 3$, the claim follows by noticing that $Vol\left(B_j\right) \to +\infty$ when $j \to +\infty$.

## 2.3 Proof of Theorem 2.1.4

The proof of Theorem 2.1.4 is an adaptation of the proof of Theorem 1.1.54, p.62, which can be found in the work by Solomon and Weiss [74, Theorem 1.4]. In the context of optical forests, the original proof is simplified.

PROOF (THEOREM 2.1.4) 1. $\implies$ 2.**:** Given $d \geq 2$, assume that there exists an optical forest $\mathfrak{F} \subseteq \mathbb{R}^d$ with growth rate bound $g$. Without loss of generality, one can take $V(\epsilon) = \epsilon^{-(d-1)}$ as a visibility function $V$ for the forest $\mathfrak{F}$. Indeed, in order to meet this condition, one can work with the set $c \cdot \mathfrak{F} = \{c \cdot \boldsymbol{x} : \boldsymbol{x} \in \mathfrak{F}\}$ if necessary, where $c > 0$ a sufficiently small constant.

Fix $\epsilon \in (0,1)$ and let $B_\infty\left(\mathbf{0}, \left(\epsilon^{-(d-1)}/2\right)\right)$ be the box centred at the origin with side-length $\epsilon^{-(d-1)}$. Set

$$Q_\epsilon = \mathfrak{F} \cap B_\infty\left(\mathbf{0}, \left(\frac{\epsilon^{-(d-1)}}{2}\right)\right) \quad \text{and} \quad \mathcal{N}_\epsilon = \epsilon^{d-1} \cdot Q_\epsilon \subseteq \mathcal{I}^d.$$

By assumption, the set $Q_\epsilon$ contains $O\left(g\left(\epsilon^{-(d-1)}\right)\right)$ points and so does the set $\mathcal{N}_\epsilon$. By assumption and from the way that $\mathcal{N}_\epsilon$ is constructed, $\mathcal{N}_\epsilon$ intersects every box in $\mathcal{B}'$ with volume $\epsilon^{(d-1)d}$. Setting $\eta = \epsilon^{(d-1)d}$ yields that the set $\mathcal{N}_\epsilon$ is an $\eta$-net in $\left(\mathcal{I}^d, \mathcal{B}'\right)$ such that $\#\mathcal{N}_\epsilon = g\left(\eta^{-\frac{1}{d}}\right)$. The choice of $\epsilon \in (0,1)$ (and thus of $\eta \in (0,1)$)

is arbitrarily. Therefore, the claim is proved.

2. $\Longrightarrow$ 1.**:** Assume that for every $\epsilon \in (0,1)$, there exists an $\epsilon$-net $\mathcal{N}_\epsilon$ in $\left(\mathcal{I}^d, \mathcal{B}'\right)$ such that $\#\mathcal{N}_\epsilon \leq g\left(\epsilon^{-\frac{1}{d}}\right)$. In particular, by abusing slightly the notation, for every $i \in \mathbb{N}$, let $\mathcal{N}_i = \mathcal{N}_{\epsilon_i}$ be an $\epsilon_i$-net with $\epsilon_i = 1/2^{i\cdot(d-1)d}$. Moreover, for every $i \in \mathbb{N}$, set $B_i = B_\infty\left(\mathbf{0}, 2^{i(d-1)-1}\right)$, $D_i = B_\infty\left(\mathbf{0}, 2^{i(d-1)}\right)$ and

$$Q_i \quad = \quad 2^{i(d-1)} \cdot \mathcal{N}_i \quad \subseteq \quad B_i \tag{2.11}$$

where, by assumption, $\#Q_i \leq g\left(2^{i(d-1)}\right)$. By construction of the set $\mathcal{N}_i$, one has that any line segment $L \subseteq B_i$ with length $\epsilon^{-(d-1)} \leq 2^{i(d-1)}$ is $O(\epsilon)$-close to the point set $Q_i$.

Given any $i \in \mathbb{N}$, the set $D_{i+1} \backslash D_i$ can be tiled with the use of $2^{d^2} - 2^d$ hypercubes where each hypercube has side-length $2^{i(d-1)}$. Let $\left\{C_j^{(i)}\right\}_{j=1}^{2^{d^2}-2^d}$ be such a tiling. Each hypercube $C_j^{(i)}$ can be identified with the hypercube $B_i$ through a translation, that is,

$$C_j^{(i)} = B_i + \boldsymbol{a}_j^{(i)}, \tag{2.12}$$

for some vector $\boldsymbol{a}_j^{(i)} \in \mathbb{R}^d$. Set

$$Q_j^{(i)} = Q_i + \boldsymbol{a}_j^{(i)},$$

where the vector $\boldsymbol{a}_j^{(i)}$ is defined in equation (2.12). Thus, for every $j \in \left\{1, ..., 2^{d^2} - 2^d\right\}$, $Q_j^{(i)}$ is a copy of the set $Q_i$ inside the set $C_j^{(i)}$.

The goal is to prove that the set

$$\mathfrak{F} \quad = \quad \bigcup_{i=1}^{+\infty} \bigcup_{j=1}^{2^{d^2}-2^d} Q_j^{(i)} \tag{2.13}$$

is an optical forest with growth rate bound $O(g(T))$. To this end, fix a line segment $L$ with length $\mathcal{C}_d \cdot \epsilon^{-(d-1)}$, where $\mathcal{C}_d = 2^{d+1}(d+1)^2\sqrt{d}$. Then, $L$ contains a line segment $L' \subseteq L$ with length $\epsilon^{-(d-1)}$ which is contained in $C_j^{(i)}$, for some $i \in \mathbb{N}$ such that $\epsilon^{-(d-1)} \leq 2^{i(d-1)}$ and $j \in \left\{1, ..., 2^{d^2} - 2^d\right\}$. Note here that the choice of the natural number $i$ depends on the choice of the line segment $L$, that is, its position and length, and not only on the choice of the real number $\epsilon \in (0, 1)$. In other words, the natural number $i$ is not necessary the smallest integer such that $\epsilon^{-(d-1)} \leq 2^{i(d-1)}$. From the construction of the sets $Q_i$ and $Q_j^{(i)}$, the point set $\mathfrak{F}$ is $O(\epsilon)$-close to the line segment $L'$ and thus to the line segment $L$. Therefore, this establishes that $\mathfrak{F}$ is an optical forest.

It is left to prove that the point set $\mathfrak{F}$ admits $O(g(T))$ as a growth rate bound. To this end, fix $i \in \mathbb{N}$. Upon setting $T_i = 2^{i(d-1)}$, it holds that

$$\begin{aligned}
\#\left(\mathfrak{F} \cap B_\infty(\mathbf{0}, T_i)\right) \quad & \underset{(2.13)}{\leq} \quad \left(2^{d^2} - 2^d\right) \cdot \sum_{k=1}^{i-1} \# Q_k \\
& \underset{(2.11)}{\leq} \quad \left(2^{d^2} - 2^d\right) \cdot \sum_{k=1}^{i-1} g\left(2^{k(d-1)}\right) \\
& \underset{(2.2)}{\ll_d} \quad \sum_{k=1}^{i-1} \frac{g\left(2^{i(d-1)}\right)}{(1+c)^{(d-1)\cdot(i-k)}} \\
& \ll_c \quad g\left(2^{i(d-1)}\right) \quad \ll_C \quad g(T_i).
\end{aligned}$$

Finally, from the upper bound of the right-hand inequality of (2.2), it follows easily that a growth rate bound for the point set $\mathfrak{F}$ is $O(g(T))$. The proof is complete. ■

## 2.4 A Conjecture on the existence of Dense Forests with a given Visibility

Theorem 2.1.2 provides a strong sufficient condition for the existence of dense forests with a given visibility. Moreover, by adapting Alon's probabilistic argument [13] to higher dimensions and by optimising his method, one can expect the existence of a Delone dense forest in any dimension with a visibility bound of the same order as the one provided by Corollary 2.1.3. In view of this, it is tempting to conjecture that the converse to Theorem 2.1.2 holds true as well:

**Conjecture 2.4.1** *Let $V : (0, 1) \mapsto \mathbb{R}^+$ be a decreasing function such that $V(\epsilon) \to +\infty$ as $\epsilon \to 0^+$. If $\mathfrak{F}$ is a dense forest in $\mathbb{R}^d$ which admits $V$ as a visibility function,* *then*

$$\sum_{j\geq 1} \frac{2^{j(d-1)}}{V\left(1/2^j\right)} < +\infty. \tag{2.14}$$

An affirmative answer to Conjecture 2.4.1 immediately implies a negative answer to Danzer's problem. Indeed, applying the criterion (2.14) with $V(\epsilon) = \epsilon^{-(d-1)}$yields that

$$\sum_{j\geq 1} \frac{2^{j(d-1)}}{V\left(1/2^j\right)} = \sum_{j\geq 1} \frac{2^{j(d-1)}}{2^{j(d-1)}} = \sum_{j\geq 1} 1 = +\infty.$$

Therefore, one infers that there do not exist dense forests in $\mathbb{R}^d$ with visibility $O\left(\epsilon^{-(d-1)}\right)$. The claim follows upon noticing that every Danzer set in $\mathbb{R}^d$ with growth rate bound $O\left(T^d\right)$ is a dense forest with visibility $O\left(\epsilon^{-(d-1)}\right)$.

# Chapter 3

# Peres-Type Forests and Super-Uniform Dispersion

## 3.1 Introduction

Peres-type forests (Definition 1.1.10, p.11) are constructed with the help of a toral sequence $\boldsymbol{a}$. Their visibility properties depend on the properties of the sequence $\boldsymbol{a}$. Indeed, as will be justified in detail in this chapter (although closely related claims are known in the literature, see for instance [1, Theorem 8]), given a toral sequence $\boldsymbol{a}$, the visibility bound of the Peres-type forest $\mathfrak{F}(\boldsymbol{a})$ is related to the dispersion of the sequence $\boldsymbol{a}$. More precisely, the visibility properties of the forest $\mathfrak{F}(\boldsymbol{a})$ are precisely captured by the following newly introduced concept, which is a strengthening of that of dispersion (see Definition 1.1.19, p.20).

**Definition 3.1.1 (Super-Uniform Dispersion)** *Let $\boldsymbol{a}$ be a sequence in $\mathbb{T}$. Given a natural number $N$, the* Super-Uniform Dispersion of order $N$ *of the sequence $\boldsymbol{a}$ is defined as*

$$\Delta_{\boldsymbol{a}}(N) = \sup_{m\in\mathbb{N}_0} \sup_{\xi\in\mathbb{T}} \delta_{\boldsymbol{a}}(N, m, \xi), \tag{3.1}$$

*where*

$$\delta_{\boldsymbol{a}}(N, m, \xi) = \sup_{\gamma\in\mathbb{T}} \min_{j\in[\![N]\!]} ||\gamma - (a_{j+m} - j\xi)||. \tag{3.2}$$

*If $\Delta_{\boldsymbol{a}}(N) \underset{N\to+\infty}{\longrightarrow} 0$, then the sequence $\boldsymbol{a}$ is said to be* Super-Uniformly Dispersed. *Moreover, given a function $V : (0,1) \mapsto \mathbb{R}^+$ such that $V(\epsilon) \to +\infty$ when $\epsilon \to 0^+$,* *the sequence* $\boldsymbol{a}$ *is* $V$*-Super-Uniformly Dispersed, if for every* $\epsilon \in (0,1)$*, it holds that* $\Delta_{\boldsymbol{a}}(V(\epsilon)) \leq \epsilon$*; that is, if for any* $m \in \mathbb{N}_0$ *and* $\gamma, \xi \in \mathbb{T}$*, there exists* $j \in [\![V(\epsilon)]\!]$ *such that* $||\gamma - (a_{j+m} - j\xi)|| \leq \epsilon$*.*

The quantities (3.1) and (3.2) in the definition of super uniform dispersion impose uniformity both in the index parameter $m$ and in the parameter $\xi$ of the linear perturbation of the sequence. The definition of a $V$-super uniformly dispersed sequence is a quantitative refinement of the concept of (just) being super-uniformly dispersed.

The following result gives the connection between a $V$-super uniformly dispersed sequence $\boldsymbol{a}$ in $\mathbb{T}$ and the visibility bound of the Peres-type forest $\mathfrak{F}(\boldsymbol{a})$. It has already been established (see for instance [1, p.18, Theorem 8]); however, its proof is given in the next section for the sake of completeness.

**Theorem 3.1.2** *Let* $\boldsymbol{a}$ *be a* $V$*-super uniformly dispersed sequence in* $\mathbb{T}$*. The Peres-type forest* $\mathfrak{F}(\boldsymbol{a})$ *defined in [Chapter 1, Equation* (1.11)*, p.27] has finite density with visibility function* $W$*, where* $W(\epsilon) = 2^{\frac{3}{2}} \cdot V(\epsilon)$*.*

In view of Theorem 3.1.2, it is natural to ask how good the visibility bounds of a Peres-type forest can be; that is, given a $V$-super uniformly dispersed sequence, how small can the function $V$ be. From the definition of dispersion (Definition 1.1.19, p.20), for any finite toral sequence $\boldsymbol{a} = (a_k)_{k=1}^{N}$, one has that $\delta_{\boldsymbol{a}}(N) \geq 1/2N$, where equality holds if, and only if, the terms of $\boldsymbol{a}$ are successively equidistant in $\mathbb{T}$. Therefore, it is clear that for any toral $V$-super-uniformly dispersed sequence, it holds that $V(\epsilon) \gg \epsilon^{-1}$. The following result implies that there exists a $V$-super-uniformly dispersed sequence with the function $V$ being almost optimal; namely with $V(\epsilon) = O_\eta(\epsilon^{-1-\eta})$ for any $\eta > 0$.

**Theorem 3.1.3 (Probabilistic Super-Uniformly Dispersed Sequence)** *There exists a* $W$*-super uniformly dispersed sequence* $\boldsymbol{a}$ *in* $\mathbb{T}$ *with*

$$W(\epsilon) \quad = \quad O\left(\epsilon^{-1} \cdot \left(\ln\left(\frac{1}{\epsilon}\right) + \ln\ln\left(\frac{1}{\epsilon}\right)\right) \cdot 2^{O\left(\sqrt{-\log_2(\epsilon)}\right)}\right).$$

*As a consequence, there exists a dense forest of the form* (1.11) *(Chapter 1, p.27) with visibility* $O\left(\epsilon^{-1}\cdot\left(\ln\left(\frac{1}{\epsilon}\right)+\ln\ln\left(\frac{1}{\epsilon}\right)\right)\cdot 2^{O\left(\sqrt{-\log_2(\epsilon)}\right)}\right)$ *in the plane.*

As far as the deterministic constructions of Peres-type forests are concerned, the best known result comes from Peres' original construction. Peres [24] specialises construction (1.11) (Chapter 1, p.27) to the case where

$$a_n \quad = \quad \begin{cases} \frac{n}{2}\cdot\phi & \text{if } n\in 2\mathbb{N} \\ 0 & \text{if } n\in 2\mathbb{N}-1 \end{cases} \tag{3.3}$$

with $\phi=\frac{1+\sqrt{5}}{2}$ the golden ratio. He then proves that the resulting dense forest $\mathfrak{F}(\boldsymbol{a})$ has visibility $O\left(\epsilon^{-4}\right)$, providing this way the first example of a deterministic dense forest in the literature (this construction was actually introduced in [24] in a problem of rectifiability of curves). A more careful analysis carried out in [4] shows that the same forest has visibility $O\left(\epsilon^{-3}\right)$, yielding this way the best known fully deterministic dense forest in the plane. In view of the generalised Peres' construction defined in (1.10) (Chapter 1, p.26) with the help of uniformly Diophantine type vectors (Definition 1.1.48, p.56), the forest $\mathfrak{F}\left(\boldsymbol{a}\right)$ corresponds to the forest $\mathfrak{F}\left(\boldsymbol{\Theta}_{2,1}\right)$ with $\boldsymbol{\Theta}_{2,1}=(0,\phi)$ (see also Figure 1.1.3, p.27).

Digital sequences are integer sequences defined from the digits in the expansion of a real number in a given integer base (see for instance [40, Chapter 1.4.3]). The following result is concerned with the effective construction of a digital $V$-super-uniformly dispersed sequence with $V(\epsilon)=O_\eta\left(\epsilon^{-2-\eta}\right)$ for every $\eta>0$. In view of Theorem 3.1.2, this yields the best known deterministic planar Peres-type forest (surpassing the bound $O\left(\epsilon^{-3}\right)$ in Peres' construction).

**Theorem 3.1.4 (Deterministic Super Uniformly Dispersed Sequence)** *There exists a deterministic $V$- super uniformly dispersed sequence in $\mathbb{T}$ with*

$$V(\epsilon) \quad = \quad O\left(\epsilon^{-2}\cdot 2^{O\left(\sqrt{-\ln(\epsilon)}\right)}\right).$$

*As a consequence, one can construct a deterministic dense forest of the form* (1.11)

*(Chapter 1, p.27) with visibility* $O\left(\epsilon^{-2} \cdot 2^{O\left(\sqrt{-\ln(\epsilon)}\right)}\right)$ *in the plane.*

We end this section by stating a result showing the existence of a large class of $V$-super uniformly dispersed sequences in the unit torus with the function $V$ equal to $O\left(\epsilon^{-1-\eta}\right)$, where $\eta > 0$ can be chosen arbitrarily.

**Theorem 3.1.5** *Given $s \geq 2$ and a vector $\mathbf{\Theta}_{s,1} = (a_1, a_2, ..., a_s) \in \mathbb{R}^n$, define the real number*

$$\theta = \sum_{j=1}^{+\infty} \sum_{i=1}^{s} \frac{\lfloor ((j-1) \cdot s + i) \cdot \{ja_i\} \rfloor}{((j-1) \cdot s + i)!}. \tag{3.4}$$

*Then, for almost every choice of the vector $\mathbf{\Theta}_{s,1}$ (with respect to the Lebesgue measure), the sequence*

$$(\theta \cdot (k-1)!)_{k=1}^{+\infty} \tag{3.5}$$

*is $O_{\theta,\eta}\left(\epsilon^{-\left(1+\frac{2}{s-1}+\eta\right)}\right)$-super uniformly dispersed for any $\eta > 0$.*

Notice that in the statement of Theorem 3.1.5, as $\eta \to 0^+$ and $s \to +\infty$, the exponent of the visibility function tends to the optimal value $-1$.

In the same way as for the notion of super-uniform dispersion, one can define the notion of super-uniform discrepancy by replacing the quantity $\delta_{\boldsymbol{a}}(N, m, \xi)$ in Definition 3.1.1 with the quantity $d_{\boldsymbol{a}}(N, m, \xi)$ given in (1.22) (Chapter 1, p.37). When it comes to the study of dense forests problems, the benefits of working with super-uniform discrepancy lie on the better understanding of the concept of discrepancy. Indeed, one can then use a variety of analytical methods (for instance Weyl's criterion, the Erdös-Turán inequality, Koksma's inequality) to study the discrepancy of a sequence. Such methods are not available to study the concept of dispersion. The interested reader is referred to Appendix A for more details on this concept of super-uniform discrepancy newly introduced here.

The chapter is organised as follows. In Section 3.2 the proof of Theorem 3.1.2 is given. In Section 3.3, we develop the apparatus which leads to the proofs of

Theorems 3.1.3 and 3.1.4 in Sections 3.4 and 3.5, respectively. Theorem 3.1.5 is proved in Section 3.6.

## 3.2 Visibility Bounds in a Peres-Type Forest

Proof (Theorem 3.1.2) Let $\boldsymbol{a}$ be a $V$-super-uniformly dispersed sequence, where $V:(0,1)\to\mathbb{R}^{+}$. Let us prove first that the point set $\mathfrak{F}(\boldsymbol{a})=\mathfrak{F}_1(\boldsymbol{a})\cup\mathfrak{F}_2(\boldsymbol{a})$ (as defined in [Chapter 1, Equation (1.11), p.27]) has finite density. To this end, it is enough to prove that the point set $\mathfrak{F}_1(\boldsymbol{a})$ has finite density, since the point set $\mathfrak{F}_2(\boldsymbol{a})$ is obtained from a $\pi/2$-rotation of the first set.

Let $B_\infty(\mathbf{0},T)$ be the ball (with respect to the sup norm) centred at the origin with radius $T>0$. From the construction of the point sets $\mathfrak{F}_j(\boldsymbol{a}),j\in\{1,2\}$, one has that

$$\#\left(\mathfrak{F}_j(\boldsymbol{a})\cap B_\infty(\mathbf{0},T)\right)\le 4\cdot(T+1)^2.$$

Since $B_2(\mathbf{0},T)\subseteq B_\infty(\mathbf{0},T)$, it holds that

$$\begin{aligned}\#\left(\mathfrak{F}(\boldsymbol{a})\cap B_2(\mathbf{0},T)\right) &\le \#\left(\mathfrak{F}(\boldsymbol{a})\cap B_\infty(\mathbf{0},T)\right)\\ &\le \#\left(\mathfrak{F}_1(\boldsymbol{a})\cap B_\infty(\mathbf{0},T)\right)+\#\left(\mathfrak{F}_2(\boldsymbol{a})\cap B_\infty(\mathbf{0},T)\right)\\ &\le 8\cdot(T+1)^2.\end{aligned}$$

Therefore, given $T\ge1$,

$$\frac{\#\left(\mathfrak{F}(\boldsymbol{a})\cap B_2(\mathbf{0},T)\right)}{T^2}\le 8\cdot\frac{(T+1)^2}{T^2}\le 32.$$

This implies that the point set $\mathfrak{F}(\boldsymbol{a})$ has finite density.

The main idea to estimate the visibility bound of the point set $\mathfrak{F}(\boldsymbol{a})$ is that the set $\mathfrak{F}_1(\boldsymbol{a})$ is a dense forest for those line segments with slope $|\xi|\le1$ and that the set $\mathfrak{F}_2(\boldsymbol{a})$ is a dense forest for those line segments with slope $|\xi|\ge1$. Indeed, denote by

$$P_+=\{(x,y):x\ge0\}\quad\text{and}\quad P_-=\{(x,y):x\le0\}$$

the right and the left semi-planes of $\mathbb{R}^2$, respectively. Fix $\xi\in[-1,1]$, $b,m'\in\mathbb{R}$

and $\epsilon \in (0,1)$. Define the line segment

$$L' \quad = \quad \{(x, \xi x + b) : x \in [m', m' + 2V(\epsilon)]\}.$$

At least half of any such line segment $L'$ lies in one of the semi-planes $P_+$ and $P_-$, say without loss of generality $P_+$. In other words, there exists $m \in \mathbb{N}_0$ such that the line segment

$$L \quad = \quad \{(x, ax + b) : x \in [m, m + V(\epsilon)]\}$$

is contained in $P_+$. From the definition of super-uniform dispersion, there exists $k \in [\![V(\epsilon)]\!]$ such that $||a_{m+k} - k\xi - (m\xi + b)|| \leq \epsilon$. Equivalently, there exists $l \in \mathbb{Z}$ such that

$$|a_{m+k} - k\xi - (m\xi + b) + l| \leq \epsilon.$$

Therefore, from the construction of the point set $\mathfrak{F}_1(\boldsymbol{a})$, one has that

$$\begin{aligned}\operatorname{dist}(\mathfrak{F}_1(\boldsymbol{a}), L) &\leq ||(m+k, a_{m+k} + l) - (m+k, (m+k)\xi + b)||_2 \\ &= |a_{m+k} - (m+k)\xi - b + l| \\ &\leq \epsilon.\end{aligned}$$

The length of the line segment $L'$ is at most $2^{\frac{3}{2}} \cdot V(\epsilon)$. The choice of $\xi, b, m$ and $\epsilon$ is arbitrary; therefore, it has just been proved that for each line segment $L'$ of length at least $2^{\frac{3}{2}} \cdot V(\epsilon)$ with slope $\xi \in [-1,1]$, it holds that $\operatorname{dist}(\mathfrak{F}_1(\boldsymbol{a}), L') \leq \epsilon$. The point set $\mathfrak{F}_2(\boldsymbol{a})$ is obtained by a $\pi/2$-rotation of the point set $\mathfrak{F}_1(\boldsymbol{a})$. Therefore, for every line segment $L'$ with length at least $2^{\frac{3}{2}} \cdot V(\epsilon)$ and slope $|\xi| \geq 1$, it holds that $\operatorname{dist}(\mathfrak{F}_2(\boldsymbol{a}), L') \leq \epsilon$. As a consequence, the point set $\mathfrak{F}(\boldsymbol{a}) = \mathfrak{F}_1(\boldsymbol{a}) \cup \mathfrak{F}_2(\boldsymbol{a})$ is a dense forest with visibility $2^{\frac{3}{2}} \cdot V(\epsilon)$.

The proof is complete. ∎

## 3.3 Construction of Super-Uniformly Dispersed Sequences

The goal of this section is to provide an efficient way to construct super-uniformly dispersed sequences. Throughout this section, an integer $n \geq 1$ will be decomposed as

$$n = k \cdot 2^i + 2^{i-1} - 2 \quad \text{with} \quad i \geq 1 \quad \text{and} \quad k \geq 0. \tag{3.6}$$

The existence and uniqueness of decomposition of $n$ is guaranteed by the dyadic decomposition of $n+2$.

The following proposition shows that, given a function $V : (0,1) \to \mathbb{R}^+$ such that $V(\epsilon) \ll \epsilon^{-\alpha}$ for some $\alpha > 0$, if for every $\epsilon > 0$, one can construct a sequence $\boldsymbol{c}_\epsilon$ in $\mathbb{T}$ such that $\Delta_{\boldsymbol{c}_\epsilon}(V(\epsilon)) \leq \epsilon$, then one can also construct a $W$-super uniformly dispersed sequence $\boldsymbol{b}$ with $W(\epsilon) = O_\eta(V(\epsilon) \cdot \epsilon^{-\eta})$, for any $\eta > 0$.

**Proposition 3.3.1** *Let $V : (0,1] \to \mathbb{R}^+$ be a decreasing function such that $V(\zeta) \geq \frac{1}{\zeta}$ for every $\zeta \in (0,1]$. Let $\boldsymbol{c}^{(i)} = \left(c_k^{(i)}\right)_{k \in \mathbb{N}} \in \mathbb{T}^{\mathbb{N}}$, $i \in \mathbb{N}$, be a family of sequences in $\mathbb{T}$ such that upon setting $V_i = V\left(\frac{1}{2^{i^2}}\right) \in \mathbb{R}^+$, it holds that*

$$\Delta_{\boldsymbol{c}^{(i)}}(V_i) \quad \leq \quad \frac{1}{2^{i^2}} \quad \text{for all } i \geq 1 \tag{3.7}$$

*(the quantities $\Delta_{c^{(i)}}(V_i)$ are defined in Definition 3.1.1). Then, the sequence $(b_n)_{n \in \mathbb{N}}$ with*

$$b_n = c_k^{(i)},$$

*where the integers $n, k, i$ are related by relation (3.6), is $W$- super uniformly dispersed. Here,*

$$W(\epsilon) \quad = \quad 2^{i+2} \cdot \frac{V_i}{V_{i-1}} \cdot V(\epsilon)$$

*with $i = i(\epsilon)$ the unique index such that $2^{-i^2} \leq \epsilon < 2^{-(i-1)^2}$.*

Proof Set $(b_n)_{n \in \mathbb{N}}$ as in the statement and $\epsilon_i = 1/2^{i^2}$ for every $i \in \mathbb{N}_0$, i.e. $i = \sqrt{-\log_2 \epsilon_i}$. The goal is to show that the sequence $\boldsymbol{b} = (b_n)_{n \in \mathbb{N}}$ is $W$- super uniform dispersed. Fix $\epsilon > 0$, $\xi, \gamma \in \mathbb{T}$ and $m \in \mathbb{N}_0$. There exists a unique

$i = i(\epsilon) \in \mathbb{N}$ such that $\epsilon_i \leq \epsilon < \epsilon_{i-1}$ and a minimal natural number $m_0 \in \mathbb{N}$ such that $m_0 \cdot 2^i + 2^{i-1} - 2 \geq m$. By assumption (3.7), there exists $j \in [\![1, V_i]\!]$ such that

$$\left|\left|c^{(i)}_{j+m_0} - \xi_0 \cdot j - \gamma_0\right|\right| \quad \leq \quad \epsilon_i,$$

where $\xi_0 = \xi \cdot 2^i$ and $\gamma_0 = \xi \cdot m_0 2^i + \xi 2^{i-1} - 2\xi - m\xi + \gamma$. In turn, by setting $j' = j \cdot 2^i + 2^{i-1} - 2 + m_0 \cdot 2^i - m$, from the way that the sequence $\boldsymbol{b}$ and the quantities $\xi_0, \gamma_0, m_0$ are defined, one infers that

$$||b_{j'+m} - \xi \cdot j' - \gamma|| \quad \leq \quad \epsilon_i.$$

Since the choice of $\xi, \gamma \in \mathbb{T}$ and $m \in \mathbb{N}_0$ is arbitrary, it is left to prove that $j' \leq W(\epsilon)$. To this end, notice that

$$1 \leq j' = j \cdot 2^i + 2^{i-1} - 2 + m_0 \cdot 2^i - m \leq 2^i \cdot V_i + 2^{i-1} + 2^i \leq 2^i \cdot V_i + \frac{3}{2} \cdot 2^i \leq 4 \cdot 2^i \cdot V_i,$$

since $m_0 \cdot 2^i - m \leq 2^i$ and $2^{i^2} \leq V_i$. Thus, $j' \in [\![2^i \cdot V_i + 3 \cdot 2^{i-1}]\!] \subseteq [\![2^{i+2} \cdot V_i]\!]$. From the monotonicity of the function $V$ it follows that

$$W(\epsilon) \leq 2^{i+2} \cdot V_i \leq 2^{i+2} \cdot \frac{V_i}{V_{i-1}} \cdot V(\epsilon).$$

The proof is complete. ■

In practice, Proposition 3.3.1 will be applied in combination with the following lemma. Given a finite sequence $\boldsymbol{a} = (a_k)_{k=1}^{V}$ and $\epsilon \in (0,1)$ such that for any $\xi \in \mathbb{T}$, $\delta_{\boldsymbol{a}}(V, 0, \xi) \leq \epsilon$, this lemma allows one to construct a sequence $\boldsymbol{c}$ such that $\Delta_{\boldsymbol{c}}(2 \cdot V) \leq \epsilon$. In other words, in view of Remark 1.1.27 (p.37) which states that, with respect to the Haar measure of $\mathbb{T}^{\mathbb{N}}$, almost no sequence is well-distributed and almost every sequence has an empty spectrum (Definition 1.1.26, p.37), one dismisses the index parameter $m \in \mathbb{N}_0$, which is the parameter that makes the construction of super-uniformly dispersed sequences difficult.

Given two finite sequences $\boldsymbol{\alpha} = \{\alpha_i\}_{i=1}^{a}$ and $\boldsymbol{\beta} = \{\beta_j\}_{j=1}^{b}$ the concatenated sequence $\boldsymbol{\alpha} \triangleright \boldsymbol{\beta}$ is defined as the finite sequence $\boldsymbol{\gamma} = \{\gamma_k\}_{k=1}^{a+b}$ where, for every

$k \in [\![1, a+b]\!]$,

$$\gamma_k = \begin{cases} \alpha_k & \text{if } k \in [\![1, a]\!] \\ \beta_{k-a} & \text{if } k \in [\![a+1, a+b]\!] \end{cases}. \tag{3.8}$$

The operation of concatenation of sequences is associative. Therefore, given a sequence of finite sequences $(\boldsymbol{a}_n)_{n\in\mathbb{N}}$, the infinite concatenation $\triangleright_{n=1}^{+\infty}\boldsymbol{a}_n$ is well defined.

**Lemma 3.3.2** *Let $V \in \mathbb{R}^+, \epsilon \in (0,1)$ be real numbers and $\boldsymbol{a} = (a_k)_{k\in[\![V]\!]}$ be a finite sequence in $\mathbb{T}$. Let also $\boldsymbol{c} = \triangleright_{n=1}^{+\infty}\boldsymbol{a}$ be the sequence obtained by concatenating the sequence $\boldsymbol{a}$ infinite many times with itself. If*

$$\sup_{\xi\in\mathbb{T}} \delta_{\boldsymbol{a}}(V, 0, \xi) \leq \epsilon,$$

*then*

$$\Delta_{\boldsymbol{c}}(2V) \leq \epsilon.$$

PROOF Assume that the finite sequence $\boldsymbol{a} = (a_k)_{k\in[\![V]\!]}$ is such that for every $\xi \in \mathbb{T}$, it holds that $\delta_{\boldsymbol{a}}(V, 0, \xi) \leq \epsilon$ for some $\epsilon \in (0,1)$.

For every $k \in \mathbb{N}$, decompose $k$ as $k = j \cdot \lfloor V \rfloor + k'$ for some $j \in \mathbb{N}_0$ and $k' \in [\![V]\!]$. This decomposition is unique. Set the sequence $\boldsymbol{c} = \triangleright_{n=1}^{+\infty}\boldsymbol{a}$; that is, $c_k = a_{k'}$ for every $k \in \mathbb{N}$. It then follows that $\Delta_{\boldsymbol{c}}(2V) \leq \epsilon$. Indeed, fix $m \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$. There exists $j' \in \mathbb{N}_0$ such that $[\![j' \cdot \lfloor V \rfloor, j' \cdot \lfloor V \rfloor + V]\!] \subseteq [\![m, m+2V]\!]$. Therefore, by assumption, there exists $k' \in [\![V]\!]$ such that

$$||a_{k'} - k'\xi - j' \lfloor V \rfloor \cdot \xi|| \leq \epsilon.$$

This implies that

$$||c_k - k\xi|| \leq \epsilon$$

with $k = j' \lfloor V \rfloor + k' \in [\![m, m+2V]\!]$. The choice of $m$ and $\xi$ was arbitrary. The proof is therefore complete. ■

An easy application of the techniques developed above (in the form of Proposition 3.3.1 and Lemma 3.3.2) yields the construction of a $V$-super uniformly dispersed sequence with $V(\epsilon) = O_\eta(\epsilon^{-3-\eta})$ for any $\eta > 0$. More precisely:

**Corollary 3.3.3** *There exists a deterministic $V$-super uniformly dispersed sequence $\boldsymbol{b} = (b_n)_{n\in\mathbb{N}}$ in $\mathbb{T}$ with $V(\epsilon) = O\left(\epsilon^{-3} \cdot 2^{O\left(\sqrt{-\log(\epsilon)}\right)}\right)$; namely,*

$$b_n \quad = \quad a_k^{(i)} \quad \text{with } k, i \quad \text{ as in } (3.6)$$

*and where, given $i \in \mathbb{N}$, $\boldsymbol{a}^{(i)} = \left(a_k^{(i)}\right)_{k\in\left[\!\left[8^{i^2+1}\right]\!\right]}$ is a finite sequence defined as*

$$a_k^{(i)} = \frac{j}{4^{i^2+1}} \cdot l + \frac{l}{2^{i^2+1}} \quad \text{with} \quad k \equiv j \cdot 2^{i^2+1} + l \pmod{8^{i^2+1}}, \tag{3.9}$$

*where $j \in \left[\!\left[0, 4^{i^2+1} - 1\right]\!\right]$ and $l \in \left[\!\left[2^{i^2+1}\right]\!\right]$.*

Proof For every $i \in \mathbb{N}$, let $\boldsymbol{a}^{(i)}$ be the sequence defined in equation (3.9). It is enough to prove that for every $i \in \mathbb{N}$, it holds that

$$\sup_{\xi\in\mathbb{T}} \delta_{\boldsymbol{a}^{(i)}}\left(8^{i^2+1}, 0, \xi\right) \quad \leq \quad \frac{1}{2^{i^2}}.$$

Indeed, the result then follows upon applying first Lemma 3.3.2 and then Proposition 3.3.1.

To this end, fix $\xi, \gamma \in \mathbb{T}$ and $i \in \mathbb{N}$. Set $\boldsymbol{a} = \boldsymbol{a}^{(i)}$ for the sake of simplicity. The goal is to show that there exists $k \in \left[\!\left[8^{i^2+1}\right]\!\right]$ such that $||a_k - k\xi - \gamma|| \leq \frac{1}{2^{i^2}}$. It is easy to check that there exists $j \in \left[\!\left[0, 4^{i^2+1} - 1\right]\!\right]$ such that

$$\left|\left|\xi - \frac{j}{4^{i^2+1}}\right|\right| \quad \leq \quad \frac{1}{4^{i^2+1}}. \tag{3.10}$$

Choose $l \in \left[\!\left[2^{i^2+1}\right]\!\right]$ such that

$$\left|\left|\frac{l}{2^{i^2+1}} - \gamma - j2^{i^2+1} \cdot \xi\right|\right| \leq \frac{1}{2^{i^2+1}}. \tag{3.11}$$

For $k = j \cdot 2^{i^2+1} + l \in \left[\!\left[8^{i^2+1}\right]\!\right]$, one has that

$$\begin{aligned} ||a_k - k\xi - \gamma|| &= \left\|\frac{j}{4^{i^2+1}} \cdot l + \frac{l}{2^{i^2+1}} - j2^{i^2+1} \cdot \xi - l\xi - \gamma\right\| \\ &\leq l \cdot \left\|\frac{j}{4^{i^2+1}} - \xi\right\| + \left\|\frac{l}{2^{i^2+1}} - \gamma - j2^{i^2+1} \cdot \xi\right\| \\ &\underset{(3.10)\&(3.11)}{\leq} \frac{l}{4^{i^2+1}} + \frac{1}{2^{i^2+1}} \underset{l \leq 2^{i^2+1}}{\leq} \frac{1}{2^{i^2}}. \end{aligned}$$

The proof is complete. ■

## 3.4 Probabilistic $O_\eta\left(\epsilon^{-1-\eta}\right)$-Super Uniformly Dispersed Sequence

The goal of this section is to prove Theorem 3.1.3. To this end, in view of Proposition 3.3.1 and Lemma 3.3.2, it is enough to establish the following proposition.

**Proposition 3.4.1** *Given $\zeta \in (0,1)$, there exists a finite sequence $\boldsymbol{a}_\zeta = (a_k)_{k=1}^{V_\zeta}$ such that*

$$\sup_{\xi \in \mathbb{T}} \delta_{\boldsymbol{a}_\zeta}(V_\zeta, 0, \xi) \leq \zeta,$$

*where*

$$V_\zeta = \frac{1}{\zeta} \cdot \left(\ln\left(\frac{500}{\zeta^3}\right) + \ln\ln\left(\frac{500}{\zeta^3}\right)\right). \tag{3.12}$$

Proof Fix $\zeta \in (0,1)$ and set $V = \lfloor V_\zeta \rfloor$. Let $(\Omega, \lambda_V)$ be a probability space, where $\Omega = \mathbb{T}^V$ and $\lambda_V$ is the product (Lebesgue) measure on $\mathbb{T}^V$. Consider the set

$$A := \left\{\frac{j}{4} \cdot \zeta \in \mathbb{T} : j \in [\![0, 4/\zeta]\!]\right\} \subseteq \mathbb{T}.$$

Given $w \in [\![0, V]\!]$ and $a', b \in A$, define the finite sequence $\boldsymbol{L} = (l_k)_{k \in [\![V]\!]}$ where, for every $k \in [\![V]\!]$,

$$l_k = ak + b \quad \text{with} \quad a = \frac{w}{V} + \frac{a'}{V}. \tag{3.13}$$

Let also $\Lambda$ be the set of all sequences of the form (3.13).

We will show below that there exists a sequence $\boldsymbol{a} = (a_k)_{k\in[\![V]\!]}$ such that for every sequence $\boldsymbol{L}$ of the form (3.13), there exists $k_0 \in [\![V]\!]$ such that

$$||a_{k_0} - l_{k_0}|| \quad \leq \quad \frac{\zeta}{2}. \tag{3.14}$$

This implies that $\sup_{\xi\in\mathbb{T}} \delta_{\boldsymbol{a}}(V,0,\xi) \leq \zeta$. Indeed, fix $\xi, \gamma \in \mathbb{T}$. There exist $p, q \in [\![0, 4/\zeta]\!]$ and $n \in [\![0, V]\!]$ such that

$$\frac{n}{V} + \frac{p}{4V}\cdot\zeta \quad \leq \quad \xi \quad \leq \quad \frac{n}{V} + \frac{p+1}{4V}\cdot\zeta \quad \text{and} \quad \frac{q}{4}\cdot\zeta \quad \leq \quad \gamma \quad \leq \quad \frac{q+1}{4}\cdot\zeta.$$

Set $a = \frac{n}{V} + \frac{p\zeta}{4V}$, $b = \frac{q\zeta}{4}$ and the sequence $\boldsymbol{L}$ as in (3.13) . By assumption, there exists $k_0 \in [\![V]\!]$ such that inequality (3.14) holds. Therefore, applying the triangle inequality, one has that

$$\begin{aligned} ||a_{k_0} - k_0\xi - \gamma|| \quad &\leq \quad ||a_{k_0} - l_{k_0}|| + k_0 \cdot \left\|\xi - \frac{n}{V} - \frac{p\zeta}{4V}\right\| + \left\|b - \frac{q}{4}\zeta\right\| \\ &\underset{(3.14)}{\leq} \quad \frac{\zeta}{2} + k_0\cdot\frac{\zeta}{4V} + \frac{\zeta}{4} \quad \underset{k\leq V}{\leq} \quad \zeta. \end{aligned}$$

The choice of $\xi \in \mathbb{T}$ is arbitrary; therefore, one has that $\sup_{\xi\in\mathbb{T}} \delta_{\boldsymbol{a}}(V,0,\xi) \leq \zeta$.

Thus, it remains to prove the existence of a sequence $\boldsymbol{a}$ satisfying inequality (3.14) for every sequence $\boldsymbol{L} \in \Lambda$. There exist at most $50V/\zeta^2$ sequences in $\Lambda$: indeed, there are at most $(4/\zeta)+1$ choices for the parameter $b$ and at most $(V+1)\cdot((4/\zeta)+1)$ choices for the parameter $a$. Therefore, one has at most

$$\left(\frac{4}{\zeta}+1\right)^2 \cdot (V+1) \quad \leq \quad \frac{50V}{\zeta^2}$$

choices for the sequence $\boldsymbol{L} \in \Lambda$.

Given $\boldsymbol{L} \in \Lambda$ define the set $E_{\boldsymbol{L}}$ of all those finite sequences $(x_k)_{k=1}^{V}$ in $\mathbb{T}$ such that for every $k \in [\![V]\!]$, it holds that $||x_k - l_k|| \geq \zeta/2$. In other words,

$$E_{\boldsymbol{L}} \quad = \quad \left\{(x_k)_{k\in[\![V]\!]} \in \mathbb{T}^{\lfloor V\rfloor} : \quad \forall k \in [\![V]\!]\,, \quad ||x_k - l_k|| \,> \frac{\zeta}{2}\right\}.$$

One has that

$$\lambda_V(E_{\boldsymbol{L}}) \quad \leq \quad (1-\zeta)^V \quad \leq \quad e^{-\zeta V}. \tag{3.15}$$

Moreover,

$$\begin{aligned}
\lambda_V\left(\bigcup_{\boldsymbol{L}\in\Lambda} E_{\boldsymbol{L}}\right) \quad &\leq \quad \sum_{\boldsymbol{L}\in\Lambda}\lambda_V(E_{\boldsymbol{L}}) \quad \leq \quad \epsilon^{-\zeta V}\cdot\left(\frac{50V}{\zeta^2}\right)\\
&\leq \quad \frac{1}{e^{\ln\left(\frac{500}{\zeta^3}\right)+\ln\ln\left(\frac{500}{\zeta^3}\right)}}\cdot\frac{50}{\zeta^3}\cdot\left(\ln\left(\frac{500}{\zeta^3}\right)+\ln\ln\left(\frac{500}{\zeta^3}\right)\right)\\
&\leq \quad \frac{\ln\left(\frac{500}{\zeta^3}\right)+\ln\ln\left(\frac{500}{\zeta^3}\right)}{10\cdot\ln\left(\frac{500}{\zeta^3}\right)} \quad < \quad 1.
\end{aligned}$$

Therefore, the set $E=\Omega\backslash\bigcup_{\boldsymbol{L}\in\Lambda}E_{\boldsymbol{L}}$ is not a null set and is thus non-empty. Since $E$ is non-empty, for every $k\in[\![V]\!]$ there exists a choice $a_k\in\mathbb{T}$ such that $(a_k)_{k\in[\![V]\!]}\in E$. In other words, for every $\boldsymbol{L}\in\Lambda$, there exist $k_0\in[\![V]\!]$ such that inequality (3.14) holds. The proof is complete. ■

Proof (Theorem 3.1.3) For every $i\in\mathbb{N}$, set $\zeta(i)=2^{-i^2}$. Given $i\in N$, let $\boldsymbol{a}_i=\boldsymbol{a}_{\zeta(i)}$ be the finite sequence obtained from Proposition 3.4.1. The theorem follows upon applying Lemma 3.3.2 and Proposition 3.3.1 to the sequences $\boldsymbol{a}_i$, $i\in\mathbb{N}$. The proof is complete. ■

## 3.5 Deterministic $O_\eta\left(\epsilon^{-2-\eta}\right)$-Super Uniformly Dispersed Sequence

The goal of this section is to prove Theorem 3.1.4. A sequence which satisfies the statement of the theorem, denote it by $\boldsymbol{u}=(u_n)_{n\geq1}$, is defined as follows: first, decompose (throughout this section) the integer $n\geq1$ into the integers $k\in\mathbb{Z}$ and $i\in\mathbb{N}$ as in relation (3.6) and the integer $k$ as

$$k \quad \equiv \quad r\cdot2^{i^2}+s \pmod{2\cdot2^{2i^2}} \quad \text{with} \quad 0\leq r\leq2\cdot2^{i^2}-1 \quad \text{and} \quad 1\leq s\leq2^{i^2}.$$

Then, $\boldsymbol{u}$ is given for all $n \geq 1$ by

$$u_n \quad = \quad \begin{cases} \frac{rs}{2^{2i^2}} & \text{if } 0 \leq r \leq 2^{i^2}-1, \\ \frac{rs}{2^{2i^2}} + \frac{s}{2^{i^2}} & \text{if } 2^{i^2} \leq r \leq 2 \cdot 2^{i^2} - 1. \end{cases} \tag{3.16}$$

To prove Theorem 3.1.4, one needs the following proposition.

**Proposition 3.5.1** *Given $i \in \mathbb{N}$, decompose the natural number $k \in [\![2^{2i+1}]\!]$ as*

$$k = r \cdot 2^i + s, \quad \textit{with} \quad 0 \leq r \leq 2^{i+1}-1 \quad \textit{and} \quad 1 \leq s \leq 2^i. \tag{3.17}$$

*The finite sequence $\boldsymbol{a}^{(i)} = \left(a_k^{(i)}\right)_{k=1}^{2^{2i+1}}$, where*

$$a_k^{(i)} \quad = \quad \begin{cases} \frac{rs}{2^{2i}} & \textit{if } 0 \leq r \leq 2^i - 1, \\ \frac{rs}{2^{2i}} + \frac{s}{2^i} & \textit{if } 2^i \leq r \leq 2 \cdot 2^i - 1 \end{cases}, \tag{3.18}$$

*is such that*

$$\sup_{\xi \in \mathbb{T}} \delta_{\boldsymbol{a}^{(i)}} \left(2^{2i+1}, 0, \xi\right) \quad \leq \quad 1/2^i.$$

Proof Fix $i \in \mathbb{N}$ and set $V_i = 2^{2i+1}$. Let $\boldsymbol{a}^{(i)} = (a_k)_{k \in [\![V_i]\!]}$ be the finite sequence defined in equation (3.18). Decompose every $k \in [\![V_i]\!]$ as in equation (3.17).

Let us prove that for every $\xi \in \mathbb{T}$, one has that $\delta_{\boldsymbol{a}^{(i)}}(V_i, 0, \xi) \leq \frac{1}{2^i}$ . The first step is to show that for every $\xi' \in \mathbb{T}$ of the form

$$\xi' \quad = \quad \frac{l}{2^i} + \frac{l'}{2^{2i}}, \quad \text{where} \quad l \in \left[\!\left[0, 2^i - 1\right]\!\right] \quad \text{and} \quad l' \in \left[\!\left[0, 2^i - 1\right]\!\right], \tag{3.19}$$

it holds that $\delta_{\boldsymbol{a}^{(i)}}(V_i, 0, \xi') \leq 1/2^{i+1}$. To see this, fix such $\xi' \in \mathbb{T}$ and also $\gamma \in \mathbb{T}$. If $l$ is odd, then for every $k'$ of the form $k' = l' \cdot 2^i + j \in [\![l' \cdot 2^i + 1, (l'+1) \cdot 2^i]\!]$, where $j \in [\![2^i]\!]$, one has

$$||k'\xi' + \gamma - a_{k'}|| \quad = \quad \left|\left|\frac{l'^2}{2^i} + \gamma + \frac{jl}{2^i}\right|\right|.$$

Since $l$ is odd, one can find $j_0 \in [\![2^i]\!]$ such that

$$\left\|\frac{l'^2}{2^i} + \gamma + \frac{j_0 \cdot l}{2^i}\right\| \quad \leq \quad \frac{1}{2^{i+1}}.$$

Similarly, if $l$ is even, then for every $k'$ of the form

$$k' \quad = \quad \left(2^i + l'\right) \cdot 2^i + j \quad \in \quad \left[\!\left[\left(2^i + l'\right) \cdot 2^i + 1, \left(2^i + l' + 1\right) \cdot 2^i\right]\!\right],$$

where $j \in [\![2^i]\!]$, one has

$$||k'\xi' + \gamma - a_{k'}|| \quad = \quad \left\|\frac{l'^2 - 1}{2^i} + \gamma + \frac{j \cdot (l-1)}{2^i}\right\|.$$

Since $l - 1$ is odd, there is a choice of $j_0 \in [\![2^i]\!]$ such that

$$\left\|\frac{l'^2 - 1}{2^i} + \gamma + \frac{j_0 \cdot (l-1)}{2^i}\right\| \quad \leq \quad \frac{1}{2^{i+1}}.$$

Fix now any $\xi, \gamma \in \mathbb{T}$. Then, there exists $\xi' = \frac{l}{2^i} + \frac{l'}{2^{2i}}$ such that $||\xi - \xi'|| \leq 2^{-(2i+1)}$. Therefore, setting $m_0 = l'$ if $l$ is odd and $m_0 = 2^i + l'$ if $l$ is even, there exists $j_0 \in [\![2^i]\!]$ such that the integer $k = m_0 \cdot 2^i + j_0$ satisfies the relation

$$||a_k - k\xi' - \gamma_0|| \quad \leq \quad \frac{1}{2^{i+1}},$$

where $\gamma_0 = m_0 2^i \cdot (\xi - \xi') + \gamma$. From the Triangle Inequality,

$$\begin{aligned}
||a_k - k\xi - \gamma|| \quad &= \quad ||a_k - k\xi' - k(\xi - \xi') - \gamma|| \\
&\leq \quad \left\|a_k - k\xi' - \gamma - m_0 \cdot 2^i \cdot (\xi - \xi')\right\| + j_0 \cdot ||\xi - \xi'|| \\
&\leq \quad \frac{1}{2^{i+1}} + \frac{2^i}{2^{2i+1}} \leq \frac{1}{2^i}.
\end{aligned}$$

Thus, it has been established that $\delta_{\boldsymbol{a}^{(i)}}(V_i, 0, \xi) \leq \frac{1}{2^i}$. The proof is complete. ■

Proof (Theorem 3.1.4) For every $i \in \mathbb{N}$, define the finite sequence $\boldsymbol{c}^{(i)} = \boldsymbol{a}^{\left(i^2\right)}$, where the sequence $\boldsymbol{a}^{\left(i^2\right)}$ is inferred from Proposition 3.5.1. The theorem follows upon applying Lemma 3.3.2 and then Proposition 3.3.1 to the sequences $\boldsymbol{c}^{(i)}, i \in \mathbb{N}$ with $V_i = 2^{2i^2+1}$. Indeed, set $V(\epsilon) = 2 \cdot \epsilon^{-2}$. The sequence obtained from Proposition 3.3.1 is the sequence $\boldsymbol{u}$ defined in (3.16). Proposition 3.3.1 yields that the sequence $\boldsymbol{u}$ is $W$-super uniformly dispersed with

$$W(\epsilon) \leq 4 \cdot 2^{i'} \cdot \frac{V_{i'+1}}{V_{i'}} \cdot V(\epsilon) = O\left(\epsilon^{-2} \cdot 2^{O\left(\sqrt{-\ln(\epsilon)}\right)}\right),$$

where $i' = i'(\epsilon)$ is the unique index such that $1/2^{i'^2} \leq \epsilon < 1/2^{(i'-1)^2}$.

The proof is complete. ■

## 3.6 Super-Uniform Dispersion and Diophantine-Type Vectors

The goal of this section is to prove Theorem 3.1.5. The proof will be achieved in a series of auxiliary results. Let $\boldsymbol{\Theta}_{s,1} = (a_1, ..., a_s)$ be a given uniformly Diophantine vector of type $\Phi$ (Definition 1.1.48, p.56) for some $s \geq 2$ and a non-increasing function $\Phi : \mathbb{R}^+ \mapsto \mathbb{R}^+$ tending to zero at infinity. In view of the bound given by Lemma 1.1.52, the following lemma allows one to construct, with the help of $\boldsymbol{\Theta}_{s,1}$, a $V$-super uniformly dispersed sequence in $\mathbb{T}$ satisfying

$$V(\epsilon) \quad = \quad O\left(\epsilon \cdot \Phi\left(2 \cdot \epsilon^{-1}\right)^{-1}\right).$$

**Lemma 3.6.1** *Let $s \geq 2$ be a natural number and $\boldsymbol{a}^{(i)} = \left(a_j^{(i)}\right)_{j\in\mathbb{N}}, i \in \{1, 2, ..., s\}$, be a family of sequences in the unit torus. Assume that for each $\epsilon > 0$, $\xi \in \mathbb{T}$ and $m \in \mathbb{N}_0$, there exists $i = i(\epsilon, \xi) \in \{1, .., s\}$ such that*

$$\delta_{\boldsymbol{a}^{(i)}}\left(V(\epsilon), m, \xi\right) \quad \leq \quad \epsilon, \tag{3.20}$$

*where $V : (0, 1) \to \mathbb{R}^+$ is such that $V(\epsilon) \underset{\epsilon\to 0}{\longrightarrow} +\infty$. Then, the sequence $\boldsymbol{b} = (b_k)_{k\in\mathbb{N}}$,* *where*

$$b_k = a_j^{(i)} \quad \text{with } k = (j-1)\cdot s + i \quad \text{for some } i \in \{1, ..., s\} \text{ and } j \in \mathbb{N}, \tag{3.21}$$

*is $O(V)$-super uniformly dispersed.*

Proof Fix $\epsilon > 0$, $m \in \mathbb{N}_0$ and $\xi, \gamma \in \mathbb{T}$. Without loss of generality, assume that $m = s \cdot m'$ for some $m' \in \mathbb{N}_0$ since, otherwise, one may work with the function $W(\epsilon) = V(\epsilon)+s$. There exist $l \in \{0, 1, ..., s-1\}$ and $\xi' \in [0, 1)$ such that $\xi = \frac{l}{s}+\frac{\xi'}{s}$. By assumption (3.20), there exist $i = i(\epsilon, \xi') \in \{1, ..., s\}$ and $j \in [\![V(\epsilon)]\!]$ such that

$$\left|\left|a_{j+m'}^{(i)} - j\xi' - \gamma_0\right|\right| \quad \leq \quad \epsilon, \tag{3.22}$$

where $\gamma_0 = \frac{li}{s} + \left(m' - 1 + \frac{i}{s}\right)\xi' + \gamma$. Therefore, by setting $k = (j + m' - 1)\, s + i$, one has that

$$||b_k - k\xi - \gamma|| \underset{(3.21)}{=} \left|\left|a_{j+m'}^{(i)} - j\xi' - \frac{li}{s} - \left(m' - 1 + \frac{i}{s}\right)\xi' - \gamma\right|\right| \underset{(3.22)}{\leq} \epsilon.$$

Obviously, $k \in [\![m+1, m+sV(\epsilon)]\!]$. Thus, it has just been proved that the sequence $\boldsymbol{b}$ is $O((s+1)\cdot V)$-super-uniformly dispersed. ■

Let $\boldsymbol{\Theta}_{s,1} = (a_1, a_2, ..., a_s)$ be a uniformly Diophantine vector of type $\Phi$. Applying Lemma 1.1.52 (p.59) to $\boldsymbol{\Theta}_{s,1}$ and Lemma 3.6.1 to $\boldsymbol{a}^{(i)} = (j \cdot a_i)_{j\in\mathbb{N}}$, $i \in \{1, .., s\}$, yields the following corollary.

