Digraph Groups and Related Groups

A Thesis submitted for the degree of Doctor of Philosophy

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Abstract

This thesis investigates finite digraph groups and related groups like the generalization of Johnson and Mennicke groups. Cuno and Williams introduced the term "digraph group" for the first time in [9], 2020. The groups are defined by non-empty presentations and each relator is in the form $R(x, y)$, where x and y are distinct generators and $R(.,.)$ is defined by some fixed cyclically reduced word $R(a, b)$ that involves both a and b. There is a directed graph associated with each of these presentations, where the vertices correspond to the generators and the arcs correspond to the relators. In Chapter 2, we investigate Cayley digraph groups to determine whether they are finite cyclic and provide formulae to calculate the order. In Chapters 3 and 4, the girth of the underlying undirected graph is at least 4. We show that the resulting groups are non-trivial and cannot be finite of rank 3 or higher under the condition $|V| = |A| - 1$ in Chapter 3. We investigate when the corresponding digraph groups are finite cyclic for $|V| \leqslant |A|$ in Chapter 4 and we are able to show that the corresponding group of strongly connected and semi-connected digraphs under certain standard conditions which are known to be necessary for the digraph group to be finite ($(i) - (iv)$ defined in Preamble 4.1). We generalise Johnson and Mennicke groups, which are non-cyclic finite groups defined in terms of a digraph that is a directed triangle to digraphs that are $n-$vertex tournaments in Chapter 5. In Chapter 6 we use *GAP* to perform a computational investigation into digraph groups with particular relators and we obtain results whether the corresponding digraph groups are cyclic, abelian, perfect or not. We also provide their size, derived series, derived length and facts about isomorphism between them. The relators used correspond to the those used in the Mennicke and Johnson groups, and some new fixed relators. We obtain digraph presentations of various 2-groups, 3-groups and perfect groups.

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CHAPTER 1

Introduction

1.1 Preamble

This chapter covers the background material and some basic concepts that are relevant in this thesis. More specific definitions, theorems and lemmas will be provided in the chapters where they are first used. The ones related to the graph theory will be given in Section 1.2.1 and the ones related to group theory will be given in Section 1.2.2. We present digraph groups in Section 1.3 and the thesis outline will be given in Section 1.4. A variety of examples, figures, and findings should assist the reader in better understanding the concepts presented in the chapter.

1.2 Basic definitions and notations

1.2.1 Graph theory

We give the essential background relating to the graph theory that we will use throughout the thesis here. Some of the definitions, lemmas and theorems will play a significant role in later chapters; others will be used to illustrate definitions. The definitions used by Bondy and Murty's book [7] are followed, unless otherwise indicated.

Definition 1.2.1. A *graph* G is a set of vertices $V(G)$ connected by *edges* $E(G)$. We call $V(G)$ the vertex set and $E(G)$ the edge set of G. We will write $G = (V, E)$ which means that V and E are the vertex set and edge set of G, respectively.

The term graph will be used to mean undirected graph throughout the thesis for distinguishing from a directed graph.

Definition 1.2.2. Two vertices u and v in an undirected graph G are called *adjacent (or neighbours)* in G if u and v are *incident* with a common edge e of G. Such an edge e is said to connect u and v.

Definition 1.2.3. We use (u, v) to denote an arc from u to v. For an arc (u, v), the first vertex u is its *tail* and the second vertex v is its *head*. The head and tail of an arc are its end-vertices. If (u, v) is an arc, we also say that u dominates v (or v is dominated by u) and denote it by $u \to v$.

Definition 1.2.4. Two undirected graphs G and H are said to be *isomorphic* if there is a bijection, Φ say, from $V(G)$ to $V(H)$ such that $g \sim h$ in G if and only if $\Phi(g) \sim \Phi(h)$ in H. If G and H are isomorphic, then we write $G \simeq H$.

Example 1.2.5. The representation of a digraph by a picture, with points for the vertices and lines for the edges, is often convenient, attractive or interesting as in Figure 1.1 where these two undirected graphs in Figure 1.1 are isomorphic to each other.

For clarity, vertices are represented by small circles throughout the thesis.

We will focus on digraphs in this thesis. Therefore, unless otherwise specified, $G = (V, E)$ mentioned in the rest of this section and chapters are for directed graphs.

Definition 1.2.6. A *directed graph (or digraph)* is an undirected graph that is made up of a set of vertices connected by arcs, where the edges have a direction associated with them.

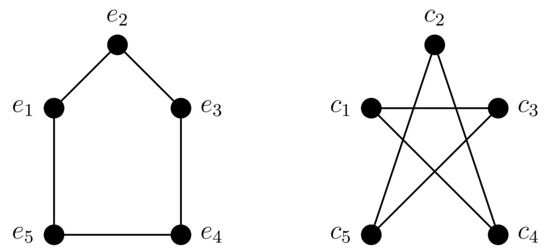

Figure 1.1: Two isomorphic undirected graphs

Definition 1.2.7. If the underlying undirected graphs of two directed graphs are both isomorphic and oriented in the same direction, then they are isomorphic to each other.

The directed digraphs at the top in Figure 1.2 are isomorphic to each other while the directed digraphs at the bottom in Figure 1.2 are non-isomorphic.

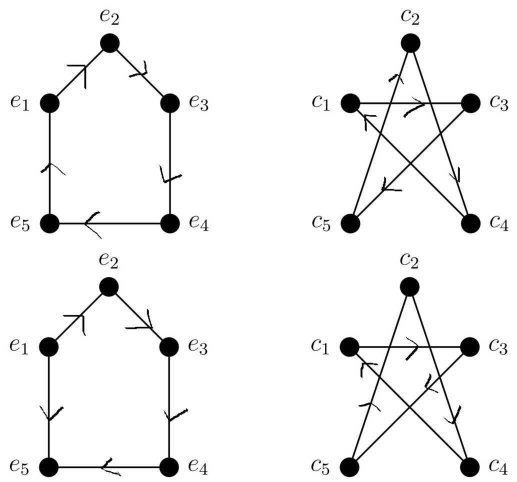

Figure 1.2: Isomorphic and non-isomorphic directed graphs

Definition 1.2.8. [17] For a directed graph, a vertex u is an *in-neighbor* of a vertex v if $(u, v) \in E$ and an out-neighbor if $(v, u) \in E$.

Definition 1.2.9. [17] Let $G = (V, E)$ and $v \in V$. The in-degree of v is denoted $deg^-(v)$ and its out-degree is denoted $deg^+(v)$. A vertex with

$deg^-(v) = 0$ is called *a source*, as it has vertices with positive out-degree and in-degree zero. Similarly, a vertex with $deg^+(v) = 0$ is called *a sink*, since it has vertices with positive in-degree and out-degree zero. Vertices whose in-degree and out-degree sum to one are called *leaves*.

We denote the number of sources as σ and the number of sinks as τ. The number of source leaves will be denoted σ_1 and the number of sink leaves as τ_1 throughout the thesis.

Definition 1.2.10. *A walk* consists of an alternating sequence of vertices and arcs consecutive elements of which are incident, that begins and ends with a vertex.

Definition 1.2.11. *A path* is a walk whose vertices are distinct. *A trail* is a walk without repeated arcs.

Definition 1.2.12. An undirected graph is said to be *connected* if any two of its vertices are joined by a path.

Definition 1.2.13. A digraph is *(weakly) connected* if its underlying graph is connected. Otherwise, it is disconnected. A digraph is *(semi) connected* if for any vertices u, v there is a $u - v$ path or a $v - u$ path. A digraph is *(strongly) connected or strong* if for any vertices u, v, there is a $u - v$ path and a $v - u$ path.

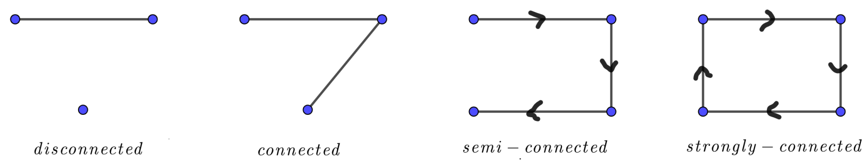

Figure 1.3: An example of disconnected, connected(weakly) graph, semi connected digraph and strongly connected digraph

Definition 1.2.14. A *Hamiltonian path* is a graph path between two vertices of a graph that visits each vertex exactly once.

Definition 1.2.15. A *Hamiltonian cycle* is a graph path that starts from one vertex and visits each vertex exactly once then returns to the original vertex.

Definition 1.2.16. An edge with identical ends is called a *loop*. A *cycle* is a path that begins and ends at the same vertex.

Definition 1.2.17. The *girth* of graph G is the length of the shortest cycle in the undirected graph G denoted by $g(G)$. A graph with $g(G) \geqslant 4$ is said to be triangle-free.

All digraphs considered in this thesis do not contain loops. (But may contain cycles of length three (Chapter 2, 5 and 6) or more (Chapter 2, 3 and 4)).

Definition 1.2.18. *A complete graph* is a graph in which each pair of graph vertices is connected by an edge. The complete graph on n vertices is denoted by K_n.

Definition 1.2.19. *A tournament* is a digraph obtained by assigning a direction for each edge in an undirected complete graph. That is, it is an orientation of a complete graph (see Figure 1.4 on page 13).

The following result concerns strong tournaments, as defined in Definition 1.2.13.

Lemma 1.2.20 ([8, Theorem 7.9])**.** *Let Γ be a non trivial-strong tournament. Then each vertex v in $V(\Gamma)$ is in some directed triangle.*

Definition 1.2.21. The *score vector* of a tournament is the ordered n-tuple $(s_1, s_2, \ldots, s_n)$, where s_i is is the out-degree of the i'th vertex. We usually assume that the vertices are labeled in such a way that $s_1 \leqslant s_2 \leqslant \ldots \leqslant s_n$.

For example, the score vector of tournament with 3 vertices at the top in Figure 1.4 is $(1, 1, 1)$ and it is $(0, 1, 2)$ at the bottom in Figure 1.4.

We provide a graphical representation of a group that Cayley invented in 1878. This group is defined by a collection of generators and relations.

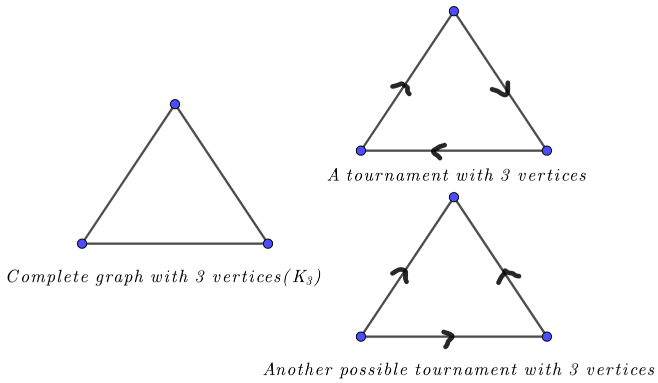

Figure 1.4: The two unlabelled tournaments on three vertices

Cayley digraphs of groups bridge two critical areas of mathematics, graph theory and group theory, and allow for a study of certain groups, such as cyclic groups. Cayley digraphs on cyclic groups are used to define Circulant digraphs. In Chapter 2, we will look at Cayley digraphs for many groups.

Definition 1.2.22 ([34, page 99])**.** Let G be a finite group and let S be a set of generators for G. We define a digraph *Cay(G:S)*, called the *Cayley digraph of* G with generating set S, as follows:

1. Each element of G is a vertex of $\mathrm{Cay}(G : S)$.

2. For x and y in G, there is an arc from x to y if and only if $xs = y$ for some $s \in S$.

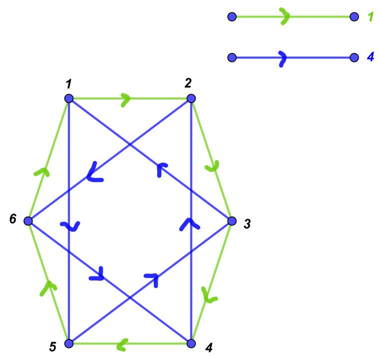

Figure 1.5: The Cayley digraph $\mathrm{Cay}(\mathbb{Z}_6, \{1, 4\})$

1.2.2 Group theory

In this section, we give some background about group theory that will be used throughout the thesis. The definitions used by [21] are followed, unless otherwise indicated.

Definition 1.2.23. We write $G = \langle X \mid R \rangle$ to denote a presentation of a group G. The elements of X are called generators and those of R defining relators. A group G is called finitely presented if it has a presentation with both X and R finite sets.

Definition 1.2.24. The *rank* of a group G, denoted by $\mathrm{rank}(G)$, is the cardinality of a smallest generating set for G.

Definition 1.2.25. A *cyclic group* G is a group that can be generated by a single element a, so every element in G has the form a^i for some integer i.

We denote the cyclic group of order n by $\mathbb{Z}_n$ since the additive group of $\mathbb{Z}_n$ is a cyclic group of order n.

Definition 1.2.26. Given any group G, recall that its derived group (or commutator subgroup) is the group G' generated by the set of all commutators $\{g^{-1}h^{-1}gh \mid g, h \in G\}$ of elements of G. It is clear that $G' \trianglelefteq G$ and that $G^{ab} := G/G'$ is abelian. G^{ab} is often called the *abelianization of* G.

Definition 1.2.27. If the abelianization of the group G is trivial ($G^{ab} = 1$), then the group G is a *perfect group*.

Definition 1.2.28. A group G is called *solvable* if it has a subnormal series whose factor groups (quotient groups) are all abelian, that is, if there are subgroups $1 = G_0 < G_1 < \ldots < G_k = G$ such that G_{j-1} is normal in G_j, and G_j/G_{j-1} is an abelian group, for $j = 1, 2, \ldots, k$.

Definition 1.2.29. Given groups $G = \langle X \mid R \rangle$ and $H = \langle Y \mid S \rangle$, their *free product* is given by the presentation

$$G * H = \langle X, Y \mid R, S \rangle.$$

Unless one of the groups G and H is trivial, the free product $(G * H)$ is always infinite.

We use Tietze transformations throughout the thesis and Tietze transformations are used to transform a given presentation of a group into another by adding or removing the relations or generators. In 1908, H. Tietze showed in [26] that given a presentation for a group G,

$$G = \langle\, a, b, c, \ldots \mid P, Q, R \ldots \rangle$$

then, any other presentation can be obtained by repeated application of the following transformations to G.

(T1) Adding a relation: If the words $S, T, \ldots$ are derivable from $P, Q, R, \ldots$ then add $S, T, \ldots$ to the defining relators in G.

(T2) Removing a relation: If some of the relators, say, $S, T, \ldots$ listed among the defining relators $P, Q, R, \ldots$ are derivable from the others, delete $S, T, \ldots$ from the defining relators in G.

(T3) Adding a generator: If $K, M, \ldots$ are words in $a, b, c, \ldots$ then adjoin the symbols $x, y, \ldots$ to the generating symbols in G and adjoin the relations $x = K, y = M, \ldots$ to the defining relators in G.

(T4) Removing a generator: If some of the defining relations in G take the form $p = V, q = W, \ldots$ where $p, q, \ldots$ are generators in G and $V, W, \ldots$ are words in the generators other than $p, q, \ldots$ then delete $p, q, \ldots$ from the generators, delete $p = V, q = W, \ldots$ from the defining relations, and replace $p, q, \ldots$ by $V, W, \ldots$ respectively, in the remaining defining relators in G.

The transformations $(T1) - (T4)$ are called Tietze transformations.

1.3 Digraph groups and related groups

The underlying undirected graphs in the thesis are connected and finite, and the groups in the thesis are defined by finite presentations where each relator is of the form $R(x, y)$, where x and y are distinct generators and $R(\cdot\,,\cdot)$ is determined by some fixed cyclically reduced word $R(a, b)$ in the free group generated by a and b that involves both a and b. Such groups were considered in the paper by Cuno and Williams [9].

We now define a construction of a group presentation from a digraph. Note that this construction is fundamental to the rest of the thesis. Let Λ be a finite *digraph* with vertex set $V(\Lambda)$ and (directed) arc set $A(\Lambda)$. The vertices $v \in V(\Lambda)$ correspond to the generators x_v and the arcs $(u, v) \in A(\Lambda)$ correspond to the relators $R(x_u, x_v)$ so that the group $G_\Lambda(R)$ is defined by the presentation

$$P_\Lambda(R) = \langle\, x_v \ (v \in V(\Lambda)) \mid R(x_u, x_v) \ ((u, v) \in A(\Lambda))\, \rangle.$$

A group is called a *digraph group* if it is isomorphic to $G_\Lambda(R)$ for some Λ and R [9].

The terminology of digraph groups were first introduced by Cuno & Williams in 2020 in the paper [9]. However, the digraph groups have a long history in the sense that many previously studied classes of groups are in fact digraph groups (although they are not referred to as such). We now have a discussion about what has been studied, when the groups can be thought as digraph groups and the new results obtained in this thesis.

Consider the free group with basis $x_0, \ldots, x_{n-1}$ and let w be a word in the free group, where $n > 0$. The shift, denoted by θ, is the free group automorphism mapping $x_i \mapsto x_{i+1}$, with subscripts mod n. Then

$$P_n(w) = \langle\, x_0, \ldots, x_{n-1} \mid w, \theta(w), \ldots, \theta^{n-1}(w)\, \rangle$$

is called a *cyclic presentation*, and we write $G_n(w)$ for the corresponding cyclically presented group [21, page 95].

If w involves exactly two generators then $G_n(w)$ is a digraph group by setting Λ to be a directed n-cycle, i.e. $V(\Lambda) = \{1, 2, \ldots, n\}$ and $A(\Lambda) = \{(1, 2), (2, 3), \ldots, (n, 1)\}$.

We will be concerned with investigating when digraph groups are finite and if a group is defined by a presentation with more generators than relators, then it is infinite [25, page 165]. Therefore, we now shall focus on the case presentations with more relators than generators or equal ($|V| \leqslant |A|$). The first step is balanced presentations which are presentations with an equal number of generators and relators. We now state a notational convention 1, partially introduced by Pride in [31] and the Lemma 1.3.3 proved by Cuno & Williams in [9, page 7] and its proof since it is used to illustrate the techniques as we will use them frequently throughout the thesis. It is also important to understand why we have these conditions in our theorems in the next chapters by the readers. That is why we are including the Lemma 1.3.3 and its proof here.

Notational convention 1 ([9]). *We use α and $-\beta$ to represent the exponent sums of a and b in a cyclically reduced word $R(a, b)$, respectively, and K is used to indicate a group defined by the presentation $\langle\, a, b \mid R(a, b)\,\rangle$. As far as cyclic permutation is considered, the word R has the form $a^{\alpha_1}b^{\beta_1}\cdots a^{\alpha_t}b^{\beta_t}$ with $t \geqslant 1$ and $\alpha_i, \beta_i \in \mathbb{Z} \smallsetminus \{0\}$ $(1 \leqslant i \leqslant t)$.*

The following property is defined by Pride in [31, page 246]: If no non-empty word of the form $a^k b^{-\ell}$ $(k, \ell \in \mathbb{Z})$ is equal to the identity in that group, then a two-generator group with generators a and b is said to have *Property* W_1 (*with respect to a and b*). Under the hypothesis that the girth of the underlying undirected graph of Λ is at least 4.

Corollary 1.3.1 ([31, Theorem 4]). *Let Λ be a non-empty finite digraph whose underlying undirected graph has $g(G) \geqslant 4$ and let $R(a, b)$ be as in notational convention 1. If K has Property W_1, then $G_\Lambda(R)$ is infinite.*

It is therefore important to study groups that do not have Property W_1.

Proposition 1.3.2 ([31, page 248]). *If there exist $k, \ell \in \mathbb{Z} \smallsetminus \{0\}$ with $a^k = b^\ell$ in K, then $\alpha \neq 0$, $\beta \neq 0$, and $a^\alpha = b^\beta$ in K.*

Therefore, K does not have Property W_1 if and only if $\alpha \neq 0$, $\beta \neq 0$, and $a^\alpha = b^\beta$ in K.

We will now state the following Lemma 1.3.3 and Lemma 1.3.3 is a specialisation of a result due to Pride, which was stated without proof in [31]. The proof was stated in proof [9], which we will frequently use by referring in the next chapters of this thesis.

Lemma 1.3.3 ([31], [9, page 7])**.** *Let Λ be a non-empty finite digraph whose underlying undirected graph has girth at least 4 and let $R(a,b)$ be a cyclically reduced word that involves both a and b. Let $R(a,b)$ be as in notational convention 1 and $|\alpha| \geqslant 2$ and $|\beta| \geqslant 2$. If $G_\Lambda(R)$ is finite then $(\alpha,\beta) = 1$ and Λ has at most one source and at most one sink.*

Proof. Assume that $u \in V(\Lambda)$. If $(u,v) \in A(\Lambda)$, then a relator $R(x_u, x_v)$ exists. The exponent sum of a in $R(a,b)$ is α as defined in notational convention 1. Thus, the relator $R(x_u, x_v)$ is transformed into $R(x_u, 1)$, i.e. to x_u^α when the terminal vertex v is killed (i.e. by adjoining the relation $x_v = 1$). Similarly, killing the initial vertex u turns the relator $R(x_u, x_v)$ into x_v^β.

Let u, w be any two fixed vertices of Λ. Killing all generators $x_v (v \neq u, w)$ shows that $G_\Lambda(R)$ maps onto

$$\langle\, x_u, x_w \mid x_u^\alpha, x_u^\beta, x_w^\alpha, x_w^\beta \,\rangle = \langle\, x_u, x_w \mid x_u^{(\alpha,\beta)}, x_w^{(\alpha,\beta)} \,\rangle \cong \mathbb{Z}_{(\alpha,\beta)} * \mathbb{Z}_{(\alpha,\beta)}$$

Now, if $(\alpha,\beta) > 1$, then $G_\Lambda(R)$ is infinite . Therefore, we have that $(\alpha,\beta) = 1$.

Assume that $u, w \in V(\Lambda)$ are a source and a sink, respectively, which are not connected by an arc. If all generators x_v ($v \in V(\Lambda) \setminus \{u,w\}$) are killed, then we have that $G_\Lambda(R)$ maps onto $\mathbb{Z}_{|\alpha|} * \mathbb{Z}_{|\beta|}$, which is infinite since $|\alpha| \geqslant 2$ and $|\beta| \geqslant 2$. As a result, we may consider that an arc exists between each source and sink.

Now, the next step is that we suppose there are distinct vertices $u, w \in V(\Lambda)$ that are both sources (resp. both sinks). It is clear that these vertices cannot be connected by an arc. $G_\Lambda(R)$ maps onto $\mathbb{Z}_{|\alpha|} * \mathbb{Z}_{|\alpha|}$ (resp. $\mathbb{Z}_{|\beta|} * \mathbb{Z}_{|\beta|}$), which is infinite when all generators x_v ($v \in V(\Lambda) \setminus \{u,w\}$) are killed. Thus, we may suppose that Λ has a maximum of one source and a maximum of one sink.

□

We will now state Lemma 1.3.4 (a),(b) proved by Cuno & Williams and $(c), (d)$. It enables us to simplify the presentations that arise in the subsequent chapters. Therefore, it is stated here for later use without further explanation throughout the thesis.

Lemma 1.3.4 ([9, Lemma 3.1]). *Let $R(a,b)$ be a word such that $a^\alpha = b^\beta$ in K and let G be a group defined by a presentation $\langle\, \mathcal{X} \mid \mathcal{R}\,\rangle$. Further suppose that there are distinct generators $x_i, x_j \in \mathcal{X}$ such that $R(x_i, x_j) \in \mathcal{R}$. Then the following hold:*

(a) *If $x_i^\gamma \in \mathcal{R}$ for some $\gamma \in \mathbb{Z}$ with $(\alpha,\gamma) = 1$, then every $p \in \mathbb{Z}$ with $p\alpha \equiv 1 \pmod{\gamma}$ yields a new presentation $\langle\, \mathcal{X} \smallsetminus \{x_i\} \mid \mathcal{S}\,\rangle$ of G. The relators $\mathcal{S}$ are obtained from $\mathcal{R}$ by removing $R(x_i, x_j)$ and x_i^γ, replacing all remaining occurrences of x_i by $x_j^{p\beta}$, and adjoining $x_j^{\beta\gamma}$.*

(b) *If $x_j^\gamma \in \mathcal{R}$ for some $\gamma \in \mathbb{Z}$ with $(\beta,\gamma) = 1$, then every $p \in \mathbb{Z}$ with $p\beta \equiv 1 \pmod{\gamma}$ yields a new presentation $\langle\, \mathcal{X} \smallsetminus \{x_j\} \mid \mathcal{S}\,\rangle$ of G. The relators $\mathcal{S}$ are obtained from $\mathcal{R}$ by removing $R(x_i, x_j)$ and x_j^γ, replacing all remaining occurrences of x_j by $x_i^{p\alpha}$, and adjoining $x_i^{\alpha\gamma}$.*

(c) if $x_i^\gamma \in \mathcal{R}$ for some $\gamma \in \mathbb{Z}$ with $(\alpha,\gamma) = 1$ then every $p \in \mathbb{Z}$ with $p\alpha \equiv 1 \bmod \gamma$ yields a new presentation $\langle\, \mathcal{X} \mid \mathcal{S}\,\rangle$ of G where $\mathcal{S} = \mathcal{R} \cup \{x_i x_j^{-p\beta}, x_j^{\beta\gamma}\}$.

(d) if $x_j^\gamma \in \mathcal{R}$ for some $\gamma \in \mathbb{Z}$ with $(\beta,\gamma) = 1$ then every $p \in \mathbb{Z}$ with $p\beta \equiv 1 \bmod \gamma$ yields a new presentation $\langle\, \mathcal{X} \mid \mathcal{S}\,\rangle$ of G where $\mathcal{S} = \mathcal{R} \cup \{x_j x_i^{-p\alpha}, x_i^{\alpha\gamma}\}$.

If Λ is a directed n-cycle ($n \geqslant 4$) and $R(a,b)$ is a cyclically reduced word that involves both a and b, then $G_\Lambda(R)$ can never be finite of rank 3 or trivial [31]. We now give precise statement and its proof in Theorem 1.3.5 that forms the cornerstone of our thesis. Since we will have directed n-cycle in the digraphs throughout the thesis mostly, the proof of the Theorem 1.3.5 is given in detail and we will only use it for referring to in the next chapters. The following Theorem 1.3.5 was stated without proof in [31, Theorem 3], a proof was given in [6, Lemma 3.4] and a different proof was given in [9, Lemma 3.4]. We repeat that proof exactly below here as it is crucial to our arguments.

Theorem 1.3.5 ([9, Lemma 3.4] [[31, Theorem 3]] , [6, Lemma 3.4])**.** *Let $R(a,b)$ be as in Notational convention 1. Further suppose that $(\alpha,\beta)=1$ and $a^\alpha=b^\beta$ in K. If $\Lambda=\Lambda(n)$, where $\Lambda(n)$ is directed $n-$cycle ($n\geqslant 2$), then $G_\Lambda(R)\cong\mathbb{Z}_{|\alpha^n-\beta^n|}$.*

Proof. Let $V(\Lambda)=\{1,\dots,n\}$ and $A(\Lambda)=\{(1,2),(2,3),\dots,(n,1)\}$. Then

$$G_\Lambda(R)=\langle\, x_1,\dots,x_n \mid R(x_1,x_2),R(x_2,x_3),\dots,R(x_n,x_1)\,\rangle.$$

Because $a^\alpha=b^\beta$ in K, if the relator $R(x_i,x_j)$ is one of the relators in the presentation above, the equation $x_i^\alpha=x_j^\beta$ holds in $G_\Lambda(R)$. Therefore,

$$x_1^{\alpha^n}=x_2^{\alpha^{n-1}\beta}=x_3^{\alpha^{n-2}\beta^2}=\dots=x_n^{\alpha\beta^{n-1}}=x_1^{\beta^n}.$$

Now let $\gamma=\alpha^n-\beta^n$, then $x_1^\gamma=1$ in $G_\Lambda(R)$. Adjoining the relator x_1^γ gives

$$G_\Lambda(R)=\langle\, x_1,\dots,x_n \mid x_1^\gamma,R(x_1,x_2),R(x_2,x_3),\dots,R(x_n,x_1)\,\rangle.$$

Since $(\alpha,\beta)=1$ we have $(\alpha,\gamma)=1$ so the presentation can be simplified by using Lemma 1.3.4 (a). Note that $(\alpha,\beta\gamma)=\dots=(\alpha,\beta^{n-2}\gamma)=1$. By choosing an integer $p\in\mathbb{Z}$ such that $p\alpha\equiv 1 \pmod{\beta^{n-2}\gamma}$, the congruence $p\alpha\equiv 1$ simultaneously holds modulo γ, $\beta\gamma$, $\dots$, $\beta^{n-2}\gamma$. Now, using Lemma 1.3.4 (a) iteratively gives

$$\begin{aligned}
G_\Lambda(R)&=\langle\, x_1,\dots,x_n \mid x_1^\gamma,R(x_1,x_2),R(x_2,x_3),R(x_3,x_4),\dots,R(x_n,x_1)\,\rangle\\
&=\langle\, x_2,\dots,x_n \mid x_2^{\beta\gamma},R(x_2,x_3),R(x_3,x_4),\dots,R(x_n,x_2^{p\beta})\,\rangle\\
&=\langle\, x_3,\dots,x_n \mid x_3^{\beta^2\gamma},R(x_3,x_4),\dots,R(x_n,x_3^{p^2\beta^2})\,\rangle\\
&=\dots\\
&=\langle\, x_n \mid x_n^{\beta^{n-1}\gamma},R(x_n,x_n^{p^{n-1}\beta^{n-1}})\,\rangle\\
&=\langle\, x_n \mid x_n^{\beta^{n-1}\gamma},x_n^{\alpha-\beta(p^{n-1}\beta^{n-1})}\,\rangle\\
&=\langle\, x_n \mid x_n^{r}\,\rangle
\end{aligned}$$

where

$$\begin{aligned} r &= (\beta^{n-1}\gamma, \alpha - \beta(p^{n-1}\beta^{n-1})) \\ &= (\gamma, \alpha - p^{n-1}\beta^n). \end{aligned}$$

Now, $p^{n-1}\beta^n \equiv (p\alpha)p^{n-1}\beta^n = \alpha p^n\beta^n = \alpha p^n(\alpha^n - \gamma) \equiv \alpha p^n\alpha^n = \alpha(p\alpha)^n \equiv \alpha 1^n = \alpha \pmod{\gamma}$. So γ divides $\alpha - p^{n-1}\beta^n$, and hence $r = \gamma = \alpha^n - \beta^n$ and so

$$G_\Lambda(R) = \langle\, x_n \mid x_n^{\alpha^n-\beta^n} \,\rangle \cong \mathbb{Z}_{|\alpha^n-\beta^n|}.$$

□

The Theorem 1.3.5 is generalized from cyclic presentations to balanced presentations (i.e. $|V| = |A|$) in [9]. We extend the theorem from balanced presentations to $|V| = |A| - 1$ in Chapter 3 and to $|V| \leqslant |A|$ for strongly connected digraphs in Chapter 4 for $|\alpha| \geqslant 2$, $|\beta| \geqslant 2$.

In many of our digraphs Γ there will be a configuration which we denote as $\Lambda(n; \xrightarrow{m})$ to mean an directed n-cycle and a m path going from the directed n-cycle or $\Lambda(n; \xleftarrow{m})$ to mean an directed n-cycle and a m path coming to the directed n-cycle (see Figure 1.6); Lemma 1.3.6 allows us to replace this sub-digraph with a vertex v and adding a corresponding relator x_v^P to the presentation. To assist the reader in Chapter 3.2(i), we will explain this reduction in detail, then in later chapters we will use this technique without further explanation.

Lemma 1.3.6 ([9, Lemma 3.5])**.** *Let $R(a, b)$ be as in notational convention 1. Further suppose that $(\alpha, \beta) = 1$ and $a^\alpha = b^\beta$ in K. Then the following hold:*

(*a*) *If $\Lambda = \Lambda(n; \xrightarrow{m})$ $(n \geqslant 2,\ m \geqslant 1)$, then $G_\Lambda(R) \cong \mathbb{Z}_{|\beta^m(\alpha^n-\beta^n)|}$.*

(*b*) *If $\Lambda = \Lambda(n; \xleftarrow{m})$ $(n \geqslant 2,\ m \geqslant 1)$, then $G_\Lambda(R) \cong \mathbb{Z}_{|\alpha^m(\alpha^n-\beta^n)|}$.*

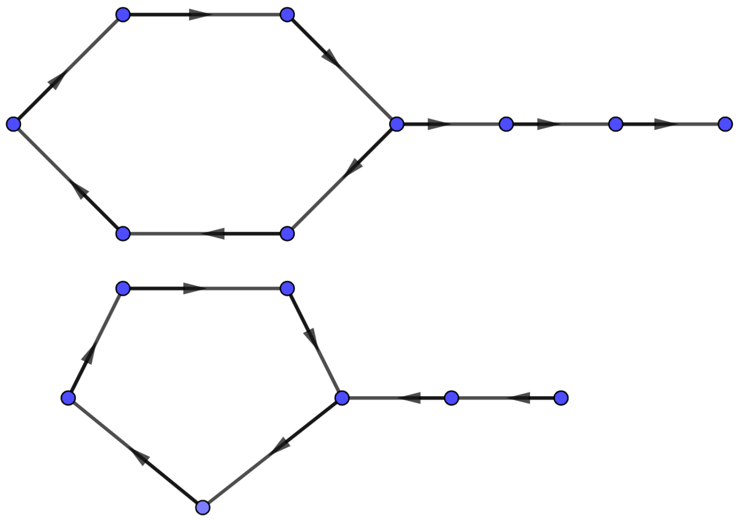

Figure 1.6: Digraphs $\Lambda(6;\xrightarrow{3})$ and $\Lambda(5;\xleftarrow{2})$

In 1962, *Baumslag-Solitar groups*

$$BS(a,b)=\langle\, x_1,x_2 \mid x_2^{-1}x_1^ax_2=x_1^b\,\rangle$$

were introduced in [5] and so $BS(a,b)=G_\Gamma(x_2^{-1}x_1^ax_2x_1^{-b})$, where Γ consists of two vertices joined by an arc and it is a digraph group. In 2012, Allcock studied *triangles of Baumslag-Solitar groups* in [1] give by the presentation

$$G(a,b;c,d;e,f)=\langle\, x_1,x_2,x_3 \mid x_2^{-1}x_1^ax_2=x_1^b, x_3^{-1}x_2^cx_3=x_2^d, x_1^{-1}x_3^ex_1=x_3^f\,\rangle.$$

Thus, triangles of Baumslag-Solitar groups are digraph groups when $a=c=e=p, b=d=f=q$ that means $G(p,q;p,q;p,q)=G_\Gamma(y^{-1}x^pyx^{-q})$, where Γ is the directed $3-$cycle, since all labels are equal. We also obtain *Mennicke's groups* when $p=1$ that means $G(1,q;1,q;1,q)=G_\Gamma(y^{-1}xyx^{-q})$, with the presentation

$$M(q,q,q)=\langle\, x_1,x_2,x_3 \mid x_1^{-1}x_2x_1=x_2^q, x_2^{-1}x_3x_2=x_3^q, x_3^{-1}x_1x_3=x_1^q\,\rangle,$$

which are studied in [27]. The group $M(q,q,q)$ is an example of cyclically-presented groups as $G_3(x_1^{-1}x_2x_1x_2^{-q})$. Thus, Mennicke groups $M(q,q,q)$ are digraph groups and cyclically presented groups. For all $q>3$ the Mennicke groups $M(q,q,q)$ are finite of rank 3 [27]. These groups have also been investigated by I.D.Macdonald and by J.W.Wamsley and they showed that $M(a,b,c)$, which are not necessarily digraph groups except for $a=b=c$, is finite whenever $|a|,|b|,|c|\geqslant 3$ in [33]. The proof of this can be found in [22] and we also stated the detailed proof in Theorem 5.3.2 since we generalise Mennicke's group and this result from directed 3-cycle to all strong tournaments in Chapter 5.

When Λ is a directed n-cycle and $q=2$ we obtain *Higman's groups* $H(n)$ [15] and the resulting group is

$$\langle\, x_0,\ldots,x_{n-1} \mid x_i^{-1}x_{i+1}x_i = x_{i+1}^2 (0\leqslant i<n)\,\rangle.$$

(where subscripts are taken mod n). When $n=3$ the resulting group $H(3)$ is the Mennicke group $M(2,2,2)$, which is trivial (see [15, Section 3]). As seen Higman groups are digraph groups, and they are also cyclically presented groups as $G_4(x_1^{-1}x_2x_1x_2^{-2})$.

Another example that we obtain is given by *Johnson's groups*

$$J(a,b,c)=\left\langle\; x_1,x_2,x_3 \;\middle|\; \begin{array}{l} x_2^{-1}x_1x_2 = x_2^{b-2}x_1^{-1}x_2^{b+2}, \\ x_3^{-1}x_2x_3 = x_3^{c-2}x_2^{-1}x_3^{c+2}, \\ x_1^{-1}x_3x_1 = x_1^{a-2}x_3^{-1}x_1^{a+2} \end{array} \right\rangle$$

considered in [18], [20] and [21, page 92], which a,b,c are non-zero even integers are finite. These are the digraph groups $G_\Lambda(R)$ where Λ is the directed 3-cycle and $a=b=c=q$, $R(a,b)=b^{-1}ab(b^{q-2}a^{-1}b^{q+2})^{-1}$. Thus, Johnson groups $J(q,q,q)$ are digraph groups and cyclically presented groups. I also generalise Johnson's groups and this result in Chapter 5.

Let $(x_i,x_j)_{m_{ij}}$ denote the word of length m_{ij} which starts with x_i and alternates between x_i and x_j. The Artin group associated to a defining graph Λ has one generator for each vertex of Λ and a relator $(x_i,x_j)_{m_{ij}}=(x_j,x_i)_{m_{ji}}$ whenever there is an edge connecting vertices x_i and x_j that has been assigned the integer m_{ij}. Thus, an *Artin group* is a group with presentation of the form

$$\langle\, x_1, x_2, \ldots, x_n \mid \underbrace{x_i x_j x_i \ldots}_{m_{ij}} = \underbrace{x_j x_i x_j \ldots}_{m_{ji}}, \quad for\ all\ (x_i, x_j) \in A(\Lambda) \,\rangle.$$

This class of Artin groups has been studied in [2], [3]. The class of Artin groups where the underlying graph is triangle-free has been studied in [30]. *Right-angled Artin groups* are Artin groups in which all relators are commutators between specified generators, commonly known as *graph groups* or *partially commutative groups*. A. Baudisch [4] initially introduced right-angled Artin groups in the 1970s, and C. Droms further developed them in the 1980s under the name graph group in [10],[11],[12]. Thus every Right Angled Artin Group is a digraph group $G_\Gamma(aba^{-1}b^{-1})$ for some digraph Γ.

Artin groups are digraph groups if each m_{ij} is the same. For example,

$$\langle\, x_1, x_2, x_3 \mid x_1x_2x_1 = x_2x_1x_2, x_2x_3x_2 = x_3x_2x_3, x_3x_1x_3 = x_1x_3x_1 \,\rangle$$

is an Artin group and a digraph group. However,

$$\langle\, x_1, x_2, x_3 \mid x_1x_2x_1x_2x_1 = x_2x_1x_2x_1x_2, x_2x_3x_2 = x_3x_2x_3, x_3x_1x_3 = x_1x_3x_1 \,\rangle$$

is an Artin group but not a digraph group since all relations do not have same form.

Remark 1.3.7 ([9, page 7])**.** *This statement introduces a* reflection principle*: Let Λ be any digraph, and $R(a, b)$ be any word. Then, the digraph Λ' may be defined as being formed by interchanging every arc with the opposite direction, and the word $R'(a, b)$ as the word that results from interchanging the letters a and b and replacing every letter with its inverse, thus α and β are also interchanged. Then, $G_\Lambda(R) \cong G_{\Lambda'}(R')$.*

1.4 Thesis outline

In Chapter 2, we investigate the digraph groups corresponding to Circulant digraphs, Cayley digraphs corresponding to direct sum of cyclic groups, quaternion groups and direct product of two groups. Circulant digraphs can have any girth but the other Cayley graphs have the girth at least 4. Therefore, Theorem 1.3.5 can be applied in all these cases. But when the girth of Circulant digraphs is less than 4, then we are supposing $(\alpha, \beta) = 1$ and $a^{\alpha} = b^{\beta}$ in $K = \langle\, a, b \mid R(a, b)\,\rangle$. We are able to show that the digraph group corresponding to each group is finite cyclic and we give the formula to calculate the order.

In Chapter 3, we specify all possible digraph families under the condition $|V| = |A| - 1$ and in most cases, determine when the corresponding groups are finite cyclic or infinite when the digraph is triangle free. The formulas are given to calculate the order if the group is finite cyclic. If it is not shown that it is finite cyclic, then we show that the group presentation can be written in terms of two generators mostly.

In Chapter 4, we present finite cyclic digraph groups when $|V| \leqslant |A|$. We prove that the corresponding group of strongly connected digraphs and semi-connected digraphs with one source and no sink, one sink and no source and no source and no sink is a finite cyclic group. In addition to this, we are able to show that the corresponding groups for more complicated digraphs are finite cyclic.

In Chapter 5, in contrast to the first three chapters, we investigate whether the corresponding group is a non-cyclic finite group or not when the girth of the digraph is exactly 3. Some examples are done by Mennicke with the word $R(a, b) = a^{-1}bab^{-q}$ for $q \geqslant 3$ and Johnson $R(a, b) = b^{-1}ab(b^{q-2}a^{-1}b^{q+2})^{-1}$ $q \geqslant 2$ and even when the digraph is 3-vertex tournament with no source and no sink (it is known as 3-vertex strong tournament which means directed triangle). We generalise Johnson's and Mennicke's theorems and their proofs from the directed triangle case to all strong tournaments.

In Chapter 6, we use computational algebraic software *GAP* [14] to look for finite non-cyclic digraph groups for tournaments with up to 12 vertices. We are giving the exact orders and derived series of the corresponding groups by creating the tables for all tournaments up to 6-vertex and some examples in between 7 and 12-vertex tournaments for some fixed words such as Mennicke when $q = 3$ that means $R(a, b) = a^{-1}bab^{-3}$ and for Johnson when $q = 2$

that means $R(a,b) = ab^{-1}ab^{-3}$. Because of computational limitations, we are unable to find out all results when $q \geqslant 4$ for Johnson and Mennicke groups. But we provide a table and some theorems in Table 6.9 on page 148, Theorem 6.4.1 and Theorem 6.4.2 for Mennicke when $q \geqslant 4$ to have some idea about the groups. We will also define some new fixed words and give the exact order $R(a,b)$ in addition to Mennicke and Johnson such as $R(a,b) = abab^3$, $R(a,b) = abab^{-2}$ and $R(a,b) = ab^2a^2b^{-2}$. An important point is that we find 2-groups with Mennicke relator, 3-groups with the new word $R(a,b) = ab^2a^2b^{-2}$ and a perfect group with the new word $R(a,b) = abab^{-2}$. We also pose some conjectures based on these experiments.

CHAPTER 2

Circulant Digraphs and Cayley Digraphs of Groups

2.1 Preamble

This chapter will turn its attention to find out whether or not digraph groups corresponding to Circulant digraphs (i.e Cayley digraphs of cyclic groups), and Cayley digraphs of some groups such as direct sum of cyclic groups, quaternion groups and direct product of two groups are finite cyclic and, if so, to determine the order. We are able to show that the corresponding groups are finite cyclic and we provide a formula to calculate the exact order for each one. We now state the main theorem and prove it section by section.

Definition 2.1.1. ([13, page 429]) The Quaternion group $\mathcal{Q}_{2n}$ is given by the presentation

$$\langle\, a, b \mid a^{2n}, a^n = b^2, b^{-1}ab = a^{-1} \,\rangle.$$

Theorem A. *Let* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $K = \langle\, a, b \mid R(a, b) \,\rangle$.

(i) If $\Gamma = Cay(\mathbb{Z}_n, \{d_1, d_2, \dots, d_t\})$,

then $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\dots,d_t-d_1)}-\beta^{(n,d_2-d_1,d_3-d_1,\dots,d_t-d_1)}|}$. *(Section 2.2)*

(ii) *If* $\Gamma = Cay(\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \ldots \oplus \mathbb{Z}_{m_t}, \{e_1, e_2, \ldots, e_t\})$,
where $\{e_1, e_2, \ldots, e_t\} = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$
then $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(m_1,m_2,\ldots,m_t)} - \beta^{(m_1,m_2,\ldots,m_t)}|}$. *(Section 2.3)*

(iii) *If* $\Gamma = Cay(\mathcal{Q}_{2n}, \{a, b\})$, *then*

$$G_\Gamma(R) \cong \begin{cases} \mathbb{Z}_{|\alpha-\beta|} & if\ n\ is\ odd. \\ \mathbb{Z}_{|\alpha^2-\beta^2|} & if\ n\ is\ even. \end{cases} \tag{2.1.1}$$

(Section 2.4)

(iv) *Let* H, K *be finite groups with generating sets* S, T, *and identity elements* e, f, *respectively. Let* $\Gamma_H = Cay(H, S), \Gamma_K = Cay(K, T)$. *Suppose* $G_{\Gamma_H}(R)$ *is finite cyclic, generated by* x_e *and* $G_{\Gamma_K}(R)$ *is finite cyclic generated by* x_f. *Let* $\Gamma = Cay(H \times K, S \times T)$. *Then* $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(N_1,\ldots,N_p)} - \beta^{(N_1,\ldots,N_p)}|}$, *where* $N_1, \ldots, N_p$ *are the lengths of the directed cycles in* Γ. *(Section 2.5)*

Now we will provide Lemma 2.1.2. Proposition 2.1.3 and Proposition 2.1.4 which we use throughout this chapter.

Lemma 2.1.2. *Let* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $K = \langle\, a, b \mid R(a, b)\,\rangle$. *Suppose* Γ *has a Hamilton path with an initial vertex* u *and a terminal vertex* v. *If* Γ *has directed cycles including* v *of length* N_i *for* $1 \leqslant i \leqslant k$ *and including* u *of length* M_j *for* $1 \leqslant j \leqslant t$ *then* $G_\Gamma(R)$ *is generated by* x_u, *which satisfies the relation* $x_u^{\alpha^{(N_1,N_2,\ldots,N_k,M_1,M_2,\ldots,M_t)} - \beta^{(N_1,N_2,\ldots,N_k,M_1,M_2,\ldots,M_t)}} = 1$.

Proof. Label the vertices of Γ as $1, 2, ..., n$ so that $1 \to 2 \to 3 \to \cdots \to (n-1) \to n$ is the Hamilton path in the statement. Then $u = 1$ and $v = n$.

Applying the argument of lines $1 - 5$ of Theorem 1.3.5 we have $x_n^{\gamma_0} = 1$ where $\gamma_0 = \alpha^{N_1} - \beta^{N_1}$. Now there is an arc $(n-1, n)$ so there is a relator $R(x_{n-1}, x_n)$ in the set of relators of $G_\Gamma(R)$. Now $(\beta, \gamma_0) = 1$ so there exists $p_0 \in \mathbb{Z}$ such that $p_0\beta \equiv 1 \bmod \gamma_0$ so using Lemma 1.3.4 (d) we may adjoin the relators $x_n x_{n-1}^{-p_0\alpha}$ and $x_{n-1}^{\alpha\gamma_0}$. Let $\gamma_1 = \alpha\gamma_0$.

Now there is an arc $(n-2, n-1)$ so there is a relator $R(x_{n-2}, x_{n-1})$. Now $(\beta, \gamma_1) = 1$ so there exists $p_1 \in \mathbb{Z}$ such that $p_1\beta \equiv 1 \bmod \gamma_1$, so by Lemma 1.3.4 (d) we may we may adjoin the relators $x_{n-1}x_{n-2}^{-p_1\alpha}$ and $x_{n-2}^{\alpha\gamma_1}$, that is $x_{n-2}^{\alpha^2\gamma_0}$. Let $\gamma_2 = \alpha\gamma_1 = \alpha^2\gamma_0$.

As before we may delete x_{n-2} and adjoin the relators $x_{n-2}x_{n-3}^{-p_2\alpha}$ and $x_{n-3}^{\alpha\gamma_2}$, that is, the relator $x_{n-3}^{\alpha^3\gamma_0}$.

Continuing in this way we obtain a presentation in which each x_s $(2 \leqslant s \leqslant n)$ can be expressed as a power of x_{s-1} through a relator of the form $x_s x_{s-1}^{-p\alpha}$ and where the generator x_1 satisfies the relator $x_1^{\alpha^{(n-1)\gamma_0}}$. Note that we have not removed any relators in this process.

Now let $\gamma_0^{(i)} = \alpha^{N_i} - \beta^{N_i}$ for each $2 \leqslant i \leqslant k$. Repeating the above argument with this latest presentation provides a new presentation which also includes the relator $x_1^{\alpha^{(n-1)\gamma_0^{(i)}}}$. Other relators of the form $x_s x_{s-1}^{-p^{(i)}\alpha}$ $(2 \leqslant i \leqslant n)$ will also have been added in this process, but they are not important to us.

Using the relators of the form $x_s x_{s-1}^{-p\alpha}$ $(2 \leqslant s \leqslant n)$ we may remove generators $x_n, x_{n-1}, \ldots, x_3, x_2$ in turn, leaving a presentation with the single generator x_1. The relators $x_1^{\alpha^{(n-1)\gamma_0}}$ and $x_1^{\alpha^{(n-1)\gamma_0^{(i)}}}$ for each $2 \leqslant i \leqslant k$ remain. Therefore $x_1^{(\alpha^{(n-1)\gamma_0}, \alpha^{(n-1)\gamma_0^{(2)}}, \ldots, \alpha^{(n-1)\gamma_0^{(k)}})} = 1$.

Now applying the argument of lines $1-5$ of Theorem 1.3.5 using the cycle of length M_j for each $1 \leqslant j \leqslant t$ involving u we have $x_1^{\bar{\gamma_0}^{(j)}} = 1$, where $\bar{\gamma}_0^{(j)} = \alpha^{M_j} - \beta^{M_j}$. Therefore $x_1^{(\bar{\gamma_0}, \bar{\gamma_0}^{(2)}, \ldots, \bar{\gamma_0}^{(t)})} = 1$.

Thus, we get

$$x_1^{(\alpha^{(n-1)\gamma_0}, \alpha^{(n-1)\gamma_0^{(2)}}, \ldots, \alpha^{(n-1)\gamma_0^{(k)}}, \bar{\gamma_0}, \bar{\gamma_0}^{(2)}, \ldots, \bar{\gamma_0}^{(t)})} = x_1^{(\gamma_0, \gamma_0^{(2)}, \ldots, \gamma_0^{(k)}, \bar{\gamma_0}, \bar{\gamma_0}^{(2)}, \ldots, \bar{\gamma_0}^{(t)})} = 1$$

so $x_1^{\alpha^{(N_1, N_2, \ldots, N_k, M_1, M_2, \ldots, M_t)} - \beta^{(N_1, N_2, \ldots, N_k, M_1, M_2, \ldots, M_t)}} = 1$, as required.

□

Proposition 2.1.3. *Let $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a, b)\,\rangle$. For any digraph Γ, there is an epimorphism between $G_\Gamma(R)$ and $\mathbb{Z}_{|\alpha-\beta|}$.*

Proof. The group $G_\Gamma(R)$ is given by the presentation $G_\Gamma(R) = \langle\, x_v(v \in V(\Gamma)) \mid R(x_u, x_v)((u, v) \in A(\Gamma))\,\rangle$.

Let $\phi : G_\Gamma(R) \to \langle\, t \mid t^{|\alpha-\beta|}\,\rangle$ be given by $\phi(x_v) = t$ for all $v \in V(\Gamma)$. Then for any arc (u, v) we have $\phi(R(x_u, x_v)) = t^{\alpha-\beta} = 1$, so ϕ is a homomorphism.

Since for each $0 \leqslant i < |\alpha - \beta|$ and any vertex $v \in V(\Gamma)$ we have $\phi(x_v^i) = t^i$, so ϕ is an epimorphism. □

Proposition 2.1.4. *In the notation of Lemma 2.1.2, if $(N_i, N_j) = 1, (M_i, M_j) = 1$ or $(N_i, M_j) = 1$ for any i, j, then $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha-\beta|}$.*

Proof. It is proved that $x_1^{\alpha(N_1,N_2,\ldots,N_k,M_1,M_2,\ldots,M_t)-\beta(N_1,N_2,\ldots,N_k,M_1,M_2,\ldots,M_t)} = 1$ by Lemma 2.1.2. Suppose now that $(N_i, N_j) = 1, (M_i, M_j) = 1$ or $(N_i, M_j) = 1$ for any i, j. Then $x_1^{|\alpha-\beta|} = 1$. Therefore $G_\Gamma(R)$ is cyclic, generated by x_1, and x_1 satisfies the relation $x_1^{|\alpha-\beta|} = 1$ so $G_\Gamma(R)$ is a quotient of the cyclic group $\mathbb{Z}_{|\alpha-\beta|}$. But by Proposition 2.1.3 $G_\Gamma(R)$ maps onto $\mathbb{Z}_{|\alpha-\beta|}$ so $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha-\beta|}$, as required. □

2.2 Circulant digraphs

Definition 2.2.1. [24] For any natural number n, we use $\mathbb{Z}_n$ to denote the additive cyclic group of integers modulo n. For any set of integers A, let $\mathrm{Cay}(\mathbb{Z}_n, A)$ be digraph whose vertex set is $\mathbb{Z}_n$, and in which there is an arc from u to $u + a \pmod{n}$, for every $u \in \mathbb{Z}_n$, and every $a \in A$. A digraph is *Circulant* if it is isomorphic to $\mathrm{Cay}(\mathbb{Z}_n, A)$, for some choice of n and A.

The following partition below covers all Circulant digraphs and we provide a proof for each one. We use different technique to prove them but all give same result as we state in main theorem. Let A be the generating set, then

(i) Theorem 2.2.3 : $A = \{d_1, d_2, \ldots, d_t\}$ and $(d_j, n) = 1$ for some $1 \leqslant j \leqslant t$,

(ii) Theorem 2.2.4: $A = \{d_1, d_2, \ldots, d_t\}$ and $(d_j, n) > 1$ for all $1 \leqslant j \leqslant t$ and $(n, d_1, d_2, \ldots, d_t) = 1, (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) = 1$

(iii) Theorem 2.2.5: $A = \{d_1, d_2, \ldots, d_t\}$ and $(d_j, n) > 1$ for all $1 \leqslant j \leqslant t$ and $(n, d_1, d_2, \ldots, d_t) = 1, (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) > 1$

Theorems 2.2.3, 2.2.4, 2.2.5 will show that if $\Gamma = \mathrm{Cay}(\mathbb{Z}_n, A)$, then, $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)} - \beta^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}|}$ for $(i) - (iii)$ as in Theorem A(i). We first provide Lemma 2.2.2 that we use throughout the sections.

Lemma 2.2.2. *Let* $\Gamma = Cay(\mathbb{Z}_n, \{1, d_1, d_2, \ldots, d_t\})$*. If* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $\langle\, a, b \mid R(a, b)\,\rangle$ *then*
$G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(n,d_1-1,d_2-1,\ldots,d_t-1)} - \beta^{(n,d_1-1,d_2-1,\ldots,d_t-1)}|}$.

Proof. The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle\ x_0, x_1 \ldots, x_{n-1}\ \middle|\ \begin{array}{l} R(x_i, x_{i+1}), (0 \leqslant i \leqslant n-1) \\ R(x_i, x_{i+d_1}), \\ R(x_i, x_{i+d_2}), \\ \vdots \\ R(x_i, x_{i+d_t}) \end{array} \right\rangle.$$

Let $\gamma = \alpha^n - \beta^n$, since $(\alpha, \beta) = 1$ there exists p, q such that $p\alpha + q\gamma = 1$ and hence $p\alpha = 1 \ (\mathrm{mod}\gamma)$. Now $x_i^\gamma = 1$ in $G_\Gamma(R)$ since $x_i^{\alpha^n} = x_{i+1}^{\alpha^{n-1}\beta} = x_{i+2}^{\alpha^{n-2}\beta^2} = \ldots = x_{i+n-1}^{\alpha\beta^{n-1}} = x_i^{\beta^n}$. So adjoin relators $x_i^\gamma = 1$ for $0 \leqslant i \leqslant n-1$. If $(i, j) \in A(\Gamma)$ then $x_i^\alpha = x_j^\beta$ in G so $x_i = x_i^{p\alpha+q\gamma} = x_i^{p\alpha} = x_j^{p\beta}$ thus $x_i = x_j^{p\beta}$. Since $(i, i+1) \in A(\Gamma)$ then we have $x_i = x_{i+1}^{p\beta}(0 \leqslant i \leqslant n-1)$ and thus $x_i = x_{i+t}^{(p\beta)^t}$ for each $t \geqslant 0$. In particular, $x_i = x_{n-1}^{(p\beta)^{n-1-i}}$. We adjoin these relators. We now show that all $R(x_i, x_{i+1})$ for $(0 \leqslant i \leqslant n-2)$ are redundant.

$$\begin{aligned}
R(x_i, x_{i+1}) &= R(x_{n-1}^{(p\beta)^{n-1-i}}, x_{n-1}^{(p\beta)^{n-1-(i+1)}}) \\
&= x_{n-1}^{\alpha(p\beta)^{n-1-i}} x_{n-1}^{-\beta(p\beta)^{n-1-(i+1)}} \\
&= x_{n-1}^{\alpha p^{n-1-i}\beta^{n-1-i} - p^{n-1-i-1}\beta^{n-1-i}} \\
&= x_{n-1}^{(p\alpha)p^{n-2-i}\beta^{n-1-i} - p^{n-2-i}\beta^{n-1-i}} \\
&= x_{n-1}^{p^{n-2-i}\beta^{n-1-i} - p^{n-2-i}\beta^{n-1-i}} \text{since } p\alpha \equiv 1 \bmod \gamma \\
&= x_{n-1}^0 \\
&= 1
\end{aligned}$$

Removing these redundant relators, we get

$$G_\Gamma(R) = \left\langle \begin{array}{c|l} x_0, x_1 \ldots, x_{n-1} & \begin{array}{l} x_i^\gamma, x_i = x_{n-1}^{(p\beta)^{n-1-i}}, (0 \leqslant i \leqslant n-1) \\ R(x_{n-1}, x_0), \\ R(x_i, x_{i+d_1}), \\ R(x_i, x_{i+d_2}), \\ \vdots \\ R(x_i, x_{i+d_t}) \end{array} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c|l} x_0, x_1 \ldots, x_{n-1} & \begin{array}{l} x_i^\gamma, x_i = x_{n-1}^{(p\beta)^{n-1-i}}, (0 \leqslant i \leqslant n-1) \\ R(x_{n-1}, x_0), \\ R(x_0, x_{d_1}), R(x_1, x_{d_1+1}), R(x_2, x_{d_1+2}), \ldots, \\ R(x_{n-1}, x_{d_1-1}), \\ R(x_0, x_{d_2}), R(x_1, x_{d_2+1}), R(x_2, x_{d_2+2}), \ldots, \\ R(x_{n-1}, x_{d_2-1}), \\ \vdots \\ R(x_0, x_{d_t}), R(x_1, x_{d_t+1}), R(x_2, x_{d_t+2}), \ldots, \\ R(x_{n-1}, x_{d_t-1}) \end{array} \end{array} \right\rangle$$

Eliminating $x_0, x_1 \ldots, x_{n-2}$ using $x_i = x_{n-1}^{(p\beta)^{n-1-i}}$ for $(0 \leqslant i \leqslant n-2)$,

$$G_\Gamma(R) = \left\langle \begin{array}{c|l} x_{n-1} & \begin{array}{l} x_{n-1}^\gamma, \\ R(x_{n-1}, x_{n-1}^{(p\beta)^{n-1}}), \\ R(x_{n-1}^{(p\beta)^{n-1}}, x_{n-1}^{(p\beta)^{n-d_1-1}}), R(x_{n-1}^{(p\beta)^{n-2}}, x_{n-1}^{(p\beta)^{n-d_1-2}}), \ldots, \\ R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_1}}), \\ R(x_{n-1}^{(p\beta)^{n-1}}, x_{n-1}^{(p\beta)^{n-d_2-1}}), R(x_{n-1}^{(p\beta)^{n-2}}, x_{n-1}^{(p\beta)^{n-d_2-2}}), \ldots, \\ R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_2}}), \\ \vdots \\ R(x_{n-1}^{(p\beta)^{n-1}}, x_{n-1}^{(p\beta)^{n-d_t-1}}), R(x_{n-1}^{(p\beta)^{n-2}}, x_{n-1}^{(p\beta)^{n-d_t-2}}), \ldots, \\ R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_t}}) \end{array} \end{array} \right\rangle.$$

We now show that all $R(x_{n-1}^{(p\beta)^{n-1-i}}, x_{n-1}^{(p\beta)^{n-1-i-d_j}})$ for $(0 \leqslant i \leqslant n-d_j)$, where $1 \leqslant j \leqslant t$, can be written in terms of $R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_j}})$ so can be eliminated.

To see this,

$$
\begin{aligned}
R(x_{n-1}^{(p\beta)^{n-1-i}}, x_{n-1}^{(p\beta)^{n-1-i-d_j}}) &= x_{n-1}^{\alpha(p\beta)^{n-1-i}-\beta(p\beta)^{n-1-i-d_j}} \\
&= x_{n-1}^{(p\beta)^{n-i-1}[\alpha-\beta(p\beta)^{n-d_j}]} \quad \text{by } (*) \\
&= (x_{n-1}^{[\alpha-\beta(p\beta)^{n-d_j}]})^{(p\beta)^{n-i-1}} \\
&= R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_j}})^{(p\beta)^{n-i-1}}.
\end{aligned}
$$

$(*)$ $(p\beta)^n = p^n\beta^n = p^n(\alpha^n-\gamma) = (p\alpha)^n - p^n\gamma = 1-0 \pmod{\gamma} = 1 \pmod{\gamma}$. Thus, we get

$$
\begin{aligned}
G_\Gamma(R) = \langle\, x_{n-1} \mid\, & x_{n-1}^{\gamma}, R(x_{n-1}, x_{n-1}^{(p\beta)^{n-1}}), R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_1}}), \\
& R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_2}}), \ldots, R(x_{n-1}, x_{n-1}^{(p\beta)^{n-d_t}})\,\rangle \\
&= \langle\, x_{n-1} \mid x_{n-1}^{\gamma}, x_{n-1}^{\alpha-\beta[(p\beta)^{n-1}]}, x_{n-1}^{\alpha-\beta[(p\beta)^{n-d_1}]}, x_{n-1}^{\alpha-\beta[(p\beta)^{n-d_2}]}, \ldots, \\
& x_{n-1}^{\alpha-\beta[(p\beta)^{n-d_t}]}\,\rangle \\
&= \langle\, x_{n-1} \mid x_{n-1}^{(\gamma,\alpha-\beta[(p\beta)^{n-1}],\alpha-\beta[(p\beta)^{n-d_1}],\alpha-\beta[(p\beta)^{n-d_2}],\ldots,\alpha-\beta[(p\beta)^{n-d_t}])}\,\rangle.
\end{aligned}
$$

Let $r = (\gamma, \alpha - \beta[(p\beta)^{n-1}])$.

$$
\begin{aligned}
\beta[(p\beta)^{n-1}] &\equiv p^{n-1}\beta^n \equiv (p\alpha)p^{n-1}\beta^n \;\text{ since } (p\alpha \equiv 1 \bmod \gamma) \\
&\equiv \alpha p^n\beta^n \equiv \alpha p^n(\alpha^n - \gamma) \;\text{ since } (\gamma = \alpha^n - \beta^n) \\
&\equiv \alpha(p\alpha)^n - \alpha p^n\gamma \equiv \alpha \pmod{\gamma}.
\end{aligned}
$$

So γ divides $\alpha - \beta[(p\beta)^{n-1}]$, and hence $r = \gamma$. Thus,

$$
G_\Gamma(R) = \langle\; x_{n-1} \;\mid\; x_{n-1}^{(\gamma,\alpha-\beta[(p\beta)^{n-d_1}]),\alpha-\beta[(p\beta)^{n-d_2}]),\ldots,\alpha-\beta[(p\beta)^{n-d_t}])}\;\rangle.
$$

We have

$$\begin{aligned}
\alpha - \beta[(p\beta)^{n-d_i}] &= \alpha - p^{n-d_i}\beta^{n-d_i+1} \\
&\equiv \alpha - (p\alpha)p^{n-d_i}\beta^{n-d_i+1} \text{ since } (p\alpha \equiv 1 \bmod \gamma) \\
&= \alpha - \alpha p^{n-d_i+1}\beta^{n-d_i+1} \\
&= \alpha(1 - p^{n-d_i+1}\beta^{n-d_i+1}) \\
&\equiv \alpha[p^{n-d_i+1}\alpha^{n-d_i+1} - p^{n-d_i+1}\beta^{n-d_i+1}] \text{ since } (p\alpha \equiv 1 \bmod \gamma) \\
&= \alpha p^{n-d_i+1}[\alpha^{n-d_i+1} - \beta^{n-d_i+1}]
\end{aligned}$$

Thus, $G_\Gamma(R) = \langle\, x_{n-1} \mid x_{n-1}^{\Delta} \,\rangle$, where $\Delta = (\gamma, \alpha p^{n-d_1+1}[\alpha^{n-d_1+1} - \beta^{n-d_1+1}], \alpha p^{n-d_2+1}[\alpha^{n-d_2+1} - \beta^{n-d_2+1}], \ldots, \alpha p^{n-d_t+1}[\alpha^{n-d_t+1} - \beta^{n-d_t+1}])$.

Let $(p,\gamma) = d$ then $d \mid p$ and $d \mid \gamma$ so $d \mid p\alpha + \beta\gamma$ and hence $d|1$ since $(p\alpha \equiv 1 \bmod \gamma)$. So $d = (p,\gamma) = 1$ and we know $(\alpha,\gamma) = 1$ by the hypothesis. Hence $(\gamma, \alpha p^{n-d_i+1}) = 1$. Then our presentation is

$$\begin{aligned}
G_\Gamma(R) &= \langle\, x_{n-1} \mid x_{n-1}^{(\gamma, \alpha^{n-d_1+1}-\beta^{n-d_1+1}, \alpha^{n-d_2+1}-\beta^{n-d_2+1}, \ldots, \alpha^{n-d_t+1}-\beta^{n-d_t+1})} \,\rangle \\
&= \langle\, x_{n-1} \mid x_{n-1}^{\alpha^{(n,n-d_1+1,n-d_2+1,\ldots,n-d_t+1)} - \beta^{(n,n-d_1+1,n-d_2+1,\ldots,n-d_t+1))}} \,\rangle \\
&= \langle\, x_{n-1} \mid x_{n-1}^{\alpha^{(n,1-d_1,1-d_2,\ldots,1-d_t)} - \beta^{(n,1-d_1,1-d_2,\ldots,1-d_t))}} \,\rangle \\
&\cong \mathbb{Z}_{|\alpha^{(n,d_1-1,d_2-1,\ldots,d_t-1)} - \beta^{(n,d_1-1,d_2-1,\ldots,d_t-1)}|}.
\end{aligned}$$

□

Theorem 2.2.3. *Let* $\Gamma = Cay(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})$. *If* $\alpha \neq 0, \beta \neq 0, (\alpha,\beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $\langle\, a, b \mid R(a,b) \,\rangle$ *and* $(d_1, n) = 1$, *then*

$$\begin{aligned}
G_\Gamma(R) &\cong G_{Cay(\mathbb{Z}_n, \{1, d_1^{-1}d_2, d_1^{-1}d_3, \ldots, d_1^{-1}d_t\})}(R) \\
&\cong \mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)} - \beta^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}|}.
\end{aligned}$$

Proof. Firstly, we will show $G_\Gamma(R) \cong G_{\mathrm{Cay}(\mathbb{Z}_n, \{1, d_1^{-1}d_2, d_1^{-1}d_3, \ldots, d_1^{-1}d_t\})}(R)$.

Since $(d_1, n) = 1$, there exists p, q such that $pd_1 + qn = 1$ and $pd_1 \equiv 1 \pmod n$.

The digraph $\mathrm{Cay}(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})$ has vertices $v_0, v_1, \ldots, v_{n-1}$ and arcs $(v_i, v_{i+d_1}), (v_i, v_{i+d_2}), \ldots, (v_i, v_{i+d_t})$, where $i = 0, 1, 2, \ldots, n-1$. We may relabel the vertices according to the rule $v_i \mapsto u_{pi}$, where $i = 0, 1, 2, \ldots, n-1$. Now, Γ has vertices $u_0, u_1, \ldots, u_{n-1}$ and arcs (u_{pi}, u_{pi+pd_1}), (u_{pi}, u_{pi+pd_2}), $\ldots$, (u_{pi}, u_{pi+pd_t}). Let $j = pi$ and we know $pd_1 \equiv 1 \pmod n$. Then the arcs are (u_j, u_{j+1}), (u_j, u_{j+pd_2}), $\ldots$, (u_j, u_{j+pd_t}). This means

$G_\Gamma(R) \cong G_{\mathrm{Cay}(\mathbb{Z}_n, \{1, pd_2, pd_3, \ldots, pd_t\})}(R) \cong G_{\mathrm{Cay}(\mathbb{Z}_n, \{1, d_1^{-1}d_2, d_1^{-1}d_3, \ldots, d_1^{-1}d_t\})}(R)$ since $p \equiv d_1^{-1} \pmod n$. Hence,

$G_{\mathrm{Cay}(\mathbb{Z}_n, \{1, d_1^{-1}d_2, d_1^{-1}d_3, \ldots, d_1^{-1}d_t\})} \cong \mathbb{Z}_{|\alpha^\Delta - \beta^\Delta|}$, where $\Delta = (n, d_1^{-1}d_2 - 1, d_1^{-1}d_3 - 1, \ldots, d_1^{-1}d_t - 1)$ by Lemma 2.2.2. Now our aim is to show

$$G_{\mathrm{Cay}(\mathbb{Z}_n, \{1, d_1^{-1}d_2, d_1^{-1}d_3, \ldots, d_1^{-1}d_t\})} \cong \mathbb{Z}_{|\alpha^{(n, d_2-d_1, d_3-d_1, \ldots, d_t-d_1)} - \beta^{(n, d_2-d_1, d_3-d_1, \ldots, d_t-d_1)}|}.$$

Therefore, we need to show $(n, d_1^{-1}d_t - 1) = (n, d_t - d_1)$ for some $2 \leqslant k \leqslant t$.

Let $(n, p) = \delta$ so $\delta \mid p$ and $\delta \mid n$ then $\delta \mid pd_1 + qn$ and hence $\delta \mid 1$ since $(d_1, n) = 1$ so $\delta = 1$.

We have

$$\begin{aligned}
(n, d_1^{-1}d_2 - 1, \ldots, d_1^{-1}d_t - 1) &= (n, pd_2 - 1, \ldots, pd_t - pd_1) \\
&\text{since } p \equiv d_1^{-1} \pmod n \\
&= (n, pd_2 - pd_1, \ldots pd_t - pd_1) \\
&\text{since } pd_1 \equiv 1 \pmod n \\
&= (n, p(d_2 - d_1), \ldots, p(d_t - d_1)) \\
&= (n, d_2 - d_1, \ldots, d_t - d_1) \\
&\text{since } (n, p) = 1 \; for \; 2 \leqslant k \leqslant t.
\end{aligned}$$

Thus, if $(d_1, n) = 1$, then

$G_{\mathrm{Cay}(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})} \cong \mathbb{Z}_{|\alpha^{(n, d_2-d_1, d_3-d_1, \ldots, d_t-d_1)} - \beta^{(n, d_2-d_1, d_3-d_1, \ldots, d_t-d_1)}|}$. □

Theorem 2.2.4. *Let $\Gamma = Cay(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})$. If $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $\langle\, a, b \mid R(a, b)\,\rangle$ and $(d_j, n) > 1$ for all $1 \leqslant j \leqslant t, (n, d_1, d_2, \ldots, d_t) = 1, (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) = 1$ then*

$G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(n, d_2-d_1, d_3-d_1, \ldots, d_t-d_1)} - \beta^{(n, d_2-d_1, d_3-d_1, \ldots, d_t-d_1)}|} \cong \mathbb{Z}_{|\alpha - \beta|}$.

Proof. If we find directed cycles and show their lengths are co-prime to each other, then we can say that $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha-\beta|}$ by Proposition 2.1.4.

Let $(d_1, d_2, \ldots, d_t) = \delta$ and $d_j = a_j\delta$ (for $1 \leqslant j \leqslant t$, $j \in \mathbb{Z}$).

If there is a directed cycle involving r_1 arcs labelled by d_1, r_2 arcs labelled by d_2, $\ldots, r_t$ arcs labelled by d_t then $r_1d_1 + r_2d_2 + \ldots + r_td_t \equiv 0 \pmod{n}$ and note that $r_1 + r_2 + \ldots + r_t$ give the length of the directed cycles.

Consider the congruence $r_1d_1 + r_2d_2 + \ldots + r_td_t \equiv 0 \pmod{n}$. This congruence has a solution $r_1 = n/(n, d_1)$, $r_j = 0$ for all j, and a solution $r_2 = n/(n, d_2)$, $r_j = 0$ for all j. That means $r_i = n/(n, d_i), r_j = 0$ for all j except for when $j = i$, $1 \leqslant j \leqslant t$.

We now seek a directed cycle where $r_j \neq 0$, for all $1 \leqslant j \leqslant t$.

$r_1d_1 + r_2d_2 + \ldots + r_td_t \equiv 0 \text{ mod } (n)$

$\Rightarrow r_1a_1\delta + r_2a_2\delta + \ldots + r_ta_t\delta \equiv 0 \text{ mod } (n)$

$\Rightarrow \delta(r_1a_1 + r_2a_2 + \ldots + r_ta_t) \equiv 0 \text{ mod } (n)$ since $(n, \delta) = 1$ by the hypothesis

$\Rightarrow r_1a_1 + r_2a_2 + \ldots + r_ta_t \equiv 0 \text{ mod } (n)$

$\Rightarrow \sum_{i=1}^{t} r_ia_i \equiv 0 \text{ mod } (n)$.

For any $2 \leqslant i \leqslant t$, let $r_1 = n - a_i$, $r_i = a_1$ and $r_j = 0$ for all $2 \leqslant j \leqslant t$, $j \neq i$.

$\Rightarrow r_1a_1 + \sum_{i=2}^{t} r_ia_i \equiv 0 \text{ mod } (n)$

$\Rightarrow (n - a_i)a_1 + \sum_{i=2}^{t} a_1a_i \equiv 0 \text{ mod } (n)$

$\Rightarrow na_1 - a_1a_i + a_1a_i \equiv 0 \text{ mod } (n)$

$\Rightarrow na_1 \equiv 0 \pmod{n}$.

Thus, we have directed $t-1$ cycle as $(n - a_i + a_1)$ for $2 \leqslant i \leqslant t$. We now claim that $(n, n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) = 1$.

To see this, we know $(n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) = 1$ by the hypothesis. Thus,

$$\begin{aligned}
1 &= (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) \\
&= (n, a_2\delta - a_1\delta, a_3\delta - a_1\delta, \ldots, a_t\delta - a_1\delta) \\
&= (n, \delta(a_2 - a_1), \delta(a_3 - a_1), \ldots, \delta(a_t - a_1)) \\
&= (n, a_2 - a_1, a_3 - a_1, \ldots, a_t - a_1) \text{ since } (n, \delta) = 1 \text{ by the hypothesis} \\
&= (n, n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1)
\end{aligned}$$

Since $(n, n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) = 1$, then $(n/(n, d_1), n -$

$a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) = 1$.

Thus we have directed cycles of length $n/(n, d_1)$ and $(n - a_i + a_1)$ for $2 \leqslant i \leqslant t$. And they are co-prime as shown above.

$(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})$ is a cyclic group so $(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})$ is abelian and it is known that every connected Cayley digraph on an abelian group has a Hamiltonian path [16, Theorem 3.1]. Then the result follows from Proposition 2.1.4. That is, we have proved $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha-\beta|}$.

$\square$

Theorem 2.2.5. *Let* $\Gamma = Cay(\mathbb{Z}_n, \{d_1, d_2, \ldots, d_t\})$. *If* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $\langle\, a, b \mid R(a, b)\,\rangle$ *and* $(d_j, n) > 1$ *for all* $1 \leqslant j \leqslant t$, $(n, d_1, d_2, \ldots, d_t) = 1, (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) > 1$ *then*

$G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\ldots,,d_t-d_1)}-\beta^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}|}$.

Proof. Applying the argument of lines $3 - 10$ of Theorem 2.2.4 we have $n/(n, d_j)$ directed cycles for $1 \leqslant j \leqslant t$, and applying the argument of lines $11 - 23$ of Theorem 2.2.4 we have $n - a_i + a_1$ for all $2 \leqslant i \leqslant t$. Let x the initial vertex, then it can be written as $x^{(n/(n,d_1),n-a_2+a_1,n-a_3+a_1,\ldots,n-a_t+a_1)}$ by Lemma 2.1.2.

We now claim that $(n/(n, d_1), n/(n, d_2), \ldots, n/(n, d_t), n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) = (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1)$.

To see this, let $(n, d_j) = k_j$ for $1 \leqslant j \leqslant t$, $(d_1, d_2, \ldots, d_t) = \delta, (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) = h$ and $(n/(n, d_1), n/(n, d_2), \ldots, n/(n, d_t), n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) = X$, where $d_j = a_j\delta$ for $1 \leqslant j \leqslant t$.

Let $(n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) = h$ and define $c_1 = n/h$ and $c_i = (d_i - d_1)/h$ for some $c_i \in \mathbb{Z}$ for $2 \leqslant j \leqslant t$.

$$\begin{aligned} X = (n/(n, d_1), \ldots, n/(n, d_t), n - a_2 + a_1, \ldots, n - a_t + a_1) \\ \mid (n, n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) \end{aligned}$$

then $X \mid (n, a_2 - a_1, a_3 - a_1, \ldots, a_t - a_1)$ and so $X \mid (n, (d_2 - d_1)/\delta, (d_3 - d_1)/\delta, \ldots, (d_t - d_1)/\delta)$.

Also $(n, (d_2 - d_1)/z, (d_3 - d_1)/z, \ldots, (d_t - d_1)/z) \mid (n, (d_2 - d_1), (d_3 -$

$d_1), \ldots, (d_t - d_1))$. Thus,

$$X \mid (n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) \Rightarrow X|h \tag{2.2.1}$$

Now,

$$\begin{aligned}
X &= (n/k_1, n/k_2, \ldots, n/k_t, n - a_2 + a_1, n - a_3 + a_1, \ldots, n - a_t + a_1) \\
&= (n/k_1, n/k_2, \ldots, n/k_t, n - d_2/\delta + d_1/\delta, n - d_3/\delta + d_1/\delta, \ldots, \\
&\quad n - d_t/\delta + d_1/\delta) \\
&= (n/k_1, n/k_2, \ldots, n/k_t, n - (d_2 - d_1)/\delta, n - (d_3 - d_1)/\delta, \ldots, \\
&\quad n - (d_t - d_1)/\delta) \\
&= (n/k_1, n/k_2, \ldots, n/k_t, hc_1 - hc_2/\delta, hc_1 - hc_3/\delta, \ldots, hc_1 - hc_t/\delta).
\end{aligned}$$

Multiplying both sides by $k_1k_2\ldots k_t\delta$ gives

$Xk_1k_2\ldots k_t\delta = (nk_2\ldots k_t\delta, nk_1k_3\ldots k_t\delta, \cdots, nk_1k_2\ldots k_{t-1}\delta, hc_1k_1k_2\ldots k_t\delta - hc_2k_1k_2\ldots k_t, hc_1k_1k_2\ldots k_t\delta - hc_3k_1k_2\ldots k_t, \cdots, hc_1k_1k_2\ldots k_t\delta - hc_tk_1k_2\ldots k_t)$.

Now, $h \mid n \Rightarrow h \mid nk_2\ldots k_t\delta, h \mid nk_1k_3\ldots k_t\delta$, up to $h \mid nk_1\ldots k_{t-1}\delta$ and

$h \mid hc_1k_1k_2\ldots k_t\delta - hc_2k_1k_2\ldots k_t$, $h \mid hc_1k_1k_2\ldots k_t\delta - hc_3k_1k_2\ldots k_t$, up to $h \mid hc_1k_1k_2\ldots k_t\delta - hc_tk_1k_2\ldots k_t$.

Therefore, h divides right hand side so $h \mid Xk_1k_2\ldots k_t\delta$.

$$\begin{aligned}
(h, k_j) &= (h, (n, d_j)) \\
&= ((n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1), (n, d_j)) \\
&= (n, d_j, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) \\
&= (n, d_1, d_2, \ldots, d_t) \\
&= 1 \ for \ 1 \leqslant j \leqslant t
\end{aligned}$$

$$\begin{aligned}
(h, \delta) &= ((n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1), (d_1, d_2, \ldots, d_t)) \\
&= (n, d_1, d_2, \ldots, d_t, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) \\
&= (1, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) \\
&= 1
\end{aligned}$$

Since $(h, k_j) = 1$ for $1 \leqslant j \leqslant t$ and $(h, \delta) = 1$, then

$$h \mid X \tag{2.2.2}$$

Thus $h = X$ by the equations 2.2.1 and 2.2.2. Hence,

$\mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}-\beta^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}|}$ maps onto $G_\Gamma(R)$.

Now we need to show that $G_\Gamma(R)$ maps onto

$\mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}-\beta^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}|}$.

Let $r = (n, d_2 - d_1, d_3 - d_1, \ldots, , d_t - d_1)$. So our aim is to show $G_\Gamma(R)$ maps onto $\langle\, x_{n-1} \mid x_{n-1}^{\alpha^r-\beta^r}\,\rangle$.

m_i is defined as $\phi(x_i) = x_{n-1}^{m_i}$ and

$$m_i = \begin{cases} \alpha^{r-1} & i \equiv \ 0 \bmod r \\ \beta^{r-1} & i \equiv \ d_1 \bmod r \\ \alpha\beta^{r-2} & i \equiv \ 2d_1 \bmod r \\ \alpha^2\beta^{r-3} & i \equiv \ 3d_1 \bmod r \\ \vdots & \vdots \\ \alpha^{r-2}\beta & i \equiv \ (r-1)d_1 \bmod r \\ \alpha^{r-1} & i \equiv \ rd_1 \bmod r \end{cases} \tag{2.2.3}$$

$d_j - d_1 \equiv 0 \bmod\ r$ for all $2 \leqslant j \leqslant t$.

$d_j \equiv d_1 \bmod r$.

Is $\{0, d_1, 2d_1, \ldots, (r-1)d_1\} (\bmod r)$ equal $\{0, 1, 2, \ldots, (r-1)\} (\bmod r)$?

We need to discuss this if $(d_1, r) = 1$, then the answer is yes. So

$(d_1, r) = (d_1, n, d_2 - d_1, d_3 - d_1, \ldots, d_t - d_1) = (n, d_1, d_2, \ldots, d_t) = 1$ by hypothesis. Thus, that is correct so $i \bmod r$ covers all generators.

We will now show $\phi(R(x_i, x_{i+d_j}) = 1$ for each i, j. So ϕ is a homomorphism.

$i \equiv 0 \bmod r \quad \phi(R(x_i, x_{i+d_1})) = (x_{n-1}^{m_i})^\alpha (x_{n-1}^{m_{i+d_1}})^{-\beta} = (x_{n-1}^{m_0})^\alpha (x_{n-1}^{m_{d_1}})^{-\beta} = (x_{n-1}^{\alpha^{r-1}})^\alpha (x_{n-1}^{\beta^{r-1}})^{-\beta} = x_{n-1}^{\alpha^r-\beta^r}$

$i \equiv d_1 \bmod r \quad (x_{n-1}^{m_{d_1}})^\alpha (x_{n-1}^{m_{2d_1}})^{-\beta} = (x_{n-1}^{\beta^{r-1}})^\alpha (x_{n-1}^{\alpha\beta^{r-2}})^{-\beta} = 1$

$i \equiv 2d_1 \bmod r \quad (x_{n-1}^{m_{2d_1}})^\alpha (x_{n-1}^{m_{3d_1}})^{-\beta} = (x_{n-1}^{\alpha\beta^{r-2}})^\alpha (x_{n-1}^{\alpha^2\beta^{r-3}})^{-\beta} = 1$

$\vdots$

$i \equiv (r-1)d_1 \bmod r \quad (x_{n-1}^{m_{(r-2)d_1}})^\alpha (x_{n-1}^{m_{(r-1)d_1}})^{-\beta} = (x_{n-1}^{\alpha^{r-3}\beta^2})^\alpha (x_{n-1}^{\alpha^{r-2}\beta})^{-\beta} = 1$

$i \equiv 0 \bmod r \quad \phi(R(x_i, x_{i+d_j})) = (x_{n-1}^{m_i})^{\alpha}(x_{n-1}^{m_{i+d_j}})^{-\beta} = (x_{n-1}^{m_0})^{\alpha}(x_{n-1}^{m_{d_j}})^{-\beta} = (x_{n-1}^{\alpha^{r-1}})^{\alpha}(x_{n-1}^{\beta^{r-1}})^{-\beta} = x_{n-1}^{\alpha^r-\beta^r}$

Since $d_j \equiv d_1 \bmod r$ *for all* $2 \leqslant j \leqslant t$

$i \equiv d_j \bmod r \quad (x_{n-1}^{m_{d_j}})^{\alpha}(x_{n-1}^{m_{2d_j}})^{-\beta} = (x_{n-1}^{\beta^{r-1}})^{\alpha}(x_{n-1}^{\alpha\beta^{r-2}})^{-\beta} = 1$

$i \equiv 2d_j \bmod r \quad (x_{n-1}^{m_{2d_j}})^{\alpha}(x_{n-1}^{m_{3d_j}})^{-\beta} = (x_{n-1}^{\alpha\beta^{r-2}})^{\alpha}(x_{n-1}^{\alpha^2\beta^{r-3}})^{-\beta} = 1$

$\vdots$

$i \equiv (r-1)d_j \bmod r \quad (x_{n-1}^{m_{(r-2)d_j}})^{\alpha}(x_{n-1}^{m_{(r-1)d_j}})^{-\beta} = (x_{n-1}^{\alpha^{r-3}\beta^2})^{\alpha}(x_{n-1}^{\alpha^{r-2}\beta})^{-\beta} = 1.$

Thus, ϕ is a homomorphism since each relator maps to the identity. Now we show that ϕ is an epimorphism.

α^{r-1} generates $\mathbb{Z}_{\alpha^r-\beta^r}$. Let $N = \alpha^r - \beta^r$, then $(\alpha^{r-1}, N) = 1$. Therefore, α^{r-1} generates $\mathbb{Z}_N$. $i = (\alpha^{r-1})^K$ for some K, $i \in \{0, 1, \ldots, N-1\}$. $(\phi(x_o))^K = (\alpha^{r-1})^K = i$. Hence, $\phi(x_o)$ generates $\mathbb{Z}_{\alpha^r-\beta^r}$. Thus, ϕ is onto.

Hence, $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}-\beta^{(n,d_2-d_1,d_3-d_1,\ldots,d_t-d_1)}|}$.

$\square$

2.3 Cayley digraphs of direct sum of Cyclic groups

Theorem 2.3.1. *Let* $\Gamma = Cay(\mathbb{Z}_{m_1} \oplus \mathbb{Z}_{m_2} \oplus \ldots \oplus \mathbb{Z}_{m_t}, \{e_1, e_2, \ldots, e_t\})$, *where* $\{e_1, e_2, \ldots, e_t\} = \{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$. *If* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $\langle\, a, b \mid R(a, b)\,\rangle$, *then* $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(m_1,m_2,\ldots,m_t)}-\beta^{(m_1,m_2,\ldots,m_t)}|}$.

Proof. The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle\ x_{(i_1,i_2,\ldots,i_t)}\ \middle|\ \begin{array}{l} R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1+1,i_2,\ldots,i_t)})\ , i_1 \in [0, m_1-1] \\ R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1,i_2+1,\ldots,i_t)})\ , i_2 \in [0, m_2-1] \\ \vdots \\ R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1,i_2,\ldots,i_t+1)})\ , i_t \in [0, m_t-1] \end{array} \right\rangle.$$

If $(i, j) \in A(\Gamma)$ then $x_i^\alpha = x_j^\beta$ in G. Thus,

We have $x_{(i_1,i_2,\ldots,i_t)}^{\alpha^{m_1}} = x_{(i_1+1,i_2,\ldots,i_t)}^{\alpha^{m_1-1}\beta} = x_{(i_1+2,i_2,\ldots,i_t)}^{\alpha^{m_1-2}\beta^2} = \ldots = x_{(i_1+m_1-1,i_2,\ldots,i_t)}^{\alpha\beta^{m_1-1}} = x_{(i_1,i_2,\ldots,i_t)}^{\beta^{m_1}}$ so we may adjoin relators $x_{(i_1,i_2,\ldots,i_t)}^{\alpha^{m_1}-\beta^{m_1}} = 1$.