**Corollary 3.6.2** *Let $s \geq 2$ be a natural number and assume that $\boldsymbol{\Theta} = (a_1, ..., a_s) \in UDT_s^d(\Phi)$ for some positive function $\Phi$ as in Definition 1.1.48 (p.56). Then, the sequence $\boldsymbol{b} = (b_k)_{k\in\mathbb{N}}$, where*

$$b_k = j \cdot a_i \quad \text{with } k = (j-1)\cdot s + i \text{ for some } i \in \{1, ..., s\} \text{ and } j \in N, \tag{3.23}$$

*is $W$-super uniformly dispersed, with $W$ such that*

$$W(\epsilon) \quad = \quad O_s\left(\Phi\left(\epsilon^{-1}\right)^{-1}\right).$$

By applying Theorem 1.1.49 (p.57) with $d = 1$, $\eta > 0$ and

$$\Phi(T) = T^{-\left(\frac{s+1}{s-1}+\eta\right)},$$

one obtains the following optimised version of Corollary 3.6.2.

**Corollary 3.6.3** *Given $s \geq 2$ and $\eta > 0$, for almost every choice $\mathbf{\Theta} = (a_1, a_2, ..., a_s) \in \mathbb{R}^s$ (with respect to the Lebesgue measure in $\mathbb{R}^s$), the sequence $\boldsymbol{b} = (b_k)_{k\in\mathbb{N}}$, where*

$$b_k = j \cdot a_i \quad \text{with} \quad k = (j-1)\cdot s + i \quad \text{for some} \quad i \in \{1, ..., s\} \quad \text{and} \quad j \in \mathbb{N}, \tag{3.24}$$

*is $W$-super uniformly dispersed, with $W$ such that*

$$W(\epsilon) = O_{\mathbf{\Theta},\eta}\left(\epsilon^{-\left(1+\frac{2}{s-1}+\eta\right)}\right).$$

The proof of Theorem 3.1.5 follows immediately from Corollary 3.6.3 upon applying the following lemma. Let $\boldsymbol{a}, \boldsymbol{b}$ be two super-uniformly dispersed sequences in $\mathbb{T}$ with dispersion bounds $V_{\boldsymbol{a}}$ and $V_{\boldsymbol{b}}$, respectively. The lemma shows that if the terms of the two sequences are close enough in a suitable sense, then the function $V_{\boldsymbol{b}}$ can be bounded in terms of the function $V_{\boldsymbol{a}}$. Furthermore, the sequence $\boldsymbol{b}$ can always be chosen to be of the form (3.5).

**Lemma 3.6.4** *Let $\boldsymbol{a} = (a_k)_{k\in\mathbb{N}}$ be a $V$-super uniformly dispersed sequence in $\mathbb{T}$. Assume that the sequence $\boldsymbol{b} = (b_k)_{k\in\mathbb{N}}$ is such that*

$$||a_k - b_k|| \leq \frac{C}{k} \tag{3.25}$$

*for some positive constant $C$. Then, the sequence $\boldsymbol{b}$ is $W$-super uniformly dispersed, where*

$$W(\epsilon) = 2 \cdot V\left(\frac{\epsilon}{2C+1}\right)$$

*for any $\epsilon > 0$. In particular, set $\boldsymbol{b} = (b_k)_{k\in\mathbb{N}}$ with*

$$b_k = \theta \cdot (k-1)!, \quad \text{where} \quad \theta = \sum_{i=1}^{+\infty} \frac{\lfloor i \cdot \{a_i\} \rfloor}{i!}. \tag{3.26}$$

*Then, the sequence $\boldsymbol{b}$ is $W$-super uniformly dispersed, with $W$ defined as*

$$W(\epsilon) \quad = \quad 2V\left(\frac{\epsilon}{9}\right).$$

Proof Fix $\epsilon > 0$, $\xi, \gamma \in \mathbb{T}$ and $m \in \mathbb{N}_0$. Since $\boldsymbol{a}$ is $V$-super uniformly dispersed, there exists $i \in [\![ V(\epsilon/2C+1), 2V(\epsilon/2C+1) ]\!]$ such that

$$||a_{m+i} - i\xi - \gamma|| \quad \leq \quad \frac{\epsilon}{2C+1}. \tag{3.27}$$

By applying the Triangle Inequality, one obtains

$$\begin{aligned}
||b_{m+i} - i\xi - \gamma|| \quad & \leq \quad ||a_{m+i} - i\xi - \gamma|| + ||b_{m+i} - a_{m+i}|| \\
& \underset{(3.25)\&(3.27)}{\leq} \quad \frac{\epsilon}{2C+1} + \frac{C}{m+i} \\
& \underset{(1.17)}{\leq} \quad \frac{\epsilon}{2C+1} + \frac{2C \cdot \epsilon}{2C+1} \quad = \quad \epsilon.
\end{aligned}$$

The choice of $\xi, \gamma$ and $m$ is arbitrary. Therefore, the sequence $\boldsymbol{b}$ is $W$-super uniformly dispersed with $W(\epsilon) = 2 \cdot V(\epsilon/(2C+1))$.

As for the special case where $\boldsymbol{b}$ is defined as in (3.26), it is enough to prove that

$$||\theta \cdot (k-1)! - a_k|| \quad \leq \quad \frac{4}{k},$$

as the result follows from the previous part. To this end, note that

$$\begin{aligned}
||\theta \cdot (k-1)! - a_k \quad || \quad & \underset{(3.26)}{=} \quad \left|\left|(k-1)! \cdot \sum_{i=1}^{+\infty} \frac{\lfloor i \cdot \{a_i\} \rfloor}{i!} - a_k\right|\right| \\
& = \quad \left|\left|(k-1)! \cdot \sum_{i=k}^{+\infty} \frac{\lfloor i \cdot \{a_i\} \rfloor}{i!} - a_k\right|\right| \\
& \leq \quad \left|\left|\frac{\lfloor k \cdot \{a_k\} \rfloor}{k} - a_k\right|\right| + \sum_{i=k+1}^{+\infty} \frac{(k-1)! \cdot \lfloor i \cdot \{a_i\} \rfloor}{i!} \\
& \leq \quad \frac{1}{k} + \frac{e}{k} \quad \leq \quad \frac{4}{k}.
\end{aligned}$$

Here, the second inequality follows from the relations

$$\begin{aligned}\left|\sum_{i=k+1}^{+\infty} \frac{(k-1)! \cdot \lfloor i \cdot \{a_i\} \rfloor}{i!}\right| \quad &\leq \quad \sum_{i=k+1}^{+\infty} \frac{(k-1)! \cdot i}{i!} \\ &\leq \quad \frac{1}{k} \cdot \sum_{i=0}^{+\infty} \frac{1}{i!} \quad = \quad \frac{e}{k}.\end{aligned}$$

The proof is complete. ∎

# Chapter 4

# Higher Dimensional Spiral Delone Sets

## 4.1 Introduction

The distance in the $d$-dimensional torus $\mathbb{T}^d$ is denoted by $||\cdot||$ while the Euclidean and the supremum norms in $\mathbb{R}^{d+1}$ are denoted by $||\cdot||_2$ and $||\cdot||_\infty$, respectively. Throughout this chapter the integer $d+1$, where $d \geq 1$, stands for the dimension of the ambient Euclidean space. The unit sphere of $\mathbb{R}^{d+1}$ is denoted by

$$\mathbb{S}^d \quad = \quad \left\{ \boldsymbol{x} \in \mathbb{R}^{d+1} : ||\boldsymbol{x}||_2 = 1 \right\}.$$

Given $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d$, write

$$\mathrm{dist}_{\mathbb{S}^d}(\boldsymbol{u}, \boldsymbol{v}) \quad = \quad \arccos(\boldsymbol{u} \cdot \boldsymbol{v}) \tag{4.1}$$

for the geodesic length between $\boldsymbol{u}$ and $\boldsymbol{v}$, where $\boldsymbol{x} \cdot \boldsymbol{y}$ stands for the usual scalar product between two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d+1}$. It is easily checked that the geodesic distance is equivalent to the one induced by the Euclidean norm $||\cdot||_2$; indeed, for all $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d$,

$$||\boldsymbol{u} - \boldsymbol{v}||_2 \quad \leq \quad \mathrm{dist}_{\mathbb{S}^d}(\boldsymbol{u}, \boldsymbol{v}) \quad \leq \quad 2\pi \cdot ||\boldsymbol{u} - \boldsymbol{v}||_2 . \tag{4.2}$$

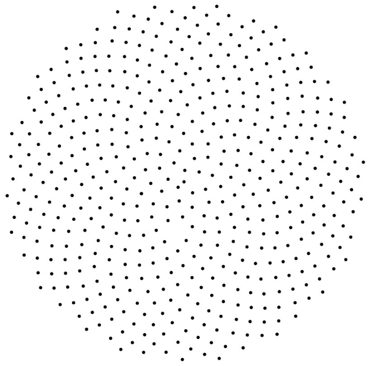



Figure 4.1: The sunflower and Fermat's spiral.

Recall that a subset $\mathfrak{Y}$ of the Euclidean space is Delone (cf. Definition 1.1.8, p.24) if it is both uniformly discrete and relatively dense. In other words, there exist constants $s, r > 0$ such that the distance between any two distinct points of $\mathfrak{Y}$ is bounded from below by $s$ (uniform discreteness) and such that every ball of the form $B_2(\boldsymbol{x}, r), \boldsymbol{x} \in \mathbb{R}^{d+1}$, intersects $\mathfrak{Y}$ non-trivially (relative density).

A spiral set in $\mathbb{R}^{d+1}$ is a point set defined by a sequence of the form

$$\mathfrak{S}(\boldsymbol{U}) = \left\{ \sqrt[d+1]{k} \cdot \boldsymbol{u}_k \right\}_{k \in \mathbb{N}}, \tag{4.3}$$

where $\boldsymbol{U} = (\boldsymbol{u}_k)_{k \in \mathbb{N}}$ is a sequence in $\mathbb{S}^d$.

Planar spirals have been particularly studied because they serve as theoretical models for phyllotaxis configurations (i.e. for configurations of leaves on a plant stem). A well-known example, used to model the *sunflower*, is *Fermat's spiral* (see Figure 4.1 [1]):

$$\left(\sqrt{k} \cdot e(k \cdot \phi)\right)_{k \in \mathbb{N}}, \quad \text{where } \phi = \frac{1+\sqrt{5}}{2} \quad \text{is the golden ratio.} \tag{4.4}$$

[1]The left-hand side photo is in colour in the original copy of the thesis.

The goal of this chapter is to study the Delone property of spiral sets. The problem is well-understood in the plane (i.e. when $d = 1$). In this case, the 1-dimensional unit sphere $\mathbb{S}^1$ can be identified with the unit torus $\mathbb{T}$ through the exponential map and thus one can exploit the underlying group structure (see [Chapter 1, Equation (1.13), p. (1.13)]). Indeed, any sequence $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ in $\mathbb{S}^1$ can be represented in a unique way as $(e(u_k))_{k\in\mathbb{N}}$, where $(u_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{T}$. Akiyama [9, Lemma 1] noticed that a necessary condition for a point set of the form $\{g(k)\cdot e(u_k)\}_{k\in\mathbb{N}}$, where $(u_k)_{k\in\mathbb{N}}$ denotes a sequence in $\mathbb{T}$ and $g$ is a strictly increasing real function, to be Delone is that $\sqrt{k} \ll g(k) \ll \sqrt{k}$. His observation was only made for the planar case but it is not hard to extend this to higher dimensions: if the point set $\{g(k)\cdot \boldsymbol{u}_k\}_{k\in\mathbb{N}}$ is Delone, where $(\boldsymbol{u}_k)_{k\in\mathbb{N}}$ is a sequence in $\mathbb{S}^d$, then

$$0 \quad < \quad \liminf_{k\to+\infty} \frac{g(k)}{\sqrt[d+1]{k}} \quad \leq \quad \limsup_{k\to+\infty} \frac{g(k)}{\sqrt[d+1]{k}} \quad < \quad +\infty. \tag{4.5}$$

This claim motivates the choice of the function $g(k) = \sqrt[d+1]{k}$ in the definition of a spiral in (4.3) and will be further justified in detail in Section 4.2.

After proving condition (4.5) for $d = 1$ and for a general toral sequence, Akiyama used the Three Distance Theorem and proved that the planar spiral $\mathfrak{S}_\alpha = \left\{\sqrt{k}\cdot e(k\alpha)\right\}_{k\in\mathbb{N}}$ is Delone if, and only if, the real number $\alpha$ is badly-approximable [9, Theorem 3] (for more details on the Three Distance Theorem see the paper of Alessandri [10]). The sufficiency of bad approximability (see Definition 1.1.36, p.45) in this equivalence was already noticed by Yudin [82] who established it with the help of analytical tools. A special case of interest, which is related to considerations in phyllotaxis, is when $\alpha$ equals the golden ratio $\phi$. Akiyama's result yields that the Fermat's spiral (4.4) illustrated in Figure 4.1 is Delone.

Soon after, Marklof [59] undertook a more general study of planar spiral sets $\mathfrak{S}_{\boldsymbol{u}} = \left\{\sqrt{k}\cdot e(u_k)\right\}_{k\in\mathbb{N}}$ and proved necessary and sufficient conditions on the sequence $\boldsymbol{u} = (u_k)_{k\in\mathbb{N}}$ for $\mathfrak{S}_{\boldsymbol{u}}$ to be Delone. To state his result, given parameters $h > 0$ and $R > 0$, let the quantities

$$g_R^h \quad \text{and} \quad G_R^h$$

denote respectively the minimal and the maximal gap in the unit torus between the points of the set $\{u_k : R^2 \leq k < (R+h)^2\}$. Then, Marklof [59, Proposition 2] establishes that $\mathfrak{S}_{\boldsymbol{u}}$ is Delone if, and only if, there exist $h, h' > 0$ such that

$$\inf_{R\geq 1} R \cdot g_R^h > 0 \quad \text{and} \quad \sup_{R\geq 1} R \cdot G_R^{h'} < +\infty.$$

Therefore, the Delone property of the spiral $\mathfrak{S}_{\boldsymbol{u}}$ is related to the spacing of the sequence $\boldsymbol{u}$ in the unit torus. Akiyama's result for spirals of the form $\left(\sqrt{k} \cdot e\left(k\alpha\right)\right)_{k\in\mathbb{N}}$ (with $\alpha$ being badly approximable) follows as a special case of this equivalence.

In [9, Section 4], Akiyama raises the question as to whether this planar theory can be extended to higher dimensions and also how to obtain higher dimensional spiral Delone sets. The same question appears in [61], where the problem of constructing an analogue of Fermat's spiral (4.4) in higher dimensions is left open.

The two main results of this chapter aim to answer this question. The first theorem (Theorem 4.1.4) generalises Marklof's work; more specifically, it characterises spiral Delone sets of the form (4.3) in terms of the spacing properties of the spherical sequence $\boldsymbol{U}$. The result will be expressed in terms of the packing and covering properties of the spherical sequence $\boldsymbol{U}$ introduced in the following definitions. In the context of this chapter, the compact metric space $(\mathcal{K}, \mathrm{dist}_{\mathcal{K}}(\cdot,\cdot))$ below will be either the $d$-dimensional unit sphere $\mathbb{S}^d$ equipped with the geodesic distance (4.1) or the $d$-dimensional torus $\mathbb{T}^d$ equipped with the toral distance $||\cdot||$.

**Definition 4.1.1 (Packing and Covering Radius)** *Let $(\mathcal{K}, dist_{\mathcal{K}}(\cdot,\cdot))$ be a compact metric space and let $A$ be a finite subset of $\mathcal{K}$. Define the* packing *(resp. the* covering*)* radius *of $A$ to be the quantity*

$$\mathtt{R}_P(A) = \inf_{\substack{\boldsymbol{x},\boldsymbol{y}\in A\\ \boldsymbol{x}\neq\boldsymbol{y}}} dist_{\mathcal{K}}(\boldsymbol{x},\boldsymbol{y}) \quad \left(resp.\quad \mathtt{R}_C(A) = \sup_{\boldsymbol{x}\in\mathcal{K}} \inf_{\boldsymbol{y}\in A} dist_{\mathcal{K}}(\boldsymbol{x},\boldsymbol{y})\right). \tag{4.6}$$

**Definition 4.1.2 (Uniform Packing and Covering Parameters)** *Let $(\mathcal{K}, dist_{\mathcal{K}}(\cdot,\cdot))$ be a compact metric space and let $\boldsymbol{V} = (\boldsymbol{v}_k)_{k\in\mathbb{N}}$ be an infinite sequence in $\mathcal{K}$.*

*The* uniform packing parameter *of* $\boldsymbol{U}$ *is defined as the quantity*

$$\mathtt{U}_P(\boldsymbol{V}) \quad = \quad \inf_{m\geq 0} \quad \inf_{N\in\mathbb{N}} \quad \sqrt[d]{N}\cdot \mathtt{R}_P\left(\{\boldsymbol{v}_{m+k}\}_{k=1}^{N}\right). \tag{4.7}$$

*Similarly, the* uniform covering parameter *of* $\boldsymbol{U}$ *is defined to be the quantity*

$$\mathtt{U}_C(\boldsymbol{V}) \quad = \quad \sup_{m\geq 0} \quad \sup_{N\geq 1} \quad \sqrt[d]{N}\cdot \mathtt{R}_C\left(\{\boldsymbol{v}_{m+k}\}_{k=1}^{N}\right). \tag{4.8}$$

**Definition 4.1.3 (Optimally Distributed Sequences)** *Let* $(\mathcal{K}, dist_{\mathcal{K}}(\cdot,\cdot))$ *be a compact metric space and let* $\boldsymbol{V} = (\boldsymbol{v}_k)_{k\in\mathbb{N}}$ *be an infinite sequence in* $\mathcal{K}$*. The sequence* $\boldsymbol{V}$ *is said to be* optimally distributed *if*

$$0 \quad < \quad \mathtt{U}_P(\boldsymbol{V}) \quad \leq \quad \mathtt{U}_C(\boldsymbol{V}) \quad < \quad +\infty. \tag{4.9}$$

**Theorem 4.1.4 (Necessary and sufficient condition for the Delone property)** *Let* $\boldsymbol{U}$ *be a sequence in* $\mathbb{S}^d$*. Then, the point set* $\mathfrak{S}(\boldsymbol{U})$ *defined in* (4.3) *is:*

1. *uniformly discrete if, and only if, there exists a constant* $h > 0$ *such that*

$$\inf_{R\geq 1} R\cdot g_R^h(\boldsymbol{U}) \quad > \quad 0, \tag{4.10}$$

   *where* $g_R^h(\boldsymbol{U}) := \mathtt{R}_P\left(\left\{\boldsymbol{u}_k : R^{d+1}\leq k < (R+h)^{d+1}\right\}\right)$.

2. *relatively dense if, and only if, there exists a constant* $h > 0$ *such that*

$$\sup_{R\geq 1} R\cdot G_R^h(\boldsymbol{U}) \quad < \quad +\infty, \tag{4.11}$$

   *where* $G_R^h(\boldsymbol{U}) := \mathtt{R}_C\left(\left\{\boldsymbol{u}_k : R^{d+1}\leq k < (R+h)^{d+1}\right\}\right)$.

It is readily checked that an optimally distributed sequence always satisfies both conditions (4.10) and (4.11) (for any choice of a positive constant $h$).

Theorem 4.1.4 reduces the problem of constructing higher dimensional spiral Delone sets to constructing a sequence in $\mathbb{S}^d$ satisfying conditions (4.10) and (4.11). The following result provides the (deterministic) construction of sequences in $\mathbb{S}^d$

satisfying the stronger condition of being optimally distributed, settling this way Akiyama's question.

**Theorem 4.1.5** *Let $d \geq 1$. Then, there exists a deterministic sequence $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ in $\mathbb{S}^d$ such that the point set $\mathfrak{S}(\boldsymbol{U})$ defined in* (4.3) *is Delone. Furthermore, the sequence $\boldsymbol{U}$ is optimally distributed.*

The proof of Theorem 4.1.5 requires the construction of an optimally distributed sequence. Such sequences can be constructed naturally in the $d$-dimensional unit torus $\mathbb{T}^d$ by taking the multiples of a badly approximable vector $\boldsymbol{\alpha}$ (see Theorem B.1.1, Appendix B, p.250). The construction of an optimally distributed sequence in $\mathbb{S}^d$ will be achieved by mapping an optimally distributed sequence in the $d$-dimensional torus to the $d$-dimensional unit sphere in such a way that the sequence obtained through the mapping is also optimally distributed. A *gap-preserving map* is the natural tool for this goal to be achieved since it preserves properties (4.9), (4.10) and (4.11).

**Definition 4.1.6 (Gap-Preserving Map)** *Let $(\mathcal{K}_1, d_1)$, $(\mathcal{K}_2, d_2)$ be two compact metric spaces. A map $\sigma : \mathcal{K}_1 \mapsto \mathcal{K}_2$ is gap-preserving if for any finite subset $A \subseteq \mathcal{K}_1$, it holds that*

$$\mathtt{R}_P\,(A) \quad \ll \quad \mathtt{R}_P\,(\sigma(A)) \quad \text{and} \quad \mathtt{R}_C\,(\sigma(A)) \quad \ll \quad \mathtt{R}_C(A). \tag{4.12}$$

To prove Theorem 4.1.5, we will take $(\mathcal{K}_1, d_1(\cdot,\cdot)) = \left(\mathbb{T}^d, ||\cdot||\right)$ and $(\mathcal{K}_2, d_2(\cdot,\cdot)) = \left(\mathbb{S}^d, \mathrm{dist}_{\mathbb{S}^d}(\cdot,\cdot)\right)$ in the definition of a gap-preserving map.

In Section 4.2, we first justify the claim (4.5) concerning the choice of the factor $\sqrt[d+1]{k}$ in the definition (4.3) of a spiral and then prove Theorem 4.1.4. The proof of Theorem 4.1.5 is given in Section 4.3 where we build two gap-preserving maps $\tau_N, \tau_S : \mathbb{T}^d \to \mathbb{S}^d$ from the $d$-dimensional torus to the northern and southern hemispheres of $\mathbb{S}^d$, respectively. The result follows by properly intertwining the maps $\tau_N, \tau_S$ to lift to the sphere $\mathbb{S}^d$ the sequence of multiples of a badly approximable vector in $\mathbb{T}^d$ provided by Theorem B.1.1 (Appendix B, p.250).

## 4.2 On the Delone Property of Spiral Sets

Let $\mathfrak{S} = \{g(k) \cdot \boldsymbol{u}_k\}_{k\in\mathbb{N}}$ be a subset of $\mathbb{R}^{d+1}$, where $g : \mathbb{N} \to \mathbb{R}^+$ is a strictly increasing function and $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ a sequences in $\mathbb{S}^d$. The claim (4.5) on the growth rate of the stretching factor $g$ for $\mathfrak{S}$ to be Delone is first established.

Proof (Bounds (4.5)) Assume that the set $\mathfrak{S}$ is relatively dense. Then, there exists a constant $r > 0$ such that

$$\bigcup_{\boldsymbol{x}\in\mathfrak{S}} B_2(\boldsymbol{x}, r) \quad = \quad \mathbb{R}^{d+1}.$$

Since

$$S(k) := \quad \{\boldsymbol{x} \in \mathfrak{S} : ||\boldsymbol{x}||_2 \leq g(k)\}$$

has cardinality $k$ and

$$\bigcup_{\boldsymbol{x}\in S(k)} B_2(\boldsymbol{x}, r) \quad \supseteq \quad B_2(\boldsymbol{0}, g(k) - r)$$

for $g(k) \gg r$, it follows that $(g(k) - r)^{d+1} \ll k \cdot r^{d+1}$, which implies

$$\frac{g(k)}{\sqrt[d+1]{k}} \quad \ll \quad r.$$

Thus, the right-hand side of inequality (4.5) is proved.

If $\mathfrak{S}$ is $2s$-uniformly discrete, then $B_2(\boldsymbol{x}, s)$ are disjoint disks for any $\boldsymbol{x} \in \mathfrak{S}$. From the inclusion $S(k) \subseteq B_2(\boldsymbol{0}, g(k))$, one obtains that

$$\bigcup_{\boldsymbol{x}\in S(k)} B_2(\boldsymbol{x}, s) \quad \subseteq \quad B_2(\boldsymbol{0}, g(k) + s),$$

which leads one to $k \cdot s^{d+1} \ll (g(k) + s)^{d+1}$. In turn, it follows that

$$s \ll \frac{g(k)}{\sqrt[d+1]{k}} \quad \text{for } k \text{ large enough.}$$

This establishes the left-hand side of inequality (4.5) and completes the proof of the claim. ■

For the proof of Theorem 4.1.4, we need the following inequality:

$$||(1+\lambda)\cdot \boldsymbol{v} - \boldsymbol{u}||_2 \quad \geq \quad ||\boldsymbol{v} - \boldsymbol{u}||_2 \quad \text{for all } \lambda \geq 0 \text{ and } \boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d. \tag{4.13}$$

This can be verified by elementary means by working in the plane defined by the points $\mathbf{0}$, $\boldsymbol{u}$ and $\boldsymbol{v}$.

Proof (Theorem 4.1.4) Let $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{S}^d$ and let $\mathfrak{S} = \mathfrak{S}(\boldsymbol{U})$ be the spiral defined in (4.3). Denote by

$$\boldsymbol{s}_k = \sqrt[d+1]{k}\cdot \boldsymbol{u}_k \quad \text{the } k\text{-th term of } \mathfrak{S}.$$

Given $R \geq 1$ and $h > 0$, let

$$\mathcal{A}(R,h) := \left\{\boldsymbol{x} \in \mathbb{R}^{d+1} : R \leq ||\boldsymbol{x}||_2 \leq R+h\right\}$$

be the annulus with inner radius $R$ and outer radius $R+h$.

**Necessary and sufficient conditions for uniform discreteness:** We prove the first part of the Theorem 4.1.4.

**$\Leftarrow$:** Assume that there exists a constant $h > 0$ such that condition (4.10) holds; that is, such that

$$s := \inf_{R\geq 1} R \cdot g_R^h(\boldsymbol{U}) > 0,$$

where $g_R^h(\boldsymbol{U})$ is defined in (4.10). It is enough to prove that for every $R \geq 1$, any two points of $\mathfrak{S}$ which belong to the annulus $\mathcal{A}(R,h)$ have distance at least $(s/2\pi)$ apart. Indeed, in this case the distance between any two distinct points of $\mathfrak{S}$ will be bounded below by $\min\{(s/2\pi), h\}$.

To this end, fix $R \geq 1$ and let $\boldsymbol{s}_m, \boldsymbol{s}_n \in \mathcal{A}(R,h)$ be two terms of $\mathfrak{S}$ with $m < n$. It follows from the definitions of $\mathfrak{S}$ and $\mathcal{A}(R,h)$ that

$$R^{d+1} \quad \leq \quad m \quad < \quad n \quad \leq (R+h)^{d+1}. \tag{4.14}$$

From assumption (4.10),

$$\operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{u}_m, \boldsymbol{u}_n) \quad \geq \quad \frac{s}{R}. \tag{4.15}$$

Therefore, one obtains that

$$\begin{aligned}
\|\boldsymbol{s}_n - \boldsymbol{s}_m\|_2 &= \left\| \sqrt[d+1]{n} \cdot \boldsymbol{u}_n - \sqrt[d+1]{m} \cdot \boldsymbol{u}_m \right\|_2 \\
&= \left\| \left( \sqrt[d+1]{n} - \sqrt[d+1]{m} \right) \cdot \boldsymbol{u}_n + \sqrt[d+1]{m} \cdot (\boldsymbol{u}_n - \boldsymbol{u}_m) \right\|_2 \\
&\underset{(4.13)}{\geq} \sqrt[d+1]{m} \cdot \|\boldsymbol{u}_n - \boldsymbol{u}_m\|_2 \\
&\underset{(4.2)}{\geq} \frac{\sqrt[d+1]{m}}{2\pi} \cdot \operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{u}_n, \boldsymbol{u}_n) \underset{(4.15)}{\geq} \frac{s}{2\pi},
\end{aligned}$$

where in the last inequality we made use of the fact $m \geq R^{d+1}$. The claim is proved.

**⇒:** Assume that the point set of $\mathfrak{S}$ is $s$-uniformly discrete for some $s > 0$. Set $h = s/2$ and fix $R \geq 1$ as well as the terms $\boldsymbol{s}_m, \boldsymbol{s}_n \in \mathcal{A}(R, h)$ with $m < n$; then, one has that

$$\begin{aligned}
\sqrt[d+1]{n} \cdot \operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{u}_m, \boldsymbol{u}_n) &\underset{(4.2)}{\geq} \left\| \sqrt[d+1]{n} \cdot \boldsymbol{u}_n - \sqrt[d+1]{n} \cdot \boldsymbol{u}_m \right\|_2 \\
&\geq \left\| \sqrt[d+1]{n} \cdot \boldsymbol{u}_n - \sqrt[d+1]{m} \cdot \boldsymbol{u}_m \right\|_2 - \left( \sqrt[d+1]{n} - \sqrt[d+1]{m} \right) \\
&\underset{(4.14)}{\geq} \|\boldsymbol{s}_n - \boldsymbol{s}_m\|_2 - \frac{s}{2} \\
&\geq \frac{s}{2},
\end{aligned}$$

where the second inequality is obtained from the Triangle Inequality. Thus, from (4.14),

$$R \cdot \operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{u}_m, \boldsymbol{u}_n) \quad \geq \quad \frac{sR}{2(R+h)} \quad \geq \quad c_s \tag{4.16}$$

where $c_s > 0$ is a constant depending only on $s$. Condition (4.10) follows upon taking the infimum on the left-hand side of inequality (4.16) over all $R \geq 1$ and over all those $m, n \in \mathbb{N}$ satisfying condition (4.14).

**Necessary and sufficient conditions for relative density:** We prove the second part of the Theorem 4.1.4.

$\Leftarrow$**:** Assume that there exists a constant $h > 0$ such that condition (4.11) holds; that is, such that

$$r := \sup_{R\geq 1} R \cdot G_R^h(\boldsymbol{U}) \quad < \quad +\infty,$$

where $G_R^h(\boldsymbol{U})$ is defined in (4.11). Then, the point set $\mathfrak{S}$ is $r'$-relative dense with $r' = \max\{2\pi \cdot r(1+h)+h, 2\}$. Indeed, fix $\boldsymbol{w} \in \mathbb{R}^{d+1}$ and set $R = ||\boldsymbol{w}||_2$. Let $\boldsymbol{v} \in \mathbb{S}^d$ be the unique unit vector such that $\boldsymbol{w} = R \cdot \boldsymbol{v}$. Without loss of generality, assume that $R \geq 1$ since otherwise a straightforward application of Triangle Inequality yields

$$||\boldsymbol{s}_1 - \boldsymbol{w}||_2 \quad \leq \quad 2.$$

From assumption (4.11), there exists $m \in \left[\!\left[R^{d+1}, (R+h)^{d+1}\right]\!\right]$ such that

$$\operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{u}_m, \boldsymbol{v}) \quad \leq \quad \frac{r}{R}. \tag{4.17}$$

Therefore, one has

$$\begin{aligned}
||\boldsymbol{s}_m - \boldsymbol{w}||_2 \quad &\leq \quad \left|\left| \sqrt[d+1]{m} \cdot \boldsymbol{u}_m - R \cdot \boldsymbol{v} \right|\right|_2 \\
&\leq \quad \sqrt[d+1]{m} \cdot ||\boldsymbol{u}_m - \boldsymbol{v}||_2 + \left(\sqrt[d+1]{m} - R\right) \cdot ||\boldsymbol{v}||_2 \\
&\underset{(4.2)\&(4.17)}{\leq} \quad 2\pi \cdot \frac{\sqrt[d+1]{m}}{R} \cdot r + h \\
&\leq \quad 2\pi \cdot \frac{R+h}{R} \cdot r + h \quad \underset{(R\geq 1)}{\leq} \quad 2\pi \cdot r(1+h) + h.
\end{aligned}$$

Since the choice of $\boldsymbol{w} \in \mathbb{R}^{d+1}$ is arbitrary, the claim is proved.

$\Rightarrow$**:** Assume that the point set $\mathfrak{S}$ is $r$-relatively dense for some $r > 0$. Set $h = 2r$ and fix $R \geq 1$ and $\boldsymbol{v} \in \mathbb{S}^d$. The point $\boldsymbol{w} = (R+r) \cdot \boldsymbol{v}$ belongs to $\mathcal{A}(R, 2r)$;

therefore, by assumption, there exists $m \in \left[\!\left[R^{d+1}, (R+h)^{d+1}\right]\!\right]$ such that

$$||\boldsymbol{s}_m - \boldsymbol{w}||_2 \quad \leq \quad r. \tag{4.18}$$

In turn, one has

$$\begin{aligned} \sqrt[d+1]{m} \cdot \mathrm{dist}_{\mathbb{S}^d}\left(\boldsymbol{u}_m, \boldsymbol{v}\right) \quad &\leq \quad 2\pi \cdot \left|\left| \sqrt[d+1]{m} \cdot \boldsymbol{u}_m - \sqrt[d+1]{m} \cdot \boldsymbol{v} \right|\right|_2 \\ &\leq \quad 2\pi \cdot \left( ||\boldsymbol{s}_m - \boldsymbol{w}||_2 + \left|\left| \left( \sqrt[d+1]{m} - (R+r) \right) \cdot \boldsymbol{v} \right|\right|_2 \right) \\ &\underset{(4.18)}{\leq} \quad 4\pi \cdot r. \end{aligned}$$

Therefore, we have proved that

$$\mathrm{dist}_{\mathbb{S}^d}\left(\boldsymbol{u}_m, \boldsymbol{v}\right) \quad \leq \quad \frac{4\pi r}{R}\cdot \tag{4.19}$$

The choice of $\boldsymbol{v}$ is arbitrary, thus inequality (4.19) implies that

$$R \cdot G_R^{2r}\left(\boldsymbol{U}\right) \quad \leq \quad 4\pi r. \tag{4.20}$$

Since the choice of $R \geq 1$ is also arbitrary, upon taking the supremum over all $R \geq 1$ in the left-hand side of inequality (4.20), one infers that condition (4.11) holds.

The proof is complete. ■

## 4.3 Lifting a Toral Sequence to the Sphere

Given a subset $A$ in a topological space $\mathcal{T}$, its topological interior, closure and boundary are denoted by $A^\circ$, $\overline{A}$ and $\mathrm{d}A$, respectively.

The goal of this section is to construct a spherical sequence $\boldsymbol{U}$ in $\mathbb{S}^d$ which satisfies both the packing condition (4.10) and the covering condition (4.11). These conditions are easier to realise when the underlying space is the torus $\mathbb{T}^d$. Indeed, given a badly approximable vector $\boldsymbol{v}$ in $\mathbb{T}^d$, Theorem B.1.1 (Appendix B, p.250) insures that the sequence $\boldsymbol{V} = (k \cdot \boldsymbol{v})_{k\in\mathbb{N}}$ is optimally distributed (condition (4.9));

that is,

$$0 \quad < \quad \mathtt{U}_P\left(\boldsymbol{V}\right) \quad < \quad \mathtt{U}_C\left(\boldsymbol{V}\right) \quad < \quad +\infty.$$

It follows immediately from the Definitions 4.1.1 (packing and covering radius) and 4.1.2 (uniform packing and covering parameters) that the sequence $\boldsymbol{V}$ satisfies both conditions (4.10) and (4.11) with constant $h = 1$.

The goal is to map an optimally distributed sequence from the torus $\mathbb{T}^d$ to the sphere $\mathbb{S}^d$ with the use of a gap-preserving mapping. Notice that all of the properties (4.9), (4.10) and (4.11) are transferable through gap-preserving maps. In other words, if $\tau : \mathcal{K}_1 \to \mathcal{K}_2$ is a gap-preserving map between two compact metric spaces and if a sequence $\boldsymbol{V} = (\boldsymbol{v}_k)_{k\in\mathbb{N}}$ in $\mathcal{K}_1$ satisfies one of these properties, then the sequence $\tau\left(\boldsymbol{V}\right) := \left(\tau\left(\boldsymbol{v}_k\right)\right)_{k\in\mathbb{N}}$ in $\mathcal{K}_2$ satisfies the same property as well. Denote the northern and the southern hemisphere of $\mathbb{S}^d$ by

$$\mathbb{S}^d_N \quad = \quad \Big\{(x_1, ..., x_{d+1}) \in \mathbb{S}^d : x_{d+1} \geq 0\Big\}$$

and

$$\mathbb{S}^d_S \quad = \quad \Big\{(x_1, ..., x_{d+1}) \in \mathbb{S}^d : x_{d+1} \leq 0\Big\},$$

respectively. Set also

$$\mathbb{K}_d = [0,1]^d, \quad \mathbb{K}'_d = [-1,1]^d \quad \text{and} \quad \mathbb{B}_d := \Big\{\boldsymbol{x} \in \mathbb{R}^d : ||\boldsymbol{x}||_2 \leq 1\Big\}$$

equiped with the induced Euclidean norm. In this section, the space $\mathcal{K}_1$ will be the torus $\mathbb{T}^d$, the space $\mathcal{K}_2$ will be either $\mathbb{S}^d_N$ or $\mathbb{S}^d_S$ and the sequence $\boldsymbol{V}$ will be the sequence

$$\boldsymbol{V} = \left(k \cdot \boldsymbol{v}\right)_{k\in\mathbb{N}}, \quad \text{where } \boldsymbol{v} \text{ is a badly approximable vector in } \mathbb{T}^d. \tag{4.21}$$

The corresponding gap-preserving maps $\tau_N : \mathbb{T}^d \to \mathbb{S}^d_N$ and $\tau_S : \mathbb{T}^d \to \mathbb{S}^d_S$ are defined with the help of the auxiliary maps (see Figure[2] 4.2)

$$\mathbb{T}^d \quad \stackrel{\pi}{\longmapsto} \quad \mathbb{K}_d \quad \stackrel{\mathfrak{d}}{\longmapsto} \quad \mathbb{K}'_d \quad \stackrel{\mathfrak{s}}{\longmapsto} \quad \mathbb{B}_d \quad \stackrel{\mathfrak{p}_{N,S}}{\longmapsto} \quad \mathbb{S}^d_{N,S} \tag{4.22}$$

as

$$\tau_N \quad = \quad \mathfrak{p}_N \circ \mathfrak{s} \circ \mathfrak{d} \circ \pi \quad \text{and} \quad \tau_S \quad = \quad \mathfrak{p}_S \circ \mathfrak{s} \circ \mathfrak{d} \circ \pi. \tag{4.23}$$

Here,

- $\pi$ is the natural projection of torus $\mathbb{T}^d$ to the $d$-dimensional unit square

$$\pi \quad : \quad \boldsymbol{x} \in \mathbb{T}^d \quad \longmapsto \quad \pi\left(\boldsymbol{x}\right) \quad = \quad \left(\left\{x_1\right\}, ..., \left\{x_d\right\}\right) \in \mathbb{K}_d,$$

- $\mathfrak{d}$ is the affine map mapping $\mathbb{K}_d$ to $\mathbb{K}'_d$:

$$\mathfrak{d} \quad : \quad \boldsymbol{x} \in \mathbb{K}_d \quad \mapsto \quad \mathfrak{d}\left(\boldsymbol{x}\right) \quad = \quad 2\boldsymbol{x} - \mathbf{1} \in \mathbb{K}'_d,$$

  where $\mathbf{1} = (1, 1, ..., 1)$ is the unit $d$-dimensional vector,

- $\mathfrak{s}$ is the mapping which maps the hypercube $\mathbb{K}'_d = [-1, 1]^d$ to the ball $\mathbb{B}_d$ in the following way: $\mathfrak{s}$ maps a vector on the boundary of the hypercube $\mathbb{K}'_d$ to the vector in the same direction normalised so as to lie on the unit sphere $\mathbb{S}^{d-1}$; in other words,

$$\mathfrak{s} \quad : \quad \boldsymbol{x} \in \mathbb{K}'_d \quad \mapsto \quad \mathfrak{s}\left(\boldsymbol{x}\right) \quad = \quad ||\boldsymbol{x}||_\infty \cdot \frac{\boldsymbol{x}}{||\boldsymbol{x}||_2} \quad \in \quad \mathbb{B}_d,$$

- $\mathfrak{p}_N$ (resp. $\mathfrak{p}_S$) is the stereographic projection with respect to the north pole $\boldsymbol{N} = (\mathbf{0}, 1)$ (resp. with respect to the south pole $\boldsymbol{S} = (\mathbf{0}, -1)$):

$$\mathfrak{p}_N \quad : \quad \boldsymbol{x} \in \mathbb{B}_d \quad \mapsto \quad \mathfrak{p}_N \quad = \quad \left(\frac{2\boldsymbol{x}}{1 + ||\boldsymbol{x}||_2^2}, \quad \frac{||\boldsymbol{x}||_2^2 - 1}{1 + ||\boldsymbol{x}||_2^2}\right) \in \mathbb{S}^d_S$$

[2]The pictures in the electronic copy of the thesis are in color.

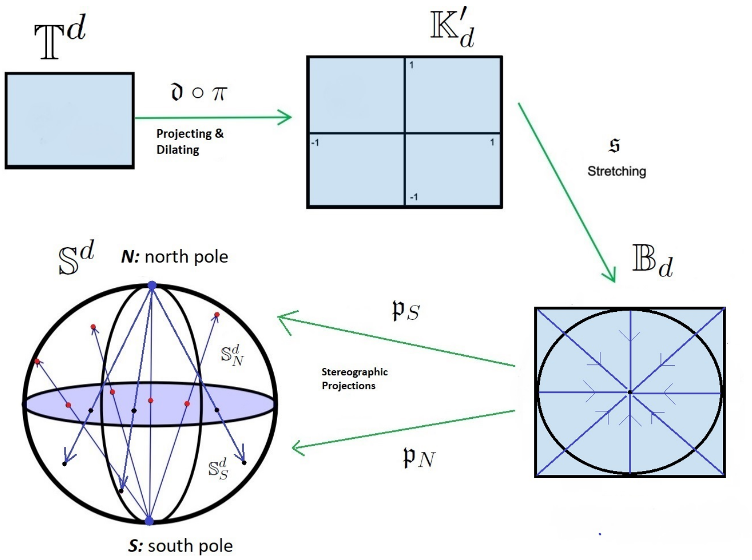



Figure 4.2: An illustration of the maps defined in (4.22).

$$\left(\text{resp.} \quad \mathfrak{p}_S \quad : \quad \boldsymbol{x} \in \mathbb{B}_d \quad \mapsto \quad \mathfrak{p}_S \quad = \quad \left( \frac{2\boldsymbol{x}}{1 + ||\boldsymbol{x}||_2^2}, \quad \frac{1 - ||\boldsymbol{x}||_2^2}{1 + ||\boldsymbol{x}||_2^2} \right) \in \mathbb{S}^d_N \right).$$

A mapping $\sigma : (\mathcal{K}_1, \text{dist}_1(\cdot, \cdot)) \mapsto (\mathcal{K}_2, \text{dist}(\cdot, \cdot))$ between two metric spaces is **bi-Lipschitz** if it is *bijective* and if for any $x, y \in \mathcal{K}_1$, it holds that

$$d_2 (\sigma(x), \sigma(y)) \quad \asymp \quad d_1(x, y). \tag{4.24}$$

All maps defined in diagram (4.22) are gap-preserving.

**Proposition 4.3.1** *The maps $\pi$, $\mathfrak{d}$, $\mathfrak{s}$, $\mathfrak{p}_N$ and $\mathfrak{p}_S$ defined in diagram* (4.22) *are gap-preserving. Furthermore, the maps $\mathfrak{d}$, $\mathfrak{s}$, $\mathfrak{p}_N$ and $\mathfrak{p}_S$ are bi-Lipschitz.*

Proposition 4.3.1 will be proved later in this section. Proposition 4.3.1 implies that the maps $\tau_N$ and $\tau_N$ are gap-preserving. Indeed, it follows from Definition 4.1.6 that the composition of two gap-preserving maps is again gap-preserving.

For the rest of this section, identify the torus $\mathbb{T}^d$ with the hypercube $\mathbb{K}_d$ and the sequence $\boldsymbol{V}$ defined in (4.21) with the sequence $\pi(\boldsymbol{V}) = (\pi(k \cdot \boldsymbol{v}))_{k \in \mathbb{N}}$. Notice that, since the map $\pi : \mathbb{T}^d \mapsto \mathbb{K}_d$ is gap-preserving, properties (4.9), (4.10) and (4.11) which are satisfied with the sequence $\boldsymbol{V}$ are also satisfied with the sequence $\pi(\boldsymbol{V})$. The following proposition provides a way to use the sequence $\boldsymbol{V}$ and the maps $\tau_N$, $\tau_S$ defined in (4.23) for the construction of a sequence in $\mathbb{S}^d$ which satisfies properties (4.9), (4.10) and (4.11).

**Proposition 4.3.2** *Let $\sigma_1, \sigma_2 : \mathbb{K}_d \to \mathbb{S}^d$ be two maps which are bi-Lipschitz and which satisfy these conditions:*

$$\sigma_1(\mathbb{K}_d)^\circ \cap \sigma_2(\mathbb{K}_d)^\circ = \emptyset, \qquad \sigma_1(\mathbb{K}_d) \cup \sigma_2(\mathbb{K}_d) = \mathbb{S}^d \quad \textit{and} \quad \sigma_1|_{d\mathbb{K}_d} = \sigma_2|_{d\mathbb{K}_d} \tag{4.25}$$

*(here, $\sigma_i|_{d\mathbb{K}_d}$ denotes the restriction of the map $\sigma_i$ to $d\mathbb{K}_d$). Let $\boldsymbol{X} = (\boldsymbol{x}_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{K}_d$ satisfying the packing condition* (4.10).

*Assume furthermore that $\boldsymbol{X}$ satisfies a covering condition; namely, that there exists a partition of the natural numbers into two sets $\mathcal{N}_1$ and $\mathcal{N}_2$ and a constant $h > 0$ such that*

$$\max_{i\in\{1,2\}} \sup_{R\geq 1} R \cdot \mathrm{R}_C \left( \left\{ \boldsymbol{x}_k : R^{d+1} \leq k < (R+h)^{d+1}, k \in \mathcal{N}_i \right\} \right) < +\infty. \tag{4.26}$$

*Then, the sequence defined for all $k \geq 1$ by*

$$\boldsymbol{u}_k = \begin{cases} \sigma_1(\boldsymbol{x}_k), & \textit{if } k \in \mathcal{N}_1 \\ \sigma_2(\boldsymbol{x}_k), & \textit{if } k \in \mathcal{N}_2 \end{cases}, \tag{4.27}$$

*satisfies the packing condition* (4.10) *and the covering condition* (4.11) *in $\mathbb{S}^d$.*

Propositions 4.3.1 and 4.3.2 will be proved at the end of this section. We assume they both hold for now to establish Theorem 4.1.5.

Proof (Theorem 4.1.5) Fix a badly approximable vector $\boldsymbol{v} \in \mathbb{T}^d$ and let $\boldsymbol{V} = (k \cdot \boldsymbol{v}_k)_{k\in\mathbb{N}}$ be the sequence of its multiples. From Theorem B.1.1 (Appendix B, p.250) one has that $\boldsymbol{V}$ is optimally distributed (Definition (4.1.3)); therefore, from Proposition 4.3.1, the sequence $\pi(\boldsymbol{V})$ in $\mathbb{K}_d$ is optimally distributed as well. It is enough to prove that the subsequences $\boldsymbol{V}_1 = ((2k-1)\cdot \boldsymbol{v})_{k\in\mathbb{N}}$ and $\boldsymbol{V}_2 = (2k\cdot \boldsymbol{v})_{k\in\mathbb{N}}$ are optimally distributed since in this case the result follows upon applying Proposition 4.3.2 with $\mathcal{N}_1 = 2\mathbb{N}$-1, $\mathcal{N}_2 = 2\mathbb{N}$, $\sigma_1 = \mathfrak{p}_N \circ \mathfrak{s} \circ \mathfrak{d}$ and $\sigma_2 = \mathfrak{p}_S \circ \mathfrak{s} \circ \mathfrak{d}$.

To apply Proposition 4.3.2, notice that assumption (4.25) follows from the construction of the maps $\mathfrak{d}$, $\mathfrak{s}$, $\mathfrak{p}_N$ and $\mathfrak{p}_S$ and assumption (4.26) holds upon proving that the sequences $\boldsymbol{V}_1$ and $\boldsymbol{V}_2$ are optimally distributed. To this end, it is enough to prove that the sequence $\boldsymbol{V}_2$ in $\mathbb{T}^d$ is optimally distributed since the sequence $\boldsymbol{V}_1$ is obtained by translating the terms of $\boldsymbol{V}_2$ by $-\boldsymbol{v}$. Set

$$s := \quad \mathtt{U}_P\left(\boldsymbol{V}\right) \ > 0 \quad \text{and} \quad r := \quad \mathtt{U}_C\left(\boldsymbol{V}\right) \ < +\infty.$$

**Proof of the Packing Condition:** Given $m \in \mathbb{N}_0$ and $N \in \mathbb{N}$, it holds that

$$\{2k\cdot \boldsymbol{v}\}_{k=m+1}^{m+N} \quad \subseteq \quad \{k\cdot \boldsymbol{v}\}_{k=2m+1}^{2m+2N}.$$

Therefore,

$$\sqrt[d]{N}\cdot \mathtt{R}_P\left(\{2k\cdot \boldsymbol{v}\}_{k=m+1}^{m+N}\right) \quad \geq \quad \sqrt[d]{N}\cdot \mathtt{R}_P\left(\{k\cdot \boldsymbol{v}\}_{k=2m+1}^{m+2N}\right) \quad \geq \quad \frac{s}{\sqrt[d]{2}}. \tag{4.28}$$

Since inequality (4.28) holds for every $m \in \mathbb{N}_0$ and every $N \in \mathbb{N}$, one obtains that

$$\mathtt{U}_P\left(\boldsymbol{V}_2\right) \quad \geq \quad \frac{s}{\sqrt[d]{2}}.$$

**Proof of the Covering Condition:** Fix $\boldsymbol{x} \in \mathbb{T}^d$. Identify the point $\boldsymbol{x}$ of the torus with the point $\pi(\boldsymbol{x})$ of the hypercube $\mathbb{K}_d$ and consider (after such an identification) the point $\frac{\boldsymbol{x}}{2} \in \mathbb{K}^d$. For any $m \in \mathbb{N}_0$ and $N \in \mathbb{N}$, it holds that

$$\inf_{1\leq k\leq N} \sqrt[d]{N}\cdot \left|\left|\frac{\boldsymbol{x}}{2} - (k+m)\cdot \boldsymbol{v}\right|\right| \quad \leq \quad \frac{r}{\sqrt[d]{N}},$$

which yields that

$$\inf_{1\leq k\leq N} \sqrt[d]{N}\cdot ||\boldsymbol{x} - 2(k+m)\cdot \boldsymbol{v}|| \quad \leq \quad \frac{2r}{\sqrt[d]{N}}. \tag{4.29}$$

Taking the supremum over all $\boldsymbol{x} \in \mathbb{T}$, $m \geq 0$ and $N \in \mathbb{N}$ in both sides of inequality (4.29) yields that

$$\mathtt{U}_C\left(\boldsymbol{V}_2\right) \quad \leq \quad 2r.$$

The proof is complete. ■

It remains to prove Propositions 4.3.1 and 4.3.2.

Proof (Proposition 4.3.1) Let the maps $\pi$, $\mathfrak{d}$, $\mathfrak{s}$, $\mathfrak{p}_N$ and $\mathfrak{p}_S$ be as defined in diagram (4.22).

**Proof that $\pi$ is gap-preserving:** For the proof of the left-hand side inequality in (4.12), notice that for any $\boldsymbol{x}, \boldsymbol{x'} \in \mathbb{T}^d$, it holds that

$$||\boldsymbol{x} - \boldsymbol{x'}|| \quad \leq \quad ||\pi\left(\boldsymbol{x}\right) - \pi\left(\boldsymbol{x'}\right)||_2 . \tag{4.30}$$

Thus, it follows immediately from inequality (4.30) that for any subset $A \subseteq \mathbb{T}^d$,

$$\mathtt{R}_P\left(A\right) \quad \leq \quad \mathtt{R}_P\left(\pi(A)\right).$$

For the proof of the right-hand side inequality in (4.12), fix a finite subset $A \subseteq \mathbb{T}^d$ and a point $\boldsymbol{y} \in \mathbb{K}^d$. Then, there exists a point $\boldsymbol{c} \in \mathbb{K}_d$ such that

$$||\boldsymbol{y} - \boldsymbol{c}||_\infty \quad \leq \quad 2 \cdot \mathtt{R}_C(A) \quad \text{and} \quad B_\infty\left(\boldsymbol{c}, 2 \cdot \mathtt{R}_C(A)\right) \quad \subseteq \quad \mathbb{K}_d.$$

Let $\boldsymbol{x}, \boldsymbol{b} \in \mathbb{T}^d$ be such that $\pi(\boldsymbol{x}) = \boldsymbol{y}$ and $\pi(\boldsymbol{b}) = \boldsymbol{c}$. Let also $\boldsymbol{a} \in A$ be such that

$$||\boldsymbol{a} - \boldsymbol{b}|| \quad \leq \quad \mathtt{R}_C(A).$$

Since $B_\infty\left(\boldsymbol{c}, 2 \cdot \mathtt{R}_C(A)\right) \subseteq \mathbb{K}_d$, this implies that $\pi\left(\boldsymbol{a}\right) \in B_\infty\left(\boldsymbol{c}, \mathtt{R}_C(A)\right)$. It follows from the way the points $\boldsymbol{a}$ and $\boldsymbol{b}$ have been chosen that

$$\begin{aligned} \frac{1}{\sqrt{d}} \cdot ||\boldsymbol{y} - \pi\left(\boldsymbol{a}\right)||_2 \quad &\leq \quad ||\boldsymbol{y} - \pi\left(\boldsymbol{a}\right)||_\infty \\ &\leq \quad ||\boldsymbol{y} - \boldsymbol{c}||_\infty + ||\pi(\boldsymbol{b}) - \pi(\boldsymbol{a})||_\infty \quad \leq \quad 3 \cdot \mathtt{R}_C(A). \end{aligned} \tag{4.31}$$

In turn, inequality (4.31) implies that

$$\mathtt{R}_C\left(\pi(A)\right) \quad \leq \quad 3\sqrt{d}\cdot \mathtt{R}_C(A).$$

The claim is proved.