We have $x^{\alpha^{m_2}}_{(i_1,i_2,\ldots,i_t)} = x^{\alpha^{m_2-1}\beta}_{(i_1,i_2+1,\ldots,i_t)} = x^{\alpha^{m_2-2}\beta^2}_{(i_1,i_2+2,\ldots,i_t)} = \ldots = x^{\alpha\beta^{m_2-1}}_{(i_1,i_2+m_2-1,\ldots,i_t)} = x^{\beta^{m_2}}_{(i_1,i_2,\ldots,i_t)}$ so we may adjoin relators $x^{\alpha^{m_2}-\beta^{m_2}}_{(i_1,i_2,\ldots,i_t)} = 1$.

$\vdots$

We have $x^{\alpha^{m_t}}_{(i_1,i_2,\ldots,i_t)} = x^{\alpha^{m_t-1}\beta}_{(i_1,i_2,\ldots,i_t+1)} = x^{\alpha^{m_t-2}\beta^2}_{(i_1,i_2,\ldots,i_t+2)} = \ldots = x^{\alpha\beta^{m_t-1}}_{(i_1,i_2,\ldots,i_t+m_t-1)} = x^{\beta^{m_t}}_{(i_1,i_2,\ldots,i_t)}$ so we may adjoin relators $x^{\alpha^{m_t}-\beta^{m_t}}_{(i_1,i_2,\ldots,i_t)} = 1$.

After we adjoin these relators, then our new presentation is

$$G_\Gamma(R) = \left\langle \quad x_{(i_1,i_2,\ldots,i_t)} \quad \middle| \quad \begin{array}{l} R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1+1,i_2,\ldots,i_t)}) \ , i_1 \in [0, m_1-1] \\ R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1,i_2+1,\ldots,i_t)}) \ , i_2 \in [0, m_2-1] \\ \vdots \\ R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1,i_2,\ldots,i_t+1)}) \ , i_t \in [0, m_t-1] \\ x^{\alpha^{m_1}-\beta^{m_1}}_{(i_1,i_2,\ldots,i_t)}, x^{\alpha^{m_2}-\beta^{m_2}}_{(i_1,i_2,\ldots,i_t)}, \ldots, x^{\alpha^{m_t}-\beta^{m_t}}_{(i_1,i_2,\ldots,i_t)} \end{array} \right\rangle.$$

Let $\gamma_s = \alpha^{m_s} - \beta^{m_s}$ for $1 \leqslant s \leqslant t$, since$(\alpha, \beta) = 1$ there exists p, q such that $p\alpha + q\gamma_s = 1$ and hence $p\alpha = 1$ (modγ_s). If $(i, j) \in A(\Gamma)$ then $x_i^\alpha = x_j^\beta$ in G so $x_i = x_i^{p\alpha+q\gamma_i} = x_i^{p\alpha} = x_j^{p\beta}$ thus $x_i = x_j^{p\beta}$ by Lemma 1.3.4 (a) so we have this equation

$$x_{(i_1,i_2,\ldots,i_t)} = x^{(p\beta)}_{(i_1+1,i_2,i_3,\ldots,i_t)} = x^{(p\beta)^2}_{(i_1+2,i_2,i_3,\ldots,i_t)} = \ldots = x^{(p\beta)^{(m_1-i_1-1)}}_{(m_1-1,i_2,i_3,\ldots,i_t)}$$

$$\begin{aligned} x^{(p\beta)^{(m_1-i_1-1)}}_{(m_1-1,i_2,i_3,\ldots,i_t)} &= x^{(p\beta)^{(m_1-i_1-1)+1}}_{(m_1-1,i_2+1,i_3,\ldots,i_t)} \\ &= x^{(p\beta)^{(m_1-i_1-1)+2}}_{(m_1-1,i_2+2,i_3,\ldots,i_t)} = \ldots = x^{(p\beta)^{(m_1-i_1-1)+(m_2-i_2-1)}}_{(m_1-1,m_2-1,i_3,\ldots,i_t)}. \end{aligned}$$

Continuing in this way, we get

$$x_{(i_1,i_2,\ldots,i_t)} = x^{(p\beta)^{(m_1+m_2+\ldots+m_t)-(i_1+i_2+\ldots+i_t)-t}}_{(m_1-1,m_2-1,\ldots,m_t-1)}. \tag{2.3.1}$$

We adjoin these relations to the presentation.

Now, let $(m_1 + m_2 + \ldots + m_t) = y$ and $(i_1 + 1 + i_2 + \ldots + i_t) = z$. Then

$$\begin{aligned}
R(x_{(i_1,i_2,\ldots,i_t)}, x_{(i_1+1,i_2,\ldots,i_t)}) &= R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}^{(p\beta)^{(m_1+m_2+\ldots+m_t)-(i_1+i_2+\ldots+i_t)-t}}, \\
&\quad x_{(m_1-1,m_2-1,\ldots,m_t-1)}^{(p\beta)^{(m_1+m_2+\ldots+m_t)-(i_1+1+i_2+\ldots+i_t)-t}}) \\
&= R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}^{(p\beta)^{y-z-t}}, x_{(m_1-1,m_2-1,\ldots,m_t-1)}^{(p\beta)^{y-(z+1)-t}}) \\
&= x_{(m_1-1,m_2-1,\ldots,m_t-1)}^{\alpha[(p\beta)^{y-z-t}]-\beta[(p\beta)^{y-(z+1)-t}]} \\
&= x_{(m_1-1,m_2-1,\ldots,m_t-1)}^{p^{y-z-t-1}\beta^{y-z-t}-p^{y-z-t-1}\beta^{y-z-t}} \\
&= x^0 \\
&= 1.
\end{aligned}$$

Similarly

$$\begin{aligned}
R(x_{(i_1,i_2,i_3,\ldots,i_t)}, x_{(i_1,i_2+1,i_3\ldots,i_t)}) &= 1 \\
R(x_{(i_1,i_2,i_3\ldots,i_t)}, x_{(i_1,i_2,i_3+1\ldots,i_t)}) &= 1 \\
&\vdots \\
R(x_{(i_1,i_2,i_3\ldots,i_t)}, x_{(i_1,i_2,i_3\ldots,i_t+1)}) &= 1
\end{aligned}$$

Thus, each of these relations are redundant and so we may remove such relations from the presentation. We then use the relations (2.3.1) to eliminate all generators except for

$x_{(m_1-1,m_2-1,m_3-1,\ldots,m_t-1)}$,

$x_{(0,m_2-1,m_3-1,\ldots,m_t-1)}$,

$x_{(m_1-1,0,m_3-1,\ldots,m_t-1)}$,

$\vdots$

$x_{(m_1-1,m_2-1,m_3-1,\ldots,m_{t-1}-1,0)}$

and the corresponding relations.

This gives

$$G_\Gamma(R) = \left\langle \begin{array}{l|l} x_{(m_1-1,m_2-1,\ldots,m_t-1)}, & R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}, x_{(0,m_2-1,\ldots,m_t-1)}), \\ x_{(0,m_2-1,\ldots,m_t-1)}, & R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}, x_{(m_1-1,0,\ldots,m_t-1)}), \\ x_{(m_1-1,0,\ldots,m_t-1)}, & \vdots \\ \vdots & R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}, x_{(m_1-1,m_2-1,\ldots,0)}), \\ x_{(m_1-1,m_2-1,\ldots,0)} & x^{\alpha^{m_1}-\beta^{m_1}}_{(i_1,i_2,\ldots,i_t)}, x^{\alpha^{m_2}-\beta^{m_2}}_{(i_1,i_2,\ldots,i_t)}, \ldots, x^{\alpha^{m_t}-\beta^{m_t}}_{(i_1,i_2,\ldots,i_t)} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l|l} & R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}, x^{(p\beta)^{m_1-1}}_{(m_1-1,m_2-1,\ldots,m_t-1)}), \\ & R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}, x^{(p\beta)^{m_2-1}}_{(m_1-1,m_2-1,\ldots,m_t-1)}), \\ x_{(m_1-1,m_2-1,\ldots,m_t-1)} & \vdots \\ & R(x_{(m_1-1,m_2-1,\ldots,m_t-1)}, x^{(p\beta)^{m_t-1}}_{(m_1-1,m_2-1,\ldots,m_t-1)}), \\ & x^{\alpha^{m_1}-\beta^{m_1}}_{(m_1-1,m_2-1,\ldots,m_t-1)}, x^{\alpha^{m_2}-\beta^{m_2}}_{(m_1-1,m_2-1,\ldots,m_t-1)}, \ldots, \\ & x^{\alpha^{m_t}-\beta^{m_t}}_{(m_1-1,m_2-1,\ldots,m_t-1)} \end{array} \right\rangle.$$

Let $h = (m_1-1, m_2-1, \ldots, m_t-1)$ and after substituting in the presentation, we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle\; x_h \;\middle|\; \begin{array}{l} R(x_h, x_h^{(p\beta)^{m_1-1}}), R(x_h, x_h^{(p\beta)^{m_2-1}}), \ldots, R(x_h, x_h^{(p\beta)^{m_t-1}}) \\ x_h^{\alpha^{m_1}-\beta^{m_1}}, x_h^{\alpha^{m_2}-\beta^{m_2}}, \ldots, x_h^{\alpha^{m_t}-\beta^{m_t}} \end{array} \right\rangle \\ &= \left\langle\; x_h \;\middle|\; \begin{array}{l} x_h^{\alpha-\beta(p\beta)^{m_1-1}}, x_h^{\alpha-\beta(p\beta)^{m_2-1}}, \ldots, x_h^{\alpha-\beta(p\beta)^{m_t-1}} \\ x_h^{\alpha^{m_1}-\beta^{m_1}}, x_h^{\alpha^{m_2}-\beta^{m_2}}, \ldots, x_h^{\alpha^{m_t}-\beta^{m_t}} \end{array} \right\rangle \\ &= \langle\, x_h \mid x_h^{(\alpha-\beta(p\beta)^{m_1-1}, \alpha^{m_1}-\beta^{m_1})}, x_h^{(\alpha-\beta(p\beta)^{m_2-1}, \alpha^{m_2}-\beta^{m_2})}, \ldots, \\ &\quad x_h^{(\alpha-\beta(p\beta)^{m_t-1}, \alpha^{m_t}-\beta^{m_t})} \rangle \\ &= \langle\, x_h \mid x_h^{(\alpha-\beta(p\beta)^{m_i-1}, \alpha^{m_i}-\beta^{m_i})},\ for\ 1 \leqslant i \leqslant t \,\rangle. \end{aligned}$$

Let $\gamma_i = \alpha^{m_i} - \beta^{m_i}$ for some $1 \leqslant i \leqslant t$. Then,

$$\begin{aligned}
\beta[(p\beta)^{m_i-1}] &= p^{m_i-1}\beta^{m_i} \\
&\equiv (p\alpha)p^{m_i-1}\beta^{m_i} \quad \textit{since } (p\alpha \equiv 1 \bmod \gamma_i) \\
&\equiv \alpha p^{m_i}\beta^{m_i} \\
&\equiv \alpha p^{m_i}(\alpha^{m_i} - \gamma_i) \quad \textit{since } (\gamma_i = \alpha^{m_i} - \beta^{m_i}) \\
&\equiv \alpha(p\alpha)^{m_i} - \alpha p^{m_i}\gamma_i \\
&\equiv \alpha \pmod{\gamma_i}.
\end{aligned}$$

So γ_i divides $\alpha - \beta[(p\beta)^{m_i-1}]$. Thus,

$$\begin{aligned}
G_\Lambda(R) &= \langle\, x_h \mid x_h^{\gamma_1}, x_h^{\gamma_2}, \ldots, x_h^{\gamma_t} \,\rangle \\
&= \langle\, x_h \mid x_h^{(\gamma_1,\gamma_2,\ldots,\gamma_t)} \,\rangle \\
&= \langle\, x_h \mid x_h^{(\alpha^{m_1}-\beta^{m_1},\alpha^{m_2}-\beta^{m_2},\ldots,\alpha^{m_t}-\beta^{m_t})} \,\rangle \\
&= \langle\, x_h \mid x_h^{\alpha^{(m_1,m_2,\ldots,m_t)}-\beta^{(m_1,m_2,\ldots,m_t)}} \,\rangle \\
&\cong \mathbb{Z}_{|\alpha^{(m_1,m_2,\ldots,m_t)}-\beta^{(m_1,m_2,\ldots,m_t)}|}.
\end{aligned}$$

□

2.4 Cayley digraphs of Quaternion groups

Example 2.4.1. The Cayley graph of the figure in the case $n = 2$ is given in ([13, page 485]). This figure is also given here.

$$H = \mathrm{Cay}(\mathcal{Q}_4, \{a, b\}) = \langle\, a, b \mid a^4, a^2 = b^2, b^{-1}ab = a^{-1} \,\rangle.$$

Theorem 2.4.2. *Let* $\Gamma = Cay(\mathcal{Q}_{2n}, \{a, b\})$*. If* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $\langle\, a, b \mid R(a, b) \,\rangle$ *then*

$$G_\Gamma(R) \cong \begin{cases} \mathbb{Z}_{|\alpha-\beta|} & if\ n\, is\, odd \\ \mathbb{Z}_{|\alpha^2-\beta^2|} & if\ n\, is\, even \end{cases} \tag{2.4.1}$$

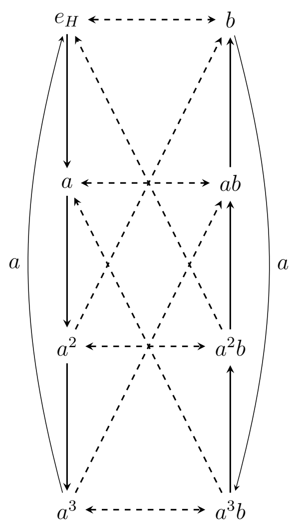

Figure 2.1: Cay($\mathcal{Q}_4, \{a, b\}$)

Proof. The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_e, x_a, x_{a^2}, \\ \ldots, x_{a^{2n-1}}, \\ x_b, x_{ab}, x_{a^2b}, \\ \ldots, x_{a^{2n-1}b} \end{array} \;\middle|\; \begin{array}{l} R(x_e, x_a), R(x_a, x_{a^2}), \ldots, R(x_{a^{2n-1}}, x_e), \\ R(x_{a^{2n-1}b}, x_{a^{2n-2}b}), R(x_{a^{2n-2}b}, x_{a^{2n-3}b}), \ldots, \\ R(x_b, x_{a^{2n-1}b}), \\ R(x_e, x_b), R(x_a, x_{ab}), \ldots, R(x_{a^{2n-1}}, x_{a^{2n-1}b}), \\ R(x_{a^nb}, x_e), R(x_{a^{n+1}b}, x_a), \ldots, R(x_{a^{2n-1}b}, x_{a^{n-1}}), \\ R(x_b, x_{a^n}), R(x_{ab}, x_{a^{n+1}}), \ldots, R(x_{a^{n-1}b}, x_{a^{2n-1}}) \end{array} \right\rangle$$

$$= \left\langle x_{a^i}, x_{a^ib} \;\middle|\; \begin{array}{l} R(x_{a^i}, x_{a^{i+1}}), R(x_{a^ib}, x_{a^{i-1}b}), R(x_{a^i}, x_{a^ib}), \\ R(x_{a^{n+i}b}, x_{a^i}) \; for \; 0 \leqslant i \leqslant 2n-1 \end{array} \right\rangle.$$

There is always a Hamiltonian path in Γ as $e \xrightarrow{\text{a}} a \xrightarrow{\text{a}} a^2 \xrightarrow{\text{a}} \ldots \xrightarrow{\text{a}} a^{2n-1} \xrightarrow{\text{b}} a^{2n-1}b \xrightarrow{\text{a}} a^{2n-1}ba = a^{2n-2}b \xrightarrow{\text{a}} \ldots \xrightarrow{\text{a}} ab \xrightarrow{\text{a}} aba = b$. Since $ba = a^{-1}b$ by hypothesis.

Firstly, we always have a directed cycle in the length of $3n$,

$$\underbrace{e \xrightarrow{a} a \xrightarrow{a} a^2 \xrightarrow{a} \dots \xrightarrow{a} a^{2n-1}}_{\text{2n-1}} \underbrace{\xrightarrow{b}}_{1}$$

$$\underbrace{a^{2n-1}b \xrightarrow{a} a^{2n-1}ba = a^{2n-2}b \xrightarrow{a} a^{2n-2}ba = a^{2n-3}b \xrightarrow{a} \dots \xrightarrow{a} a^n b}_{\text{n-1}}$$

$$\underbrace{\xrightarrow{b}}_{1} a^n b^2 = a^n a^n = a^{2n} = e$$

Secondly, we always have a directed cycle in the length of $2n$,
$\underbrace{e \xrightarrow{a} a \xrightarrow{a} a^2 \xrightarrow{a} \dots \xrightarrow{a} a^{2n-1} \xrightarrow{a} a^{2n} = e}_{\text{2n}}$.
Thirdly, we always have a directed cycle in the length of 4,
$\underbrace{e \xrightarrow{b} b \xrightarrow{b} b^2 = a^n \xrightarrow{b} a^n b \xrightarrow{b} a^n b^2 = a^n a^n = a^{2n} = e}_{4}$.
Lastly, we always have a directed cycle in the length of $n+2$,
$\underbrace{e \xrightarrow{a} a \xrightarrow{a} a^2 \xrightarrow{a} \dots \xrightarrow{a} a^n}_{\text{n}} \underbrace{\xrightarrow{b}}_{1} \underbrace{a^n b \xrightarrow{b} a^n b^2 = a^n a^n = a^{2n} = e}_{1}$

Let v be the terminal vertex of the Hamiltonian path and $G_\Gamma(R)$ is generated by x_v, so it is cyclic. Since v is a vertex of directed cycle of length $3n$, then $x_v^{\alpha^{3n}-\beta^{3n}}$ and v is a vertex of directed cycle of length $2n$, then $x_v^{\alpha^{2n}-\beta^{2n}}$ and v is a vertex of directed cycle of length 4, then $x_v^{\alpha^4-\beta^4}$ and lastly v is a vertex of directed cycle of length $n+2$, then $x_v^{\alpha^{n+2}-\beta^{n+2}}$ by Lemma 2.1.2.

Hence, the group $G_\Gamma(R)$ is a quotient of

$$\begin{aligned} &\langle\, x_v \mid x_v^{\alpha^{3n}-\beta^{3n}}, x_v^{\alpha^{2n}-\beta^{2n}}, x_v^{\alpha^4-\beta^4}, x_v^{\alpha^{n+2}-\beta^{n+2}} \,\rangle \\ &= \langle\, x_v \mid x_v^{\alpha^{(3n,2n,4,n+2)}-\beta^{(3n,2n,4,n+2)}} \,\rangle \cong \mathbb{Z}_{|\alpha^{(n,4,n+2)}-\beta^{(n,4,n+2)}|}. \end{aligned}$$

Now, $(n,4,n+2) = (n,4,2) = (n,2)$ so $\mathbb{Z}_{|\alpha^{(n,2)}-\beta^{(n,2)}|}$ maps onto $G_\Gamma(R)$.

If n is odd then $\mathbb{Z}_{|\alpha-\beta|}$ maps onto $G_\Gamma(R)$ and therefore $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha-\beta|}$ by Proposition 2.1.4.

If n is even then $\mathbb{Z}_{|\alpha^2-\beta^2|}$ maps onto $G_\Gamma(R)$. Now we claim that if n is even, $G_\Gamma(R)$ maps onto $\mathbb{Z}_{|\alpha^2-\beta^2|}$.

Let $G_\Gamma(R) = \langle\, x_v\ (v \in V(\Lambda)) \mid R(x_u, x_v)\ ((u,v) \in A(\Lambda)) \,\rangle$.

We define the map $G_\Gamma(R) \xrightarrow{\phi} \langle\, x_{n-1} \mid x_{n-1}^{\alpha^2-\beta^2} \,\rangle \cong \mathbb{Z}_{\alpha^2-\beta^2}$ by

$$\phi(x_{a^i}) = \begin{cases} x_{n-1}^{\alpha} & i \textit{ even} \\ x_{n-1}^{\beta} & i \textit{ odd} \end{cases} \tag{2.4.2}$$

$$\phi(x_{a^ib}) = \begin{cases} x_{n-1}^{\beta} & i \textit{ even} \\ x_{n-1}^{\alpha} & i \textit{ odd} \end{cases} \tag{2.4.3}$$

Then

$$\phi(R(x_{a^i}, x_{a^{i+1}})) = \phi(x_{a_i})^{\alpha}\phi(x_{a_{i+1}})^{-\beta} = \begin{cases} (x_{n-1}^{\alpha})^{\alpha}(x_{n-1}^{\beta})^{-\beta} & i \textit{ even} \\ (x_{n-1}^{\beta})^{\alpha}(x_{n-1}^{\alpha})^{-\beta} & i \textit{ odd} \end{cases} = 1 \tag{2.4.4}$$

$$\phi(R(x_{a^ib}, x_{a^{i+1}b})) = \phi(x_{a_i})^{\alpha}\phi(x_{a_{i+1}})^{-\beta} = \begin{cases} (x_{n-1}^{\beta})^{\alpha}(x_{n-1}^{\alpha})^{-\beta} & i \textit{ even} \\ (x_{n-1}^{\alpha})^{\alpha}(x_{n-1}^{\beta})^{-\beta} & i \textit{ odd} \end{cases} = 1 \tag{2.4.5}$$

So ϕ is a homomorphism since each relator maps to the identity. Now $(\alpha, \beta) = 1$ so there exist p, q such that $p\alpha + q\beta = 1$. Therefore,

$x_{n-1} = x_{n-1}^1 = x_{n-1}^{p\alpha+q\beta} = (x_{n-1}^{\alpha})^p(x_{n-1}^{\beta})^q = \phi(x_{a^2}^p)\phi(x_a^q) = \phi(x_{a^2}^p x_a^q)$. Hence, ϕ is an epimorphism. Thus, $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^2-\beta^2|}$. □

2.5 Cayley digraphs of direct product of two groups

Theorem 2.5.1. *Let H, K be finite groups with generating sets S, T, and identity elements e, f, respectively. Let $\Gamma_H = Cay(H, S), \Gamma_K = Cay(K, T)$. Let $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $\langle\, a, b \mid R(a, b)\,\rangle$. Suppose $G_{\Gamma_H}(R)$ is finite cyclic, generated by x_e and $G_{\Gamma_K}(R)$ is finite cyclic generated by x_f. Let $\Gamma = Cay(H \times K, S \times T)$. Then $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^{(N_1,\ldots,N_p)} - \beta^{(N_1,\ldots,N_p)}|}$, where $N_1, \ldots, N_p$ are the length of the directed cycles in Γ.*

Proof. Since G_{Γ_K} is finite cyclic generated by x_f, any generator x_k of G_{Γ_K}

can be written in terms of x_f. The arcs of Γ are labelled (s,t) $s \in S,\ t \in T$

$$V(\Gamma) = \{(h,k) \mid h \in H, k \in K\} \text{ and } G_\Gamma(R) = \langle\, x_{(h,k)}\ (h \in H, k \in K) \mid \dots \rangle$$

Now, fix h in H. There is a path from (h,f) to (h,k) for any k in Γ_K. Thus, since $a^\alpha = b^\beta$ in $K = \langle\, a,b \mid R(a,b)\,\rangle$ as in [9] any generator $x_{(h,k)}$ can be written in terms of the generator $x_{(h,f)}$. There is also a path from (e,f) to (h,f) for each h in Γ_H. Thus, any generator $x_{(h,f)}$ can be written in terms of $x_{(e,f)}$. Therefore any generator $x_{(h,k)}$ can be written in terms of $x_{(e,f)}$. Thus, G_{Γ_R} is cyclic.

Since Γ is a Cayley digraph it has a Hamiltonian path with (e,f) as the terminal vertex. By hypothesis (e,f) is a vertex of directed cycles of length $N_1,\dots,N_p$. Then $x_{(e,f)}^{\alpha^{N_1}-\beta^{N_1}} = 1, x_{(e,f)}^{\alpha^{N_2}-\beta^{N_2}} = 1, \dots, x_{(e,f)}^{\alpha^{N_p}-\beta^{N_p}} = 1$ by Lemma 2.1.2. Hence, we get

$$\begin{aligned}&\langle\, x_{(e,f)} \mid x_{(e,f)}^{\alpha^{N_1}-\beta^{N_1}}, x_{(e,f)}^{\alpha^{N_2}-\beta^{N_2}}, \dots, x_{(e,f)}^{\alpha^{N_p}-\beta^{N_p}} \,\rangle \\ &= \langle\, x_{(e,f)} \mid x_{(e,f)}^{\alpha^{(N_1,N_2,N_3,\dots,N_p)}-\beta^{(N_1,N_2,N_3,\dots,N_p)}} \,\rangle \cong \mathbb{Z}_{|\alpha^{(N_1,N_2,N_3,\dots,N_p)}-\beta^{(N_1,N_2,N_3,\dots,N_p)}|}.\end{aligned}$$

□

Example 2.5.2. Let $H = \mathrm{Cay}(\mathcal{Q}_4, \{a,b\}) = \langle\, a,b \mid a^4, a^2 = b^2, b^{-1}ab = a^{-1}\,\rangle$, $K = \mathrm{Cay}(\mathbb{Z}_2, \{1\})$ and

$\Gamma = \mathrm{Cay}(H \times K, S \times T) = \mathrm{Cay}(\mathcal{Q}_4 \oplus \mathbb{Z}_2, \{(a,0),(b,0),(e,1)\})$, then the figure of this digraph is in Figure 2.2, below.

In this example, we can find directed cycles in the length of $2,4,6,8,\dots$ but it is not possible to find a directed cycle in odd length. For example, $[(e,0),(b,0)], [(b,0),(e,0)]$, which is length 2, and $[(e,0),(a,0)], [(a,0),(a^2,0)]$ $[(a^2,0),(a^2b,0)], [(a^2b,0),(e,0)]$, which is length 4. Hence, $G_\Gamma(R) \cong \mathbb{Z}_{|\alpha^2-\beta^2|}$.

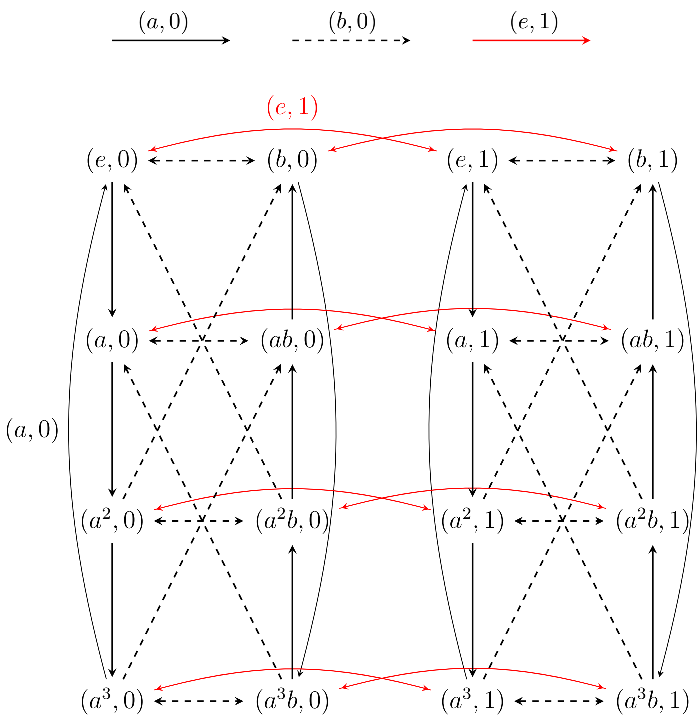

Figure 2.2: $\mathrm{Cay}(\mathcal{Q}_4 \oplus \mathbb{Z}_2, \{(a, 0), (b, 0), (e, 1)\})$

CHAPTER 3

Digraph groups with $|V| = |A| - 1$

3.1 Introduction

Presentations with more generators than relators necessarily define infinite groups, which can be seen by abelianizing the groups [21, page 84]. Cuno & Williams [9] investigated digraph groups $G_\Gamma(R)$ where $|V(\Gamma)| = |A(\Gamma)|$ (i.e with equal number of generators as relators) and where the undirected graph is triangle free (i.e. $g(\Gamma) \geqslant 4$) and in most cases they proved that the corresponding group $G_\Gamma(R)$ is either finite cyclic or infinite. Therefore, in this chapter we investigate the case $|V| = |A| - 1$. Before defining the classes of digraphs, we construct the graphs under the following conditions.

(i) Γ connected (if Γ is disconnected with components $\Gamma_1, \Gamma_2, \ldots \Gamma_N$ then $G_\Gamma(R) \cong G_{\Gamma_1}(R) * \ldots * G_{\Gamma_N}(R)$. Thus we may assume that Γ is connected) [7, page 13].

(ii) $|V| = |A| - 1$ (it is known if $|V| > |A|$, then the group is infinite [21, page 84], and Cuno and Williams investigated possible graphs when $|V| = |A|$ in [9]).

(iii) $|\alpha| \geqslant 2$, $|\beta| \geqslant 2$ (when $\alpha = 1$ and β arbitrary or $\beta = 1$ and α is arbitrary enables us recursively prune the graphs but we will focus on $|\alpha| \geqslant 2$, $|\beta| \geqslant 2$ which gives us interesting digraphs).

(iv) $0 \leqslant \sigma_1 \leqslant \sigma \leqslant 1$ and $0 \leqslant \tau_1 \leqslant \tau \leqslant 1$, where $\sigma, \tau, \sigma_1, \tau_1$ are the number of sources, sinks, source leaves and sink leaves respectively, (Cuno and Williams showed why we look between 0 and 1 [9]. It is also explained in Lemma 1.3.3).

(v) $\sigma \geqslant \tau$ and if $\sigma = \tau$, then $\sigma_1 \geqslant \tau_1$ (by reflection principle (see Remark 1.3.7)).

Lemma 3.1.1. *Under these circumstances* $(i), (ii), (iv), (v)$*, there are* 35 *possible digraph families as indicated in Figure 3.4 on page 58 and Figure 3.5 on page 59.*

Proof. Since entire digraph is connected (i), there are two possibilities. First consider the case where the graph has no leaves. There are two possibilities for the form of the underlying graph. One possibility is that the underlying graph is constructed by fusing two cycles together with a path between them (as in Γ_1); the other is to connect two cycles along a path common to both cycles (as in Γ_2). The figures are given in Figure 3.1, below where we label two particular vertices k, l. We now need to direct Γ_1 and Γ_2 to specify the possible digraph families.

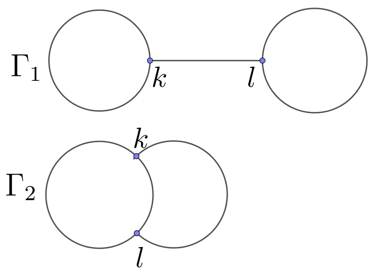

Figure 3.1: Two possible undirected graphs with $|V| = |A| - 1$.

By conditions (iv) and (v) we have $(\sigma, \tau) = (0, 0), (1, 0)$ or $(1, 1)$.

Case 1: $(\sigma, \tau) = (0, 0)$**.** Then $(\sigma_1, \tau_1) = (0, 0)$.

For Γ_1, these two cycles are the directed cycles since they have neither source nor sink and the path between them could be in any direction (basically from k to l or l to k) which gives isomorphic to each other and this is (i) in Figure 3.4 on page 58.

For Γ_2, there are $2 \cdot 2 \cdot 2 = 8$ cases to direct the undirected graph as 2 for first cycle, 2 for intersection part of the cycles and 2 for second cycle (basically from k to l or l to k).

(1) If k to l for first cycle, k to l for intersection part of the cycles and k to l for second cycle, then it is not possible since k is a source which is not possible.

(2) If k to l for first cycle, k to l for intersection part of the cycles and l to k for second cycle then this gives the one which is isomorphic to (vii) in Figure 3.4 on page 58.

(3) If k to l for first cycle, l to k for intersection part of the cycles and k to l for second cycle then this is (vii) in Figure 3.4.

(4) k to l for first cycle, l to k for intersection part of the cycles and l to k for second cycle, then this gives the one which is isomorphic to (vii) in Figure 3.4.

(5) l to k for first cycle, k to l for intersection part of the cycles and k to l for second cycle, then this gives the one which is isomorphic to (vii) in Figure 3.4.

(6) l to k for first cycle, k to l for intersection part of the cycles and l to k for second cycle, then this gives the one which is isomorphic to (vii) in Figure 3.4.

(7) l to k for first cycle, l to k for intersection part of the cycles and k to l for second cycle, then this gives the one which is isomorphic to (vii) in Figure 3.4.

(8) l to k for first cycle, l to k for intersection part of the cycles and l to k for second cycle, then it is not possible since k is a sink which is not possible.

Case 2: $(\sigma, \tau) = (1, 0)$**.** Then $(\sigma_1, \tau_1) = (0, 0)$ or $(1, 0)$

Case 2(a): $(\sigma, \tau) = (1, 0)$ and $(\sigma_1, \tau_1) = (0, 0)$.

For Γ_1, this source can be either on one of the cycle (does not matter which one since they are isomorphic to each other) or on the path between the cycles. If the source is on the cycle (say on first cycle), then the path between the cycles is from k to l since k cannot be a sink and the second cycle has to be a directed cycle since it cannot have one more source or sink. As a result, this is (ii) in Figure 3.4 on page 58. If the source is on the path between the cycles (let t be a source $k \leqslant t \leqslant l$), then there is a path from t to k and t to l and the cycles have to be directed cycles since they cannot have one more source or sink. As a result, this is (iii) in Figure 3.4 on page 58.

For Γ_2, this source can be on the first cycle, intersection of them or the second cycle. All of them are isomorphic to each other. Therefore, it will be enough to consider a source on the first cycle. There is a path from the source to k and l. Then there are $2 \cdot 2 = 4$ cases to direct the undirected graph as 2 for intersection part of the cycles and 2 for second cycle (basically from k to l or l to k).

(1) If k to l for intersection part of the cycles and k to l for second cycle, then it is not possible since l is a sink which is not possible.

(2) If k to l for intersection part of the cycles and l to k for second cycle then this gives the one which is isomorphic to $(viii)$ in Figure 3.4 on page 58.

(3) If l to k for intersection part of the cycles and k to l for second cycle then this is $(viii)$ in Figure 3.4 on page 58.

(4) l to k for intersection part of the cycles and l to k for second cycle, then it is not possible since k is a sink which is not possible.

Case 2(b): $(\sigma, \tau) = (1, 0)$ **and** $(\sigma_1, \tau_1) = (1, 0)$**.**

Now, there is a source leaf so to obtain the digraphs in this case we need to add a source leaf to the Figure 3.1 on page 51. The possibilities are listed in Figure 3.2 on page 54, below where we label three particular vertices t, k, l.

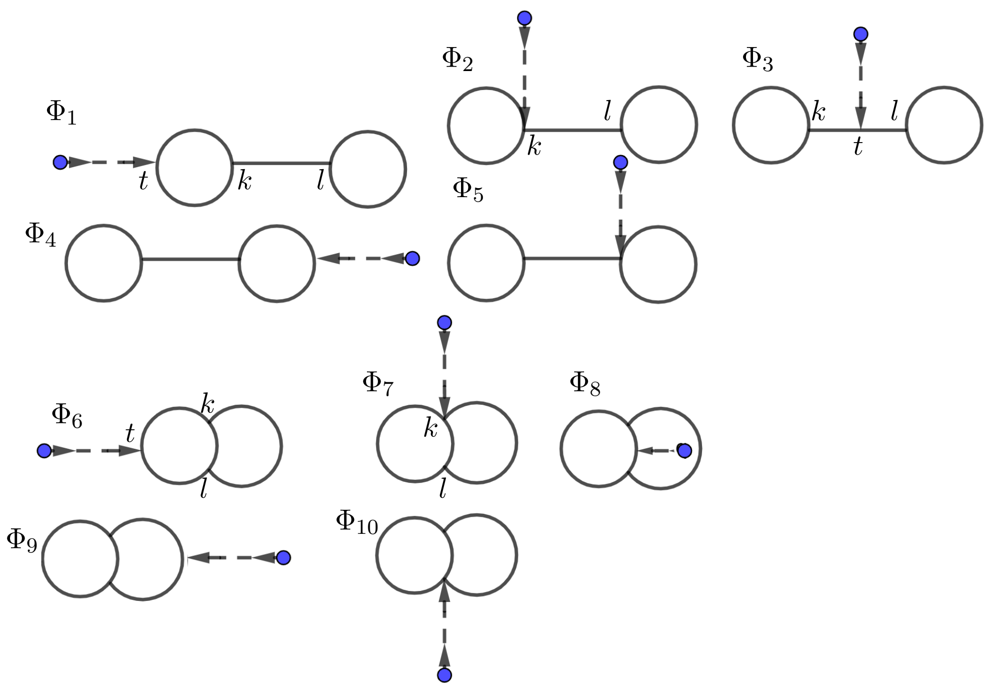

Figure 3.2: Possible undirected graphs with a source leaf for $|V| = |A| - 1$.

It can be seen that [Φ_1 and Φ_4], [Φ_2 and Φ_5], [Φ_6,Φ_8 and Φ_9], [Φ_7 and Φ_{10}] are isomorphic. Therefore, in assigning directions to arcs, it is enough to consider the graphs $\Phi_1, \Phi_2, \Phi_3, \Phi_6, \Phi_7$. Note that there cannot be another source or sink.

To direct the graph Φ_1 in Figure 3.2 on page 54, there are $4 \cdot 2 \cdot 1 = 8$ cases to direct the undirected graph as 4 for first cycle, 2 for intersection part of the cycles and 1 for second cycle since it has to be a directed cycle.

(1) If t to k for upper part of the cycle and t to k for lower part of the cycle in first cycle, k to l for intersection part of the cycles, then this is (xii) in Figure 3.5 on page 59.

(2) If t to k for upper part of the cycle and t to k for lower part of the cycle in first cycle, l to k for intersection part of the cycles, then it is not possible since k is a sink which is not possible.

(3) If t to k for upper part of the cycle and k to t for lower part of the cycle in first cycle, k to l for intersection part of the cycles, then this gives

the one which is isomorphic to $(xiii)$ in Figure 3.5 on page 59.

(4) If t to k for upper part of the cycle and k to t for lower part of the cycle in first cycle, l to k for intersection part of the cycles, then this gives the one which is isomorphic to (xiv) in Figure 3.5.

(5) If k to t for upper part of the cycle and t to k for lower part of the cycle in first cycle, k to l for intersection part of the cycles, then this gives the one which is isomorphic to $(xiii)$ in Figure 3.5.

(6) If k to t for upper part of the cycle and t to k for lower part of the cycle in first cycle, l to k for intersection part of the cycles, then this gives the one which is isomorphic to (xiv) in Figure 3.5.

(7) If k to t for upper part of the cycle and k to t for lower part of the cycle in first cycle, k to l for intersection part of the cycles,then it is not possible since k is a source which is not possible.

(8) If k to t for upper part of the cycle and k to t for lower part of the cycle in first cycle, l to k for intersection part of the cycles, then it is not possible since k is a source, which is not possible.

Using same technique, after directing the graphs in Figure 3.2 on page 54, we get $(xvii)$ and $(xxxii)$ for the graph Φ_2, $(xv), (xvi)$ for the graph Φ_3, $(xxviii)$,
$(xxix), (xxxiv)$ for the graph Φ_6 and $(xxiv), (xxv)$ for the graph Φ_7.

Case 3: $(\sigma, \tau) = (1, 1)$**.** Then $(\sigma_1, \tau_1) = (0, 0), (1, 0)$ or $(1, 1)$.

Case 3(a): $(\sigma, \tau) = (1, 1)$ **and** $(\sigma_1, \tau_1) = (0, 0)$**.** The resulting digraphs are $(iv), (v), (vi)$ in Figure 3.4 on page 58 after directing the graphs Γ_1 in Figure 3.1 on page 51 and $(ix), (x), (xi)$ in Figure 3.4 after directing the graphs Γ_2 in Figure 3.1.

Case 3(b): $(\sigma, \tau) = (1, 1)$ **and** $(\sigma_1, \tau_1) = (1, 0)$**.** The resulting digraphs are $(xviii), (xix), (xx)$ in Figure 3.5 on page 59 after directing the graphs Φ_1 in Figure 3.2 on page 54, $(xxiii)$ and $(xxxiii)$ in Figure 3.5 after directing the graphs Φ_2 in Figure 3.2, $(xxi), (xxii)$ in Figure 3.5 after directing the graphs Φ_3 in Figure 3.2, $(xxx), (xxxi), (xxxv)$ in Figure 3.5 after directing the graphs Φ_6 in Figure 3.2 and $(xxvi), (xxvii)$ in Figure 3.5 after directing the graphs Φ_7 in Figure 3.2.

Case 3(c): $(\sigma, \tau) = (1, 1)$ **and** $(\sigma_1, \tau_1) = (1, 1)$**.** This case cannot occur for otherwise the restriction that there is an arc between every source and every sink implies that it is the digraph consisting of two vertices and one arc between them, and thus has more vertices than arcs.

□

We now show that if l, k are vertices of a directed cycle (see Figure 3.3) then the generator x_k can be written in terms of generator x_l. Then we will use this relation in our presentations. We set $\gamma = \alpha^l - \beta^l$ and $\zeta = \beta(p\alpha - 1)$, where $p\alpha \equiv 1 \pmod{\gamma}$.

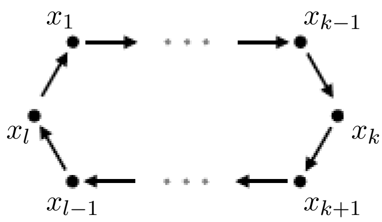

Figure 3.3: Γ_1: set up a relation between x_k and x_l

Lemma 3.1.2. *Suppose that Γ_1 is the directed cycle in Figure 3.3 and k, l are vertices of Γ_1 and suppose $(\alpha, \beta) = 1$. Let p, q be integers such that $p\alpha + q\beta = 1$. Then $x_k = x_l^{p^{l-k}\beta^{l-k}}$.*

Proof. In this case, we have the presentation of $G_{\Gamma_1}(R)$ for Figure 3.3,

$$\left\langle \begin{array}{c|c} x_1, x_2, \ldots, x_k \ldots, x_l & \begin{array}{l} x_1^{\gamma}, R(x_1, x_2), R(x_2, x_3), \ldots, R(x_{k-1}, x_k), \\ R(x_k, x_{k+1}) \ldots, R(x_{l-1}, x_l), R(x_l, x_1) \end{array} \end{array} \right\rangle.$$

Note that $p\alpha \equiv 1 \pmod{\gamma}$, we continue simplifying this presentation by

Lemma 1.3.4 (a),

$$\begin{aligned}
G_{\Gamma_1}(R) &= \langle\, x_2,\ldots,x_k\ldots,x_l \mid x_2^{\beta\gamma}, R(x_2,x_3),\ldots,R(x_{k-1},x_k),R(x_k,x_{k+1}),\\
&\ldots,R(x_l,x_2^{p\beta})\,\rangle\\
&= \langle\, x_3,\ldots,x_k\ldots,x_l \mid x_3^{\beta^2\gamma}, R(x_3,x_4),\ldots,R(x_{k-1},x_k),R(x_k,x_{k+1}),\\
&\ldots,R(x_l,x_3^{p^2\beta^2})\,\rangle\\
&\vdots\\
&= \langle\, x_k,x_{k+1},\ldots,x_l \mid x_k^{\beta^{k-1}\gamma}, R(x_k,x_{k+1}),R(x_{k+1},x_{k+2}),\ldots,\\
&R(x_l,x_k^{p^{k-1}\beta^{k-1}})\,\rangle\\
&= \langle\, x_k,x_{k+1},\ldots,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_{k+1}^{p\beta}, R(x_k,x_{k+1}),R(x_{k+1},x_{k+2}),\\
&\ldots,R(x_l,x_k^{p^{k-1}\beta^{k-1}})\,\rangle\\
&= \langle\, x_k,x_{k+1},\ldots,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_{k+1}^{p\beta}, R(x_{k+1}^{p\beta},x_{k+1}),R(x_{k+1},x_{k+2}),\\
&\ldots,R(x_l,x_{k+1}^{p^k\beta^k})\,\rangle
\end{aligned}$$

$$\begin{aligned}
&= \langle\, x_k,x_{k+1},\ldots,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_{k+1}^{p\beta}, x_{k+1}^{\beta(p\alpha-1)}, R(x_{k+1},x_{k+2}),\\
&\ldots,R(x_l,x_{k+1}^{p^k\beta^k})\,\rangle\\
&= \langle\, x_k,x_{k+2},\ldots,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_{k+2}^{p^2\beta^2}, x_{k+2}^{\beta\zeta}, R(x_{k+2},x_{k+3}),\\
&\ldots,R(x_l,x_{k+2}^{p^{k+1}\beta^{k+1}})\,\rangle\\
&= \langle\, x_k,x_{k+3},\ldots,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_{k+3}^{p^3\beta^3}, x_{k+3}^{\beta^2\zeta}, R(x_{k+3},x_{k+4}),\\
&\ldots,R(x_l,x_{k+3}^{p^{k+2}\beta^{k+2}})\,\rangle\\
&\vdots\\
&= \langle\, x_k,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_l^{p^{l-k}\beta^{l-k}}, x_l^{\beta^{l-k-1}\zeta}, R(x_l,x_l^{p^{l-1}\beta^{l-1}})\,\rangle\\
&= \langle\, x_k,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_l^{p^{l-k}\beta^{l-k}}, x_l^{\beta^{l-k-1}\zeta}, x_l^{p^{l-1}\beta^{l-1}-\alpha})\,\rangle\\
&= \langle\, x_k,x_l \mid x_k^{\beta^{k-1}\gamma}, x_k=x_l^{p^{l-k}\beta^{l-k}}, x_l^{\gamma}\,\rangle.
\end{aligned}$$

Hence $x_k = x_l^{p^{l-k}\beta^{l-k}}$. □

Remark 3.1.3. *Suppose* $(\alpha,\beta)=1$, $l,m \geqslant 1$ *and let* $\gamma=\alpha^l-\beta^l, \eta=\alpha^m-\beta^m$. *Then* $(\alpha,\gamma)=(\beta,\eta)=1$.

This chapter is organised in the following fashion. It will first indicate the classes of digraphs in Figure 3.4 and 3.5, and state the main theorem. Afterwards, it will turn its attention to prove whether these corresponding groups are finite cyclic group or not.

In most cases we are able to determine if $G_\Gamma(R)$ is a finite or infinite; where we show that it is finite we show that it is cyclic group (i.e. $\mathrm{rank}(G_\Gamma(R)) = 1$) in which case we give the order. In the case we are unable to determine if the groups is finite, then we show that $\mathrm{rank}(G_\Gamma(R)) \in \{1, 2\}$ except for 3 cases (x), (xi) and $(xxxiv)$.

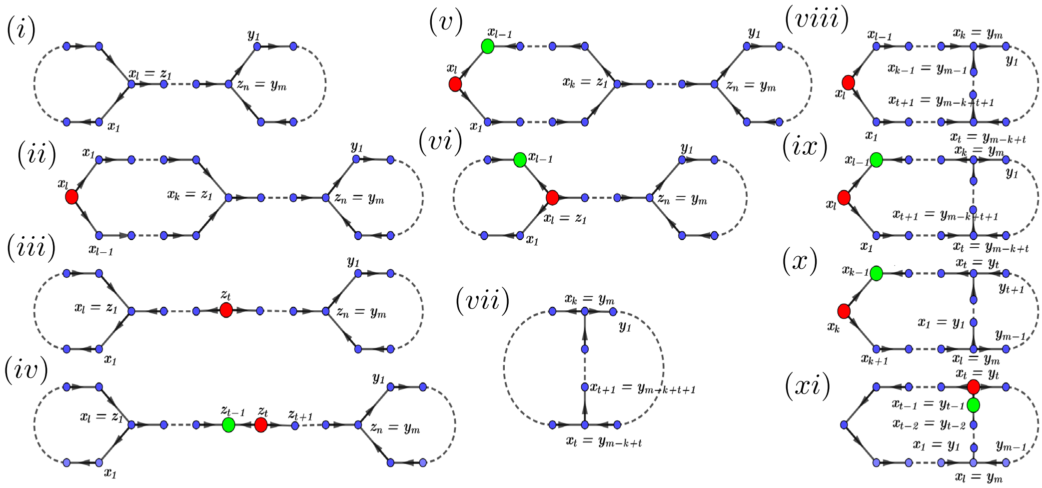

Figure 3.4: Classes of digraphs without leaf referred to in the statement of Theorem B

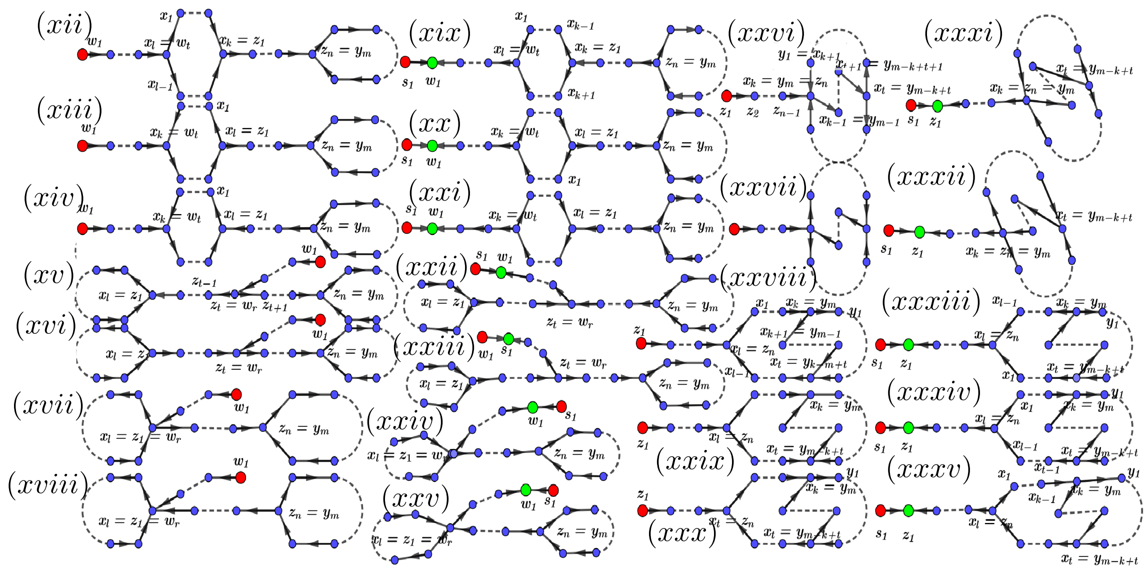

Figure 3.5: Classes of digraphs with leaf referred to in the statement of Theorem B

Before going to Theorem B, we will explain how we produce Figure 3.4 and Figure 3.5 by Figure 3.6 in an example. We can present all possible digraphs with one graph family in Figure 3.6 on page 60. Note that Figure 3.6 could cover more digraphs than we have. The idea is just to cover all our possible digraphs with one graph family. We also see what l, n, m, t, r represent in that example. Note that if a_i is not specified, it means $a_i = 0$.

When we get $a_3, a_5, a_8, a_9, a_{18}, a_{19}, a_{21}, a_{23}, (a_{17} = 1)$ and all other $a_i = 0$, then we create one of the digraph in Figure 3.7 on page 61 (an example of the digraph families $(xxii)$ in Figure 3.6). In that way, we can produce all digraphs families in Figure 3.4 and Figure 3.5 by Figure 3.6 . When we investigate digraph in terms of l, n, m, t, r, then we see that l is the number of arcs of the first directed cycle, which is $l = 6$, t is the number of arcs plus one from first directed cycle to the common point in the middle of the path between two directed cycle. which is $t = 4$, in that way $n = 6, m = 5, r = 3$. $a_3 + a_5 + a_8 + a_{10} = 6, a_{17} = 1, a_{18} = 2, a_{19} = 2, a_{21} = 2$ and $a_{23} = 5$ in that example in Figure 3.7.

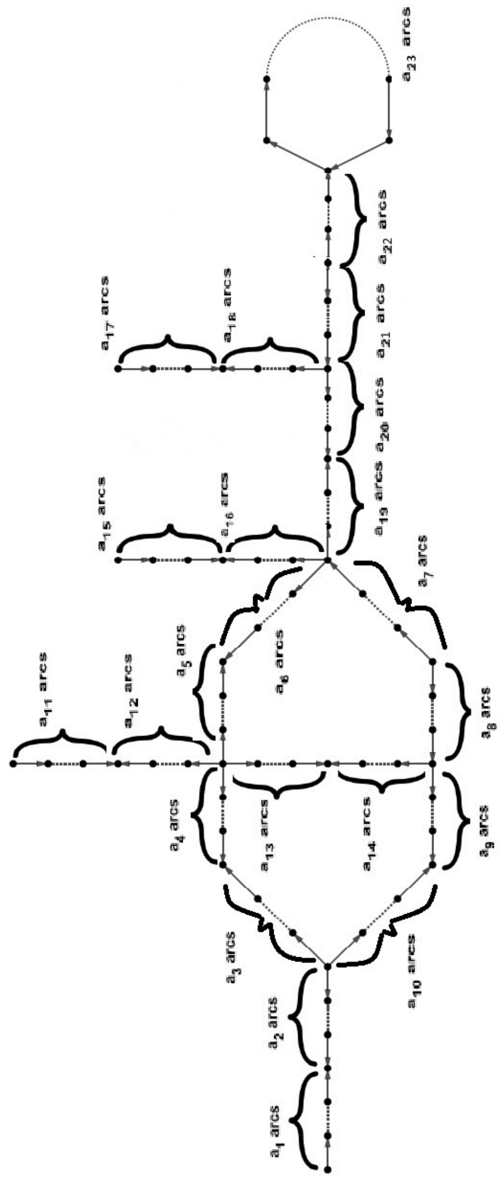

Figure 3.6: A digraph family that covers all possible digraphs of Theorem B

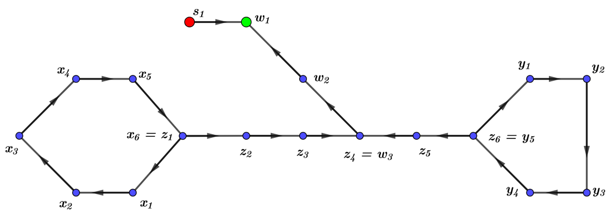

Figure 3.7: An example how to produce digraphs by Figure 3.6

Theorem B. *Let Γ be a non-empty finite digraph such that the number of generators is one less than the number of relators ($|V| = |A| - 1$) whose underlying undirected graph has girth n ($n \geqslant 4$) and let $R(a,b)$ be a cyclically reduced word that involves both a and b with exponent sums α and $-\beta$ in a and b, respectively where $|\alpha| \geqslant 2$, $|\beta| \geqslant 2|$. If $G_\Gamma(R)$ is finite, then $\alpha \neq 0$, $\beta \neq 0$, $(\alpha,\beta) = 1$, $\alpha^n - \beta^n \neq 0$, $a^\alpha = b^\beta$ in $K = \langle\, a,b \mid R(a,b)\,\rangle$, $G_\Gamma(R)$ is non-trivial, and Γ is the graph in Figure 3.4 and 3.5, where the non-zero a_i's in Figure 3.6 are one of the following (note that in cases (i) – (xi) the digraphs do not have a leaf, and in cases (xii) – (xxxv) the digraphs have a leaf).*

	in which case
(i) $a_3, a_5, a_8, a_9, a_{19}, a_{22}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{(l,m)} - \beta^{(l,m)}\|}$,
(ii) $a_3, a_5, a_7, a_{10}, a_{19}, a_{22}, a_{23}$ $(l < 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{\min\{l-k,\|n+l-k-1\|\}}(\alpha^{(m,2k-l)} - \beta^{(m,2k-l)})\|}$,
$(l > 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{\min\{k,\|n+l-k-1\|\}}(\alpha^{(m,l-2k)} - \beta^{(m,l-2k)})\|}$,
$(l = 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{n+l-k-1}(\alpha^m - \beta^m)\|}$,
(iii) $a_3, a_5, a_8, a_9, a_{20}, a_{22}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{\min\{\|n-1\|,\|t-1\|\}}(\alpha^{(l,m)} - \beta^{(l,m)})\|}$,
(iv) $a_3, a_5, a_8, a_9, a_{19}, a_{22}, a_{23}(a_{21} = 1)$ ..	$\mathrm{rank}(G_\Gamma(R)) \in \{1,2\}$,
(v) $a_4, a_6, a_7, a_{10}, a_{19}, a_{22}, a_{23}, (a_3 = 1)$	$\mathrm{rank}(G_\Gamma(R)) \in \{1,2\}$,
(vi) $a_3, a_5, a_8, a_9, a_{19}, a_{22}, a_{23}(a_6 = 1)$ ..	$\mathrm{rank}(G_\Gamma(R)) \in \{1,2\}$,
(vii) $a_4, a_5, a_8, a_{10}, a_{14}$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{(l,m)} - \beta^{(l,m)}\|}$,
(viii) $a_3, a_5, a_8, a_{10}, a_{14}(l < 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^{\min\{k,l-k\}}(\alpha^{(m,2k-l)} - \beta^{(m,2k-l)})\|}$,
$(l > 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^k(\alpha^{(m,l-2k)} - \beta^{(m,l-2k)})\|}$,
$(l = 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\|\alpha^k(\alpha^m - \beta^m)\|}$,
(ix) $a_4, a_5, a_8, a_{10}, a_{14}, (a_3 = 1)$	$\mathrm{rank}(G_\Gamma(R)) \in \{1,2\}$,
(x) $a_4, a_6, a_7, a_{10}, a_{14}, (a_3 = 1)$...........	?
(xi) $a_4, a_5, a_8, a_{10}, a_{14}, (a_{13} = 1)$	?

		in which case
(xii)	$a_1, a_3, a_5, a_7, a_{10}, a_{19}, a_{22}, a_{23}$ $(l < 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{(t-1)}\alpha^{\min\{l-k,n+l-k-1\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})\rvert}$,
	$(l > 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{(t-1)}\alpha^{\min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})\rvert}$,
	$(l = 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{(t-1)}\alpha^{n+l+t-k-2}(\alpha^{m}-\beta^{m})\rvert}$,
(xiii)	$a_1, a_4, a_6, a_7, a_{10}, a_{19}, a_{22}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{t-1}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xiv)	$a_1, a_4, a_6, a_7, a_{10}, a_{20}, a_{21}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{t-1}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xv)	$a_4, a_6, a_7, a_{10}, a_{17}, a_{20}, a_{22}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{r-1}\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xvi)	$a_4, a_6, a_7, a_{10}, a_{17}, a_{19}, a_{22}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{r-1}(\alpha^{(l,m)}-\beta^{l,m)})\rvert}$,
(xvii)	$a_4, a_6, a_7, a_{10}, a_{15}, a_{19}, a_{22}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{r-1}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xviii)	$a_4, a_6, a_7, a_{10}, a_{15}, a_{20}, a_{21}, a_{23}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{r-1}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xix)	$a_2, a_4, a_6, a_8, a_9, a_{20}, a_{21}, a_{23}, (a_1 = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xx)	$a_3, a_5, a_8, a_9, a_{19}, a_{22}, a_{23}, (a_1 = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxi)	$a_3, a_5, a_8, a_9, a_{20}, a_{21}, a_{23}, (a_1 = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxii)	$a_3, a_5, a_8, a_9, a_{18}, a_{19}, a_{21}, a_{23}, (a_{17} = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxiii)	$a_3, a_5, a_8, a_9, a_{18}, a_{19}, a_{22}, a_{23}, (a_{17} = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxiv)	$a_3, a_5, a_8, a_9, a_{16}, a_{19}, a_{22}, a_{23}, (a_{15} = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxv)	$a_3, a_5, a_8, a_9, a_{16}, a_{20}, a_{21}, a_{23}, (a_{15} = 1).$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxvi)	$a_3, a_6, a_7, a_9, a_{11}, a_{13}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{n-1}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xxvii)	$a_4, a_5, a_8, a_{10}, a_{11}, a_{14}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{n-1}(\alpha^{(l,m)}-\beta^{(l,m)})\rvert}$,
(xxviii)	$a_1, a_3, a_5, a_8, a_9, a_{13}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{n-1}(\alpha^{l}-\beta^{l})\rvert}$,
(xxix)	$a_1, a_3, a_5, a_8, a_{10}, a_{14}(l \leqslant 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{n-1}\alpha^{k}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})\rvert})$,
	$(l > 2k)$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{n-1}\alpha^{\min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})\rvert})$,
(xxx)	$a_1, a_3, a_5, a_8, a_9, a_{14}$	$G_\Gamma(R) \cong \mathbb{Z}_{\lvert\alpha^{k+l+n-t-1}(\alpha^{m}-\beta^{m})\rvert})$,
(xxxi)	$a_3, a_6, a_7, a_9, a_{12}, a_{13}, (a_{11} = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxxii)	$a_4, a_5, a_8, a_{10}, a_{12}, a_{14}, (a_{11} = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxxiii)	$a_2, a_4, a_5, a_8, a_9, a_{14}, (a_1 = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$,
(xxxiv)	$a_2, a_3, a_5, a_8, a_9, a_{14}, (a_1 = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2, \ldots, n\}$,
(xxxv)	$a_2, a_3, a_5, a_8, a_9, a_{13}, (a_1 = 1)$	$\operatorname{rank}(G_\Gamma(R)) \in \{1, 2\}$.

3.2 Proving the main theorem

Recall that we can always suppose that $\alpha \neq 0$, $\beta \neq 0$, $|\alpha| \geqslant 2$, $|\beta| \geqslant 2$, $(\alpha, \beta)= 1$ and $a^\alpha = b^\beta$ in K. Otherwise, the group K has Property W_1 and thus $G_\Gamma(R)$ is infinite by Corollary 1.3.1 and Proposition 1.3.2. By Lemma 3.1.1, the digraphs to consider are those in Figure 3.5 and Figure 3.4, for otherwise $G_\Gamma(R)$ is infinite.

(i) a_3, a_8, a_{19}, a_{23}

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{c} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} R(x_1,x_2), R(x_2,x_3), \dots, R(x_l,x_1), \\ R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{array} \right\rangle$$

$a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a,b) \,\rangle$ by Proposition 1.3.2, thus we get

$$x_1^{\alpha^l} = x_2^{\alpha^{l-1}\beta} = x_3^{\alpha^{l-2}\beta^2} = \dots = x_l^{\alpha\beta^{l-1}} = x_1^{\beta^l}.$$

$$y_1^{\alpha^m} = y_2^{\alpha^{m-1}\beta} = y_3^{\alpha^{m-2}\beta^2} = \dots = y_m^{\alpha\beta^{m-1}} = y_1^{\beta^m}.$$

We set $\gamma = \alpha^l - \beta^l$ and $\eta = \alpha^m - \beta^m$ obtain that $x_1^\gamma = 1$, and $y_1^\eta = 1$ in $G_\Gamma(R)$. Adjoining the relator x_1^γ and y_1^η yield

$$G_\Gamma(R) = \left\langle \begin{array}{c} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} x_1^\gamma, R(x_1,x_2), R(x_2,x_3), \dots, R(x_l,x_1), \\ y_1^\eta, R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{array} \right\rangle.$$

Applying precisely the same transformations as in the proof of Theorem 1.3.5, we get

$$G_\Gamma(R) = \left\langle \begin{array}{c} x_2, \dots, x_l, \\ y_2, \dots, y_m, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} x_2^{\beta\gamma}, R(x_2,x_3), \dots, R(x_l,x_2^{p\beta}), \\ y_2^{\beta\eta}, R(y_2,y_3), \dots, R(y_m,y_2^{p\beta}), \\ R(z_1,z_2), R(z_2,z_3), \dots, R(z_{n-1},z_n), \\ x_l = z_1, z_n = y_m \end{array} \right\rangle.$$

Simplifying this presentations in that way, what remains is

$$G_{\Gamma_1}(R) = \left\langle \begin{array}{l} x_l, y_m, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} x_l^\gamma \\ y_m^\eta \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), \\ x_l = z_1, z_n = y_m \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} x_l, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} x_l^\gamma, \\ \boxed{z_n^\eta, R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n),} \\ x_l = z_1 \end{array} \right\rangle.$$

Since $(\beta, \eta) = 1$ by Remark 3.1.3 and an iterated application of Lemma 1.3.4(b) for the relation inside the box yields

$$\begin{aligned} G_\Gamma(R) &= \langle\, x_l, z_1 \mid x_l^\gamma, z_1^{\alpha^{n-1}\eta}, x_l = z_1 \,\rangle \\ &= \langle\, x_l \mid x_l^\gamma, x_l^{\alpha^{n-1}\eta} \,\rangle \\ &= \langle\, x_l \mid x_l^{((\alpha^l - \beta^l), \alpha^{n-1}(\alpha^m - \beta^m))} \,\rangle \\ &= \langle\, x_l \mid x_l^{((\alpha^l - \beta^l), (\alpha^m - \beta^m))} \,\rangle \\ &= \langle\, x_l \mid x_l^{\alpha^{(l,m)} - \beta^{(l,m)}} \,\rangle. \end{aligned}$$

So $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(l,m)} - \beta^{(l,m)}$.

(ii) a_3, a_7, a_{19}, a_{23}

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_1), R(x_1, x_2), R(x_2, x_3), \dots, R(x_{k-1}, x_k), \\ R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ \boxed{R(y_1, y_2), R(y_2, y_3), \dots, R(y_m, y_1),} \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), \\ x_k = z_1, z_n = y_m \end{array} \right\rangle.$$

We set as $\gamma = \alpha^m - \beta^m$ and after applying precisely the same transformations as in the proof of Lemma 1.3.4(b) for the relation inside the box yields

$$G_\Gamma(R) = \left\langle \begin{array}{l|l} & R(x_l,x_1), R(x_1,x_2), R(x_2,x_3), \ldots, R(x_{k-1},x_k), \\ x_1,\ldots,x_l, & R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}), \ldots, R(x_{k+1},x_k), \\ y_m, & y_m^\gamma, \\ z_1,\ldots,z_n & R(z_1,z_2), R(z_2,z_3), \ldots, R(z_{n-1},z_n), \\ & x_k = z_1, z_n = y_m \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l|l} & R(x_l,x_1), R(x_1,x_2), R(x_2,x_3), \ldots, R(x_{k-1},x_k), \\ x_1,\ldots,x_l, & R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}), \ldots, R(x_{k+1},x_k), \\ z_1,\ldots,z_n & z_n^\gamma, R(z_1,z_2), R(z_2,z_3), \ldots, R(z_{n-1},z_n), \\ & x_k = z_1 \end{array} \right\rangle.$$

Since $(\beta,\gamma) = 1$ and (see Remark 3.1.3), an iterated application of Lemma 1.3.4 (b) yields

$$G_\Gamma(R) = \left\langle \begin{array}{l|l} x_1,\ldots,x_l, & R(x_l,x_1), R(x_1,x_2), R(x_2,x_3), \ldots, R(x_{k-1},x_k), \\ z_1 & R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}), \ldots, R(x_{k+1},x_k), \\ & z_1^{\alpha^{n-1}\gamma}, x_k = z_1 \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l|l} & R(x_l,x_1), R(x_1,x_2), R(x_2,x_3), \ldots, \\ x_1,\ldots,x_k,\ldots,x_l & R(x_{k-1},x_k), \\ & x_k^{\alpha^{n-1}\gamma}, \\ & R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}), \ldots, R(x_{k+1},x_k) \end{array} \right\rangle.$$

Since $(\beta, \alpha^{n-1}\gamma) = (\beta,\gamma) = 1$ (see Remark 3.1.3) there exists integers p, q such that $p\beta + q\gamma = 1$ and hence $p\beta = 1 \pmod{\gamma}$, an iterated application of

Lemma 1.3.4 (b) yields

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1,\dots,x_{k-1}, \\ x_{k+1},\dots,x_l \end{array} \,\middle|\, \begin{array}{l} R(x_l,x_1),R(x_1,x_2),R(x_2,x_3),\dots,R(x_{k-1},x_{k+1}^{p\alpha}), \\ x_{k+1}^{\alpha^n\gamma},R(x_l,x_{l-1}),R(x_{l-1},x_{l-2}),\dots,R(x_{k+2},x_{k+1}) \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} x_1,\dots,x_{k-1}, \\ x_{k+2},\dots,x_l \end{array} \,\middle|\, \begin{array}{l} R(x_l,x_1),R(x_1,x_2),R(x_2,x_3),\dots,R(x_{k-1},x_{k+2}^{p\alpha^2}), \\ x_{k+2}^{\alpha^{n+1}\gamma},R(x_l,x_{l-1}),R(x_{l-1},x_{l-2}),\dots,R(x_{k+3},x_{k+2}) \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l} x_1,\dots,x_{k-1}, \\ x_l \end{array} \,\middle|\, \begin{array}{l} x_l^{\alpha^{n+l-k-1}\gamma}, \\ R(x_l,x_1),R(x_1,x_2),R(x_2,x_3),\dots,R(x_{k-1},x_l^{(p\alpha)^{l-k}}) \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} x_1,\dots,x_{k-1}, \\ x_l \end{array} \,\middle|\, \begin{array}{l} x_l^{\alpha^{n+l-k-1}\gamma}, \\ R(x_l,x_1),R(x_1,x_2),R(x_2,x_3),\dots,R(x_{k-1},x_l^{(p\alpha)^{l-k}}), \\ x_i = x_l^{(p\alpha)^i} (1 \leqslant i \leqslant k-1) \text{ by Lemma 1.3.4 (b)} \end{array} \right\rangle$$
$$= \left\langle x_l \,\middle|\, \begin{array}{l} x_l^{\alpha^{n+l-k-1}\gamma},R(x_l,x_l^{p\alpha}),R(x_l^{p\alpha},x_l^{(p\alpha)^2}),R(x_l^{(p\alpha)^2},x_l^{(p\alpha)^3}),\dots, \\ R(x_l^{(p\alpha)^{k-1}},x_l^{(p\alpha)^{l-k}}) \end{array} \right\rangle$$
$$= \left\langle x_l \,\middle|\, \begin{array}{l} x_l^{\alpha^{n+l-k-1}\gamma},x_l^{\alpha-\beta p\alpha},x_l^{p\alpha^2-\beta(p\alpha)^2},\dots,x_l^{\alpha(p\alpha)^{k-2}-\beta(p\alpha)^{k-1}}, \\ x_l^{\alpha(p\alpha)^{k-1}-\beta(p\alpha)^{l-k}} \end{array} \right\rangle.$$

We can remove redundant relators $x_l^{\alpha-\beta p\alpha}, x_l^{p\alpha^2-\beta(p\alpha)^2}, \dots, x_l^{\alpha(p\alpha)^{k-2}-\beta(p\alpha)^{k-1}}$ since $p\beta \equiv 1 \bmod \gamma$. Thus, we get

$$\begin{aligned} G_\Gamma(R) &= \langle\, x_l \mid x_l^{\alpha^{n+l-k-1}\gamma}, x_l^{p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta} \,\rangle \\ &= \langle\, x_l \mid x_l^{(\alpha^{n+l-k-1}\gamma,\, p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta)} \,\rangle \\ &= \langle\, x_l \mid x_l^d \,\rangle, \text{ where } d = (\alpha^{n+l-k-1}\gamma, p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta). \end{aligned}$$

$$\begin{aligned} d &= (\alpha^{n+l-k-1}\gamma, p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta) \\ &= (\alpha^{n+l-k-1}\gamma, (p\beta)p^{k-1}\alpha^k - p^{l-k}\alpha^{l-k}\beta) \text{ since } p\beta \equiv 1 \bmod \gamma \\ &= (\alpha^{n+l-k-1}\gamma, \beta(p^k\alpha^k - p^{l-k}\alpha^{l-k})) \\ &= (\alpha^{n+l-k-1}\gamma, p^k\alpha^k - p^{l-k}\alpha^{l-k}) \text{ since } (\beta, \alpha\gamma) = 1 \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k - (p\alpha)^{l-k}). \end{aligned}$$

After that supposing $k < l-k$ and continue to simplify the equation above, we get

$$\begin{aligned} d &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k(1-(p\alpha)^{l-2k})) \text{ since } p\beta \equiv 1 \bmod \gamma \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k((p\beta)^{l-2k}-(p\alpha)^{l-2k})) \text{ since } p\beta \equiv 1 \bmod \gamma \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^k(p^{l-2k}(\beta^{l-2k}-\alpha^{l-2k}))) \\ &= (\alpha^{n+l-k-1}\gamma, p^{l-k}\alpha^k(\beta^{l-2k}-\alpha^{l-2k})) \\ &= (\alpha^{min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)}). \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{\min\{k,|n+l-k-1|\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})$.
Now supposing $k > l-k$ and simplifying the equation, we get

$$\begin{aligned} d &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^{l-k}((p\alpha)^{2k-l}-1)) \text{ since } p\beta \equiv 1 \bmod \gamma \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^{l-k}((p\alpha)^{2k-l}-(p\beta)^{2k-l})) \text{ since } p\beta \equiv 1 \bmod \gamma \\ &= (\alpha^{n+l-k-1}\gamma, (p\alpha)^{l-k}(p^{2k-l}(\beta^{2k-l}-\alpha^{2k-l}))) \\ &= (\alpha^{n+l-k-1}\gamma, p^{k}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l})) \\ &= (\alpha^{min\{l-k,n+l-k-1\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)}). \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{\min\{l-k,|n+l-k-1|\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})$.
Now, supposing $k = l-k$, we get

$$x_l^{p^{k-1}\alpha^k-p^{l-k}\alpha^{l-k}\beta} = x_l^{(p\alpha)^k-(p\alpha)^{l-k}} = x_l^{(p\alpha)^k-(p\alpha)^k} = x^0 = 1.$$

Thus, we can remove redundant relators from the presentation. Hence, we get $G_\Gamma(R) = \langle\, x_l \mid x_l^{\alpha^{n+l-k-1}\gamma} \,\rangle$.
Therefore, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n+l-k-1}(\alpha^m-\beta^m)$.

(iii) $a_3, a_8, a_{19}, a_{20}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \;\middle|\; \begin{array}{l} x_1^\gamma, R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ y_1^\eta, R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_t, z_{t-1}), R(z_{t-1}, z_{t-2}), \ldots, R(z_2, z_1), \\ R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-1}, z_n), \\ x_l = z_1, y_m = z_n \end{array} \right\rangle.$$

We set $\gamma = \alpha^l - \beta^l$ and $\eta = \alpha^m - \beta^m$, and apply precisely the same transformations as in the proof of Theorem 1.3.5. Then, what remains is

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} x_l, y_m, \\ z_1, \ldots, z_n, \end{array} \;\middle|\; \begin{array}{l} x_l^\gamma, y_m^\eta, \\ R(z_t, z_{t-1}), R(z_{t-1}, z_{t-2}), \ldots, R(z_2, z_1), \\ R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-1}, z_n), \\ x_l = z_1, y_m = z_n \end{array} \right\rangle \\ &= \left\langle \; z_1, \ldots, z_n \;\middle|\; \begin{array}{l} z_1^\gamma, R(z_t, z_{t-1}), R(z_{t-1}, z_{t-2}), \ldots, R(z_2, z_1), \\ z_n^\eta, R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-1}, z_n) \end{array} \right\rangle. \end{aligned}$$

Since $(\beta, \gamma) = 1$ and $(\beta, \eta) = 1$ (see Remark 3.1.3) and an iterated application of Lemma 1.3.4 (b) yields

$$\begin{aligned} G_\Gamma(R) &= \left\langle \; z_2, \ldots, z_{n-1}, \;\middle|\; \begin{array}{l} z_2^{\alpha\gamma}, R(z_t, z_{t-1}), R(z_{t-1}, z_{t-2}), \ldots, R(z_3, z_2), \\ z_{n-1}^{\alpha\eta}, R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-2}, z_{n-1}) \end{array} \right\rangle \\ &= \langle \; z_t \mid z_t^{\alpha^{t-1}\gamma}, z_t^{\alpha^{n-t}\eta} \; \rangle \\ &= \langle \; z_t \mid z_t^{\alpha^{\min\{n-t, t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)})} \; \rangle. \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{\min\{n-t, t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(iv) $a_3, a_8, a_{19}, a_{22}, a_{23}, (a_{20} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{t-2}, z_{t-1}), \\ x_l = z_1, \end{array}} \\ \boxed{\begin{array}{l} R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-1}, z_n), \\ y_m = z_n, \end{array}} \\ R(z_t, z_{t-1}) \end{array} \right\rangle .$$

We set $\gamma = \alpha^l - \beta^l$ and $\eta = \alpha^m - \beta^m$, and apply precisely the same transformations as we have in the form $\Gamma(n; \xrightarrow{m})$ for the first box and $\Gamma(n; \xleftarrow{m})$ for the second box, by Lemma 1.3.6, we get

$$G_\Gamma(R) = \langle \; z_{t-1}, z_t \;\mid\; z_{t-1}^{\beta^{t-2}\gamma}, z_t^{\alpha^{n-t}\eta}, R(z_t, z_{t-1}) \; \rangle .$$

After we get this presentation, we cannot eliminate z_{t-1} or z_t from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\alpha^{n-t}\eta, \alpha) \neq 1$ and $(\beta^{t-2}\gamma, \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(v) $a_4, a_7, a_{19}, a_{23}, (a_3 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_k, x_{k+1}), \ldots, R(x_{l-2}, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{k-1}, x_k), \\ \boxed{\begin{array}{l} R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), \\ x_k = z_1, z_n = y_m \end{array}} \end{array} \right\rangle .$$

We set $\gamma = \alpha^m - \beta^m$ and apply precisely the same transformations as in the proof of (ii) to obtain that

$$G_\Gamma(R) = \left\langle \; x_1, \ldots, x_l, \; \middle| \; \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_k, x_{k+1}), \ldots, R(x_{l-2}, x_{l-1}), \\ \boxed{x_k^{\alpha^{n-1}\gamma}, R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{k-1}, x_k)} \end{array} \right\rangle.$$

Since $(\beta, \alpha^{n-1}\gamma) = 1$ (see Remark 3.1.3) and an iterated application of Lemma 1.3.4 (b) yields

$$G_\Gamma(R) = \left\langle \; x_k, , x_{k+1}, \ldots, x_l \; \middle| \; \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_k, x_{k+1}), \ldots, R(x_{l-2}, x_{l-1}), \\ x_l^{\alpha^{n+k-1}\gamma} \end{array} \right\rangle.$$

Adjoin the relations $x_i = x_{l-1}^{(p\beta)^{l-1-i}} \; for \; k+1 \leqslant i \leqslant l-1$, where $p \in \mathbb{Z}$, to the presentation so we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle \; x_k, \ldots, x_{l-1}, x_l \; \middle| \; \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_k, x_{k+1}), \ldots, R(x_{l-2}, x_{l-1}), \\ x_l^{\alpha^{n+k-1}\gamma}, x_i = x_{l-1}^{(p\beta)^{l-1-i}} \; for \; k+1 \leqslant i \leqslant l-1 \end{array} \right\rangle \\ &= \left\langle \; x_{l-1}, x_l \; \middle| \; \begin{array}{l} R(x_l, x_{l-1}), \\ \cancel{R(x_{l-1}^{(p\beta)^{l-k-1}}, x_{l-1}^{(p\beta)^{l-k-2}})}, \ldots, \cancel{R(x_{l-1}^{p\beta}, x_{l-1})}, \\ x_l^{\alpha^{n+k-1}\gamma}, \end{array} \right\rangle \\ &= \langle \; x_{l-1}, x_l \; | \; x_l^{\alpha^{n+k-1}\gamma}, R(x_l, x_{l-1}) \; \rangle. \end{aligned}$$

After we get this presentation, we cannot eliminate x_{l-1} or x_l from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\alpha^{n+k-1}\gamma, \alpha) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(vi) $a_8, a_{19}, a_{23}, (a_6 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{c} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \left| \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{l-2}, x_{l-1}), \\ \boxed{\begin{array}{l} y_1^\gamma, R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), \\ x_l = z_1, z_n = y_m \end{array}} \end{array} \right. \right\rangle.$$

We set $\gamma = \alpha^m - \beta^m$ and apply precisely the same transformations as in the proof (v) for the relation inside the box to obtain that

$$G_\Gamma(R) = \left\langle \ x_1, \ldots, x_l, \ \left| \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{l-2}, x_{l-1}), \\ x_l^{\alpha^{n-1}\gamma} \end{array} \right. \right\rangle.$$

Now, adjoin these relations $x_l = x_1^{p\beta}, x_i = x_{l-1}^{(p\beta)^{l-1-i}}$ for $1 \leqslant i \leqslant l-1$, where $p \in \mathbb{Z}$, to the presentation and we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle \ x_1, \ldots, x_l, \ \left| \begin{array}{l} R(x_l, x_{l-1}), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{l-2}, x_{l-1}), \\ x_l^{\alpha^{n-1}\gamma}, \\ x_l = x_1^{p\beta}, x_i = x_{l-1}^{(p\beta)^{l-1-i}} \ \ for\ 1 \leqslant i \leqslant l-1 \end{array} \right. \right\rangle \\ &= \left\langle \ x_1, x_{l-1}, x_l, \ \left| \begin{array}{l} R(x_l, x_{l-1}), \\ \cancel{R(x_l^{(p\beta)^{l-1}}, x_l^{(p\beta)^{l-2}})}, \cancel{R(x_{l-1}^{(p\beta)^{l-2}}, x_{l-1}^{(p\beta)^{l-3}})}, \ldots, \\ \cancel{R(x_{l-1}^{(p\beta)^2}, x_{l-1}^{p\beta})}, R(x_{l-1}^{p\beta}, x_{l-1}), \\ x_l^{\alpha^{n-1}\gamma} \end{array} \right. \right\rangle. \end{aligned}$$

Since $R(x_l^{(p\beta)^i}, x_l^{(p\beta)^{i-1}}) = x_l^{\alpha(p\beta)^i - \beta(p\beta)^{i-1}} = x_l^{(p\beta)^{i-1}(\alpha p\beta - \beta)}$ for $1 \leqslant i \leqslant l-1$, and $\alpha p\beta - \beta = 0 \bmod \gamma$ since $p\alpha \equiv 1 \bmod \gamma$. Thus, these relations are redundant so can be removed.

$$G_\Gamma(R) = \langle\ x_{l-1}, x_l \ \mid\ x_l^{\alpha^{n-1}\gamma}, R(x_l, x_{l-1})\ \rangle.$$

After we get this presentation, we cannot eliminate x_{l-1} or x_l from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\alpha^{n-1}\gamma, \alpha) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(vii) $a_4, a_5, a_8, a_{10}, a_{14}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{c} x_1, \ldots, x_l, \\ y_1, \ldots, y_m \end{array} \;\middle|\; \begin{array}{l} x_1^\gamma, R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1) \\ y_1^\eta, R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m \end{array} \right\rangle.$$

We set $\gamma = \alpha^l - \beta^l$, $\eta = \alpha^m - \beta^m$.

Since $p_1\alpha \equiv 1 \pmod{\gamma}$, there is an integer $q_1 \in \mathbb{Z}$ such that $p_1\alpha + q_1\gamma = 1$. Moreover, $p_1\alpha \equiv 1 \pmod{\gamma}$ implies that $y_i = y_i^{p_1\alpha} = y_j^{p_1\beta}$ in G. This allows us to adjoin the relation $y_i = y_j^{p_1\beta}$ and to eliminate the generator y_i, and since $p_2\alpha \equiv 1 \pmod{\eta}$, there is an integer $q_2 \in \mathbb{Z}$ such that $p_2\alpha + q_2\eta = 1$. Moreover, $p_2\alpha \equiv 1 \pmod{\eta}$ implies that $x_i = x_i^{p_2\alpha} = x_j^{p_2\beta}$ in G. This allows us to adjoin the relation $x_i = x_j^{p_2\beta}$ and to eliminate the generator x_i as follows:

$$\begin{array}{l} y_1 = y_2^{(p_1\beta)} = y_3^{(p_1\beta)^2} = \ldots = y_t^{(p_1\beta)^{t-1}} \ldots = y_m^{(p_1\beta)^{m-1}} \\ x_1 = x_2^{(p_2\beta)} = x_3^{(p_2\beta)^2} = \ldots = x_t^{(p_2\beta)^{t-1}} = \ldots = x_l^{(p_2\beta)^{l-1}} \end{array}$$

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{c} x_2, \ldots, x_l, \\ y_2, \ldots, y_m \end{array} \;\middle|\; \begin{array}{l} x_2^{\beta\gamma}, R(x_2, x_3), \ldots, R(x_l, x_2^{p_2\beta}) \\ y_2^{\beta\eta}, R(y_2, y_3), \ldots, R(y_m, y_2^{p_1\beta}), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m \end{array} \right\rangle \\ &= \left\langle \begin{array}{c} x_3, \ldots, x_l, \\ y_3, \ldots, y_m \end{array} \;\middle|\; \begin{array}{l} x_3^{\beta^2\gamma}, R(x_3, x_4), \ldots, R(x_l, x_3^{(p_2\beta)^2)}, \\ y_3^{\beta^2\eta}, R(y_3, y_4), \ldots, R(y_m, y_3^{(p_1\beta)^2}), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m \end{array} \right\rangle. \end{aligned}$$

Simplifying in that way, what remains is

$$\begin{aligned}
G_\Gamma(R) &= \left\langle \begin{array}{c} x_k, \\ y_m \end{array} \;\middle|\; \begin{array}{l} x_k^\gamma, y_m^\eta \\ x_k^{(p_2\beta)^{k-t}} = x_k^{(p_1\beta)^{k-t}}, x_k^{(p_2\beta)^{k-t-1}}, = x_k^{(p_1\beta)^{k-t-1}} \ldots, x_k = y_m \end{array} \right\rangle \\
&= \left\langle \; x_k \;\middle|\; \begin{array}{l} x_k^\gamma, x_k^\eta \\ x_k^{(p_2\beta)^{k-t}-(p_1\beta)^{k-t}}, x_k^{(p_2\beta)^{k-t-1}-(p_1\beta)^{k-t-1}}, \ldots, x_k^{p_2\beta-p_1\beta} \end{array} \right\rangle \\
&= \left\langle \; x_k \;\middle|\; \begin{array}{l} x_k^\gamma, x_k^\eta \\ x_k^{\beta^{k-t}(p_2^{k-t}-p_1^{k-t})}, x_k^{\beta^{k-t-1}(p_2^{k-t-1}-p_1^{k-t-1})}, \ldots, x_k^{\beta(p_2-p_1)} \end{array} \right\rangle \\
&= \left\langle \; x_k \;\middle|\; \begin{array}{l} x_k^\gamma, x_k^\eta \\ x_k^{(\beta^{k-t},\beta^{k-t-1},\ldots,\beta)(p_2^{k-t}-p_1^{k-t},p_2^{k-t-1}-p_1^{k-t-1},\ldots,p_2-p_1)} \end{array} \right\rangle \\
&= \langle \; x_k \mid x_k^\gamma, x_k^\eta, x_k^{\beta(p_2-p_1)} \; \rangle \\
&= \langle \; x_k \mid x_k^{(\gamma,\eta,\beta(p_2-p_1))} \; \rangle \\
&= \langle \; x_k \mid x_k^{(\gamma,\eta,p_2-p_1)} \; \rangle.
\end{aligned}$$

Now, $p_1\alpha \equiv 1 \pmod{\gamma}$, and we can say $p_1\alpha \equiv 1 \pmod{(\gamma,\eta)}$,

$p_2\alpha \equiv 1 \pmod{\eta}$, and we can say $p_2\alpha \equiv 1 \pmod{(\gamma,\eta)}$. So, $p_1\alpha - p_2\alpha \equiv 0 \pmod{(\gamma,\eta)}$.

Since $(\alpha,\gamma) = 1$ and $(\alpha,\eta) = 1$, $\Delta = (\gamma,\eta,(p_1-p_2)) = (\gamma,\eta,(p_1-p_2)\alpha) = (\gamma,\eta)$. Then the presentation is

$$\begin{aligned}
G_\Gamma(R) &= \langle \; x_k \mid x_k^{(\gamma,\eta)} \; \rangle \\
&= \langle \; x_k \mid x_k^{\alpha^{(l,m)}-\beta^{(l,m)}} \; \rangle.
\end{aligned}$$

So $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(l,m)} - \beta^{(l,m)}$.