Since a bi-Lipschitz map is always gap-preserving, it is enough to prove the bi-Lipschitz property for the other maps. It follows trivially that the maps $\mathfrak{d}$, $\mathfrak{s}$ and $\mathfrak{p}_N$ (resp. $\mathfrak{p}_S$) are bijective. Also, it is easy to check that the map $\mathfrak{d}$ is bi-Lipschitz. Therefore, it remains to prove that the maps $\mathfrak{s}$, $\mathfrak{p}_N$ and $\mathfrak{p}_S$ satisfy property (4.24).

To this end, given $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^{d+1}$ expressed in polar coordinates as $\boldsymbol{x} = \rho_{\boldsymbol{x}}\boldsymbol{v}_{\boldsymbol{x}}$ and $\boldsymbol{y} = \rho_{\boldsymbol{y}}\boldsymbol{v}_{\boldsymbol{y}}$ with $\rho_{\boldsymbol{x}}, \rho_{\boldsymbol{y}} > 0$ and $\boldsymbol{v}_{\boldsymbol{x}}, \boldsymbol{v}_{\boldsymbol{y}} \in \mathbb{S}^d$, it holds that[3]

$$||\rho_{\boldsymbol{x}}\boldsymbol{v}_{\boldsymbol{x}} - \rho_{\boldsymbol{y}}\boldsymbol{v}_{\boldsymbol{y}}||_2 \quad \asymp \quad |\rho_{\boldsymbol{x}} - \rho_{\boldsymbol{y}}| + \sqrt{\rho_{\boldsymbol{x}}\rho_{\boldsymbol{y}}}\cdot ||\boldsymbol{v}_{\boldsymbol{x}} - \boldsymbol{v}_{\boldsymbol{y}}||_2\,. \tag{4.32}$$

**Proof that $\mathfrak{s}$ is bi-Lipschitz:** For every $\boldsymbol{x} \in \mathbb{K}'_d$, one has that

$$\mathfrak{s}\left(\boldsymbol{x}\right) \quad = \quad ||\boldsymbol{x}||_\infty \cdot \boldsymbol{v}_x \quad = \quad \rho_{\boldsymbol{x}}\cdot ||\boldsymbol{v}_x||_\infty \cdot \boldsymbol{v}_x,$$

where $\rho_{\boldsymbol{x}} \geq 0$ and $\boldsymbol{v}_{\boldsymbol{x}} \in \mathbb{S}^d$ are the polar coordinates of $\boldsymbol{x}$. Fix $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{K}_d$ expressed in polar coordinates as $\boldsymbol{a} = \rho\cdot\boldsymbol{u}$ and $\boldsymbol{b} = r\cdot\boldsymbol{v}$ for some $\rho, r \in \left[0, \sqrt{d}\right]$ and $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d$. The goal is to show that

$$||\rho\boldsymbol{u} - r\boldsymbol{v}||_2 \quad \asymp \quad ||\ \ \rho\cdot||\boldsymbol{u}||_\infty\cdot\boldsymbol{u} - r\cdot||\boldsymbol{v}||_\infty\cdot\boldsymbol{v}\ \ ||_2$$

or, equivalently, from relation (4.32), that

$$|\rho - r| + \sqrt{\rho r}\cdot||\boldsymbol{u} - \boldsymbol{v}||_2 \quad \asymp \quad \sqrt{uv}\cdot\sqrt{\rho r}\cdot||\boldsymbol{u} - \boldsymbol{v}||_2 + |\rho u - r v|\,, \tag{4.33}$$

with

$$u = ||\boldsymbol{u}||_\infty \quad \text{and} \quad v = ||\boldsymbol{v}||_\infty\,.$$

[3]One has that

$$||\rho_{\boldsymbol{x}}\boldsymbol{v}_{\boldsymbol{x}} - \rho_{\boldsymbol{y}}\boldsymbol{v}_{\boldsymbol{y}}||_2^2 \quad = \quad \left(\rho_{\boldsymbol{x}} - \rho_{\boldsymbol{y}}\right)^2 + \rho_{\boldsymbol{x}}\rho_{\boldsymbol{y}}\cdot||\boldsymbol{v}_{\boldsymbol{x}} - \boldsymbol{v}_{\boldsymbol{y}}||_2^2\,,$$

which yields the claim.

Since $\boldsymbol{u}, \boldsymbol{v} \in \mathbb{S}^d$, one has that

$$u, v \in \left[\frac{1}{\sqrt{d}}, 1\right] \tag{4.34}$$

and thus

$$\sqrt{\rho r} \cdot ||\boldsymbol{u} - \boldsymbol{v}||_2 \quad \asymp \quad \sqrt{uv} \cdot \sqrt{\rho r} \cdot ||\boldsymbol{u} - \boldsymbol{v}||_2 . \tag{4.35}$$

Consequently, from relation (4.35), to establish relation (4.33) it is enough to prove the inequalities

$$|\rho - r| \quad \ll \quad |\rho u - rv| + \sqrt{\rho r} \cdot ||\boldsymbol{u} - \boldsymbol{v}||_2 \tag{4.36}$$

and

$$|\rho u - rv| \quad \ll \quad |\rho - r| + \sqrt{\rho r} \cdot ||\boldsymbol{u} - \boldsymbol{v}||_2 . \tag{4.37}$$

To this end, without loss of generality, assume that

$$r \quad \leq \quad \rho.$$

Inequality (4.36) is proved as follows:

$$\begin{aligned} |\rho - r| \quad & \underset{(4.34)}{\ll} \quad r\,|u - v| + |\rho u - rv| \\ & \ll r\,||\boldsymbol{u} - \boldsymbol{v}||_2 + |\rho u - rv| \\ & \underset{(r \leq \rho)}{\ll} \sqrt{\rho r} \cdot ||\boldsymbol{u} - \boldsymbol{v}||_2 + |\rho u - rv| . \end{aligned}$$

Inequality (4.37) is proved by splitting cases. Note that, since

$$|\rho u - rv| \quad \leq \quad \rho\,|u - v| + v\,|\rho - r| ,$$

it is enough to prove that

$$\rho\,|u - v| \quad \ll \quad \sqrt{\rho r}\,||\boldsymbol{u} - \boldsymbol{v}||_2 + |\rho - r| .$$

We split the following cases.

- If $r = 0$, then $\rho\,|u - v| \leq |\rho - r|$ which yields the claim.

- If $2r \leq \rho$, then
$$\rho\,|u-v| \quad \ll \quad \rho \quad \leq \quad 2\,|\rho - r|\,.$$
Otherwise, if $\rho \leq 2r$, then
$$\rho\,|u-v| \quad \leq \quad 2\sqrt{\rho r}\cdot ||\boldsymbol{u}-\boldsymbol{v}||_2\,.$$
In both cases, the claim follows.

Thus, inequality (4.37) is proved and this completes the proof of relation (4.33).

**Proof that $\mathfrak{p}_N$ and $\mathfrak{p}_S$ are bi-Lipschitz:** It is enough to prove the claim only for the map $\mathfrak{p}_N$ since the proof for $\mathfrak{p}_S$ is identical. To this end, fix $\boldsymbol{a} = (a_1, ..., a_d)\,, \boldsymbol{b} = (b_1, ..., b_d) \in \mathbb{B}_d$ and set $a = ||\boldsymbol{a}||_2$ and $b = ||\boldsymbol{b}||_2$. It then holds that

$$\begin{aligned} ||\mathfrak{p}_N(\boldsymbol{a}) - \mathfrak{p}_N(\boldsymbol{b})||_2^2 \quad &= \quad \sum_{i=1}^{d}\left(\frac{2a_i}{1+a^2} - \frac{2b_i}{1+b^2}\right)^2 + \left(\frac{1-||\boldsymbol{a}||_2^2}{1+a^2} - \frac{1-||\boldsymbol{b}||_2^2}{1+b^2}\right)^2 \\ &= \quad \frac{4}{(1+a^2)\cdot(1+b^2)}\cdot ||\boldsymbol{a}-\boldsymbol{b}||_2^2\,, \end{aligned}$$

whence
$$||\mathfrak{p}_N(\boldsymbol{a}) - \mathfrak{p}_N(\boldsymbol{b})||_2 \quad = \quad \frac{2}{\sqrt{1+a^2}\cdot\sqrt{1+b^2}}\cdot ||\boldsymbol{a}-\boldsymbol{b}||_2\,.$$
Taking into consideration that $\sqrt{1+a^2}, \sqrt{1+b^2} \in \left[1, \sqrt{2}\right]$, one obtains

$$||\boldsymbol{a}-\boldsymbol{b}||_2 \quad \leq \quad ||\boldsymbol{p}_N(\boldsymbol{a}) - \boldsymbol{p}_N(\boldsymbol{b})||_2 \quad \leq \quad 2\cdot ||\boldsymbol{a}-\boldsymbol{b}||\,.$$

Thus, $\mathfrak{p}_N$ is bi-Lipschitz. The proof is complete. ■

Proof (Proposition 4.3.2) Let $\sigma_1, \sigma_2 : \mathbb{K}_d \mapsto \mathbb{S}^d$ be two bi-Lipschitz maps satisfying assumption (4.25) and let $\boldsymbol{X} = (\boldsymbol{x}_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{K}_d$ satisfying the packing condition (4.10), the covering condition (4.11) and assumption (4.26) for some partition of $\mathbb{N}$ into sets $\mathcal{N}_1$ and $\mathcal{N}_2$. Let also $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ be the sequence defined in (4.27).

We prove first that $\boldsymbol{U}$ satisfies the covering condition (4.11) in the space $\mathbb{S}^d$. Fix $R \geq 1$, $\boldsymbol{v} \in \mathbb{S}^d$ and let $h > 0$ be as in the statement. Let $r > 0$ equal the quantity defined in the left-hand side of inequality (4.26). If $\boldsymbol{v} \in \sigma_1(\mathbb{K}_d)$, then set $j = 1$, otherwise set $j = 2$. Let also $\boldsymbol{x} \in \mathbb{K}_d$ be such that $\boldsymbol{v} = \sigma_j(\boldsymbol{x})$. By assumption (4.26), there exists $k \in \left[\!\left[R^{d+1}, (R+h)^{d+1}\right]\!\right]$ with $k \in \mathcal{N}_j$ such that

$$||\boldsymbol{x}_k - \boldsymbol{x}||_2 \quad \leq \quad \frac{r}{R}.$$

Thus, using the fact that $\sigma_j$ is bi-Lipschitz, one gets

$$\mathrm{dist}_{\mathbb{S}^d}\left(\sigma_j\left(\boldsymbol{x}_k\right), \boldsymbol{v}\right) \quad \ll \quad \frac{r}{R}. \tag{4.38}$$

The choice of $\boldsymbol{v}$ is arbitrary, therefore,

$$G_R^h(\boldsymbol{U}) = \mathtt{R}_C\left(\{\boldsymbol{u}_k\}_{k=R^{d+1}}^{(R+h)^{d+1}}\right) \quad \underset{(4.38)}{\ll} \quad \frac{r}{R}.$$

Since the choice of $R$ is arbitrary, one obtains that

$$\sup_{R\geq 1} R \cdot G_R^h(\boldsymbol{U}) \quad \ll \quad r.$$

The claim is proved.

The proof of the packing condition (4.10) is more involved. The sequence $\boldsymbol{X}$ satisfies condition (4.10), therefore, there exists $h > 0$ such that for any $R \geq 1$ and for any $k, l \in \left[\!\left[R^{d+1}, (R+h)^{d+1}\right]\!\right]$,

$$||\boldsymbol{x}_k - \boldsymbol{x}_l||_2 \quad \gg \quad \frac{1}{R}. \tag{4.39}$$

Let

$$A(R,h) := \quad \left[\!\left[R^{d+1}, (R+h)^{d+1}\right]\!\right].$$

The goal is to show that for every $k, l \in A(R,h)$, it holds that

$$\mathrm{dist}_{\mathbb{S}^d}\left(\boldsymbol{u}_k, \boldsymbol{u}_l\right) \quad \gg \quad \frac{1}{R}. \tag{4.40}$$

Indeed, inequality (4.40) implies condition (4.10); that is, $\inf_{R\geq 1} R\cdot g_R^h(\boldsymbol{U}) \ > 0$.

To this end, fix $R\geq 1$ and $k,l\in A(R,h)$. Relation (4.40) follows easily when $k,l$ both belong to either $\mathcal{N}_1$ or $\mathcal{N}_2$. Indeed, assume without loss of generality that $k,l\in\mathcal{N}_1$. Then, inequality (4.39) holds and since $\sigma_1$ is bi-Lipschitz, one infers that (4.40) is also true.

To conclude the proof, assume that $k,l$ belong to different sets of the partition, say $k\in\mathcal{N}_i$ and $l\in\mathcal{N}_j$ with $i,j\in\{1,2\}$ and $i\neq j$. Since $\sigma_1$ and $\sigma_2$ coincide on the boundary of $\mathbb{K}_d$, in view of the previous case, one can furthermore assume without loss of generality that $\boldsymbol{x}_k,\boldsymbol{x}_l\in\mathbb{K}_d^\circ$. Then, $\boldsymbol{u}_k\in\sigma_i(\mathbb{K}_d)$ and $\boldsymbol{u}_l\in\sigma_j(\mathbb{K}_d)$. Consider the geodesic arc $t\in[0,1]\mapsto\boldsymbol{v}(t)$ joining $\boldsymbol{v}(0)=\boldsymbol{u}_k$ to $\boldsymbol{v}(1)=\boldsymbol{u}_l$ and note that assumption (4.25) implies that the decomposition

$$\mathbb{S}^d \quad = \quad \sigma_1(\mathbb{K}_d)^\circ\cup\sigma_2(\mathbb{K}_d)^\circ\cup\sigma_1(\mathrm{d}\mathbb{K}_d)$$

holds, where the unions are pairwise disjoint. The Intermediate Value Theorem applied to the continuous map

$$t\in(0,1)\quad\mapsto\quad d(t):=\quad \mathrm{dist}_{\mathbb{S}^d}\left(\boldsymbol{v}(t),\sigma_1\left(\mathbb{K}_d^\circ\right)\right)-\mathrm{dist}_{\mathbb{S}^d}\left(\boldsymbol{v}(t),\sigma_2\left(\mathbb{K}_d^\circ\right)\right)$$

then implies the existence of a value $t_0\in(0,1)$ such that $d(t_0)=0$ and of a vector $\boldsymbol{z}\in\mathrm{d}\mathbb{K}_d$ such that $\boldsymbol{v}(t_0)=\sigma_1(\boldsymbol{z})$.

Therefore, using the fact that the maps $\sigma_1$ and $\sigma_2$ are bi-Lipschitz, one gets

$$\begin{aligned}
\mathrm{dist}_{\mathbb{S}^d}(\boldsymbol{u}_k,\boldsymbol{u}_l) &= \mathrm{dist}_{\mathbb{S}^d}\left(\sigma_i(\boldsymbol{x}_k),\sigma_j(\boldsymbol{x}_l)\right)\\
&= \mathrm{dist}_{\mathbb{S}^d}\left(\sigma_i(\boldsymbol{x}_k),\sigma_i(\boldsymbol{z})\right)+\mathrm{dist}_{\mathbb{S}^d}\left(\sigma_j(\boldsymbol{x}_l),\sigma_j(\boldsymbol{z})\right)\\
&\gg \|\boldsymbol{x}_k-\boldsymbol{z}\|_2+\|\boldsymbol{x}_l-\boldsymbol{z}\|_2\\
&\geq \|\boldsymbol{x}_k-\boldsymbol{x}_l\|_2 \underset{(4.39)}{\gg} \frac{1}{R}.
\end{aligned}$$

Thus, inequality (4.40) is established in this case as well. The proof is complete. ■

# Chapter 5

# Visibility Properties of Spiral Sets

Given $\boldsymbol{x} \in \mathbb{S}^d$ and $\rho > 0$, $\mathcal{C}_d(\boldsymbol{x}, \rho)$ stands for the spherical cap centred at $\boldsymbol{x}$ with geodesic radius $\rho$; that is,

$$\mathcal{C}_d(\boldsymbol{x}, \rho) \quad = \quad \left\{ \boldsymbol{v} \in \mathbb{S}^d : \quad \operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{x}, \boldsymbol{v}) \leq \rho \right\},$$

where the geodesic distance $\operatorname{dist}_{\mathbb{S}^d}(\cdot, \cdot)$ is defined in [Chapter 4, Equation (4.1), p.115].

## 5.1 Introduction

In chapter 4, we studied the Delone property of discrete spiral sets of the form given in [Chapter 4, Equation (4.3), p.116]. Recall that a spiral in $\mathbb{R}^{d+1}$ is a sequence of the form

$$\mathfrak{S}(\boldsymbol{U}) = \left\{ \sqrt[d+1]{k} \cdot \boldsymbol{u}_k \right\}_{k \in \mathbb{N}}, \quad \text{where } \boldsymbol{U} = (\boldsymbol{u}_k)_{k \in \mathbb{N}} \text{ is a sequence in } \mathbb{S}^d. \tag{5.1}$$

In the planar case, Akiyama [9] and Marklof [59] established necessary and also sufficient conditions on the sequence $\boldsymbol{U}$ for the spiral $\mathfrak{S}(\boldsymbol{U})$ to be Delone. The resulting question of determining whether this theory can be extended to higher dimensions is settled in Chapter 4 (and in its published version [5]): there, a necessary and sufficient condition is indeed established for a spiral set to be Delone in $\mathbb{R}^{d+1}$ (Theorem 4.1.4, p.119). Furthermore, an explicit construction is provided to show that such point sets do exist in any dimension (Theorem 4.1.5, p.120).

The goal of this chapter is to study the distribution properties of spiral sets from a complementary standpoint; namely, from that of so-called visibility problems in discrete geometry, which quantify in suitable senses the density of a point set. More precisely, the goal is to establish necessary and sufficient conditions for a spiral set to become arbitrarily close to line segments, provided they are long enough. This can be formalised in several distinct ways.

A first possible formalisation is motivated by Pólya's orchard problem [67, Problem 239]. To motivate this approach, the reader is referred to the definition of an orchard's, Definition 1.1.17, p.33, and to the discussion surrounding it. Recall that an orchard $\mathcal{O} \subseteq \mathbb{R}^{d+1}$ is a discrete point set with finite density (see equation (1.2), p.17) which becomes $\epsilon$-close to all line segments with length $V = V(\epsilon)$ which have the origin as one of their end points. The function $V$ is called a *visibility function* of the orchard. As is not hard to see, a visibility function $V$ in an orchard in dimension $(d+1)$ has to satisfy the bound

$$V(\epsilon) \quad \geq \quad c \cdot \epsilon^{-d} \tag{5.2}$$

for some constant $c > 0$. This will be justified in detail in Section 5.2.

Removing the assumption of the origin being one of the end points of the line segments under consideration, while keeping the constraint that the line segments must be supported on directions passing through the origin leads one to the concept of a *uniform orchard*, which is introduced for the first time here.

**Definition 5.1.1 (Uniform Orchard)** *A subset $\mathcal{O} \subseteq \mathbb{R}^{d+1}$ is a uniform orchard if it has finite density and if there exists a function $V : \epsilon \in (0,1) \to V(\epsilon) \in \mathbb{R}^{+}$, increasing as $\epsilon \to 0^{+}$, such that the following holds: for every $\epsilon > 0$, every $t_0 \in \mathbb{R}$ and every direction $\boldsymbol{v} \in \mathbb{S}^d$, there exists a point $\boldsymbol{o} \in \mathcal{O}$ and a real number $t_0 < t < t_0 + V(\epsilon)$ such that $||\boldsymbol{o} - t \cdot \boldsymbol{v}||_2 \leq \epsilon$.*

Clearly, a uniform orchard is also an orchard (just take $t_0 = 0$ in the above definition). The converse, however, does not hold and, despite the apparent similarity in their definitions, these two concepts should be thought of as being rather different in nature: indeed, Section 5.2 provides examples of an orchard and of a uniform orchard which have drastically different visibility properties (in a sense made precise therein).

Removing the assumption in Definitions 1.1.17 (orchard, p.33) and 5.1.1 (uniform orchard) that the line segments should be supported on directions passing through the origin leads one to further concepts of visibility such as the notions of visible or hidden points (Definition 1.1.15, p.32) and dense forests (Definition 1.1.7, p.22). These concepts have previously appeared and been studied in the literature −− see, e.g., [1, 4, 25, 74].

Clearly, a dense forest is both a set with an empty set of visible points and a uniform orchard (and therefore also an orchard). The latter implies in particular that the lower bound (5.2) still holds for the visibility function in a dense forest.

Some of the visibility properties of the spiral $\mathfrak{S}(\boldsymbol{U})$ defined in (5.1) will be expressed in terms of the covering radius and the uniform covering parameter given in Definition 4.1.1 (Chapter 4, p.118).

**Theorem 5.1.2** *Let $\mathfrak{S}(\boldsymbol{U})$ be a spiral in $\mathbb{R}^{d+1}$ defined from a spherical sequence $\boldsymbol{U} = (\boldsymbol{u}_k)_{k\in\mathbb{N}}$ as in* (5.1)*. Let $V : \epsilon \in (0,1) \to V(\epsilon) \in \mathbb{R}^+$ be a function increasing as $\epsilon \to 0^+$ with polynomial growth rate*[1].

*Given real numbers $R \geq 1$ and $h > 0$, let $G_R^h(\boldsymbol{U})$ be the quantity defined in* (4.11) *(p.119).*

1. ***(Spirals and orchards)** The spiral $\mathfrak{S}(\boldsymbol{U})$ forms an orchard with visibility $c_{\boldsymbol{U}} \cdot V$ for some constant $c_{\boldsymbol{U}} > 0$ if, and only if, the following holds for some constant $\kappa > 0$: for all $\epsilon > 0$ small enough and all $\boldsymbol{v} \in \mathbb{S}^d$, there exists an integer $1 \leq n \leq V(\epsilon)^{d+1}$ such that*

$$dist_{\mathbb{S}^d}(\boldsymbol{u}_n, \boldsymbol{v}) \quad \leq \quad \kappa \cdot \frac{\epsilon}{\sqrt[d+1]{n}}.$$

[1]A function $V : \mathbb{R}^+ \mapsto \mathbb{R}^+$ has polynomial growth rate if

$$\limsup_{\epsilon\to 0} \frac{\log V(\epsilon)}{\log(1/\epsilon)} \quad < \quad +\infty.$$

2. ***(Spirals and uniform orchards)*** *The following two claims are equivalent:*

   (a) *There exists a constant $c_{\boldsymbol{U}} \geq 1$ such that the spiral $\mathfrak{S}(\boldsymbol{U})$ is a uniform orchard with visibility $c_{\boldsymbol{U}} \cdot V$.*

   (b) *There exists a constant $K > 0$ such that*

$$\sup_{\epsilon \in (0,1)} \sup_{R \geq V(\epsilon)} R \cdot \epsilon^{-1} \cdot G_R^{W(\epsilon)}(\boldsymbol{U}) \quad < \quad +\infty, \tag{5.3}$$

   *where $W(\epsilon) = K \cdot V(\epsilon)$.*

   *Furthermore:*

   (i) *whenever $\mathtt{U}_C(\boldsymbol{U}) < +\infty$, the spiral $\mathfrak{S}(\boldsymbol{U})$ is a uniform orchard with optimal visibility $c_{\boldsymbol{U}} \cdot \epsilon^{-d}$ for some constant $c_{\boldsymbol{U}} > 0$, where $\mathtt{U}_C(\boldsymbol{U})$ is defined in* (4.8) *(p.119).*

   (ii) *a sufficient condition for the set $\mathfrak{S}(\boldsymbol{U})$ to be a uniform orchard is that the relation*

$$\lim_{x \to +\infty} \left( \sup_{R \geq x} R \cdot G_R^x(\boldsymbol{U}) \right) \quad = \quad 0$$

   *should hold. In this case, there exists a constant $c_{\boldsymbol{U}} > 1$ such that $\mathfrak{S}(\boldsymbol{U})$ admits $W(\epsilon) = c_{\boldsymbol{U}} \cdot V(\epsilon)$ as a visibility function, where*

$$V(\epsilon) \quad = \quad \sup \left\{ x \geq 0 : \sup_{R > x} R \cdot G_R^x(\boldsymbol{U}) \geq \epsilon \right\}. \tag{5.4}$$

3. ***(Visible Points in Spirals)*** *The following two claims hold:*

   (i) *The spiral $\mathfrak{S}(\boldsymbol{U})$ has an empty set of visible points if, and only if, the following holds for some constant $c > 0$: for any $\epsilon > 0$, any $\lambda \geq 0$ and any choice of orthogonal vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{S}^d$, there exists a positive real number $t > 0$ and an integer $n \geq 1$ such that the inequalities*

$$\left| \sqrt[d+1]{n} - \sqrt{\lambda^2 + t^2} \right| \quad \leq \quad \epsilon \quad \textit{and} \quad \mathit{dist}_{\mathbb{S}^d}\left( \boldsymbol{u}_n, \frac{\lambda \boldsymbol{v} + t \boldsymbol{w}}{\sqrt{\lambda^2 + t^2}} \right) \quad \leq \quad \frac{c \cdot \epsilon}{\sqrt{\lambda^2 + t^2}} \tag{5.5}$$

   *are satisfied.*

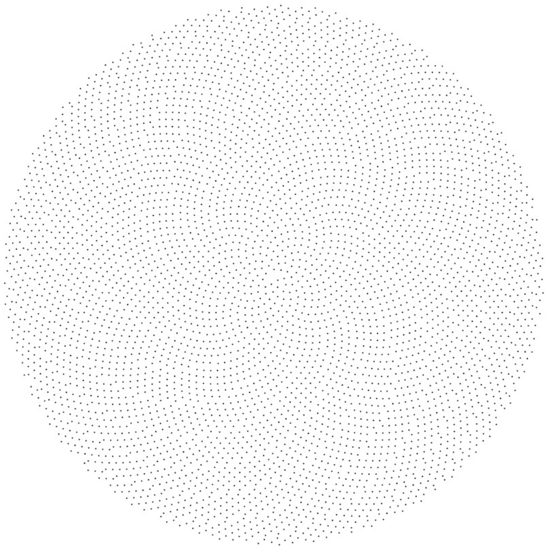


Figure 5.1: The Fermat (sunflower) spiral (4.4), p.116, at a scale different from the one in Figure 4.1, p.116.

*(ii) A spiral set which is a uniform orchard has an empty set of visible points (in other words, all points in $\mathbb{R}^{d+1}$ are hidden). The converse, however, does not hold.*

4. ***(Spirals and dense forests)*** *The spiral $\mathfrak{S}(\boldsymbol{U})$ is a dense forest with visibility function $W = c_{\boldsymbol{U}} \cdot V$ for some constant $c_{\boldsymbol{U}} \geq 1$ if, and only if, there exists a constant $c > 0$ such that for any $\epsilon > 0$, $\lambda \geq 0$, $t_0 \in \mathbb{R}$ and any choice of orthogonal vectors $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{S}^d$, there exists $t \in [t_0, t_0 + V(\epsilon)]$ and an integer $n \geq 1$ such that the inequalities* (5.5) *are simultaneously met.*

Explicit spherical sequences with finite uniform covering parameters are constructed in any dimension in Theorem 4.1.5 (Chapter 4, p.120). As a consequence of Points 2 and 3(ii), this establishes the existence of spiral sets which are uniform orchards with optimal visibility and have an empty set of visible points. Furthermore, the constructions provided in Theorem 4.1.5 (p.120) ensure that the resulting spirals enjoy the additional property of being Delone. An example of such a spiral in the plane is the Fermat (or sunflower) spiral defined in (4.4) (p.116) (see also Figures 4.1, p.116, and 5.1, p.141).

By contrast, the existence of a spiral set which is a dense forest remains an open question. As a matter of fact, it is conjectured in [6] that there is no such set. The case of the sunflower is already elusive and specialising Point 4 in Theorem 5.1.2 above to this situation leads one to a new kind of moving and shrinking target problem modulo one (for more details on shrinking target problems, the reader is referred to the papers [45, 52, 71]).

## 5.2 Proof of the Main Theorem

The claim on the lower bound (5.2) for a visibility function in an orchard is first established.

Proof (Lower Bound (5.2)) Let $\mathcal{O} \subseteq \mathbb{R}^{d+1} \setminus \{\mathbf{0}\}$ be an orchard with visibility function $V$. Enumerate the points in $\mathcal{O}$ in a sequence $(\boldsymbol{x}_k)_{k\in\mathbb{N}} = (\rho_k \boldsymbol{u}_k)_{k\in\mathbb{N}}$ in such a way that

$$||\boldsymbol{x}_k||_2 = \rho_k \quad \leq \quad ||\boldsymbol{x}_{k+1}||_2 = \rho_{k+1} \quad \text{for all } k \geq 1.$$

In particular, $\boldsymbol{u}_k \in \mathbb{S}^d$ for all $k \geq 1$. Since $\mathcal{O}$ has finite density,

$$\rho_k \quad \gg \quad \sqrt[d+1]{k}. \tag{5.6}$$

Fix $\epsilon \in (0, 1)$. Without loss of generality, one can assume that

$$\epsilon \leq \min \{\rho_k : k \geq 1\} . \tag{5.7}$$

By assumption, given a direction $\boldsymbol{v} \in \mathbb{S}^d$, there exists a real number

$$\rho \in (0, V(\epsilon)) \tag{5.8}$$

and an index $k \geq 1$ such that $||\rho\boldsymbol{v} - \rho_k\boldsymbol{u}_k||_2 \leq \epsilon$. From relation (4.32), this implies that $|\rho - \rho_k| \ll \epsilon$. Thus, from inequality (5.7), one obtains

$$\rho_k \quad \asymp \quad \rho. \tag{5.9}$$

Consequently, from relations (5.9), (4.32) (p.132) and (4.2) (p.115),

$$\mathrm{dist}_{\mathbb{S}^d}\left(\boldsymbol{u}_k, \boldsymbol{v}\right) \quad \asymp \quad ||\boldsymbol{u}_k - \boldsymbol{v}||_2 \quad \ll \quad \frac{\epsilon}{\rho_k}\cdot \tag{5.10}$$

As inequality (5.10) holds for any direction $\boldsymbol{v} \in \mathbb{S}^d$, it follows that the successive spherical caps

$$\mathcal{C}_d\left(\boldsymbol{u}_k, \quad C\cdot\frac{\epsilon}{\rho_k}\right), \quad \text{where } 1 \leq k \ll V(\epsilon)^{d+1}$$

and where $C > 0$ is a constant independent of $k$, cover the sphere $\mathbb{S}^d$. Here, the upper bound for the index $k$ is obtained by combining relations (5.6), (5.8) and (5.9).

Denoting by $A_d$ the surface area of $\mathbb{S}^d$, one thus deduces that, for some constant $C > 0$,

$$A_d \quad \ll \quad \sum_{k=1}^{C\cdot V(\epsilon)^{d+1}} \frac{\epsilon^d}{\rho_k^d} \quad \ll \quad \sum_{k=1}^{V(\epsilon)^{d+1}} \frac{\epsilon^d}{k^{d/d+1}} \quad \asymp \quad \epsilon^d \cdot V(\epsilon).$$

The claim follows. ■

The direct implication in Point 3(ii) in Theorem 5.1.2 holds in a generality greater than that of spiral point sets. This is why it is proved in the following proposition separately from the rest of the points.

**Proposition 5.2.1** *Let $\mathcal{O} \subseteq \mathbb{R}^{d+1}\backslash\{\mathbf{0}\}$ be a uniform orchard. Then, its set of visible points is empty.*

Proof Let $V$ denote a visibility function for the uniform orchard $\mathcal{O}$. Fix a point $\boldsymbol{x} \in \mathbb{R}^{d+1}$, a direction $\boldsymbol{v} \in \mathbb{S}^d$ and a real number $\epsilon > 0$. The goal is to show that $\mathrm{dist}_2\left(L\left(\boldsymbol{x}, \boldsymbol{v}\right), \mathcal{O}\right) \leq \epsilon$, where $L\left(\boldsymbol{x}, \boldsymbol{v}\right)$ is the ray emanating from $\boldsymbol{x}$ in direction $\boldsymbol{v}$ as defined in (1.15), p.31. Let $\boldsymbol{u} \in \mathbb{S}^d$ be a direction such that $L\left(\boldsymbol{x}, \boldsymbol{v}\right) \cap L\left(\mathbf{0}, \boldsymbol{u}\right) \neq \emptyset$ and

$$||\boldsymbol{v} - \boldsymbol{u}||_2 \quad = \quad \frac{\epsilon}{4\pi\cdot V\left(\epsilon/2\right)}\cdot \tag{5.11}$$

Set $\{\boldsymbol{w}\} = L\left(\boldsymbol{x}, \boldsymbol{v}\right) \cap L\left(\mathbf{0}, \boldsymbol{u}\right)$ and let $t_0 \geq 0$ be such that $\boldsymbol{w} = t_0 \cdot \boldsymbol{u}$. Since $\mathcal{O}$ is a uniform orchard, there exists

$$0 \quad \leq \quad t \quad \leq \quad V\left(\frac{\epsilon}{2}\right) \tag{5.12}$$

and $\boldsymbol{o} \in \mathcal{O}$ such that

$$||(t_0 + t) \cdot \boldsymbol{u} - \boldsymbol{o}||_2 \quad = \quad ||\boldsymbol{w} + t\boldsymbol{u} - \boldsymbol{o}||_2 \quad \leq \quad \frac{\epsilon}{2}. \tag{5.13}$$

Consequently,

$$\begin{aligned} \operatorname{dist}_2\left(L\left(\boldsymbol{x}, \boldsymbol{v}\right), \boldsymbol{0}\right) \quad &\leq \quad ||(\boldsymbol{w} + t \cdot \boldsymbol{v}) - \boldsymbol{o}||_2 \\ &\leq \quad ||\boldsymbol{w} + t \cdot \boldsymbol{u} - \boldsymbol{o}||_2 + t \cdot ||\boldsymbol{v} - \boldsymbol{u}||_2 \\ &\underset{(5.11),(5.12)\&(5.13)}{\leq} \quad \epsilon. \end{aligned}$$

The proof is complete. ■

Proposition 5.2.1 justifies the claim made in the introduction to this chapter, namely that the concepts of an orchard on the one hand and that of a uniform orchard on the other are rather different in nature when considering their visibility properties. This can already be seen in the case of spiral sets, which can be both orchards and have a non-empty set of visible points, in contrast with uniform orchards the set of visible points of which is always empty, as seen in Proposition 5.2.1. As established in the following statement, an example of a spiral in the plane which is an orchard but not a uniform orchard is obtained by considering the point set

$$\boldsymbol{\Lambda} \quad = \quad \left\{\sqrt{n} \cdot e\left(\frac{p}{q}\right)\right\}_{n \in \mathbb{N}}. \tag{5.14}$$

Here, given an integer $n \geq 1$, the integers $q$ and $p$ are defined by the unique decomposition of $n$ as

$$n = \frac{q(q+1)}{2} + p \qquad \text{with} \quad q \geq 1 \quad \text{and} \quad 0 \leq p \leq q. \tag{5.15}$$

**Proposition 5.2.2** *The spiral* $\boldsymbol{\Lambda}$ *defined in* (5.14) *is an orchard with visibility* $V(\epsilon) \ll \epsilon^{-1}$ *but has a non-empty set of visible points.*

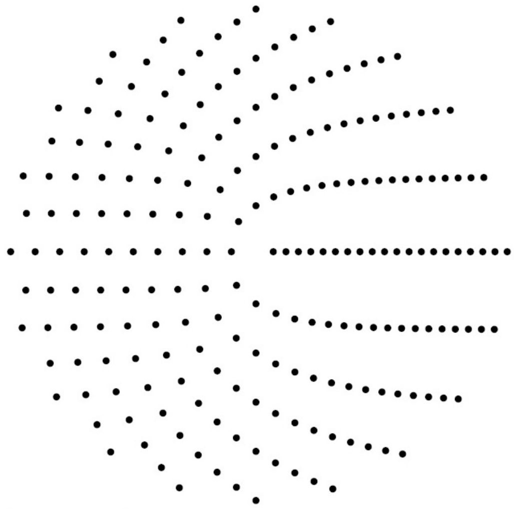


Figure 5.2: The Spiral $\mathbf{\Lambda}$.

The spiral $\mathbf{\Lambda}$ is depicted in Figure 5.2. In the same way as for the standard lattice in Pólya's original orchard problem, the points close to the origin play a preponderant role in blocking rays emanating from the origin. By contrast, horizontal half-lines not passing through the origin but close to it determine visible points.

Proof (Proposition 5.2.2) Let $\theta \in [0,1]$ and $N \geq 3$. From Dirichlet's Theorem in Diophantine approximation (Theorem 1.1.33, p.43), there exists a rational $p/q \in [0,1]$ with $1 \leq q \leq N$ such that

$$\left|\theta - \frac{p}{q}\right| \quad \leq \quad \frac{1}{qN}.$$

Therefore,

$$\left|e\left(\theta\right) - e\left(\frac{p}{q}\right)\right| \quad \leq \quad \frac{2\pi}{qN}.$$

Setting $n = q(q+1)/2 + p$, one thus gets that

$$\left|\sqrt{n} \cdot e\left(\theta\right) - \sqrt{n} \cdot e\left(\frac{p}{q}\right)\right| \quad \leq \quad \sqrt{\left(\frac{q(q+1)}{2} + p\right)} \cdot \frac{2\pi}{qN} \quad \leq \quad \frac{4\pi}{N}.$$

The bound $V(\epsilon) \ll \epsilon^{-1}$ for the visibility function follows upon noticing that

$\sqrt{n} \leq N$ when $N \geq 3$.

The claim that the set of visible points is non-empty is implied by the existence of a vacant strip of the form

$$S := \quad \left\{(x, y) \in \mathbb{R}^2 : \quad x > 0, \quad 0 < y < \delta\right\}$$

for some $\delta > 0$: this is saying that the set of points $(x, y) \in \mathbb{R}^2$ such that their second coordinate satisfies the constraint $0 < y < \delta$ does not contain any point of the spiral. To see this, note that the distance between a point $\sqrt{n} \cdot e\left(\frac{p}{q}\right)$ in the spiral set $\boldsymbol{\Lambda}$ (identified with the planar point $\left(\sqrt{n} \cdot \cos\left(2\pi p/q\right), \sqrt{n} \cdot \sin\left(2\pi p/q\right)\right)$ and the axis $\{(x, y) \in \mathbb{R}^2 : y = 0\}$ is $\sqrt{n} \cdot |\sin\left(2\pi p/q\right)|$. Since $0 \leq p \leq q$, this distance is non-zero if, and only if, $p \notin \{0, q\}$, in which case

$$\sqrt{n} \cdot \left|\sin\left(2\pi \cdot \frac{p}{q}\right)\right| \quad \geq \quad \frac{\sqrt{n}}{q} \quad \underset{(5.15)}{\gg} \quad 1.$$

This concludes the proof. $\blacksquare$.

The rest of this section is devoted to the proof of the remaining statements in Theorem 5.1.2.

Proof Let $\boldsymbol{U} = \left(\boldsymbol{u}_k\right)_{k \in \mathbb{N}}$ be a sequence in $\mathbb{S}^d$.

**Proof of Point 1.** By definition, the spiral $\mathfrak{S}\left(\boldsymbol{U}\right) = \left(\sqrt[d+1]{k} \cdot \boldsymbol{u}_k\right)_{k \in \mathbb{N}}$ is an orchard with visibility $V$ if, and only if, for any vector $\boldsymbol{v} \in \mathbb{S}^d$ and any real $\epsilon > 0$, there exists a real $0 < t < V(\epsilon)$ such that for some index $n \geq 1$,

$$\left\|t \cdot \boldsymbol{v} - \sqrt[d+1]{n} \cdot \boldsymbol{u}_n\right\|_2 \quad \leq \quad \epsilon.$$

Assume first that $\mathfrak{S}\left(\boldsymbol{U}\right)$ is an orchard and fix $\epsilon \in (0, 1)$ and $0 < t < V(\epsilon)$. From relations (4.2), p.115, and (4.32), p.132, one has that there exists $n \geq 1$ such that

$$\left|t - \sqrt[d+1]{n}\right| + \sqrt{t \cdot \sqrt[d+1]{n}} \cdot \operatorname{dist}_{\mathbb{S}^2}\left(\boldsymbol{u}_n, \boldsymbol{v}\right) \quad \underset{\substack{(4.2),\\(4.32)}}{\asymp} \quad \left\|t \cdot \boldsymbol{v} - \sqrt[d+1]{n} \cdot \boldsymbol{u}_n\right\|_2 \quad \leq \quad \epsilon.$$

One infers from this identity, on the one hand that for $\epsilon > 0$ small enough, $\sqrt[d+1]{n} \asymp t \ll V(\epsilon)$ and, on the other, that $\mathrm{dist}_{\mathbb{S}^d}(\boldsymbol{u}_n, \boldsymbol{v}) \ll \epsilon / \sqrt[d+1]{n}$. Conversely, assuming that one can find an index $n \ll V(\epsilon)^{d+1}$ such that $\mathrm{dist}_{\mathbb{S}^d}(\boldsymbol{u}_n, \boldsymbol{v}) \ll \epsilon / \sqrt[d+1]{n}$, setting $t = \sqrt[d+1]{n}$ proves that $\mathfrak{S}(\boldsymbol{U})$ is an orchard with visibility a function $W$ satisfying

$$W(\epsilon) \quad = \quad c_{\boldsymbol{U}} \cdot V(\eta \cdot \epsilon)$$

for some large constant $c_{\boldsymbol{U}} > 0$ and a small constant $\eta > 0$, both depending only on $\boldsymbol{U}$. Since the function $V$ has been assumed to have polynomial growth rate, the constant $\eta$ can be absorbed in the constant $c_{\boldsymbol{U}}$, yielding the claim.

**Proof that 2(a) implies 2(b).** Assume that the spiral $\mathfrak{S}(\boldsymbol{U})$ is a uniform orchard with visibility $W = c \cdot V$ for some constant $c = c_{\boldsymbol{U}}$. Fix a direction $\boldsymbol{v} \in \mathbb{S}^d$, a real $\epsilon \in (0,1)$ and an integer

$$R \geq V(\epsilon).$$

Since $\mathfrak{S}(\boldsymbol{U})$ is a uniform orchard, there exists a real $t \in [R, R + W(\epsilon)]$ and integer $k \geq 1$ such that

$$\left|\left| \sqrt[d+1]{k} \cdot \boldsymbol{u}_k - t \cdot \boldsymbol{v} \right|\right|_2 \quad \leq \quad \epsilon.$$

It is clear that $k > (R-1)^{d+1}$. Therefore, from relations (4.2), p.115, and (4.32), p.132, one obtains

$$\sqrt[d+1]{k} \asymp t \qquad \text{and} \qquad R\epsilon^{-1} \cdot \mathrm{dist}_{\mathbb{S}^d}(\boldsymbol{u}_k, \boldsymbol{v}) \ll 1. \tag{5.16}$$

The choice of $\boldsymbol{v} \in \mathbb{S}^d$ and $R \geq 1$ is arbitrary. Inequality (5.3) then follows upon taking the supremum over $\boldsymbol{v}$ and $R$ in both sides of the right-hand inequality in (5.16). The claim is proved.

**Proof that 2(b) implies 2(a).** Assume that there exists a constant $K > 0$ such that assumption (5.3) holds for $W(\epsilon) = K \cdot V(\epsilon)$. Fix a direction $\boldsymbol{v} \in \mathbb{S}^d$ and real numbers $\epsilon \in (0,1)$ and $R \geq V(\epsilon)$, and define the line segment

$$L \quad := \quad \left\{ t \cdot \boldsymbol{v} : t \in \left[ R^{d+1}, (R + W(\epsilon))^{d+1} \right] \right\}.$$

By assumption (5.3), there exists $k \in \left[\!\left[R^{d+1}, (R + W(\epsilon))^{d+1}\right]\!\right]$ such that

$$\operatorname{dist}_{\mathbb{S}^d}(\boldsymbol{u}_k, \boldsymbol{v}) \quad \ll \quad \frac{\epsilon}{R}. \tag{5.17}$$

Inequality (5.17) implies that

$$\operatorname{dist}_2(\mathfrak{S}, L) \quad \leq \quad C_1 \cdot \epsilon$$

with $C_1 = 2\pi \cdot (K+1)$, since

$$\operatorname{dist}_2(\mathfrak{S}, L) \quad \leq \quad \left\| \sqrt[d+1]{k} \cdot \boldsymbol{u}_k - \sqrt[d+1]{k} \cdot \boldsymbol{v} \right\|_2 \quad \underset{(4.2)}{\leq} \quad 2\pi \cdot (K+1) \cdot \epsilon.$$

In other words, the point set $\mathfrak{S}(\boldsymbol{U})$ is a uniform orchard with visibility

$$V_0(\epsilon) \quad \leq \quad 2 \cdot W\left(\frac{\epsilon}{C_1}\right) \quad \leq \quad \boldsymbol{c_U} \cdot V(\epsilon),$$

where the last inequality holds for some constant $\boldsymbol{c_U}$ under the assumption that the function $V$ has polynomial growth rate.

**Proof of Point 2(i).** Assume that $\mathtt{U}_C(\boldsymbol{U}) < +\infty$. From the definitions of the quantities $\mathtt{U}_C(\boldsymbol{U})$ and $G_R^N(\boldsymbol{U})$, one infers that for all $R, N \geq 1$ with $R \geq N$,

$$G_R^N \quad \leq \quad C \cdot \frac{1}{R \cdot \sqrt[d]{N}}$$

for some constant $C > 0$. Specialising this relation in the case when, given $\epsilon \in (0,1)$ and $R \geq C \cdot \epsilon^{-d}$, $N$ is the integer part of $\left\{2C \cdot \epsilon^{-d}\right\}$ shows that Point 2($b$) holds with $W(\epsilon) = A \cdot \epsilon^{-d}$ for some $A > 0$, whence the claim.

**Proof of Point 2(ii).** The assumption on the limit implies that the function $V$ in (5.4) is well-defined. It is then immediate that the condition stated in Point 2($b$) is satisfied with $W = V$, whence the claim.

**Proof of Point 3(i).** A half-line $L$ in $\mathbb{R}^{d+1}$ can be parametrised as the set $\{\lambda \boldsymbol{v} + t\boldsymbol{w} : t \geq t_0\}$, where $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{S}^d$ are orthogonal, $t_0 \in \mathbb{R}$ and where $\lambda \geq 0$ is the distance from $L$ to the origin. Without loss of generality, set $t_0 = 0$. Given $\epsilon > 0$, a point in $\mathfrak{S}(\boldsymbol{U}) = \left\{ \sqrt[d+1]{k} \cdot \boldsymbol{u}_k \right\}_{k \in \mathbb{N}} \subseteq \mathbb{R}^{d+1}$ lies $\epsilon$-close to the half-line if, and only if, there exists an integer $n \geq 1$ and a parameter $t \geq 0$ such that

$$\left\| \sqrt[d+1]{n} \cdot \boldsymbol{u}_n - (\lambda \boldsymbol{v} + t \boldsymbol{w}) \right\|_2 \quad \leq \quad \epsilon.$$

From relations (4.2) and (4.32), up to multiplicative constants, this is equivalent to asking that

$$\left| \sqrt[d+1]{n} - \sqrt{\lambda^2 + t^2} \right| \ll \epsilon \quad \text{and} \quad \sqrt{\sqrt[d+1]{n} \cdot \sqrt{\lambda^2 + t^2}} \cdot \operatorname{dist}_{\mathbb{S}^d} \left( \boldsymbol{u}_n, \frac{\lambda \boldsymbol{v} + t \boldsymbol{w}}{\sqrt{\lambda^2 + t^2}} \right) \ll \epsilon.$$

Since $n \geq 1$, the first relation implies that $\sqrt[d+1]{n} \asymp \sqrt{\lambda^2 + t^2}$ which, together with the second relation, is easily seen to yield the claimed equivalence.

**Proof that converse does not hold in Point 3(ii).** The goal is to construct a spiral $\mathfrak{S}(\boldsymbol{U})$ which has an empty set of visible points but which is not a uniform orchard. To this end, consider first a uniform orchard $\mathfrak{S}(\boldsymbol{U}') = \left( \sqrt[d+1]{k} \cdot \boldsymbol{u}'_k \right)_{k \in \mathbb{N}}$ with visibility, say, $V(\epsilon) \ll \epsilon^{-d}$ (such a point set exists from the comments following the statement of Theorem 5.1.2). Let $\rho_n = 2 \cdot V(2^{-n})$, in such a way that the quantity $n! - \rho_n$ is positive for any $n$ larger than some integer $n_0 \geq 1$. For $n \geq n_0$ define $\mathcal{A}_n$ as the annulus with outer radius $n!$ and inner radius $n! - \rho_n$ (see Figure[2] 5.3).

Let $\delta > 0$ and $\boldsymbol{v}_0 \in \mathbb{S}^d$. Let $\mathcal{D}$ be the intersection between $\mathbb{R}^{d+1} \setminus (\cup_{n \geq n_0} \mathcal{A}_n)$ and the $\delta$-neighbourhood of the ray emanating from the origin in direction $\boldsymbol{v}_0 \in \mathbb{S}^d$. The spiral $\mathfrak{S}(\boldsymbol{U})$ is then defined from the uniform orchard $\mathfrak{S}(\boldsymbol{U}')$ as follows: if the index $k \geq 1$ is such that $\sqrt[d+1]{k} \cdot \boldsymbol{u}'_k$ lies in $\mathcal{D}$, then set $\boldsymbol{u}_k = \boldsymbol{u}'_{m_k}$, where $m_k \geq k$ is the smallest index such that $\sqrt[d+1]{m_k} \cdot \boldsymbol{u}_{m_k} \notin \mathcal{D}$ (the existence of this index is guaranteed by the uniform orchard property of $\mathfrak{S}(\boldsymbol{U}')$). Otherwise, set $\boldsymbol{u}_k = \boldsymbol{u}'_k$.

[2] The picture in the electronic copy of the thesis is in colour.

Clearly, $\mathfrak{S}(\boldsymbol{U})$ is not a uniform orchard since it has no point in the region $\mathcal{D}$ which contains arbitrarily long line segments (supported in the direction determined by $\boldsymbol{v}_0$). To show that it has an empty set of visible points, consider, given a point $\boldsymbol{x} \in \mathbb{R}^{d+1}$ and a direction $\boldsymbol{v} \in \mathbb{S}^d$, the ray $L(\boldsymbol{x}, \boldsymbol{v})$ as defined in (1.15). For $n \geq n_0$ large enough, there exist exactly two points $\boldsymbol{a}_n, \boldsymbol{b}_n \in L(\boldsymbol{x}, \boldsymbol{v})$ with smallest and largest norms, respectively, intersecting the annulus $\mathcal{A}_n$. Denote by $\boldsymbol{w} \in \mathbb{S}^d$ the direction of the half-line $L'_n$ joining the origin to $\boldsymbol{b}_n$ and let $\boldsymbol{c}_n$ be the point with minimal norm in $\mathcal{A}_n$ lying in $L'_n$. Clearly, the largest distance between a point in the intersection $L(\boldsymbol{x}, \boldsymbol{v}) \cap \mathcal{A}_n$ and a point in $L'_n \cap \mathcal{A}_n$ is, for $n$ large enough, the quantity $\epsilon_n = ||\boldsymbol{a}_n - \boldsymbol{c}_n||_2$.

Elementary trigonometric considerations then show that $\epsilon_n$ is at most a constant multiple (depending on $\boldsymbol{x}$ and $\boldsymbol{v}$) of

$$\frac{||\boldsymbol{b}_n - \boldsymbol{c}_n||_2}{||\boldsymbol{b}_n||_2} \quad = \quad \frac{\rho_n}{n!}.$$

Thus, $\lim_{n\to+\infty} \epsilon_n = 0$.

Since $\mathfrak{S}(\boldsymbol{U})$ coincides with $\mathfrak{S}(\boldsymbol{U}')$ on $\mathcal{A}_n$ and since $\mathfrak{S}(\boldsymbol{U}')$ is a uniform orchard, there exists a point in $\mathfrak{S}(\boldsymbol{U}')$ which is $2^{-n}$ close to $L'_n \cap \mathcal{A}_n$, and therefore $(2^{-n} + \epsilon_n)$-close to $L(\boldsymbol{x}, \boldsymbol{v})$. Upon letting $n$ tend to infinity, this shows that $\mathrm{dist}_2(L(\boldsymbol{x}, \boldsymbol{v}), \mathfrak{S}(\boldsymbol{U})) = 0$, which concludes the proof.