(viii) $a_3, a_5, a_8, a_{10}, a_{14}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ y_1^\eta, R(y_1, y_2), R(y_2, y_3), \dots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \dots, x_k = y_m \end{array} \right\rangle.$$

We set $\eta = \alpha^m - \beta^m$ and since $p_1\alpha \equiv 1 \pmod{\eta}$, there is an integer $q_1 \in \mathbb{Z}$ such that $p_1\alpha + q_1\eta = 1$. Moreover, $p_1\alpha \equiv 1 \pmod{\eta}$ implies that $y_i = y_i^{p_1\alpha} = y_j^{p_1\beta}$ in G and after applying precisely the same transformations as in the proof of Lemma 1.3.4(a), we get

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_m \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ y_m^\eta, \\ x_t = y_m^{(p_1\beta)^{k-t}}, x_{t+1} = y_m^{(p_1\beta)^{k-t-1}}, \dots, x_{k-1} = y_m^{p_1\beta}, \\ x_k = y_m \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} x_1, \dots, x_l \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_k), \\ x_k^\eta, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ x_t = x_k^{(p_1\beta)^{k-t}}, x_{t+1} = x_k^{(p_1\beta)^{k-t-1}}, \dots, x_{k-1} = x_k^{p_1\beta} \end{array} \right\rangle.$$

Since $(\beta, \eta) = 1$, there are integers $p_2, q_2 \in \mathbb{Z}$ such that $p_2\beta + q_2\eta = 1$ so $p_2\beta \equiv 1 \pmod{\eta}$. We can thus apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, x_2, \dots, x_{k-1}, \\ x_{k+1}, \dots, x_l \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_{k-1}^{p_2\alpha}) \\ x_{k-1}^{\alpha\eta}, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-2}, x_{k-1}) \\ x_t = x_k^{(p_1\beta)^{k-t}}, x_{t+1} = x_k^{(p_1\beta)^{k-t-1}}, \dots, \\ \cancel{x_{k-1} = x_{k-1}^{p_2\alpha\, p_1\beta}} \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} x_1, x_2, \dots, x_{k-2}, \\ x_{k+1}, \dots, x_l \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \dots, R(x_{k+1}, x_{k-2}^{(p_2\alpha)^2}), \\ x_{k-2}^{\alpha^2\eta}, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-3}, x_{k-2}), \\ x_t = x_k^{(p_1\beta)^{k-t}}, x_{t+1} = x_k^{(p_1\beta)^{k-t-1}}, \dots, \\ \cancel{x_{k-2} = x_{k-2}^{(p_2\alpha)^2 (p_1\beta)^2}} \end{array} \right\rangle.$$

Simplifying in that way, what remains is

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, x_2, \ldots, x_t, \\ x_{k+1}, \ldots, x_l \end{array} \;\middle|\; \begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \ldots, R(x_{k+1}, x_t^{(p_2\alpha)^{k-t}}), \\ x_t^{\alpha^{k-t}\eta}, R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{t-1}, x_t), \\ \cancel{x_t = x_t^{(p_2\alpha)^{k-t} \cdot (p_1\beta)^{k-t}}} \end{array} \right\rangle.$$

Since $(\beta, \alpha^{k-t}\eta) = 1$, we can thus apply Lemma 1.3.4 (b),

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} x_1, \\ x_{k+1}, x_{k+2}, \ldots, x_l \end{array} \;\middle|\; \begin{array}{l} x_1^{\alpha^{k-1}\eta}, R(x_l, x_1), \\ R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \ldots, \\ R(x_{k+1}, x_1^{(p_2\alpha)^{k-1}}) \end{array} \right\rangle \\ &= \left\langle \; x_{k+1}, x_{k+2}, \ldots, x_l \;\middle|\; \begin{array}{l} x_l^{\alpha^k\eta}, R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \ldots, \\ R(x_{k+1}, x_l^{(p_2\alpha)^k}), \\ x_i = x_l^{(p_2\alpha)^{l-i}} \;\; for \;\; k+1 \leqslant i \leqslant l-1 \end{array} \right\rangle \\ &= \left\langle \; x_l \;\middle|\; \begin{array}{l} x_l^{\alpha^k\eta}, R(x_l, x_l^{p_2\alpha}), R(x_l^{p_2\alpha}, x_l^{(p_2\alpha)^2}), \ldots, R(x_l^{(p_2\alpha)^{l-k-2}}, \\ x_l^{(p_2\alpha)^{l-k-1}}), R(x_l^{(p_2\alpha)^{l-k-1}}, x_l^{(p_2\alpha)^k}) \end{array} \right\rangle \\ &= \left\langle \; x_l \;\middle|\; \begin{array}{l} x_l^{\alpha^k\eta}, x_l^{\alpha-\beta p_2\alpha}, x_l^{p_2\alpha^2-\beta(p_2\alpha)^2}, \ldots, \\ x_l^{\alpha(p_2\alpha)^{l-k-2}-\beta(p_2\alpha)^{l-k-1}}, x_l^{\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k} \end{array} \right\rangle \\ &= \left\langle \; x_l \;\middle|\; \begin{array}{l} x_l^{\alpha^k\eta}, x_l^{\alpha-\beta p_2\alpha}, x_l^{p_2\alpha(\alpha-\beta p_2\alpha)}, \ldots, \\ x_l^{(p_2\alpha)^{l-k-2}(\alpha-\beta p_2\alpha)}, x_l^{\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k} \end{array} \right\rangle. \end{aligned}$$

Since $x_l^{\alpha-\beta p_2\alpha} = x_l^{\alpha-\alpha} = 1 \bmod \eta$, we get

$$G_\Gamma(R) = \langle \; x_l \mid x_l^{\alpha^k\eta}, x_l^{\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k} \; \rangle.$$

Supposing $l-k<k$, then

$$\begin{aligned}
\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k &= p_2^{l-k-1}\alpha^{l-k}-\beta p_2^k\alpha^k\\
&= p_2^{l-k-1}\alpha^{l-k}-p_2^{k-1}\alpha^k\\
&= p_2^{l-k-1}\alpha^{l-k}(1-p_2^{2k-l}\alpha^{2k-l})\\
&= p_2^{l-k-1}\alpha^{l-k}(p_2^{2k-l}\beta^{2k-l}-p_2^{2k-l}\alpha^{2k-l})\\
&\text{since } p_2\beta\equiv 1 \bmod \eta\\
&= p_2^{l-k-1}\alpha^{l-k}p_2^{2k-l}(\beta^{2k-l}-\alpha^{2k-l})\\
&= p_2^{k-1}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l}).
\end{aligned}$$

Thus, we get

$$\begin{aligned}
G_\Gamma(R) &= \langle\ x_l \mid x_l^{\alpha^k\eta}, x_l^{p_2^{k-1}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l})}\ \rangle\\
&= \langle\ x_l \mid x_l^{\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}\ \rangle.
\end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{min\{k,l-k\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})$.
Supposing $l-k>k$, then

$$\begin{aligned}
\alpha(p_2\alpha)^{l-k-1}-\beta(p_2\alpha)^k &= p_2^{l-k-1}\alpha^{l-k}-\beta p_2^k\alpha^k\\
&= p_2^{l-k-1}\alpha^{l-k}-p_2^{k-1}\alpha^k\\
&= p_2^{k-1}\alpha^k(p_2^{l-2k}\alpha^{l-2k}-1)\\
&= p_2^{k-1}\alpha^k(p_2^{l-2k}\alpha^{l-2k}-p_2^{l-2k}\beta^{l-2k})\\
&\text{since } p_2\beta\equiv 1 \bmod \eta\\
&= p_2^{l-k-1}\alpha^k(\alpha^{l-2k}-\beta^{l-2k}).
\end{aligned}$$

Therefore, we get

$$\begin{aligned}
G_\Gamma(R) &= \langle\ x_l \mid x_l^{\alpha^k\eta}, x_l^{p_2^{l-k-1}\alpha^k(\alpha^{l-2k}-\beta^{l-2k})}\ \rangle\\
&= \langle\ x_l \mid x_l^{\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}\ \rangle.
\end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})$.
Supposing $k=l-k$, then

$x_l^{p_2^{k-1}\alpha^{l-k}(\beta^{2k-l}-\alpha^{2k-l})} = x_l^{p_2^{k-1}\alpha^k(p_2^{l-2k}\alpha^{l-2k}-1)} = x_l^0 = 1.$

Therefore, it can be removed from the presentation. Thus, we get $G_\Gamma(R) = \langle\, x_l \mid x_l^{\alpha^k\eta}\,\rangle$.

Hence, $G_\Gamma(R)$ is finite cyclic of order $\mathbb{Z}_{|\alpha^k(\alpha^m-\beta^m)|}$.

(ix) $a_4, a_5, a_8, a_{10}, a_{14}, (a_3 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{c} x_1,\ldots,x_l, \\ y_1,\ldots,y_m \end{array} \;\middle|\; \begin{array}{l} R(x_l,x_{l-1}), \\ R(x_k,x_{k+1}), R(x_{k+1},x_{k+2})\ldots, R(x_{l-2},x_{l-1}), \\ \boxed{\begin{array}{l} R(x_l,x_1), R(x_1,x_2),\ldots,R(x_{k-1},x_k), \\ y_1^\eta, R(y_1,y_2), R(y_2,y_3),\ldots,R(y_m,y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1},\ldots,x_k = y_m \end{array}} \end{array} \right\rangle.$$

We set $\eta = \alpha^m - \beta^m$, $p_1\alpha \equiv 1 \pmod{\eta}$ and $p_2\beta \equiv 1 \pmod{\eta}$ as in $(viii)$ then we apply precisely the same transformations as in the proof of $(viii)$ for the relations inside the box to obtain that

$$\begin{aligned}
G_\Gamma(R) &= \left\langle x_{k+1},x_{k+2},\ldots,x_l \;\middle|\; \begin{array}{l} R(x_l,x_{l-1}), \\ x_l^{\alpha^k\eta}, R(x_l^{(p_2\alpha)^k},x_{k+1}), R(x_{k+1},x_{k+2}),\ldots, \\ R(x_{l-2},x_{l-1}) \end{array} \right\rangle \\
&= \left\langle x_{k+1},x_{k+2},\ldots,x_l \;\middle|\; \begin{array}{l} R(x_l,x_{l-1}), \\ x_l^{\alpha^k\eta}, R(x_l^{(p_2\alpha)^k},x_{k+1}), R(x_{k+1},x_{k+2}),\ldots, \\ R(x_{l-2},x_{l-1}), \\ x_i = x_{l-1}^{(p_2\beta)^{l-1-i}} \text{ for } k+1 \leqslant i \leqslant l-1 \end{array} \right\rangle \\
&= \left\langle x_{l-1},x_l \;\middle|\; \begin{array}{l} R(x_l,x_{l-1}), \\ x_l^{\alpha^k\eta}, R(x_l^{(p_2\alpha)^k},x_{k+1}), R(x_{k+1},x_{k+2}),\ldots,R(x_{l-2},x_{l-1}), \\ x_i = x_{l-1}^{(p_2\beta)^{l-1-i}} \text{ for } k+1 \leqslant i \leqslant l-1 \end{array} \right\rangle \\
&= \left\langle x_{l-1},x_l \;\middle|\; \begin{array}{l} R(x_l,x_{l-1}), \\ x_l^{\alpha^k\eta}, R(x_l^{(p_2\alpha)^k},x_{l-1}^{(p_2\beta)^{l-k-2}}), R(x_{l-1}^{(p_2\beta)^{l-k-2}},x_{l-1}^{(p_2\beta)^{l-k-3}}), \\ \ldots, R(x_{l-1}^{p_2\beta},x_{l-1}) \end{array} \right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle\ x_{l-1}, x_l \ \mid\ R(x_l, x_{l-1}), x_l^{\alpha^k\eta}, R(x_l^{(p_2\alpha)^k}, x_{l-1}^{(p_2\beta)^{l-k-2}})\ \rangle \\
&= \langle\ x_{l-1}, x_l \ \mid\ x_l^{\alpha^k\eta}, x_l^{\alpha} = x_{l-1}^{\beta}, x_l^{\alpha(p_2\alpha)^k} = x_{l-1}^{\beta(p_2\beta)^{l-k-2}}\ \rangle \\
&= \langle\ x_{l-1}, x_l \ \mid\ x_l^{\alpha^k\eta}, x_l^{\alpha} = x_{l-1}^{\beta}, x_l^{\alpha(p_2\alpha)^k} = x_l^{\alpha(p_2\beta)^{l-k-2}}\ \rangle \\
&= \langle\ x_{l-1}, x_l \ \mid\ x_l^{(\alpha^k\eta, \alpha(p_2\alpha)^k - \alpha(p_2\beta)^{l-k-2})}, x_l^{\alpha} = x_{l-1}^{\beta}\ \rangle.
\end{aligned}
$$

After we get this presentation, we cannot eliminate x_{l-1} or x_l from the presentation by our limited knowledge now (it is because we cannot apply Lemma 1.3.4 further). Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(x) $a_4, a_6, a_7, a_{10}, a_{14}, (a_3 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$
G_\Gamma(R) = \left\langle \begin{array}{c} x_1, \ldots, x_l, \\ y_1, \ldots, y_m \end{array} \;\middle|\; \begin{array}{l} R(x_k, x_{k+1}), R(x_{k+1}, x_{k+2}), \ldots, R(x_{l-1}, x_l), \\ R(x_k, x_{k-1}), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{k-2}, x_{k-1}), \\ R(y_m, y_1), R(y_1, y_2), \ldots, R(y_{t-1}, y_t), \\ R(y_m, y_{m-1}), R(y_{m-1}, y_{m-2}), \ldots, R(y_{t+1}, y_t), \\ x_l = y_m, x_1 = y_1, x_2 = y_2, \ldots, x_t = y_t \end{array} \right\rangle.
$$

There are no directed cycles in that graph. Therefore, we can not apply Theorem 1.3.5.

(xi) $a_4, a_5, a_8, a_{10}, a_{14}, (a_{13} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$
G_\Gamma(R) = \left\langle \begin{array}{c} x_1, \ldots, x_l, \\ y_1, \ldots, y_m \end{array} \;\middle|\; \begin{array}{l} R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{t-2}, x_{t-1}), \\ R(x_t, x_{t-1}), \\ R(y_t, y_{t+1}), R(y_{t+1}, y_{t+2}), \ldots, \\ R(y_{m-1}, y_m), \\ x_l = y_m, x_1 = y_1, x_2 = y_2, \ldots, x_t = y_t \end{array} \right\rangle.
$$

There are no directed cycles in that graph. Therefore, we can not apply Theorem 1.3.5.

(xii) $a_1, a_5, a_7, a_{19}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1,\dots,x_l, \\ y_1,\dots,y_m, \\ z_1,\dots,z_n, \\ w_1,\dots,w_t \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_l,x_1), R(x_1,x_2), R(x_2,x_3),\dots,R(x_{k-1},x_k), \\ R(x_l,x_{l-1}), R(x_{l-1},x_{l-2}),\dots,R(x_{k+1},x_k), \\ R(y_1,y_2), R(y_2,y_3),\dots,R(y_m,y_1), \\ R(z_1,z_2), R(z_2,z_3),\dots,R(z_{n-1},z_n), \\ x_k = z_1, z_n = y_m, \end{array}} \\ R(w_1,w_2), R(w_2,w_3),\dots,R(w_{t-1},w_t), \\ x_l = w_t, \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (ii) for the relations inside the box to simplify the presentation

Supposing $k < l - k$, we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} x_l, \\ w_1,\dots,w_t \end{array} \;\middle|\; \begin{array}{l} x_l^{\alpha^{\min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}, \\ R(w_1,w_2), R(w_2,w_3),\dots,R(w_{t-1},w_t), \\ x_l = w_t, \end{array} \right\rangle \\ &= \left\langle \; w_1,\dots,w_t \;\middle|\; \begin{array}{l} w_t^{\alpha^{\min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}, \\ R(w_1,w_2), R(w_2,w_3),\dots,R(w_{t-1},w_t) \end{array} \right\rangle. \end{aligned}$$

Let $\phi = \alpha^{\min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)} - \beta^{(m,l-2k)})$. Since $(\phi, \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle \; w_1 \mid w_1^{\alpha^{t-1}\alpha^{\min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})} \; \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(t-1)}\alpha^{\min\{k,n+l-k-1\}}(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})$.

Applying same transformations when $k > l - k$

$$G_\Gamma(R) = \langle \; w_1 \mid w_1^{\alpha^{t-1}\alpha^{\min\{l-k,n+l-k-1\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})} \; \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(t-1)}\alpha^{\min\{l-k,n+l-k-1\}}(\alpha^{(m,2k-l)}-\beta^{(m,2k-l)})$.

Supposing $k = l-k$, we have

$$\begin{aligned}
G_\Gamma(R) &= \left\langle\ w_1,\ldots,w_t\ \middle|\ \begin{array}{l} w_t^{\alpha^{n+l-k-1}(\alpha^m-\beta^m)}, \\ R(w_1,w_2),R(w_2,w_3),\ldots,R(w_{t-1},w_t) \end{array} \right\rangle \\
&= \langle\ w_1\ \mid\ w_1^{\alpha^{t-1}\alpha^{n+l-k-1}(\alpha^m-\beta^m)}\ \rangle \\
&= \langle\ w_1\ \mid\ w_1^{\alpha^{n+l+t-k-2}(\alpha^m-\beta^m)}\ \rangle.
\end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{(t-1)}\alpha^{n+l+t-k-2}(\alpha^m-\beta^m)$.

(xiii) $a_1, a_6, a_7, a_{19}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1,\ldots,x_l, \\ y_1,\ldots,y_m, \\ z_1,\ldots,z_n, \\ w_1,\ldots,w_t \end{array} \middle| \begin{array}{l} \boxed{\begin{array}{l} R(x_1,x_2),R(x_2,x_3),\ldots,R(x_l,x_1), \\ R(y_1,y_2),R(y_2,y_3),\ldots,R(y_m,y_1), \\ R(z_1,z_2),R(z_2,z_3),\ldots,R(z_{n-1},z_n), \\ x_l=z_1, z_n=y_m \end{array}} \\ R(w_1,w_2),R(w_2,w_3),\ldots,R(w_{t-1},w_t), \\ x_k=w_t, \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (i) for the relations inside the box to obtain that

$$\begin{aligned}
G_\Gamma(R) &= \left\langle\ x_k w_1,\ldots,w_t\ \middle|\ \begin{array}{l} x_k^{\alpha^{(l,m)}-\beta^{(l,m)}}, \\ R(w_1,w_2),R(w_2,w_3),\ldots,R(w_{t-1},w_t), \\ x_k=w_t \end{array} \right\rangle \\
&= \langle\ w_1,\ldots,w_t\ \mid\ w_t^{\alpha^{(l,m)}-\beta^{(l,m)}},R(w_1,w_2),R(w_2,w_3),\ldots,R(w_{t-1},w_t)\ \rangle.
\end{aligned}$$

Since $(\alpha^{(l,m)}-\beta^{(l,m)},\beta)=1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle\ w_1\ \mid\ w_1^{\alpha^{t-1}(\alpha^{(l,m)}-\beta^{(l,m)})}\ \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{t-1}(\alpha^{(l,m)}-\beta^{(l,m)})$.

(xiv) $a_1, a_6, a_7, a_{20}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_t \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_n, z_{n-1}), R(z_{n-1}, z_{n-2}), \ldots, R(z_2, z_1), \\ x_l = z_1, z_n = y_m \end{array}} \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{t-1}, w_t), \\ x_k = w_t, \end{array} \right\rangle$$

We apply precisely the same transformations as in the proof of (i) for the relations inside the box to obtain that

$$\begin{aligned} G_\Gamma(R) &= \left\langle \; x_k w_1, \ldots, w_t \;\middle|\; \begin{array}{l} x_k^{\alpha^{(l,m)} - \beta^{(l,m)}}, \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{t-1}, w_t), \\ x_k = w_t \end{array} \right\rangle \\ &= \langle \; w_1, \ldots, w_t \mid w_t^{\alpha^{(l,m)} - \beta^{(l,m)}}, R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{t-1}, w_t) \; \rangle. \end{aligned}$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}, \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle \; w_1 \mid w_1^{\alpha^{t-1}(\alpha^{(l,m)} - \beta^{(l,m)})} \; \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{t-1}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(xv) $a_6, a_7, a_{17}, a_{20}, a_{22}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n, \\ w_1, \dots, w_r \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} x_1^\gamma, R(x_1, x_2), R(x_2, x_3), \dots, R(x_l, x_1), \\ y_1^\eta, R(y_1, y_2), R(y_2, y_3), \dots, R(y_m, y_1), \\ R(z_t, z_{t-1}), R(z_{t-1}, z_{t-2}), \dots, R(z_2, z_1), \\ R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \dots, R(z_{n-1}, z_n), \\ x_l = z_1, y_m = z_n, \end{array}} \\ R(w_1, w_2), R(w_2, w_3), \dots, R(w_{r-1}, w_r), \\ z_t = w_r \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (iii) for the relations inside the box to obtain that

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} z_t, \\ w_1, \dots, w_r \end{array} \;\middle|\; \begin{array}{l} z_t^{\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_1, w_2), R(w_2, w_3), \dots, R(w_{r-1}, w_r), \\ z_t = w_r \end{array} \right\rangle \\ &= \left\langle \; w_1, \dots, w_r \;\middle|\; \begin{array}{l} w_r^{\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_1, w_2), R(w_2, w_3), \dots, R(w_{r-1}, w_r) \end{array} \right\rangle. \end{aligned}$$

Since $(\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle \; w_1 \; | \; w_1^{\alpha^{r-1}\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)})} \; \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{r-1}\alpha^{\min\{n-t,t-1\}}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(xvi) $a_6, a_7, a_{17}, a_{19}, a_{22}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_r \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{t-1}, z_t), \\ x_l = z_1, \end{array}} \\ \boxed{\begin{array}{l} R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-1}, z_n), \\ z_n = y_m, \end{array}} \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r), \\ z_t = w_r \end{array} \right\rangle.$$

We apply Lemma 1.3.6 (a) for the relation inside the first box and Lemma 1.3.6 (b) for the relation inside the second box

$$G_\Gamma(R) = \left\langle \begin{array}{l} z_t, \\ w_1, \ldots, w_r \end{array} \;\middle|\; \begin{array}{l} z_t^{\beta^{t-1}(\alpha^l - \beta^l)}, z_t^{\alpha^{n-t}(\alpha^m - \beta^m)}, \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r), \\ z_t = w_r \end{array} \right\rangle.$$

Since $(\beta^{t-1}, \alpha^{n-t}) = 1$, we get

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} z_t, \\ w_1, \ldots, w_r \end{array} \;\middle|\; \begin{array}{l} z_t^{\alpha^{(l,m)} - \beta^{(l,m)}}, \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r), \\ z_t = w_r \end{array} \right\rangle \\ &= \langle\, w_1, \ldots, w_r \mid w_r^{\alpha^{(l,m)} - \beta^{(l,m)}}, R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r) \,\rangle. \end{aligned}$$

We apply Lemma 1.3.4 (b), we get

$$G_\Gamma(R) = \langle\, w_1 \mid w_1^{\alpha^{r-1}(\alpha^l - \beta^l)} \,\rangle$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(xvii) $a_6, a_7, a_{15}, a_{19}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_r \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), \\ x_l = z_1, z_n = y_m, \end{array}} \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r), \\ x_l = w_r \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (i) for the relations inside the box to obtain that

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} x_l, \\ w_1, \ldots, w_r \end{array} \;\middle|\; \begin{array}{l} x_l^{(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r), \\ x_l = w_r \end{array} \right\rangle \\ &= \left\langle \; w_1, \ldots, w_r \;\middle|\; \begin{array}{l} w_r^{(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_1, w_2), R(w_2, w_3), \ldots, R(w_{r-1}, w_r) \end{array} \right\rangle. \end{aligned}$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) = 1$ by Remark 3.1.3, we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle \; w_1 \mid w_1^{\alpha^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})} \; \rangle$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})|$.

(xviii) $a_6, a_7, a_{15}, a_{20}, a_{23}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n, \\ w_1, \dots, w_r \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1,x_2), R(x_2,x_3), \dots, R(x_l,x_1), \\ R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ R(z_n,z_{n-1}), R(z_{n-1},z_{n-2}), \dots, R(z_2,z_1), \\ x_l = z_1, z_n = y_m, \end{array}} \\ R(w_1,w_2), R(w_2,w_3), \dots, R(w_{r-1},w_r), \\ x_l = w_r \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of $(xvii)$ for the relations inside the box to obtain that

$$\begin{aligned} G_\Gamma(R) &= \left\langle \begin{array}{l} x_l, \\ w_1, \dots, w_r \end{array} \;\middle|\; \begin{array}{l} x_l^{(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_1,w_2), R(w_2,w_3), \dots, R(w_{r-1},w_r), \\ x_l = w_r \end{array} \right\rangle \\ &= \left\langle \; w_1, \dots, w_r \;\middle|\; \begin{array}{l} w_r^{(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_1,w_2), R(w_2,w_3), \dots, R(w_{r-1},w_r) \end{array} \right\rangle. \end{aligned}$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle \; w_1 \;|\; w_1^{\alpha^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})} \; \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(xix) $a_2, a_6, a_8, a_{20}, a_{23}, (a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1,\dots,x_l, \\ y_1,\dots,y_m, \\ z_1,\dots,z_n, \\ w_1,\dots,w_t, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_k,x_{k+1}),R(x_{k+1},x_{k+2}),\dots,R(x_{l-1},x_l), \\ R(x_k,x_{k-1}),R(x_{k-1},x_{k-2}),\dots,R(x_1,x_l), \\ R(y_1,y_2),R(y_2,y_3),\dots,R(y_m,y_1), \\ R(z_n,z_{n-1}),R(z_{n-2},z_{n-3}),\dots,R(z_2,z_1), \\ R(w_t,w_{t-1}),R(w_{t-1},w_{t-2}),\dots,R(w_2,w_1), \\ x_k = z_1, z_n = y_m, x_l = w_t, \end{array}} \\ R(s_1,w_1), \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (xii) for the relations inside the box but we need to interchange α and β by reflection principle since the direction of each arc are reversed according to (xii). Thus we get

Supposing $k < l-k$, we get

$$G_\Gamma(R) = \left\langle\; w_1, s_1 \;\middle|\; w_1^{\beta^{t-1}\beta^{\min\{k,n+l-k-1\}}(\beta^{(m,l-2k)}-\alpha^{(m,l-2k)})}, R(s_1,w_1) \;\right\rangle.$$

Supposing $k > l-k$, we get

$$G_\Gamma(R) = \left\langle\; w_1, s_1 \;\middle|\; w_1^{\beta^{t-1}\beta^{\min\{l-k,n+l-k-1\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}, R(s_1,w_1) \;\right\rangle.$$

Supposing $k = l-k$, we get

$$G_\Gamma(R) = \left\langle\; w_1, s_1 \;\middle|\; w_1^{\beta^{n+l+t-k-2}\eta}, R(s_1,w_1) \;\right\rangle.$$

After we get those presentation, we cannot eliminate w_1 or s_1 from the presentations. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{t-1}\beta^{\min\{k,n+l-k-1\}}(\beta^{(m,l-2k)}-\alpha^{(m,l-2k)}), \beta) \neq 1$, $(\beta^{t-1}\beta^{\min\{k,n+l-k-1\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)}), \beta) \neq 1$ and $(\beta^{n+l+t-k-2}\eta, \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xx) $a_2, a_6, a_7, a_{19}, a_{23}, (a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_t, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), \\ x_l = z_1, z_n = y_m, x_k = w_t, \end{array}} \\ R(w_t, w_{t-1}), R(w_{t-1}, w_{t-2}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of $(xiii)$ for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} w_1, \ldots, w_t, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} w_t^{\alpha^{(l,m)} - \beta^{(l,m)}}, \\ R(w_t, w_{t-1}), R(w_{t-1}, w_{t-2}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right\rangle.$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}, \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$G_\Gamma(R) = \left\langle \; w_1, s_1 \;\middle|\; w_1^{\beta^{t-1}(\alpha^{(l,m)} - \beta^{(l,m)})}, R(s_1, w_1) \; \right\rangle.$$

After we get this presentation, we cannot eliminate s_1 or w_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{t-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxi) $a_2, a_6, a_7, a_{20}, a_{23}, (a_1 = 1)$

This is exactly same result with (xx) since the only difference for the digraph is the reverse of the direction in the bridge between these two directed cycles in Figure 3.5 and the direction of this bridge between these two directed cycles does not affect the result since it is also isomorphic to (i).

Thus, the group $G_\Gamma(R)$ has the presentation.

$$G_\Gamma(R) = \left\langle\ w_1, s_1\ \middle|\ w_1^{\beta^{t-1}(\alpha^{(l,m)}-\beta^{(l,m)})}, R(s_1, w_1)\ \right\rangle.$$

After we get this presentation, we cannot eliminate s_1 or w_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{t-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxii) $a_6, a_7, a_{18}, a_{19}, a_{21}, a_{23}, (a_{17} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_r, \\ s_1 \end{array} \middle| \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{t-1}, z_t), \\ R(z_n, z_{n-1}), R(z_{n-1}, z_{n-2}), \ldots, R(z_{t+1}, z_t), \\ R(w_r, w_{r-1}), R(w_{r-1}, w_{r-2}), \ldots, R(w_2, w_1), \\ x_l = z_1, y_m = z_n, z_t = w_r, \end{array}} \\ R(s_1, w_1) \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (xv) for the relations inside the box but we need to interchange α and β by reflection principle since the direction of each arc are reversed according to (xv). Thus we get

$$G_\Gamma(R) = \left\langle\ w_1, s_1\ \middle|\ w_1^{\beta^{r-1}\beta^{min\{n-t,t-1\}}(\beta^{(l,m)}-\alpha^{(l,m)})}, R(s_1, w_1)\ \right\rangle.$$

After we get this presentation, we cannot eliminate s_1 or w_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{r-1}\beta^{min\{n-t,t-1\}}(\beta^{(l,m)} - \alpha^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxiii) $a_6, a_7, a_{18}, a_{19}, a_{22}, a_{23}, (a_{17} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_r, \\ s_1 \end{array} \middle| \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{t-1}, z_t), \\ x_l = z_1, \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_t, z_{t+1}), R(z_{t+1}, z_{t+2}), \ldots, R(z_{n-1}, z_n), \\ z_n = y_m, z_t = w_r \end{array}} \\ R(w_r, w_{r-1}), R(w_{r-2}, w_{r-3}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (xvi) for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} w_1, \ldots, w_r, \\ s_1 \end{array} \middle| \begin{array}{l} w_r^{\alpha^{(l,m)} - \beta^{(l,m)}}, \\ R(w_r, w_{r-1}), R(w_{r-2}, w_{r-3}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right\rangle.$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}, \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$G_\Gamma(R) = \langle\ w_1, s_1 \mid w_1^{\beta^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})}, R(s_1, w_1)\ \rangle.$$

After we get this presentation, we cannot eliminate s_1 or w_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxiv) $a_6, a_7, a_{16}, a_{19}, a_{23}, (a_{15} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_r, \\ s_1 \end{array} \left| \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), \\ x_l = z_1 = w_r, z_n = y_m, \end{array}} \\ R(w_r, w_{r-1}), R(w_{r-2}, w_{r-3}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right. \right\rangle.$$

We apply precisely the same transformations as in the proof of $(xvii)$ for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} w_1, \ldots, w_r, \\ s_1 \end{array} \left| \begin{array}{l} w_r^{(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_r, w_{r-1}), R(w_{r-2}, w_{r-3}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right. \right\rangle.$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}), \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$G_\Gamma(R) = \langle\ w_1, s_1 \ \mid\ w_1^{\beta^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})}, R(s_1, w_1)\ \rangle.$$

After we get this presentation, we cannot eliminate s_1 or w_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxv) $a_6, a_7, a_{16}, a_{20}, a_{23}, (a_{15} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ w_1, \ldots, w_r, \\ s_1 \end{array} \left| \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ R(z_n, z_{n-1}), R(z_{n-1}, z_{n-2}), \ldots, R(z_2, z_1), \\ x_l = z_1 = w_r, z_n = y_m, \end{array}} \\ R(w_r, w_{r-1}), R(w_{r-2}, w_{r-3}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right. \right\rangle.$$

We apply precisely the same transformations as in the proof of $(xxiv)$ for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} w_1, \ldots, w_r, \\ s_1 \end{array} \left| \begin{array}{l} w_r^{(\alpha^{(l,m)} - \beta^{(l,m)})}, \\ R(w_r, w_{r-1}), R(w_{r-2}, w_{r-3}), \ldots, R(w_2, w_1), \\ R(s_1, w_1) \end{array} \right. \right\rangle.$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}, \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$G_\Gamma(R) = \langle\ w_1, s_1 \mid w_1^{\beta^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)})}, R(s_1, w_1)\ \rangle.$$

After we get this presentation, we cannot eliminate s_1 or w_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{r-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxvi) $a_4, a_6, a_7, a_{10}, a_{11}, a_{13}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \left| \begin{array}{l} \boxed{\begin{array}{l} R(x_1, x_2), R(x_2, x_3), \ldots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m, \end{array}} \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_k = z_n \end{array} \right. \right\rangle.$$

We apply precisely the same transformations as in the proof of (vii) for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_k, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} x_k^{\alpha^{(l,m)}-\beta^{(l,m)}}, \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), x_k = z_n \end{array} \right\rangle$$
$$= \langle\ z_1, \dots, z_n \ \mid\ z_n^{\alpha^{(l,m)}-\beta^{(l,m)}}, R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n)\ \rangle.$$

Since $(\alpha^{(l,m)}-\beta^{(l,m)}), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle\ z_1 \ \mid\ z_1^{\alpha^{n-1}(\alpha^{(l,m)}-\beta^{(l,m)})}\ \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n-1}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(xxvii) $a_4, a_6, a_7, a_{10}, a_{11}, a_{14}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n \end{array} \;\middle|\; \begin{array}{l} R(x_1, x_2), R(x_2, x_3), \dots, R(x_l, x_1), \\ R(y_1, y_2), R(y_2, y_3), \dots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \dots, x_k = y_m, \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), x_k = z_n \end{array} \right\rangle.$$

This is exactly same result with $(xxvi)$ since there is no difference in terms of presentation. Thus

$G_\Gamma(R) = \langle\ z_1 \ \mid\ z_1^{\alpha^{n-1}(\alpha^{(l,m)}-\beta^{(l,m)})}\ \rangle$.

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n-1}(\alpha^{(l,m)} - \beta^{(l,m)})$.

(xxviii) $a_1, a_3, a_5, a_8, a_9, a_{13}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_k, \\ x_{k+1}, \dots, x_t, \\ x_{t+1}, \dots, x_l, \\ y_1, \dots, y_{k+m-t}, \\ y_{k+m-t+1}, \dots, y_m, \\ z_1, \dots, z_n \end{array} \middle| \begin{array}{l} R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k), \\ R(x_k, x_{k+1}), R(x_{k+1}, x_{k+2}), \dots, R(x_{t-1}, x_t), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \dots, R(x_{l-1}, x_l), \\ R(y_m, y_1), R(y_1, y_2), \dots, R(y_{k+m-t-1}, y_{k+m-t}), \\ x_k = y_m, x_{k+1} = y_{m-1}, x_{k+2} = y_{m-2} \dots, \\ x_{t-1} = y_{k+m-t+1}, x_t = y_{k+m-t}, \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

Adjoin $x_k^{\alpha^l - \beta^l}$ to the presentation by Theorem 1.3.5.

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_k, \\ x_{k+1}, \dots, x_t, \\ x_{t+1}, \dots, x_l, \\ y_1, \dots, y_{k+m-t}, \\ y_{k+m-t+1}, \dots, y_m, \\ z_1, \dots, z_n \end{array} \middle| \begin{array}{l} \boxed{x_k^{\alpha^l-\beta^l}, R(x_l, x_1), R(x_1, x_2), \dots, R(x_{k-1}, x_k),} \\ R(x_k, x_{k+1}), R(x_{k+1}, x_{k+2}), \dots, R(x_{t-1}, x_t), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \dots, R(x_{l-1}, x_l), \\ R(y_m, y_1), R(y_1, y_2), \dots, R(y_{k+m-t-1}, y_{k+m-t}), \\ \boxed{x_k^{\alpha^l-\beta^l}, x_k = y_m}, x_{k+1} = y_{m-1}, x_{k+2} = y_{m-2}, \\ \dots, x_{t-1} = y_{k+m-t+1}, x_t = y_{k+m-t}, \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

Since $(\alpha^l - \beta^l, \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b) for the first box,

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_{k+1}, \dots, x_t, \\ x_{t+1}, \dots, x_l, \\ y_1, \dots, y_{k+m-t}, \\ y_{k+m-t+1}, \dots, y_m, \\ z_1, \dots, z_n \end{array} \middle| \begin{array}{l} x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ R(y_m, x_{k+1}), R(x_{k+1}, x_{k+2}), \dots, R(x_{t-1}, x_t), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \dots, R(x_{l-1}, x_l), \\ \boxed{\begin{array}{l} y_m^{\alpha^l-\beta^l}, R(y_m, y_1), R(y_1, y_2), \dots, \\ R(y_{k+m-t-1}, y_{k+m-t}), \end{array}} \\ x_{k+1} = y_{m-1}, x_{k+2} = y_{m-2} \dots, \\ x_{t-1} = y_{k+m-t+1}, x_t = y_{k+m-t}, \\ R(z_1, z_2), R(z_2, z_3), \dots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

We set $\gamma = \alpha^l - \beta^l$, $\eta = \beta^{t-k}(\alpha^l - \beta^l)$.

Since $p_1\alpha \equiv 1 \pmod{\gamma}$, there is an integer $q_1 \in \mathbb{Z}$ such that $p_1\alpha + q_1\gamma = 1$. Moreover, $p_1\alpha \equiv 1 \pmod{\gamma}$ implies that $y_i = y_i^{p_1\alpha} = y_j^{p_1\beta}$ in G. This allows us to adjoin the relation $y_i = y_j^{p_1\beta}$ and to eliminate the generator y_i.

Since $p_2\alpha \equiv 1 \pmod{\eta}$, there is an integer $q_2 \in \mathbb{Z}$ such that $p_2\alpha + q_2\eta = 1$. Moreover, $p_2\alpha \equiv 1 \pmod{\eta}$ implies that $x_i = x_i^{p_2\alpha} = x_j^{p_2\beta}$ in G. This allows us to adjoin the relation $x_i = x_j^{p_2\beta}$ and to eliminate the generator x_i.

Since $(\gamma, \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_{k+1}, \ldots, x_t, \\ x_{t+1}, \ldots, x_l, \\ y_{k+m-t}, \\ y_{k+m-t+1}, \ldots, y_{m-1}, \\ z_1, \ldots, z_n \end{array} \left| \begin{array}{l} x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ R(y_{k+m-t}^{(p_1\beta)^{t-k-m}}, x_{k+1}), R(x_{k+1}, x_{k+2}), \\ \ldots, R(x_{t-1}, x_t), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ y_{k+m-t}^{\beta^{t-k}(\alpha^l-\beta^l)} \\ x_{k+1} = y_{m-1}, x_{k+2} = y_{m-2}, \ldots, \\ x_{t-1} = y_{k+m-t+1}, x_t = y_{k+m-t}, \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right. \right\rangle.$$

$$= \left\langle \begin{array}{l} x_t, \\ x_{t+1}, \ldots, x_l, \\ y_{k+m-t}, \\ z_1, \ldots, z_n \end{array} \left| \begin{array}{l} x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ R(y_{k+m-t}^{(p_1\beta)^{t-k-m}}, x_t^{(p_2\beta)^{t-k-1}}), R(x_t^{(p_2\beta)^{t-k-1}}, x_t^{(p_2\beta)^{t-k-2}}) \\ , \ldots, R(x_t^{p_2\beta}, x_t), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ y_{k+m-t}^{\beta^{t-k}(\alpha^l-\beta^l)}, \\ x_t^{(p_2\beta)^{t-k-1}} = y_{k+m-t}^{(p_1\beta)^{t-k-1}}, x_t^{(p_2\beta)^{t-k-2}} = y_{k+m-t}^{(p_1\beta)^{t-k-2}}, \\ \ldots, x_t^{p_2\beta} = y_{k+m-t}^{p_1\beta}, \boxed{x_t = y_{k+m-t}}, \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right. \right\rangle$$

$$= \left\langle \begin{array}{l|l} x_t, & x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ x_{t+1}, \ldots, x_l, & \boxed{\begin{array}{l} R(x_t^{(p_1\beta)^{t-k-m}}, x_t^{(p_2\beta)^{t-k-1}}), R(x_t^{(p_2\beta)^{t-k-1}}, x_t^{(p_2\beta)^{t-k-2}}) \\ , \ldots, R(x_t^{p_2\beta}, x_t), \end{array}} \\ z_1, \ldots, z_n & R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ & x_t^{\beta^{t-k}(\alpha^l-\beta^l)}, \\ & x_t^{(p_2\beta)^{t-k-1}} = x_t^{(p_1\beta)^{t-k-1}}, x_t^{(p_2\beta)^{t-k-2}} = x_t^{(p_1\beta)^{t-k-2}}, \\ & \ldots, x_t^{p_2\beta} = x_t^{p_1\beta}, \\ & R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

Since $R(x_t^{(p_2\beta)^i}, x_t^{(p_2\beta)^{i-1}}) = x_t^{\alpha(p_2\beta)^i - \beta(p_2\beta)^{i-1}} = x_t^{(p_2\beta)^{i-1}(\alpha p_2\beta-\beta)}$, and $\alpha p_2\beta - \beta = 0 \mod \gamma$ since $p_2\alpha \equiv 1 \mod \gamma$. Thus, these relations are redundant so can be removed.

$$G_\Gamma(R) = \left\langle \begin{array}{l|l} x_t, & x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ x_{t+1}, \ldots, x_l, & R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ z_1, \ldots, z_n & x_t^{\beta^{t-k}(\alpha^l-\beta^l)}, \\ & x_t^{(p_2\beta)^{t-k-1}} = x_t^{(p_1\beta)^{t-k-1}}, x_t^{(p_2\beta)^{t-k-2}} = x_t^{(p_1\beta)^{t-k-2}}, \\ & \ldots, x_t^{p_2\beta} = x_t^{p_1\beta}, \\ & R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l|l} x_t, x_{t+1}, \ldots, x_l, & x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ z_1, \ldots, z_n & R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ & x_t^{\beta^{t-k}(\alpha^l-\beta^l)}, \\ & x_t^{(p_2\beta)^{t-k-1}-(p_1\beta)^{t-k-1}}, x_t^{(p_2\beta)^{t-k-2}-(p_1\beta)^{t-k-2}}, \ldots, x_t^{p_2\beta-p_1\beta}, \\ & R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

$$= \left\langle \begin{array}{l|l} x_t, x_{t+1}, \ldots, x_l, & x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ z_1, \ldots, z_n & R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ & x_t^{\beta^{t-k}(\alpha^l-\beta^l)}, \\ & x_t^{(\beta^{t-k-1},\beta^{t-k-2},\ldots,\beta^2,\beta)(p_2^{t-k-1}-p_1^{t-k-1},p_2^{t-k-2}-p_1^{t-k-2},\ldots,p_2^2-p_1^2,p_2-p_1)}, \\ & R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} x_t, x_{t+1}, \ldots, x_l, \\ z_1, \ldots, z_n \end{array} \middle| \begin{array}{l} x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ x_t^{(\beta^{t-k}(\alpha^l-\beta^l), \beta(p_2-p_1))}, \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle$$

$$= \left\langle \begin{array}{l} x_t, x_{t+1}, \ldots, x_l, \\ z_1, \ldots, z_n \end{array} \middle| \begin{array}{l} x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ \boxed{\begin{array}{l} R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ x_t^{\beta(\alpha^l-\beta^l)}, \end{array}} \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

Since $(\beta(\alpha^l - \beta^l), \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$= \left\langle \begin{array}{l} x_l, \\ z_1, \ldots, z_n \end{array} \middle| \begin{array}{l} x_l^{\alpha^k(\alpha^l-\beta^l)}, \\ x_l^{\beta^{l-t}(\alpha^l-\beta^l)}, \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

Since $(\alpha^k(\alpha^l - \beta^l), \beta^{l-t}(\alpha^l - \beta^l)) = (\alpha^l - \beta^l)$, we get

$$\begin{aligned} &= \left\langle \begin{array}{l} x_l, \\ z_1, \ldots, z_n \end{array} \middle| \begin{array}{l} x_l^{\alpha^l-\beta^l}, \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle \\ &= \langle\, z_1, \ldots, z_n \mid z_n^{\alpha^l-\beta^l}, R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n) \,\rangle \\ &= \langle\, z_1 \mid z_1^{\alpha^{n-1}(\alpha^l-\beta^l)} \,\rangle. \end{aligned}$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n-1}(\alpha^l - \beta^l)$.

(xxix) $a_1, a_3, a_5, a_8, a_{10}, a_{14}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \middle| \begin{array}{l} \boxed{\begin{array}{l} R(x_l, x_{l-1}), R(x_{l-1}, x_{l-2}), \ldots, R(x_{k+1}, x_k), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{k-1}, x_k), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m, \end{array}} \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_l = z_n \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of $(viii)$ for the relations inside the box to obtain that

Supposing $k < l - k$, we get

$$G_\Gamma(R) = \left\langle \begin{array}{l|l} x_l, & x_l^{\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}, \\ z_1,\ldots,z_n & R(z_1,z_2),R(z_2,z_3),\ldots,R(z_{n-1},z_n), x_l = z_n \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l|l} z_1,\ldots,z_n & \begin{array}{l} z_n^{\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}, \\ R(z_1,z_2),R(z_2,z_3),\ldots,R(z_{n-1},z_n) \end{array} \end{array} \right\rangle.$$

Since $(\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)} - \alpha^{(m,2k-l)}), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle\ z_1 \mid z_1^{\alpha^{n-1}\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}\ \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n-1}\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)} - \alpha^{(m,2k-l)})$.

Supposing $k \geqslant l - k$, we get

$$G_\Gamma(R) = \left\langle \begin{array}{l|l} x_l, & x_l^{\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}, \\ z_1,\ldots,z_n & R(z_1,z_2),R(z_2,z_3),\ldots,R(z_{n-1},z_n), x_l = z_n \end{array} \right\rangle$$
$$= \left\langle \begin{array}{l|l} z_1,\ldots,z_n & \begin{array}{l} z_n^{\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}, \\ R(z_1,z_2),R(z_2,z_3),\ldots,R(z_{n-1},z_n) \end{array} \end{array} \right\rangle.$$

Since $(\alpha^k(\alpha^{(m,l-2k)} - \beta^{(m,l-2k)}), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle\ z_1 \mid z_1^{\alpha^{n-1}\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}\ \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{n-1}\alpha^k(\alpha^{(m,l-2k)} - \beta^{(m,l-2k)})$.

(xxx) $a_1, a_3, a_5, a_8, a_9, a_{14}$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n \end{array} \;\middle|\; \begin{array}{l} R(x_t, x_{t-1}), R(x_{t-1}, x_{t-2}), \ldots, R(x_{l-1}, x_l), \\ \boxed{\begin{array}{l} \mathrm{R}(\mathrm{x}_l, x_1), R(x_1, x_2), \ldots, R(x_{k-1}, x_k), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m, \end{array}} \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_t = z_n \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (ix) for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_l, x_{l-1}, \ldots, x_t \\ z_1, \ldots, z_n \end{array} \;\middle|\; \begin{array}{l} R(x_t, x_{t-1}), R(x_{t-1}, x_{t-2}), \ldots, R(x_{l-1}, x_l), \\ x_l^{\alpha^k(\alpha^m - \beta^m)}, \\ R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n), x_t = z_n \end{array} \right\rangle.$$

Since $(\alpha^k(\alpha^m - \beta^m), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle\ z_1, \ldots, z_n \mid z_n^{\alpha^{k+l-t}(\alpha^m - \beta^m)}, R(z_1, z_2), R(z_2, z_3), \ldots, R(z_{n-1}, z_n)\ \rangle.$$

Since $(\alpha^{k+l-t}(\alpha^m - \beta^m), \beta) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (b),

$$G_\Gamma(R) = \langle\ z_n \mid z_n^{\alpha^{k+l+n-t-1}(\alpha^m - \beta^m)}\ \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic of order $\alpha^{k+l+n-t-1}(\alpha^m - \beta^m)$.

(xxxi) $a_4, a_6, a_7, a_{10}, a_{12}, a_{13}, (a_{11} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \dots, x_l, \\ y_1, \dots, y_m, \\ z_1, \dots, z_n, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1,x_2), R(x_2,x_3), \dots, R(x_l,x_1), \\ R(y_1,y_2), R(y_2,y_3), \dots, R(y_m,y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \dots, x_k = \\ y_m, x_k = z_n, \end{array}} \\ R(z_n,z_{n-1}), R(z_{n-2},z_{n-3}), \dots, R(z_2,z_1), \\ R(s_1,z_1) \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of $(xxvi)$ for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle \begin{array}{l} z_1, \dots, z_n, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} z_n^{\alpha^{(l,m)} - \beta^{(l,m)}}, \\ R(z_n,z_{n-1}), R(z_{n-2},z_{n-3}), \dots, R(z_2,z_1), \\ R(s_1,z_1) \end{array} \right\rangle.$$

Since $(\alpha^{(l,m)} - \beta^{(l,m)}), \alpha) = 1$ (see Remark 3.1.3), we can apply Lemma 1.3.4 (a),

$$G_\Gamma(R) = \langle\; z_1, s_1 \mid z_1^{\beta^{n-1}(\alpha^{(l,m)} - \beta^{(l,m)})}, R(s_1,z_1) \;\rangle.$$

After we get this presentation, we cannot eliminate s_1 or z_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{n-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxxii) $a_4, a_6, a_7, a_{10}, a_{12}, a_{14}, (a_{11} = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1,\ldots,x_l, \\ y_1,\ldots,y_m, \\ z_1,\ldots,z_n, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_1,x_2), R(x_2,x_3), \ldots, R(x_l,x_1), \\ R(y_1,y_2), R(y_2,y_3), \ldots, R(y_m,y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = \\ y_m, x_k = z_n, \end{array}} \\ R(z_n,z_{n-1}), R(z_{n-2},z_{n-3}), \ldots, R(z_2,z_1), \\ R(s_1,z_1) \end{array} \right\rangle.$$

This is exactly same result with $(xxxi)$ since there is no difference in terms of presentation. Hence

$$G_\Gamma(R) = \langle\ z_1, s_1 \ \mid\ z_1^{\beta^{n-1}(\alpha^{(l,m)}-\beta^{(l,m)})}, R(s_1,z_1)\ \rangle.$$

After we get this presentation, we cannot eliminate s_1 or z_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{n-1}(\alpha^{(l,m)} - \beta^{(l,m)}), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxxiii) $a_2, a_4, a_5, a_8, a_9, a_{14}, (a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1,\ldots,x_l, \\ y_1,\ldots,y_m, \\ z_1,\ldots,z_n, \\ s_1 \end{array} \;\middle|\; \begin{array}{l} \boxed{\begin{array}{l} R(x_k,x_{k+1}), R(x_{k+1},x_{k+2}), \ldots, R(x_{l-1},x_l), \\ R(x_k,x_{k-1}), R(x_{k-1},x_{k-2}), \ldots, R(x_1,x_l), \\ R(y_1,y_2), R(y_2,y_3), \ldots, R(y_m,y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m, \\ R(z_n,z_{n-1}), R(z_{n-1},z_{n-2}), \ldots, R(z_2,z_1), x_l = z_n, \end{array}} \\ R(s_1,z_1) \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of $(xxix)$ for the relations inside the box but we need to interchange α and β by reflection principle since the direction of each arc are reversed according to $(xxix)$.

Thus we get

Supposing $k < l - k$

$$G_\Gamma(R) = \langle\ z_1, s_1 \mid z_1^{\alpha^{n-1}\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)}-\alpha^{(m,2k-l)})}, R(s_1, z_1)\ \rangle.$$

Supposing $k \geqslant l - k$, we get

$$G_\Gamma(R) = \langle\ z_1, s_1 \mid z_1^{\alpha^{n-1}\alpha^k(\alpha^{(m,l-2k)}-\beta^{(m,l-2k)})}, R(s_1, z_1)\ \rangle.$$

After we get those presentations, we cannot eliminate s_1 or z_1 from the presentations. It is because we are not able to apply Lemma 1.3.4 further since $(\alpha^{n-1}\alpha^{min\{k,l-k\}}(\beta^{(m,2k-l)} - \alpha^{(m,2k-l)}), \alpha) \neq 1$ and $(\alpha^{n-1}\alpha^k(\alpha^{(m,l-2k)} - \beta^{(m,l-2k)}), \alpha) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

(xxxiv) $a_2, a_3, a_5, a_8, a_9, a_{14}, (a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_l, \\ y_1, \ldots, y_m, \\ z_1, \ldots, z_n, \\ s_1 \end{array} \middle| \begin{array}{l} \boxed{\begin{array}{l} \mathrm{R}(\mathrm{x}_t, x_{t-1}), R(x_{t-1}, x_{t-2}), \ldots, R(x_{l-1}, x_l), \\ R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{k-1}, x_k), \\ R(y_1, y_2), R(y_2, y_3), \ldots, R(y_m, y_1), \\ x_t = y_{m-k+t}, x_{t+1} = y_{m-k+t+1}, \ldots, x_k = y_m, \\ x_t = z_n \end{array}} \\ R(z_n, z_{n-1}), R(z_{n-2}, z_{n-3}), \ldots, R(z_2, z_1), \\ R(s_1, z_1) \end{array} \right\rangle.$$

We apply precisely the same transformations as in the proof of (xxx) for the relations inside the box to obtain that

$$G_\Gamma(R) = \left\langle\ s_1, z_1, \ldots, z_n\ \middle|\ \begin{array}{l} z_n^{\alpha^{k+l-t}(\alpha^m-\beta^m)}, \\ R(z_n, z_{n-1}), R(z_{n-2}, z_{n-3}), \ldots, R(z_2, z_1), \\ R(s_1, z_1) \end{array} \right\rangle.$$

After we get this presentation, we cannot eliminate $s_1, z_1, z_2, \ldots, z_n$ from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\alpha^{k+l-t}(\alpha^m - \beta^m), \alpha) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a $(n+1)$-generator presentation.

(xxxv) $a_2, a_3, a_5, a_8, a_9, a_{13}$ $(a_1 = 1)$

The group $G_\Gamma(R)$ is defined by the presentation

$$G_\Gamma(R) = \left\langle \begin{array}{l} x_1, \ldots, x_k, \\ x_{k+1}, \ldots, x_t, \\ x_{t+1}, \ldots, x_l, \\ y_1, \ldots, y_{k+m-t}, \\ y_{k+m-t+1}, \ldots, y_m, \\ z_1, \ldots, z_n, \\ s_1 \end{array} \left| \begin{array}{l} \boxed{\begin{array}{l} R(x_l, x_1), R(x_1, x_2), \ldots, R(x_{k-1}, x_k), \\ R(x_k, x_{k+1}), R(x_{k+1}, x_{k+2}), \ldots, R(x_{t-1}, x_t), \\ R(x_t, x_{t+1}), R(x_{t+1}, x_{t+2}), \ldots, R(x_{l-1}, x_l), \\ R(y_m, y_1), R(y_1, y_2), \ldots, \\ R(y_{k+m-t-1}, y_{k+m-t}), \\ x_k = y_m, x_{k+1} = y_{m-1}, x_{k+2} = y_{m-2} \ldots, \\ x_{t-1} = y_{k+m-t+1}, x_t = y_{k+m-t}, \end{array}} \\ R(z_n, z_{n-1}), R(z_{n-2}, z_{n-3}), \ldots, R(z_2, z_1), \\ x_l = z_n, R(s_1, z_1), \end{array} \right. \right\rangle.$$

Inside the box is exactly same result with $(xxviii)$ since there is no difference in terms of presentation. Hence

$$G_\Gamma(R) = \left\langle \begin{array}{l} z_1, z_2, \ldots, z_n, \\ s_1 \end{array} \left| \begin{array}{l} z_n^{\alpha^l - \beta^l}, \\ R(z_n, z_{n-1}), R(z_{n-2}, z_{n-3}), \ldots, R(z_2, z_1), \\ R(s_1, z_1), \end{array} \right. \right\rangle.$$

$$G_\Gamma(R) = \langle\ z_1, s_1 \mid z_1^{\beta^{n-1}(\alpha^l - \beta^l)}, R(s_1, z_1), \rangle.$$

After we get this presentation, we cannot eliminate s_1 or z_1 from the presentation. It is because we are not able to apply Lemma 1.3.4 further since $(\beta^{n-1}(\alpha^l - \beta^l), \beta) \neq 1$. Therefore, we cannot go further. Thus, the group $G_\Gamma(R)$ has a 2-generator presentation.

CHAPTER 4

Digraph groups when $|V| \leqslant |A|$

4.1 Preamble

In this chapter we consider digraphs Γ with $|V(\Gamma)| \leqslant |A(\Gamma)|$, and girth $g(\Gamma) \geqslant 4$ and one of the following holds:

(i) Γ has no source and no sink; or

(ii) Γ has exactly one source and no sink; or

(iii) Γ has exactly one sink and no source; or

(iv) Γ has one source and one sink and they are adjacent.

It is already known that the corresponding group is infinite if none of (i)-(iv) hold by Lemma 1.3.3. For Case (iv) the finiteness of the corresponding groups remains unresolved both in [9] for $|V| = |A|$ and in Chapter 3 for $|V| = |A| - 1$.

In contrast the Chapter 3 we now focus on determining when the group is finite and will no longer be concerned with calculating the order and structure of the group. As corollaries to our main theorem (Theorem 4.1.1) we prove that $G_\Gamma(R)$ is finite cyclic when Γ is strongly connected (Corollary 4.2.1) or semi-connected defined by $(i) - (iii)$ in Preamble 4.1 (Corollary 4.3.2 and 4.3.4).

Theorem 4.1.1. *Suppose $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a, b)\,\rangle$. If Γ, where $g(\Gamma) \geqslant 4$, has a trail including all vertices and the terminal vertex v is in a directed cycle of length N, then $G_\Gamma(R)$ is generated by x_u, where u is the initial vertex of this trail.*

Proof. Consider a trail in Γ that includes every vertex of Γ, in which the initial vertex is u, say, and the terminal vertex is v, say, and v is the vertex of some directed cycle. We now claim that, given an arc (ι, τ) in this trail, generator x_τ can be expressed as a power of x_ι. Using all such expressions, every generator x_τ (where τ is a vertex of this trail) can be expressed as a power of x_u. Therefore, every generator of $G_\Gamma(R)$, except x_u, can be eliminated and so $G_\Gamma(R)$ is cyclic, generated by x_u.

We now prove the claim. Let $[w, v]$ be the last arc of the trail. Since v is in some directed cycle (of length N, say) there is a relator x_v^γ where $\gamma = \alpha^N - \beta^N$. Using Lemma 1.3.4(d) we may write x_v as a power of x_w and adjoin a relator o the form $x_w^{\alpha\gamma}$. Repeating this procedure for the remaining vertices of the trail in turn (from the 2nd last, 3rd last, to the second vertex), for each arc (ι, τ) of the trail we may express x_τ as a power of x_ι and thus as a power of x_u, as claimed.

□

Corollary 4.1.2. *Suppose $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a, b)\,\rangle$. If Γ, where $g(\Gamma) \geqslant 4$, has a trail including all vertices and the terminal vertex u is in a directed cycle of length N, then $G_\Gamma(R)$ is generated by x_v, where v is the initial vertex of this trail.*

Proof. This is a corollary of Theorem 4.1.1 by reflection principle addressed in Remark 1.3.7. □

4.2 Finite cyclic groups for strongly-connected digraphs

If a digraph is strongly connected, then it cannot have a source or sink. Therefore, it can be thought that we only have Case (i): no source, no sink.

Corollary 4.2.1. *Let* Γ *be a strongly connected digraph with* $g(\Gamma) \geqslant 4$. *Then* $G_\Gamma(R)$ *is finite if and only if* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $K = \langle\, a, b \mid R(a, b)\,\rangle$, *in which case* $G_\Gamma(R)$ *is cyclic.*

Proof. If $G_\Gamma(R)$ is finite then (as explained in the Preamble), the stated conditions on α, β, and K hold.

Γ is strongly connected, then there exists a path $i \to j$ for all i, j. Thus, there are paths $1 \to 2, 2 \to 3, \ldots, n-1 \to n$. Hence, there is a trail $1 \to n$ includes all vertices of the digraph. There is also a trail $n \to 1$ with same technique. Therefore, n is in a closed trail and so it is in a directed cycle. Since Theorem 4.1.1 holds, $G_\Gamma(R)$ is finite cyclic.

□

4.3 Finite cyclic groups for semi-connected digraphs

If a digraph is semi-connected, then we have 4 cases $(i)-(iv)$ that we described in Preamble. Therefore, we need to investigate these cases separately here.

Lemma 4.3.1. *For any semi-connected digraph with no source no sink, there is a directed trail which includes all vertices of the digraph.*

Proof. We prove this by induction.

Inductive Hypothesis. Any semi-connected digraph with n vertices and no source no sink has a directed trail that includes every vertex.

Anchor case: this is true for $n = 2$, as such a digraph consists of two vertices joined to each other by an arc in each direction.

For the inductive step we must show that any semi-connected digraph with $n+1$ vertices and no source and no sink have a trail that includes every vertex.

Let Γ be such a digraph and let u be some vertex of Γ and let Λ be the induced sub-digraph of Γ with vertex set $V(\Gamma)\backslash\{u\}$. Then by the inductive hypothesis there is a trail that includes every vertex of Λ. Relabel the vertices of Λ(if necessary) such that the trail is $1 \to 2 \to 3 \to \ldots \to n-1 \to n$.

Consider vertices n, u. If there is an arc $n \to u$ then there is a trail $1 \to 2 \to 3 \to \ldots \to n \to u$ so assume there is an arc $u \to n$.

Consider vertices $n-1, u$. If there is an arc $n-1 \to u$ then there is a trail $1 \to 2 \to 3 \to \ldots \to n-1 \to u \to n$ so assume there is an arc $u \to n-1$.

Consider vertices $n-2, u$. If there is an arc $n-2 \to u$ then there is a trail $1 \to 2 \to 3 \to \ldots \to n-2 \to u \to n-1 \to n$ so assume there is an arc $u \to n-2$.

Continue in this way. Consider vertices $2, u$. If there is an arc $2 \to u$ then there is a trail $1 \to 2 \to u \to 3 \to \ldots \to n-1 \to n$ so assume there is an arc $u \to 2$.

Continue in this way. Consider vertices $1, u$. If there is an arc $1 \to u$ then there is a trail $1 \to u \to 2 \to 3 \to \ldots \to n-1 \to n$ so assume there is an arc $u \to 1$ and then there is a trail $u \to 1 \to 2 \to 3 \to \ldots \to n-1 \to n$ which is a trail through every vertex of Γ.

$\square$

For Case (i) we have the following:

Corollary 4.3.2. *Suppose* $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $K = \langle\, a, b \mid R(a, b) \,\rangle$ *and the* $g(\Gamma) \geqslant 4$*. If* Γ *is any semi-connected digraph with no source and no sink then* $G_\Gamma(R)$ *is finite cyclic.*

Proof. Let $V(\Gamma) = \{1, \ldots, n\}$ then by Lemma 4.3.1 there exists a trail $1 \to 2 \to 3 \to \ldots \to n-1 \to n$ (relabel some of the vertices if necessary). Since n is not a sink there is a trail n to j for some $1 < j < n$. Thus, n is in a closed trail and so it is in a directed cycle. Therefore, Theorem 4.1.1 holds and $G_\Gamma(R)$ is finite cyclic.

$\square$

Lemma 4.3.3. *For any semi-connected digraph with exactly one source, there is a trail which includes all vertices of the digraph.*

Proof. We prove this by induction.

Inductive Hypothesis. Any semi-connected digraph with n vertices and exactly one source has a trail that includes every vertex.

Anchor case: this is true for $n = 2$, as such a digraph consists of two vertices joined by an arc.

For the inductive step we must show that any semi-connected digraph with $n + 1$ vertices and exactly one source has a trail that includes every vertex.

Let Γ be such a digraph and let u be some vertex of Γ that is not the source, and let Λ be the induced sub-digraph of Γ with vertex set $V(\Gamma)\backslash\{u\}$. Then by the inductive hypothesis there is a trail that includes every vertex of Λ. Relabel the vertices of Λ such that the trail is $1 \to 2 \to 3 \to \ldots \to n-1 \to n$ (and therefore the source is 1).

Consider vertices n, u. If there is an arc $n \to u$ then there is a trail $1 \to 2 \to 3 \to \ldots \to n \to u$ so assume there is an arc $u \to n$.

Consider vertices $n-1, u$. If there is an arc $n-1 \to u$ then there is a trail $1 \to 2 \to 3 \to \ldots \to n-1 \to u \to n$ so assume there is an arc $u \to n-1$.

Consider vertices $n-2, u$. If there is an arc $n-2 \to u$ then there is a trail $1 \to 2 \to 3 \to \ldots \to n-2 \to u \to n-1 \to n$ so assume there is an arc $u \to n-2$.

Continue in this way. Consider vertices $2, u$. If there is an arc $2 \to u$ then there is a trail $1 \to 2 \to u \to 3 \to \ldots \to n-1 \to n$ so assume there is an arc $u \to 2$.

Now since 1 is a source there is an arc $1 \to u$, so there is a trail $1 \to u \to 2 \to 3 \to \ldots \to n-1 \to n$, which is a trail through every vertex of Γ. □

For Case $(ii), (iii)$ we have the following:

Corollary 4.3.4. *Let Γ be a semi-connected digraph with $g(\Gamma) \geqslant 4$, and suppose that if Γ has exactly one source and exactly one sink then they are not adjacent. Then $G_\Gamma(R)$ is finite if and only if $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) =$*

$1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a,b)\, \rangle$, and one of the following holds:

(ii) Γ has exactly one source and no sink; or

(iii) Γ has exactly one sink and no source;

in which case $G_\Gamma(R)$ is cyclic.

Proof. By the reflection principle we may assume that Γ has exactly one source and no sinks. Let $V(\Gamma) = \{1, \ldots, n\}$ and 1 is the source. By Lemma 4.3.3 there exists a trail $1 \to 2 \to 3 \to \ldots \to n-1 \to n$ (relabel some of the vertices if necessary).

Since n is not a sink there is a trail n to j for some $1 < j < n$. Thus n is in a closed trail and so it is in a directed cycle. Therefore, Theorem 4.1.1 holds and $G_\Gamma(R)$ is finite cyclic.

□

It is natural to ask if Theorem 4.1.1 can be applied to Case (iv), i.e. semi-connected digraphs with exactly one source and exactly one sink that are adjacent. The answer to this is no, since the digraph in Figure 4.1 is semi-connected and has a path $1 \to 2 \to 3 \to 4 \to 5 \to 6$ through every vertex, but the terminal vertex is not in a directed cycle.

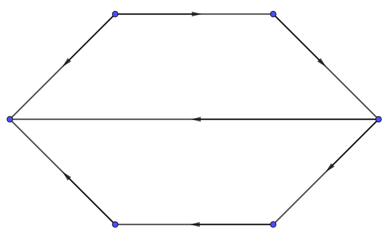

Figure 4.1: A digraph with exactly one source and one sink that are adjacent

Corollaries 4.3.2 and 4.3.4 imply that if Γ is semi-connected with (i) no source and no sink, or (ii) exactly one source and no sink or (iii) exactly one sink and no source (note that $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a,b)\, \rangle$ and the $g(\Gamma) \geqslant 4$), then $G_\Gamma(R)$ is finite cyclic.

4.4 More digraph families

In this section, we consider digraph groups $G_\Gamma(R)$ where Γ is formed as a combination of other digraphs. Recall that u_i is an in-neighbour of v_j if there is an arc (u_i, v_j) for some i and j by Definition 1.2.8.

Theorem 4.4.1. *Suppose $\alpha \neq 0, \beta \neq 0, (\alpha, \beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ in $K = \langle\, a, b \mid R(a, b)\,\rangle$. Let $\Gamma = \Gamma_1 \cup \Gamma_2$ and $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, the $g(\Gamma) \geqslant 4$ and Γ is connected and the conditions $(i) - (iii)$ in Preamble hold. If Γ_1 has a trail containing every vertex $u_1 \to u_2 \to \ldots \to u_{N-1} \to u_N$ of Γ_1 where u_N is in a directed cycle, Γ_2 has a trail containing every vertex $v_1 \to v_2 \to \ldots \to v_{M-1} \to v_M$ of Γ_2 where v_M is in a directed cycle, then $G_\Gamma(R)$ is finite cyclic.*

Proof. u_1, v_1 cannot both be sources by hypothesis, so without loss of generality assume v_1 is not a source. Then there are two cases:
Case 1: there is an arc (u_i, v_1) for some $1 \leqslant i \leqslant N$.
Case 2: there is an arc (v_j, v_1) for some $4 \leqslant j \leqslant M$

Case 1 is connected but Case 2 is not yet connected. Therefore it splits into 4 cases:
Case 2(a): there is an arc (u_i, v_m) for some $1 \leqslant i \leqslant N$, $m \leqslant j$
Case 2(b): there is an arc (u_i, v_m) for some $1 \leqslant i \leqslant N$, $m > j$
Case 2(c): there is an arc (v_m, u_i) for some $1 \leqslant i \leqslant N$, $m \leqslant j$
Case 2(d): there is an arc (v_m, u_i) for some $1 \leqslant i \leqslant N$, $m > j$

Now we will give the proof for each cases. Note that we set up $\gamma = \alpha^N - \beta^N$ and $\eta = \alpha^M - \beta^M$.
Case 1: By Theorem 4.1.1, we can eliminate all generators leaving only the generator x_{u_1} for $G_{\Gamma_1}(R)$ and x_{v_1} for $G_{\Gamma_2}(R)$. Now the aim is to to write the generator x_{v_1} in terms of the generator x_{u_1}. Then $G_\Gamma(R)$ is finite cyclic.

There is a trail $u_1 \to u_2 \to \ldots \to u_i \to v_1 \to v_2 \to \ldots \to v_M$ which means the generator x_{v_1} can be written in terms of x_{u_1} since v_M is in a directed cycle. That means leaving only the generator x_{u_1} and the relator $x_{u_1}^r$, where $r \in \mathbb{Z}$. Hence, $G_\Gamma(R)$ is finite cyclic.

Case 2(a): By Theorem 4.1.1, we can eliminate all generators leaving only the generator x_{u_1} for $G_{\Gamma_1}(R)$ and x_{v_1} for $G_{\Gamma_2}(R)$. Now the aim is to to write the generator x_{v_1} in terms of the generator x_{u_1}. Then $G_\Gamma(R)$ is finite cyclic.

There is a trail $u_1 \to u_2 \to \ldots \to u_i \to v_m \to v_{m+1} \to \ldots \to v_j \to v_1 \to v_2 \to v_3 \to \ldots \to v_m$ which means the generators x_{v_1} can be written in terms of x_{u_1} since v_m is in a directed cycle. That means leaving only the generator x_{u_1} and the relator $x_{u_1}^r$, where r. Hence, $G_\Gamma(R)$ is finite cyclic.

Case 2(b): By Theorem 4.1.1, we can eliminate all generators leaving only the generator x_{u_1} for $G_{\Gamma_1}(R)$.

We now demonstrate the presentation in terms of x_{v_m} for $G_{\Gamma_2}(R)$.

There is a directed cycle $v_1 \to v_2 \to \ldots \to v_j \to v_1$ so the generators $x_{v_1}, x_{v_2}, \ldots, x_{v_{j-1}}$ are eliminated leaving the generator x_{v_j} and the relator $x_{v_j}^\gamma$ by Theorem 1.3.5. Since $(\beta, \gamma) = 1$ and an iterated application of Lemma 1.3.4 (b) yields that we can eliminate the generators $x_{v_j}, x_{v_{j+1},\ldots,x_{v_{m-1}}}$ leaving only the generator x_{v_m} and the relator $x_{v_m}^{\alpha^{m-j}\gamma}$. There is another path $v_m \to v_{m+1} \to \ldots \to v_M$. Since $(\alpha, \eta) = 1$ and an iterated application of Lemma 1.3.4 (a) yields that the generators $x_{v_{m+1}}, x_{v_{m+2}}, \ldots, x_{v_M}$ are eliminated leaving the generator x_{v_m} for some m and the relator $x_{v_m}^{\beta^r\eta}$. Now we have one generator and two relators which are $x_{v_m}^{\alpha^{m-j}\gamma}$ and $x_{v_m}^{\beta^r\eta}$. Then we only have the relators $x_{v_m}^{(\gamma,\eta)}$ for some m since $(\alpha, \beta) = 1$ for $G_{\Gamma_2}(R)$.

Now, there is a path as $u_1 \to u_2 \to \ldots \to u_i \to v_m$ and $x_{v_m}^{(\gamma,\eta)} = 1$. Since $(\beta, (\gamma, \eta)) = 1$, x_{v_m} can be eliminated leaving the generator x_{u_1} by Lemma 1.3.4 (b). Hence, $G_\Gamma(R)$ is finite cyclic.

Case 2(c): For Γ_2, there is a path $v_j \to v_{j+1} \to \ldots \to v_M$ and v_M is in a directed cycle. That means the generators $x_{v_{j+1}}, x_{v_{j+2}}, \ldots, x_{v_M}$ can be eliminated leaving the generator x_{v_j} and the relator $x_{v_j}^{\alpha^r(\alpha^s-\beta^s)}$, where $r, s \in \mathbb{Z}$, by Lemma 1.3.6 (b). There is a directed cycle $v_j \to v_1 \to v_2 \to \ldots \to v_{j-1} \to v_j$. That means the generators $x_{v_1}, x_{v_2}, \ldots, x_{v_m}, x_{v_{m+1}} \ldots, x_{v_{j-1}}$ can be eliminated leaving the generator x_{v_j} and the relator $x_{v_j}^{\alpha^j-\beta^j}$ by Theorem 1.3.5.

Thus, $G_{\Gamma_2}(R) = \langle\, x_{v_j} \mid x_{v_j}^{\alpha^r(\alpha^s-\beta^s)}, x_{v_j}^{\alpha^j-\beta^j} \,\rangle = \langle\, x_{v_j} \mid x_{v_j}^{\alpha^{(j,s)}-\beta^{(j,s)}} \,\rangle$.

By Lemma 3.1.2, x_{v_j} can be written as $x_{v_m}^{p^{j-m}\beta^{j-m}}$ so $G_{\Gamma_2}(R) = \langle\, x_{v_m} \mid x_{v_m}^{p^{j-m}\beta^{j-m}(\alpha^{(j,s)}-\beta^{(j,s)})} \,\rangle$.

Now, for Γ_1, there is a path as $u_i \to u_{i+1} \to \ldots \to u_N$ and u_N is in a directed cycle. That means the generators $x_{u_{i+1}}, x_{u_{i+2}}, \ldots, x_{u_N}$ can be eliminated leaving the generator x_{u_i} and the relator $x_{u_i}^{\alpha^k(\alpha^h-\beta^h)}$ where $k, h \in \mathbb{Z}$, by Lemma 1.3.6 (b). Since $\Gamma = \Gamma_1 \cup \Gamma_2$, we get

$$G_\Gamma(R) = \left\langle \; x_{u_1}, x_{u_2}, \ldots, x_{u_i}, x_{v_m} \;\middle|\; \begin{array}{l} x_{u_i}^{\alpha^k(\alpha^h-\beta^h)}, \\ \boxed{x_{v_m}^{p^{j-m}\beta^{j-m}(\alpha^{(j,s)}-\beta^{(j,s)})}, R(v_m, u_i),} \\ R(u_1, u_2), R(u_2, u_3), \ldots, R(u_{i-1}, u_i) \end{array} \right\rangle$$

by Lemma 1.3.6 (a)

$$= \left\langle \; x_{u_1}, x_{u_2}, \ldots, x_{u_i} \;\middle|\; \begin{array}{l} x_{u_i}^{\alpha^k(\alpha^h-\beta^h)}, x_{u_i}^{p^{j-m}\beta^{j-m+1}(\alpha^{(j,s)}-\beta^{(j,s)})}, \\ R(u_1, u_2), R(u_2, u_3), \ldots, R(u_{i-1}, u_i) \end{array} \right\rangle$$

$$= \left\langle \; x_{u_1}, x_{u_2}, \ldots, x_{u_i} \;\middle|\; \begin{array}{l} x_{u_i}^{\alpha^{(j,s,h)}-\beta^{(j,s,h)}}, \\ R(u_1, u_2), R(u_2, u_3), \ldots, R(u_{i-1}, u_i) \end{array} \right\rangle$$

by Lemma 1.3.6 (b)

$$\vdots$$

$$= \langle \; x_{u_1} \mid x_{u_1}^{\alpha^{i-1}(\alpha^{(j,s,h)}-\beta^{(j,s,h)})} \; \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic.

Case 2(d): There is a path $v_m \to v_{m+1} \ldots \to v_M$ and v_M is in a directed cycle. That means the generators $x_{v_{m+1}}, x_{v_{m+2}}, \ldots, x_{v_M}$ can be eliminated leaving the generator x_{v_m} and the relator $x_{v_m}^{\alpha^r(\alpha^s-\beta^s)}$, where $r, s \in \mathbb{Z}$, by Lemma 1.3.6 (a). There is a trail $v_j \to v_1 \to v_2 \to \ldots \to v_{j-1} \to v_j \to v_{j+1} \to v_m$. That means the generators $x_{v_1}, x_{v_2}, \ldots, x_{v_j}, x_{v_{j+1}}, v_{m-1}$ can be eliminated leaving the generator $x_{v_m}^{\beta^{m-j}(\alpha^j-\beta^j)}$ by Corollary 4.1.2.

Thus, $G_{\Gamma_2}(R) = \langle \, x_{v_m} \mid x_{v_m}^{\alpha^r(\alpha^s-\beta^s)}, x_{v_m}^{\beta^{m-j}(\alpha^j-\beta^j)} \, \rangle = \langle \, x_{v_m} \mid x_{v_m}^{\alpha^{(j,s)}-\beta^{(j,s)}} \, \rangle$.

Now, there is a trail as $u_i \to u_{i+1} \to \ldots \to u_N$ and u_N is in a directed cycle. That means the generators $x_{u_{i+1}}, x_{u_{i+2}}, \ldots, x_{u_N}$ can be eliminated leaving the generator x_{u_i} and the relator $x_{u_i}^{\alpha^k(\alpha^h-\beta^h)}$ where $k, h \in \mathbb{Z}$, by Theorem 4.1.1.

Thus, we get

$$G_\Gamma(R) = \left\langle\ x_{u_1}, x_{u_2}, \ldots, x_{u_i}, x_{v_m}\ \middle|\ \begin{array}{l} x_{u_i}^{\alpha^k(\alpha^h-\beta^h)}, \boxed{x_{v_m}^{\alpha^{(j,s)}-\beta^{(j,s)}}, R(v_m,u_i),} \\ R(u_1,u_2), R(u_2,u_3), \ldots, R(u_{i-1},u_i) \end{array}\right\rangle$$

by Lemma 1.3.6 (a)

$$= \left\langle\ x_{u_1}, x_{u_2}, \ldots, x_{u_i}\ \middle|\ \begin{array}{l} x_{u_i}^{\alpha^k(\alpha^h-\beta^h)}, x_{u_i}^{\beta\alpha^{(j,s)}-\beta^{(j,s)}}, \\ R(u_1,u_2), R(u_2,u_3), \ldots, R(u_{i-1},u_i) \end{array}\right\rangle$$

$$= \left\langle\ x_{u_1}, x_{u_2}, \ldots, x_{u_i}\ \middle|\ \begin{array}{l} x_{u_i}^{\alpha^{(j,s,h)}-\beta^{(j,s,h)}}, \\ R(u_1,u_2), R(u_2,u_3), \ldots, R(u_{i-1},u_i) \end{array}\right\rangle$$

by Lemma 1.3.6 (b)

$$\vdots$$

$$= \langle\ x_{u_1}\ |\ x_{u_1}^{\alpha^{i-1}(\alpha^{(j,s,h)}-\beta^{(j,s,h)})}\ \rangle.$$

Hence, $G_\Gamma(R)$ is finite cyclic.

□

Corollary 4.4.2. *Suppose* $\alpha \neq 0, \beta \neq 0, (\alpha,\beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $K = \langle a, b \mid R(a,b) \rangle$. *Let* $\Gamma = \Gamma_1 \cup \Gamma_2$ *and* $V(\Gamma_1) \cap V(\Gamma_2) = \emptyset$, *the* $g(\Gamma) \geqslant 4$ *and* Γ *is connected and* Γ_1, Γ_2 *are either strongly connected or are semi-connected and satisfy* $(i)-(iii)$. *Then* $G_\Gamma(R)$ *is a finite cyclic group.*

Proof. The digraphs Γ_1 and Γ_2 which are described in Theorem 4.4.1 can be replaced by any strongly connected or semi-connected digraph with $(i)-(iii)$ in Preamble as we proved in Corollary 4.2.1, 4.3.2 and 4.3.4, respectively. Thus, $G_\Gamma(R)$ is finite cyclic.

□

Theorem 4.4.3. *Suppose* $\alpha \neq 0, \beta \neq 0, (\alpha,\beta) = 1, \alpha^n - \beta^n \neq 0, a^\alpha = b^\beta$ *in* $K = \langle a, b \mid R(a,b) \rangle$ *and* $a^\alpha = b^\beta$. *Let* Γ *be a digraph with* $(i)-(iii)$ *in Preamble and the* $g(\Gamma) \geqslant 4$. *If* $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \ldots \cup \Gamma_t$ *and* $V(\Gamma_1) \cap V(\Gamma_i) \neq \emptyset$ *for* $2 \leqslant i \leqslant t$, *where* Γ_1 *has a path as* $v_1 \to v_2 \to \ldots \to v_M$ *and the terminal vertex* v_M *is in a directed cycle and each* Γ_i *for* $2 \leqslant i \leqslant t$ *has a path* P_i *that contains every vertex of* Γ_i *whose terminal vertex (respectively initial vertex)* *is in a directed cycle, and whose initial vertex (respectively terminal vertex) is v_j for some j. Then, $G_\Gamma(R)$ is finite cyclic.*

Proof. Γ_1 has $v_1 \to v_2 \to \ldots \to v_M$ and the terminal vertex v_M is in a directed cycle. By hypothesis, there are two possible cases as v_j terminal vertex or initial vertex for each Γ_i $(2 \leqslant i \leqslant M)$. We show that for each $2 \leqslant i \leqslant M$, if u is a vertex of Γ_i then x_u can be written in terms of x_{v_1}, so can be eliminated, so $G_\Gamma(R)$ is cyclic.

Case 1: If v_j is initial vertex of P_i then the terminal vertex of P_i is either some vertex v_k (case 1(a)) or it is not equal to v_k for any k (case (1(b)), and is a vertex of a directed cycle.

Case 1(a): There is a path as $v_1 \to v_2 \to \ldots \to v_j \to v_k \to v_{k+1} \to \ldots \to v_M$ (where $v_j \to v_k$ is the path P_i). That means all generator can be eliminated leaving only the generator x_{v_1} and thus $G_\Gamma(R)$ is finite cyclic.

Case 1(b): There is a path as $v_1 \to v_2 \to \ldots \to v_j$ followed by the path P_i ending with a different directed cycle. That means all generator x_u, where $u \in V(\Gamma_i)$, can be eliminated leaving only the generator x_{v_1} and thus $G_\Gamma(R)$ is finite cyclic.

Case 2: If v_j is terminal vertex of P_i then the initial vertex of P_i is either some vertex v_k (case 2(a)) or it is not equal to v_k for any k (case (2(b)), and is a vertex of a directed cycle.

Case 2(a): There is a path as $v_1 \to v_2 \to \ldots \to v_k \to v_j \to v_{j+1} \to \ldots \to v_M$ (where $v_k \to v_j$ is the path P_i). That means all generator can be eliminated leaving only the generator x_{v_1} and thus $G_\Gamma(R)$ is finite cyclic.

Case 2(b): All generators $x_{v_j}, x_{v_{j+1}}, \ldots, x_{v_M}$ of path P_1 can be eliminated leaving the only generator $x_{v_j}^{\alpha^{M-j+1}\gamma_1}$ by Lemma 1.3.6 (b).

All generators correspond to Γ_i $(2 \leqslant i \leqslant M)$ can be eliminated leaving only the generator $x_{v_j}^{\beta^{s_i}\gamma_i}$ by Lemma 1.3.6 (a), where s_i is the number of arcs between v_j and the directed cycle of the path P_i. Thus, we have the presentation

$$\begin{aligned} G_\Gamma(R) &= \left\langle \; x_{v_1}, x_{v_2}, \ldots, x_{v_j} \;\middle|\; \begin{array}{l} x_{v_j}^{\beta^{s_i}\gamma_i} \; (2 \leqslant i \leqslant M), x_{v_j}^{\alpha^{M-j+1}\gamma_1}, \\ R(x_{v_1}, x_{v_2}), R(x_{v_2}, x_{v_3}), \ldots, R(x_{v_{j-1}}, x_{v_j}) \end{array} \right\rangle \\ &= \left\langle \; x_{v_1}, x_{v_2}, \ldots, x_{v_j} \;\middle|\; \begin{array}{l} x_{v_j}^{(\gamma_1,\gamma_2,\ldots,\gamma_M)}, \\ R(x_{v_1}, x_{v_2}), R(x_{v_2}, x_{v_3}), \ldots, R(x_{v_{j-1}}, x_{v_j}) \end{array} \right\rangle. \end{aligned}$$

Let $\Lambda = (\gamma_1, \gamma_2, \ldots, \gamma_M)$ and we want to eliminate $x_{v_2}, \ldots, x_{v_j}$ from $G_\Gamma(R)$. So using the arc (v_{j-1}, v_j) and the relator $x_{v_j}^{\Lambda}$ we can eliminate x_{v_j} and add the relation $x_{v_{j-1}}^{\beta\Lambda} = 1$ by Lemma 1.3.4 (a). Thus,

$\langle\, x_{v_1}, x_{v_2}, \ldots, x_{v_{j-1}} \mid x_{v_{j-1}}^{\beta\Lambda}, R(x_{v_1}, x_{v_2}), R(x_{v_2}, x_{v_3}), \ldots, R(x_{v_{j-2}}, x_{v_{j-1}})\,\rangle$.

Continuing in this way we can use the arcs (v_{j-i}, v_{j-i+1}) to eliminate $x_{v_2}, \ldots, x_{v_{j-2}}$ leaving only the generator x_{v_1} and relator $x_{v_1}^{\beta^r\Lambda}$, where r is an integer.

Hence, $G_\Gamma(R)$ is finite cyclic.

□

Corollary 4.4.4. *For Theorem 4.4.3, if each Γ_i is replaced with a strongly connected or semi-connected digraph that satisfies $(i) - (iii)$ in Preamble 4.1 then $G_\Gamma(R)$ is finite cyclic.*

Proof. The digraphs Γ_i which are described in Theorem 4.4.3 can be replaced by any strongly connected or semi-connected digraph that satisfies $(i) - (iii)$ in Preamble 4.1 as we proved in Corollary 4.2.1, 4.3.2 and 4.3.4, respectively. Thus, $G_\Gamma(R)$ is finite cyclic.

□

CHAPTER 5

Generalization of Johnson's and Mennicke's group

5.1 Preamble

In 1959, Mennicke [27] provided an example of a group defined by the presentation

$$M(a,b,c) = \langle\, x, y, z \mid y^{-1}xy = x^a, z^{-1}yz = y^b, x^{-1}zx = z^c \,\rangle$$

which is finite in the case $a = b = c \geqslant 3$. These groups have also been investigated by I.D.Macdonald and by J.W.Wamsley and they showed that $M(a,b,c)$, which is not necessarily digraph groups except for $a = b = c$, is finite whenever $|a|, |b|, |c| \geqslant 3$ in [33]. The proof of this can be found in [22] and we also stated the detailed proof in Theorem 5.3.2.

In 1997, Johnson [21] provided another group needing exactly 3 generators which is presented by

$$J(a,b,c) = \langle\, x, y, z \mid x^y = y^{b-2}x^{-1}y^{b+2}, y^z = z^{c-2}y^{-1}z^{c+2}, z^x = x^{a-2}z^{-1}x^{a+2} \,\rangle$$

and finite in the case a, b, c are non-zero even integers.

These are important since they are able to construct examples of finite groups needing exactly 3 generators. The groups $M(a,b,c)$, $J(a,b,c)$ can be expressed as groups M_Γ, J_Γ that we will define in Definitions 5.2.1,5.3.1, where Γ is a directed triangle. We will generalize Mennicke [27] and Johnson [21] theorems from a directed triangle to all strong tournaments in this chapter.

5.2 Generalization of Johnson's group

5.2.1 Strategy

Before giving the proof of the main Theorem, we sketch the strategy here. Firstly, we will state Johnson's Theorem and we reproduce its proof in details in Theorem 5.2.2 since it forms a crucial ingredient to our methods. The underlying digraph is a directed triangle. It will be important to know that all vertices in a strong tournament belong to a directed triangle which is stated in Lemma 1.2.20. Thus, we will combine this Theorem 5.2.2 and Lemma 1.2.20 to prove the main Theorem in the Theorem 5.2.4. We begin with the definition of the Johnson group to generalise from 3 generators to n generators.

Definition 5.2.1. Let Γ be a digraph with arcs (u,v) labelled by even integers $q_{(u,v)} \geqslant 2$. We define the *generalized Johnson group* to be the group

$$J_\Gamma = \langle\, x_v \ (v \in V(\Gamma)) \mid x_u^{x_v} = x_v^{q_{(u,v)}-2} x_u^{-1} x_v^{q_{(u,v)}+2} \quad (u,v) \in A(\Gamma)\,\rangle$$

where x^y denotes $y^{-1}xy$.