**Proof of Point 4.** Let $\epsilon > 0$. A line segment $L$ with length $V(\epsilon)$ in $\mathbb{R}^{d+1}$ can be parametrised as the set

$$L \quad = \quad \{\lambda \boldsymbol{v} + t\boldsymbol{w} : t_0 \leq t \leq t_0 + V(\epsilon)\},$$

where $\boldsymbol{v}, \boldsymbol{w} \in \mathbb{S}^d$ are orthogonal and where $\lambda \geq 0$ and $t_0$ are reals. A point in $\mathfrak{S}(\boldsymbol{U}) = \left\{\sqrt[d+1]{k} \cdot \boldsymbol{u}_k\right\}_{k\in\mathbb{N}} \subseteq \mathbb{R}^{d+1}$ lies $\epsilon$-close to this line segment if, and only if, there exist an integer $n \geq 1$ and a parameter $t \in [t_0, t_0 + V(\epsilon)]$ such that (5.5) holds. The argument is then concluded in the same way as in the proof of Point 3($i$). ■

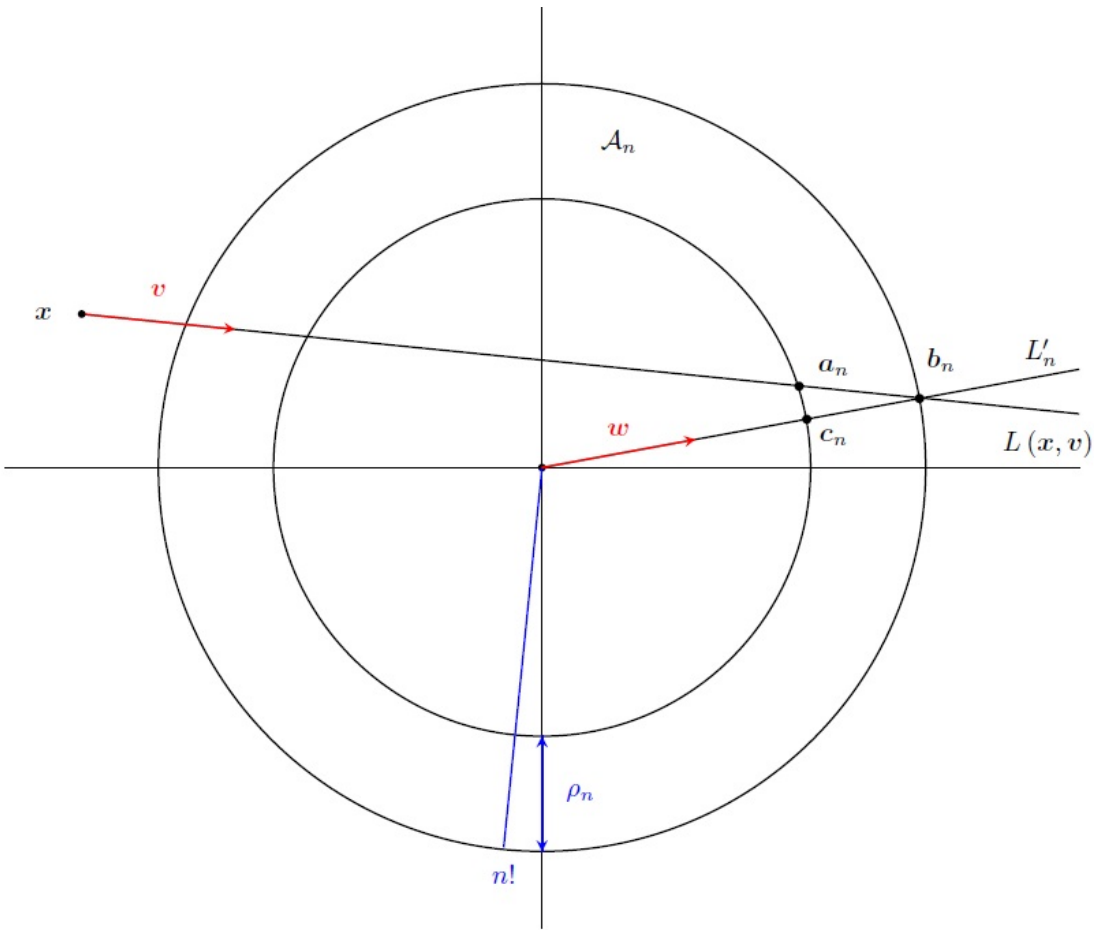



Figure 5.3: An illustration of the proof of the converse implication of Point 3$(ii)$.

# Chapter 6

# Density of Oscillating Sequences in the Real Line

## 6.1 Introduction

Given a real number $x$, denote by $\{x\}_2$ the signed fractional part of $x$, which is the unique real number in $\left[-\frac{1}{2}, \frac{1}{2}\right)$ such that $x - \{x\}_2 \in \mathbb{Z}$. Recall that, $\{x\}$ stands for the fractional part of $x$ and that $||x||$ denotes its distance from the nearest integer: $||x|| = \min_{n\in\mathbb{Z}} |x - n|$.

It is asked in [60] whether the sequence $(k \cdot \sin(k))_{k\in\mathbb{N}}$ is dense in $\mathbb{R}$. More generally, it is natural to determine the values of the parameters $\beta > 0$ and $\alpha \in \mathbb{R}$ for which the oscillating sequence $\left(k^\beta \cdot \sin(2\pi \cdot k\alpha)\right)_{k\in\mathbb{N}}$ is dense in $\mathbb{R}$. In this chapter, we answer this question by studying the density properties in $\mathbb{R}$ of the more general class of oscillating sequences of the form

$$(g(k) \cdot F(k\alpha))_{k\in\mathbb{N}}, \tag{6.1}$$

where

$$g(t) \quad = \quad t^\beta + o\left(t^\beta\right) \quad \text{as } t \to +\infty \tag{6.2}$$

for some $\beta > 0$, and where the function $F$ is a real, 1-periodic, continuous function with only isolated roots. We assume further that, if $r \in \mathbb{R}$ is a root of $F$, then $F$

admits an expansion of the form

$$F(r+x) \quad = \quad c_r \cdot \epsilon(x) \cdot |x|^{\gamma(r)} + o\left(|x|^{\gamma(r)}\right) \quad \text{as } x \to 0 \tag{6.3}$$

for some $\gamma(r) > 0$ and some $c_r \in \mathbb{R}\setminus\{0\}$. Here, the function $\epsilon : \mathbb{R} \mapsto \{-1, 0, 1\}$ stands for the sign function

$$\epsilon(x) \quad = \quad \begin{cases} 1, & \text{if } x > 0 \\ 0, & \text{if } x = 0 \\ -1, & \text{if } x < 0. \end{cases} \tag{6.4}$$

A study of the density of oscillating sequences in the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ has been done by Berend, Boshernitzan and Kolesnik $--$ see [19, 20, 21]. In this body of work, the authors consider oscillating sequences of the form

$$\left(P(k) \cdot f\left(Q(k)\right)\right)_{k\in\mathbb{N}}, \tag{6.5}$$

where $P, Q$ are polynomials and $f$ is a (highly differentiable) periodic function with period $T > 0$. In particular, they consider three aspects of the problem: the problem of small values modulo 1 of such sequences, that of their density modulo 1, and that of their uniform distribution.

More precisely, in [19], the authors deal with the above-stated three problems by providing in each case sufficient conditions on the degree of differentiability of the function $f$ at the point $Q(0)$ for the related properties to hold. In [20], they generalise the results regarding the small values and the density of the sequence (6.5) in two directions. On the one hand, they allow the function $f$ to be quasi-periodic; that is $f(x) = f_0(x, x, ..., x)$, where $f_0 : \mathbb{R}^n \to \mathbb{R}$ is a periodic function of several variables. On the other hand, they study a more general family of sequences, namely sequences of the form $\left(P(k) \cdot f\left(Q(k)\right) \cdot g\left(R(k)\right)\right)_{k\in\mathbb{N}}$, where $R(k)$ is a polynomial and the function $g$ is periodic. For instance, they prove that, given integers $d$ and $l$, there exists $r = r(d, l)$ having the following properties: for any polynomial $P$ of degree $d$, any function $f$ with $f^{(s)}(0) \neq 0$ for some $s \geq r$ and any real number $\alpha$ with $\frac{\alpha}{T}$ irrational, the sequence $\left(P(k) \cdot f\left(k^l \cdot \frac{\alpha}{T}\right)\right)_{k\in\mathbb{N}}$ is dense modulo 1.

Other results regarding the distribution of the sine function in the real line are given for instance in [3]. In this paper, Adiceam establishes a result concerning rational approximations of irrationals with the numerators and the denominators of the rational approximants restricted to prescribed arithmetic progressions, and proves that for every $\rho \in \mathbb{R}$ and irrational $\alpha$, it holds that

$$\limsup_{k\to+\infty} \left(\sin\left(2\pi k\alpha+\rho\right)\right)^k \quad = \quad -\liminf_{k\to+\infty} \left(\sin\left(2\pi k\alpha+\rho\right)\right)^k \quad = \quad 1.$$

Our approach to study the sequence (6.1) makes a connection between its density properties in $\mathbb{R}$ and the density properties of auxiliary sequences of the form

$$\left(k^{\beta}\cdot\{k\alpha-\rho\}_2\right)_{k\in\mathbb{N}}, \tag{6.6}$$

where $\rho$ is a real number (see Proposition 6.2.1 in Section 6.2 below for details). Working with the signed fractional part instead of the distance from the nearest integer, which may seem more natural, is a consequence of working in the real line as one has to consider separately the positive and the negative values of the function (6.3).

Little seems to be known regarding the density of oscillating sequences *in the real line*. One of the goals of this paper is to relate the density of (6.1) with the approximation properties of $\alpha$. Here, by approximation properties we are referring to the irrationality measure $\mu(\alpha)$ of $\alpha$:

$$\mu(\alpha)=\sup\left\{v\ >0:\left|\alpha-\frac{p}{q}\right|\le\frac{1}{q^v}\quad \begin{array}{r}\text{holds for infinitely many rationals } \dfrac{p}{q}\\ \text{with } \gcd(p,q)=1\end{array}\right\}.$$

It can readily be checked that every rational number $r$ has irrationality measure $\mu(r)=1$ while, from Dirichlet's theorem in Diophantine approximation, for every irrational $x$ it holds that $\mu(x)\ge 2$. We consider more precisely some additional quantities which refine the notion of irrationality measure. To define them, one needs the concepts of the continued fraction expansion of an irrational number $\alpha$ [Chapter 1, Equation (1.43), p.47] and of the Ostrowski expansion of a real number

$\rho$ with base $\alpha$ (Chapter 1, Definition 1.1.42, Equation (1.52), p.51). Throughout this chapter, the sequence of the denominators of the convergents of an irrational $\alpha$ defined in (1.45), p.48, is denoted by $(q_n)_{n\in\mathbb{N}}$.

**Definition 6.1.1 (Signed Irrationality Evaluation)** *Given an irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, a positive real number $\beta > 0$ and a real number $\rho \in \mathbb{R}$, denote by $\mu_+(\alpha,\beta,\rho)$ and $\mu_-(\alpha,\beta,\rho)$ the quantities*

$$\mu_+(\alpha,\beta,\rho) = \liminf_{\substack{k\to+\infty,\\ \{k\alpha-\rho\}_2>0}} k^\beta \cdot \{k\alpha-\rho\}_2 \geq 0$$

*and*

$$\mu_-(\alpha,\beta,\rho) = \liminf_{\substack{k\to+\infty,\\ \{k\alpha-\rho\}_2<0}} -k^\beta \cdot \{k\alpha-\rho\}_2 \geq 0.$$

*Moreover, denote by $\mu(\alpha,\beta,\rho) = \min\{\mu_+(\alpha,\beta,\rho), \mu_-(\alpha,\beta,\rho)\}$ the minimum of the above two quantities. When $\rho = 0$, one may write $\mu_+(\alpha,\beta), \mu_-(\alpha,\beta)$ and $\mu(\alpha,\beta)$ to simplify notation.*

Given the Ostrowski expansion (1.52) of $\rho$, set further

$$\tau_+(\alpha,\beta,\rho) := \liminf_{n\to+\infty} \max\left\{1, \min\left\{e_{2n}(\rho)^\beta, (a_{2n+1}-e_{2n}(\rho))^{\frac{\beta+1}{2}}\right\}\right\}\cdot q_{2n}^\beta \cdot \{q_{2n}\alpha\}_2 \geq 0$$

and

$$\tau_-(\alpha,\beta,\rho) := \liminf_{n\to+\infty} -\max\left\{1, \min\left\{e_{2n-1}(\rho)^\beta, (a_{2n}-e_{2n-1}(\rho))^{\frac{\beta+1}{2}}\right\}\right\}\cdot q_{2n-1}^\beta \cdot \{q_{2n-1}\alpha\}_2 \geq 0.$$

Our main result provides necessary and also sufficient conditions on the oscillating sequence (6.1) to be dense in $\mathbb{R}$.

**Theorem 6.1.2** *Denote by $(y_k)_{k\in\mathbb{N}}$ the sequence defined in* (6.1)*. Let the function $F$ satisfy assumption* (6.3) *and let $g$ satisfy assumption* (6.2)*.*

1. *If the sequence* (6.1) *is dense in $\mathbb{R}^+$ (resp. in $\mathbb{R}^-$), then there exists a root $r$ of $F$ such that either $c_r > 0$ (resp. $c_r < 0$) and $\mu_+\left(\alpha, \frac{\beta}{\gamma(r)}, r\right) = 0$, or else $c_r < 0$ (resp. $c_r > 0$) and $\mu_-\left(\alpha, \frac{\beta}{\gamma(r)}, r\right) = 0$. Moreover, if the root $r$ is rational, then this condition is also sufficient.*

2. *If there exists a root $r$ of $F$ such that either $c_r > 0$ (resp. $c_r < 0$) and $\tau_+\left(\alpha, \frac{\beta}{\gamma(r)}, r\right) = 0$, or else $c_r < 0$ (resp. $c_r > 0$) and $\tau_-\left(\alpha, \frac{\beta}{\gamma(r)}, r\right) = 0$, then the sequence* (6.1) *is dense in $\mathbb{R}^+$ (resp. in $\mathbb{R}^-$).*

Under the assumptions of Theorem 6.1.2, the density of the oscillating sequence (6.1) depends only on the local properties of $F$ around its isolated roots. In order to prove Theorem 6.1.2, the results are first established for the auxiliary sequence (6.6). Thus, in Section 6.3, is proved that if $\rho$ is rational, then the sequence (6.6) is dense in $\mathbb{R}^+$ (resp. $\mathbb{R}^-$) if and only if $\mu_+(\alpha, \beta, \rho) = 0$ (resp. if and only $\mu_-(\alpha, \beta, \rho) = 0$). In Section 6.4, we will use the Ostrowski expansion in order to prove that, if $\tau_+(\alpha, \beta, \rho) = 0$ (resp. $\tau_-(\alpha, \beta, \rho) = 0$), then the squence (6.6) is dense in $\mathbb{R}^+$ (resp. in $\mathbb{R}^-$).

In the special case where $F(x) = \sin(2\pi \cdot x)$, one obtains the following corollary answering the opening question of the paper.

**Corollary 6.1.3** *Given $\beta > 0$ and $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, the sequence*

$$\left(k^\beta \cdot \sin(2\pi \cdot k\alpha)\right)_{k\in\mathbb{N}}$$

*is dense in $\mathbb{R}$ if and only if at least one of the following holds:*

1. $\mu_+(\alpha, \beta) = 0$ *and* $\mu_-(\alpha, \beta) = 0$,
2. $\mu_+(\alpha, \beta) = 0$ *and* $\mu_+\left(\alpha, \beta, \frac{1}{2}\right) = 0$,
3. $\mu_-(\alpha, \beta) = 0$ *and* $\mu_-\left(\alpha, \beta, \frac{1}{2}\right) = 0$.

For instance, one can apply Corollary 6.1.3 when $\alpha$ is badly approximable; that is, when there exists $c > 0$ such that for every $k \in \mathbb{N}$, it holds $k \cdot ||k\alpha|| \geq c$. In this case, for every $\beta \geq 1$, it holds that $\mu(\alpha, \beta) > 0$ and therefore $\left(k^\beta \cdot \sin(2\pi \cdot k\alpha)\right)_{k\in\mathbb{N}}$ is not dense in $\mathbb{R}$. Similarly, if $\beta < 1$ it holds that $\mu_\pm(\alpha, \beta) = 0$ and the same sequence is dense in $\mathbb{R}$.

**Remark 6.1.4** *From Definition 6.1.1, it follows immediately that*

$$\mu(\alpha, \beta, r) = \liminf_{k\to+\infty} k^\beta \cdot ||k\alpha - r|| \,.$$

*However more natural this quantity may seem, as proved in Section 6.5, it does not hold that $\mu_+(\alpha, \beta, r) = 0$ if and only if $\mu_-(\alpha, \beta, r) = 0$. This is the reason why the results are not stated in terms of the quantity $\mu(\alpha, \beta, r)$ alone.*

Theorem 6.1.2 also yields the following corollary stating some cases where the sequence (6.1) is trivially dense in $\mathbb{R}$.

**Corollary 6.1.5** *Let $(y_k)_{k\in\mathbb{N}}$ be the sequence defined in* (6.1) *with the function $F$ satisfying assumption* (6.3) *and $g$ satisfying assumption* (6.2)*. If there exists a root $r \in \mathbb{R}$ of $F$ such that $\frac{\beta}{\gamma(r)} \in (0, 1)$, then the sequence* (6.1) *is dense in $\mathbb{R}$.*

Note that the sufficient condition stated in Theorem 6.1.2 is not necessary. This is proved in Section 6.4 by explicitly constructing a suitable sequence $(e_n)_{n\geq 0}$ in the Ostrowski expansion (1.52), p.51.

In addition to Theorem 6.1.2, the following result is established which, in the case where $\rho \in \mathbb{Q}$, characterizes the quantities $\mu_\pm(\alpha, \beta, \rho)$ in terms of the sequence of denominators of the convergents to the irrational $\alpha$.

**Theorem 6.1.6** *Given $\beta \geq 1$, an irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and a rational number $\theta \in \mathbb{Q}$, where $\theta = \frac{p}{q}$ for some $p \in \mathbb{Z}, q \in \mathbb{N}$ with $(p, q) = 1$, it holds that*

$$\mu_+\left(\alpha, \beta, \frac{p}{q}\right) = 0 \qquad \left(\textit{resp.} \quad \mu_-\left(\alpha, \beta, \frac{p}{q}\right) = 0\right)$$

*if and only if*

$$\liminf_{\substack{n\to+\infty,\\ q\mid q_{2n}}} q_{2n}^{\beta} \cdot \{q_{2n}\alpha\}_2 = 0 \qquad \left(\textit{resp.} \quad \liminf_{\substack{n\to+\infty,\\ q\mid q_{2n-1}}} q_{2n-1}^{\beta} \cdot \{q_{2n-1}\alpha\}_2 = 0\right).$$

Finally, we provide results regarding the density of oscillating sequences (6.1) in $\mathbb{R}$ when the parameters $\alpha$ and $\beta$ satisfy $\mu(\alpha, \beta) = +\infty$. Note that the inequalities $\mu_+(\alpha, \beta) \leq \tau_+(\alpha, \beta, \rho)$ and $\mu_-(\alpha, \beta) \leq \tau_-(\alpha, \beta, \rho)$ hold for every choice of $\alpha, \beta$ and $\rho$ (see Lemma 6.5.1). The aforementioned assumption therefore implies that, for every real $\rho$, $\tau_+(\alpha, \beta, \rho) = \tau_-(\alpha, \beta, \rho) = +\infty$. Thus, the sufficient condition in the statement of Theorem 6.1.2 does not hold. Before stating the result, recall the definition of inhomogeneous Bohr sets (see [31, 32] and [76, March 2021] for more details): given a real number $\rho$, an irrational number $\alpha$, a natural number $N$ and a positive number $\epsilon > 0$, let

$$\mathcal{N}_\rho\left(N, \alpha, \epsilon\right) \quad = \quad \left\{k \in \mathbb{N} : \quad k \leq N, \quad ||k\alpha - \rho|| \leq \epsilon\right\}. \tag{6.7}$$

We use Bohr sets in order to capture the terms of the sequence (6.6) which affect its density properties in $\mathbb{R}$. Given the Ostrowski expansion (1.52) of $\rho$, p.51, define a sequence of natural numbers by setting

$$\kappa_n := \quad \sum_{j=0}^{n} e_j(\rho) \cdot q_j \quad \text{for all } n \geq 0. \tag{6.8}$$

**Theorem 6.1.7** *Let $\alpha$ be an irrational number and let $\beta > 0$ be such that $\mu(\alpha, \beta) = +\infty$. Denote by $(w_k)_{k\in\mathbb{N}}$ the sequence defined in* (6.6)*. Let $\rho$ be a real number and let $(e_j(\rho))_{j\geq 0}$ be the digits in its Ostrowski expansion (Definition 1.1.42, Equation* (1.52)*, p.51). Also, let $(\kappa_n)_{n\in\mathbb{N}}$ be the sequence defined in* (6.8)*. Then, the sequence $(w_k)_{k\in\mathbb{N}}$ is dense in $\mathbb{R}$ if and only if the subsequence $(w_k)_{k\in\mathfrak{D}}$ is dense in $\mathbb{R}$, where*

$$\mathfrak{D} \quad = \quad \bigcup_{n=0}^{+\infty} \left(\mathcal{N}_\rho\left(n\right) \cup \mathcal{N}'_\rho\left(n\right)\right) \tag{6.9}$$

*with $\mathcal{N}_\rho(n) = \mathcal{N}_\rho\left(\kappa_{n+1}, \alpha, ||q_{n+1}\alpha||\right)$ and $\mathcal{N}'_\rho(n) = \mathcal{N}_\rho\left(\kappa_n + q_{n+1}, \alpha, ||q_n\alpha||\right)$. Moreover, the inclusions*

$$\left\{\kappa_n\right\}_{n\in\mathbb{N}} \quad \subseteq \quad \mathfrak{D} \quad \textit{and} \quad \mathfrak{D} \quad \subseteq \quad \bigcup_{n=0}^{+\infty} \left(\mathcal{M}_\rho(n) \cup \mathcal{M}'_\rho(n)\right) \tag{6.10}$$

*hold, where*

$$\mathcal{M}_\rho(n) := \bigcup_{l=0}^{2} \left\{\kappa_n + (e_{n+1} - l) \cdot q_{n+1}\right\}$$

*and*

$$\mathcal{M}'_\rho(n) := \bigcup_{l=0}^{1} \left\{\kappa_n + (l+1)q_n, \kappa_n + q_{n+1} - lq_n\right\}.$$

This work leaves open the question of determining the density properties of the oscillating sequence (6.1) defined by more general growth functions than those of the form (6.2).

The chapter is organized as follows. In Section 6.2, the study of (6.1) is reduced to that of the auxiliary sequences (6.6). In Section 6.3, we study the case where $\rho$ is rational and establish in this case the first statement in Theorem 6.1.2. In Section 6.4, the Ostrowski expansion [Definition 1.1.42, Equality (1.52), p.51] is used to prove sufficient conditions for (6.6) to be dense in $\mathbb{R}$ when the root $r$ is irrational. Moreover, given parameters $\alpha$ and $\beta$ and a prescribed positive quantity $\gamma$, we provide an effective construction of the sequence $(e_n)_{n\geq 0}$ in the expansion (1.52), p.51, ensuring that oscillating sequences of the form (6.1) are dense in $\mathbb{R}$ and satisfy $\gamma(r) = \gamma$, for some root $r$ of $F$. In Section 6.5, the results from the previous sections are used to complete the proof of Theorem 6.1.2 and to prove Theorem 6.1.6. Theorem 6.1.7 is proved in Section 6.6.

## 6.2 Some Auxiliary Results

The goal of this section is to reduce the study of the density of sequence (6.1) to that of the sequence (6.6).

**Proposition 6.2.1** *Let $F : \mathbb{R} \to \mathbb{R}$ be a 1-periodic function satisfying assumption* (6.3)*. Let also $g : \mathbb{R}^+ \to \mathbb{R}^+$ satisfy assumption* (6.2) *and let $(a_k)_{k\in\mathbb{N}}$ be a sequence of real numbers. Then, a real number $h \in \mathbb{R}$ is a limit point of the sequence $(g(k) \cdot F(a_k))_{k\in\mathbb{N}}$ if and only if there exists a root $r$ of $F$ such that $h$ lies in the closure of the set*

$$\left\{\epsilon\left(\{a_k - r\}_2\right) \cdot c_r \cdot k^{\beta} \cdot ||a_k - r||^{\gamma(r)}\right\}_{k\in\mathbb{N}},$$

*where $\epsilon : \mathbb{R} \mapsto \{-1, 0, 1\}$ is the sign function defined in* (6.4)*.*

To prove Proposition 6.2.1, one needs the following lemma which allows us to remove the error terms from the definitions of the growth rate function in (6.2) and the periodic function in (6.3). Its proof, which is elementary, is left to the reader.

**Lemma 6.2.2** *Let $\boldsymbol{f} = (f_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{R}$ such that $f_k \underset{k\to+\infty}{\longrightarrow} 0$. Let $g : \mathbb{R}^+ \to \mathbb{R}^+$ be an increasing function such that $g(t) \underset{t\to+\infty}{\longrightarrow} +\infty$. Let also $u, v$ be* *real functions such that*

$$\lim_{t\to+\infty} u(t) = 0 \quad \textit{and} \quad \lim_{x\to 0} v(x) = 0.$$

*Then, the sequences*

$$(g(k)\cdot f_k)_{k\in\mathbb{N}}\,, \quad ((g(k)+u(k)\cdot g(k))\cdot f_k)_{k\in\mathbb{N}} \quad \textit{and} \quad (g(k)\cdot(f_k+v\,(f_k)\cdot f_k))_{k\in\mathbb{N}}$$

*are pairwise asymptotically equal*[1] *and have therefore the same limit points.*

We now deduce Proposition 6.2.1.

PROOF (PROPOSITION 6.2.1:) By assumption, the function $F$ is 1-periodic, continuous in $\mathbb{R}$ and has only isolated roots in $[0,1)$. Therefore, it admits only finitely many roots in the interval $[0,1)$. Let $r_0 < r_1 < .... < r_m$ be the finitely many distinct roots of $F$ in $[0,1)$. Fix $h\in\mathbb{R}$, where $h$ is a limit point of the sequence $(g(k)\cdot F\,(a_k))_{k\in\mathbb{N}}$. Thus, there exists a sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that $\lim\limits_{n\to+\infty} g\,(k_n)\cdot F\,(a_{k_n}) = h$. This implies that $\lim\limits_{n\to+\infty} F\,(a_{k_n}) = 0$ because $g(t) \underset{t\to+\infty}{\longrightarrow} +\infty$. By passing to a subsequence if necessary, the sequence $(a_{k_n})_{n\in\mathbb{N}}$ converges modulo 1 to some $r\in[0,1)$ which, by continuity, is a root of $F$. In particular, $\{a_{k_n}-r\}_2 \underset{n\to+\infty}{\longrightarrow} 0$.

Set

$$u(t) = \frac{t^\beta - g(t)}{g(t)} \quad \text{and} \quad v(x) = \frac{c_r\cdot\epsilon(x)\cdot|x|^{\gamma(r)} - F(r+x)}{F(r+x)}\cdot$$

Assumptions (6.2) and (6.3) imply that $\lim_{t\to+\infty} u(t) = 0$ and $\lim_{x\to 0} v(x) = 0$, respectively. Applying Lemma 6.2.2 to $\boldsymbol{f} = (f_{k_n})_{n\in\mathbb{N}} = (F\,(a_{k_n}))_{n\in\mathbb{N}}$, $u$ and $v$ yields that $h$ lies in the closure of the set

$$\left\{\epsilon\left(\{a_k-r\}_2\right)\cdot c_r\cdot k^\beta\cdot||a_k-r||^{\gamma(r)}\right\}_{k\in\mathbb{N}}.$$

The converse follows similarly from Lemma 6.2.2 and assumption (6.3). The proof is complete. ■

[1] Two real sequences $(a_n)_{n\in\mathbb{N}}\,,(b_n)_{n\in\mathbb{N}}$ such that $a_n\neq 0$ and $b_n\neq 0$ for all $n$ large enough are called asymptotically equal if and only if $a_n/b_n \underset{n\to+\infty}{\longrightarrow} 1$.

## 6.3 Rational Values of the Parameter $\rho$

The goal of this section is to study the sequence (6.6) in the case where $\rho \in \mathbb{Q}$. To this end, the following proposition is proved which relates the quantities $\mu_{\pm}(\alpha, \beta, \rho)$ with the density in $\mathbb{R}$ of the sequence (6.6).

**Proposition 6.3.1** *Let $\beta > 0$ be a positive real number. Given an irrational number $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and a rational number $\rho$, it holds that the sequence* (6.6) *is dense in $\mathbb{R}^+$ (resp.in $\mathbb{R}^-$) if and only if*

$$\mu_+(\alpha, \beta, \rho) = 0 \quad (\textit{resp.} \quad \mu_-(\alpha, \beta, \rho) = 0)\,. \tag{6.11}$$

Proof We prove the claim concerning the quantity $\mu_+(\alpha, \beta, \rho)$ and the density of the sequence (6.6) in $\mathbb{R}^+$, as the claim related to $\mu_-(\alpha, \beta, \rho)$ and $\mathbb{R}^-$ is established in the same way.

From assumption (6.11), one has that, for every $n \in \mathbb{N}$, there exists $m = m(n) \in \mathbb{N}$ such that

$$0 \leq \{m\alpha - \rho\}_2 = \frac{\epsilon_n}{m^\beta} < \frac{1}{2} \quad \text{for some} \quad 0 \leq \epsilon_n \leq \frac{1}{n}\cdot$$

Without loss of generality, assume that

$$\{m\alpha\}_2 \quad = \quad \{\rho\}_2 + \frac{\epsilon_n}{m^\beta}$$

as otherwise

$$\{m\alpha\}_2 \quad = \quad -1 + \{\rho\}_2 + \frac{\epsilon_n}{m^\beta},$$

in which case one works similarly. Let us assume that $\{\rho\}_2 = \frac{p}{q}$ for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$ with $(p, q) = 1$. Then, for every $l \in \mathbb{N}_0$ such that

$$(lq + 1) \cdot \frac{\epsilon_n}{n^\beta} \quad < \quad \frac{1}{2}, \tag{6.12}$$

it holds that $\left\{(lq+1) \cdot m\alpha - \frac{p}{q}\right\}_2 = (lq+1) \cdot \frac{\epsilon_n}{m^\beta}$. For those $l \in \mathbb{N}$ which satisfy inequality (6.12), set

$$Q_\beta(m,l) \quad = \quad (lq+1)^\beta \cdot m^\beta \cdot \left\{(lq+1)\cdot m\alpha - \frac{p}{q}\right\}_2 \quad = \quad (lq+1)^{1+\beta}\cdot\epsilon_n, \quad (6.13)$$

where recall that $\epsilon_n$ depends on the choice $m$. Fix $h > 0$. Notice that, for $n$ large enough, that is, for $\epsilon_n \leq \frac{1}{n}$ sufficiently small and $m = m(n)$ sufficiently large, the natural number

$$l_h \quad = \quad \left\lceil \frac{h^{\frac{1}{1+\beta}}\cdot \epsilon_n^{-\frac{1}{1+\beta}} - 1}{q} \right\rceil$$

satisfies inequality (6.12). The quantity $Q_\beta(m,l_h)$ is therefore a term in the sequence (6.6).

The density of sequence (6.6) follows upon noticing that

$$h \quad = \quad \left(\left(\frac{h^{\frac{1}{1+\beta}}\cdot \epsilon_n^{-\frac{1}{1+\beta}} - 1}{q}\right)\cdot q + 1\right)^{1+\beta}\cdot\epsilon_n \quad = \quad Q_\beta\left(m(n), l_h\right) + O\left(h^\beta \cdot \epsilon_n^{\frac{1}{1+\beta}}\right)$$

and upon letting $n \to +\infty$.

In the other direction, assume that $\mu_+(\alpha,\beta,\rho) > 0$. From Definition 6.1.1 of the quantity $\mu_+(\alpha,\beta,\rho)$, one has that for every $k \gg_\alpha 1$ such that $\{k\alpha-\rho\}_2 > 0$, it holds that

$$k^\beta \cdot \{k\alpha - \rho\}_2 \quad \geq \quad C$$

for some positive constant $C$. Therefore, the sequence (6.6) cannot be dense in $\mathbb{R}^+$.

The proof is complete. ■

An immediate consequence of Proposition 6.3.1 is the following corollary which deals with the case when the exponent $\beta$ takes values in $(0,1)$.

**Corollary 6.3.2** *Given $\beta \in (0,1)$, $\alpha$ irrational and $\rho$ a rational number, the sequence*

$$\left(k^\beta \cdot \{k\alpha-\rho\}_2\right)_{k\in\mathbb{N}}$$

*is dense in* $\mathbb{R}$*. Equivalently,*

$$\mu_+(\alpha, \beta, \rho) \quad = \quad \mu_-(\alpha, \beta, \rho) \quad = \quad 0.$$

PROOF Let $\beta, \alpha, \rho$ be as in the statement of the corollary. By the theory of continued fractions, if $\frac{p_n}{q_n}$ is one of the convergents of $\alpha$, then it holds that $\left\|\alpha - \frac{p_n}{q_n}\right\| \leq \frac{1}{q_n^2}$. Thus, the finite sequence $(k\alpha)_{k=1}^{q_n}$ is $\frac{2}{q_n}$-dense in $\mathbb{T}$. This implies that

$$\mu_+(\alpha, 1, \rho) \quad = \quad \liminf_{\substack{k \to +\infty, \\ \{k\alpha - r\}_2 > 0}} k \cdot \{k\alpha - \rho\}_2 \quad \leq \quad 2,$$

which in turn implies that $\mu_+(\alpha, \beta, \rho) = \liminf_{\substack{k \to +\infty, \\ \{k\alpha - \rho\}_2 > 0}} k^\beta \cdot \{k\alpha - \rho\}_2 = 0$. One works similarly with the quantities $\mu_-(\alpha, 1, \rho)$ and $\mu_-(\alpha, \beta, \rho)$. Proposition 6.3.1 now implies the result. ■

## 6.4 Real Values of the Parameter $\rho$

The goal of this section is to use the Ostrowski expansion of a real number $\rho$ in order to obtain sufficient conditions for the sequence (6.6) to be dense in $\mathbb{R}$. This will lead us to the proof of the second statement in Theorem 6.1.2.

### 6.4.1 Sufficient Conditions for Density in $\mathbb{R}$

It is now proved that, if $\tau_+(\alpha, \beta, \rho) = \tau_-(\alpha, \beta, \rho) = 0$, then the sequence (6.6) is dense in $\mathbb{R}$. Moreover, in the case where $\mu_+(\alpha, \beta) = \mu_-(\alpha, \beta) = 0$, the proof provides an effective way to construct the coefficients in the Ostrowski expansion (1.52), and thus the parameter $\rho$, for the sequence (6.6) to enjoy the density property.

**Proposition 6.4.1** *Given* $\beta > 0$, $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ *and* $\rho \in \mathbb{R}$*, if*

$$\tau_+(\alpha, \beta, \rho) = 0 \quad (\textit{resp.} \quad \tau_-(\alpha, \beta, \rho) = 0),$$

*then the sequence* (6.6) *is dense in* $\mathbb{R}^+$ *(resp. in* $\mathbb{R}^-$*).*

Before continuing, recall some facts which will be used extensively in the forthcoming proofs. For every $x \in \mathbb{R}$, it holds that $-\,||x|| \leq \{x\}_2 \leq ||x||$. Also, given an irrational $\alpha$ and the Ostrowski expansion (1.52), p.51, of a real number $\rho$, one has that, for every $n \in \mathbb{N}$

$$\left|\sum_{j=n+1}^{+\infty} e_j(\rho) \cdot \{q_j\alpha\}_2\right| \quad \leq \quad ||q_n\alpha|| \,. \tag{6.14}$$

Indeed, by the definition of the continued fraction expansion of a real number $\alpha$, one has that $a_1 \cdot \{\alpha\} - 1 = \{q_1\alpha\}_2$, $\{\alpha\} + a_2 \cdot \{q_1\alpha\}_2 = \{q_2\alpha\}_2$ and, for every $n \geq 1$, it holds that $\{q_n\alpha\}_2 + a_{n+2} \cdot \{q_{n+1}\alpha\}_2 = \{q_{n+2}\alpha\}_2$ (see [22, Section 3]). This implies that

$$\{q_n\alpha\}_2 \quad = \quad -\sum_{i=1}^{+\infty} a_{n+2i} \cdot \{q_{n+2i-1} \cdot \alpha\}_2 \quad \text{for every } n \geq 1. \tag{6.15}$$

In turn, this implies that $\left|\sum_{j=n+1}^{+\infty} e_j(\rho) \cdot \{q_j\alpha\}_2\right| \leq \left|\sum_{i=1}^{+\infty} a_{n+2i} \cdot \{q_{n+2i-1}\alpha\}_2\right| = ||q_n\alpha||$, whence the claim.

Proof (Proposition 6.4.1:) We prove the result regarding the quantity $\tau_+(\alpha, \beta, \rho)$ and the density of (6.6) in $\mathbb{R}^+$. The other case follows in the same way. To this end, assume that $\tau_+(\alpha, \beta, \rho) = 0$. Given $j \geq 0$, set $e_j = e_j(\rho)$ to be the digits in the Ostrowski expansion of $\rho$ as defined in (1.52), p.51.

**Case 1:** Assume that

$$\liminf_{j \to +\infty} \max\left\{1, e_{2j}^{\beta}\right\} \cdot q_{2j}^{\beta} \cdot \{q_{2j}\alpha\}_2 \quad = \quad 0. \tag{6.16}$$

Fix $n \in \mathbb{N}$. There exists $m = m(n) \in 2\mathbb{N}$ such that

$$\epsilon_n := \quad q_m^{\beta} \cdot \{q_m\alpha\}_2 \quad \leq \quad \frac{1}{n} \qquad \text{and} \qquad e_m^{\beta} \cdot \epsilon_n \quad \leq \quad \frac{1}{n} \cdot \tag{6.17}$$

Since $m \in 2\mathbb{N}$, one has that $\{q_m\alpha\}_2 = ||q_m\alpha||$ and thus inequality (6.14) yields that $\left|\sum_{j=m+1}^{+\infty} e_j \cdot \{q_j\alpha\}_2\right| \leq \{q_m\alpha\}_2$. Set

$$\eta \quad = \quad q_m^\beta \cdot \sum_{j=m+1}^{+\infty} e_j \cdot \{q_j\alpha\}_2\,, \tag{6.18}$$

so that $|\eta| \leq \epsilon_n$. Given $l \in \mathbb{N}$ such that $l\epsilon_n - \eta \ < \frac{1}{2}$, set

$$\begin{aligned} Q_\beta\,(m,l) \quad &= \quad \left(\sum_{j=0}^{m-1} e_j \cdot q_j + e_m \cdot q_m + l \cdot q_m\right)^\beta \cdot \\ &\qquad \left\{\left(\sum_{j=0}^{m-1} e_j \cdot q_j + e_m \cdot q_m + l \cdot q_m\right) \cdot \alpha - \rho\right\}_2 \\ &\underset{(1.52),(6.17),(6.18)}{=} \quad \left(\frac{\sum_{j=0}^{m-1} e_j \cdot q_j}{q_m} + e_m + l\right)^\beta \cdot (l \cdot \epsilon_n - \eta)\,. \end{aligned}$$

Here, the second equality holds because, for $n \in \mathbb{N}$ large enough, (that is, for $m = m(n) \in 2\mathbb{N}$ large enough), one has that

$$0 \leq \left\{\left(\sum_{j=0}^{m-1} e_j \cdot q_j + e_m \cdot q_m + l \cdot q_m\right) \cdot \alpha - \rho\right\}_2 \ < \frac{1}{2}$$

and thus

$$\left\{\left(\sum_{j=0}^{m-1} e_j \cdot q_j + e_m \cdot q_m + l \cdot q_m\right) \cdot \alpha - \rho\right\}_2 = l \cdot \{q_m\alpha\}_2 - \sum_{j=m+1}^{+\infty} e_j \cdot \{q_j\alpha\}_2\,.$$

From the Ostrowski expansion of a natural number (Definition 1.1.42, Equation (1.51), p.51) it follows that

$$\frac{\sum_{j=0}^{m-1} e_j \cdot q_j}{q_m} \quad \leq \quad 1$$

(see also [22, Lemma 3.1]). In turn, one infers from inequalities (6.17) that $|Q_\beta(m,0)| \ll 1/n$.

Fix $h \ > 0$. Note that for $l' = 2 \cdot \left\lfloor h^{\frac{1}{\beta+1}} \cdot \epsilon_n^{-\frac{1}{\beta+1}} \right\rfloor$ and for $n$ large enough, it holds that $l'\epsilon_n - \eta \leq \frac{1}{4}$ thanks to relations (6.17) and (6.18). Thus, for every $l \in [\![0, l']\!]$, the quantity $Q_\beta(m,l)$ is a term of the sequence (6.6). Moreover, it holds that $Q(m,l') > h$. It then follows from the relations in (6.17) that, for every $l \in [\![0,l']\!]$,

$$\begin{aligned}
|Q_\beta\,(m(n),l+1) - Q_\beta\,(m(n),l)| &\leq (2+e_m+l)^\beta \cdot ((l+1)\cdot \epsilon_n - \eta) \\
&\qquad - (e_m+l)^\beta \cdot (l\epsilon_n - \eta) \\
&\ll \left((3+e_m+l)^\beta - (e_m+l)^\beta\right)\cdot (l\epsilon_n - \eta) \\
&\qquad + l^\beta \cdot \epsilon_n + e_m^\beta \cdot \epsilon_n \\
&\underset{(|\eta|\leq\epsilon_n)}{\ll} e_m^{\beta-1}\cdot l\epsilon_n + l^\beta\cdot\epsilon_n + e_m^\beta\cdot\epsilon_n \\
&\underset{\substack{(6.17),\\(l\leq l')}}{\ll} h^{\frac{1}{1+\beta}}\cdot\left(\frac{1}{n}\right)^{\frac{\beta}{\beta+1}} + h^{\frac{\beta}{\beta+1}}\cdot\left(\frac{1}{n}\right)^{\frac{1}{\beta+1}} + \frac{1}{n} \\
&=: \eta(n,h).
\end{aligned}$$

Since $|Q_\beta\,(m(n),0)| \ll 1/n$ and $Q_\beta\,(m(n),l') > h$, the last inequality yields that the terms $\{Q_\beta\,(m(n),l)\}_{l\in[\![0,l']\!]}$ partition the interval $[0,h]$ into subintervals with length at most $O\,(\eta(n,h))$. Since $\eta(n,h) \to 0$ when $n \to +\infty$ and since the choice of $h > 0$ is arbitrary, one infers that the sequence (6.6) is dense in $\mathbb{R}^+$.

**Case 2:** Let us assume that

$$\liminf_{j\to+\infty} \max\left\{1, (a_{2j+1}-e_{2j})^{\frac{\beta+1}{2}}\right\}\cdot q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 = 0. \tag{6.19}$$

Without loss of generality, assume that $\beta \geq 1$ as otherwise assumption (6.16) holds and the claim reduces to Case 1 (this will be proved in detail in Corollary 6.4.2). The following argument is similar to the first case. Fix $n \in \mathbb{N}$. Then, there exists $m = m(n) \in 2\mathbb{N}$ such that

$$\epsilon_n := q_m^\beta \cdot \{q_m\alpha\}_2 \leq \frac{1}{n} \quad\text{and}\quad (a_{m+1}-e_m)^{\frac{\beta+1}{2}}\cdot\epsilon_n \leq \frac{1}{n}\cdot \tag{6.20}$$

Define $\eta$ as in (6.17) in such a way that $|\eta| \leq \epsilon_n$. Also, set $\eta' = q_m^\beta \cdot \{q_{m+1}\alpha\}_2$, wherefrom it follows that $|\eta'| \leq \epsilon_n$.

Given $l \geq 1$ such that $l \cdot \epsilon_n + (a_{m+1} - e_m) \cdot \epsilon_n - \eta' - \eta \ < \frac{1}{2}$, let

$$\begin{aligned}
P_\beta(m,l) \quad &= \quad \left( \sum_{j=0}^{m-1} e_j \cdot q_j + l \cdot q_m - q_{m-1} \right)^\beta \cdot \\
&\qquad \left\{ \left( \sum_{j=0}^{m-1} e_j \cdot q_j + l \cdot q_m - q_{m-1} \right) \cdot \alpha - \rho \right\}_2 \\
&\underset{(1.52)}{=} \left( \sum_{j=0}^{m-1} e_j \cdot q_j + l \cdot q_m - q_{m-1} \right)^\beta \cdot \\
&\qquad \left\{ (l + a_{m+1} - e_m) \cdot \{q_m \alpha\}_2 - \{q_{m+1}\alpha\}_2 - \sum_{j=m+1}^{+\infty} e_j \cdot \{q_j \alpha\}_2 \right\}_2 \\
&\underset{(6.18),(6.20)}{=} \left( \frac{\sum_{j=0}^{m-1} e_j \cdot q_j - q_{m-1}}{q_m} + l \right)^\beta \cdot \\
&\qquad (l\epsilon_n + (a_{m+1} - e_m) \cdot \epsilon_n - \eta' - \eta) .
\end{aligned}$$

As in the previous case, the Ostrowski expansion of a natural number (Definition 1.1.42, Equation (1.51), p.51) yields

$$\left| \frac{\sum_{j=0}^{m-1} e_j \cdot q_j - q_{m-1}}{q_m} \right| \quad \leq \quad 1.$$

In turn, one infers from inequalities (6.20) that $|P_\beta(m,0)| \ll 1/n$.

Fix $h \ > 0$. For $l' = 4 \cdot \left\lfloor h^{\frac{1}{\beta+1}} \cdot \epsilon_n^{-\frac{1}{\beta+1}} \right\rfloor$ and $n$ large enough, inequalities (6.20) and (6.18) imply that $l' \cdot \epsilon_n + (a_{n+1} - e_n) \cdot \epsilon_n - \eta' - \eta \leq \frac{1}{4}$. Thus, given $l \in [\![0, l']\!]$, the quantity $P_\beta(m,l)$ is a term in the sequence (6.6). Moreover, it holds that $P_\beta\left(m, l'\right) \ > h$. One can then use the relations in (6.20) in order to prove that, for every $l \in [\![0, l']\!]$,

$$\begin{aligned}
|P_\beta(m(n), l+1) - P_\beta(m(n), l)| \quad & \underset{\substack{(|\eta| \leq \epsilon_n), \\ (|\eta'| \leq \epsilon_n)}}{\leq} \quad (l+2)^\beta \cdot (l\epsilon_n + (a_{m+1} - \epsilon_m)\,\epsilon_n + 3\epsilon_n) \\
& \qquad - (l-1)^\beta \cdot (l\epsilon_n + (a_{m+1} - e_m)\,\epsilon_n - 2\epsilon_n) \\
& \ll \quad l^\beta \cdot \epsilon_n + l^{\beta-1} \cdot (a_{m+1} - e_m) \cdot \epsilon_n \\
& \underset{\substack{(6.20), \\ (l \leq l')}}{\ll} \quad h^{\frac{\beta}{\beta+1}} \cdot \left(\frac{1}{n}\right)^{\frac{1}{\beta+1}} + h^{\frac{\beta-1}{\beta+1}} \cdot \left(\frac{1}{n}\right)^{\frac{2}{\beta+1}} \\
& =: \quad \eta(n, h).
\end{aligned}$$

Since $|P_\beta(m(n), 0)| \ll 1/n$ and $P_\beta(m(n), l') \; > h$, the last inequality yields that the terms $\{P_\beta(m(n), l)\}_{l \in [\![0, l']\!]}$ partition the interval $[0, h]$ into subintervals with length at most $O(\eta(n, h))$. Since $\eta(n, h) \to 0$ when $n \to +\infty$ and since the choice of $h \; > 0$ is arbitrary, one infers that the sequence (6.6) is dense in $\mathbb{R}^+$.

The proof is complete. ■

The following corollary is a straightforward consequence of Proposition 6.4.1.

**Corollary 6.4.2** *Let $\beta \in (0, 1)$ be a real number. Let also $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ be an irrational and let $\rho$ be a real number. Then, the sequence*

$$\left(k^\beta \cdot \{k\alpha - \rho\}_2\right)_{k \in \mathbb{N}}$$

*is dense in $\mathbb{R}$.*

Proof Let $\beta, \alpha, \rho$ be as in the statement. Assume that $(e_j)_{j \geq 0}$ is the sequence of the digits in the Ostrowski expansion of $\rho$ (Definition 1.1.42, Equation (1.52), p.51). From the theory of continued fractions, for every $n \in \mathbb{N}$, it holds that $||q_n\alpha|| \leq \frac{1}{a_{n+1}q_n}$. Consequently,

$$\begin{aligned}
\liminf_{j \to +\infty} q_{2j}^\beta \cdot \max\left\{1, e_{2j}^\beta\right\} \cdot \{q_{2j}\alpha\}_2 \quad & \leq \quad \liminf_{j \to +\infty} \max\left\{\frac{q_{2j}^\beta}{a_{2j+1} \cdot q_{2j}}, \frac{e_{2j}^\beta \cdot q_{2j}^\beta}{a_{2j+1} \cdot q_{2j}}\right\} \\
& \leq \quad \liminf_{j \to +\infty} \frac{1}{q_{2j}^{1-\beta}} \quad = \quad 0.
\end{aligned}$$

Similarly, one can show that $\liminf_{j\to+\infty} -q_{2j+1}^{\beta}\cdot\max\left\{1, e_{2j+1}^{\beta}\right\}\cdot\{q_{2j+1}\alpha\}_2 = 0$. Therefore, Proposition 6.4.1 implies that the sequence (6.6) is dense in $\mathbb{R}$. The proof is complete. ■

Proposition 6.2.1 and Corollary 6.4.2 immediately imply Corollary 6.1.5.

### 6.4.2 Effective Construction of the Parameter $\rho$

The sufficient condition in the statement of Proposition 6.4.1 is not necessary. Indeed, in the following proposition we construct real numbers $\rho\in\mathbb{R}$ such that the sequence (6.6) is dense in $\mathbb{R}$ but with $\tau_{\pm}(\alpha,\beta,\rho)=+\infty$.

**Proposition 6.4.3** *Let $\beta>0$ be a positive number and $\alpha$ be an irrational such that both $\mu_+(\alpha,\beta)$ and $\mu_-(\alpha,\beta)$ equal either zero or infinity. Then, there exists an effectively constructible sequence of digits $(e_j)_{j\geq 0}$ in the Ostrowski expansion* (1.52) *of the real number $\rho$ such that the sequence $(w_k)_{k\in\mathbb{N}}$ defined in* (6.6) *is dense in $\mathbb{R}$. Moreover, there exist uncountably many such numbers $\rho$.*

Proof The proof is split into three cases depending on the values of $\mu_{\pm}(\alpha,\beta)$.

**Case 1:** Assume that $\mu_+(\alpha,\beta)=\mu_-(\alpha,\beta)=0$. Then, the result follows easily from Proposition 6.4.1. For instance, for every $j\geq 0$, one can choose $e_j\in\{0,1\}$ so that the resulting sequence is dense in $\mathbb{R}$.

**Case 2:** Assume that $\mu_+(\alpha,\beta)=\mu_-(\alpha,\beta)=+\infty$; that is, that

$$\liminf_{n\to+\infty} q_{2n}^{\beta}\cdot\{q_{2n}\alpha\}_2 \quad=\quad \liminf_{n\to+\infty} -q_{2n+1}^{\beta}\cdot\{q_{2n+1}\alpha\}_2 \quad=\quad +\infty. \tag{6.21}$$

Fix a sequence $\boldsymbol{b}=(b_j)_{j\in\mathbb{N}}$ of real numbers which is dense in $\mathbb{R}$. The goal is to define the sequence $(e_j)_{j\geq 0}$ recursively in such a way that

$$\left|b_j - w_{\kappa_{m(j)}}\right| \quad \underset{j\to+\infty}{\longrightarrow} \quad 0, \tag{6.22}$$

where $\left(\kappa_{m(j)}\right)_{j\in\mathbb{N}}$ is a proper subsequence of the sequence (6.8) defined in the course of the proof below. Relation (6.22) then yields the density of sequence (6.6).

Choose $e_0 \in [\![0, a_1 - 1]\!]$ arbitrary and fix $j \in \mathbb{N}$. If $j = 1$, then, without loss of generality, assume that $b_1 > 0$. From equation (6.21), there exists $m(1) \in 2\mathbb{N}$ such that $q_{m(1)}^{\beta} \cdot \left\{ q_{m(1)} \cdot \alpha \right\}_2 \geq 5b_1$. Given $n \in [\![1, m(1) - 1]\!]$, set $e_n = 0$ and choose $e_{m(1)} \in \left[\!\left[1, a_{m(1)+1}\right]\!\right]$ arbitrary. Fix $l(1) \in \mathbb{N}$ large enough. From equation (6.15) and the choice of $m(1)$, for every $n \in [\![m(1) + 1, m(1) + l(1)]\!]$, the digits $e_n \in [\![0, a_{n+1}]\!]$ can be chosen in such a way that

$$\left| b_1 + \kappa_{m(1)}^{\beta} \cdot \sum_{n=m(1)+1}^{m(1)+l(1)} e_n \cdot \{q_n \alpha\}_2 \right| \quad \leq \quad \frac{1}{2}.$$

The choice of the natural number $l(1)$ and of the digits $\{e_n\}_{m(1)+1}^{m(1)+l(1)}$ is possible because, from equation (6.15), one can choose the digits $\{e_n\}_{n=m(1)+1}^{\infty}$ in such a way that the number $\sum_{n=m(1)+1}^{+\infty} e_n \cdot \{q_n \alpha\}_2$ approximates arbitrary well any given number $x \in \left[ -\left\{ q_{m(1)} \alpha \right\}_2, 0 \right]$. The assumption $q_{m(1)}^{\beta} \cdot \left\{ q_{m(1)} \alpha \right\}_2 \geq 5b_1$ guarantees the existence of such a choice upon noticing that $\kappa_{m(1)} \geq q_{m(1)}$.

If $j \geq 2$, then assume that the numbers $m(j-1), l(j-1) \in \mathbb{N}$ have been chosen in such a way that, for every $n \in [\![1, m(j-1) + l(j-1)]\!]$, the digits $e_n \in [\![0, a_{n+1}]\!]$ are such that for every $j' \in [\![1, j-1]\!]$,

$$\left| b_{j'} + \kappa_{m(j')}^{\beta} \cdot \sum_{n=m(j')+1}^{m(j')+l(j')} e_n \cdot \{q_n \alpha\}_2 \right| \leq \frac{1}{2j'} \quad \text{and} \quad \kappa_{m(j'-1)}^{\beta} \cdot \left\| q_{m(j')-2} \cdot \alpha \right\| \leq \frac{1}{2(j'-1)}. \tag{6.23}$$

Without loss of generality, assume that $b_j \geq 0$. From equation (6.21), there exists $m(j) \in 2\mathbb{N}$ such that $m(j) \geq m(j-1) + l(j-1) + 1$ and

$$q_{m(j)}^{\beta} \cdot \left\{ q_{m(j)} \cdot \alpha \right\}_2 \quad \geq \quad 5b_j \quad \text{and} \quad \kappa_{m(j-1)}^{\beta} \cdot \left\| q_{m(j)-2} \cdot \alpha \right\| \quad \leq \quad \frac{1}{2(j-1)}, \tag{6.24}$$

where the last inequality holds if $m(j)$ is chosen large enough. Here, the constant 5 in the first inequality ensures that the choice of the digits $e_n$ in the next step of the induction satisfies the properties of the Ostrowski expansion (as given in relation (1.52), p.51).