Theorem 5.2.2 (([20, page 70]))**.** *Let Γ be the digraph with vertex set $V(\Gamma) = \{1,2,3\}$ and arc set $A(\Gamma) = \{(1,2),(2,3),(3,1)\}$. Then $J_\Gamma = \langle\, x_1, x_2, x_3 \mid x_1^{x_2} = x_2^{q(1,2)-2} x_1^{-1} x_2^{q(1,2)+2}, x_2^{x_3} = x_3^{q(2,3)-2} x_2^{-1} x_3^{q(2,3)+2}, x_3^{x_1} = x_1^{q(3,1)-2} x_3^{-1} x_1^{q(3,1)+2}\,\rangle$ is finite.*

Proof. ([20, page 71, 72]) The first step is to show that

$$x_1 \text{ commutes with } x_2^2,\ x_2 \text{ commutes with } x_3^2 \text{ and } x_3 \text{ commutes with } x_1^2. \tag{5.2.1}$$

To see this,

$$\begin{aligned} x_1^{x_2^2} = x_2^{-2}x_1x_2^2 = x_2^{-1}x_2^{-1}x_1x_2x_2 &= x_2^{-1}x_1^{x_2}x_2 \\ &= x_2^{-1}x_2^{q_{(1,2)}-2}x_1^{-1}x_2^{q_{(1,2)}+2}x_2 \\ &\text{by the first relator} \\ &= x_2^{q_{(1,2)}-2}x_2^{-1}x_1^{-1}x_2x_2^{q_{(1,2)}+2} \\ &= x_2^{q_{(1,2)}-2}(x_2^{-1}x_1x_2)^{-1}x_2^{q_{(1,2)}+2} \\ &= x_2^{q_{(1,2)}-2}(x_1^{x_2})^{-1}x_2^{q_{(1,2)}+2} \\ &= x_2^{q_{(1,2)}-2}(x_2^{q_{(1,2)}-2}x_1^{-1}x_2^{q_{(1,2)}+2})^{-1}x_2^{q_{(1,2)}+2} \\ &= x_2^{q_{(1,2)}-2}x_2^{-q_{(1,2)}-2}x_1x_2^{-q_{(1,2)}+2}x_2^{q_{(1,2)}+2} \\ &= x_2^{-4}x_1x_2^4 \end{aligned}$$

Thus, $x_2^{-2}x_1x_2^2 = x_2^{-4}x_1x_2^4$ so $x_1 = x_2^{-2}x_1x_2^2$ so $x_2^2x_1 = x_1x_2^2$. Thus, x_1 commutes with x_2^2. From the other two relations in the definition by using same method, we can get that x_2 commutes with x_3^2 and x_3 commutes with x_1^2. Hence, the subgroup $H = \langle x_1^2, x_2^2, x_3^2\rangle$ of J is abelian.

Furthermore, $x_1^{x_2} = x_2^{q_{(1,2)}-2}x_1^{-1}x_2^{q_{(1,2)}+2}$ by the first relator. We already showed that x_1 commutes with x_2^2 so x_1 commutes with $x_2^{q_{(1,2)}-2}$ since $q_{(1,2)}$ is even integer. As a consequence of this $x_1^{x_2} = x_1^{-1}x_2^{2q_{(1,2)}}$ and from the other original relations, $x_2^{x_3} = x_2^{-1}x_3^{2q_{(2,3)}}$ and $x_3^{x_1} = x_3^{-1}x_1^{2q_{(3,1)}}$. That is

$$x_1^{x_2} = x_1^{-1}x_2^{2q_{(1,2)}}, x_2^{x_3} = x_2^{-1}x_3^{2q_{(2,3)}} \text{ and } x_3^{x_1} = x_3^{-1}x_1^{2q_{(3,1)}} \tag{5.2.2}$$

Now we show that $(x_1^2)^w \in H, (x_2^2)^w \in H, (x_3^2)^w \in H$ for any $w \in J$. To see this observe the following

$(x_1^2)^{x_1} = x_1^{-1}x_1^2x_1 = x_1^2$
$(x_1^2)^{x_2} = (x_1^{x_2})^2 = (x_1^{-1}x_2^{2q_{(1,2)}})^2 = x_1^{-1}x_2^{2q_{(1,2)}}x_1^{-1}x_2^{2q_{(1,2)}} = x_1^{-2}x_2^{4q_{(1,2)}}$
$(x_1^2)^{x_3} = x_3^{-1}x_1^2x_3 = x_1^2$

$(x_2^2)^{x_1} = x_1^{-1}x_2^2x_1 = x_2^2$
$(x_2^2)^{x_2} = x_2^{-1}x_2^2x_2 = x_2^2$
$(x_2^2)^{x_3} = (x_2^{x_3})^2 = (x_2^{-1}x_3^{2q_{(2,3)}})^2 = x_2^{-1}x_3^{2q_{(2,3)}}x_2^{-1}x_3^{2q_{(2,3)}} = x_2^{-2}x_3^{4q_{(2,3)}}$

$(x_3^2)^{x_1} = (x_3^{x_1})^2 = (x_3^{-1}x_1^{2q_{(3,1)}})^2 = x_3^{-1}x_1^{2q_{(3,1)}}x_3^{-1}x_1^{2q_{(3,1)}} = x_3^{-2}x_1^{4q_{(3,1)}}$
$(x_3^2)^{x_2} = x_2^{-1}x_3^2x_2 = x_3^2$
$(x_3^2)^{x_3} = x_3^{-1}x_3^2x_3 = x_3^2$

So for any $w \in J_\Gamma$, we have $(x_1^2)^w \in H, (x_2^2)^w \in H, (x_3^2)^w \in H$. Therefore, H is normal in J.

The quotient J/H is given by adjoining the relations $x_1^2 = x_2^2 = x_3^2 = 1$ to those defining J [21, Proposition 4.2, page 93]. If $x_2^2 = 1$, then $(x_2^2)^{q_{(1,2)}/2} = 1$ so $x_2^{q_{(1,2)}} = 1$. Thus $x_2^{q_{(1,2)}+2} = 1$ and $x_2^{q_{(1,2)}-2} = 1$. By the first relator, $x_2^{-1}x_1x_2 = x_2^{q_{(1,2)}-2}x_1^{-1}x_2^{q_{(1,2)}+2}$ so $x_2^{-1}x_1x_2 = x_1^{-1}$. Thus, $x_1x_2^{-1}x_1x_2 = 1$ and $x_1x_2x_1x_2 = 1$ since $x_2^2 = 1$. Hence, $(x_1x_2)^2 = 1$. By using same technique, we get $x_3^2 = 1$ with the second relator, and $x_1^2 = 1$ with the third relator. Thus, we get $(x_2x_3)^2 = 1$ and $(x_3x_1)^2 = 1$ respectively.

Now, we can write

$J/H = \langle\, x_1, x_2, x_3 \mid x_1^2 = x_2^2 = x_3^2 = (x_1x_2)^2 = (x_2x_3)^2 = (x_3x_1)^2 = 1\,\rangle.$

Thus, $J/H \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $|J : H| = 8$.

We now show that H is a finite abelian group. We have

$$
\begin{aligned}
[[x_1, x_2], x_3^{x_1}] &= [x_1^{-1}x_2^{-1}x_1x_2, x_3^{-1}x_1^{2q_{(3,1)}}] \quad \text{by} \quad (5.2.3) \\
&= [x_1^{-1}x_1^{x_2}, x_3^{-1}x_1^{2q_{(3,1)}}] \\
&= [x_1^{-1}x_1^{-1}x_2^{2q_{(1,2)}}, x_3^{-1}x_1^{2q_{(3,1)}}] \quad \text{by} \quad (5.2.3) \\
&= [x_1^{-2}x_2^{2q_{(1,2)}}, x_3^{-1}x_1^{2q_{(3,1)}}] \\
&= x_2^{-2q_{(1,2)}}x_1^2x_1^{-2q_{(3,1)}}x_3x_1^{-2}x_2^{2q_{(1,2)}}x_3^{-1}x_1^{2q_{(3,1)}} \\
&= x_2^{-2q_{(1,2)}}x_3x_1^2x_1^{-2q_{(3,1)}}x_1^{-2}x_1^{2q_{(3,1)}}x_2^{2q_{(1,2)}}x_3^{-1} \text{ by } (5.2.1) \\
&= x_2^{-2q_{(1,2)}}x_3x_2^{2q_{(1,2)}}x_3^{-1} \\
&= x_2^{-2q_{(1,2)}}x_3^{-1}x_3^2x_2^{2q_{(1,2)}}x_3^{-2}x_3 \\
&= x_2^{-2q_{(1,2)}}x_3^{-1}x_2^{2q_{(1,2)}}x_3^2x_3^{-2}x_3 \quad \text{by } (5.2.1) \\
&= x_2^{-2q_{(1,2)}}x_3^{-1}x_2^{2q_{(1,2)}}x_3 \\
&= x_2^{-2q_{(1,2)}}(x_2^{x_3})^{2q_{(1,2)}} \\
&= x_2^{-2q_{(1,2)}}(x_2^{-1}x_3^{2q_{(2,3)}})^{2q_{(1,2)}} \quad \text{by} \quad (5.2.3) \\
&= x_2^{-4q_{(1,2)}}x_3^{4q_{(2,3)}q_{(1,2)}}.
\end{aligned}
$$

By using same method, we can get

$$[[x_2,x_3],x_1^{x_2}] = x_3^{-4q_{(2,3)}} x_1^{4q_{(3,1)}q_{(2,3)}} \text{ and } [[x_3,x_1],x_2^{x_3}] = x_1^{-4q_{(3,1)}} x_2^{4q_{(1,2)}q_{(3,1)}}.$$

Substituting this into Witt identity (see [21, Exercise 4.13, page 56]) and since x_1^2, x_2^2, x_3^2 all commute,

$$\begin{aligned} e &= [[x_1,x_2],x_3^{x_1}] \,.\, [[x_2,x_3],x_1^{x_2}] \,.\, [[x_3,x_1],x_2^{x_3}] \\ &= x_2^{-4q_{(1,2)}} x_3^{4q_{(2,3)}q_{(1,2)}} \,.\, x_3^{-4q_{(2,3)}} x_1^{4q_{(3,1)}q_{(2,3)}} \,.\, x_1^{-4q_{(3,1)}} x_2^{4q_{(1,2)}q_{(3,1)}} \\ &= x_1^{4q_{(3,1)}(q_{(2,3)}-1)} \,.\, x_2^{4q_{(1,2)}(q_{(3,1)}-1)} \,.\, x_3^{4q_{(2,3)}(q_{(1,2)}-1)} \end{aligned}$$

To see a power x_1 can be written in terms of a power x_2,
Since x_2^2, x_3^2 commute with x_2 so does $x_1^{4q_{(3,1)}(q_{(2,3)}-1)}$, and thus by (5.2.3)

$$\begin{aligned} x_1^{4q_{(3,1)}(q_{(2,3)}-1)} &= (x_1^{4q_{(3,1)}(q_{(2,3)}-1)})^{x_2} \\ &= (x_1^{x_2})^{4q_{(3,1)}(q_{(2,3)}-1)} \\ &= (x_1^{-1}x_2^{2q_{(1,2)}})^{4q_{(3,1)}(q_{(2,3)}-1)} \\ &= x_1^{-4q_{(3,1)}(q_{(2,3)}-1)} x_2^{8q_{(1,2)}q_{(3,1)}(q_{(2,3)}-1)} \end{aligned}$$

Thus, $x_1^{8q_{(3,1)}(q_{(2,3)}-1)} = x_2^{8q_{(1,2)}q_{(3,1)}(q_{(2,3)}-1)}$ and similarly can be found $x_2^{8q_{(1,2)}(q_{(3,1)}-1)} = x_3^{8q_{(2,3)}q_{(1,2)}(q_{(3,1)}-1)}$ and $x_3^{8q_{(2,3)}(q_{(1,2)}-1)} = x_1^{8q_{(2,3)}q_{(3,1)}(q_{(1,2)}-1)}$.

Using these relations, we get that

$$\begin{aligned} x_1^{8q_{(3,1)}(q_{(1,2)}-1)(q_{(2,3)}-1)(q_{(3,1)}-1)} &= x_2^{8q_{(3,1)}q_{(1,2)}(q_{(1,2)}-1)(q_{(2,3)}-1)(q_{(3,1)}-1)} \\ &= x_3^{8q_{(3,1)}q_{(1,2)}q_{(2,3)}(q_{(1,2)}-1)(q_{(2,3)}-1)(q_{(3,1)}-1)} \\ &= x_1^{8q_{(3,1)}^2q_{(1,2)}q_{(2,3)}(q_{(1,2)}-1)(q_{(2,3)}-1)(q_{(3,1)}-1)}. \end{aligned}$$

It is therefore x_1 has order dividing $|8q_{(3,1)}(q_{(1,2)}-1)(q_{(2,3)}-1)(q_{(3,1)}-1)(q_{(1,2)}q_{(2,3)}q_{(3,1)}-1)|$ which is non-zero. Thus x_1^2 has finite order similarly so do x_2^2 and x_3^2 . Following that $H = \langle x_1^2, x_2^2, x_3^2\rangle$ is a finite abelian group. Since $J/H \cong \mathbb{Z}_2^3$, the group J is finite. $\square$

5.2.2 Proving the main theorem

Lemma 5.2.3. *Let Γ be a simple digraph where each arc $(u,v) \in A(\Gamma)$ is labelled by an even integer $q_{(u,v)}$. If J_Γ is finite then Γ is a non-trivial tournament.*

Proof. Suppose that Γ is not a tournament. Then Γ is non-trivial so it has at least two vertices and there is a pair of vertices $w_1, w_2 \in V(\Gamma)$ that are not joined by an arc. Adjoining relators x_u to the defining presentation of J_Γ for all $u \neq w_1, w_2$ and adjoining the relators $x_{w_1}^2, x_{w_2}^2$ shows that J_Γ has the infinite quotient $\langle\, x_{w_1}, x_{w_2} \mid x_{w_1}^2, x_{w_2}^2 \,\rangle \cong \mathbb{Z}_2 * \mathbb{Z}_2$. □

Theorem 5.2.4. *Let Γ be a non-trivial strong tournament, then J_Γ is finite.*

Proof. Firstly, we will show the subgroup $H = \langle x_v^2 \;\; v \in V(\Gamma) \;\rangle$ of J is finite abelian. Secondly, H is normal subgroup of J ($H \triangleleft J$). Lastly, $J/H = \langle\, x_v \;\, v \in V(\Gamma) \mid x_v^2,\; x_u^{x_v} = x_v^{q_{(u,v)}-2} x_u^{-1} x_v^{q_{(u,v)}+2} \;\;\; v \in V(\Gamma) \;\;\; (u,v) \in A(\Gamma) \,\rangle$ is isomorphic to $\mathbb{Z}_2^n$.

We first show that if we have an arc (u,v), then x_u commutes with x_v^2. To see this

$$\begin{aligned}
x_u^{x_v^2} = x_v^{-2} x_u x_v^2 = x_v^{-1} x_v^{-1} x_u x_v x_v &= x_v^{-1} x_u^{x_v} x_v \\
&= x_v^{-1} x_v^{q_{(u,v)}-2} x_u^{-1} x_v^{q_{(u,v)}+2} x_v \\
&= x_v^{q_{(u,v)}-2} x_v^{-1} x_u^{-1} x_v x_v^{q_{(u,v)}+2} \\
&= x_v^{q_{(u,v)}-2} (x_v^{-1} x_u x_v)^{-1} x_v^{q_{(u,v)}+2} \\
&= x_v^{q_{(u,v)}-2} (x_u^{x_v})^{-1} x_v^{q_{(u,v)}+2} \\
&= x_v^{q_{(u,v)}-2} (x_v^{q_{(u,v)}-2} x_u^{-1} x_v^{q_{(u,v)}+2})^{-1} x_v^{q_{(u,v)}+2} \\
&= x_v^{q_{(u,v)}-2} x_v^{-q_{(u,v)}-2} x_u x_v^{-q_{(u,v)}+2} x_v^{q_{(u,v)}+2} \\
&= x_v^{-4} x_u x_v^4 \\
&= x_u^{x_v^4}
\end{aligned}$$

Thus, $x_v^{-2} x_u x_v^2 = x_v^{-4} x_u x_v^4$ so $x_u = x_v^{-2} x_u x_v^2$ so $x_v^2 x_u = x_u x_v^2$. Thus,

$$x_u \;\text{ commutes with } x_v^2 \;\text{ whenever }\; (u,v) \in A(\Gamma). \tag{5.2.3}$$

If Γ is a tournament, then there is exactly one of (u, v) or (v, u). If $(u, v) \in A(\Gamma)$ then x_u commutes with x_v^2 or if $(v, u) \in A(\Gamma)$ then x_v commutes with x_u^2, in both cases x_u^2 commutes with x_v^2. Therefore, $H = \langle x_v^2 \ \ v \in V(\Gamma) \rangle$ of J is abelian. Furthermore, if $(u, v) \in A(\Gamma)$, then $x_u^{x_v} = x_v^{q_{(x_u,x_v)}-2} x_u^{-1} x_v^{q_{(x_u,x_v)}+2}$ by definition of J_Γ. We already showed that x_u commutes with x_v^2 so x_u commutes with $x_v^{q_{(x_u,x_v)}-2}$ since $q_{(x_u,x_v)}$ is even integer. As a consequence of this,

$$x_u^{x_v} = x_u^{-1} x_v^{2q_{(x_u,x_v)}} (u, v) \in A(\Gamma). \tag{5.2.4}$$

We now show that H is a normal subgroup of J_Γ.

Let x_v^2 be a generator of H and let x_u be a generator of J (where $u, v \in V(\Gamma)$). Given an arc $(u, v) \in A(\Gamma)$, then we showed that x_u commutes with x_v^2 above. Thus, $(x_v^2)^{x_u} = x_u^{-1} x_v^2 x_u = x_u^{-1} x_u x_v^2 = x_v^2 \in H$. If we have (v, u), then $(x_v^2)^{x_u} = (x_v^{x_u})^2 = (x_u^{-1} x_v^{2q_{(u,v)}})^2 (\textit{ by } (5.2.4)) = x_u^{-1} x_v^{2q_{(u,v)}} x_u^{-1} x_v^{2q_{(u,v)}} = x_u^{-2} x_v^{4q_{(u,v)}} \in H$. Thus $H \triangleleft J$.

Now, we will show that

$$J/H = \langle\, x_v(v \in V(\Gamma)) \mid x_v^2, x_u^{x_v} = x_v^{q_{(u,v)}-2} x_u^{-1} x_v^{q_{(u,v)}+2} (v \in V(\Gamma)(u, v) \in A(\Gamma)) \,\rangle$$

is isomorphic to $\mathbb{Z}_2^{|V(\Gamma)|}$. To see, this observe

$$\begin{aligned} J/H &= \langle\, x_v(v \in V(\Gamma)) \mid x_v^2, x_u^{x_v} = x_v^{q_{(u,v)}-2} x_u^{-1} x_v^{q_{(u,v)}+2} \ (u, v) \in A(\Gamma) \,\rangle \\ &= \langle\, x_v(v \in V(\Gamma)) \mid x_v^2, x_u x_v = x_v x_u \ (u, v) \in A(\Gamma) \,\rangle \text{ since } q_{(u,v)} \text{ is even} \\ &= \langle\, x_v(v \in V(\Gamma)) \mid x_v^2 \,\rangle^{ab} \\ &= \mathbb{Z}_2^n = \mathbb{Z}_2^{|V(\Gamma)|}. \end{aligned}$$

Thus, $J/H \cong \mathbb{Z}_2^n$ and $|J : H| = 2^n$.

As shown above H is abelian and every vertex $v \in V(\Gamma)$ is in some directed triangle with vertices x_1, x_2, x_3 and arcs $(x_1, x_2), (x_2, x_3), (x_3, x_1)$ by Lemma 1.2.20. As in the proof Theorem 5.2.2, we obtain the relations $x_v^A = x_u^B$ so x_v has order dividing something which is a non-zero. Thus, x_v^2 has finite order for each $v \in V(\Gamma)$ so H is finite abelian group. Since J/H is finite and H is finite abelian, so the group J is finite. $\square$

5.3 Generalization of Mennicke's group

5.3.1 Strategy

Before giving the proof of the main Theorem, we sketch the strategy here. Firstly, we will state Mennicke's Theorem and we reproduce its proof in detail in Theorem 5.3.2. As in the proof of Theorem 5.2.2, we will use Lemma 1.2.20. Thus, we will combine the Theorem 5.3.2 and Lemma 1.2.20 to prove the main Theorem in the Theorem 5.3.4. We begin with the definition of the Mennicke group to generalise from 3 generators to n generators.

Definition 5.3.1. Let Γ be a digraph with arcs (u, v) labelled by integers $q_{(u,v)} \geqslant 2$. We define the *generalized Mennicke group* to be the group

$$M_\Gamma = \langle\, x_v (v \in V(\Gamma)) \mid x_u^{x_v} = x_u^{q_{(u,v)}} (u, v) \in A(\Gamma) \,\rangle$$

where x^y denotes $y^{-1}xy$. If, for each $(u, v) \in A(\Gamma)$ we have $q_{(u,v)} = q$ for some fixed $q \geqslant 2$ then M_Γ is an example of a digraph group.

Theorem 5.3.2 ([22])**.** *Let Γ be the digraph with vertex set $V(\Gamma) = \{1, 2, 3\}$ and arc set $A(\Gamma) = \{(1, 2), (2, 3), (3, 1)\}$ Then*
$M_\Gamma = \langle\, x_1, x_2, x_3 \mid x_1^{x_2} = x_1^{q(1,2)}, x_2^{x_3} = x_2^{q(2,3)}, x_3^{x_1} = x_3^{q(3,1)} \,\rangle$ *is finite.*

Proof. [22] By the defining relation of the groups $x_2^{-1}x_1x_2 = x_1^{q(1,2)}$. So

$$\begin{aligned}
x_2^{-2}x_1x_2^2 = &= x_2^{-1}(x_2^{-1}x_1x_2)x_2 \\
&= x_2^{-1}x_1^{q(1,2)}x_2 \\
&= (x_2^{-1}x_1x_2)^{q(1,2)} \\
&= (x_1^{q(1,2)})^{q(1,2)} \\
&= x_1^{{q(1,2)}^2}
\end{aligned}$$

An inductive argument then shows that for all $u \geqslant 0$

$$x_2^{-u}x_1x_2^u = x_1^{q^u_{(1,2)}} \tag{5.3.1}$$

Raising the equation (5.3.1) with power $v \in \mathbb{Z}$ given

$$x_2^{-u} x_1^v x_2^u = x_1^{vq_{(1,2)}^u} \tag{5.3.2}$$

By same techniques, we get

$$x_3^{-u} x_2^v x_3^u = x_2^{vq_{(2,3)}^u} \tag{5.3.3}$$

$$x_1^{-u} x_3^v x_1^u = x_3^{vq_{(3,1)}^u} \tag{5.3.4}$$

Substituting this to Witt identity and it follows that

$$\begin{aligned}
1 &= [x_1, x_2, x_3^{x_1}] \quad [x_3, x_1, x_2^{x_3}] \quad [x_2, x_3, x_1^{x_2}] \\
&= [[x_1, x_2], x_3^{x_1}] \quad [[x_3, x_1], x_2^{x_3}] \quad [[x_2, x_3], x_1^{x_2}] \\
&= [x_1, x_2]^{-1} (x_3^{x_1})^{-1} [x_1, x_2] x_3^{x_1} \quad [x_3, x_1]^{-1} (x_2^{x_3})^{-1} [x_3, x_1] x_2^{x_3} \quad [x_2, x_3]^{-1}, \\
&(x_1^{x_2})^{-1} [x_2, x_3] x_1^{x_2} \\
&= (x_1^{-1} x_2^{-1} x_1 x_2)^{-1} (x_1^{-1} x_3 x_1)^{-1} x_1^{-1} x_2^{-1} x_1 x_2 x_1^{-1} x_3 x_1 \\
&(x_3^{-1} x_1^{-1} x_3 x_1)^{-1} (x_3^{-1} x_2 x_3)^{-1} x_3^{-1} x_1^{-1} x_3 x_1 x_3^{-1} x_2 x_3 \\
&(x_2^{-1} x_3^{-1} x_2 x_3)^{-1} (x_2^{-1} x_1 x_2)^{-1} x_2^{-1} x_3^{-1} x_2 x_3 x_2^{-1} x_1 x_2 \\
&= x_2^{-1} x_1^{-1} x_2 \cancel{x_1 x_1^{-1}} x_3^{-1} \cancel{x_1 x_1^{-1}} x_2^{-1} x_1 x_2 x_1^{-1} x_3 x_1 \\
&x_1^{-1} x_3^{-1} x_1 \cancel{x_3 x_3^{-1}} x_2^{-1} \cancel{x_3 x_3^{-1}} x_1^{-1} x_3 x_1 x_3^{-1} x_2 x_3 \\
&x_3^{-1} x_2^{-1} x_3 \cancel{x_2 x_2^{-1}} x_1^{-1} \cancel{x_2 x_2^{-1}} x_3^{-1} x_2 x_3 x_2^{-1} x_1 x_2 \\
&= \underbrace{x_2^{-1} x_1^{-1} x_2}\, x_3^{-1} \,\underbrace{x_2^{-1} x_1 x_2}\, x_1^{-1} x_3 x_1 \\
&\underbrace{x_1^{-1} x_3^{-1} x_1}\, x_2^{-1} \,\underbrace{x_1^{-1} x_3 x_1}\, x_3^{-1} x_2 x_3 \\
&\underbrace{x_3^{-1} x_2^{-1} x_3}\, x_1^{-1} \,\underbrace{x_3^{-1} x_2 x_3}\, x_2^{-1} x_1 x_2 \\
&= x_1^{-q_{(1,2)}} x_3^{-1} x_1^{q_{(1,2)}} x_1^{-1} x_3 x_1 \quad \text{by (5.3.2)} \\
&x_3^{-q_{(3,1)}} x_2^{-1} x_3^{q_{(3,1)}} x_3^{-1} x_2 x_3 \quad \text{by (5.3.4)} \\
&x_2^{-q_{(2,3)}} x_1^{-1} x_2^{q_{(2,3)}} x_2^{-1} x_1 x_2 \quad \text{by (5.3.3)}
\end{aligned}$$

$$\begin{aligned}
&= x_1^{-q_{(1,2)}} x_3^{-1} \underbrace{x_1^{q_{(1,2)}-1} x_3 x_1^{1-q_{(1,2)}}} x_1^{q_{(1,2)}} \\
&x_3^{-q_{(3,1)}} x_2^{-1} \underbrace{x_3^{q_{(3,1)}-1} x_2 x_3^{1-q_{(3,1)}}} x_3^{q_{(3,1)}} \\
&x_2^{-q_{(2,3)}} x_1^{-1} \underbrace{x_2^{q_{(2,3)}-1} x_1 x_2^{1-q_{(2,3)}}} x_2^{q_{(2,3)}} \\
&= x_1^{-q_{(1,2)}} x_3^{-1} x_3^{(q_{(3,1)})^{(1-q_{(1,2)})}} x_1^{q_{(1,2)}} \quad \text{by (5.3.4)} \\
&x_3^{-q_{(3,1)}} x_2^{-1} x_2^{(q_{(2,3)})^{(1-q_{(3,1)})}} x_3^{q_{(3,1)}} \quad \text{by (5.3.3)} \\
&x_2^{-q_{(2,3)}} x_1^{-1} x_1^{(q_{(1,2)})^{(1-q_{(2,3)})}} x_2^{q_{(2,3)}} \quad \text{by (5.3.2)} \\
&= \underbrace{x_1^{-q_{(1,2)}} x_3^{(q_{(3,1)})^{(1-q_{(1,2)})}-1} x_1^{q_{(1,2)}}} \quad \underbrace{x_3^{-q_{(3,1)}} x_2^{(q_{(2,3)})^{(1-q_{(3,1)})}-1} x_3^{q_{(3,1)}}} \\
&\underbrace{x_2^{-q_{(2,3)}} x_1^{(q_{(1,2)})^{(1-q_{(2,3)})}-1} x_2^{q_{(2,3)}}} \\
&= x_3^{(q_{(3,1)}^{(1-q_{(1,2)})}-1)q_{(3,1)}^{q_{(1,2)}}} \; x_2^{(q_{(2,3)}^{(1-q_{(3,1)})}-1)q_{(2,3)}^{q_{(3,1)}}} \; x_1^{(q_{(1,2)}^{(1-q_{(2,3)})}-1)q_{(1,2)}^{q_{(2,3)}}} \\
&= x_3^{q_{(3,1)}-q_{(3,1)}^{q_{(1,2)}}} \; x_2^{q_{(2,3)}-q_{(2,3)}^{q_{(3,1)}}} \; x_1^{q_{(1,2)}-q_{(1,2)}^{q_{(2,3)}}} \\
&= x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \; x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} \; x_3^{q_{(3,1)}^{q_{(1,2)}}-q_{(3,1)}}
\end{aligned}$$

So that

$$x_3^{q_{(3,1)}-q_{(3,1)}^{q_{(1,2)}}} = x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \; x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} \tag{5.3.5}$$

Premultiplying by x_3^{-1} and postmultiplying x_3, we get

$$\begin{aligned}
x_3^{-1} \, x_3^{q_{(3,1)}-q_{(3,1)}^{q_{(1,2)}}} \, x_3 &= x_3^{-1} x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \; x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} x_3 \\
x_3^{q_{(3,1)}-q_{(3,1)}^{q_{(1,2)}}} &= x_3^{-1} x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \; x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} x_3
\end{aligned}$$

After applying the equation (5.3.5) for left hand side, we get

$$x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \; x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} = x_3^{-1} x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \; x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} x_3$$

$$\Rightarrow x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} = \underbrace{x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} x_1^{q_{(1,2)}-q_{(1,2)}^{q_{(2,3)}}}}_{1} x_3^{-1} x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \underbrace{x_3 x_3^{-1}}_{1} x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} x_3$$

$$\Rightarrow x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} = x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}} \underbrace{x_1^{q_{(1,2)}-q_{(1,2)}^{q_{(2,3)}}} x_3^{-1} x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}}} x_3 \underbrace{x_3^{-1} x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} x_3}$$

$$\Rightarrow \cancel{x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}}} x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} = \cancel{x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}}} (x_1^{q_{(1,2)}-q_{(1,2)}^{q_{(2,3)}}} x_3 x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}})^{-1} x_3 (x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}})^{x_3}$$

$$\Rightarrow x_2^{q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)}} = \underbrace{(x_1^{q_{(1,2)}-q_{(1,2)}^{q_{(2,3)}}} x_3 x_1^{q_{(1,2)}^{q_{(2,3)}}-q_{(1,2)}})^{-1} x_3}_{\text{a power of } x_3 \text{ by } (5.3.4)} \underbrace{x_2^{q_{(2,3)}(q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)})}}_{\text{by } (5.3.3)}$$

Hence, it follows that $x_2^{(q_{(2,3)}-1)(q_{(2,3)}^{q_{(3,1)}}-q_{(2,3)})}$ is a power of x_3, and thus, so is $x_2^{(q_{(2,3)}-1)(q_{(2,3)}^{q_{(3,1)}{}^{-1}}-1)}$ (conjugating by x_3^{-1}), whence $x_2^{(q_{(2,3)}-1)^2(q_{(2,3)}^{q_{(3,1)}{}^{-1}}-1)} = 1$ (conjugating by x_3). Similarly, x_1 and x_3 have finite order as $x_1^{(q_{(1,2)}-1)^2(q_{(1,2)}^{(q_{(2,3)}-1)}-1)} = 1$, and $x_3^{(q_{(3,1)}-1)^2(q_{(3,1)}^{(q_{(1,2)}-1)}-1)} = 1$. Thus, we say that if u is a vertex of a directed triangle (u,v,w) then $x_u^{\Phi(u,v,w)} = 1$ in M_Γ, where $\Phi(u,v,w) = (q_{(u,v)}-1)^2(q_{(u,v)}^{q_{(v,w)}{}^{-1}}-1)$.

We now set up $x_1^{A_1} = 1$, $x_2^{A_2} = 1$ and $x_3^{A_3} = 1$, where $A_1 = (q_{(1,2)} - 1)^2(q_{(1,2)}^{(q_{(2,3)}-1)}-1)$, $A_2 = (q_{(2,3)}-1)^2(q_{(2,3)}^{(q_{(3,1)}-1)}-1)$ and $A_3 = (q_{(3,1)}-1)^2(q_{(3,1)}^{(q_{(1,2)}-1)}-1)$ and we will use the following notation in this section. Given elements $a, b \in G$, if $ab = ba^t$ for some $t \in \mathbb{Z}$ we write $a \hookrightarrow b$ to denote that we can "pull a through b"; if $a \hookrightarrow b$ and $b \hookrightarrow a$ we write $a \sim b$.

Since $(q_{(1,2)}, q_{(1,2)} - 1) = 1$, $(q_{(1,2)}, A_1) = 1$ there exists α_1 such that

$q_{(1,2)}\alpha_1 = 1 \bmod A_1$. $x_1^{x_2} = x_1^{q_{(1,2)}}$ by the first relator. So $x_2^{-1}x_1x_2 = x_1^{q_{(1,2)}}$ then $(x_2^{-1}x_1x_2)^{\alpha_1} = x_1^{\alpha_1 q_{(1,2)}} = x_1$ since $x_1^{A_1} = 1$. Thus, $x_2^{-1}x_1^{\alpha_1}x_2 = x_1$ and so $x_1^{\alpha_1}x_2 = x_2x_1$ and do $x_2 \hookrightarrow x_1$. By the first relator, $x_2^{-1}x_1x_2 = x_1^{q_{(1,2)}}$ so $x_1x_2 = x_2x_1^{q_{(1,2)}}$ and so $x_1 \hookrightarrow x_2$. Thus, $x_1 \sim x_2$.

By using same technique, we get $x_2^{\alpha_2}x_3 = x_3x_2$ and $x_3^{\alpha_3}x_1 = x_1x_3$ from the second and third relators respectively and by these relators, we have $x_3^{-1}x_2x_3 = x_2^{q_{(2,3)}}$ so $x_2x_3 = x_3x_2^{q_{(2,3)}}$ and $x_1^{-1}x_3x_1 = x_3^{q_{(3,1)}}$ so $x_3x_1 = x_1x_3^{q_{(3,1)}}$. Hence we say that $x_2 \sim x_3$.

Therefore, if any $g \in M$ then $g = x_1^{p_1}x_2^{p_2}x_3^{p_3}$ for some $0 \leqslant p_1 \leqslant A_1$, $0 \leqslant p_2 \leqslant A_2$ and $0 \leqslant p_3 \leqslant A_3$. Thus,

$|M(q_{(1,2)}, q_{(2,3)}, q_{(3,1)})| \leqslant (q_{(1,2)} - 1)^2(q_{(1,2)}^{(q_{(2,3)}-1)} - 1)(q_{(2,3)} - 1)^2(q_{(2,3)}^{(q_{(3,1)}-1)} - 1)(q_{(3,1)} - 1)^2(q_{(3,1)}^{(q_{(1,2)}-1)} - 1) = A_1A_2A_3$.

□

5.3.2 Proving the main theorem

Lemma 5.3.3. *Let Γ be a simple digraph where each arc $(u, v) \in A(\Gamma)$ is labelled by an integer $q_{(u,v)} \geqslant 1$ and suppose $\gcd\{q_{(u,v)} - 1 \mid (u, v) \in A(\Gamma)\} > 1$. If M_Γ is finite then Γ is a tournament without sinks.*

Proof. Suppose that Γ is not a tournament and let $d = \gcd\{q_{(u,v)} - 1 \mid (u, v) \in A(\Gamma)\} > 1$. Then Γ is non-trivial so there is a pair of distinct vertices $w_1, w_2 \in V(\Gamma)$ that are not joined by an arc. Adjoining the relators $x_{w_1}^d, x_{w_2}^d$ and the relators x_u for all $u \neq w_1, w_2$ to the defining presentation of M_Γ shows that M_Γ has the infinite quotient $\langle\, x_{w_1}, x_{w_2} \mid x_{w_1}^d, x_{w_2}^d \,\rangle \cong \mathbb{Z}_d * \mathbb{Z}_d$, so M_Γ is infinite. Suppose then that Γ is a tournament with a sink, t, say. Adjoining relators x_u for all $u \in V(\Gamma)$ where $u \neq t$ shows that M_Γ maps onto $\langle\, x_t \mid \ \rangle \cong \mathbb{Z}$, so M_Γ is infinite. □

Theorem 5.3.4. *Let Γ be a non-trivial strong tournament, then M_Γ is finite.*

Proof. If $(u, v) \in A(\Gamma)$ then $x_u^{x_v} = x_u^{q_{(u,v)}}$ by the defining relation. Thus, $x_v^{-1}x_ux_v = x_u^{q_{(u,v)}}$. Hence, $x_ux_v = x_vx_u^{q_{(u,v)}}$ holds in M_Γ, and so $x_u \hookrightarrow x_v$.

Conversely, we now show that if $(u, v) \in A(\Gamma)$ and x_v has finite order in M_Γ then $x_v \hookrightarrow x_u$. Our argument is essentially that given in [29, page 1293].

Suppose x_v has order $P < \infty$ in M_Γ. Repeated applications of the relation $x_v^{-1}x_ux_v = x_u^{q_{(u,v)}}$ gives $x_v^{-P}x_ux_v^P = x_u^{q_{(u,v)}^P}$. Therefore $x_u^{q_{(u,v)}^P-1} = e$ (where e is the identity of M_Γ) and so x_u has finite order, Q, say, which divides $q_{(u,v)}^P - 1$, and so is co-prime to $q_{(u,v)}$. Thus there exists $\bar{q}_{(u,v)} \in \mathbb{Z}$ such that $q_{(u,v)}\bar{q}_{(u,v)} \equiv 1 \bmod Q$. Raising the defining relation of M_Γ that involves x_u, x_v to the power $\bar{q}_{(u,v)}$ gives $(x_v^{-1}x_ux_v)^{\bar{q}_{(u,v)}} = x_u^{q_{(u,v)}\bar{q}_{(u,v)}}$; that is, $x_v^{-1}x_u^{\bar{q}_{(u,v)}}x_v = x_u$ or $x_vx_u = x_u^{\bar{q}_{(u,v)}}x_v$ so $x_v \hookrightarrow x_u$.

We are now in a position to prove Theorem 5.3.4; our proof is a generalization of the argument in [22].

Since Γ is a non-trivial, strongly connected tournament, each vertex $u \in V(\Gamma)$ is in some directed triangle, so by Theorem 5.3.2 $x_u^{\Phi(u,v,w)} = 1$ in M_Γ for each directed triangle $[u, v, w]$ in Γ. Thus $x_u^{\phi(u)} = 1$ in M_Γ for all $u \in V$, so each generator has finite order. Therefore, if $(u, v) \in A(\Gamma)$ then $x_u \sim x_v$, and since Γ is a tournament $x_u \sim x_v$ for all $u, v \in V(\Gamma)$. Writing $V(\Gamma) = \{1, 2, \ldots, n\}$, each element of M_Γ can therefore be written in the form $x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}$ where $0 \leqslant \alpha_v < \phi(v)$ $(1 \leqslant v < n)$. Hence $|M_\Gamma| \leqslant \prod_{v\in V}\phi(v)$, as required.

□

CHAPTER 6

Digraph groups for some tournaments

6.1 Introduction

In this Chapter, we use the computational algebraic software *GAP* [14] to look for finite non-cyclic digraph groups where the graphs are tournaments. All known examples of finite non-cyclic digraph groups that currently appear to be in the literature are obtained by Mennicke with the word $R(a, b) = a^{-1}bab^{-q}$ for $q \geqslant 3$ [27] and Johnson with the word $R(a, b) = b^{-1}ab(b^{q-2}a^{-1}b^{q+2})^{-1}$ [21]. We generalized that Mennicke [27] and Johnson [21] groups from a directed triangle to all strong tournaments without giving the order in Chapter 5. Thus, in this chapter, we investigate digraph groups $G_\Gamma(R)$, where R is the Johnson or Mennicke relator and Γ is a tournament, in terms of the order, derived series and the structure of the group such as abelian, solvable, cyclic and perfect as far as the software *GAP* allows computationally. I will also define some new fixed relators $R(a, b)$ in addition to Mennicke and Johnson relators.

The number of possible non-isomorphic tournaments with up to 6 vertices was given in [28]. The table summarizing these results is also given in Table 6.1 on page 133. By this table, there are 4 non-isomorphic 4-vertex tournaments, 12 non-isomorphic 5−vertex tournaments and 56 non-isomorphic 6-vertex tournaments. We investigate all possible non-isomorphic tournaments up to a 6-vertex tournaments. However, it is computationally infeasible to investigate digraphs groups for all possible 7-vertex tournaments and 8-vertex tournaments since there are 456 and 6880 non-isomorphic tournaments respectively. However, we provide some results up to 12-vertex tournament to see the patterns based on the conjectures that we made. We are unable to provide an example of a finite digraph group for an n-vertex tournament when $n \geqslant 13$ since these cases are beyond the reach of the available computational power.

Firstly, we investigate the group of Mennicke (corresponding to the word $R(a,b) = a^{-1}bab^{-3}$) for all tournaments on $3 \leqslant n \leqslant 6$ vertices and some on $7 \leqslant n \leqslant 12$ vertices. For other values of q for $4 \leqslant q \leqslant 10$, we are able to provide two theorems for G/G' in theorem 6.4.1 and 6.4.2 and a table for G'/G'' in Table 6.9 on page 148. Note that we are unable to find out the order when $q \geqslant 4$ because of computational limitations. Secondly, we investigate the group of Johnson (when $q = 2$ which is corresponding to the word $R(a,b) = b^{-1}ab^{-3}a$) for all tournaments on $3 \leqslant n \leqslant 6$ vertices and some on $7 \leqslant n \leqslant 12$ vertices. For other values of q, we can obtain results similar to those for the Mennicke group. Lastly, we had many experiments using *GAP* and these experiments indicate that when $R(a,b) = abab^3$, $R(a,b) = abab^{-2}$ and $R(a,b) = ab^2a^2b^{-2}$, we have some finite non-cyclic digraph groups as new examples. The possible tournaments with these words are also investigated. Interestingly, the word $R(a,b) = ab^2a^2b^{-2}$ gives us 3-groups and the word $R(a,b) = abab^{-2}$ gives us a perfect group for certain tournaments Γ. We also state some conjectures throughout the chapter based on our experimental results.

6.2 Preliminary observations

There are some restrictions that are proved theoretically. In this section, I will give these proofs. We also provide some important tables and restrictions here.

Lemma 6.2.1. *Let Γ be a digraph and $R(a,b)$ is of the form $a^{\alpha_1}b^{\beta_1}\cdots a^{\alpha_t}b^{\beta_t}$ with $t \geqslant 1$ and $\alpha_i, \beta_i \in \mathbb{Z}\setminus\{0\}$ $(1 \leqslant i \leqslant t)$, where $\alpha = \sum_1^t \alpha_i$ and $\beta = -\sum_1^t \beta_i$.*

If $\alpha = 0$ and Γ contains a source or $\beta = 0$ and Γ contains a sink, then $G_\Gamma(R)$ is infinite.

Proof. Let $\beta = 0$ and suppose that Γ contains a sink. If v is a sink, then all relations involving x_v are in the form $R(x_u, x_v) = x_u^{\alpha_1}x_v^{\beta_1}x_u^{\alpha_2}x_v^{\beta_2}\ldots x_u^{\alpha_t}x_v^{\beta_t}$.

Consider a map $\phi : G_\Gamma(R) \to \langle y \mid \ \rangle \cong \mathbb{Z}$ given by $\phi(x_v) = y$ and $\phi(x_u) = y^0 = 1$ if $u \neq v$. Then $\phi(R(x_u,x_v)) = e^{\alpha_1}y^{\beta_1}e^{\alpha_2}y^{\beta_2}\ldots e^{\alpha_t}y^{\beta_t} = y^{\sum_1^t \beta_i} = y^{-\beta} = y^0 = 1$. Therefore ϕ is a homomorphism. It is also an epimorphism since $\phi(x_v^n) = y^n$ for all $n \in \mathbb{Z}$. Thus, the corresponding group is infinite.