Given $n \in [\![m(j-1) + l(j-1) + 1, m(j) - 1]\!]$, set $e_n = 0$ and choose $e_{m(j)} \in \left[\!\left[1, a_{m(j)+1}\right]\!\right]$ arbitrary. Fix $l(j) \in \mathbb{N}$ large enough. From equation (6.15) and the first inequality in (6.24), for every $n \in [\![m(j)+1, m(j)+l(j)]\!]$, the digits $e_n \in [\![0, a_{n+1}]\!]$ can be chosen in such a way that

$$\left| b_j + \kappa^{\beta}_{m(j)} \cdot \sum_{n=m(j)+1}^{m(j)+l(j)} e_n \cdot \{q_n \alpha\}_2 \right| \quad \leq \quad \frac{1}{2j}.$$

Here, the existence of such a choice of a natural number $l(j)$ and of the digits $\{e_n\}_{m(j)+1}^{m(j)+l(j)}$ is guaranteed in the same way as in the case $j = 1$. In the case where $b_j < 0$, one works in a similar way by choosing $m(j) \in 2\mathbb{N} + 1$ large enough. Therefore, the sequence $(e_n)_{n\in\mathbb{N}}$ has been defined and one can thus set

$$\rho = e_0 \cdot \{\alpha\} + \sum_{n=1}^{+\infty} e_n \cdot \{q_n \alpha\}_2 .$$

It is not hard to check that for every $j \in \mathbb{N}$, it holds that

$$\begin{aligned}
\left| b_j - w_{\kappa_{m(j)}} \right| \quad &= \quad \left| b_j + \kappa^{\beta}_{m(j)} \cdot \sum_{n=m(j)+1}^{+\infty} e_n \cdot \{q_n \alpha\}_2 \right| \\
&\leq \quad \left| b_j + \kappa^{\beta}_{m(j)} \cdot \sum_{n=m(j)+1}^{m(j)+l(j)} e_n \cdot \{q_n \alpha\}_2 \right| + \kappa^{\beta}_{m(j)} \cdot \left| \sum_{n=m(j+1)}^{\infty} e_n \cdot \{q_n \alpha\}_2 \right| \\
&\underset{\substack{(6.23),\\(6.15)}}{\leq} \quad \frac{1}{2j} + \kappa^{\beta}_{m(j)} \cdot \left|\left| q_{m(j+1)-2} \cdot \alpha \right|\right| \quad \underset{(6.23)}{\leq} \quad \frac{1}{j}.
\end{aligned}$$

The claim is thus proved.

**Case 3:** Assume that one of the quantities $\mu_{\pm}(\alpha, \beta)$ equals zero and that the other one equals infinity. For instance, assume that $\mu_+(\alpha, \beta) = +\infty$ and $\mu_-(\alpha, \beta) = 0$. Fix a sequence $\boldsymbol{b} = (b_j)_{j\in\mathbb{N}}$ of real numbers which is dense in $\mathbb{R}^+$. We follow the steps in the proof of the second case but this time, one chooses $m(j) \in 2\mathbb{N}$ large enough so that $q^{\beta}_{m(j)-1} \cdot \left|\left| q_{m(j)-1}\alpha \right|\right| \underset{j\to+\infty}{\longrightarrow} 0$ and $e_{m(j)-1} \in \{0, 1\}$. The density in $\mathbb{R}^+$ follows from the arguments presented in the second case, and the density in $\mathbb{R}^-$ follows from Proposition 6.4.1. When $\mu_+(\alpha, \beta) = 0$ and $\mu_-(\alpha, \beta) = +\infty$, one works similarly.

The arguments in all three cases imply easily the existence of uncountably many such numbers $\rho$. The proof is complete. $\blacksquare$.

**Remark 6.4.4** *Given $\beta > 0$ and an irrational $\alpha$, it can be shown that there exist (uncountably many) real numbers $\rho$ such that the sequence* (6.6) *is dense in* $\mathbb{R}$. *However, if at least one of the quantities $\mu_{\pm}(\alpha,\beta)$ is positive and finite, then, it is not clear how one can effectively construct the digits $(e_j)_{j\in\mathbb{N}}$ of the Ostrowski expansion* (1.52)*, p.51, of the real $\rho$. Note that given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, there exists at most one real number $\beta_+ > 0$ (resp. $\beta_- > 0$) such that $\mu_+(\alpha,\beta_+) \in (0,+\infty)$ (resp. such that $\mu_-(\alpha,\beta_-) \in (0,+\infty)$).*

## 6.5 Proof of Theorems 6.1.2 and 6.1.6

We are now ready to prove Theorem 6.1.2 and Corollary 6.1.3.

Proof (Theorem 6.1.2) As far as the first part of the theorem is concerned, assume that the sequence $(y_k)_{k\in\mathbb{N}}$ defined in (6.1) is dense in $\mathbb{R}^+$. Then, there exists an increasing sequence of natural numbers $(k_n)_{n\in\mathbb{N}}$ such that, for every $n \in \mathbb{N}$, $F(k_n\alpha) > 0$ and $g(k_n)\cdot F(k_n\alpha) \leq \frac{1}{n}$. By passing to a subsequnce if necessary, the continuity of $F$ yields that $||k_n\alpha - r|| \underset{n\to+\infty}{\longrightarrow} 0$ for some root $r$ of $F$, and the claim follows. Work similarly in the case where $(y_k)_{k\in\mathbb{N}}$ is dense in $\mathbb{R}^-$. In the special case where the root $r$ is rational, an immediate application of Propositions 6.3.1 and 6.2.1 implies the claim.

The second part of the theorem follows from Propositions 6.4.1 and 6.2.1. $\blacksquare$

Proof (Corollary 6.1.3) The function $F(x) = \sin(2\pi\cdot x)$ is easily seen to satisfy assumption (6.3). Moreover, all its roots are rationals. The result follows by applying Theorem 6.1.2 upon noticing that, given $\alpha$ irrational, $\beta > 0$ and $\rho$ a rational number, $\mu_+(\alpha,\beta,\rho) = 0$ (resp. $\mu_-(\alpha,\beta,\rho) = 0$) implies $\mu_+(\alpha,\beta) = 0$ (resp. $\mu_-(\alpha,\beta) = 0$). This claim follows easily from the definition of the quantities $\mu_{\pm}(\alpha,\beta,\rho)$. Indeed, assume for instance that $\mu_+(\alpha,\beta,\rho) = 0$ for some $\alpha \in \mathbb{R}\backslash\mathbb{Q}$, $\beta > 0$ and $\rho = p/q$ with $p, q \in \mathbb{Z}$, that is,

$$\liminf_{\substack{k\to+\infty,\\ \left\{k\alpha-\frac{p}{q}\right\}_2>0}} k^{\beta}\cdot\left\{k\alpha-\frac{p}{q}\right\}_2 = 0.$$

This yields that

$$\begin{aligned} 0 &= \liminf_{\substack{k\to+\infty \\ \left\{k\alpha-\frac{p}{q}\right\}_2>0}} q^{1+\beta}\cdot k^{\beta}\cdot\left\{k\alpha-\frac{p}{q}\right\}_2 \\ &= \liminf_{\substack{k\to+\infty \\ \{qk\alpha\}_2>0}} (qk)^{\beta}\cdot\{qk\alpha\}_2 \\ &\geq \mu_+(\alpha,\beta). \end{aligned}$$

Therefore, it has been proved that $\mu_+(\alpha,\beta)=0$. Similarly for the case where $\mu_-\left(\alpha,\beta,\frac{p}{q}\right)=0$. The proof of the corollary is complete. ■

Theorem 6.1.6 is now proved.

Proof (Theorem 6.1.6) Fix $\alpha\in\mathbb{R}\backslash\mathbb{Q}$, $\beta\geq 1$ and $\theta=p/q\in\mathbb{Q}$ with $\gcd(p,q)=1$. We will prove only the case dealing with the quantities

$$\lim_{\substack{j\to+\infty, \\ q|q_{2j}}} q_j^{\beta}\cdot\{q_j\alpha\}_2$$

and $\mu_+(\alpha,\beta,\theta)$. The other case is similar.

$\Leftarrow$: Fix $\epsilon'>0$ and let $(q_n)_{n\in\mathbb{N}}$ be the sequence of the denominators of convergents of $\alpha$. Assume that

$$\lim_{\substack{j\to+\infty, \\ q|q_{2j}}} q_j^{\beta}\cdot\{q_j\alpha\}_2 = 0.$$

Then, there exists $n\in 2\mathbb{N}$ such that $q|q_n$ and, for this $q_n$, it holds that

$$0 < q_n^{\beta}\cdot\{q_n\alpha\}_2 =: \epsilon \leq \epsilon'. \tag{6.25}$$

Since $n$ is even, the theory of continued fractions implies that $\{q_n\alpha\}_2>0$. One obtains immediately that

$$\alpha = \frac{p_n}{q_n}+\frac{\epsilon}{q_n\cdot q_n^{\beta}} \tag{6.26}$$

for some $p_n \in \mathbb{Z}$ with $(p_n, q_n) = 1$. Write $q_n = q \cdot q'_n$ for some $q'_n \in \mathbb{N}$ and choose $p'_n \in \{1, ..., q-1\}$ such that $p'_n \cdot p_n \equiv p \pmod{q}$. Then,

$$\begin{aligned}(p'_n \cdot q'_n)^\beta \cdot \{p'_n q'_n \cdot \alpha - \theta\}_2 \quad &\underset{(6.26)}{=} \quad (p'_n \cdot q'_n)^\beta \cdot \left\{p'_n \cdot q'_n \cdot \left(\frac{p_n}{q_n} + \frac{\epsilon}{q_n \cdot q_n^\beta}\right) - \frac{p}{q}\right\}_2 \\ &\underset{\left(\substack{p'_n \cdot p_n \equiv p \\ \pmod{q}}\right)}{=} \left(\frac{p'_n \cdot q'_n}{q_n}\right)^{1+\beta} \cdot \epsilon \quad \underset{(p'_n < q)}{\leq} \quad \epsilon.\end{aligned}$$

This implies that $\mu_+\left(\alpha, \beta, \frac{p}{q}\right) \leq \epsilon'$. Therefore, $\mu_+\left(\alpha, \beta, \frac{p}{q}\right) = 0$ as $\epsilon'$ is chosen arbitrary.

$\Rightarrow$: Assume that $\mu_+\left(\alpha, \beta, \frac{p}{q}\right) = 0$. Without loss of generality, assume that $p/q \in [0, 1)$. We prove first the case $q \geq 2$. Fix

$$0 \quad < \quad \epsilon_0 \quad < \quad \frac{1}{2 \cdot q^{2+\beta}}. \tag{6.27}$$

By assumption, there exists $k \in \mathbb{N}$ such that

$$0 \quad < \quad k^\beta \cdot \{k\alpha - \theta\}_2 \quad \leq \quad \epsilon_0.$$

Set

$$\epsilon = k^\beta \cdot \{k\alpha - \theta\}_2 .$$

Then,

$$\{k\alpha\} = \theta + \frac{\epsilon}{k^\beta} \quad = \quad \frac{p}{q} + \frac{\epsilon}{k^\beta}. \tag{6.28}$$

It follows from inequality (6.27) that $q\epsilon/k^\beta \in \left[-\frac{1}{2}, \frac{1}{2}\right)$. Therefore, equation (6.28) yields

$$\{qk \cdot \alpha\}_2 \quad = \quad \frac{q\epsilon}{k^\beta}.$$

Let $n \in \mathbb{N}$ be such that $q_n \leq qk < q_{n+1}$. Then, from the theory of continued fractions one deduces that $||q_n\alpha|| \leq q\epsilon/k^\beta$ (see Chapter 1, p.50, Inequality (1.49) and the discussion around it). Also, one can write

$$\alpha = \frac{p_n}{q_n} + (-1)^n \cdot \frac{\eta}{q_n^2}$$

for some $p_n, q_n \in \mathbb{N}$ with $(p_n, q_n) = 1$ and some $\eta > 0$. Setting $\epsilon' = q_n^{\beta-1} \cdot \eta$ yields that

$$\alpha = \frac{p_n}{q_n} + (-1)^n \cdot \frac{\epsilon'}{q_n^{1+\beta}}$$

which in turn implies that

$$\frac{\epsilon'}{q_n^{\beta}} = ||q_n\alpha|| \leq \frac{q\epsilon}{k^{\beta}}. \tag{6.29}$$

Let us prove that $q|q_n$ and $n \in 2\mathbb{N}$. Choosing $\epsilon_0$ sufficiently small yields that $q_n \geq q$. Therefore, without loss of generality, assume for the rest of the proof that $q_n \geq q$.

Assume that $q \not| q_n$. Then, for every $j \in \mathbb{N}$,

$$\left|\left|\frac{j}{q_n} - \frac{p}{q}\right|\right| \geq \frac{1}{qq_n}$$

since $\frac{j}{q_n} \neq \frac{p}{q} \pmod 1$ for all $j \in \mathbb{Z}$. It holds that

$$\frac{\epsilon}{k^{\beta}} \underset{(q_n \leq \quad qk)}{\leq} q^{\beta} \cdot \frac{\epsilon}{q_n^{\beta}} \underset{\substack{(6.27),\\(\beta \geq 1)}}{<} \frac{1}{2q \cdot q_n}, \tag{6.30}$$

therefore, in order for at least one of the relations

$$k \cdot \left(\frac{p_n}{q_n} + \frac{\epsilon'}{q_n^{1+\beta}}\right) = \theta + \frac{\epsilon}{k^{\beta}} \pmod 1 \quad \text{or} \quad k \cdot \left(\frac{p_n}{q_n} - \frac{\epsilon'}{q_n^{1+\beta}}\right) = \theta + \frac{\epsilon}{k^{\beta}} \pmod 1 \tag{6.31}$$

to hold, it is necessary that

$$k \cdot \frac{\epsilon'}{q_n^{1+\beta}} > \frac{1}{2q \cdot q_n}. \tag{6.32}$$

However,

$$k \cdot \frac{\epsilon'}{q_n^{1+\beta}} \underset{(6.29)}{\leq} kq \cdot \frac{\epsilon}{q_n \cdot k^{\beta}} \underset{(\beta \geq 1)}{\leq} q \cdot \frac{\epsilon}{q_n} \underset{(\epsilon < \epsilon_0)}{<} \frac{1}{2q_n \cdot q^{1+\beta}} \leq \frac{1}{2q_n \cdot q}.$$

This contradiction establishes that $q|q_n$.

Assume that $n$ is odd. One may notice that, in this case, inequality (6.30) implies that if the right-hand side of the relation (6.31) holds, then the inequality (6.32) is satisfied. However, we have already proved that inequality (6.32) does not hold yielding this way a contradiction. Therefore, it has been established that $n$ is even and, in particular, that

$$\alpha \quad = \quad \frac{p_n}{q_n} + \frac{\epsilon'}{q_n^{1+\beta}} \quad \text{with} \quad k \cdot \frac{\epsilon'}{q_n^{1+\beta}} \quad = \quad \frac{\epsilon}{k^\beta}.$$

Note here that, since $q|q_n$, for the left-hand side equation of the relation (6.31) to be satisfied, it is not necessary that inequality (6.32) holds.

Finally, one has that

$$\begin{aligned}\liminf_{\substack{j\to+\infty,\\ q|q_{2j}}} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 \quad &\leq \quad q_n^\beta \cdot \{q_n\alpha\}_2 \quad \underset{(6.29)}{\leq} \quad q_n^\beta \cdot \frac{q\epsilon}{k^\beta} \\ &\underset{(q_n\leq qk)}{\leq} \quad q^{1+\beta} \cdot \epsilon \quad \leq \quad q^{1+\beta} \cdot \epsilon_0.\end{aligned}$$

By letting $\epsilon_0 \to 0$, one obtains that

$$\lim_{\substack{j\to+\infty,\\ q|q_{2j}}} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 \quad = \quad 0.$$

It remains to establish the case $q = 1$; that is, the case when $\theta \in \mathbb{Z}$. Assume that $\mu_+(\alpha, \beta) = 0$. The goal is to prove that

$$\lim_{j\to+\infty} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 \quad = \quad 0.$$

The following lemma immediately implies the claim.

**Lemma 6.5.1** *Given $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\beta \geq 1$, the relations*

$$\mu_+(\alpha,\beta) = \liminf_{j\to+\infty} q_{2j}^\beta \cdot \{q_{2j}\alpha\}_2 \quad \text{and} \quad \mu_-(\alpha,\beta) = \liminf_{j\to+\infty} -q_{2j-1}^\beta \cdot \{q_{2j-1}\alpha\}_2$$

*hold.*

Proof We prove the first relation as the second one follows in the same way. To this end, fix an even integer $n$. It is enough to prove that

$$(q_n + l \cdot q_{n+1})^\beta \cdot \{(q_n + l \cdot q_{n+1}) \cdot \alpha\}_2 \quad \geq \quad q_n^\beta \cdot \{q_n\alpha\}_2$$

for every $l \in [\![1, a_{n+2} - 1]\!]$ since

$$0 \quad < \quad \{q_{n+2}\alpha\}_2 \quad < \quad \{k\alpha\}_2 \quad \leq \quad \{q_n\alpha\}_2 \tag{6.33}$$

with $k < q_{n+2}$ if and only if $k = q_n + l \cdot q_{n+1}$ for some $l \in [\![0, a_{n+2} - 1]\!]$. This claim is proved at the end of the proof. Assuming it, for every $l \in [\![1, a_{n+2} - 1]\!]$, it holds that

$$\begin{aligned}
(q_n + l \cdot q_{n+1})^\beta \cdot \{(q_n + l \cdot q_{n+1}) \cdot \alpha\}_2 \quad &= \quad (q_n + l \cdot q_{n+1})^\beta \cdot (\{q_n\alpha\}_2 + l \cdot \{q_{n+1}\alpha\}_2) \\
&\geq \quad (1 + l a_{n+1})^\beta \cdot q_n^\beta \cdot \left(\frac{a_{n+2} - l}{a_{n+2}}\right) \cdot \{q_n\alpha\}_2 \\
&\underset{(\beta \geq 1)}{\geq} \quad \frac{(1 + l) \cdot (a_{n+2} - l)}{a_{n+2}} \cdot q_n^\beta \cdot \{q_n\alpha\}_2 \\
&\geq \quad q_n^\beta \cdot \{q_n\alpha\}_2 \, ,
\end{aligned}$$

where the first inequality follows upon noticing that

$$a_{n+2} \cdot ||q_{n+1}\alpha|| \quad \leq \quad ||q_n\alpha|| \quad \leq \quad (a_{n+2} + 1) \cdot ||q_{n+1}\alpha||$$

and thus, by setting $||q_n\alpha|| = (a_{n+2} + \eta) \cdot ||q_{n+1}\alpha||$ for some $\eta \in [0, 1]$, one obtains that

$$\begin{aligned}
\{q_n\alpha\}_2 + l \cdot \{q_{n+1}\alpha\}_2 \quad &= \quad (a_{n+2} + \eta - l) \cdot ||q_{n+1}\alpha|| \\
&= \quad \left(\frac{a_{n+2} + \eta - l}{a_{n+2} + \eta}\right) \cdot \{q_n\alpha\}_2 \\
&\geq \quad \left(\frac{a_{n+2} - l}{a_{n+2}}\right) \cdot \{q_n\alpha\} \, .
\end{aligned}$$

It thus remains to prove (6.33).

Fix $k \in [\![1, q_{n+2} - 1]\!]$ such that $0 < \{k\alpha\} \leq \{q_n\alpha\}$. The Ostrowski expansion of the integer $k$ (Definition 1.1.42, p.51, Equation (1.51)) is of the form $k =$

$\sum_{j=0}^{n+1} e_j \cdot q_j$. Let $m$ be the minimal natural number in $[\![0, n+1]\!]$ such that $e_m \geq 1$. We show that $k = q_n + lq_{n+1}$ for some $l \in [\![0, a_{n+2} - 1]\!]$ in three steps.

**Step 1:** We prove that $m$ is even. Assume for a contradiction that $m$ is odd. Then, from equation (6.15), it can easily be deduced that

$$\{k\alpha\}_2 = \left\{ \sum_{j=m}^{n+1} e_j \cdot q_j \alpha \right\}_2 < 0.$$

This contradicts the assumption that $\{k\alpha\}_2 > 0$.

**Step 2:** We prove that $e_m = 1$. Since $k = \sum_{j=m}^{n+1} e_j \cdot q_j$ and $m, n \in 2\mathbb{N}$, one has that $m \leq n$. Assume for a contradiction that $e_m \geq 2$. Then, from equation (6.15), one infers that

$$\{k\alpha\}_2 = \left\{ \sum_{j=m}^{n+1} e_j \cdot q_j \alpha \right\}_2 > \{q_m \alpha\}_2 \underset{(m \leq n)}{\geq} \{q_n \alpha\}_2 .$$

This contradicts the assumption that $\{k\alpha\}_2 \leq \{q_n \alpha\}_2$.

**Step 3:** We prove that $m = n$. Assume for a contradiction that $m \leq n - 2$. Then, since $e_m > 0$, from the definition of the Ostrowski expansion (Chapter 1, p.51, Equation (1.52)) one has that $e_{m+1} \leq a_{m+2} - 1$. Thus, from equation (6.15) one deduces that

$$\{k\alpha\}_2 = \left\{ \sum_{j=m}^{n+1} e_j \cdot q_j \alpha \right\}_2 > \{q_{m+2} \alpha\}_2 \geq \{q_n \alpha\}_2 .$$

This contradicts the assumption that $\{k\alpha\}_2 \leq \{q_n \alpha\}_2$.

Therefore inequality (6.33) holds if and only if $k = q_n + lq_{n+1}$ with $l \in [\![0, a_{n+2} - 1]\!]$. The claim is established, which completes the proof of Lemma 6.5.1. ■

This concludes the proof of Theorem 6.1.6. ■

We end this section by showing that the quantities $\mu_{\pm}(\alpha,\beta,\rho)$ (cf. Definition 6.1.1) cannot be replaced in the statements of Theorems 6.1.2 and 6.1.6 with $\mu(\alpha,\beta,\rho)$ (cf. Definition 6.1.1).

**Proposition 6.5.2** *Given $\beta \geq 1$ and a rational number $\rho$, there exists a real $\alpha$ such that $\mu_+(\alpha,\beta,\rho) = 0$ and $\mu_-(\alpha,\beta,\rho) > 0$. Conversely, there exists a real $\alpha$ such that $\mu_-(\alpha,\beta,\rho) = 0$ and $\mu_+(\alpha,\beta,\rho) > 0$.*

Proof From Theorem 6.1.6, it is enough to prove the claim when $\rho = 0$.

Let $\alpha = [a_0; a_1, a_2, ...] \in \mathbb{R}\backslash\mathbb{Q}$ be an irrational number whose partial quotients will be defined recursively. Set

$$y_n = [0; a_{n+1}, a_{n+2}, ...] \in [0,1), \quad n \in \mathbb{N}_0.$$

Let $(p_n/q_n)_{n\in\mathbb{N}}$ be the sequence of convergents of $\alpha$. A standard analysis of the continued fraction expansion of $\alpha$ yields that

$$\{q_{n-1}\cdot\alpha\}_2 = \frac{(-1)^{n-1}}{q_n + y_n \cdot q_{n-1}}$$

and

$$\frac{1}{q_n + y_n\cdot q_{n-1}} \asymp \frac{1}{q_n} \asymp \frac{1}{a_n \cdot q_{n-1}}.$$

Therefore,

$$|q_{n-1}\cdot\{q_{n-1}\cdot\alpha\}_2| \asymp \frac{1}{a_n}.$$

Define the sequence $(a_n)_{n\in\mathbb{N}}$ as follows: for odd $n \in \mathbb{N}$, choose $a_n = \left\lfloor n\cdot q_{n-1}^{\beta-1}\right\rfloor$ and for even $n \in \mathbb{N}$, choose $1 \leq a_n \leq C$ for some arbitrary predefined positive constant $C \geq 1$. Then, $\mu_+(\alpha,\beta) = 0$ and $\mu_-(\alpha,\beta) > 0$. ∎

# 6.6 Proof of Theorem 6.1.7

Proof (Theorem 6.1.7) Fix $\alpha \in \mathbb{R}\backslash\mathbb{Q}$ and $\beta > 0$ such that $\mu(\alpha,\beta) = +\infty$.

Given a real number $\rho \in \mathbb{R}$, for each $n \geq 0$ write $e_n = e_n(\rho)$ and set $\mathfrak{D}$ as defined in (6.9), that is,

$$\mathfrak{D} = \bigcup_{n=0}^{+\infty} \left(\mathcal{N}_\rho(n) \cup \mathcal{N}'{}_\rho(n)\right),$$

where the sequence $(\kappa_n)_{n\in\mathbb{N}_0}$ as defined in (6.8) and

$$\mathcal{N}_\rho(n) = \mathcal{N}_\rho(\kappa_{n+1}, \alpha, ||q_{n+1}\alpha||) \qquad \text{and} \qquad \mathcal{N}'_\rho(n) = \mathcal{N}'_\rho(\kappa_n + q_{n+1}, \alpha, ||q_n\alpha||).$$

Notice also that, from the way that the sequence $(\kappa_n)_{n\in\mathbb{N}_0}$ is defined, relation (6.15) yields that for every $n \in \mathbb{N}_0$

$$||\kappa_n\alpha - \rho|| \leq ||q_n\alpha||. \tag{6.34}$$

Let $(w_k)_{k\in\mathbb{N}}$ be the sequence defined in (6.6). The goal is to prove that $\lim\limits_{\substack{k\to+\infty,\\ k\notin\mathfrak{D}}} |w_k| = +\infty$. This, in turn, implies that the sequence $(w_k)_{k\in\mathbb{N}}$ and the subsequence $(w_k)_{k\in\mathfrak{D}}$ have the same set of finite limit points.

First of all notice that if the non-decreasing sequence of natural numbers $(\kappa_n)_{n\in\mathbb{N}_0}$ does not tend to infinity, then the sequence $(w_k)_{k\in\mathbb{N}}$ has an empty set of limit points. Indeed, assume that $\lim_{n\in\mathbb{N}} \kappa_n = \kappa$ for some $\kappa \in \mathbb{N}$, that is, there exists $n_0 \in \mathbb{N}$ such that $e_n = 0$ for every $n \geq n_0$. In this case, Proposition 6.2.1 yields that the sequence $(w_k)_{k\in\mathbb{N}}$ has the same set of limit points with the sequence $\left((k+\kappa)^\beta \cdot \{k\alpha\}_2\right)_{k\in\mathbb{N}}$. In turn, from the same proposition one obtains that the sequence $\left((k+\kappa)^\beta \cdot \{k\alpha\}_2\right)_{k\in\mathbb{N}}$ has the same set of limit points with the sequence $\left(k^\beta \cdot \{k\alpha\}_2\right)_{k\in\mathbb{N}}$. From the assumption that $\mu(\alpha,\beta) = +\infty$ one infers that the set of limit points of the sequence $\left(k^\beta \cdot \{k\alpha\}_2\right)_{k\in\mathbb{N}}$ is empty and the claim follows.

Therefore, one may assume without loss of generality that the sequence $(\kappa_n)_{n\in\mathbb{N}}$ tends to infinity, that is, $\lim_{n\to+\infty} \kappa_n = +\infty$. We prove that $\lim\limits_{\substack{k\to+\infty,\\ k\notin\mathfrak{D}}} |w_k| = +\infty$ by showing that for every $h > 0$ there exists $n_h \in \mathbb{N}$ such that, for every $n \geq n_h$ and every $k \in [\![\kappa_n + 1, \kappa_{n+1}]\!] \setminus \mathfrak{D}$, it holds that

$$|w_k| \geq 4h. \tag{6.35}$$

To this end, fix $h \in \mathbb{R}^+$. By assumption, there exist $n_h \in \mathbb{N}$ such that

$$\kappa_{n_h} \geq q_{n_h} \qquad \text{and} \qquad q_n^\beta \cdot ||q_n\alpha|| \geq 4h \quad \text{for every } n \geq n_h.$$

Fix $n \geq n_h$ and $k \in [\![\kappa_n, \kappa_{n+1}]\!] \setminus \mathfrak{D}$. If $k \in [\![\kappa_n + q_{n+1} + 1, \kappa_{n+1}]\!] \setminus \mathcal{N}_\rho(n)$, then

$$|w_k| \quad \geq \quad q_{n+1}^\beta \cdot ||q_{n+1}\alpha|| \quad \geq \quad 4h.$$

If $k \in [\![\kappa_n + 1, \kappa_n + q_{n+1}]\!] \setminus \mathfrak{D}$, then let $m = m(n) \in \mathbb{N}$ be the minimum natural number such that $\kappa_m = \kappa_n$ and set $n' = n - m$. One has that

$$\begin{aligned}&[\![\kappa_n + 1, \kappa_n + q_{n+1}]\!] \setminus \mathfrak{D}\\ \subseteq\; &\Big([\![\kappa_m + 1, \kappa_m + q_{m+1}]\!] \setminus \mathcal{N}'_\rho(m)\Big) \cup \left(\bigcup_{j=1}^{n'} \Big([\![\kappa_m + q_{m+j}, \kappa_m + q_{m+j+1}]\!] \setminus \mathcal{N}'_\rho(m+j)\Big)\right).\end{aligned}$$

If $k \in [\![\kappa_m, \kappa_m + q_{m+1}]\!] \setminus \mathcal{N}'_\rho(m)$, then, from the way that the natural number $m$ was chosen, one has that $e_m \neq 0$ and $m \geq n_h$. Thus, one obtains that

$$|w_k| \quad \geq \quad q_m^\beta \cdot ||q_m\alpha|| \quad \underset{(m \geq n_h)}{\geq} \quad 4h.$$

Similarly, if $k \in [\![\kappa_m + q_{m+j}, \kappa_m + q_{m+j+1}]\!] \setminus \mathcal{N}'_\rho(m+j)$ for some $1 \leq j \leq n'$, then it holds that

$$||w_k|| \quad \geq \quad q_{m+j}^\beta \cdot ||q_{m+j}\alpha|| \quad \geq \quad 4h.$$

Therefore, inequality (6.35) has been proved. Since the choice of $h \in \mathbb{R}^+$ is arbitrary, one infers that the sequences $(w_k)_{k\in\mathbb{N}}$ and $(w_k)_{k\in\mathfrak{D}}$ have the same set of limit points.

We now prove inclusions (6.10) in the statement of Theorem 6.1.7. It follows from the definition of the Bohr set (6.7) that

$$\mathcal{N}_\rho(\kappa_{n+1}, \alpha, ||q_{n+1}\alpha||) \cap [\![1, \kappa_n]\!] \quad \subseteq \quad \mathcal{N}_\rho(\kappa_n, \alpha, ||q_n\alpha||),$$

$$\mathcal{N}_\rho(\kappa_n + q_{n+1}, \alpha, ||q_n\alpha||) \cap [\![1, \kappa_n]\!] \quad \subseteq \quad \mathcal{N}_\rho(\kappa_n, \alpha, ||q_n\alpha||)$$

and

$$\mathcal{N}_\rho\left(\kappa_{n+1},\alpha,||q_{n+1}\alpha||\right)\cap[\![\kappa_n+q_{n+1}]\!] \quad\subseteq\quad \mathcal{N}_\rho\left(\kappa_n+q_{n+1},\alpha,||q_n\alpha||\right).$$

Therefore, it is enough to show, on the one hand that

$$\{\kappa_{n+1}\} \quad\subseteq\quad \mathcal{N}_\rho\left(\kappa_{n+1},\alpha,||q_{n+1}\alpha||\right) \tag{6.36}$$

and

$$\mathcal{N}_\rho\left(\kappa_{n+1},\alpha,||q_{n+1}\alpha||\right)\cap[\![\kappa_n+q_{n+1}+1,\kappa_{n+1}]\!] \quad\subseteq\quad \kappa_n+\bigcup_{l=0}^{2}\{(e_{n+1}-l)\cdot q_{n+1}\} \tag{6.37}$$

and, on the other, that

$$\mathcal{N}_\rho\left(\kappa_n+q_{n+1},\alpha,||q_n\alpha||\right)\cap[\![\kappa_n+1,\kappa_n+q_{n+1}]\!] \quad\subseteq\quad \kappa_n+\bigcup_{l=0}^{1}\{(l+1)q_n\}\cup\{q_{n+1}-lq_n\}. \tag{6.38}$$

As far as inclusion (6.36) is concerned, inequality (6.34) immediately implies that, for every $n\in\mathbb{N}_0$, it holds that $\kappa_{n+1}\in\mathcal{N}_\rho\left(\kappa_{n+1},\alpha,||q_{n+1}\alpha||\right)$. As for inclusion (6.37), there is nothing to prove if $e_{n+1}\leq 1$. Therefore, without loss of generality, assume that $e_{n+1}\geq 2$. We show that

$$\begin{aligned}\mathcal{N}_\rho\left(\kappa_{n+1},\alpha,||q_{n+1}\alpha||\right)&\cap[\![\kappa_n+q_{n+1}+1,\kappa_{n+1}]\!]\\ &\subseteq\quad \kappa_{n+1}-\left(\mathcal{N}_0\left(e_{n+1}q_{n+1},\alpha,2\cdot||q_{n+1}\alpha||\right)\cup\{0\}\right).\end{aligned} \tag{6.39}$$

To this end, fix $k\in\mathcal{N}_\rho\left(\kappa_{n+1},\alpha,||q_{n+1}\alpha||\right)\cap[\![\kappa_n+q_{n+1},\kappa_{n+1}]\!]$ and set

$$s_{n+1}=\sum_{j=n+2}^{+\infty}e_j(\rho)\left\{q_j\alpha\right\}_2.$$

Inequality (6.14) yields that $|s_{n+1}|\leq||q_{n+1}\alpha||$. By applying the triangle inequality one obtains that

$$\begin{aligned} ||(\kappa_{n+1} - k)\,\alpha|| &= ||\kappa_{n+1}\alpha + s_{n+1} - k\alpha - s_{n+1}|| \\ &\leq ||k\alpha - \rho|| + ||s_{n+1}|| \\ &\leq 2\,||q_{n+1}\alpha||\,, \end{aligned}$$

which implies inclusion (6.39). Inclusion (6.37) follows from inclusion (6.39) upon noticing that

$$(\{0\} \cup \mathcal{N}_0\,(e_{n+1}q_{n+1}, \alpha, 2 \cdot ||q_{n+1}\alpha||)) \cap [\![0, (e_{n+1} - 1)\,q_{n+1}]\!] \quad \subseteq \quad \bigcup_{l=0}^{2} \{l \cdot q_{n+1}\}\,.$$

As for the inclusion in (6.38), one has that

$$\mathcal{N}_\rho\,(\kappa_n + q_{n+1}, \alpha, ||q_n\alpha||) \cap [\![\kappa_n + 1, \kappa_n + q_{n+1}]\!] \quad \subseteq \quad \kappa_n + \mathcal{N}_0\,(q_{n+1}, \alpha, 2\,||q_n\alpha||)\,.$$

It easily follows that

$$\mathcal{N}_0\,(q_{n+1}, \alpha, 2\,||q_n\alpha||) \quad \subseteq \quad \{q_n, 2q_n, q_{n+1} - q_n, q_{n+1}\}\,.$$

Therefore, it has been established that

$$\{\kappa_n\}_{n\in\mathbb{N}} \quad \subseteq \quad \mathfrak{D}$$

and

$$\mathfrak{D} \quad \subseteq \quad \bigcup_{n=0}^{+\infty} \left(\kappa_n + \bigcup_{l=0}^{2} \{(e_{n+1} - l) \cdot q_{n+1}\}\right) \cup \left(\kappa_n + \bigcup_{l=0}^{1} \{(l+1)q_n, q_{n+1} - lq_n\}\right).$$

The proof is complete. ■

# Chapter 7

# A Generalisation of Sárközy's Theorem in more Variables

Given integers $a_1, a_2, ..., a_s$, define the function $\mathcal{L} : \mathbb{N}^s \to \mathbb{Z}$ by

$$\mathcal{L}(x_1, x_2, ..., x_s) \quad = \quad a_1x_1 + a_2x_2 + \cdots + a_sx_s. \tag{7.1}$$

Given $t \in \mathbb{N}$, a **quadratic form** $\mathcal{Q} \in \mathbb{Z}[y_1, y_2, \cdots, y_t]$ is a map $\mathcal{Q} : \mathbb{R}^t \to \mathbb{R}$ of the form

$$\mathcal{Q}(y_1, y_2, \cdots, y_t) \quad = \quad \sum_{j=1}^{t} b_j y_j^2 + \sum_{1 \le l < k \le t} d_{l,k} y_l y_k. \tag{7.2}$$

For every $l, k \in [\![t]\!]$ with $l > k$, define $d_{l,k} = d_{k,l}$. This serves technical purposes; in particular, it will allow us to avoid mentioning the ordering between the two parameters $l, k$.

## 7.1 Introduction

The goal of this chapter is to prove the following result which generalises Theorem 1.2.1, p.70, both in the number of variables and in the form of the configurations under consideration.

**Theorem 7.1.1** *Let $a_1, a_2, ..., a_s$ be integer numbers such that*

$$a_1 + a_2 + \cdots + a_s = 0 \tag{7.3}$$

*and let* $\mathcal{Q} \in \mathbb{Z}[y_1, y_2, ..., y_t]$ *be a quadratic form as in equation* (7.2)*.*

*Then, there exist constants* $0 < c_1 < 1 \leq C_1$ *(depending on* $a_1, a_2, \cdots, a_s, \mathcal{Q}$*) satisfying the following property: for every* $N \in \mathbb{N}$ *and for every* $A \subseteq [\![N]\!]$ *with*

$$\#A \quad \geq \quad C_1 \cdot \frac{N}{(\log\log N)^{c_1}}, \tag{7.4}$$

*there exist distinct* $x_1, x_2, \cdots, x_s \in A$ *and distinct* $y_1, y_2, \cdots, y_t \in \mathbb{N}$ *such that*

$$\mathcal{L}(x_1, ..., x_s) \; = \; \mathcal{Q}(y_1, y_2, \cdots, y_t), \tag{7.5}$$

*where* $\mathcal{L}$ *is as in equation* (7.1)*.*

As in Sárközy's original proof of Theorem 1.2.1 (p.70), the proof of Theorem 7.1.1 is based on a Fourier analytic density increment argument which proceeds as follows: fix the parameters $a_1, ..., a_s \in \mathbb{Z}$ and fix $\mathcal{Q} \in \mathbb{Z}[y_1, ..., y_t]$ as in Theorem 7.1.1. Given $N \in \mathbb{N}$, denote by $2^{[\![N]\!]}$ the power-set of $[\![N]\!]$ and let

$$\mathtt{T}_N : \quad 2^{[\![N]\!]} \ni A \quad \mapsto \quad \mathtt{T}_N(A) \in \{\text{true, false}\}$$

denote the truth-value of the statement: there exist distinct $x_1, ..., x_s \in A \subseteq [\![N]\!]$ and distinct $y_1, ..., y_t \in \mathbb{N}$ satisfying equation (7.5). Fix $\delta \in (0, 1)$ and $N \gg_\delta 1$ large enough. Assume that one wants to prove that $\mathtt{T}_N(A) = $ true for every subset $A$ of $[\![N]\!]$ with density $\delta(A; [\![N]\!]) \geq \delta$ (as defined in [Chapter 1, Equation (1.86), p.78]). By fixing such a subset $A \subseteq [\![N]\!]$, one can distinguish two cases:

1. The set $A$ is $\epsilon$-Fourier uniform (Definition 1.2.14, p.78) for some $\epsilon = \epsilon(\delta)$ depending on $\delta$; that is,

$$\left|\left|\hat{\chi}_A - \mathbb{E}_{n \in [\![N]\!]}\chi_A(n) \cdot \hat{\chi}_{[\![N]\!]}\right|\right|_\infty \leq \epsilon \cdot N.$$

   Then one is able to show that $\mathtt{T}_N(A)$ is true.

2. The set $A$ is not $\epsilon$-Fourier uniform for the same $\epsilon = \epsilon(\delta)$ as in the first case. Then one is able to show the existence of a long arithmetic subprogression $P \subseteq [\![N]\!]$ with step equal to a perfect square $q^2$, for some $q \in \mathbb{N}$, in which

the density of $A$ is increased in the sense that

$$\delta(A;P) \ \geq \ \delta(A;[\![N]\!]) + c(\delta).$$

Here, $c(\delta) > 0$ is a constant depending only on $\delta$.

The second case is relevant when the arithmetic progression $P$ has the form

$$P \ = \ \left\{a + q^2 n : n \in [\![M]\!]\right\}$$

for some $a, M \in \mathbb{N}$. Indeed, the quadratic equation (7.5) is invariant on these kind of subsets in the following sense: fix $A \subseteq [\![N]\!]$ and a subset $P \subseteq [\![N]\!]$ of the form $P = \{a + q^2 n : n \in [\![N']\!]\}$ for some $N' \in \mathbb{N}$. Set

$$A' := \left\{n \in \mathbb{N} : a + q^2 n \in A\right\}.$$

The quadratic equation (7.5) admits a solution in $A'$ for some $x_1', ..., x_s' \in A'$ and $y_1', ..., y_t' \in \mathbb{N}$ if, and only if, equation (7.5) admits a solution in $A$ with

$$x_i := \quad q^2 x_i' + a, \quad i \in \{1, ..., s\} \quad \text{and} \quad y_j := \quad q y_j', \quad j \in \{1, ..., t\}. \qquad (7.6)$$

This is clear from the following equivalences:

$$\begin{aligned}
& & \mathcal{L}(x_1', ..., x_s') &= \mathcal{Q}(y_1', ..., y_t') \\
&\Leftrightarrow & a_1 x_1' + ... + a_s x_s' &= \sum_{j=0}^{t} b_j y_j'^2 + \sum_{1 \leq k < l \leq t} d_{k,l} y_k' y_l' \\
&\underset{(7.3)}{\Leftrightarrow} & a(a_1 + ... + a_s) + a_1 q^2 x_1' + ... + a_s q^2 x_s' &= \sum_{j=0}^{t} b_j q^2 y_j'^2 + \sum_{1 \leq k < l \leq t} d_{k,l} q^2 y_k' y_l' \\
&\Leftrightarrow & \mathcal{L}\left(a + q^2 x_1', ..., a + q^2 x_s'\right) &= \mathcal{Q}(q y_1', ..., q y_t') \\
&\Leftrightarrow & \mathcal{L}(x_1, ..., x_s) &= \mathcal{Q}(y_1, ..., y_t).
\end{aligned}$$

Provided that $N \in \mathbb{N}$ is large enough, the density increment argument is concluded by iterating the aforementioned two steps, leading one to eventually infer that the value of $\mathtt{T}_N(A)$ is true; that is, that there exists distinct $x_1, ..., x_s \in A$ and distinct $y_1, ..., y_t \in \mathbb{N}$ satisfying the quadratic equation (7.5). This follows upon noticing that the iteration can happen at most $1/c(\delta)$-times: indeed, on the one hand, in each step the density of $A$ over the obtained subsequence $P$ increases at least by $c(\delta)$ and, on the other, the density of $A$ cannot be larger than 1 so that the Case 2 is ruled out.

The density increment argument leads one to the following theorem (proved in Section 7.4) from which one derives Theorem 7.1.1.

**Theorem 7.1.2 (Fourier Uniformity Lemma)** *There exists an absolute constant $C > 0$ such that, for each $\epsilon \in (0,1)$ and for each $N \geq \exp\left(\exp\left(\frac{C}{\epsilon}\right)\right)$, the following holds: for any $A \subseteq [\![N]\!]$, there exists an arithmetic progression $P \subseteq [\![N]\!]$ with a square common difference $q^2$, for some $q \in \mathbb{N}$, such that*

1. $\mathop{\mathbb{E}}_{n\in P}\left(\chi_A(n)\right) \quad \geq \quad \mathop{\mathbb{E}}_{n\in [\![N]\!]}\left(\chi_A(n)\right),$
2. $\left|\left|\hat{\chi}_{A\cap P} - \mathop{\mathbb{E}}_{n\in P}\chi_A(n)\hat{\chi}_P\right|\right|_\infty \quad \leq \quad \epsilon\,|P|$ *and*
3. $|P| \quad \geq \quad N^{\exp(-C/\epsilon)}.$

The chapter is organised as follows. In Section 7.2, we prove Theorem 7.1.1 under two assumptions: first, that Theorem 7.1.2 holds true and also, that an assumption concerning the number of solutions to equation (7.5), namely Proposition 7.2.1, holds. The latter assumption (Proposition 7.2.1) is proved in Section 7.3. Theorem 7.1.2 is established in Section 7.4. A crucial result from the literature due to Bourgain which is part of the proof of Proposition 7.2.1, namely Lemma 7.3.3, is proved in Section 7.5 for the sake of completeness.

## 7.2 Proof of Theorem 7.1.1

For the rest of the exposition, we consider only quadratic forms $\mathfrak{Q}$ which have $y_1$ as an independent variable in the sense that for every $l \in \{2,...,t\}$, $d_{1,l} = 0$. In other words,

$$\mathfrak{Q}\left(y_1,...,y_t\right) \quad = \quad b_1 y_1^2 + \sum_{j=2}^{t} b_j y_j^2 + \sum_{2\leq l<k\leq t} d_{l,k} y_l y_k. \tag{7.7}$$

There is no loss of generality in doing so because, given an arbitrary quadratic form $\mathcal{Q}$, one can find a quadratic form $\mathfrak{Q}$ as in (7.7) such that every solution of the equation $\mathcal{L}\left(x_1,...,x_t\right) = \mathfrak{Q}\left(y_1,...,y_t\right)$ corresponds to a solution of the equation $\mathcal{L}\left(x_1,...,x_s\right) = \mathcal{Q}\left(y_1',...,y_t'\right)$. To see this, define the following two changes of variables:

$$V_1\left(y_1,...,y_t\right) := \quad \left(y_1 - \sum_{2\leq l\leq t} d_{1,l} y_l, \quad 2b_1 y_2, \quad ... \quad , \quad 2b_1 y_t\right) \tag{7.8}$$

and

$$V_2\left(y_1,...,y_t\right) := \quad \left(y_1 + y_2, \quad y_1 - y_2, \quad y_3, \quad ... \quad , \quad y_t\right). \tag{7.9}$$

Given a quadratic form $\mathcal{Q} \in \mathbb{Z}\left[y_1,...,y_t\right]$ as in (7.2), define the quadratic form $\mathfrak{Q} \in \mathbb{Z}\left[y_1,...,y_t\right]$ obtained by applying the following steps:

1. **Step 1:** If there exist $j \in \{1,...,t\}$ such that $b_j \neq 0$, then assume, without loss of generality, that $b_1 \neq 0$ and go to Step 3. Otherwise, go to Step 2.

2. **Step 2:** If for every $j \in \{1,...,t\}$ it holds $b_j = 0$, then assume, without loss of generality, that $d_{\{1,2\}} \neq 0$ and go to Step 3.

3. **Step 3:** Set

$$\mathfrak{Q}\left(y_1,...,y_t\right) := \begin{cases} \mathcal{Q}\left(V_1\left(y_1,...,y_t\right)\right) & \text{if Step 1 holds,} \\ \mathcal{Q}\left(V_1 \circ V_2\left(y_1,...,y_t\right)\right) & \text{otherwise.} \end{cases} \tag{7.10}$$

The resulting quadratic form $\mathfrak{Q}$ has the form (7.7). For instance, if $\mathcal{Q}\left(y_1,y_2\right) = y_1^2 + 2y_2^2 + 3y_1y_2$, then Step 1 holds and $V_1\left(y_1,y_2\right) = \left(y_1 - 3y_2, 2y_2\right)$. Therefore, from Step 3, one has that $\mathfrak{Q}\left(y_1,y_2\right) = \mathcal{Q}\left(V_1\left(y_1,y_2\right)\right) = y_1^2 - y_2^2$. To stress their generality, whenever possible, the results in the following sections will be proved for the general quadratic forms (7.2).

Theorem 7.1.2 shows that, given $\epsilon \in (0,1)$, $N \in \mathbb{N}$ large enough and a subset $A \subseteq [\![N]\!]$, one can find a square-difference arithmetic progression $P \subseteq [\![N]\!]$ such that the set $A$ is $\epsilon$-Fourier uniform over $P$. In turn, the Fourier-uniformity of the set $A$ over the arithmetic progression $P$ allows one to count the number of solutions of the quadratic equation (7.5) with $x_1, ..., x_s \in A$. This estimation is provided by the following proposition which, in combination with Theorem 7.1.2, yields the proof of Theorem 7.1.1.

Before stating it, given integers $a_1, ..., a_s$ and a quadratic form $\mathcal{Q} \in \mathbb{Z}[y_1, ..., y_t]$, define the operator

$$\mathcal{E}(f_1, f_2, ..., f_2) = \sum_{\mathcal{L}(x_1,...,x_s)=\mathcal{Q}(y_1,...,y_t)} f_1(x_1) \cdots f_s(x_s) \cdot \chi_{[\![\sqrt{N}]\!]}(y_1) \cdots \chi_{[\![\sqrt{N}]\!]}(y_t), \tag{7.11}$$

where $f_1, ..., f_s : [\![N]\!] \to \mathbb{C}$ are 1-bounded[1] functions and $\mathcal{L} : \mathbb{N}^s \to \mathbb{Z}$ is as in (7.1). When $f_1 = ... = f_s = \chi_A$, where $\chi_A$ is the characteristic function of $A$, the value $\mathcal{E}(\chi_A, ..., \chi_A)$ corresponds to the number of solutions of the quadratic equation (7.5) with $x_1, .., x_s \in A$ and $y_1, ..., y_t \leq \sqrt{N}$.

**Proposition 7.2.1** *Let $A$ be a subset of $[\![N]\!]$ with density $\delta$. Given non-zero integers $a_1, ..., a_s \in \mathbb{Z}\setminus\{0\}$ and a quadratic form $\mathfrak{Q} \in \mathbb{Z}[y_1, ..., y_t]$ defined as in* (7.7)*, the following holds: there exists a small absolute constant* $0 < c < 1$ *and a large absolute constant $C_0 \geq 1$ such that if $A \subseteq [\![N]\!]$ is a $c \cdot \delta^{C_0}$-Fourier uniform set, then*

$$|\mathcal{E}(\chi_A, ..., \chi_A)| \quad \geq \quad \frac{c\delta^s}{2} \cdot N^{s+\frac{t}{2}-1}, \tag{7.12}$$

*where $\mathcal{E}$ is the operator defined in equation* (7.11)*.*

*Furthermore, there exists an absolute constant $C_1 > 0$ such that for any $N \geq C_1 \cdot \delta^{-2s}$, the quadratic equation* (7.5) *admits a solution with distinct $x_1, ..., x_s \in A$ and distinct $y_1, ..., y_t \in \left[\!\left[\sqrt{N}\right]\!\right]$.*

Proof (Theorem 7.1.1) Given a quadratic form $\mathcal{Q}$, let $\mathfrak{Q}$ be the quadratic form obtained from the algorithm in (7.10).

[1] A complex function $f : [\![N]\!] \mapsto \mathbb{C}$ is 1-bounded if, for every $n \in [\![N]\!]$, $|f(n)| \leq 1$.

Fix $\delta \in (0,1)$ and set $\epsilon = c \cdot \delta^{C_0}$, where the constants $c, C_0$ are provided by Proposition 7.2.1. Fix also $N \geq \exp\left(\exp\left(\frac{C}{\epsilon}\right)\right)$ (where the constant $C$ is obtained from the statement of Theorem 7.1.2) and a subset $A \subseteq [\![N]\!]$ with density

$$\delta_A \;=\; \frac{\#A}{N} \;\geq\; \delta.$$

From Theorem 7.1.2, there exists a square-difference arithmetic subprogression $P = \{a + q^2 m : m \in [\![M]\!]\}$ (for some $a, q \in \mathbb{N}$) of length $M \in \mathbb{N}$ such that

1. the following relation holds:

$$M \;=\; |P| \;\geq\; N^{\exp(-C/\epsilon)},$$

2. the density of $A$ in $P$ is at least $\delta$ and
3. $A \cap P$ is $c\delta^{C_0}$-Fourier uniform in $P$.

Set

$$A' \quad = \quad \left\{ n \in [\![M]\!] : a + nq^2 \in A \right\}.$$

It follows easily from the density of the set $A \cap P$ in the arithmetic progression $P$ and from the definition of Fourier-uniformity (Definition 1.2.14, p.78) that the density of $A'$ is at least $\delta$ over the set $[\![M]\!]$ and that $A'$ is $c\delta^{C_0}$-Fourier uniform. Upon chosing the constant $C_0 > 0$ large enough and the constant $0 < c \leq 1$ small enough if necessary, Proposition 7.2.1 implies that there exist distinct $x'_1, ..., x'_s \in A'$ and distinct $y'_1, ..., y'_t \in \mathbb{N}$ such that

$$\mathcal{L}\left(x'_1, ..., x'_s\right) \;=\; \mathfrak{Q}\left(y'_1, ..., y'_t\right). \tag{7.13}$$

In turn, equation (7.13) implies that

$$\mathcal{L}\left(x_1, ..., x_t\right) \;=\; \mathfrak{Q}\left(y_1, .., y_t\right)$$

with $x_i = a + q^2 x'_i \in A$ for every $i \in \{1, ..., s\}$ and $y_j = q y'_j$ for every $j \in \{1, .., t\}$. From the way the quadratic form $\mathfrak{Q}$ is defined in (7.10), one obtains a solution for the quadratic equation (7.5) with distinct $x_1, x_2, ..., x_s \in A$ and $y_1, ..., y_t \in \mathbb{N}$.

As for the size of the set $A$, one has that

$$N \quad \geq \quad \exp\exp\left(\frac{C}{c\delta^{C_0}}\right),$$

which implies that

$$\delta \quad \geq \quad \frac{(C/c)^{\frac{1}{C_0}}}{\ln\ln(N)^{\frac{1}{C_0}}}.$$

Therefore,

$$\#A \quad \geq \quad \delta N \quad \geq \quad \frac{(C/c)^{\frac{1}{C_0}}}{\ln\ln(N)^{\frac{1}{C_0}}} \cdot N.$$

Setting $C_1 = (C/c)^{\frac{1}{C_0}}$ and $c_1 = 1/C_0$ completes the proof. ■

## 7.3 Proof of Proposition 7.2.1

Given a quadratic form $\mathfrak{Q} \in \mathbb{Z}[y_1, ..., y_t]$ as defined in (7.7) and integers $a_1, ..., a_s \in \mathbb{Z}$ such that the zero sum condition (7.3) holds, Proposition 7.2.1 shows that an $\epsilon$-Fourier uniform set admits solutions to the equation

$$\mathcal{L}(x_1, ..., x_s) = \mathfrak{Q}(y_1, ..., y_t) \quad \text{with} \quad x_1, ..., x_s \in [\![N]\!] \quad \text{and} \quad y_1, ..., y_t \in \left[\!\left[\sqrt{N}\right]\!\right] \tag{7.14}$$

when $N$ is large enough, where $\mathcal{L}(\cdot, ..., \cdot)$ as defined in (7.1). To prove Proposition 7.2.1, one first seeks to count the numbers of solutions to equation (7.14).

**Proposition 7.3.1** *Given non-zero integers $a_1, ..., a_s$ such that $a_1 + \cdots + a_s = 0$ and a general quadratic form $\mathcal{Q} \in \mathbb{Z}[y_1, ..., y_t]$, it holds that*

$$\sum_{\mathcal{L}(x_1,...,x_s)=\mathcal{Q}(y_1,...,y_t)} \chi_{[\![N]\!]}(x_1) \cdots \chi_{[\![N]\!]}(x_s) \cdot \chi_{[\![\sqrt{N}]\!]}(y_1) \cdots \chi_{[\![\sqrt{N}]\!]}(y_t) \quad \gg \quad N^{s+\frac{t}{2}-1},$$

*where $A : \mathbb{N}^s \to \mathbb{Z}$ is defined in* (7.1)*. Equivalently,*

$$\left|\mathcal{E}\left(\chi_{[\![N]\!]}, ..., \chi_{[\![N]\!]}\right)\right| \quad \gg \quad N^{s+\frac{t}{2}-1}$$

*with the operator $\mathcal{E}$ as defined in* (7.11).

Proof (Proposition 7.3.1) Given $j \in \{1, 2, ..., t\}$ and $k, l \in \{1, 2, ..., t\}$ with $k < l$, let $b_j$, $d_{l,k}$ be the coefficients of $\mathcal{Q}$ as defined in equation (7.2).

Since the non-zero integers $a_1, ..., a_s$ sum to zero, there exist at least one positive and one negative number between them. By changing the indices if necessary, assume, without loss of generality, that $a_1 > 0$ and $a_2 < 0$. Let $d = \gcd(a_1, a_2)$ be the greatest common divisor of $a_1$ and $a_2$. The Diophantine equation $a_1x + a_2y = n$ admits solutions if, and only if, $n$ is a multiple of $d$. In view of this, fix natural numbers $x_1', x_2' \in \mathbb{N}$ such that

$$a_1x_1' + a_2x_2' \quad = \quad d.$$

The goal is to bound from below the number of solutions to the equation

$$a_1x_1 + a_2x_2 \quad = \quad \mathcal{Q}(y_1, ..., y_t) - a_3x_3 - \cdots - a_sx_s \tag{7.15}$$

with $x_1, ..., x_s \in [\![N]\!]$ and $y_1, ..., y_t \in \left[\!\left[\sqrt{N}\right]\!\right]$. To this end, for each $i \in \{3, 4, .., s\}$, $j \in \{1, 2, ..., t\}$, set

$$x_i = d \cdot z_i \qquad \text{and} \qquad y_j = d \cdot w_j \tag{7.16}$$

with

$$z_i \in \left[\!\!\left[\frac{N}{C \cdot sd\,|a_i| \cdot (x_1' + x_2')}\right]\!\!\right] \qquad \text{and} \qquad w_j \in \left[\!\!\left[\frac{1}{dt} \cdot \sqrt{\frac{N}{Cb\,(x_1' + x_2')}}\right]\!\!\right], \tag{7.17}$$

where

$$b = \max\{\max\{|b_j|, |d_{j,k}|\} : j, k \in \{1, 2, ..., t\}, j \neq k\}$$

and $C \geq 10 \cdot \max\{|a_1|, |a_2|\}$ is a large constant. Making the change of variables (7.16) in equation (7.15), one obtains

$$a_1x_1 + a_2x_2 \;=\; d^2 \cdot \mathcal{Q}(w_1, ..., w_t) - d \cdot (a_3z_3 + \cdots + a_sz_s)\,. \tag{7.18}$$

Taking further into consideration the restrictions (7.17), one obtains that

$$\frac{-2N}{C \cdot d \cdot (x_1' + x_2')} \ \leq \ d\mathcal{Q}\left(w_1, ..., w_t\right) - \left(a_3 z_3 + \cdots + a_s z_s\right) \ \leq \ \frac{2N}{C \cdot d \cdot (x_1' + x_2')}. \tag{7.19}$$

The restrictions in (7.17) imply that there are $\Omega\left(N^{s+\frac{t}{2}-2}\right)$ choices of the parameters $w_1, ..., w_t,\ z_3, ..., z_s$ satisfying inequality (7.19). Thus, it is enough to prove that for every such choice of the parameters $w_j,\ j \in \{1, ..., t\}$ and $z_i,\ i \in \{3, ..., s\}$, equation (7.18) has $\Omega\left(N\right)$ pairs of solutions $x_1$ and $x_2$. Establishing this implies that equation (7.15) admits $\Omega\left(N^{s+\frac{t}{2}-2}\right) \cdot \Omega\left(N\right) \ = \ \Omega\left(N^{s+\frac{t}{2}-1}\right)$ solutions with $x_i \in [\![N]\!],\ i \in \{1, 2, ..., t\}$ and $y_j \in \left[\!\left[\sqrt{N}\right]\!\right],\ j \in \{1, ..., t\}$.

To this end, fix $w_1, ..., w_t,\ y_3, ..., y_s$ satisfying the restrictions (7.17). Set

$$n \ = \ d \cdot \mathcal{Q}\left(w_1, ..., w_t\right) - \left(a_3 z_3 + \cdots + a_s z_s\right).$$

From inequality (7.19), one has that

$$n \in \left[\!\!\left[-\frac{2N}{C \cdot d \cdot (x_1' + x_2')}, \quad \frac{2N}{C \cdot d \cdot (x_1' + x_2')}\right]\!\!\right] \tag{7.20}$$

and equation (7.15) then becomes

$$a_1 x_1 + a_2 x_2 \ = \ dn. \tag{7.21}$$

By setting $X_1 = n \cdot x_1'$ and $X_2 = n \cdot x_2'$, one obtains a solution to equation (7.21). From the way that the quantities $X_1, X_2$ are defined, one has that

$$X_1, X_2 \quad \in \quad \left[\!\!\left[-\frac{2\max\left\{x_1', x_2'\right\} \cdot N}{C\left(x_1' + x_2'\right)}, \frac{2\max\left\{x_1', x_2'\right\} \cdot N}{C\left(x_1' + x_2'\right)}\right]\!\!\right] \quad \subseteq \quad \left[\!\!\left[-\frac{2N}{Cd}, \frac{2N}{Cd}\right]\!\!\right]. \tag{7.22}$$

Moreover, every solution $x_1, x_2$ to equation (7.18) has the form

$$x_1 = X_1 - a_2 m \quad \text{and} \quad x_2 = X_2 + a_1 m \quad \text{with } m \in \mathbb{Z}.$$

The goal is to show that there exist $\Omega(N)$ values of $m \in \mathbb{Z}$ such that $0 \ < x_1, x_2 \leq$

$N$. It is easy to check that this requirement holds true when

$$m \in I_N := \left[\!\left[\frac{X_1}{a_2}, \frac{N - X_1}{-a_2}\right]\!\right] \cap \left[\!\left[\frac{-X_2}{a_1}, \frac{N - X_2}{a_1}\right]\!\right].$$

In turn, from inclusions (7.20) and (7.22), it follows that

$$\left[\!\left[\frac{2N}{Cd \min\{|a_1|, |a_2|\}}, \quad \frac{(Cd - 2)\, N}{Cd \max\{|a_1|, |a_2|\}}\right]\!\right] \quad \subseteq \quad I_N.$$

Therefore, $|I_N| \gg N$. The proof is complete. ■

Proposition 7.3.1 provides a lower bound for the operator $\mathcal{E}$ defined in (7.11) when all the input functions equal the characteristic function $\chi_A$ of a subset $A \subseteq [\![N]\!]$. To conclude the proof of Proposition 7.2.1, one needs also an upper bound for $L$. It is given by the following result when the quadratic form $\mathfrak{Q}$ is as in (7.7).

**Proposition 7.3.2** *Let $f_1, f_2, \cdots, f_s : [\![N]\!] \to \mathbb{C}$ be 1-bounded functions. Given $t, s \in \mathbb{N}$, a quadratic form $\mathfrak{Q} \in \mathbb{Z}[y_1, y_2, \cdots, y_t]$ (as defined in equation (7.7)) and non-zero integers $a_1, a_2, \cdots, a_s \in \mathbb{Z} \backslash \{\mathbf{0}\}$, it holds that*

$$\left| \sum_{\mathcal{L}(x_1, \cdots, x_s) = \mathfrak{Q}(y_1, \cdots, y_s)} f_1(x_1) \cdot f_2(x_2) \cdots f_s(x_2) \cdot \chi_{[\![\sqrt{N}]\!]}(y_1) \cdots \chi_{[\![\sqrt{N}]\!]}(y_t) \right| \ll$$

$$N^{s + \frac{t}{2} - 1} \cdot \left( \frac{\left|\left|\hat{f}_s\right|\right|_\infty}{N} \right)^{\frac{2}{5}},$$

*where $\mathcal{L} : \mathbb{N}^s \to \mathbb{Z}$ is defined in equation (7.1) and $||\cdot||_\infty$ stands for the (well-defined) supremum norm.*

In order to prove Proposition 7.3.2, one needs the following lemma due to Bourgain [4], which will be reproved in detail for the sake of completeness in Section 7.5. Given $N \in \mathbb{N}$, let the function $\mathtt{S} : \mathbb{T} \to \mathbb{C}$ be such that

$$\mathtt{S}(\alpha) \quad = \quad \sum_{x=1}^{\sqrt{N}} e\left(\alpha x^2\right). \tag{7.23}$$

**Lemma 7.3.3** *Let* $\mathtt{S}$ *be the function defined in* (7.23). *It holds that*

$$\int_{\alpha\in\mathbb{T}} |\mathtt{S}(\alpha)|^5 \, d\alpha \quad \ll \quad N^{\frac{3}{2}}. \tag{7.24}$$

Recall that, given two real integrable functions $f, g : \mathbb{T} \to \mathbb{C}$, *Hölder's inequality* states that if $p, q$ are positive real numbers such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_{\alpha\in\mathbb{T}} |f(\alpha)\cdot g(\alpha)| \, \mathrm{d}\alpha \quad \leq \quad \left(\int_{\alpha\in\mathbb{T}} |f(\alpha)|^p \, \mathrm{d}\alpha\right)^{\frac{1}{p}} \cdot \left(\int_{\alpha\in\mathbb{T}} |g(\alpha)|^q \, \mathrm{d}\alpha\right)^{\frac{1}{q}}. \tag{7.25}$$

Proof (Proposition 7.3.2) Set the function $\sum_{\mathfrak{Q}} : \mathbb{T} \mapsto \mathbb{C}$ with formula

$$\sum_{\mathfrak{Q}} (\alpha) \;=\; \sum_{(y_1,\cdots,y_t)\in[\![\sqrt{N}]\!]^t} e\left(\alpha \cdot \mathfrak{Q}\left(y_1, \cdots, y_t\right)\right) \tag{7.26}$$

and

$$J \quad = \quad \left| \sum_{\mathcal{L}(x_1,x_2,\cdots,x_s)=\mathfrak{Q}(y_1,\ldots,y_t)} f_1\left(x_1\right)\cdot f_2\left(x_2\right)\cdots f_s\left(x_s\right)\cdot \chi_{[\![\sqrt{N}]\!]}\left(y_1\right)\cdots \chi_{[\![\sqrt{N}]\!]}\left(y_t\right) \right|.$$

From the orthogonality lemma (Lemma 1.2.13, p.77) one has that

$$J \quad = \quad \left| \int_{\alpha\in\mathbb{T}} \hat{f}_1\left(a_1\alpha\right)\cdots \hat{f}_s\left(a_s\alpha\right)\cdot \sum_{\mathfrak{Q}} (-\alpha) \, \mathrm{d}\alpha \right|. \tag{7.27}$$

Therefore, it is enough to bound the integral in equation (7.27). To this end, note that

$$\begin{aligned} J \quad &\leq \quad \int_{\alpha\in\mathbb{T}} \left| \hat{f}_1\left(a_1\alpha\right)\cdots \hat{f}_s\left(a_s\alpha\right)\cdot \sum_{\mathfrak{Q}} (-\alpha) \right| \mathrm{d}\alpha \\ &\underset{(7.25)}{\leq} \quad \left(\int_{\alpha\in\mathbb{T}} \left|\hat{f}_1\left(a_1\alpha\right)\right|^{\frac{5}{4}} \cdots \left|\hat{f}_s\left(a_s\alpha\right)\right|^{\frac{5}{4}} \mathrm{d}\alpha\right)^{\frac{4}{5}} \cdot \left(\int_{\alpha\in\mathbb{T}} \left|\sum_{\mathfrak{Q}} (-\alpha)\right|^5 \mathrm{d}\alpha\right)^{\frac{1}{5}}, \end{aligned} \tag{7.28}$$

where the first inequality is the Triangle Inequality and the second one follows from Hölder's Inequality (7.25) with $p = 5/4$ and $q = 5$.

Let us first establish that

$$\left(\int_{\alpha\in\mathbb{T}}\left|\sum_{\mathfrak{Q}}(-\alpha)\right|^5 \mathrm{d}\alpha\right)^{\frac{1}{5}} \quad \ll \quad N^{\frac{t-1}{2}+\frac{3}{10}}. \tag{7.29}$$

Since $\mathfrak{Q}$ is of the form (7.7), one can write

$$\sum_{\mathfrak{Q}}(\alpha) \quad = \quad \sum_{x=1}^{\sqrt{N}} e\left(\alpha\cdot b_1x^2\right)\cdot \sum_{(y_2,\ldots,y_t)\in\left[\!\left[\sqrt{N}\right]\!\right]^{t-1}} e\left(\alpha\cdot\left(\sum_{j=2}^{t} b_jy_j^2+\sum_{2\le l<k\le t} d_{l,k}y_ly_k\right)\right).$$

This yields that

$$\left(\int_{\alpha\in\mathbb{T}}\left|\sum_{\mathfrak{Q}}(-\alpha)\right|^5 \mathrm{d}\alpha\right)^{\frac{1}{5}} \quad \le \quad N^{\frac{t-1}{2}}\cdot\left(\int_{\alpha\in\mathbb{T}}\left|\mathtt{S}\left(-b_1\alpha\right)\right|^5 \mathrm{d}\alpha\right)^{\frac{1}{5}} \quad \ll \quad N^{\frac{t-1}{2}+\frac{3}{10}},$$

where $\mathtt{S}(-b_1\alpha)$ is as in (7.24). Here, the first inequality follows from the Triangle Inequality and the second one from Lemma 7.3.3.

Secondly, we prove that

$$\left(\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^{\frac{5}{4}}\cdots\left|\hat{f}_s\left(a_s\alpha\right)\right|^{\frac{5}{4}} \mathrm{d}\alpha\right)^{\frac{4}{5}} \quad \ll \quad N^{s-\frac{6}{5}}\cdot\left\|\hat{f}_s\right\|_{\infty}^{\frac{2}{5}}. \tag{7.30}$$

To see this, notice first the following trivial bound:

$$\begin{aligned}&\left(\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^{\frac{5}{4}}\cdots\left|\hat{f}_s\left(a_s\alpha\right)\right|^{\frac{5}{4}} \mathrm{d}\alpha\right)^{\frac{4}{5}}\\ &\quad\le\quad \left(\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^{\frac{5}{4}}\cdots\left|\hat{f}_{s-1}\left(a_{s-1}\alpha\right)\right|^{\frac{5}{4}}\cdot\left|\hat{f}_s\left(a_s\alpha\right)\right|^{\frac{3}{4}} \mathrm{d}\alpha\right)^{\frac{4}{5}}\cdot\left\|\hat{f}_s\right\|_{\infty}^{\frac{2}{5}}.\end{aligned} \tag{7.31}$$

Applying Hölder's inequality (7.25) with $p = 8/5$ and $q = 8/3$ then yields

$$\begin{aligned}&\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^{\frac{5}{4}}\cdots\left|\hat{f}_{s-1}\left(a_{s-1}\alpha\right)\right|^{\frac{5}{4}}\cdot\left|\hat{f}_s\left(a_s\alpha\right)\right|^{\frac{3}{4}} \mathrm{d}\alpha\\ &\quad\le\quad \left(\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^{2}\cdots\left|\hat{f}_{s-1}\left(a_{s-1}\alpha\right)\right|^{2} \mathrm{d}\alpha\right)^{\frac{5}{8}}\cdot\left(\int_{\alpha\in\mathbb{T}}\left|\hat{f}_s\left(a_s\alpha\right)\right|^{2} \mathrm{d}\alpha\right)^{\frac{3}{8}}.\end{aligned} \tag{7.32}$$

From Parseval's Identity (1.85), one obtains that

$$\left(\int_{\alpha\in\mathbb{T}}\left|\hat{f}_s\left(a_s\alpha\right)\right|^2 \mathrm{d}\alpha\right)^{\frac{3}{8}} \quad\leq\quad N^{\frac{3}{8}}. \tag{7.33}$$

By assumption, $f_j$ is 1-bounded for every $j\in\{1,2,...,s\}$. Thus, from the definition of the Fourier transform, one obtains that for every $j\in\{1,...,s\}$, $\left|\hat{f}_j\left(a_j\alpha\right)\right|^2\leq N^2$ and, therefore,

$$\begin{aligned}\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^2\cdots\left|\hat{f}_{s-1}\left(a_{s-1}\alpha\right)\right|^2\mathrm{d}\alpha \quad&\leq\quad N^{2s-4}\cdot\int_{\alpha\in\mathbb{T}}\left|\hat{f}_{s-1}\left(a_{s-1}\alpha\right)\right|^2\mathrm{d}\alpha\\ &\underset{(1.85)}{\leq}\quad N^{2s-3}.\end{aligned} \tag{7.34}$$

Combining relations (7.32),(7.33) and (7.34), one obtains that

$$\int_{\alpha\in\mathbb{T}}\left|\hat{f}_1\left(a_1\alpha\right)\right|^{\frac{5}{4}}\cdots\left|\hat{f}_{s-1}\left(a_{s-1}\alpha\right)\right|^{\frac{5}{4}}\cdot\left|\hat{f}_s\left(a_s\alpha\right)\right|^{\frac{3}{4}}\mathrm{d}\alpha \quad\leq\quad N^{\frac{5s}{4}-\frac{3}{2}}. \tag{7.35}$$

Inequalities (7.31) and (7.35) then imply (7.30).

Finally, relations (7.28), (7.29) and (7.30) yield that

$$J \quad\leq\quad N^{s+\frac{t}{2}-1}\cdot\left(\frac{\left|\left|\hat{f}_s\right|\right|}{N}\right)^{\frac{2}{5}}.$$

The proof is complete. ■

Proof (Proposition 7.2.1) Fix a subset $A$ of $[\![N]\!]$ with density $\delta=\underset{n\in[\![N]\!]}{\mathbb{E}}\left(\chi_A(n)\right)$ and a quadratic form $\mathfrak{Q}$ as in (7.7). Assume that $A$ is $\epsilon$-Fourier uniform for some $\epsilon>0$.

From the way the operator $\mathcal{E}$ is defined (see equation (7.11)) and from the orthogonality lemma (Lemma 1.2.13, p.77), one has that

$$\mathcal{E}\left(f_1,...,f_s\right) \quad=\quad \int_{\alpha\in\mathbb{T}}\hat{f}_1\left(a_1\alpha\right)\cdots\hat{f}_s\left(a_s\alpha\right)\cdot\sum_{\mathfrak{Q}}\left(-\alpha\right)\mathrm{d}\alpha,$$

with $\sum_{\mathfrak{Q}}(\alpha)$ defined as in (7.26).

By setting $f_1 = f_2 = \cdots = f_s = \chi_A$, one has that the quantity $\mathcal{E}(\chi_A, ..., \chi_A)$ equals the number of solutions to the equation (7.5) with $x_1, ..., x_s \in A$ and $y_1, ..., y_t \in \left[\!\left[\sqrt{N}\right]\!\right]$. From the multilinearity of the operator $\mathcal{E}$ and upon applying the Generalised Triangle Inequality to the right-hand side of the equation

$$\mathcal{E}(\chi_A, ..., \chi_A) \; = \; \mathcal{E}\left(\chi_A - \delta \cdot \chi_{[\![N]\!]} + \delta \cdot \chi_{[\![N]\!]}, \quad ... \quad , \quad \chi_A - \delta \cdot \chi_{[\![N]\!]} + \delta \cdot \chi_{[\![N]\!]}\right),$$

one gets that

$$|\mathcal{E}(\chi_A, ..., \chi_A)| \quad \geq \quad \delta^s \cdot \left|\mathcal{E}\left(\chi_{[\![N]\!]}, ..., \chi_{[\![N]\!]}\right)\right| - \sum_{\text{finite sum}} \left|\mathcal{E}\left(..., \chi_A - \delta \cdot \chi_{[\![N]\!]}, ...\right)\right|, \tag{7.36}$$

where the last sum is over all those $s$-tuples of entries of the operator $\mathcal{E}$ where the function $\chi_A - \delta \cdot \chi_{[\![N]\!]}$ appears at least once.

From Proposition 7.3.1, one has that

$$\left|\mathcal{E}\left(\chi_{[\![N]\!]}, ..., \chi_{[\![N]\!]}\right)\right| \quad \gg \quad N^{s+\frac{t}{2}-1}. \tag{7.37}$$

Also, since the set $A$ is $\epsilon$-Fourier uniform, the function $f := \chi_A - \delta \cdot \chi_{[\![N]\!]}$ is $\epsilon$-Fourier uniform. Therefore, it follows from Proposition 7.3.2 that if at least one of the entries of $\mathcal{E}$ equals $f$, say without loss of generality the first one, then

$$|\mathcal{E}(f, g_2, ..., g_s)| \quad \ll \quad \epsilon^{\frac{2}{5}} \cdot N^{s+\frac{t}{2}-1}, \tag{7.38}$$

where $g_2, ..., g_s \in \{f, \chi_A\}$.

Thus, inequalities (7.36), (7.37) and (7.38) yield the existence of a small constant $0 < c' < 1$ and of a large constant $C' \geq 1$ such that

$$|\mathcal{E}(\chi_A, ..., \chi_A)| \quad \geq \quad c' \cdot \delta^s \cdot N^{s+\frac{t}{2}-1} - C' \cdot \epsilon^{\frac{2}{5}} \cdot N^{s+\frac{t}{2}-1}. \tag{7.39}$$

For a proper choice of a small constant $0 < c < 1$ and of a large constant $C_0 \geq 1$, one has that, if $\epsilon = c \cdot \delta^{C_0}$, then inequality (7.39) implies that

$$|\mathcal{E}(\chi_A, ..., \chi_A)| \quad \geq \quad \frac{c\delta^s}{2} \cdot N^{s+\frac{t}{2}-1}.$$

As for the existence of distinct solutions to the equation (7.5), notice that the number of solutions with at least two of the variables $x_1, ..., x_s \in A$ being equal and with $y_1, ..., y_t \in \left[\!\left[\sqrt{N}\right]\!\right]$ is $O\left(N^{s+\frac{t}{2}-2}\right)$. Similarly, the number of solutions with at least two of the variables $y_1, ..., y_t \in \left[\!\left[\sqrt{N}\right]\!\right]$ being equal is $O\left(N^{s+\frac{t}{2}-\frac{3}{2}}\right)$. Therefore, the total number of solutions $\Omega\left(\delta^s \cdot N^{s+\frac{t}{2}-1}\right)$ has an order larger than the number of non-distinct solutions, which is $O\left(N^{s+\frac{t}{2}-2} + N^{s+\frac{t}{2}-\frac{3}{2}}\right)$. One infers that there exists a constant $C_1 > 0$ such that if $N \geq C_1 \cdot \delta^{-2s}$, then equation (7.5) admits a solution with distinct $x_1, ..., x_s \in A$ and distinct $y_1, ..., y_t \in \sqrt{N}$.

The proof is complete. ∎

## 7.4 Proof of the Fourier Uniformity Lemma

To complete the proof of Theorem 7.1.1, it is left to establish Theorem 7.1.2. To this end, one needs the following result which provides the induction step of the density increment argument. In other words, it shows that, given a subset $A \subseteq [\![N]\!]$ with density $\delta$, if $A$ is not $\epsilon$-Fourier uniform (for a proper choice of $\epsilon$), then there exists a square-difference finite arithmetic progression $P \subseteq [\![N]\!]$ such that the density of $A \cap P$ in $P$ is increased.

**Proposition 7.4.1** *There exists an absolute constant $C_0 > 0$ such that, for every $\epsilon \in (0,1)$ and for every $N \geq C_0 \cdot \epsilon^{-C_0}$, the following holds: if a subset $A \subseteq [\![N]\!]$ is not $\epsilon$-Fourier uniform; that is, if*

$$\left|\left|\hat{\chi}_A - \mathbb{E}_{n\in[\![N]\!]}\left(\chi_A(n)\hat{\chi}_{[\![N]\!]}\right)\right|\right|_\infty \quad > \quad \epsilon \cdot N,$$

*then there exist absolute constants $0 < c < 1 \leq C$ and an arithmetic progression $P \subseteq [\![N]\!]$ with square common difference $q^2$, for some $q \in \mathbb{N}$, such that*

$$\#P \quad \geq \quad \frac{\epsilon^2}{32\,(8C+1)} \cdot N^{\frac{c}{4}}.$$

*Moreover, the density of $A$ in $P$ is estimated by the relation*

$$\mathop{\mathbb{E}}_{n\in P}\left(\chi_A(n)\right) \quad \geq \quad \mathop{\mathbb{E}}_{n\in[\![N]\!]}\left(\chi_A(n)\right) + \frac{C}{4\,(8C+1)} \cdot \epsilon.$$

The following lemma due to Heilbronn [14] is needed of Proposition 7.4.1.

**Lemma 7.4.2 (Single Recurrence for Squares)** *[14] There exist absolute constants $C, c > 0$ such that, for every $\alpha \in \mathbb{T}$ and for every $N \in \mathbb{N}$, there is $n \in [\![N]\!]$ such that*

$$\left|\left|n^2\alpha\right|\right| \quad \leq \quad \frac{C}{N^c},$$

*where $||\cdot||$ stands for the distance of a real number to the set of integers.*

**Remark 7.4.3** *Vinogradov [36] proved the following more general result: given $d \in \mathbb{N}$, there exists a small constant $0 < \eta_d < 1$ such that for every $\alpha \in \mathbb{T}$ and for every $N \in \mathbb{N}$, it holds that*

$$\min_{n\in[\![N]\!]} \left|\left|n^d\alpha\right|\right| \quad \ll_d \quad N^{-\eta_d}. \tag{7.40}$$

*Recently, Maynard [19] proved the following stronger version of Vinogradov's result:*

**Theorem 7.4.4** *[19, Theorem 1.1 and Corollary 1.2] Let $k, d$ be positive integers. There is a constant $C_d > 2$ depending only on $d$ and a constant $C_{d,k} > 2$ depending only on $d$ and $k$ such that the following holds.*

*Let $f_1, ..., f_k \in \mathbb{R}[x]$ be polynomials of degree at most $d$ such that $f_1(0) = ... = f_k(0) = 0$. Let $\epsilon_1, ..., \epsilon_k \in (0, 1/100]$, and put $\Delta = \prod_{i=1}^k \epsilon_i$. If $\Delta^{-1} \leq N^{1/C_d}$ and $N > C_{d,k}$, then there is a positive integer $n < N$ such that*

$$||f_i(n)|| \leq \epsilon_i \quad \text{for all} \quad i \in \{1, ..., k\}.$$

*In particular, there is a positive integer $n < N$ such that*

$$||f_i(n)|| \ll_{d,k} N^{-\frac{c_d}{k}} \quad \text{for all} \quad i \in \{1, ..., k\}.$$

*Here, $c_d > 0$ is a constant depending only on $d$, and the implied constant depends only on $d$ and $k$.*

*The constant $c_d$ in Theorem 7.4.4 (resp. the constant $\eta_d$ in Equation (7.40)) can be chosen as $c_d = 10^{-d}$ (resp. $\eta_d = 10^{-d}$) [19, Discussion after Corollary 1.2].*

PROOF (PROPOSITION 7.4.1) Fix $\epsilon \in (0,1)$ and $N \in \mathbb{N}$ with $N \geq C_0\epsilon^{-C_0}$, where $C_0$ is an absolute constant which will be determined at the end of the proof. Let $A$ be a subset of $[\![N]\!]$ with density $\delta = \mathbb{E}_{n\in[\![N]\!]}\left(\chi_A(n)\right)$ such that

$$\left|\left|\hat{\chi}_A - \delta \cdot \hat{\chi}_{[\![N]\!]}\right|\right|_{\infty} \quad > \quad \epsilon N.$$

In particular, $A$ is not $\epsilon$-Fourier uniform. Set

$$f \;=\; \chi_A - \delta \cdot \chi_{[\![N]\!]} \tag{7.41}$$

for the average function of $A$. The non-uniformity assumption for the set $A$ implies that there exists $\alpha \in \mathbb{T}$ such that $\left|\hat{f}(\alpha)\right| > \epsilon N$. Fix such an $\alpha \in \mathbb{T}$.

The main idea is to partition the interval $[\![N]\!]$ into subprogressions $P_i$ on which the function $x \to e\left(\alpha x\right)$ is approximately constant. To this end, let $M, Q$ be natural numbers which will be determined later. From Lemma 7.4.2, there exists $q \in [\![Q]\!]$ such that

$$\left|\left|q^2\alpha\right|\right| \quad \leq \quad \frac{C}{Q^c} \quad \text{for some absolute constants } c, C\ > 0. \tag{7.42}$$

Given $a \in \mathbb{Z}$, for each $x, y \in a + q^2\,[\![M]\!]$, it holds that

$$|e(\alpha x) - e\left(\alpha y\right)| \underset{(4.2),\text{ p.115}}{\leq} 2\pi \cdot ||\alpha(x-y)|| \underset{(7.42)}{\leq} 2\pi \cdot \frac{C}{Q^c} \cdot M. \tag{7.43}$$

Partition the interval $[\![N]\!]$ into subprogressions of the form

$$P_i \;=\; \left(a_i + q^2\,[\![M]\!]\right) \cap [\![N]\!], \tag{7.44}$$

where the set of indexes $i$ and the choice of the integers $a_i$ depend on the choice of $q, M, Q$. Here, without loss of generality, one can assume that for every subprogression $P_i$ in (7.44), it holds that $\#P_i = M$. Otherwise, one may allow some subprogressions to contain at most $2M$ terms. In this case, the following argument still works if one replaces the constant $C$ by $2C$. For each such subprogression, $P_i$

it holds that

$$\begin{aligned}
\left|\sum_{x\in P_i} f(x)\cdot e(\alpha x)\right| &\leq \left|\sum_{x\in P_i} f(x)e(\alpha y)\right| + \left|\sum_{x\in P_i} f(x)\left(e(\alpha x)-e(\alpha y)\right)\right| \\
&\leq \left|\sum_{x\in P_i} f(x)\right| + \left|\sum_{x\in P_i} f(x)\left(e(\alpha x)-e(\alpha y)\right)\right| \\
&\underset{\substack{(7.43),\\ \left(\|f\|_\infty\leq 1\right)}}{\leq} \left|\sum_{x\in P_i} f(x)\right| + 2\pi\cdot\frac{C}{Q^c}\cdot M\cdot \#P_i.
\end{aligned} \tag{7.45}$$

In turn, inequality (7.45) implies that

$$\epsilon N \leq \sum_i \left|\sum_{x\in P_i} f(x)\right| + 7\cdot\frac{C}{Q^c}\cdot MN, \tag{7.46}$$

where the first sum is over all those $i$ indexing the partition of $[\![N]\!]$ into subprogressions of the form (7.44). Inequality (7.46) is indeed proved as follows:

$$\begin{aligned}
\epsilon N &\leq \left|\hat{f}(\alpha)\right| = \left|\sum_{x\in[\![N]\!]} f(x)e(\alpha x)\right| \\
&\underset{(7.45)}{\leq} \sum_i \left(\left|\sum_{x\in P_i} f(x)\right| + 2\pi\cdot\frac{C}{Q^c}M\cdot\#P_i\right) \\
&= \sum_i \left|\sum_{x\in P_i} f(x)\right| + 2\pi\cdot\frac{C}{Q^c}M\cdot\sum_i \#P_i \\
&= \sum_i \left|\sum_{x\in P_i} f(x)\right| + 7\cdot\frac{C}{Q^c}\cdot MN,
\end{aligned}$$

where the last equality follows from the fact that $\sum_i \#P_i = N$.

Furthermore, one can remove the absolute values from inequality (7.46). Indeed, since $\sum_{x\in[\![N]\!]} f(x) = 0$, it holds that

$$\sum_i \left|\sum_{x\in P_i} f(x)\right| = \sum_i \max\left\{0, 2\sum_{x\in P_i} f(x)\right\}.$$

Therefore, inequality (7.44) becomes

$$\epsilon N \quad \leq \quad \sum_i \max\left\{0, 2\sum_{x\in P_i} f(x)\right\} + 7\frac{C}{Q^c}\cdot MN. \tag{7.47}$$

A trivial counting argument yields that the number of progressions of the form (7.44) which partition the interval $[\![N]\!]$ is at most $\frac{N}{M}+Q^2+1$, that is, $i \leq \frac{N}{M}+Q^2+1$. Set $j$ to be such that

$$\sum_{x\in P_j} f(x) \quad = \quad \max_i \left(\sum_{x\in P_i} f(x)\right).$$

Then, inequality (7.47) becomes

$$\epsilon N \quad \leq \quad 7\frac{C}{Q^c}\cdot MN + 2\left(\frac{N}{M}+Q^2+1\right)\cdot \sum_{x\in P_j} f(x). \tag{7.48}$$

Set

$$M = \left\lfloor \frac{\epsilon}{8C}\cdot N^{\frac{c}{4}}\right\rfloor \quad \text{and} \quad Q = N^{\frac{1}{4}}. \tag{7.49}$$

Substituting equations (7.49) in inequality (7.48) yields that

$$\begin{aligned}
\epsilon N \quad &\leq \quad 7C\frac{\epsilon N^{\frac{c}{4}}}{8CN^{\frac{c}{4}}}\cdot N + 2\left(\frac{N}{\left\lfloor \epsilon N^{\frac{c}{4}}/8C\right\rfloor} + N^{\frac{1}{2}} + 1\right)\cdot \sum_{x\in P_j} f(x) \\
&\leq \quad \frac{7\epsilon}{8}\cdot N + 4\left(\frac{8CN^{1-\frac{c}{4}}}{\epsilon} + N^{\frac{1}{2}}\right)\cdot \sum_{x\in P_j} f(x) \\
&\underset{(c<1)}{\leq} \quad \frac{7\epsilon}{8}\cdot N + \frac{4}{\epsilon}\left(8CN^{1-\frac{c}{4}} + N^{1-\frac{c}{4}}\right)\cdot \sum_{x\in P_j} f(x),
\end{aligned} \tag{7.50}$$

where the second inequality is satisfied if one chooses the constant $C_0 \geq 1$ large enough in the lower bound $C_0\epsilon^{-C_0}$ of $N$. In turn, since $\epsilon \in (0,1)$ one obtains that

$$\frac{\epsilon^2 N^{\frac{c}{4}}}{32\,(8C+1)} \quad \leq \quad \sum_{x\in P_j} f(x)$$

By taking further into account relation (7.49), one infers that

$$\left(\frac{C}{4\,(8C+1)}\right)\cdot \epsilon M \quad \leq \quad \sum_{x\in P_j} f(x). \tag{7.51}$$

From the definition of the function $f$ in (7.41), one has that

$$\sum_{x\in P_j} f(x) \quad = \quad \#\,(A\cap P_j) - \frac{\#A}{N}\cdot \#P_j. \tag{7.52}$$

From (7.51) and (7.52), it holds that

$$\frac{\#\,(A\cap P_j)}{\#P_j} \quad \geq \quad \frac{\#A}{N} + \frac{\epsilon M}{\#P_j}\cdot\left(\frac{C}{4\,(8C+1)}\right) \quad \underset{(M\geq \#P_j)}{\geq} \quad \delta + \left(\frac{C}{4\,(8C+1)}\right)\cdot \epsilon.$$

As for the length of the progression $P_j$, one has that

$$\#P_j \;\geq\; \left(\frac{C}{4\,(8C+1)}\right)\cdot \epsilon M \underset{(7.49)}{\geq} \frac{\epsilon^2}{32\,(8C+1)}\cdot N^{\frac{c}{4}}. \tag{7.53}$$

This follows immediately from inequality (7.51) upon noticing that $\#P_j \geq \sum_{x\in P_j} f(x)$ because $f$ is 1-bounded.

Finally, it remains to define the absolute constant $C_0$ in the statement, which is the constant in the lower bound $C_0\epsilon^{-C_0}$ for $N$. It is determined by asking that two conditions should hold. Firstly, as already stated, it must be large enough for the second inequality in (7.50) to hold. Secondly, it is required that the length of the progression $P_j$ should be greater or equal than one. To this end, by substituting equation (7.49) in inequality (7.53), it is enough to require that

$$\left(\frac{C}{32C\,(8C+1)}\right)\cdot \epsilon^2 \cdot N^{\frac{c}{4}} \quad \geq \quad 1.$$

Since $N \geq C_0\epsilon^{-C_0}$ it is enough to take $C_0$ large enough so that

$$\left(\frac{C}{32C\,(8C+1)}\right)\epsilon^2 \cdot C_0^{c/4}\cdot \epsilon^{\frac{-cC_0}{4}} \geq 1.$$

The proof is complete. ■

PROOF (THEOREM 7.1.2) The main idea is to repeatedly apply Proposition 7.4.1; that is, to apply the same proposition on arithmetic progressions of the form $P' = a+q^2 [\![N']\!]$ for some $a, q, N' \in \mathbb{N}$. This is done by identifying the arithmetic progression $P'$ with the interval $[\![N']\!]$ through the map $P' \ni a + q^2 n \mapsto n \in [\![N']\!]$. Recall that proving the existence of a solution of the equation (7.5) with $x'_1, ..., x'_s \in [\![N']\!]$ implies the existence of a solution of (7.5) with $x_1, ..., x_s \in P'$ and $x_j = a + q^2 x'_j$, $j \in \{1, ..., s\}$ (see relation (7.6) and the discussion below it).

Clearly, no matter the number of iterations, the arithmetic progression given from the application of Proposition 7.4.1 will have a square common difference.

It is enough to show that, for some $m \in \mathbb{N}$, the arithmetic progression $P_m$ obtained from the $m$-th iteration enjoys the properties of the statement. To this end, let $N$ be sufficiently large. It is left to the end of the proof to determine how large $N \in \mathbb{N}$ has to be. Assume that $P_1 \supset P_2 \supset ... \supset P_{m'}$ are the arithmetic progressions obtained from the first $m'$ iterations of Proposition 7.4.1. It follows from the conclusion of the same proposition that for any $j \in \{1, 2, ..., m' - 1\}$,

$$\mathop{\mathbb{E}}_{n \in P_j} (\chi_A(n)) + \frac{C'}{4\,(8C' + 1)} \cdot \epsilon \quad \leq \quad \mathop{\mathbb{E}}_{n \in P_{j+1}} (\chi_A(n))\,, \tag{7.54}$$

where the constants $0 \ < c' \ < 1 \leq C'$ are given by Proposition 7.4.1. Since the density of a set cannot be larger than 1, equation (7.54) implies that

$$m' \quad \leq \quad \frac{32C' + 4}{C'\epsilon}\cdot$$

Moreover, since the maximum number of times one can apply Proposition 7.4.1 is bounded by $(32C' + 4)\,/C'\epsilon$, Equation (7.54) implies that there exists

$$m \in [\![(32C' + 4)\,/C'\epsilon]\!] \tag{7.55}$$

such that

$$\left\|\hat{\chi}_{A\cap P_m} - \mathop{\mathbb{E}}_{n \in P_m} (\chi_A(n)\hat{\chi}_{P_m})\right\|_\infty \quad \leq \quad \epsilon \cdot \#P_m.$$

Finally, Proposition 7.4.1 implies that for any $j \in \{1, 2, ..., m-1\}$,

$$\#P_{j+1} \quad \geq \quad \frac{\epsilon^2}{32\,(8C'+1)} \cdot (\#P_j)^{\frac{c'}{4}}. \tag{7.56}$$

Iterating inequality (7.56) yields that

$$\#P_m \quad \geq \quad \left(\frac{\epsilon^2}{32\,(8C'+1)}\right)^{\sum_{j=1}^{m}\left(\frac{c'}{4}\right)^{j-1}} \cdot N^{\left(\frac{c'}{4}\right)^m}.$$

In turn, since $C' \geq 1$, it holds that

$$\frac{\epsilon^2}{32\,(8C'+1)} < 1$$

and thus one has that

$$\begin{aligned}\left(\frac{\epsilon^2}{32\,(8C'+1)}\right)^{\sum_{j=1}^{m}\left(\frac{c'}{4}\right)^{j-1}} \cdot N^{\left(\frac{c'}{4}\right)^m} &\underset{(7.55)}{\geq} \left(\frac{\epsilon^2}{32\,(8C'+1)}\right)^{\sum_{j=1}^{+\infty}\left(\frac{c'}{4}\right)^{j-1}} \cdot N^{\left(\frac{c'}{4}\right)^{\frac{32C'+4}{C'\epsilon}}} \\ &\underset{(N\geq 2)}{\geq} N^{\exp\left(-\frac{C_1}{\epsilon}\right)}\end{aligned} \tag{7.57}$$

for a large absolute constant $C_1 \geq 1$. The existence of the constant $C_1$ follows upon comparing the growths of $N^{\exp(-C_1/\epsilon)}$, $N^{(c'/4)^{(32C'+4)/C'\epsilon}}$ and $B \cdot \epsilon^{2A}$, where $A = \sum_{j=1}^{+\infty} (c'/4)^{j-1}$ and $B = (32(8C'+1))^{-A}$. In particular, inequality (7.57) holds for every $C_1 \geq C_2 + C_3$, where $C_2$ satisfies $(c'/4)^{(32C'+4)/C'} \geq e^{-C_2}$ and $C_3$ is such that $B \cdot \epsilon^{2A} \geq 2^{-C_3/\epsilon}$ for every $\epsilon \in (0, 1)$. Finally, one infers from inequality (7.57) that

$$\#P_m \quad \geq \quad N^{\exp\left(-\frac{C_1}{\epsilon}\right)}. \tag{7.58}$$

As for the size of $N$, it has to be large enough for Proposition 7.4.1 to be applied $m$ times. To this end, it is sufficient to ask that for every $j \in \{1, 2, ..., m-1\}$,

$$\#P_j \quad \geq \quad C_0\epsilon^{-C_0},$$

where the constant $C_0 \geq 1$ is given by the statement of Proposition 7.4.1. For this condition to be satisfied, it is enough to ask that the last-obtained arithmetic progression $P_m$ contains at least $C_0\epsilon^{-C_0}$ terms. Thus, it is enough to ask $\#P_m \geq C_0\epsilon^{-C_0}$. It follows from inequality (7.58) that this occurs if $N^{\exp(-C_1/\epsilon)} \geq C_0\epsilon^{-C_0}$. It is readily checked that this holds true if one chooses $N \geq \exp(\exp(C/\epsilon))$ for a sufficiently large absolute constant $C > C_1$. The proof is complete. ■

## 7.5 Proof of the Lemma 7.3.3

Bourgain's lemma 7.3.3 is proved in this section.

Proof (Lemma 7.3.3) Given a natural number $N \in \mathbb{N}$, set $\mathtt{S} : \mathbb{T} \mapsto \mathbb{C}$ with

$$\mathtt{S}(\alpha) := \sum_{n \in [\![N]\!]} e\left(n^2\alpha\right).$$

The goal is to prove that

$$\int_{\alpha\in\mathbb{T}} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha \quad \ll \quad N^3. \tag{7.59}$$

The proof proceeds by splitting the integral into a minor and a major arc

$$\int_{\alpha\in\mathbb{T}} |\mathtt{S}(a)|^5 \, \mathrm{d}\alpha \quad = \quad \int_{|\mathtt{S}(\alpha)|\leq\delta N} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha + \int_{|\mathtt{S}(\alpha)| > \delta N} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha, \tag{7.60}$$

and by proving that each arc has size $O(N^3)$. Here, $\delta \in (0,1)$ is a parameter which will be chosen in an efficient way.

**Proof of the minor arc estimate:** The first goal is to prove that

$$\int_{|\mathtt{S}(\alpha)|\leq\delta N} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha \quad \ll \quad N^{3-\eta} \tag{7.61}$$

with $\delta = N^{-c}$ for some small constant $c > 0$ and for some $\eta \in (0,1)$.

For this purpose one needs the following Hua-type lemma. Hua's lemma [15], is an estimate for exponential sums. It states that if $P$ is an **integral-valued**

**polynomial** (i.e. such that for every $n \in \mathbb{Z}, P(n) \in \mathbb{Z}$) of degree $k$, if $\epsilon$ is a positive real number, and if $f$ is the function defined by

$$f(x) \quad = \quad \sum_{k=1}^{N} e^{2\pi i \cdot P(k) \cdot x} \quad \text{for some } N \in \mathbb{N},$$

then

$$\int_0^1 |f(x)|^\lambda \, \mathrm{d}x \quad \ll_{P,\epsilon} \quad N^{\mu(\lambda)},$$

where the point $(\lambda, \mu(\lambda)) \in \mathbb{R}^2$ lies in the polygonal line with vertices $\{(2^n, 2^n - n + \epsilon)\}_{n=1}^k$.

**Lemma 7.5.1 (Hua-type Lemma)** *For any* $\epsilon > 0$*,*

$$\int_{\alpha \in \mathbb{T}} |\mathtt{S}(\alpha)|^4 \, d\alpha \quad \ll_\epsilon \quad N^{2+\epsilon}.$$

Proof (Lemma 7.5.1) Let $\mathtt{d} : \mathbb{N} \mapsto \mathbb{N}$ be the divisor function

$$\mathtt{d}(n) \ = \ \# \{d \in \mathbb{N} : d | n\}.$$

Observe that if $x^2 - y^2 = n$ for some $x, y \in \mathbb{N}$, then

$$(x+y) \, | n \quad \text{and} \quad x - y = \frac{n}{x+y},$$

hence, there are at most $\mathtt{d}(|n|)$ pairs of positive integers $(x, y)$ such that $x^2 - y^2 = n$. From the orthogonality lemma (Lemma 1.2.13, p.77) and the bound $\mathtt{d}(n) \ll_\epsilon n^\epsilon$ [34, Equation 3], one obtains

$$\begin{aligned}
\int_{\alpha \in \mathbb{T}} |\mathtt{S}(\alpha)|^4 \, \mathrm{d}\alpha \quad &= \quad \# \left\{ \{x_1, y_1, x_2, y_2\} \in [\![N]\!]^4 : x_1^2 - y_1^2 = x_2^2 - y_2^2 \right\} \\
&\leq \quad N^2 + 2 \sum_{1 \leq n < N^2} \mathtt{d}(n)^2 \quad \ll_\epsilon \quad N^2 + 2 \sum_{1 \leq n < N^2} n^\epsilon \\
&\ll_\epsilon \quad N^{2+\epsilon}.
\end{aligned}$$

The proof of the lemma is complete. ∎

Inequality (7.61) follows upon choosing

$$\delta \quad = \quad N^{-c}$$

and applying Lemma 7.5.1 with $\epsilon = c/2$:

$$\int_{|\mathtt{S}(\alpha)|\leq\delta N} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha \quad \leq \quad \delta N \cdot \int_{\alpha\in\mathbb{T}} |\mathtt{S}(\alpha)|^4 \, \mathrm{d}\alpha \quad \ll \quad N^{3-\frac{c}{2}}.$$

**Proof of the major arc estimate:** The second goal is to prove that

$$\int_{|\mathtt{S}(\alpha)|\ >\delta N} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha \quad \ll \quad N^3 \tag{7.62}$$

for some small constant $c\ > 0$. To this end, notice that

$$\int_{|\mathtt{S}(\alpha)|\ >\delta N} |\mathtt{S}(\alpha)|^5 \, \mathrm{d}\alpha \quad \leq \quad \sum_{\substack{1\leq p\leq q\leq C\cdot\delta^{-C},\\ \gcd(p,q)=1}} \int_{-\frac{C\delta^{-C}}{N^2}}^{\frac{C\delta^{-C}}{N^2}} \left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right|^5 \mathrm{d}\beta. \tag{7.63}$$

Inequality (7.63) follows from the inclusion

$$\{\alpha\in\mathbb{T}: \quad |\mathtt{S}(\alpha)|\ >\delta N\} \quad \subseteq \quad \bigcup_{\substack{1\leq p\leq q\leq C\delta^{-C},\\ \gcd(p,q)=1}} \left\{\alpha\in\mathbb{T}: \quad \left\|\alpha-\frac{p}{q}\right\| \quad \leq \quad \frac{C\delta^{-C}}{N^2}\right\}$$

which, in turn, is implied by the following result due to Green and Tao [12].

**Lemma 7.5.2** *[12] There exists a constant $C\geq 1$ satisfying the following property. If for some $\alpha,\beta\in\mathbb{T}$ and $N\in[\![N]\!]$, it holds that*

$$\left|\mathop{\mathbb{E}}_{n\in[\![N]\!]} e\left(\alpha n^2+\beta n\right)\right| \quad \geq \quad \delta, \tag{7.64}$$

*then there exists a natural number $q\leq C\cdot\epsilon^{-C}$ such that*

$$||q\alpha|| \quad \leq \quad \frac{C\delta^{-C}}{N^2}.$$

In view of inequality (7.63), the problem has been reduced to estimating the quantity $S\left((p/q)+\beta\right)$ with $\gcd(p,q)=1$ and $\beta\in\mathbb{T}$. Such an estimate is given by the following lemma.

**Lemma 7.5.3 (Local Approximation on the Major Arcs)** *For any $\beta \in \mathbb{T}$ and any integers $p, q, N \in \mathbb{N}$ such that* $\gcd(p,q) = 1$,

$$\left| \mathtt{S}\left(\frac{p}{q} + \beta\right) \right| \quad = \quad |s_q(p) I(\beta)| + O\left(q + |\beta| q N^2 + |\beta| q^2 N\right),$$

*where*

$$s_q(p) := \quad \mathop{\mathbb{E}}_{n \in [\![q]\!]} e_q\left(pn^2\right), \quad \textit{with} \quad e_q(x) := \quad e^{\frac{2\pi i x}{q}} \tag{7.65}$$

*and*

$$I(\beta) := \quad \int_0^N e\left(\beta t^2\right) dt. \tag{7.66}$$

Proof For any two integers $p, q$ such that $\gcd(p,q) = 1$ and any $\beta \in \mathbb{T}$, one has

$$\mathtt{S}\left(\frac{p}{q} + \beta\right) \quad = \quad \sum_{n \in [\![q]\!]} \left( e_q\left(p \cdot n^2\right) \cdot \sum_{-\frac{n}{q} < r \leq \frac{N-n}{q}} e\left(\beta \cdot (n + qr)^2\right) \right). \tag{7.67}$$

This follows from straightforward computations:

$$\begin{aligned}
\mathtt{S}\left(\frac{p}{q} + \beta\right) \quad &= \quad \sum_{r \in [\![N]\!]} e\left(\left(\frac{p}{q} + \beta\right) \cdot r^2\right) \quad = \quad \sum_{n \in [\![q]\!]} \sum_{r=0}^{\frac{N-n}{q}} e\left(\left(\frac{p}{q} + \beta\right) \cdot (n + qr)^2\right) \\
&= \quad \sum_{n \in [\![q]\!]} \sum_{r=0}^{\frac{N-n}{q}} e\left(\frac{p}{q} \cdot n^2\right) \cdot e\left(\beta \cdot (n + qr)^2\right) \\
&= \quad \sum_{n \in [\![q]\!]} \left( e_q\left(p \cdot n^2\right) \cdot \sum_{-\frac{n}{q} < r \leq \frac{N-n}{q}} e\left(\beta \cdot (n + qr)^2\right) \right).
\end{aligned}$$

The main idea is to approximate the inner sum on the right-hand side of relation (7.67) by the integral $I(\beta)$ defined in (7.66). Specifically, for every $n \in [\![q]\!]$, it holds that

$$\left| \sum_{-\frac{n}{q} < r \leq \frac{N-n}{q}} e\left(\beta \cdot (n + qr)^2\right) \right| \quad = \quad \left| \frac{I(\beta)}{q} \right| + O\left(|\beta| N^2 + |\beta| \cdot qN + 1\right). \tag{7.68}$$

To see this, fix $n \in [\![q]\!]$ and set

$$\begin{aligned} J := & \left| \sum_{-\frac{n}{q} < r \le \frac{N-n}{q}} e\left(\beta \cdot (n+qr)^2\right) - \frac{I(\beta)}{q} \right| \\ = & \left| \sum_{-\frac{n}{q} < r \le \frac{N-n}{q}} e\left(\beta \cdot (n+qr)^2\right) - \frac{\int_0^N e\left(\beta x^2\right) \mathrm{d}x}{q} \right|. \end{aligned}$$

Recall that for every $x, y \in \mathbb{T}$ it holds that

$$|e(x) - e(y)| \le 2\pi \cdot ||x - y|| . \tag{7.69}$$

By making the change of variables $x = n + qt$, one has

$$\begin{aligned} J = & \left| \sum_{-\frac{n}{q} < r \le \frac{N-n}{q}} e\left(\beta \cdot (n+qr)^2\right) - \int_{-\frac{n}{q}}^{\frac{N-n}{q}} e\left(\beta \cdot (n+qt)^2\right) \mathrm{d}t \right| \\ \underset{\substack{\text{Triangle} \\ \text{Inequality}}}{\le} & \left| \sum_{r=1}^{(N-n)/q} \int_{r-1}^{r} \left(e\left(\beta \cdot (n+qr)^2\right) - e\left(\beta \cdot (n+qt)^2\right)\right) \mathrm{d}t \right| \\ & + \left| \sum_{-n/q < r \le 0} e\left(\beta\,(n+qr)^2\right) \right| + \left| \int_{-\frac{n}{q}}^{0} e\left(\beta \cdot (n+qt)^2\right) \mathrm{d}t \right| \\ & + \left| \int_{\left\lfloor \frac{N-n}{q} \right\rfloor}^{\frac{N-n}{q}} e\left(\beta \cdot (n+qt)^2\right) \mathrm{d}t \right| \\ \underset{(n \le q)}{\le} & \sum_{r=1}^{(N-n)/q} \int_{r-1}^{r} \left| e\left(\beta \cdot (n+qr)^2\right) - e\left(\beta \cdot (n+qt)^2\right) \right| \mathrm{d}t + 3 \\ \underset{(7.69)}{\le} & 2\pi|\beta| \cdot \sum_{r=1}^{(N-n)/q} \int_{r-1}^{r} \left| q^2\left(r^2 - t^2\right) + 2qn\,(r-t) \right| \mathrm{d}t + 3 \\ \le & 2\pi|\beta| \cdot \sum_{r=1}^{(N-n)/q} \left(q^2 \cdot \frac{3r-1}{3} + qn\right) + 3 \ll |\beta| \cdot \sum_{r=1}^{(N-n)/q} \left(q^2 r + qn\right) + 3 \\ \ll & |\beta| \cdot N^2 + |\beta| \cdot nN + 3 \underset{(n \le q)}{\ll} |\beta| \cdot N^2 + |\beta| \cdot qN + 1. \end{aligned}$$

From relations (7.67) and (7.68), one obtains

$$
\begin{aligned}
\left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right| &= \left|\sum_{n\in[\![q]\!]} e_q\left(p\cdot n^2\right)\cdot\left(\left|\frac{I(\beta)}{q}\right| + O\left(|\beta|\cdot N^2+|\beta|\cdot qN+1\right)\right)\right| \\
&= \left|s_q(p)\cdot I(\beta)+\sum_{n\in[\![q]\!]} e_q\left(p\cdot n^2\right)\cdot O\left(|\beta|\cdot N^2+|\beta|\cdot qN+1\right)\right|
\end{aligned}
$$

which implies that

$$
\left|\left|s_q(p)\cdot I(\beta)\right| - \left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right|\right| \quad \ll \quad |\beta|\cdot qN^2+|\beta|\cdot q^2N+q.
$$

The proof of Lemma (7.5.3) is complete. ■

From Lemma 7.5.3, to estimate the quantity $\mathtt{S}\left((p/q)+\beta\right)$, it suffices to estimate the integral $I(\beta)$ defined in (7.66).

**Lemma 7.5.4** *For any $\beta\in\mathbb{R}$ and $N\in\mathbb{N}$ the following two statements hold:*

1. *If $|\beta|\le N^{-2}$, then*
$$|I(\beta)| \quad \le \quad N.$$

2. *If $|\beta| > N^{-2}$, then*
$$|I(\beta)| \quad \ll \quad |\beta|^{-\frac{1}{2}}.$$

*Moreover,*

$$\int_{\mathbb{R}} |I(\beta)|^5\, d\beta \quad \ll \quad N^3. \tag{7.70}$$

Proof Fix $\beta\in\mathbb{R}$ and $N\in\mathbb{N}$.

**Proof of Part 1:** By applying the Triangle Inequality, one obtains

$$
|I(\beta)| \quad = \quad \left|\int_0^N e\left(\beta t^2\right)\mathrm{d}t\right| \quad \le \quad \int_0^N 1\mathrm{d}t \quad = \quad N.
$$

**Proof of Part 2:** Assume that $|\beta| > N^2$. Without loss of generality, one can assume that $\beta > N^2$ since $|I(\beta)| = \left|\overline{I(-\beta)}\right|$. Making the change of variables

$t = \beta^{-\frac{1}{2}}x$ in the integral (7.66),

$$|I(\beta)| \quad = \quad |\beta|^{-\frac{1}{2}} \cdot \left| \int_0^{\beta^{\frac{1}{2}}N} e\left(x^2\right) \mathrm{d}x \right|.$$

The goal is to show that for any $T \geq 1$, it holds that

$$\left| \int_0^T e\left(t^2\right) \mathrm{d}t \right| \quad \ll \quad 1. \tag{7.71}$$

Notice that

$$\left| \int_0^T e\left(t^2\right) \mathrm{d}t \right| \quad \leq \quad \left| \int_0^1 e\left(t^2\right) \mathrm{d}t \right| + \left| \int_1^T e\left(t^2\right) \mathrm{d}t \right| \quad \leq \quad 1 + \left| \int_1^T e\left(t^2\right) \mathrm{d}t \right|.$$

Therefore, to prove inequality (7.71), it is enough to prove that

$$\left| \int_1^T e\left(t^2\right) \mathrm{d}t \right| \quad \ll \quad 1. \tag{7.72}$$

To this end, make the change of variables $t = \sqrt{u}$:

$$\begin{aligned}
\left| \int_1^T e\left(t^2\right) \mathrm{d}t \right| \quad &= \quad \frac{1}{2} \cdot \left| \int_1^{T^2} \frac{e(u)}{u^{\frac{1}{2}}} \mathrm{d}u \right| \\
&= \quad \left| \frac{1}{4\pi i} \cdot \left[ \frac{e(u)}{\sqrt{u}} \right]_{u=1}^{u=T^2} + \frac{1}{8\pi i} \cdot \int_1^{T^2} \frac{e(u)}{u^{\frac{3}{2}}} \mathrm{d}u \right| \\
&\ll \quad 1 + \int_1^{T^2} \left| \frac{e(u)}{u^{\frac{3}{2}}} \right| \mathrm{d}u \\
&\ll \quad 1 + \int_1^{T^2} \frac{1}{u^{\frac{3}{2}}} \mathrm{d}u \quad \ll \quad 1,
\end{aligned}$$

where in the second equality we integrate by parts and in the first inequality we make use of the Triangle Inequality.

Finally, it remains to prove inequality (7.70):

$$\begin{aligned}\int_{\mathbb{R}} |I(\beta)|^5 \,\mathrm{d}\beta \quad &= \quad \int_{|\beta|\leq N^{-2}} |I(\beta)|^5 \,\mathrm{d}\beta + \int_{|\beta|\, > N^{-2}} |I(\beta)|^5 \,\mathrm{d}\beta \\ &\underset{\text{Parts 1\&2}}{\ll} \quad 2N^{-2}\cdot N^5 + \int_{|\beta|\, > N^{-2}} |\beta|^{-\frac{5}{2}} \mathrm{d}\beta \quad \ll \quad N^3.\end{aligned}$$

The proof of the Lemma 7.5.4 is complete. ■

As for the quantity $s_q(p)$, it holds that for any $p, q \in \mathbb{N}$ with $\gcd(p,q) = 1$,

$$|s_q(p)| \;\ll\; q^{-\frac{1}{2}}. \tag{7.73}$$

This is proved as follows:

$$\begin{aligned}|s_q(p)|^2 \underset{(7.65)}{=} \left|\mathop{\mathbb{E}}_{n\in[\![q]\!]} e_q\left(p\cdot n^2\right)\right|^2 \;&=\; \frac{1}{q^2}\cdot \sum_{n\in[\![q]\!]} e_q\left(p\cdot n^2\right)\cdot \sum_{m\in[\![q]\!]} e_q\left(-p\cdot m^2\right) \\ &\leq \frac{1}{q^2}\cdot \sum_{m,n\in[\![q]\!]} e_q\left(p\cdot\left(n^2-m^2\right)\right) \\ &\underset{(n=m+r)}{=} \frac{1}{q^2}\cdot \sum_{m\in[\![q]\!]}\sum_{r\in[\![q]\!]} e_q\left(p\cdot\left(2mr+r^2\right)\right) \;\leq\; \frac{1}{q^2}\cdot\sum_{m\in[\![q]\!]}\left|\sum_{r\in[\![q]\!]} e_q\left(p\cdot 2mr\right)\right| \\ &= \frac{1}{q^2}\cdot\sum_{m\in[\![q]\!]} \left|e_q\left(p\cdot 2mq\right)\right| + \frac{1}{q^2}\cdot\left|\sum_{r\in[\![q-1]\!]} e_q\left(p\cdot 2qr\right)\right| \\ &\qquad + \frac{1}{q^2}\cdot \sum_{m\in[\![q-1]\!]}\left|\sum_{r\in[\![q-1]\!]} e_q\left(p\cdot 2mr\right)\right| \\ &= \frac{1}{q^2}\cdot\left(q+(q-1)+\sum_{m\in[\![q]\!]}\left|\frac{e_q\left(p\cdot 2mq\right)-1}{e_q\left(2pm\right)-1}\right|\right) \;=\; \frac{2q-1}{q^2} \;\leq\; \frac{2}{q}.\end{aligned}$$

Moreover,

$$\begin{aligned}\sum_{\substack{1\leq p\leq q\,<+\infty,\\ \gcd(p,q)=1}} |s_q(p)|^5 \quad &\underset{(7.73)}{\ll} \quad \sum_{\substack{1\leq p\leq q\,<+\infty,\\ \gcd(p,q)=1}} q^{-\frac{5}{2}} \quad = \quad \sum_{q\in\mathbb{N}}\sum_{\substack{1\leq p\leq q,\\ \gcd(p,q)=1}} q^{-\frac{5}{2}} \\ &\leq \quad \sum_{q\in\mathbb{N}} q\cdot q^{-\frac{5}{2}} \quad = \quad \sum_{q\in\mathbb{N}} q^{-\frac{3}{2}} \quad \ll \quad 1.\end{aligned} \tag{7.74}$$

Set
$$\delta = N^{-c} \quad \text{for some sufficiently small constant } c \in (0,1).$$

Given $\beta \in \mathbb{T}$ with $||\beta|| \leq \left(C\delta^{-C}/N^2\right)$ and $q \in \left[\left[C\delta^{-C}\right]\right]$, where the $C \geq 1$ is a constant sufficiently large, from the statement of the Lemma 7.5.3, one obtains that
$$\left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right| \quad = \quad \left|s_q\left(p\right) I(\beta)\right| + O\left(N^{\kappa}\right) \tag{7.75}$$

where $\kappa = \kappa(c)$ is a positive constant sufficiently small provided that the constant $c$ is sufficiently small. Moreover, from Lemma 7.5.4 and inequality (7.73),
$$\left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right| \quad \ll \quad \frac{N}{q^{\frac{1}{2}}} \quad \ll \quad N. \tag{7.76}$$

In turn, inequalities (7.75) and (7.76) yield
$$\left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right|^5 \quad \ll \quad \left|s_q(p)I(\beta)\right|^5 + N^{5\kappa} \quad \ll \quad \left|s_q(p)I(\beta)\right|^5 + N^{\frac{9}{2}} \tag{7.77}$$

where the last inequality holds since the constant $\kappa \ > 0$ has be chosen small enough.

One now deduces inequality (7.61) as follows:
$$\begin{aligned}
\int_{|\mathtt{S}(\alpha)|\ >\delta N} |\mathtt{S}(\alpha)|\,\mathrm{d}\alpha \quad &\underset{(7.63)}{\leq} \quad \sum_{\substack{1\leq p\leq q\leq C\cdot\delta^{-C},\\ \gcd(p,q)=1}} \int_{-\frac{C\delta^{-C}}{N^2}}^{-\frac{C\delta^{-C}}{N^2}} \left|\mathtt{S}\left(\frac{p}{q}+\beta\right)\right|^5 \mathrm{d}\beta \\
&\underset{(7.77)}{\ll} \quad \sum_{\substack{1\leq p\leq q\leq C\cdot\delta^{-C},\\ \gcd(p,q)=1}} \int_{-\frac{C\delta^{-C}}{N^2}}^{-\frac{C\delta^{-C}}{N^2}} \left(\left|s_q(p)I(\beta)\right|^5 + N^{\frac{9}{2}}\right) \mathrm{d}\beta \\
&\underset{(7.70),(7.74)}{\ll} \quad N^3 + N^{\frac{5}{2}+\eta} \quad \ll \quad N^3,
\end{aligned}$$

where in the last inequality $\eta = \eta(c)$ is a small positive constant.

Fixing a sufficiently small constant $c \ > 0$ and substituting inequalities (7.61) and (7.62) in relation (7.60) (with $\delta = N^{-c}$) yields inequality (7.59). The proof is complete. ■

# Appendix A

# Super-Uniform Discrepancy

## A.1 Introduction

In Section 1.1.6 of Chapter 1, the notions of dispersion (Definition 1.1.19, p.34) and discrepancy (Definition 1.1.20, p.34) were introduced. Dispersion is a measure of density of the terms of a given sequence and is of a metric nature. In contrast, discrepancy is a measure of the uniform distribution of a given sequence and it is of a measure-theoretic nature. In Chapter 3, it was seen that one can use the concept of super-uniform dispersion (Definition 3.1.1, p.95) to construct planar Peres-type forests (Definition 1.1.10, p.26) with visibility close to the optimal $O(\epsilon^{-1})$. More precisely, from [Theorem 3.1.3, p. 96, Chapter 3], one obtains the existence a Peres-type forest with (almost optimal) visibility $O_\eta(\epsilon^{-1-\eta})$ for any $\eta > 0$ and from [Theorem 3.1.4, p.97, Chapter 3] one obtains a deterministic Peres-type forest with very good visibility bounds, i.e. $O_\eta(\epsilon^{-2-\eta})$. However, the estimation of the dispersion of a given sequence is not an easy task as there is not sufficient "machinery" for this purpose. In contrast, there are many analytic methods for the estimation of the discrepancy of a sequence (see Theorems 1.1.21, 1.1.22 and 1.1.23, p.36). In view of the analysis undertaken in Chapter 3, one can define the following strengthening of the notions of discrepancy and equidistribution (Definition 1.1.20, p.34) in a way similar to the definition of super-uniform dispersion (Definition 3.1.1, p.95).

**Definition A.1.1** (**Super-Uniform Discrepancy & Super-Uniform Equidistribution**)

*Let $\boldsymbol{u} = (u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{T}$. Given a natural number $N$, the Super-Uniform Discrepancy of $N$ terms of the sequence $\boldsymbol{u}$ is defined as*

$$D_{\boldsymbol{u}}(N) \quad = \quad \sup_{m\in\mathbb{N}_0} \ \sup_{\xi\in\mathbb{T}} \ d_{\boldsymbol{u}}(N,m,\xi), \tag{A.1}$$

*where $d_{\boldsymbol{u}}(N,m,\xi)$ is as in (1.22), p.37; that is,*

$$d_{\boldsymbol{u}}(N,m,\xi) \quad = \quad \sup_{I\subseteq\mathbb{T}} \left| \frac{1}{N} \sum_{k=1}^{N} \chi_I(u_{m+k} - k\xi) - \lambda(I) \right|,$$

*where the supremum is taken over all intervals $I \subseteq \mathbb{T}$.*

*If $D_{\boldsymbol{u}}(N) \underset{N\to+\infty}{\longrightarrow} 0$, then the sequence $\boldsymbol{u}$ is said to be Super-Uniformly Equidistributed. Moreover, the sequence $\boldsymbol{u}$ is $V$-Super Uniformly Equidistributed, where $V:(0,1)\to\mathbb{R}^+$, if for every $\epsilon\in(0,1)$, it holds that $D_{\boldsymbol{u}}(V(\epsilon)) \leq \epsilon$.*

Given a $V$-super uniformly equidistributed sequence $\boldsymbol{u}$ in $\mathbb{T}$, Inequality (1.20) (Chapter 1, p.35) yields that the sequence $\boldsymbol{u}$ is $V$-super uniformly dispersed. Therefore, from Theorem 3.1.2 (p.96) one has that the Peres-type forest $\mathfrak{F}(\boldsymbol{u})$ has visibility $O(V)$. Working with (super-uniform) discrepancy may be more convenient but this comes with a cost. The concept of discrepancy is genuinely more restrictive than that of dispersion. Indeed, for any $V$-super uniformly equidistributed sequence, the function $V$ always satisfies the bound

$$V(\epsilon) \gg \epsilon^{-2} \tag{A.2}$$

whereas $V(\epsilon) \gg \epsilon^{-1}$ is the corresponding lower bound for the super-uniform dispersion. This will be justified in detail in Section A.3.

The goal of this appendix is to provide a reasonably complete study of the concept of super-uniform equidistribution which is introduced here for first time. This will be achieved (1) by providing a characterization in terms of exponential sums of the super-uniform equidistribution property, (2) by giving an example of a super uniformly equidistributed sequence enjoying good super-uniform discrepancy bounds and (3) by giving explicit examples of families of sequences which are (not) super-uniformly equidistributed. The statements below will be proved in the upcoming sections.

**(1) Criterion for Super-Uniform Equidistribution.** Analogously to Weyl's Criterion for equidistribution (Theorem 1.1.21, p.35, Chapter 1), the super-uniform equidistribution property can be characterised analytically in terms of exponential sums as stated below. This will be the main tool to study examples of super-uniformly equidistributed sequences later in this section.

**Theorem A.1.2** *Let $\boldsymbol{u} = (u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{T}$. The sequence $\boldsymbol{u}$ is super-uniformly equidistributed if, and only if, for every $h \in \mathbb{N}$, it holds that*

$$S_h(N) = o(1) \quad \text{as } N \to +\infty, \tag{A.3}$$

*where*

$$S_h(N) = \frac{1}{N} \cdot \sup_{m\in\mathbb{N}_0} \sup_{\xi\in\mathbb{T}} \left| \sum_{k=1}^{N} e\left(h \cdot (u_{k+m} - k\xi)\right)\right|. \tag{A.4}$$

**(2) A $O\left(\epsilon^{-3}\right)$-Super-Uniformly Equidistributed Sequence.** The following result provides a family of $V$-super uniformly equidistributed sequence with $V(\epsilon) = O\left(\epsilon^{-3}\right)$. Given $M \in \mathbb{N}$, denote by $Bad(M)$ the set of real numbers with partial quotients bounded by $M$ in their continued fraction expansion:

$$Bad(M) = \left\{\alpha = [a_0; a_1, a_2 ...] \in \mathbb{R} : a_0 \in \mathbb{Z} \quad \text{and for every } i \in \mathbb{N}, \quad 1 \leq a_i \leq M\right\}. \tag{A.5}$$

**Theorem A.1.3** *Given $M \in \mathbb{N}$, let $\alpha = [a_0; a_1, a_2, ...] \in Bad(M)$ be a badly approximable number. Then, the sequence $\boldsymbol{\alpha} = (\alpha \cdot k^2)_{k\in\mathbb{N}}$ satisfies the estimate*

$$D_{\boldsymbol{\alpha}}\left(V(\epsilon)\right) \quad \leq \quad \epsilon,$$

*that is, the sequence $\boldsymbol{\alpha}$ is $V$-super uniformly equidistributed where $V(\epsilon) = O_M\left(\epsilon^{-3}\right)$.*

The proof of Theorem A.1.3 is based on applying the Erdös-Turán inequality (Theorem 1.1.23, p.36, Chapter 1). For this application to be optimised, one needs an estimation for the partial quotients of the real number $h \cdot \alpha$, where $\alpha \in Bad(M)$ and $h \in \mathbb{N}$. The following proposition, which is the main ingredient in the proof of Theorem A.1.3, is proved in Section A.3.1.

**Theorem A.1.4** *Given $M \in \mathbb{N}$, let $\alpha = [a_0; a_1, a_2, ...] \in Bad(M)$ be a badly approximable number with partial quotients bounded by $M$. Let $h \in \mathbb{N}$ be a natural number and $h \cdot \alpha = [b_0; b_1, b_2 ...] \in \mathbb{R}$ be the continued fraction expansion of $h\alpha$. Then, for every $i \in \mathbb{N}$, it holds that*

$$b_i \quad \leq \quad 40 \cdot hM;$$

*that is, $h\alpha \in Bad\,(40hM)$.*

**(3) Examples of Super-Uniformly Equidistributed Sequences** More examples of super-uniformly equidistributed sequences can be constructed with the use of strongly $q$-additive functions (Definition 1.1.28, p.38, Chapter 1). Indeed, given a strongly $q$-additive function $g : \mathbb{N}_0 \to \mathbb{N}_0$ and given $x \in \mathbb{R}\backslash\mathbb{Q}$ an irrational, define the sequence $\boldsymbol{g}$ by

$$g_k = x \cdot g(k), \tag{A.6}$$

for every $k \in \mathbb{N}$. Sequences of the form (A.6) are always super-uniformly equidistributed:

**Theorem A.1.5** *Let $x \in \mathbb{R}\backslash\mathbb{Q}$ be an irrational number and let $g : \mathbb{N}_0 \to \mathbb{N}_0$ be a strongly $q$-additive function. Assume that there exists $b \in [\![1, q-1]\!]$ such that $g(b) > 0$. Then, the sequence $\boldsymbol{g} = (x \cdot g(k))_{k\in\mathbb{N}}$ satisfies the relation*

$$D_{\boldsymbol{g}}\,(N) \to 0 \qquad \text{as } N \to +\infty;$$

*that is, $\boldsymbol{g}$ is super-uniformly equidistributed.*

The last result of this chapter shows that sequences obtained from too regular functions do not satisfy the super-uniform equidistribution property. More precisely, given a real function $f : [0, +\infty) \to \mathbb{R}$, define the sequence

$$\boldsymbol{f} = (f(k))_{k\in\mathbb{N}}\,. \tag{A.7}$$

The following theorem provides sufficient conditions on the first and the second derivatives of $f$ (when they exist) for the sequence $\boldsymbol{f}$ not to be super-uniformly equidistributed. Below, we denote by $\mathcal{C}^1(\mathbb{R}^+)$ (resp. $\mathcal{C}^2(\mathbb{R}^+)$) the set of functions $f : \mathbb{R}^+ \mapsto \mathbb{R}$ with one continuous derivative (resp. with two continuous derivatives).

**Theorem A.1.6** *Let $f : (0, +\infty) \to \mathbb{R}$ be a real-valued function.*

1. *If $f \in \mathcal{C}^1(\mathbb{R}^+)$ and for every $x \geq 0, f'(x) > 0$ and $\lim_{x\to+\infty} f'(x) = l$ for some real $l \in \mathbb{R}$, then the sequence $\boldsymbol{f}$ defined in* (A.7) *is not super-uniformly equidistributed.*

2. *If $f \in \mathcal{C}^2(\mathbb{R}^+)$ and $\lim_{x\to+\infty} f''(x) = 0$, then the sequence $\boldsymbol{f}$ defined in* (A.7) *is not super-uniformly equidistributed.*

For instance, the sequence $\left(\sqrt{k}\right)_{k\in\mathbb{N}}$ is equidistributed mod 1 [65, p.238, Exercise 3] but it is not super-uniformly equidistributed. Indeed, the claim follows from the first part of Theorem A.1.6. Similarly, the sequence $(k \cdot \log(k))_{k\in\mathbb{N}}$ is equidistributed modulo 1 [55, p.18, Example 2.8] but not super-uniformly equidistributed. The claim follows from the second part of Theorem A.1.6.

## A.2 Criterion for Super-Uniform Equidistribution

Proof (Theorem A.1.2) Let $\boldsymbol{u} = (u_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{T}$.

**$\Rightarrow$:** Assume that there exists $h \in \mathbb{N}$ satisfying

$$\limsup_{N\to+\infty} S_h(N) \quad \geq \quad c,$$

for some $c > 0$. Thus, for every $N_0 \in \mathbb{N}$ there exists $N \geq N_0$, $m = m(N) \in \mathbb{N}_0$ and $\xi = \xi(N) \in \mathbb{T}$ such that

$$\frac{1}{N} \cdot \left|\sum_{k=0}^{N} e\left((h \cdot u_{k+m} - \xi k)\right)\right| \quad \geq \quad c. \tag{A.8}$$

A straightforward application of Koksma's inequality (Theorem 1.1.22, p.35, Chapter 1) yields that

$$\begin{array}{rcl} D_{\boldsymbol{u}}(N) & \geq & d_{\boldsymbol{u}}(N, m, \xi) \\ & \geq & \dfrac{1}{4hN} \cdot \left| \displaystyle\sum_{k=1}^{N} e\left(h \cdot (u_{k+m} - \xi k)\right) \right| \underset{(\mathrm{A}.8)}{\geq} \dfrac{c}{4h}, \end{array}$$

where the quantities $D_{\boldsymbol{u}}(N)$ and $d_{\boldsymbol{u}}(N, m, \xi)$ are defined in (A.1) and (1.22) (p.37), respectively.

Therefore, there exist infinitely many $N \in \mathbb{N}$ such that $D_{\boldsymbol{u}}(N) \geq c/4h > 0$. Hence the sequence $\boldsymbol{u}$ is not super-uniformly equidistributed modulo 1.

**$\Leftarrow$:** Assume now that equation (A.3) holds for every $h \in \mathbb{N}$. Fix $\epsilon > 0$, $\xi \in \mathbb{T}$ and $m \in \mathbb{N}_0$ and set $H = \lfloor 12\epsilon^{-1} + 1 \rfloor$. By assumption, one has that for every $h \in [\![H]\!]$,

$$S_h(N) \leq \frac{\pi}{4} \cdot \frac{\epsilon}{2\left(2 + \ln(H)\right)}, \quad \text{whenever } N \gg_H 1. \tag{A.9}$$

Therefore, for $N \in \mathbb{N}$ sufficiently large, an immediate application of the Erdös-Turán inequality (Theorem 1.1.23, p.36, Chapter 1) yields that

$$\begin{array}{rcl} d_{\boldsymbol{u}}(N, m, \xi) & \leq & \dfrac{6}{H+1} + \dfrac{4}{\pi} \cdot \displaystyle\sum_{h=1}^{H} \left(\frac{1}{h} - \frac{1}{H+1}\right) \cdot \left| \frac{1}{N} \cdot \sum_{k=1}^{N} e\left(h \cdot (u_{k+m} - \xi k)\right) \right| \\ & \underset{H \geq 12\epsilon^{-1}}{\leq} & \dfrac{\epsilon}{2} + \dfrac{4}{\pi} \cdot \displaystyle\sum_{h=1}^{H} \frac{1}{h} S_h(N) \\ & \underset{(\mathrm{A}.9)}{\leq} & \dfrac{\epsilon}{2} + \dfrac{\epsilon}{2} = \epsilon. \end{array}$$

The choice of $\epsilon, m, \xi$ is arbitrary and the choice of $N$ does not depend on the choice of $m$ and $\xi$. Therefore, $D_{\boldsymbol{u}}(N) \leq \epsilon$ for $N$ sufficiently large. Thus, $D_{\boldsymbol{u}}(N) \to 0$ as $N \to +\infty$.

The proof is complete. ■

## A.3 A $O\left(\epsilon^{-3}\right)$-Super Uniformly Equidistributed Sequence

The claim on the lower bound (A.2) for a $V$-super uniformly equidistributed sequence is first established. The proof rests on the orthogonality relations (1.83), p.77, which are restated here,

$$\int_0^1 e(n\xi)\mathrm{d}\xi \quad = \quad 0, \quad \text{for any } n \in \mathbb{Z}\setminus\{0\}. \tag{A.10}$$

Let $\boldsymbol{c} = (c_k)_k$ be a given sequence in $\mathbb{T}$. The goal is to prove that for every $N \in \mathbb{N}$, there exists $\xi_N \in \mathbb{T}$ such that

$$\left|\sum_{k=1}^{N} e\left(c_k - k\xi_N\right)\right| \quad \geq \quad \sqrt{N}. \tag{A.11}$$

Indeed, in this case, by applying Koksma's inequality [Chapter 1, Theorem 1.1.22, p.35], one has that

$$D_{\boldsymbol{c}}(N) \quad \geq \quad d_{\boldsymbol{c}}\left(N, 0, \xi_N\right) \quad \geq \quad \frac{1}{4\cdot\sqrt{N}}.$$

In turn, one obtains that for a $V$-super uniformly equidistributed sequence, it always holds that

$$V(\epsilon) \quad \geq \quad \frac{\epsilon^{-2}}{16}.$$

To conclude the proof, inequality (A.11) follows upon noticing that

$$\begin{aligned}
\int_0^1 \left|\sum_{k=1}^{N} e\left(c_k - k\xi\right)\right|^2 \mathrm{d}\xi \quad &= \quad \int_0^1 \left(\sum_{k=1}^{N} e\left(c_k - k\xi\right)\right) \cdot \left(\sum_{l=1}^{N} e\left(-\left(c_l - l\xi\right)\right)\right) \mathrm{d}\xi \\
&= \quad \sum_{1\leq k,l\leq N} \left(e\left(c_k - c_l\right) \cdot \int_0^1 e\left((l-k)\cdot\xi\right)\mathrm{d}\xi\right) \\
&\underset{(A.10)}{=} \quad N.
\end{aligned}$$

The claim is proved. ■

We continue with the proof of Theorem A.1.3. To this end, one needs the following estimate concerning exponential sums of quadratic polynomial sequences.

**Theorem A.3.1** *[44, Theorem 6] Let $P(x) = \alpha x^2 + \beta x$, $\alpha, \beta \in \mathbb{R}$ be a quadratic polynomial (in particular, $\alpha \neq 0$). Assume that $p \in \mathbb{Z}$, $q \in \mathbb{N}$ are such that $gcd(p, q) = 1$ and*

$$\left|\alpha - \frac{p}{q}\right| \quad \leq \quad \frac{1}{q^2}.$$

*Then*[1]*, for every natural number $N \in \mathbb{N}$ it holds that*

$$\left|\sum_{k \in [\![N]\!]} e\left(\alpha k^2 + \beta k\right)\right| \quad = \quad O\left(\frac{N}{\sqrt{q}} + \sqrt{q}\right).$$

*Here, the implicit constant is absolute, that is, it does not depend on the choice of the parameters $\alpha, \beta \in \mathbb{R}$ with $\alpha \neq 0$.*

Proof (Theorem A.1.3) Given $M \in \mathbb{N}$, let $\alpha = [a_0; a_1, a_2, ...] \in Bad(M)$ be an irrational real number with partial quotients bounded by the natural number $M$. Set $\boldsymbol{\alpha} = (\alpha \cdot k^2)_{k \in \mathbb{N}}$ and consider it as a sequence in $\mathbb{T}$.

Applying the Erdös-Turán inequality (Theorem 1.1.23, p.36, Chapter 1) yields that for any $H, N \in \mathbb{N}$,

$$D_{\boldsymbol{\alpha}}(N) \quad \leq \quad \frac{6}{H+1} \quad + \quad \frac{4}{\pi} \sum_{h=1}^{H} \frac{1}{h} \left|S_h(N)\right|, \tag{A.12}$$

[1] A fundamental result in the theory of exponential sums is the following estimate concerning exponential sums of quadratic polynomial sequences: Let $\alpha \neq 0$ and $\beta$ be real numbers with $\alpha$ satisfying

$$\left|\alpha - \frac{p}{q}\right| \quad \leq \quad \frac{1}{q^2},$$

for some $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Then, for every natural number $N \in \mathbb{N}$ it holds that

$$\left|\sum_{k \in [\![N]\!]} e\left(\alpha k^2 + \beta k\right)\right| \quad \ll \quad \frac{N}{\sqrt{q}} + \sqrt{q} \log q.$$

Here, the implicit constant is absolute.

This bound is slightly weaker than the one provided by Theorem A.3.1 and is proved by exploiting Weyl's method (see [48, Chapter 8, Theorem 8.1] and [64, Chapter 3, Theorem 1]).

where $S_h(N)$ is as in (A.4). It is enough to prove that for any $h, N \in \mathbb{N}$,

$$|S_h(N)| \quad \ll_M \quad \sqrt{\frac{h}{N}}. \tag{A.13}$$

Indeed, in this case, one has that

$$\sum_{h=1}^{H} \frac{1}{h} |S_h(N)| \quad \underset{(A.13)}{\ll_M} \quad \sum_{h=1}^{H} \frac{1}{\sqrt{hN}} \quad \ll \quad \sqrt{\frac{H}{N}}. \tag{A.14}$$

By substituting inequality (A.14) in (A.12) and by setting further $H = \lfloor 12 \cdot \epsilon^{-1} + 1 \rfloor$ and $N \geq C_M \cdot \epsilon^{-3}$, where $C_M > 0$ is a large constant depending only on $M$, one obtains that

$$D_{\boldsymbol{\alpha}}(N) \quad \leq \quad \epsilon. \tag{A.15}$$

It thus remains to prove inequality (A.13). To this end, fix $h, N \in \mathbb{N}$, $\xi \in \mathbb{T}$ and $m \in \mathbb{N}_0$. Without loss of generality assume that $N \geq 40 \cdot hM + 1$. From Theorem A.1.4, one has that $h\alpha \in Bad\,(40hM)$. From Equation (1.46) (Chapter 1, p.49) this implies that for every $n \in \mathbb{N}$, it holds that $q_n/q_{n-1} \leq 40hM + 1$, where $(q_n)_{n\in\mathbb{N}_0}$ is the sequence of denominators of the convergents of $h\alpha$. Therefore, it follows that there exists a convergent $p/q$ of $h\alpha$ such that

$$\left| h\alpha - \frac{p}{q} \right| \quad \leq \quad \frac{1}{q^2}$$

with

$$gcd(p, q) = 1 \quad \text{ and } \quad \frac{N}{40hM + 1} - 1 \leq q \leq N. \tag{A.16}$$

Applying Theorem A.3.1 then yields that

$$\left| \sum_{k=1}^{N} e\left( h\alpha(k + m)^2 - h\xi k \right) \right| \quad \ll \quad \frac{N}{\sqrt{q}} + \sqrt{q} \quad \underset{(A.16)}{\leq} \quad 3\sqrt{40M} \cdot \sqrt{hN} \quad \ll_M \sqrt{hN}. \tag{A.17}$$

The choice of $m$ and $\xi$ is arbitrary. Taking the supremum over $m$ and $\xi$ on the left-hand side of inequality (A.17) yields inequality (A.13). The proof is complete. ■

## A.3.1 Bound of Partial Quotients

In this section, we prove Theorem A.1.4; namely that, given a natural number $M \in \mathbb{N}$ and the continued fraction expansion of a badly approximable real number $\alpha$, if $\alpha \in Bad(M)$, then for any $h \in \mathbb{N}$, the partial quotients of $h\alpha$ are bounded by $40hM$. The main idea underlying this result lies in the connection between the partial quotients of a badly approximable number $\alpha$ and the dispersion $\delta_{\boldsymbol{\alpha}}(N)$ of the sequence of rotations of $\alpha$ (see Equation (1.21), Chapter 1, p.36). This connection is captured in the following statement which is proved at the end of this section:

**Proposition A.3.2** *Let $M \in \mathbb{N}$ be a natural number and let $\alpha = [a_0; a_1, a_2, ...] \in Bad(M)$ be a badly approximable number with partial quotients bounded by $M$. Denote by $\boldsymbol{\alpha} = (k\alpha)_{k\in\mathbb{N}}$ the sequence of multiples of $\alpha$. The following claims hold:*

1. *If $V(\epsilon) = 2(M+1) \cdot \epsilon^{-1}$, then for every $\epsilon > 0$,*

$$\delta_{\boldsymbol{\alpha}}\left(V(\epsilon)\right) \quad \leq \quad \epsilon. \tag{A.18}$$

2. *Let $T \geq \frac{1}{2}$ be a real number and set $V(\epsilon) = T \cdot \epsilon^{-1}$. If, for any $\epsilon \in (0,1)$, it holds that $\delta_{\boldsymbol{\alpha}}\left(V(\epsilon)\right) \leq \epsilon$, then $\alpha \in Bad\,(10T)$.*

The proof of Proposition A.3.2 is given after the proof of Theorem A.1.4.

Proof (Theorem A.1.4) Fix $M, h \in \mathbb{N}$ and $\alpha \in Bad(M)$. Set $\beta = h\alpha$, $T_\alpha = 2(M+1)$ and $T_\beta = h \cdot T_\alpha$. Denote by $\boldsymbol{\alpha} = (k\alpha)_{k\in\mathbb{N}}$ and $\boldsymbol{\beta} = (k\beta)_{k\in\mathbb{N}}$ the sequences of multiples of $\alpha$ and $\beta$, respectively. The goal is to show that $\delta_{\boldsymbol{\beta}}\left(W(\epsilon)\right) \leq \epsilon$, where $W(\epsilon) = T_\beta \cdot \epsilon^{-1}$. This is enough since from the second part of Proposition A.3.2, it follows that $\beta \in Bad\,(10T_\beta)$. One can then conclude the result upon noticing that $10T_\beta \leq 40hM$.

To this end, fix $\epsilon \in (0,1)$ and $x \in \mathbb{T}$. From the first part of Proposition A.3.2, given $V(\epsilon) = T_{\boldsymbol{\alpha}} \cdot \epsilon^{-1}$, one has that $\delta_{\boldsymbol{\alpha}}\left(V(\epsilon)\right) \leq \epsilon$. In other words, there exists $k' \in [\![V\left(\epsilon/h\right)]\!]$ such that

$$\left|\left|k'\alpha - \frac{x}{h}\right|\right| \quad \leq \quad \frac{\epsilon}{h},$$

which yields in turn that

$$||k' \cdot h\alpha - x|| \quad \leq \quad \epsilon.$$

It has just been established that $\delta_{\beta}\left(V(\epsilon/h)\right) \leq \epsilon$. Since $W(\epsilon) = V(\epsilon/h)$, the proof of Theorem A.1.4 is complete. ■

It remains to prove Proposition A.3.2. For the proof of its first part, we need the following more general result.

**Lemma A.3.3** *Let $\alpha \in \mathbb{T}$. Assume that a sequence of rational numbers $(p_n/d_n)_{n\in\mathbb{N}_0}$ with gcd $(p_n, d_n) = 1$ is such that:*

1. $\lim_{n\to+\infty} (p_n/d_n) = \alpha$ *and the sequence $(d_n)_{n\in\mathbb{N}_0}$ is strictly increasing.*
2. *for every $n \in \mathbb{N}$,*

$$\alpha \in B\left(\frac{p_n}{d_n}, \frac{C}{d_n^2}\right) \qquad \text{for some absolute constants } C > 0,$$

*where $B(x,r)$ stands for the ball in $\mathbb{T}$ centred at $x$ with radius $r$.*

*Then, for every $n \in \mathbb{N}$, it holds that*

$$\delta_{\boldsymbol{\alpha}}(d_n) \quad \leq \quad \frac{C+1}{d_n},$$

*where $\boldsymbol{\alpha} = (k\alpha)_{k\in\mathbb{N}}$ is the sequence of multiples of $\alpha$. Moreover, defining $f : \mathbb{N} \to \mathbb{R}^+$ by $f(n) = d_n/d_{n-1}$, it holds that for any $\epsilon \in (0,1)$,*

$$\delta_{\boldsymbol{\alpha}}\left((C+1)f(n_\epsilon) \cdot \epsilon^{-1}\right) \quad \leq \quad \epsilon. \tag{A.19}$$

*Here, $n_\epsilon$ is the unique natural number $n \in \mathbb{N}$ such that*

$$\frac{C+1}{d_n} \quad \leq \quad \epsilon \quad < \quad \frac{C+1}{d_{n-1}}.$$

Proof Fix an irrational $\alpha \in \mathbb{T}$ and assume that the sequence of rationals $(p_n/d_n)_{n\in\mathbb{N}_0}$ is as in the statement of the lemma. It is easy to check that for a given $n \in \mathbb{N}$ and for any $\xi \in B\left(\frac{p_n}{d_n}, \frac{C}{d_n^2}\right)$, the sequence $(k\xi)_{k=1}^{d_n}$ is $e_n$-dense, where

$$e_n \quad = \quad \frac{C+1}{d_n}.$$

By assumption, one has that

$$\{\alpha\} \quad = \quad \bigcap_{n=1}^{+\infty} B\left(\frac{p_n}{d_n}, \frac{C}{d_n^2}\right) \quad \subseteq \quad \mathbb{T};$$

therefore, for any $n \in \mathbb{N}$, the sequence $(k\alpha)_{k=1}^{d_n}$ is $e_n$-dense.

Fix $\epsilon \in (0,1)$ and set $n_\epsilon \in \mathbb{N}$ and $f : \mathbb{N} \to \mathbb{R}^+$ as in the statement. Let

$$N \quad = \quad (C+1) \cdot f(n_\epsilon) \cdot \epsilon^{-1}.$$

One has that $N \geq d_{n_\epsilon}$ and therefore the sequence $(k\alpha)_{k=1}^{N}$ is $e_{n_\epsilon}$-dense. This yields inequality (A.19). The proof is complete. ■

Proof (Proposition A.3.2) Let $\alpha = [a_0; a_1, a_2, ...]$ be an irrational number such that $\alpha \in Bad(M)$ for some fixed $M \in \mathbb{N}$. Denote by $(p_n/q_n)_{n\in\mathbb{N}_0}$ the sequence of convergents of $\alpha$ as defined in (1.45), p.48.

**1.** Given $n \in \mathbb{N}$, it holds that

$$\left\|\alpha - \frac{p_n}{q_n}\right\| \quad \leq \quad \frac{1}{q_n^2}.$$

Define the function $f : \mathbb{N} \to \mathbb{R}^+$ by setting

$$f(n) \quad = \quad \frac{q_n}{q_{n-1}}.$$

From Equation (1.46) (Chapter 1, p.49), it follows that for every $n \in \mathbb{N}$ it holds that $(q_n/q_{n-1}) \leq M+1$, therefore, the claim follows upon applying Lemma A.3.3 to the sequence $(p_n/q_n)_{n\in\mathbb{N}_0}$ with $C = 1$.

**2.** Let $\alpha = [a_0; a_1, a_2, ...]$ be as in the assumption. Set for every $n \in \mathbb{N}$

$$x_n \quad = \quad [0; a_{n+1}, a_{n+2}...]\,. \tag{A.20}$$

With the notation from equation (A.20), given $n \in \mathbb{N}$, one can rewrite the continued fraction expansion of $\alpha$ as

$$\alpha = [a_0; a_1, ..., a_{n-1}, a_n + x_n] \,. \tag{A.21}$$

It should be noted here that in the above expression, the term $a_n + x_n$ is not integer. Since relations (1.46), p.49, hold true for general continued fractions [2] as well [51, p.4, Theorem 1], from (A.21) and from the left-hand side equation in [Chapter 1, p.49, Equation (1.47)], it is easily proved that for every $n \in \mathbb{N}$,

$$\alpha = \frac{p_{n-1}}{q_{n-1}} + \frac{(-1)^{n-1}}{q_{n-1} \cdot (q_n + x_n q_{n-1})} \quad \text{with} \quad \frac{1}{q_{n-1} \cdot (q_n + x_n q_{n-1})} < \frac{1}{a_n q_{n-1}^2}. \tag{A.22}$$

Identify the point $x \in \mathbb{T}$ with the point $\{x\} \in [0,1)$. Fix $n \in \mathbb{N}$ and set $s : [\![q_{n-1}]\!] \to \{\{i\alpha\} : i \in [\![q_{n-1}]\!]\}$, which orders the set $S_n = \{\{i\alpha\} : i \in [\![q_{n-1}]\!]\}$. In particular,

$$s(1) < s(2) < \cdots < s(q_{n-1})$$

with $s(i) = \{k_i \alpha\}$ for some $k_i \in [\![q_{n-1}]\!]$. If $q_{n-1} \geq 2$, then one can distinguish two cases. In the first one, where $n$ is odd, one concludes from equation (A.22) that

$$\begin{aligned} 0 < \{q_{n-1}\alpha\} = s(1) < \frac{1}{q_{n-1}} &< \ldots \\ &< \frac{i-1}{q_{n-1}} < s(i) < \frac{i}{q_{n-1}} < \cdots \\ &< \frac{q_{n-1}-1}{q_{n-1}} < s(q_{n-1}) < 1. \end{aligned} \tag{A.23}$$

In the second case, where $n$ is even, one concludes again from equation (A.22) that

[2] A general continued fraction is an expression of the form $[y_0; y_1, y_2, ...]$ with $y_0 \in \mathbb{R}$ and $y_i \in \mathbb{R}^+, i \geq 1$. See the discussion after [Theorem 1.1.41, Chapter 1, p.50] for the convergence of such expressions.

$$\begin{aligned}0 &< s(1) < \frac{1}{q_{n-1}} < \dots \\ &< \frac{i-1}{q_{n-1}} < s(i) < \frac{i}{q_{n-1}} < \cdots \\ &< \frac{q_{n-1}-1}{q_{n-1}} < s\left(q_{n-1}\right) = \{q_{n-1}\alpha\} < 1.\end{aligned}$$

We prove first the case where $q_{n-1} \geq 2$ (the proof for $q_{n-1} = 1$ will be given at the end). Without loss of generality, assume that $n$ is odd as the proof for $n$ even is the same. Then, equation (A.22) becomes

$$\alpha = \frac{p_{n-1}}{q_{n-1}} + \frac{1}{q_{n-1} \cdot (q_n + x_n q_{n-1})}.$$

Given $j \in [\![a_n]\!]$, we prove that the finite sequence $\boldsymbol{\alpha}_j = (k\alpha)_{k \in [\![jq_{n-1}]\!]}$ cannot be $\delta_j$-dense when

$$\delta_j = \frac{a_n - j}{2 \cdot q_{n-1} a_n} \leq \frac{||s(2) - jq_{n-1} \cdot \alpha||}{2}.$$

More precisely, it will be established that there is no term of the sequence $\boldsymbol{\alpha}_j$ in the interval

$$\begin{aligned}I_j := &\left(\{jq_{n-1} \cdot \alpha\}, s(2)\right) \\ = &\left\{\{jq_{n-1} \cdot \alpha\} + t \in \mathbb{T} : \quad 0 < t < \{s(2) - \{jq_{n-1} \cdot \alpha\}\}\right\}.\end{aligned}$$

The claim concerning the density of the sequence $\boldsymbol{\alpha}_j$ follows upon noticing that $\delta_j \leq \left(\lambda\left(I_j\right)/2\right)$, where $\lambda(\cdot)$ is the 1-dimensional Lebesgue measure. Indeed, from equation (A.22), one has that

$$0 < \{jq_{n-1} \cdot \alpha\} = \frac{j}{q_n + x_n q_{n-1}} \underset{(j \leq a_n)}{<} \frac{1}{q_{n-1}} \underset{\text{(A.23)}}{<} s(2) < \frac{2}{q_{n-1}}$$

which implies that

$$\begin{aligned}
||s(2) - \{jq_{n-1}\cdot\alpha\}_2|| &> \frac{1}{q_{n-1}} - \frac{j}{q_n + x_n q_{n-1}} \\
&= \frac{q_n + x_n q_{n-1} - jq_{n-1}}{q_{n-1}\cdot(q_n + x_n q_{n-1})} \\
&= \frac{a_n q_{n-1} + q_{n-2} + x_n q_{n-1} - jq_{n-1}}{q_{n-1}\cdot(a_n q_{n-1} + q_{n-2} + x_n q_{n-1})} \\
&\geq \frac{a_n q_{n-1} - jq_{n-1}}{q_{n-1}\cdot q_n} \\
&\geq \frac{a_n - j}{q_{n-1}a_n} \quad = \quad 2\delta_j.
\end{aligned}$$

Given an integer $k \in [\![jq_{n-1}]\!]$, decompose it as

$$k = lq_{n-1} + m \quad \text{with} \quad l \in [\![0, j-1]\!] \quad \text{and} \quad m \in [\![q_{n-1}]\!]\,. \tag{A.24}$$

One obtains that

$$\{k\alpha\} \underset{(A.22)}{=} \frac{l}{q_n + x_n q_{n-1}} + s(\mathfrak{m}) \pmod 1, \tag{A.25}$$

where $\mathfrak{m} = s^{-1}(\{m\alpha\}) \in [\![q_{n-1}]\!]$. In other words,

$$S_n(j) := \{k\alpha : k \in [\![jq_{n-1}]\!]\} = \left\{ s(\mathfrak{m}) + \frac{l}{q_n + x_n q_{n-1}} : l \in [\![0, j-1]\!]\,, \mathfrak{m} \in [\![q_{n-1}]\!] \right\}.$$

Fix $j \in [\![a_n]\!]$. In order to prove that the sequence $\boldsymbol{\alpha}_j$ is not $\delta_j$-dense, it is enough to show that $S_n(j) \cap I_j = \emptyset$. This is done by distinguishing three cases for the values of $k \in [\![jq_{n-1}]\!]$:

**Case 1:** Assume that $\mathfrak{m} = 1$ and $l \in [\![0, j-1]\!]$. Since $n$ is odd, it holds that $s(\mathfrak{m}) = q_{n-1}$ and thus, from equality (A.25), one has that

$$0 < \{k\alpha\} = \frac{l+1}{q_n + x_n q_{n-1}} \underset{l \leq j-1}{\leq} \frac{j}{q_n + x_n q_{n-1}} = \{jq_{n-1}\cdot\alpha\}\,.$$

Therefore, $\{k\alpha\} \notin I_j$.

**Case 2:** Assume that $\mathfrak{m} \in [\![2, q_{n-1} - 1]\!]$ and $l \in [\![0, j-1]\!]$. From equality (A.25) one has that

$$s(2) \quad \leq \quad \{k\alpha\} \quad = \quad s\left(\mathfrak{m}\right) + \frac{l}{q_n + x_n q_{n-1}} \quad < \quad 1,$$

where the last inequality follows because $l \leq j \leq a_n$ and $s\left(\mathfrak{m}\right) \; < \left(\left(q_{n-1} - 1\right)/q_{n-1}\right)$. Thus, $\{k\alpha\} \notin I_j$.

**Case 3:** Assume that $\mathfrak{m} = q_{n-1}$ and $l \in [\![0, j-1]\!]$. If it holds that

$$\frac{q_{n-1} - 1}{q_{n-1}} \quad < \quad \{k\alpha\} \quad < \quad 1,$$

then there is nothing to prove since clearly it holds that $\{k\alpha\} \notin I_j$. If it holds that

$$0 \quad \leq \quad \{k\alpha\} \quad < \quad \frac{q_{n-1} - 1}{q_{n-1}},$$

then, from equality (A.25) and inequality (A.23), one obtains that

$$0 \quad \leq \quad \{k\alpha\} \quad = \quad \frac{l}{q_n + x_n q_{n-1}} - \left(1 - s\left(q_{n-1}\right)\right) \quad < \quad \frac{l}{q_n} \quad < \quad \frac{j}{q_n + x_n q_{n-1}}.$$

Therefore, $\{k\alpha\} \notin I_j$.

Thus, this establishes that the finite sequence $\boldsymbol{\alpha}_j$ is not $\delta_j$-dense.

Given $N \in \mathbb{N}$, define the quantity

$$\epsilon_N \quad = \quad \inf \left\{ \epsilon \in (0,1) : (k\alpha)_{k \in [\![N]\!]} \text{ is } \epsilon - \text{dense} \right\}.$$

By assumption, there is $T \geq \frac{1}{2}$ such that $\delta_{\boldsymbol{\alpha}}\left(T \cdot \epsilon^{-1}\right) \leq \epsilon$, which implies that

$$\epsilon_N \cdot N \quad \leq \quad T. \tag{A.26}$$

For $N = jq_{n-1}$, we have established that

$$\epsilon_{jq_{n-1}} \quad \geq \quad \frac{a_n - j}{2 \cdot q_{n-1} a_n},$$

thus, from inequality (A.26) one obtains that

$$T \quad \geq \quad \frac{j\,(a_n - j)}{2a_n}. \tag{A.27}$$

If $a_n \geq 3$, then setting $j = \lfloor a_n/2 \rfloor$ and substituting this value of $j$ in inequality (A.27) yields

$$a_n \quad \leq \quad 10T. \tag{A.28}$$

If $a_n \in \{1, 2\}$, then inequality (A.28) holds trivially. Therefore, we have proved the desired bound for the partial quotient $a_n$ of $\alpha$.

To complete the proof, it remains to establish inequality (A.28) in the case where $q_{n-1} = 1$. For any $n \geq 3$, it holds that $q_{n-1} \geq 2$; therefore, we have to split two cases.

**Case $n = 1$:** Assume that $n = 1$, that is, $q_{n-1} = q_0 = 1$ (see [Chapter 1, p.49, Equation (1.46)]). Recall that $q_1 = a_1$. In this case, equation (A.22) becomes

$$\alpha \quad = \quad p_0 + \frac{1}{a_1 + x_1}.$$

Without loss of generality, assume that $a_1 \geq 3$; otherwise inequality (A.28) holds trivially. As before, in order to prove that $a_1 \leq 10T$, it is enough to prove that for every $j \in [\![q_1 - 1]\!]$, the finite sequence $(k\alpha)_{k \in [\![j]\!]}$ is not $\delta_j$-dense, where

$$\delta_j \quad = \quad \frac{a_1 - j}{2a_1}.$$

This claim follows from inequality (A.26) upon setting $N = \lfloor a_1/2 \rfloor$. To see this, note that

$$\{j\alpha\} = \frac{j}{a_1 + x_1} \quad \text{and} \quad 0 \quad < \quad \frac{j}{a_1 + x_1} \quad < \quad \frac{j}{a_1} \quad < \quad 1.$$

The claim on the density of the sequence $(k\alpha)_{k\in[\![j]\!]}$ follows upon noticing that the interval $\left(\frac{j}{a_1}, 1\right)$ contains no term of the sequence.

**Case $n = 2$:** Assume that $n = 2$ and $q_1 = 1$, that is, $a_1 = 1$ and $q_2 = a_2 + 1$ (see [Chapter 1, p.49, Equation (1.46)]). In this case, equation (A.22) becomes

$$\alpha \quad = \quad p_1 - \frac{1}{a_2 + 1 + x_2}.$$

Without loss of generality, assume that $a_2 \geq 3$; otherwise inequality (A.28) holds trivially. To prove that $a_2 \leq 10T$, it is enough to prove that for every $j \in [\![q_2 - 1]\!]$, the finite sequence $(k\alpha)_{k\in[\![j]\!]}$ is not $\delta_j$-dense, where

$$\delta_j \quad = \quad \frac{a_2 + 1 - j}{2\,(a_2 + 1)}.$$

Then, the claim follows from inequality (A.26) upon setting $N = [(a_2 + 1)/2]$. The claim on the density of the sequence $(k\alpha)_{k\in[\![j]\!]}$ follows upon noticing that the interval $\left(0, 1 - \frac{j}{a_2+1}\right)$ contains no term of the sequence. Indeed, one has that

$$\{j\alpha\} = 1 - \frac{j}{a_2 + 1 + x_2} \quad \text{and} \quad 0 \quad < \quad 1 - \frac{j}{a_2 + 1} \quad < \quad 1 - \frac{j}{a_2 + 1 + x_2} \quad < \quad 1.$$

The proof is complete. ■

## A.4 Examples of Super-Uniformly Equidistributed Sequences

The goal of this section is to provide examples both of sequences satisfying and not satisfying the super-uniform equidistribution property. This will be achieved upon establishing respectively Theorems A.1.5 and A.1.6.

### A.4.1 Proof of Theorem A.1.5

The proof of Theorem A.1.5 is achieved by applying the criterion for super-uniform equidistribution (Theorem A.1.2). The main idea is that for one to apply the criterion to the sequence $\boldsymbol{g}$ defined in (A.6), it is enough to show that the Fourier-Bohr spectrum of $\boldsymbol{g}$ (cf. Definition 1.1.31, p.41) is empty. The application is based on an argument due to Drmota and Tichy [40, Theorem 1.108] who proved that the sequence $\boldsymbol{g}$ has empty-spectrum (cf. Definition 1.1.26, p.37). To this end, the following proposition is needed.

**Proposition A.4.1** *Let $g$ be a $q$-additive function (cf. Definition 1.1.28, p.38) and let $x \in \mathbb{R}$ be a real number. Assume that the Fourier-Bohr spectrum of $\boldsymbol{g} = (x \cdot g(k))_{k\in\mathbb{N}}$ is empty; that is, that for any $\xi \in \mathbb{T}$,*

$$\left|\sum_{k=0}^{N-1} e\left(x \cdot g(k) - k\xi\right)\right| \quad = \quad o(N) \quad \text{as } N \to +\infty. \tag{A.29}$$

*Then,*

$$\sup_{m\in\mathbb{N}_0} \quad \sup_{\xi\in\mathbb{T}} \quad \left|\sum_{k=0}^{N-1} e\left(x \cdot g(k+m) - k\xi\right)\right| \quad = \quad o(N).$$

The following lemma due to Tichy and Turnwald [78] allows one to simplify further assumption (A.29) in the statement of Proposition A.4.1.

**Lemma A.4.2** *[78, p.70, Lemma 1] Let $q \geq 2$ be a natural number and let $G : \mathbb{N}_0 \to \mathbb{C}$ be a function such that $G(0) = 1$, $|G(k)| \leq 1$ and such that for every $k \in \mathbb{N}$,*

$$G(k) \quad = \quad \prod_{n=0}^{+\infty} G\left(d_n(q,k) \cdot q^n\right). \tag{A.30}$$

*Assume that for any $n \in \mathbb{N}$, it holds that*

$$\left|\frac{1}{q^n} \cdot \sum_{j=0}^{q^n-1} G(k)\right| \quad \leq \quad \frac{1}{f\left(q^n\right)}, \tag{A.31}$$

*where $f : [1, +\infty) \to (0, +\infty)$ is a continuous non-decreasing function such that $f(x) \leq x$. Then, for any $N \in \mathbb{N}$,*

$$\left|\frac{1}{N} \cdot \sum_{k=0}^{N-1} G(k)\right| \quad \leq \quad \frac{q+1}{f\left(\sqrt{N}\right)}.$$

The proofs of Proposition A.4.1 and of Lemma A.4.2 are given at the end of this section. The proof of Lemma A.4.2 is provided for the sake of completeness.

Proof (Theorem A.1.5) Fix $h \in \mathbb{N}$ and set

$$S_h(N) \quad = \quad \frac{1}{N} \cdot \sup_{m \in \mathbb{N}_0} \quad \sup_{\xi \in \mathbb{T}} \quad \left| \sum_{k=1}^{N} e\left(hxg(k+m) - hk\xi\right) \right|.$$

The goal is to show that

$$S_h(N) \quad = \quad o(1) \quad \text{ as } N \to +\infty. \tag{A.32}$$

Then, the result follows upon applying Theorem A.1.2.

From Proposition A.4.1, to prove equation (A.32), it is enough to show that for every $\xi \in \mathbb{T}$, it holds that

$$\lim_{N \to +\infty} \quad \frac{1}{N} \sum_{k=0}^{N-1} e\left(hxg(k) - hk\xi\right) \quad = \quad 0. \tag{A.33}$$

Fix $\xi \in \mathbb{T}$ and set $G(k) = e\left(hxg(k) - hk\xi\right)$. Since $g$ is $q$-additive, one has that for every $k \in \mathbb{N}_0$,

$$G(k) = \prod_{n=0}^{+\infty} G\left(d_n(q,k) \cdot q^n\right) \quad \text{ and } \quad |G(k)| \quad \leq \quad 1.$$

Therefore, from Lemma A.4.2, to prove equation (A.33), it is enough to show that

$$\lim_{n \to +\infty} \quad \frac{1}{q^n} \sum_{k=0}^{q^n-1} e\left(hxg(k) - hk\xi\right) \quad = \quad 0. \tag{A.34}$$

By using again the assumption that $g$ is strongly $q$-additive, one obtains that

$$\frac{1}{q^n} \sum_{k=0}^{q^n-1} e\left(hxg(k) - hk\xi\right) \quad = \quad \prod_{j=0}^{n-1} \frac{1}{q} \left( \sum_{b=0}^{q-1} e\left(hxg(b) - h\xi b q^j\right) \right). \tag{A.35}$$

Since

$$\left|\frac{1}{q}\left(\sum_{b=0}^{q-1} e\left(hxg(b) - h\xi bq^j\right)\right)\right| \quad \leq \quad 1,$$

the modulus of the right-hand side quantity in relation (A.35) is a decreasing sequence of the parameter $n \in \mathbb{N}$. Therefore, the limit

$$\lim_{n\to+\infty} \left|\prod_{j=0}^{n-1} \frac{1}{q}\left(\sum_{b=0}^{q-1} e\left(hxg(b) - h\xi bq^j\right)\right)\right|$$

exists. Assume for a contradiction that this limit does not equal zero. Then, from equality (A.35), it follows that

$$\lim_{j\to+\infty} \quad \left|\frac{1}{q}\cdot\sum_{b=0}^{q-1} e\left(hxg(b) - h\xi bq^j\right)\right| \quad = \quad 1$$

or, equivalently, that

$$\lim_{j\to+\infty} \quad \left|\sum_{b=0}^{q-1} e\left(hxg(b) - h\xi bq^j\right)\right| \quad = \quad q. \tag{A.36}$$

Since $e\left(hxg(0)\right) = 1$, it follows that for every $1 \leq b \leq q-1$,

$$\lim_{j\to+\infty} h\xi bq^j \quad = \quad hxg(b) \pmod 1. \tag{A.37}$$

In order to prove that relation (A.37) cannot hold for every $b \in [\![1, q-1]\!]$, one needs the following lemma.

**Lemma A.4.3** *Let $y \in \mathbb{R}$ be a real number and let $q \geq 2$ be an integer. The sequence $\left(y \cdot q^j\right)_{j=1}^{+\infty}$ converges modulo 1 if, and only if, there exist integers $n_0 \in \mathbb{N}$ and $d, y_n \in \{0, 1, .., q-1\}$ for $n \leq n_0$, such that the q-adic expansion of $y$ has the form*

$$y \quad = \quad \lfloor y \rfloor + \sum_{n=1}^{n_0} \frac{y_n}{q^n} + \sum_{n=n_0+1}^{+\infty} \frac{d}{q^n}\cdot \tag{A.38}$$

Proof Let $y \in \mathbb{R}$ be a real number such that the sequence $\left(y \cdot q^j\right)_{j=1}^{+\infty}$ converges modulo 1 to a real number $l$;

$$\lim_{j\to+\infty} y\cdot q^j \quad = \quad l \qquad \pmod 1. \tag{A.39}$$

It is first proved that $l$ is rational. To this end, assume that $l$ is irrational. Thus, on the one hand, it holds that $ql \neq l \pmod 1$ and, on the other, it holds that

$$ql \underset{(A.39)}{=} \lim_{j\to+\infty} y\cdot q^{j+1} \quad = \quad \lim_{j\to+\infty} y\cdot q^j \quad = \quad l \qquad \pmod 1.$$

This yields a contradiction, therefore, $l \in \mathbb{Q}$.

The same argument impliess that for every $m \in \mathbb{N}$

$$q^m l = l \pmod 1, \tag{A.40}$$

which in turn, by expanding $l$ in its $q$-adic expansion

$$l \quad = \quad \sum_{n=1}^{+\infty} \frac{l_n}{q^n} \qquad \pmod 1,$$

where for all $j \in \mathbb{N}$, $l_j \in \{0, 1, ..., q-1\}$, yields that $l_m = l_1$ for any $m \in \mathbb{N}$. Thus, by setting $d = l_1$ one has that

$$l \quad = \quad \sum_{n=1}^{+\infty} \frac{d}{q^n} \qquad \pmod 1. \tag{A.41}$$

Also, one has that

$$\lim_{j\to+\infty} (y-l)\cdot q^j \quad = \quad \lim_{j\to+\infty} \left(y\cdot q^j - l\cdot q^j\right) \underset{(A.40)}{=} \lim_{j\to+\infty} y\cdot q^j - l \quad = \quad 0. \tag{A.42}$$

Notice that if $x \in \mathbb{R}$ is a real number with $q$-adic expansion

$$x \quad = \quad \lfloor x \rfloor + \sum_{n=1}^{+\infty} \frac{x_n}{q^n}$$

such that

$$\lim_{j\to+\infty} x\cdot q^j \quad = \quad 0, \tag{A.43}$$

then there exists $n_0 \in \mathbb{N}$ such that

$$\text{either} \quad x_n = 0 \text{ for any } n \geq n_0 \quad \text{or} \quad x_n = q-1 \text{ for any } n \geq n_0. \tag{A.44}$$

To see this, notice that if $n \in \mathbb{N}$ is such that $1 \leq x_{n+1} \leq q-2$, then $||x\cdot q^n|| \geq 1/q$. Therefore, from assumption (A.43), it follows that there exists $n_1 \in \mathbb{N}$ such that for every $n \geq n_1$, it holds that either $x_n = 0$ or $x_n = q-1$. If there exist infinitely many $k, l \in \mathbb{N}$ such that $x_k = 0$ and $x_l = q-1$, then there exist infinitely many $n \in \mathbb{N}$ such that $x_{n+1} = 0$ and $x_{n+2} = q-1$. In turn, for each such $n \in \mathbb{N}$, one has that

$$\frac{q-1}{q^2} \quad \leq \quad ||x\cdot q^n|| \quad \leq \quad \frac{1}{q},$$

which contradicts assumption (A.43). This proves the existence of a natural number $n_0$ for which the condition (A.44) holds.

Equation (A.38) now follows from equations (A.41), (A.42), (A.43) and (A.44) upon setting $x = y - l$.

The converse claim is immediate. The proof is complete. ■

In order to complete the proof of Theorem A.1.5, it remains to establish that relation (A.37) cannot hold for $b \in [\![1, q-1]\!]$. By assumption, there exists $b_0 \in [\![1, q-1]\!]$ such that $g(b_0) > 0$. Assume for a contradiction that the equation (A.37) holds for $b = b_0$. Since $g(b_0) > 0$ and $x$ is irrational, the right-hand side of equation (A.37) is irrational. The contradiction follows upon noticing that the left-hand side of equation (A.37) is rational. Indeed, applying Lemma A.4.3 to $y = h\xi b_0$ yields that there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$, it holds that

$$y\cdot q^n \quad = \quad \frac{d}{q-1} \pmod 1,$$

where the integer $d \in \{0, 1, ..., q-1\}$ is provided by the same lemma. Therefore, equality (A.37) cannot hold for $b = b_0$ which, in turn, implies equality (A.34). This completes the proof of Theorem A.1.5. ■

It remains, first, to prove Proposition A.4.1 and, then, Lemma A.4.2. As far as Proposition A.4.1 is concerned, one needs the following lemma[3] due to Spiegelhofer [75]. Its proof is given for the sake of completeness.

**Lemma A.4.4** *[75, p.4, Theorem 2.4] Assume that $(f_i)_{i\geq 0}$ is a sequence of non-negative continuous functions on $[0,1]$ converging pointwise to the zero function. Assume that for any $i \in \mathbb{N}$, it holds that*

$$|f_{i+1}(x)| \quad \leq \quad \max\{|f_i(x), f_{i-1}(x)|\}. \tag{A.45}$$

*Then, the sequence $(f_i)_{i\geq 0}$ converges uniformly to the zero function.*

Proof Fix $\epsilon > 0$. For any $N \in \mathbb{N}$, set

$$A_N = \{t \in [0,1] : f_N(t) < \epsilon \text{ and } f_{N+1}(t) < \epsilon\}.$$

Since the functions $f_i$ are continuous for every $i \in \mathbb{N}_0$, one has that the set $A_N$ is open in the topological space $([0,1], |\cdot|)$. Moreover, from assumption (A.45), one obtains that

$$A_N = \{t \in [0,1] : f_n(t) < \epsilon \text{ for all } n \geq N\}. \tag{A.46}$$

Also, one has that $A_N \subseteq A_{N+1}$.

From equality (A.46), it is enough to prove that there exists $N_\epsilon \in \mathbb{N}$ such that $A_{N_\epsilon} = [0,1]$. To this end, note that by assumption, for every $x \in [0,1]$, there exists $N_x \in \mathbb{N}$ such that for all $n \geq N_x$, it holds that $f_n(x) \leq \epsilon$. Therefore, the set $\{A_{N_x} : x \in [0,1]\}$ is an open covering of $[0,1]$. Since $[0,1]$ is compact, one can choose points $x_1, x_2, ..., x_k \in [0,1]$ such that

$$A_{N_{x_1}} \cup A_{N_{x_2}} \cup ... \cup A_{N_{x_k}} \quad = \quad [0,1].$$

The sequence of sets $(A_n)_{n\in\mathbb{N}}$ is increasing; therefore, setting $N_\epsilon = \max\{N_{x_1}, .., N_{x_k}\}$ yields that $A_{N_\epsilon} = [0,1]$. The proof is complete. ■

[3]Lemma A.4.4 is a version of *Dini's theorem* [18, p.238, Theorem 8.2.6] which states the folllowing: if $X$ is a compact topological space and $(f_k)_{k\in\mathbb{N}}$ is a **monotonically increasing sequence** (meaning $f_n(x) \leq f_{n+1}(x)$ for all $x \in X$) of continuous real-valued functions on $X$ which converges pointwise to a continuous function $f : X \mapsto \mathbb{R}$, then the convergence is *uniform*. The same conclusion holds if $(f_k)_{k\in\mathbb{N}}$ is a monotonically decreasing sequence.

PROOF (PROPOSITION A.4.1) Fix $x \in \mathbb{R}$ and let $g$ be a $q$-additive function. Assume that for any $\xi \in \mathbb{T}$, equation (A.29) holds. The proof is divided into the following two steps.

**Step 1:** First, it is established that

$$\limsup_{i\to+\infty} \sup_{\xi\in\mathbb{T}} \frac{1}{q^i} \cdot \left| \sum_{k=0}^{q^i-1} e\left(xg(k) - k\xi\right) \right| = 0. \tag{A.47}$$

To this end, set $G(\xi, k) = e\left(xg(k) - k\xi\right)$ and

$$f_i(\xi) = \frac{1}{q^i} \cdot \left| \sum_{k=0}^{q^i-1} G(\xi, k) \right|. \tag{A.48}$$

It is clear that the functions $f_i, i \geq 1$ are all continuous and, furthermore, it follows from assumption (A.29) that the sequence tends pointwise to the zero function. Showing that this convergence is uniform in $\xi$ is equivalent to showing that equality (A.47) holds. In view of Lemma A.4.4, it is enough to prove that for every $\xi \in \mathbb{T}$, it holds that $|f_{i+1}(\xi)| \leq |f_i(\xi)|$. To establish this, note that for a fixed $i \in \mathbb{N}$,

$$f_{i+1}(\xi) = \frac{1}{q^{i+1}} \cdot \left| \sum_{b=0}^{q-1} \sum_{u=0}^{q^i-1} G\left(\xi, u + bq^i\right) \right| = \frac{q^i}{q^{i+1}} \cdot \left| \sum_{b=0}^{q-1} G\left(\xi, b \cdot q^i\right) \cdot f_i\left(\xi\right) \right|, \tag{A.49}$$

where the last equality follows from the assumption that $g$ is $q$-additive. Applying the Triangle Inequality to the right-hand side of (A.49) yields that

$$|f_{i+1}(\xi)| \leq |f_i(\xi)|.$$

Therefore, equality (A.47) indeed holds.

**Step 2:** In this step, we conclude that

$$\sup_{m\in\mathbb{N}_0} \sup_{\xi\in\mathbb{T}} \left| \sum_{k=1}^{N} e\left(x \cdot g(k+m) - k\xi\right) \right| = o(N). \tag{A.50}$$

Given $i, N \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$ set

$$\psi_i \quad = \quad \sup_{\xi \in \mathbb{T}} \quad \frac{1}{q^i} \cdot \left| \sum_{k=0}^{q^i - 1} G(\xi, k) \right|$$

and

$$F_N(m, \xi) = \left| \sum_{k=0}^{N-1} e\left(xg\left(k+m\right) - k\xi\right) \right|. \tag{A.51}$$

From equality (A.47), it follows that

$$\lim_{i \to +\infty} \psi_i \quad = \quad 0 \quad \text{and for every } i \in \mathbb{N}, \quad F_{q^i}(0, \xi) \quad \leq \quad \psi_i \cdot q^i. \tag{A.52}$$

The goal is to prove that

$$\lim_{N \to +\infty} \quad \frac{1}{N} \cdot \sup_{m \in \mathbb{N}_0} \quad \sup_{\xi \in \mathbb{T}} \quad F_N(m, \xi) \quad = \quad 0. \tag{A.53}$$

Since the function $\boldsymbol{g}$ is $q$-additive, given $t, m, n \in \mathbb{N}$ such that $q^t | m$ and $n \leq q^t - 1$, it holds that $g(m+n) = g(m) + g(n)$. Therefore, for every $N \leq q^t$ and $\xi \in \mathbb{T}$, one has that

$$F_N(m, \xi) \quad = \quad F_N(0, \xi). \tag{A.54}$$

To complete the proof, fix $\epsilon \in (0,1)$, $m \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$. From the left-hand relation in (A.52), one can choose $i = i(\epsilon) \in \mathbb{N}$ such that $\psi_i \leq (\epsilon/2)$. It is enough to show that for every $k \cdot q^i \leq N \ < (k+1) \cdot q^i$ with $k \in \mathbb{N}, k \geq 2$,

$$F_N(m, \xi) \quad \leq \quad 2 \cdot q^i \quad + \quad k \cdot \psi_i q^i. \tag{A.55}$$

Indeed, upon choosing $k \geq (4/\epsilon) + 4$ (in particular, $N \geq ((4/\epsilon) + 4) \cdot q^i$), equality (A.55) yields that

$$\frac{1}{N} \cdot F_N(m, \xi) \quad \leq \quad \epsilon.$$

The choice of $m$ and $\xi$ is arbitrary, whence equality (A.53).

To prove inequality (A.55), fix $N \in [\![k \cdot q^i, (k+1) \cdot q^i]\!]$ for some $k \geq 2$ and define the number $M \in [\![0, q^i - 1]\!]$ to be such that $q^i | (m + M)$. Set $J = \left\lfloor \frac{N-M}{q^i} \right\rfloor - 1$ and, for every $j \in [\![0, J+1]\!]$, set further $M_j = m + M + jq^i$. Partition the interval

$[\![m, m+N-1]\!]$ into the following subintervals:

$$I_j = \begin{cases} [\![m, M_0 - 1]\!] & \text{if } j = -1, \\ [\![M_j, M_{j+1} - 1]\!] & \text{if } j \in [\![0, J]\!], \\ [\![M_{J+1}, m+N-1]\!] & \text{if } j = J+1. \end{cases}$$

By splitting the sum in the definition of $F_N(m,\xi)$ (see equation (A.51)) according to the intervals $(I_j)_{j\in[\![-1,J+1]\!]}$, and by applying the Triangle Inequality, one obtains that

$$\begin{aligned} F_N(m,\xi) &\leq F_M(m,\xi) + \sum_{j=0}^{J} F_{q^j}(M_j,\xi) + F_{(m+N-M_{J+1})}(M_{J+1},\xi) \\ &\underset{\substack{(A.54) \\ q^i \mid M_j}}{=} F_M(m,\xi) + \sum_{j=0}^{J} F_{q^j}(0,\xi) + F_{(m+N-M_{J+1})}(M_{J+1},\xi) \\ &\underset{(A.52)}{\leq} q^i + k \cdot \psi_i q^i + q^i, \end{aligned}$$

where in the last inequality follows from the facts that $J \leq k+1$, $\psi_i \leq (\epsilon/2)$, $M \leq q^i - 1$ and $m + N - M_{J+1} < q^i$. Therefore, inequality (A.55) is proved and the result follows. ∎

Proof (Lemma A.4.2) Let $q$ be a given natural number and let $G : \mathbb{N}_0 \to \mathbb{C}$ be a function with $G(0) = 1$, $|G(k)| \leq 1$, which satisfies assumptions (A.30) and (A.31) for some continuous non-decreasing function $f : [1, +\infty) \to (0, +\infty)$ such that $f(x) \leq x$.

Fix $N \in \mathbb{N}$ and let $M \in \mathbb{N}_0$ be the largest index $n \in \mathbb{N}_0$ such that $d_n \neq 0$ with $d_n = d_n(q,N)$. Given $j \in [\![0, M]\!]$, set

$$N(j) = \sum_{n=j}^{M} d_n(q,N)\, q^n.$$

Since the sequence $(N(j))_{j=0}^{M}$ is decreasing, one can rewrite

$$\sum_{k=0}^{N-1} G(k) \quad = \quad \sum_{k=0}^{N(M)-1} G(k) \quad + \quad \sum_{j=0}^{m-1} \sum_{k=N(j+1)}^{N(j)-1} G(k). \tag{A.56}$$

Furthermore, it holds that

$$\sum_{k=0}^{N(M)-1} G(k) \quad = \quad \sum_{l=0}^{d_M-1} \quad \sum_{k=lq^M}^{(l+1)q^M-1} G(k) \quad \underset{(A.30)}{=} \quad \sum_{l=0}^{d_M-1} G\left(lq^M\right) \cdot \sum_{k=0}^{q^M-1} G(k)$$

and

$$\begin{aligned} \sum_{k=N(j+1)}^{N(j)-1} G(k) \quad &= \quad G\left(N(j+1)\right) \cdot \sum_{k=0}^{d_j q^j - 1} G(k) \\ &\underset{(A.30)}{=} \quad G\left(N(j+1)\right) \cdot \sum_{l=0}^{d_j-1} G\left(lq^j\right) \cdot \sum_{k=0}^{q^j-1} G(k). \end{aligned}$$

Substituting these two formulae in equality (A.56) yields

$$\begin{aligned} \left| \sum_{k=0}^{N-1} G(k) \right| \quad &\underset{||G||_\infty \leq 1}{\leq} \quad \sum_{j=0}^{M} \left| \sum_{l=0}^{d_j-1} G\left(lq^j\right) \right| \cdot \left| \sum_{k=0}^{q^j-1} G(k) \right| \\ &\underset{||G||_\infty \leq 1}{\leq} \quad \sum_{j=0}^{M} d_j q^j \cdot \frac{1}{q^j} \left| \sum_{k=0}^{q^j-1} G(k) \right| \\ &\underset{(A.31)}{\leq} \quad \sum_{j=0}^{r-1} d_j q^j \quad + \quad \sum_{j=r}^{M} d_j q^j \frac{1}{f\left(q^r\right)} \\ &\leq \quad q^r + \frac{N}{f\left(q^r\right)} \end{aligned}$$

for an arbitrary $r \in \mathbb{N}$, where $||G||_\infty$ stands for the supremum-norm of $G$. In other words, given any $r \in \mathbb{N}$

$$\left| \sum_{k=0}^{N-1} G(k) \right| \quad \leq \quad q^r + \frac{N}{f\left(q^r\right)}. \tag{A.57}$$

Since $f$ is non-decreasing, there exists a unique real number $t \in [1, +\infty)$ such that

$$\frac{t}{q} \cdot f\left(\frac{t}{q}\right) = N.$$

Since $f(t/q) \leq (t/q)$ it follows that $(t/q) \geq \sqrt{N}$. By choosing $r$ such that $t \in [\![q^r, q^{r+1}-1]\!]$, inequality (A.57) becomes

$$\begin{aligned}
\left|\sum_{k=0}^{N-1} G(k)\right| \quad &\leq \quad q^r + \frac{N}{f\left(q^r\right)} \\
&\underset{(q^r \leq t < q^{r+1})}{\leq} \quad t + \frac{N}{f\left(\frac{t}{q}\right)} \\
&= \quad \frac{qN}{f\left(\frac{t}{q}\right)} + \frac{N}{f\left(\frac{t}{q}\right)} \\
&\underset{(\sqrt{N} \leq \frac{t}{q})}{\leq} \quad (q+1) \cdot \frac{N}{f\left(\sqrt{N}\right)}.
\end{aligned}$$

The proof is complete. ■

### A.4.2 Proof of Theorem A.1.6

The proof of Theorem A.1.6 will rely on the following proposition. The first part shows that, if the function $f \in C^1(\mathbb{R}^+)$ is such that its first derivative converges to a real number $\lambda$ as $x \to +\infty$, then the sequence $\boldsymbol{f} = (f(k))_{k\in\mathbb{N}}$ tends to behave like the sequence $(\lambda \cdot k)_{k\in\mathbb{N}}$ (in a suitable sense). The second part shows, that if $f$ is two times differentiable with its second derivative tending to zero as $x \to +\infty$, then given $N \in \mathbb{N}$ and $m \gg_N 1$, the terms $(f(k))_{k=m+1}^{m+N}$ behave like the terms of the sequence $(f'(m) \cdot k)_{k\in\mathbb{N}}$ (in a suitable sense). In both cases, this is enough to prevent the super-uniform equidistribution property of $\boldsymbol{f}$ from happening. The proof of the following proposition is given at the end of this section, after the proof of Theorem A.1.6.

**Proposition A.4.5** *Let $f : (0, +\infty) \to \mathbb{R}$ be a real function.*

1. *Assume that $f \in \mathcal{C}^1(\mathbb{R}^+)$ with $f'(x) > 0$ for $x \in \mathbb{R}^+$ and $\lim_{x\to+\infty} f'(x) = \lambda$ for some $\lambda \in \mathbb{R}$. Given $N \in \mathbb{N}$, there exist $m_N \in \mathbb{N}$ and $\epsilon_N \in (0,1)$ such that*

*for any $m \geq m_N$ and $\xi \in \mathbb{T}$ with $||\lambda - \xi|| \leq \epsilon_N$, it holds that*

$$\left|\sum_{k=m+1}^{m+N} e\left(f(k) - \xi k\right)\right| \quad \geq \quad N-1. \tag{A.58}$$

2. *Assume that $f \in \mathcal{C}^2\left(\mathbb{R}^+\right)$ with $\lim_{x\to+\infty} f''(x) = 0$. Given $N \in \mathbb{N}$, there exist $m_N \in \mathbb{N}$ and $\epsilon_N \in (0,1)$ such that for any $m \geq m_N$ and $\xi \in \mathbb{T}$ with $||\xi - f'(m)|| \leq \epsilon_N$, it holds that*

$$\left|\sum_{k=m+1}^{m+N} e\left(f(k) - k\xi\right)\right| \quad \geq \quad N-1. \tag{A.59}$$

Assuming Proposition A.4.5, one obtains the following:

Proof (Theorem A.1.6) Let $f : (0, +\infty) \to \mathbb{R}$ be a real function.

**1.** Assume that the function $f$ satisfies the assumptions of the first part in the statement of Theorem A.1.6. The goal is to show that the sequence $\boldsymbol{f} = \left(f(k)\right)_{k\in\mathbb{N}}$ is not super-uniformly equidistributed. This will be done by applying Theorem A.1.2 upon proving that $S_1(N)$ (defined in (A.4)) is greater than $1/2$ for any $N \geq 2$. To see this, fix $N \geq 2$. From the first part of Proposition A.4.5, there exists $m = m(N) \in \mathbb{N}$ and $\xi_N \in \mathbb{R}$ such that

$$\left|\sum_{k=m+1}^{m+N} e\left(f(k) - k\xi_N\right)\right| \quad \geq \quad N-1. \tag{A.60}$$

In turn, one has that

$$\begin{aligned} S_1(N) \quad &\geq \quad \frac{1}{N} \cdot \left|\sum_{k=m+1}^{m+N} e\left(f(k) - k\xi_N\right)\right| \\ &\geq \quad \frac{N-1}{N} \quad \geq \quad \frac{1}{2}. \end{aligned}$$

The claim follows.

**2.** The proof is the same as in the first part. ■

Proof (Proposition A.4.5) Let $f : (0,1) \to \mathbb{R}$ be a real function.

**1.** Assume that $f$ satisfies the assumptions of the first part of Proposition A.4.5. Given $N \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\xi \in \mathbb{R}$ and a finite sequence $\boldsymbol{\delta} = (\delta_k)_{k=1}^N$ in $(-1,1)$, define

$$S(N,m,\xi) \quad = \quad \sum_{k=m+1}^{m+N} e\left(f(k) - \xi k\right) \quad \text{and} \quad s(\boldsymbol{\delta}, N, m, \xi) = \sum_{k=m+1}^{m+N} e\left((\lambda + \delta_k)k - \xi k\right). \tag{A.61}$$

Given $N \in \mathbb{N}$, the goal is to show that $|S(N,m,\xi)| \geq N - 1$ when $m$ is sufficiently large and $\xi$ sufficiently close to $\lambda$. Fix $N \in \mathbb{N}$ and set

$$\delta = \frac{1}{4\pi N^3}. \tag{A.62}$$

Since for any $x, y \in \mathbb{T}$, $|e(x) - e(y)| \leq 2\pi \cdot ||x - y||$, it is easy to check that for any finite sequence $\boldsymbol{\delta} = (\delta_k)_{k\in\mathbb{N}}$ with $|\delta_k| \leq \delta$, it holds that

$$|s(\mathbf{0}, N, m, \xi)| - \frac{1}{2} \quad \leq \quad |s(\boldsymbol{\delta}, N, m, \xi)| \quad \leq \quad |s(\mathbf{0}, N, m, \xi)| + \frac{1}{2} \tag{A.63}$$

for all $m \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$, where $\mathbf{0} = (0)_{k=1}^N$ is the zero sequence.

By assumption, $\lim_{x\to+\infty} f'(x) = \lambda$ for some $\lambda \in \mathbb{R}$. Thus, there exists $m_N \in \mathbb{N}$ such that for any $x \geq m_N$,

$$|f'(x) - \lambda| \quad \leq \quad \frac{\delta}{N}. \tag{A.64}$$

Fix $m \geq m_N$. Integrating both sides of inequality (A.64) in the interval $[m, m+x]$ with $0 \leq x \leq N$ yields that

$$f(x) \quad = \quad \lambda x + \delta_x \cdot x + c \tag{A.65}$$

where $|\delta_x| \leq \delta$ for any $x \in [m, m+N]$ and $c \in \mathbb{R}$ is a constant. Substituting equality (A.65) in inequality (A.63) implies that for any $\xi \in \mathbb{T}$,

$$|s(\mathbf{0}, N, m, \xi)| - \frac{1}{2} \quad \leq \quad |S(N, m, \xi)| \quad \leq \quad |s(\mathbf{0}, N, m, \xi)| + \frac{1}{2}. \tag{A.66}$$

A trivial estimation yields that $s(\mathbf{0}, N, m, \lambda) = N$. From the continuity of the function $s(\mathbf{0}, N, m, \xi)$ in the variable $\xi \in \mathbb{T}$, it follows that there exists $\epsilon_N > 0$ such that for any $\xi \in \mathbb{T}$ with $||\lambda - \xi|| \leq \epsilon_N$, it holds that

$$N - \frac{1}{2} \quad \leq \quad |s(\mathbf{0}, N, m, \xi)| \quad \leq \quad N + \frac{1}{2}. \tag{A.67}$$

Substituting inequality (A.67) in inequality (A.66) yields that

$$|S(N, m, \xi)| \quad \geq \quad N - 1.$$

The claim is proved.

**2.** Assume that $f$ satisfies the assumptions of the second part of Proposition A.4.5. Given $N \in \mathbb{N}$, $m \in \mathbb{N}_0$, $\xi \in \mathbb{R}$ and a finite sequence $\boldsymbol{\delta} = (\delta_k)_{k=1}^{N}$ in $(-1, 1)$, define

$$\sigma(\boldsymbol{\delta}, N, m, \xi) \quad = \quad \sum_{k=m+1}^{m+N} e\left((f'(m) + \delta_k)k - \xi k\right) \tag{A.68}$$

and let $S(N, m, \xi)$ be as in (A.61). Given $N \in \mathbb{N}$ and $m \gg_N 1$ sufficiently large, the goal is to show that $|S(N, m, \xi)| \geq N - 1$ when $\xi$ is sufficiently close to $f'(m)$.

Fix $N \in \mathbb{N}$ and set $\delta$ as in (A.62). Repeating the argument used in the proof of inequality (A.63), one has that for any sequence $(\delta_k)_{k \in \mathbb{N}}$ with $|\delta_k| \leq \delta$,

$$|\sigma(\mathbf{0}, N, m, \xi)| - \frac{1}{2} \quad \leq \quad |\sigma(\boldsymbol{\delta}, N, m, \xi)| \quad \leq \quad |\sigma(\mathbf{0}, N, m, \xi)| + \frac{1}{2} \tag{A.69}$$

for all $m \in \mathbb{N}_0$ and $\xi \in \mathbb{T}$, where $\mathbf{0} = (0)_{k=1}^{N}$ is the zero sequence.

By assumption, $\lim_{x \to +\infty} f''(x) = 0$. Thus, there exists $m_N \in \mathbb{N}$ such that for any $x \geq m_N$,

$$|f''(x)| \quad \leq \quad \frac{\delta}{N^2}. \tag{A.70}$$

Fix $m \geq m_N$. By integrating both sides of inequality (A.70) in the interval

$[m, m+x]$, where $0 \leq x \leq N$, one obtains that

$$|f'(x) - \lambda| \quad \leq \quad \frac{\delta}{N}. \tag{A.71}$$

Integrating inequality (A.71) once again in the interval $[m, m+x]$ yields

$$f(x) \quad = \quad f'(m)x + \delta_x x + c_m \tag{A.72}$$

where $|\delta_x| \leq \delta$ for any $x \in [m, m+N]$ and $c_m \in \mathbb{R}$ is a constant depending only on the choice of $m$. Substituting equality (A.72) in inequality (A.69) implies that for any $\xi \in \mathbb{T}$

$$|\sigma(\mathbf{0}, N, m, \xi)| - \frac{1}{2} \quad \leq \quad |S(N, m, \xi)| \quad \leq \quad |\sigma(\mathbf{0}, N, m, \xi)| + \frac{1}{2}. \tag{A.73}$$

A trivial estimation shows that $\sigma\left(\mathbf{0}, N, m, f'(m)\right) = N$. From the continuity of the function $\sigma\left(\mathbf{0}, N, m, \xi\right)$ in the variable $\xi \in \mathbb{T}$, there exists $\epsilon_N > 0$ such that for any $\xi \in \mathbb{T}$ with $|f'(m) - \xi| \leq \epsilon_N$,

$$N - \frac{1}{2} \quad \leq \quad |\sigma\left(\mathbf{0}, N, m, \xi\right)| \quad \leq \quad N + \frac{1}{2}. \tag{A.74}$$

Finally, substituting inequality (A.74) in inequality (A.73) yields

$$|S(N, m, \xi)| \quad \geq \quad N - 1.$$

The proof is complete. ■

# Appendix B

# Effective Dispersion in the Torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$

## B.1 Introduction

The main conclusions in Chapters 4 and 5 follow from the existence of optimally distributed sequences (a concept defined in Definition 4.1.1, Equation (4.9), p.119) in the $d$-dimensional unit sphere $\mathbb{S}^d$. This allows for the construction of spiral Delone sets in any dimension (Theorem 4.1.5, p.120) which furthermore enjoy optimal visibility properties (Theorem 5.1.2, p.139) such as being (uniform) orchards (cf. Definition 5.1.1, p.138) or having an empty set of visible points (cf. Definition 1.1.15, p.32). The construction of an optimally distributed sequence $\boldsymbol{U}$ in $\mathbb{S}^d$ has been achieved by lifting to the sphere, through a gap-preserving map (Definition 4.1.6, p.120), an optimally distributed sequence $\boldsymbol{V}$ in $\mathbb{T}^d$.

The goal of this appendix is to prove in detail the existence of optimally distributed sequences in the $d$-dimensional torus $\mathbb{T}^d$. In particular, the main result shows that such sequences are given by the multiples of a badly approximable vector $\boldsymbol{v} \in \mathbb{T}^d$ (see Definition 1.1.44, p.52). This result is the key to the proof of Theorem 4.1.5 (Chapter 4, p.120) as indicated therein.

**Theorem B.1.1** *Let $\boldsymbol{v} \in \mathbb{T}^d$ be a badly approximable vector. Then, the sequence*

$\boldsymbol{V} = (k \cdot \boldsymbol{v})_{k \in \mathbb{N}}$ *is optimally distributed; that is,*

$$0 \quad < \quad \mathtt{U}_P(\boldsymbol{V}) \quad \leq \quad \mathtt{U}_C(\boldsymbol{V}) \quad < \quad +\infty,$$

*where the quantities* $\mathtt{U}_P(\boldsymbol{V})$ *and* $\mathtt{U}_C(\boldsymbol{V})$ *are the uniform packing and the uniform covering parameters of* $\boldsymbol{V}$*, respectively, as defined in Definition 4.1.2, p.118.*

In particular, given a badly approximable vector $\boldsymbol{v} \in \mathbb{T}^d$, Theorem B.1.1 implies that for any $N \in \mathbb{N}$,

$$\frac{c_{\boldsymbol{v}}}{N^{\frac{1}{d}}} \quad \leq \quad \mathtt{R}_P\left(\{k \cdot \boldsymbol{v}\}_{k=1}^N\right) \quad \leq \quad \mathtt{R}_C\left(\{k \cdot \boldsymbol{v}\}_{k=1}^N\right) \quad \leq \quad \frac{C_{\boldsymbol{v}}}{N^{\frac{1}{d}}},$$

where the quantities $\mathtt{R}_P\left(\{k \cdot \boldsymbol{v}\}_{k=1}^N\right)$ and $\mathtt{R}_C\left(\{k \cdot \boldsymbol{v}\}_{k=1}^N\right)$ stand for the packing and covering radii (see Definition 4.1.1, p.118) of the first $N$ terms of $\boldsymbol{V}$, with the constants $c_{\boldsymbol{v}}, C_{\boldsymbol{v}} > 0$ depending only on the choice of $\boldsymbol{v}$.

Given $\epsilon \in (0,1)$, the following result, which is a version stronger than Proposition 1.1.51 (p.58) when specialised to the case $d = 1$, characterises (in the sense stated in the theorem below) those real numbers $\alpha \in \mathbb{R}$ for which the finite sequence $(k \cdot \alpha)_{k=1}^N$ is not $\epsilon$-dense in $\mathbb{T}$.

**Theorem B.1.2** *Fix a positive real number* $\epsilon \in (0,1)$*. Given an integer* $N \in \mathbb{N}$ *with* $N \geq \epsilon^{-1}$*, define the sets*

$$C(\epsilon, N) := \left\{\xi \in \mathbb{T} : (k \cdot \xi)_{k=1}^N \text{ is not } \epsilon\text{-dense in } \mathbb{T}\right\} \subseteq \mathbb{T} \tag{B.1}$$

*and*

$$S(\epsilon, N) := \bigcup_{\substack{1 \leq q < \epsilon^{-1}, \\ \gcd(p,q)=1}} B\left(\frac{p}{q}, \frac{1}{qN}\right) \subseteq \mathbb{T}, \tag{B.2}$$

*where the balls are taken modulo one. Then, the following two statements hold:*

1. *For any* $N \in \mathbb{N}$ *with* $N \geq \epsilon^{-1}$*,*

$$C(\epsilon, N) \quad \subseteq \quad S(\epsilon, N). \tag{B.3}$$

*2. For any* $\eta \in (0, 1/2)$ *and any* $N > \frac{2}{\eta(1-2\eta)} \cdot \epsilon^{-1} + \frac{1}{\eta}$,

$$S(\epsilon, N) \quad \subseteq \quad C\left(\left(\frac{1-2\eta}{2}\right) \cdot \epsilon, \lfloor \eta N \rfloor\right). \tag{B.4}$$

*In particular, for* $\eta = 1/4$ *and for any* $N > 16\epsilon^{-1} + 4$,

$$S(\epsilon, N) \quad \subseteq \quad C\left(\frac{\epsilon}{4}, \left\lfloor \frac{N}{4} \right\rfloor\right). \tag{B.5}$$

The proofs of Theorems B.1.1 and B.1.2 are given in Sections B.2 and B.3, respectively.

## B.2 Dispersion of the Multiples of a Badly Approximable Vector Modulo One

Given a subset $A \subseteq \mathbb{T}^d$, recall the definitions of the packing radius $\mathtt{R}_P(A)$ and of the covering radius $\mathtt{R}_C(A)$ of $A$ (Definition 4.1.1, p.118).

Proof (Theorem B.1.1) Let $\boldsymbol{v} = (v_1, ..., v_d) \in \mathbb{T}^d$ be a badly approximable vector. The quantities $c_S(\boldsymbol{v})$ (as defined in (1.54), p.53) and $c_L(\boldsymbol{v})$ (as defined in (1.56), p.53) are thus both positive. Denote by $\boldsymbol{V} = (k \cdot \boldsymbol{v})_{k \in \mathbb{N}}$ the sequence of multiples of $\boldsymbol{v}$.

The goal is to show that for any $m \in \mathbb{N}_0$ and $N \in \mathbb{N}$, it holds that

$$\mathtt{R}_P\left(\{k \cdot \boldsymbol{v}\}_{k=m+1}^{m+N}\right) \quad \gg \quad \frac{1}{N^{\frac{1}{d}}} \quad \text{and} \quad \mathtt{R}_C\left(\{k \cdot \boldsymbol{v}\}_{k=m+1}^{m+N}\right) \quad \ll \quad \frac{1}{N^{\frac{1}{d}}}. \tag{B.6}$$

Clearly, for any subset $A \subseteq \mathbb{T}^d$ and any $\boldsymbol{x} \in \mathbb{T}^d$, one has that

$$\mathtt{R}_P(A) \quad = \quad \mathtt{R}_P(\boldsymbol{x} + A) \quad \text{and} \quad \mathtt{R}_C(A) \quad = \quad \mathtt{R}_C(\boldsymbol{x} + A).$$

Therefore, it is enough to prove inequality (B.6) in the case $m = 0$; that is, to prove that

$$\mathtt{R}_P\left(\{k \cdot \boldsymbol{v}\}_{k=1}^N\right) \quad \gg \quad \frac{1}{N^{\frac{1}{d}}} \quad \text{and} \quad \mathtt{R}_C\left(\{k \cdot \boldsymbol{v}\}_{k=1}^N\right) \quad \ll \quad \frac{1}{N^{\frac{1}{d}}}\cdot \tag{B.7}$$

**Proof of the Bound on the Uniform Packing Parameter $\mathtt{U}_P(\boldsymbol{V})$ of $\boldsymbol{V}$:** Fix $N \in \mathbb{N}$ and $k, l \in [\![N]\!]$ with $k < l$. Then, one has that

$$||l \cdot \boldsymbol{v} - k \cdot \boldsymbol{v}|| \quad = \quad ||(l-k) \cdot \boldsymbol{a}|| \quad \underset{(1.54)}{\geq} \quad \frac{c_S(\boldsymbol{v})}{(l-k)^{\frac{1}{d}}} \quad \geq \quad \frac{c_S(\boldsymbol{v})}{N^{\frac{1}{d}}}\cdot$$

Since $c_S(\boldsymbol{v}) > 0$, the first inequality in (B.7) is established.

**Proof of the Bound on the Uniform Covering Parameter $\mathtt{U}_C(\boldsymbol{V})$ of $\boldsymbol{V}$:** Set

$$\beta := \quad \sup_{n \in \mathbb{N}} \frac{\mathfrak{q}_{n+1}}{\mathfrak{q}_n} \quad \text{and} \quad \gamma := \quad \sup_{n \in \mathbb{N}} \frac{M_{n+1}}{M_n},$$

where the sequences $(\mathfrak{q}_n)_{n \in \mathbb{N}}$ and $(M_n)_{n \in \mathbb{N}}$ are defined in (1.57), p.53, and (1.60), p.54, respectively. Since the sequences $(\mathfrak{q}_n)_{n \in \mathbb{N}}$ and $(M_n)_{n \in \mathbb{N}}$ are increasing, from Theorem 1.1.46 [Chapter 1, p.55], one has that

$$1 \quad \leq \quad \beta, \gamma \quad < \quad +\infty$$

Fix $\epsilon \in (0,1)$. From the way that the quantities $\beta, \gamma$ are defined, there exist $n, l \in \mathbb{N}$ such that

$$\begin{gathered} d \cdot \epsilon^{-1} \quad < \quad M_l \quad \leq \quad \gamma d \cdot \epsilon^{-1} \\ \text{and} \\ \left(\frac{2C_d d}{c_L(\boldsymbol{v})}\right)^{\frac{d}{d+1}} \cdot M_l^d \quad \leq \quad \mathfrak{q}_n \quad \leq \quad \beta \cdot \left(\frac{2C_d d}{c_L(\boldsymbol{v})}\right)^{\frac{d}{d+1}} \cdot M_l^d, \end{gathered} \tag{B.8}$$

where the constant $C_d > 0$ is the one provided by Theorem 1.1.43 (p.52) and depends only on the choice of $d$.

From the definition of a badly approximable vector and from the Perron-Khintchin transference principle (Theorem 1.1.45, p.53), one has that for any integer vector $\boldsymbol{u} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ with $1 \leq ||\boldsymbol{u}||_\infty \leq +\infty$,

$$||\boldsymbol{v} \cdot \boldsymbol{u}|| \quad \geq \quad \frac{c_L(\boldsymbol{v})}{M_l^d} \quad > \quad 0. \tag{B.9}$$

Also, by applying Theorem 1.1.43 (p.52) with $N = q_n$, one obtains

$$\left\| v_i - \frac{p_i^{(n)}}{\mathsf{q}_n} \right\| \quad \leq \quad \frac{C_d}{\mathsf{q}_n^{1+\frac{1}{d}}}, \quad \text{for all } i \in \{1, 2, ..., d\}. \tag{B.10}$$

Here, the integer vector $\boldsymbol{p}_n = \left(p_1^{(n)}, ..., p_d^{(n)}\right)$ is given by the definition of the best approximation vector for simultaneous approximation (1.57), p.53. Combining these two inequalities yields that for any $\boldsymbol{u} \in \mathbb{Z}^d \backslash \{\mathbf{0}\}$ with $1 \leq ||\boldsymbol{u}||_\infty \leq M_l$,

$$\begin{aligned}
\frac{c_L(\boldsymbol{v})}{M_l^d} \quad & \underset{(\text{B.9})}{\leq} \quad ||\boldsymbol{v} \cdot \boldsymbol{u}|| \\
& \leq \quad \left\| \frac{\boldsymbol{p}_n}{\mathsf{q}_n} \cdot \boldsymbol{u} \right\| + \left\| \left( \boldsymbol{v} - \frac{\boldsymbol{p}_n}{\mathsf{q}_n} \right) \cdot \boldsymbol{u} \right\| \\
& \underset{\substack{(\text{B.10}),\\ ||\boldsymbol{u}||_\infty \leq M_l}}{\leq} \quad \left\| \frac{\boldsymbol{p}_n}{\mathsf{q}_n} \cdot \boldsymbol{u} \right\| + \frac{Cd \cdot M_l}{\mathsf{q}_n^{1+\frac{1}{d}}} \\
& \underset{(\text{B.8})}{\leq} \quad \left\| \frac{\boldsymbol{p}_n}{\mathsf{q}_n} \cdot \boldsymbol{u} \right\| + \frac{c_L(\boldsymbol{v})}{2M_l^d}.
\end{aligned}$$

Thus, for any integer vector $\boldsymbol{u}$ with $1 \leq ||\boldsymbol{u}||_\infty \leq M_l$,

$$\left\| \frac{\boldsymbol{p}_n}{\mathsf{q}_n} \cdot \boldsymbol{u} \right\| \quad \geq \quad \frac{c_L(\boldsymbol{v})}{2 \cdot M_l^d}. \tag{B.11}$$

Let $\Lambda(\boldsymbol{p}_n, \mathsf{q}_n)$ be the lattice defined in (1.79), p.68, and let $\Lambda^*(\boldsymbol{p}_n, \mathsf{q}_n)$ be its dual lattice given by relation (1.80), p.69. One infers from inequality (B.11) that

$$\lambda_1 (\Lambda^* (\boldsymbol{p}_n, \mathsf{q}_n)) \quad > \quad M_l \quad \underset{(\text{B.8})}{>} \quad d \cdot \epsilon^{-1}, \tag{B.12}$$

where $\lambda_1 (\Lambda^* (\boldsymbol{p}_n, \mathsf{q}_n))$ is the first successive minimum of $\Lambda^* (\boldsymbol{p}_n, \mathsf{q}_n)$ (Definition 1.1.64, p.67).

From Lemma 1.1.66, p.69, and inequality (B.12), one obtains that the finite sequence $\left(k \cdot \frac{\boldsymbol{p}_n}{\mathsf{q}_n}\right)_{k=1}^{\mathsf{q}_n}$ is $c_d \cdot \epsilon$-dense in $\mathbb{T}^d$, where the existence of such a constant $c_d > 0$ is guaranteed by the same lemma. Moreover, for any $k \in [\![\mathsf{q}_n]\!]$ and $j \in \{1, ..., d\}$, it holds that

$$\begin{aligned} \left\| k \cdot v_j - k \cdot \frac{p_j^{(n)}}{\mathsf{q}_n} \right\| &\leq k \cdot \left\| v_j - \frac{p_j^{(n)}}{\mathsf{q}_n} \right\| \\ &\underset{(B.10)}{\leq} \frac{C_d \cdot k}{\mathsf{q}_n^{1+\frac{1}{d}}} \\ &\underset{k \leq \mathsf{q}_n}{\leq} \frac{C_d}{\mathsf{q}_n^{\frac{1}{d}}} \\ &\underset{(B.8)}{\leq} \kappa_1 \cdot \epsilon, \end{aligned}$$

where

$$\kappa_1 = \left(\frac{c_L(\boldsymbol{v})}{2C_d \cdot d}\right)^{\frac{1}{d+1}} \cdot \frac{1}{d}.$$

Therefore, it follows that the finite sequence $(k \cdot \boldsymbol{v})_{k=1}^{\mathsf{q}_n}$ is $(\kappa_2 \cdot \epsilon)$-dense, where $\kappa_2 = \kappa_1 + c_d$. From inequalities (B.8), one has that

$$\mathsf{q}_n \leq \kappa_3 \cdot \epsilon^{-d}$$

with

$$\kappa_3 = \beta \cdot \left(\frac{2C_d d}{c_L(\boldsymbol{v})}\right)^{\frac{d}{d+1}} \cdot (\gamma d)^d.$$

Thus, the sequence $(k \cdot \boldsymbol{v})_{k=1}^{\kappa_3 \cdot \epsilon^{-d}}$ is $(\kappa_2 \cdot \epsilon)$-dense. The choice of $\epsilon \in (0, 1)$ is arbitrary, therefore, the right-hand side of inequality (B.7) follows. The proof is complete. ■

## B.3 Distribution of the Multiples of Real Numbers Modulo One

Proof (Theorem B.1.2) From Dirichlet's theorem on Diophantine approximation (Theorem 1.1.33, p.43), one has that

$$\mathbb{T} \quad = \quad \bigcup_{\substack{1\le q\le N,\\ \gcd(p,q)=1}} B\left(\frac{p}{q}, \frac{1}{qN}\right),$$

where the balls are taken modulo one.

**Part 1:** Fix a real number $\epsilon \in (0,1)$ and a natural number $N \geq \epsilon^{-1}$. The goal is to prove that for any $\xi \in B\left(\frac{p}{q}, \frac{1}{qN}\right)$, with $\epsilon^{-1} \leq q \leq N$ and $\gcd(p,q) = 1$, the sequence $(k\cdot\xi)_{k=1}^{N}$ is $\epsilon$-dense in $\mathbb{T}$. To this end, fix such $p,q \in \mathbb{N}$, $\xi \in \mathbb{T}$ and $\theta \in (-1,1)$ so that the relation

$$\xi \quad = \quad \frac{p}{q} + \frac{\theta}{qN}$$

holds. Without loss of generality, assume that $\theta \geq 0$. The case $\theta < 0$ is proved in the same way. Since $\gcd(p,q) = 1$ and $q \geq \epsilon^{-1}$, it is clear that the sequence $(kp/q)_{k=1}^{N}$ is $\frac{\epsilon}{2}$-dense in $\mathbb{T}$. Note also that for any $m \in [\![N]\!]$,

$$0 \quad \leq \quad \left\{m\xi - m\frac{p}{q}\right\} \quad = \quad m\cdot\frac{\theta}{qN} \quad \leq \quad \epsilon. \tag{B.13}$$

Fix $x \in \mathbb{T}$. Since the sequence $\left(k\frac{p}{q}\right)_{k=1}^{N}$ is $\frac{\epsilon}{2}$-dense in $\mathbb{T}$, there exists $k \in [\![N]\!]$ such that $x \in B\left(k\cdot\frac{p}{q}, \frac{\epsilon}{2}\right)$. If

$$0 \quad \leq \quad \left\{x - k\cdot\frac{p}{q}\right\} \quad \leq \quad \frac{\epsilon}{2},$$

then one obtains from inequality (B.13) that $x \in B(k\xi, \epsilon)$. If

$$0 \quad \leq \quad \left\{k\cdot\frac{p}{q} - x\right\} \quad \leq \frac{\epsilon}{2},$$

then define $k' \in [\![N]\!]$ to be such that

$$\left\{k' \cdot \frac{p}{q}\right\} \quad = \quad \left\{\frac{kp-1}{q}\right\}.$$

Then, $x \in B\left(k' \cdot \frac{p}{q}, \epsilon\right)$ and, from inequality (B.13), one infers that $x \in B(k'\xi, \epsilon)$. The choice of $x \in \mathbb{T}$ is arbitrary, therefore, the sequence $(k\xi)_{k=1}^{N}$ is $\epsilon$-dense in $\mathbb{T}$.

**Part 2:** Fix $\eta \in \left(0, \frac{1}{2}\right)$, $\epsilon \in (0,1)$, $\xi \in S(\epsilon, N)$, $N \geq \frac{2}{\eta(1-2\eta)} \cdot \epsilon^{-1} + \frac{1}{\eta}$ and set

$$\epsilon_\eta := \quad \left(\frac{1-2\eta}{2}\right) \cdot \epsilon.$$

The goal is to show that

$$\xi \quad \in \quad C\left(\epsilon_\eta, \lfloor \eta N \rfloor\right);$$

that is, that the sequence $(k\xi)_{k=1}^{\lfloor \eta N \rfloor}$ is not $\epsilon_\eta$-dense in $\mathbb{T}$. Note that the lower bound on $N$ implies that the set $C\left(\epsilon_\eta, \lfloor \eta N \rfloor\right)$ is well-defined.

Since $\xi \in S(\epsilon, N)$, there exist $p, q \in \mathbb{N}$ with $\gcd(p,q) = 1$ and $q \leq \epsilon^{-1}$ such that

$$\xi \quad \in \quad B\left(\frac{p}{q}, \frac{1}{qN}\right).$$

Thus, one has that for any $1 \leq k \leq \lfloor \eta N \rfloor$,

$$k\xi \quad \in \quad B\left(k\frac{p}{q}, \frac{\eta}{q}\right).$$

In turn, this relation yields that the sequence $(k\xi)_{k=1}^{\lfloor \eta N \rfloor}$ is not $\epsilon_\eta$- dense in $\mathbb{T}$. Indeed, the interval between two consecutive balls of the form $B(kp/q, \eta/q)$ and $B((k+1)p/q, \eta/q)$ has length

$$\frac{1}{q} - \frac{2\eta}{q} \quad \underset{(q \leq \epsilon^{-1})}{\geq} \quad (1-2\eta)\epsilon.$$

The proof is complete. ■

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