The proof is similar when $\alpha = 0$ and Γ contains a source. □

Lemma 6.2.2. *Let Γ be a digraph and $R(a,b)$ is of the form $a^{\alpha_1}b^{\beta_1}\cdots a^{\alpha_t}b^{\beta_t}$ with $t \geqslant 1$ and $\alpha_i, \beta_i \in \mathbb{Z}\setminus\{0\}$ $(1 \leqslant i \leqslant t)$, where $\alpha = \sum_1^t \alpha_i$ and $\beta = -\sum_1^t \beta_i$.*

If $\alpha - \beta = 0$, then $G_\Gamma(R)$ is infinite.

Proof. Consider a map $G_\Gamma(R) \xrightarrow{\phi} \langle y \mid \ \rangle \cong \mathbb{Z}$ given by $\phi(x_u) = y$ for all $u \in V(\Gamma)$. Then $\phi(R(x_u,x_v)) = y^{\alpha_1}y^{\beta_1}y^{\alpha_2}y^{\beta_2}\ldots y^{\alpha_t}y^{\beta_t} = y^{\sum_1^t \alpha_i + \sum_1^t \beta_i} = y^{\alpha-\beta} = y^0 = 1$. Therefore, ϕ is homomorphism. It is also an epimorphism since $\phi(x_u^n) = y^n$ for all $n \in Z$ and any $u \in V(\Gamma)$. Thus, $G_\Gamma(R)$ is infinite.

□

Lemma 6.2.3. *Let Γ be a digraph, and $R(a,b) = a^{m_1}b^{n_1}a^{m_2}b^{n_2}$, where $m_1, m_2, n_1, n_2 \in \mathbb{Z}$ be a relator. Then*

(i) $G_\Gamma(R(a,b)) \cong G_\Gamma(R(a^{-1},b))$ *if* $n_1 = -n_2$

(ii) $G_\Gamma(R(a,b)) \cong G_\Gamma(R(a,b^{-1}))$ *if* $m_1 = -m_2$

(iii) $G_\Gamma(R(a,b)) \cong G_\Gamma(R(a^{-1},b^{-1}))$

Proof.

(i)

$$\begin{aligned} G_\Gamma(R(a,b)) &= G_\Gamma(a^{m_1}b^{n_1}a^{m_2}b^{n_2}) \\ &\cong G_\Gamma(a^{m_2}b^{n_2}a^{m_1}b^{n_1}) \text{ by cyclically permuting relators} \\ &\cong G_\Gamma(b^{-n_1}a^{-m_1}b^{-n_2}a^{-m_2}) \text{ by inverting permuting relators} \\ &\cong G_\Gamma(a^{-m_1}b^{-n_2}a^{-m_2}b^{-n_1}) \text{ by cyclically permuting relators} \\ &\cong G_\Gamma(a^{-m_1}b^{n_1}a^{-m_2}b^{n_2}) \text{ since } n_1 = -n_2 \\ &= G_\Gamma(R(a^{-1},b)) \end{aligned}$$

(ii)

$$\begin{aligned} G_\Gamma(R(a,b)) &= G_\Gamma(a^{m_1}b^{n_1}a^{m_2}b^{n_2}) \\ &\cong G_\Gamma(b^{n_1}a^{m_2}b^{n_2}a^{m_1}) \text{ by cyclically permuting relators} \\ &\cong G_\Gamma(a^{-m_1}b^{-n_2}a^{-m_2}b^{-n_1}) \text{ by inverting permuting relators} \\ &\cong G_\Gamma(a^{-m_2}b^{-n_1}a^{-m_1}b^{-n_2}) \text{ by cyclically permuting relators} \\ &\cong G_\Gamma(a^{m_1}b^{-n_1}a^{m_2}b^{-n_2}) \text{ since } m_1 = -m_2 \\ &= G_\Gamma(R(a,b^{-1})) \end{aligned}$$

(iii)

$$\begin{aligned} G_\Gamma(R(a,b)) &=< x_v(v \in V) \mid R(x_u,x_v)((u,v) \in A) > \\ &=< y_v, x_v(v \in V) \mid R(x_u,x_v)((u,v) \in A), y_v = x_v^{-1}(v \in V) > \\ &=< y_v, x_v(v \in V) \mid R(x_u,x_v)((u,v) \in A), x_v = y_v^{-1}(v \in V) > \\ &=< y_v(v \in V) \mid R(y_u^{-1},y_v^{-1})((u,v) \in A) > \\ &= G_\Gamma(R(a^{-1},b^{-1})) \end{aligned}$$

□

In principle we had many experiments to find out the possible words $R(a,b)$ in addition to Mennicke's and Johnson's word using GAP. Lemma 6.2.3 allows us to reduce the number of relators considered. For example, if we perform the experiments when $R(a,b) = a^2b^{-3}$, then there is no need to perform the same experiments with $R(a,b) = a^{-2}b^{-3}$ by (i), $R(a,b) = a^2b^3$ by (ii) and $R(a,b) = a^{-2}b^3$ by (iii).

We will use *score vectors* to define the tournaments. If there is 0 in score vector that means the graph has a sink and $n-1$ in score vector that means the graph has a source. We add the number of possible non-isomorphic tournaments into the Table 6.1 on page 133 with no source no sink case, just one source and no sink case, just one sink and no source case and a source, a sink but they are adjacent case. The corresponding groups are infinite if we have any cases except for these four cases by Lemma 1.3.3.

Note that for a table (i.e Table 6.4 on page 143), a tick (✓) in a cell means that the group G in the column header has a subgroup whose abelianization is the group in the row header, a cross (✗) means it does not. *DL* in a cell means Derived Length. Also, we will write $[n,j]$ to denote the j'th group of order n in the Small Groups Library. When we have Small group ids as column headings, we can actually say what the group is, which would be more useful to know. For example the group $[32,51]$ is the group $\mathbb{Z}_2^5$ and $[64,260]$ is $\mathbb{Z}_2^4 \oplus \mathbb{Z}_4$. It can be found this out using *StructureDescription(SmallGroup(32,51));* into the *GAP*. We use subscript on the score vector if there is more than 1 non-isomorphic tournament with same score vector. For example, there are 12 non-isomorphic tournaments with the score vector $(1,2,2,3,3,4)$ and we denote then $(1,2,2,3,3,4)$, $(1,2,2,3,3,4)_2$, $(1,2,2,3,3,4)_3, \ldots, (1,2,2,3,3,4)_{12}$ (Note that we do not add a subscript in the first case).

Conjecture 6.2.4. *Let Γ_1, Γ_2 be two n-vertex tournaments each of which contains a Hamilton cycle. Then $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$.*

If a tournament Γ contains a Hamiltonian cycle, then Γ does not have a source or a sink. We see that the corresponding digraph groups have same order by the tables that we created.

Also, if Γ is n-vertex tournament which has 3 vertices of out-degree $n-2$ and assume that these vertices are x,y,z. The only way we can keep all of them down to in-degree 1 is for them to be a directed 3-cycle, say $x \to y \to z \to x$. However then all other arcs out of these vertices have to go elsewhere, so the tournament is not irreducible, so is not strong, so does not have a Hamilton cycle though Γ does not have a source or a sink (An example is with the score vector 111444).

Conjecture 6.2.4 is confirmed for all tournaments up to 6-vertex tournaments and for some 7 and 8-vertex tournaments (see the possible tournaments by the tables 6.3, 6.10, 6.12, 6.14, 6.15).

	K_3	K_4	K_5	K_6	K_7	K_8
total number	2	4	12	56	456	6880
No source No sink	1	1	6	36	?	?
One source, no sink	0	1	2	8	?	?
One sink, no source	0	1	2	8	?	?
1 source, 1 sink and they are adjacent	1	1	2	4	?	?

Table 6.1: The number of non-isomorphic tournaments with up to 8 vertices

	K_3	K_4	K_5	K_6	K_7	K_8
total number	2	4	9	22	59	167
No source No sink	1	1	3	8	16	?
Just source	0	1	2	5	?	?
Just sink	0	1	2	5	?	?
1 source, 1 sink and they are adjacent	1	1	2	4	?	?

Table 6.2: The number of score vectors of size n

6.3 The Mennicke relator with $q = 3$: 2-groups

In this section, we investigate the Mennicke relator with $q = 3$, that means $R(a, b) = a^{-1}bab^{-3}$ for n-vertex tournaments with $n \leqslant 12$. If there is a source in the tournaments and $\alpha = 0$, then the corresponding digraph groups are infinite by Lemma 6.2.1 since $\alpha = -1 + 1 = 0$ in Mennicke relators $(R(a, b) = a^{-1}bab^{-q})$. Thus, $G_\Gamma(R)$ is infinite if there is a source, which is showed as $n - 1$ in the score vector.

Table 6.3: All possible n-vertex tournaments, where all for $3 \leqslant n \leqslant 6$ and for some $7 \leqslant n \leqslant 12$.

$R(a,b) = a^{-1}bab^{-3}$				
Score Vectors	$\|G\|$	G/G'	G'/G''	DL
$(0,1,2)$	∞	$\mathbb{Z} \oplus \mathbb{Z}_2^2$		
$(1,1,1)$	2^{11}	$\mathbb{Z}_2^3$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8^2$	2
$(0,1,2,3)$ $(1,1,1,3)$	∞	$\mathbb{Z} \oplus \mathbb{Z}_2^3$		
$(0,2,2,2)$	2^{14}	$\mathbb{Z}_2^4$	$\mathbb{Z}_4^2 \oplus \mathbb{Z}_8^2$	2
$(1,1,2,2)$	2^{12}	$\mathbb{Z}_2^4$	$\mathbb{Z}_4^4$	2
$(0,1,2,3,4)$ $(0,2,2,2,4)$ $(1,1,1,3,4)$ $(1,1,2,2,4)$	∞	$\mathbb{Z} \oplus \mathbb{Z}_2^4$		
$(0,1,3,3,3)$	2^{17}	$\mathbb{Z}_2^5$	$\mathbb{Z}_4^3 \oplus \mathbb{Z}_8^2$	2
$(0,2,2,3,3)$ $(1,1,2,3,3)$ $(1,1,2,3,3)_2$ $(1,2,2,2,3)$ $(1,2,2,2,3)_2$ $(1,2,2,2,3)_3$ $(2,2,2,2,2)$	2^{15}	$\mathbb{Z}_2^5$	$\mathbb{Z}_4^5$	2

$(0,1,2,3,4,5)$ $(0,1,3,3,3,5)$ $(0,2,2,2,4,5)$ $(0,2,2,3,3,5)$ $(1,1,1,3,4,5)$ $(1,1,2,2,4,5)$ $(1,1,2,3,3,5)$ $(1,1,2,3,3,5)_2$ $(1,2,2,2,3,5)$ $(1,2,2,2,3,5)_2$ $(1,2,2,2,3,5)_3$ $(2,2,2,2,2,5)$	∞	$\mathbb{Z} \oplus \mathbb{Z}_2^5$		
$(0,1,2,4,4,4)$ $(1,1,1,4,4,4)$	2^{20}	$\mathbb{Z}_2^6$	$\mathbb{Z}_4^4 \oplus \mathbb{Z}_8^2$	2

$(0,1,3,3,4,4)$	2^{18}	$\mathbb{Z}_2^6$	$\mathbb{Z}_4^6$	2
$(0,2,2,3,4,4)$				
$(0,2,2,3,4,4)_2$				
$(0,2,3,3,3,4)$				
$(0,2,3,3,3,4)_2$				
$(0,2,3,3,3,4)_3$				
$(0,3,3,3,3,3)$				
$(1,1,3,3,3,4)$				
$(1,1,3,3,3,4)_2$				
$(1,1,3,3,3,4)_3$				
$(1,1,2,3,4,4)$				
$(1,1,2,3,4,4)_2$				
$(1,1,2,3,4,4)_3$				
$(1,1,2,3,4,4)_4$				
$(1,2,2,3,3,4)$				
$(1,2,2,3,3,4)_2$				
$(1,2,2,3,3,4)_3$				
$(1,2,2,3,3,4)_4$				
$(1,2,2,3,3,4)_5$				
$(1,2,2,3,3,4)_6$				
$(1,2,2,3,3,4)_7$				
$(1,2,2,3,3,4)_8$				
$(1,2,2,3,3,4)_9$				
$(1,2,2,3,3,4)_{10}$				
$(1,2,2,3,3,4)_{11}$				
$(1,2,2,3,3,4)_{12}$				
$(1,2,2,2,4,4)$				
$(1,2,2,2,4,4)_2$				
$(1,2,2,2,4,4)_3$				
$(1,2,3,3,3,3)$				
$(1,2,3,3,3,3)_2$				
$(1,2,3,3,3,3)_3$				
$(1,2,3,3,3,3)_4$				
$(2,2,2,3,3,3)$				
$(2,2,2,3,3,3)_2$				
$(2,2,2,3,3,3)_3$				
$(2,2,2,3,3,3)_4$				
$(2,2,2,3,3,3)_5$				
$(2,2,2,2,3,4)$				
$(2,2,2,2,3,4)_2$				
$(2,2,2,2,3,4)_3$				
$(2,2,2,2,3,4)_4$				

$(1,1,3,3,3,4,6)$	∞	$\mathbb{Z} \oplus \mathbb{Z}_2^6$		
$(1,2,3,3,3,4,5)$ $(0,3,3,3,3,4,5)$	2^{21}	$\mathbb{Z}_2^7$	$\mathbb{Z}_4^7$	2
$(0,1,2,3,5,5,5)$ $(0,2,2,2,5,5,5)$ $(1,1,2,2,5,5,5)$ $(1,1,1,3,5,5,5)$	2^{23}	$\mathbb{Z}_2^7$	$\mathbb{Z}_4^5 \oplus \mathbb{Z}_8^2$	2
$(2,2,3,3,3,4,5,6)$	2^{24}	$\mathbb{Z}_2^8$	$\mathbb{Z}_4^8$	2
$(0,1,2,3,4,6,6,6)$ $(0,2,2,3,3,6,6,6)$	2^{26}	$\mathbb{Z}_2^8$	$\mathbb{Z}_4^6 \oplus \mathbb{Z}_8^2$	2
$(1,2,3,4,4,4,5,6,7)$	2^{27}	$\mathbb{Z}_2^9$	$\mathbb{Z}_4^9$	2
$(2,2,2,3,3,3,7,7,7)$ $(0,1,2,3,4,5,7,7,7)$	2^{29}	$\mathbb{Z}_2^9$	$\mathbb{Z}_4^7 \oplus \mathbb{Z}_8^2$	2
$(0,1,2,3,4,5,6,8,8,8)$	2^{32}	$\mathbb{Z}_2^{10}$	$\mathbb{Z}_4^8 \oplus \mathbb{Z}_8^2$	2
$(0,1,2,3,4,5,6,7,9,9,9)$	2^{35}	$\mathbb{Z}_2^{11}$	$\mathbb{Z}_4^9 \oplus \mathbb{Z}_8^2$	2
$(0,1,2,3,4,5,6,7,8,10,10,10)$	2^{38}	$\mathbb{Z}_2^{12}$	$\mathbb{Z}_4^{10} \oplus \mathbb{Z}_8^2$	2

We now state some lemmas and conjectures that were motivated by the table.

Lemma 6.3.1. *Let $R(a,b) = a^{-1}bab^{-3}$. If Γ_1, Γ_2 are non-isomorphic 5-vertex tournaments which have no sources and do not have 3 vertices of out-degree 3, then $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ and $G_{\Gamma_1}(R) \ncong G_{\Gamma_2}(R)$.*

Proof. There are 7 non-isomorphic tournaments with 5 vertices which have no sources and do not have 3 vertices of out-degree 3. By Table 6.3 the orders of the corresponding digraph groups are all equal to 2^{15}. We will give a computational proof here that the corresponding digraph groups are pairwise non-isomorphic. We compare abelianization of low index subgroups to understand whether the digraph groups are isomorphic or not. We use the command *IdGroup(G)* in *GAP*.

Let $(I), (II), (III), (IV), (V), (VI), (VII)$ denote the tournaments with score vectors

$(1,2,2,2,3), (1,2,2,2,3)_2, (1,2,2,2,3)_3, (1,1,2,3,3), (1,1,2,3,3)_2,$ $(2,2,2,2,2), (0,2,2,3,3)$, respectively.

By studying their index 2 subgroups we see that $G_{(VII)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 192]$ whereas $G_{(I)}(R)$,$G_{(II)}(R)$, $G_{(III)}(R)$,$G_{(IV)}(R)$,$G_{(V)}(R)$,$G_{(VI)}(R)$ do not. Thus, $G_{(VII)}(R)$ is not isomorphic to any of $G_{(I)}(R)$,$G_{(II)}(R)$,$G_{(III)}(R)$, $G_{(IV)}(R)$,$G_{(V)}(R)$, $G_{(VI)}(R)$.

By studying their index 2 subgroups we see that $G_{(I)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 1601]$ whereas $G_{(II)}(R), G_{(III)}(R)$, $G_{(IV)}(R), G_{(VI)}(R)$ do not and $G_{(V)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 246]$ whereas $G_{(I)}(R)$ does not. Thus, $G_{(I)}(R)$ is not isomorphic to any of $G_{(II)}(R), G_{(III)}(R), G_{(IV)}(R), G_{(V)}(R), G_{(VI)}(R)$.

By studying their index 2 subgroups we see that $G_{(II)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 260]$ whereas $G_{(III)}(R), G_{(V)}(R)$, $G_{(VI)}(R)$ do not and by studying their index 4 subgroups we see that $G_{(II)}(R)$ has an index 4 subgroup whose abelianization is the group $[64, 267]$ whereas $G_{(IV)}(R)$ does not. Thus, $G_{(II)}(R)$ is not isomorphic to any of $G_{(III)}(R), G_{(IV)}(R), G_{(V)}(R), G_{(VI)}(R)$.

By studying their index 2 subgroups we see that $G_{(IV)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 260]$ whereas $G_{(III)}(R)$ does not, $G_{(V)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 1601]$ whereas $G_{(III)}(R)$ does not and $G_{(III)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 2301]$ whereas $G_{(VI)}(R)$ does not. Thus, $G_{(III)}(R)$ is not isomorphic to any of $G_{(IV)}(R), G_{(V)}(R), G_{(VI)}(R)$.

By studying their index 2 subgroups we see that $G_{(IV)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 260]$ whereas $G_{(V)}(R), G_{(VI)}(R)$ do not. Thus, $G_{(IV)}(R)$ is not isomorphic to any of $G_{(V)}(R), G_{(VI)}(R)$.

By studying their index 2 subgroups we see that $G_{(V)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 1601]$ whereas $G_{(VI)}(R)$ does not. Thus, $G_{(V)}(R)$ is not isomorphic to any of $G_{(VI)}(R)$.

Thus, corresponding digraph groups of these 7 tournaments are non-isomorphic though the corresponding digraph groups have same order. The *GAP* code is provided in Appendix A.3.

□

Lemma 6.3.2. *Let* $R(a, b) = a^{-1}bab^{-3}$. *If* Γ_1, Γ_2 *are non-isomorphic* 6*-vertex tournaments which have no sources and have* 3 *vertices of out-degree* 4*, then*

$|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ *and* $G_{\Gamma_1}(R) \ncong G_{\Gamma_2}(R)$.

Proof. There are 2 non-isomorphic tournaments with 6 vertices which have no sources and have 3 vertices of out-degree 4. By Table 6.3 on page 134 the orders of the corresponding digraph groups are all equal to 2^{20}. We will give a computational proof here that the corresponding digraph groups are pairwise non-isomorphic. We compare abelianization of low index subgroups to understand whether the digraph groups are isomorphic or not. We use the command *IdGroup(G)* in *GAP*.

Let $(I), (II)$ denote the tournaments with score vectors $(1, 1, 1, 4, 4, 4)$, $(0, 1, 2, 4, 4, 4)$, respectively. By studying their index 2 subgroups we see that $G_{(I)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 260]$ whereas $G_{(II)}(R)$ does not. Thus, $G_{(I)}(R)$ is not isomorphic to $G_{(II)}(R)$. Hence, corresponding digraph groups of these 2 tournaments are non-isomorphic though the corresponding digraph groups have same order. □

Lemma 6.3.3. *Let* $R(a, b) = a^{-1}bab^{-3}$. *If* Γ_1, Γ_2 *are non-isomorphic* 6*-vertex tournaments which have no sources and do not have* 3 *vertices of out-degree* 4*, then* $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ *and* $G_{\Gamma_1}(R) \ncong G_{\Gamma_2}(R)$ *except possibly for these pair of tournaments:*

$\{\Gamma_1, \Gamma_2\} = \{(7), (33)\}, \{(11), (24)\}, \{(11), (40)\}, \{(24), (40)\}, \{(17), (42)\}, \{(19), (20)\}, \{(22), (34)\}$.

Proof. There are 42 non-isomorphic tournaments with 6 vertices which have no sources and do not have 3 vertices of out-degree 4. By Table 6.3 on page 134, the orders of the corresponding digraph groups are all equal to 2^{18}. To prove that the corresponding groups are not isomorphic to each other, we need to compare all of them, which means making $41 \cdot 40/2 = 820$ comparisons for just index 2 subgroups. This seems infeasible and we know the technique since we already proved in Lemma 6.3.1 and Lemma 6.3.2. Therefore, we create the tables and if they are not identical, then it means the corresponding digraph groups are not isomorphic to each other.

Let $(1), (2), \ldots, (42)$ denote the tournaments with score vectors in the order of the Table 6.4 on page 143 as $(0, 1, 3, 3, 4, 4), (0, 2, 2, 3, 4, 4), \ldots, (2, 2, 2, 2, 3, 4)_4$, respectively. Note that we use identical if the rows in the Table 6.4 or the columns in the Table 6.5, 6.6, 6.7 and 6.8 are exactly same to each other and if a row or a column in the related table does not same with others, then we say it is unique. Also note that if the related rows or columns are identical, then we cannot decide whether it is isomorphic or not but if it is unique then we say that it is not isomorphic to others (to decide the rows or columns are identical or unique, check the related tables)

By Table 6.4 on page 143 (abelianization of index 2 subgroups), we have that

(1) and (37) are unique. That means $G_{(1)}(R)$ and $G_{(37)}(R)$ are not-isomorphic to other digraph groups.

(2) and (9) are identical to each other.

$(3), (4), (6), (8), (13), (14), (15), (21), (26), (27), (31)$ are identical to each other.

$(5), (10), (12), (16), (23), (28), (41)$ are identical to each other.

$(7), (30), (33), (35), (36), (38)$ are identical to each other.

$(11), (17), (18), (19), (20), (22), (24), (25), (29), (32), (34), (39), (40), (42)$ are identical to each other

Therefore it is not possible to distinguish the corresponding groups. However all other tournaments unique and so groups corresponding to these tournaments are pairwise not isomorphic.

By repeating abelianization of index 4 subgroups, we see that the index 4 subgroups (2) has $[128, 1601]$ and (9) does not. Thus, $G_{(2)}(R)$ is not isomorphic to $G_{(9)(}R)$.

By Table 6.5 on page 144 (abelianization of index 4 subgroups), we have that (8) and (27) are identical to each other. Others are not isomorphic to each other since they are unique. By abelianization of index 5 subgroups, (27) has $[128, 2150]$ and (8) does not. Thus, $G_{(8)}(R)$ is not isomorphic to $G_{(27)(}R)$. That means the corresponding digraph groups for $(3), (4), (6), (8), (13), (14), (15), (21), (26), (27), (31)$ are not isomorphic to each other.

By Table 6.6 on page 144 (abelianization of index 4 subgroups), all are unique. Thus, the corresponding digraph groups for $(5), (10), (12), (16), (23),$ $(28), (41)$ are not isomorphic to each other.

By Table 6.7 on page 145 (abelianization of index 4 subgroups), we have that (7) and (33) are identical to each other. Others are not isomorphic to each other since they are unique.

By Table 6.8 on page 145 (abelianization of index 4 subgroups), we have that

$(11), (24)$ and (40) are identical to each other.

(17) and (42) are identical to each other.

(19) and (20) are identical to each other.

(22) and (34) are identical to each other.

Others are not isomorphic to each other since they are unique.

By studying their index subgroups up to index 7, the pair of non-isomorphic tournaments: $\{(7), (33)\}, \{(11), (24)\}, \{(11), (40)\}, \{(24), (40)\}, \{(17), (42)\}, \{(19), (20)\}, \{(22), (34)\}$ are identical and *GAP* computations do not complete with index 8 or bigger. Therefore, we have been unable to determine if $G_{(7)} \cong G_{(33)}, G_{(11)} \cong G_{(24)}, G_{(11)} \cong G_{(40)}, G_{(24)} \cong G_{(40)}, G_{(17)} \cong G_{(42)}, G_{(19)} \cong G_{(20)}, G_{(22)} \cong G_{(34)}$. $\square$

Conjecture 6.3.4. *Let $R(a, b) = a^{-1}bab^{-3}$, suppose that Γ does not have a source and Γ be an n-vertex tournament $(n \geqslant 3)$ and let $G = G_\Gamma(R)$, then the derived length of G is 2 (and so G is solvable, not abelian and not cyclic).*

We can say that the derived length of $G_\Gamma(R)$, when this is finite, is equal to 2 for all n-vertex tournaments up to $n = 6$. Thus, $G_\Gamma(R)$ is solvable for all n-vertex tournaments up to $n = 6$ if $G_\Gamma(R)$ is finite. $G_\Gamma(R)$ for all n-vertex tournaments up to $n = 6$ is also not abelian since derived length is not 1. It is well known that if the group is not abelian, then it is not cyclic. Thus, we can say that $G_\Gamma(R)$ for all n-vertex tournaments up to $n = 6$ is also non cyclic. It seems that it also holds for $8 \leqslant n \leqslant 12$ in our limited examples (see the Table 6.3 on page 134).

The structure of the digraph groups conjectured in Conjectures 6.3.5, 6.3.6, 6.5.4, 6.5.5, 6.6.4, 6.8.3, 6.8.4 are more complicated that those encountered in related Sections (which were cyclic). Therefore new techniques are likely to be needed to tackle these conjectures.

Conjecture 6.3.5. *Let $R(a,b) = a^{-1}bab^{-3}$ and let Γ be an n-vertex tournament ($n \geqslant 3$) which has no sources and does not have 3 vertices of out-degree $n-2$. Then $|G_\Gamma(R)| = 2^{3n}$, $G/G' \cong \mathbb{Z}_2^n$ and $G'/G'' \cong \mathbb{Z}_4^n$.*

We confirmed that it is true for $3 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament ($7 \leqslant n \leqslant 12$) by using *GAP*.

Conjecture 6.3.6. *Let $R(a,b) = a^{-1}bab^{-3}$ and let Γ be an n-vertex tournament ($n \geqslant 3$) which has no sources and has 3 vertices of out-degree $n-2$. Then $|G_\Gamma(R)| = 2^{3n+2}$, $G/G' \cong \mathbb{Z}_2^n$ and $G'/G'' \cong \mathbb{Z}_4^{n-2} \oplus \mathbb{Z}_8^2$.*

We confirmed that it is true for $3 \leqslant n \leqslant 7$ for all tournaments and for some n-vertex tournament ($8 \leqslant n \leqslant 12$) by using *GAP*.

By Index 2	[32,51]	[64,260]	[64,267]	[128,2150]	[128,2301]	[128,2319]	[256,53038]	[256,56059]
(1)=(0,1,3,3,4,4)	✓	✓	✓	✓	✗	✓	✓	✓
(2)=(0,2,2,3,4,4)	✓	✓	✓	✗	✓	✓	✓	✓
(3)= (0,2,2,3,4,4)_2	✓	✓	✓	✗	✓	✗	✓	✓
(4)=(0,2,3,3,3,4)	✓	✓	✓	✗	✓	✗	✓	✓
(5)=(0,2,3,3,3,4)_2	✓	✓	✓	✗	✓	✓	✗	✓
(6)= (0,2,3,3,3,4)_3	✓	✓	✓	✗	✓	✗	✓	✓
(7)=(0,3,3,3,3,3)	✓	✗	✓	✗	✓	✗	✗	✗
(8)=(1,1,3,3,3,4)	✓	✓	✓	✗	✓	✗	✓	✓
(9)=(1,1,3,3,3,4)_2	✓	✓	✓	✗	✓	✓	✓	✓
(10)=(1,1,3,3,3,4)_3	✓	✓	✓	✗	✓	✓	✗	✓
(11)=(1,1,2,3,4,4)	✓	✓	✓	✗	✓	✗	✗	✓
(12)=(1,1,2,3,4,4)_2	✓	✓	✓	✗	✓	✓	✗	✓
(13)=(1,1,2,3,4,4)_3	✓	✓	✓	✗	✓	✗	✓	✓
(14)=(1,1,2,3,4,4)_4	✓	✓	✓	✗	✓	✗	✓	✓
(15)=(1,2,2,3,3,4)	✓	✓	✓	✗	✓	✗	✓	✓
(16)=(1,2,2,3,3,4)_2	✓	✓	✓	✗	✓	✓	✗	✓
(17)=(1,2,2,3,3,4)_3	✓	✓	✓	✗	✓	✗	✗	✓
(18)=(1,2,2,3,3,4)_4	✓	✓	✓	✗	✓	✗	✗	✓
(19)=(1,2,2,3,3,4)_5	✓	✓	✓	✗	✓	✗	✗	✓
(20)=(1,2,2,3,3,4)_6	✓	✓	✓	✗	✓	✗	✗	✓
(21)=(1,2,2,3,3,4)_7	✓	✓	✓	✗	✓	✗	✓	✓
(22)=(1,2,2,3,3,4)_8	✓	✓	✓	✗	✓	✗	✗	✓
(23)=(1,2,2,3,3,4)_9	✓	✓	✓	✗	✓	✓	✗	✓
(24)=(1,2,2,3,3,4)_10	✓	✓	✓	✗	✓	✗	✗	✓
(25)=(1,2,2,3,3,4)_11	✓	✓	✓	✗	✓	✗	✗	✓
(26)=(1,2,2,3,3,4)_12	✓	✓	✓	✗	✓	✗	✓	✓
(27)=(1,2,2,2,4,4)	✓	✓	✓	✗	✓	✗	✓	✓
(28)=(1,2,2,2,4,4)_2	✓	✓	✓	✗	✓	✓	✗	✓
(29)=(1,2,2,2,4,4)_3	✓	✓	✓	✗	✓	✗	✗	✓
(30)=(1,2,3,3,3,3)	✓	✗	✓	✗	✓	✗	✗	✗
(31)=(1,2,3,3,3,3)_2	✓	✓	✓	✗	✓	✗	✓	✓
(32)=(1,2,3,3,3,3)_3	✓	✓	✓	✗	✓	✗	✗	✓
(33)=(1,2,3,3,3,3)_4	✓	✗	✓	✗	✓	✗	✗	✗
(34)=(2,2,2,3,3,3)	✓	✓	✓	✗	✓	✗	✗	✓
(35)=(2,2,2,3,3,3)_2	✓	✗	✓	✗	✓	✗	✗	✗
(36)=(2,2,2,3,3,3)_3	✓	✗	✓	✗	✓	✗	✗	✗
(37)=(2,2,2,3,3,3)_4	✓	✓	✓	✗	✗	✗	✓	✓
(38)=(2,2,2,3,3,3)_5	✓	✗	✓	✗	✓	✗	✗	✗
(39)=(2,2,2,2,3,4)	✓	✓	✓	✗	✓	✗	✗	✓
(40)=(2,2,2,2,3,4)_2	✓	✓	✓	✗	✓	✗	✗	✓
(41)=(2,2,2,2,3,4)_3	✓	✓	✓	✗	✓	✓	✗	✓
(42)=(2,2,2,2,3,4)_4	✓	✓	✓	✗	✓	✗	✗	✓

Table 6.4: Abelianization of index 2 subgroups.

By Index 4	(3)	(4)	(6)	(8)	(13)	(14)	(15)	(21)	(26)	(27)	(31)
[16,14]	✓	✗	✓	✓	✓	✓	✓	✗	✓	✓	✓
[32,45]	✓	✗	✓	✓	✓	✓	✓	✗	✓	✓	✓
[32,51]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[64,192]	✗	✗	✗	✓	✓	✓	✓	✗	✗	✓	✗
[64,246]	✗	✗	✓	✓	✓	✓	✓	✗	✗	✓	✓
[64,260]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[64,267]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[128,1601]	✗	✗	✓	✓	✓	✓	✓	✓	✗	✓	✗
[128,2150]	✗	✗	✗	✓	✗	✗	✗	✗	✗	✓	✓
[128,2301]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[128,2319]	✓	✗	✓	✗	✓	✓	✗	✗	✗	✗	✗
[256,13313]	✗	✗	✗	✗	✓	✗	✗	✗	✗	✗	✗
[256,53038]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[256,56059]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 6.5: Abelianization of index 4 subgroups.

By Index 4	(5)	(10)	(12)	(16)	(23)	(28)	(41)
[16,14]	✓	✓	✓	✓	✓	✓	✓
[32,45]	✓	✓	✓	✓	✓	✓	✓
[32,51]	✓	✓	✓	✓	✓	✓	✓
[64,192]	✗	✗	✓	✓	✓	✓	✓
[64,246]	✓	✓	✓	✓	✓	✓	✓
[64,260]	✓	✓	✓	✓	✓	✓	✓
[64,267]	✓	✓	✓	✓	✓	✓	✓
[128,1601]	✓	✓	✓	✓	✓	✓	✓
[128,2150]	✓	✗	✗	✗	✗	✗	✓
[128,2301]	✓	✓	✓	✓	✓	✓	✓
[128,2319]	✓	✓	✗	✓	✓	✗	✓
[256,10298]	✓	✗	✗	✗	✓	✗	✗
[256,13313]	✗	✗	✓	✗	✓	✗	✗
[256,53038]	✓	✓	✓	✓	✓	✓	✓
[256,56059]	✓	✓	✓	✓	✓	✓	✓

Table 6.6: Abelianization of index 4 subgroups .

By Index 4	(7)	(30)	(33)	(35)	(36)	(38)
[16,14]	✓	✓	✓	✓	✓	✓
[32,45]	✓	✓	✓	✓	✓	✓
[32,51]	✓	✓	✓	✓	✓	✓
[64,192]	✗	✓	✗	✓	✓	✓
[64,246]	✓	✓	✓	✓	✓	✓
[64,260]	✓	✓	✓	✓	✓	✓
[64,267]	✓	✓	✓	✓	✓	✓
[128,1601]	✓	✓	✓	✓	✓	✓
[128,2150]	✓	✓	✓	✗	✗	✗
[128,2301]	✓	✓	✓	✓	✓	✓
[128,2319]	✗	✓	✗	✓	✓	✓
[128,2328]	✗	✗	✗	✗	✓	✓
[256,10298]	✓	✓	✓	✓	✓	✓
[256,13313]	✗	✗	✗	✓	✓	✗
[256,53038]	✓	✓	✓	✓	✓	✗
[256,56059]	✓	✓	✓	✓	✓	✓

Table 6.7: Abelianization of index 4 subgroups.

By Index 4	(11)	(17)	(18)	(19)	(20)	(22)	(24)	(25)	(29)	(32)	(34)	(39)	(40)	(42)
[16,14]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[32,45]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[32,51]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[64,192]	✓	✓	✗	✓	✓	✓	✓	✓	✗	✗	✓	✓	✓	✓
[64,246]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
[64,260]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[64,267]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[128,1601]	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓
[128,2150]	✗	✗	✗	✗	✗	✓	✗	✗	✗	✗	✓	✗	✗	✗
[128,2301]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[128,2319]	✗	✓	✓	✓	✓	✗	✗	✓	✗	✓	✗	✓	✗	✓
[256,10298]	✗	✗	✗	✗	✗	✓	✗	✗	✗	✗	✓	✓	✗	✗
[256,13313]	✗	✗	✗	✓	✓	✓	✗	✓	✗	✗	✓	✓	✗	✗
[256,53038]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
[256,56059]	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓

Table 6.8: Abelianization of index 4 subgroups.

6.4 The Mennicke relator with $4 \leqslant q \leqslant 10$

We will investigate Mennicke relator with other values of q for $4 \leqslant q \leqslant 10$ in this section. We will provide two Theorems for G/G' in Theorem 6.4.1 and Theorem 6.4.2 and a table for G'/G'' in Table 6.9 on page 148. This table does not include the tournaments with source because it is proved that G/G' is infinite in Theorem 6.4.1.

Theorem 6.4.1. *Let $R(a,b) = a^{-1}bab^{-q}$ and let Γ be an n-vertex tournament $(n \geqslant 3)$ which has a source. Then, $G/G' \cong \mathbb{Z} \oplus \mathbb{Z}_{q-1}^{n-1}$.*

Proof. Let $G = G_\Gamma(R)$. Then

$$G = \langle\, x_v(v \in V(\Gamma)) \mid x_u^{-1}x_vx_ux_v^{-q} \;\; (u,v) \in A(\Gamma)\,\rangle.$$

Therefore,

$$\begin{aligned}
G^{ab} &= \langle\; x_v(v \in V(\Gamma)) \;\mid\; x_u^{-1}x_vx_ux_v^{-q}, x_ux_v = x_vx_u \;\; (u,v) \in A(\Gamma)\;\rangle \\
&= \langle\, x_v(v \in V(\Gamma)) \mid x_v^{1-q}, x_ux_v = x_vx_u \;\; (u,v) \in A(\Gamma)\,\rangle \\
&= \Big\langle\; x_v(v \in V(\Gamma)) \;\Big|\, x_v^{1-q}, x_ux_v = x_vx_u \;\Big\rangle \\
&= \oplus_{v\in V(\Gamma)}\langle\, x_v \mid x_v^{1-q}(\text{whenever } v \text{ is the terminal vertex of some arc})\,\rangle.
\end{aligned}$$

Now since Γ has a source s, then every other vertex is the terminal vertex of some arc. Thus,

$$\begin{aligned}
G^{ab} &= \oplus_{v\in V(\Gamma)}\langle\, x_v \mid x_v^{1-q}(\text{whenever } v \text{ is the terminal vertex of some arc})\,\rangle \\
&= \oplus_{v\in V(\Gamma)}\langle\, x_v \mid x_v^{1-q}(\text{whenever } v \neq s)\,\rangle \\
&= \langle\, x_s \mid \,\rangle \oplus \oplus_{v\in V(\Gamma),\; v\neq s}\langle\, x_v \mid x_v^{1-q}\,\rangle \\
&= \mathbb{Z} \oplus \oplus_{v\in V(\Gamma),\; v\neq s}\mathbb{Z}_{|q-1|} \\
&= \mathbb{Z} \oplus \mathbb{Z}_{|q-1|}^{n-1}.
\end{aligned}$$

□

Theorem 6.4.2. *Let $R(a,b) = a^{-1}bab^{-q}$ and let Γ be an n-vertex tournament $(n \geqslant 3)$ which does not have a source. Then, $G/G' \cong \mathbb{Z}_{q-1}^{n}$.*

Proof. Let $G = G_\Gamma(R)$. Then

$$G = \langle\, x_v(v \in V(\Gamma)) \mid x_u^{-1}x_vx_ux_v^{-q} \quad (u,v) \in A(\Gamma)\,\rangle.$$

Therefore,

$$\begin{aligned}
G^{ab} &= \langle\ x_v(v \in V(\Gamma)) \ \mid\ x_u^{-1}x_vx_ux_v^{-q} \quad (u,v) \in A(\Gamma), x_ux_v = x_vx_u \quad (u,v) \in A(\Gamma)\ \rangle \\
&= \langle\, x_v(v \in V(\Gamma)) \mid x_v^{1-q} \quad (u,v) \in A(\Gamma), x_ux_v = x_vx_u \quad (u,v) \in A(\Gamma)\,\rangle \\
&= \langle\, x_v(v \in V(\Gamma)) \mid x_v^{1-q}, x_ux_v = x_vx_u\,\rangle \\
&= \oplus_{v\in V(\Gamma)}\langle\, x_v \mid x_v^{1-q}(\text{whenever } v \text{ is the terminal vertex of some arc})\,\rangle.
\end{aligned}$$

Now since Γ does not have a source then every vertex is the terminal vertex of some arc. Therefore,

$$G^{ab} = \oplus_{v\in V(\Gamma)}\langle\, x_v \mid x_v^{1-q}\,\rangle = \oplus_{v\in V(\Gamma)}\mathbb{Z}_{|1-q|} = \mathbb{Z}_{|1-q|}^{|V|} = \mathbb{Z}_{|q-1|}^{|V|} = \mathbb{Z}_{|q-1|}^{n}.$$

□

$R(a,b) = a^{-1}bab^{-q}$							
	$q=4$	$q=5$	$q=6$	$q=7$	$q=8$	$q=9$	$q=10$
Score Vectors	G'/G''	G'/G''	G'/G''	G'/G''	G'/G''	G'/G''	G'/G''
$(1,1,1)$	$\mathbb{Z}_3 \oplus \mathbb{Z}_7^3 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3^3 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_{13}^3 \oplus \mathbb{Z}_{16}^2$	$\mathbb{Z}_5 \oplus \mathbb{Z}_{25}^2 \oplus \mathbb{Z}_{311}^3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_9^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{19}^3 \oplus \mathbb{Z}_{43}^3$	$\mathbb{Z}_7 \oplus \mathbb{Z}_{49}^2 \oplus \mathbb{Z}_{127}^3 \oplus \mathbb{Z}_{337}^3$	$\mathbb{Z}_5^3 \oplus \mathbb{Z}_8 \oplus \mathbb{Z}_{17}^3 \oplus \mathbb{Z}_{41}^3 \oplus \mathbb{Z}_{64}^2 \oplus \mathbb{Z}_{193}^3$	$\mathbb{Z}_9 \oplus \mathbb{Z}_{37}^3 \oplus \mathbb{Z}_{81}^2 \oplus \mathbb{Z}_{33667}^3$
$(0,2,2,2)$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_7^4 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_{13}^4 \oplus \mathbb{Z}_{16}^2$	$\mathbb{Z}_5^2 \oplus \mathbb{Z}_{25}^2 \oplus \mathbb{Z}_{311}^4$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{19}^4 \oplus \mathbb{Z}_{43}^4$	$\mathbb{Z}_7^2 \oplus \mathbb{Z}_{49}^2 \oplus \mathbb{Z}_{127}^4 \oplus \mathbb{Z}_{337}^4$	$\mathbb{Z}_5^4 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_{17}^4 \oplus \mathbb{Z}_{41}^4 \oplus \mathbb{Z}_{64}^2 \oplus \mathbb{Z}_{193}^4$	$\mathbb{Z}_9 \oplus \mathbb{Z}_{37}^4 \oplus \mathbb{Z}_{81}^2 \oplus \mathbb{Z}_{333667}^4$
$(1,1,2,2)$		$\mathbb{Z}_3^4 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_{13}^4$	$\mathbb{Z}_5^4 \oplus \mathbb{Z}_{311}^4$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_{19}^4 \oplus \mathbb{Z}_{43}^4$	$\mathbb{Z}_7^4 \oplus \mathbb{Z}_{127}^4 \oplus \mathbb{Z}_{337}^4$	$\mathbb{Z}_5^4 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_{17}^4 \oplus \mathbb{Z}_{41}^4 \oplus \mathbb{Z}_{193}^4$	$\mathbb{Z}_9^4 \oplus \mathbb{Z}_{37}^4 \oplus \mathbb{Z}_{333667}^4$
$(0,1,3,3,3)$		$\mathbb{Z}_3^5 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_{13}^5 \oplus \mathbb{Z}_{16}^2$	$\mathbb{Z}_5^3 \oplus \mathbb{Z}_{25}^2 \oplus \mathbb{Z}_{311}^5$	$\mathbb{Z}_3^3 \oplus \mathbb{Z}_8^3 \oplus \mathbb{Z}_9^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{19}^5 \oplus \mathbb{Z}_{43}^5$	$\mathbb{Z}_7^3 \oplus \mathbb{Z}_{49}^2 \oplus \mathbb{Z}_{127}^5 \oplus \mathbb{Z}_{337}^5$	$\mathbb{Z}_5^5 \oplus \mathbb{Z}_8^3 \oplus \mathbb{Z}_{17}^5 \oplus \mathbb{Z}_{41}^5 \oplus \mathbb{Z}_{64}^2 \oplus \mathbb{Z}_{193}^5$	memory limit
$(0,2,2,3,3)$ $(1,1,2,3,3)$ $(1,1,2,3,3)_2$ $(1,2,2,2,3)$ $(1,2,2,2,3)_2$ $(1,2,2,2,3)_3$ $(2,2,2,2,2)$		$\mathbb{Z}_3^5 \oplus \mathbb{Z}_4^5 \oplus \mathbb{Z}_{13}^5$	$\mathbb{Z}_5^5 \oplus \mathbb{Z}_{311}^5$	$\mathbb{Z}_3^5 \oplus \mathbb{Z}_8^5 \oplus \mathbb{Z}_{19}^5 \oplus \mathbb{Z}_{43}^5$	$\mathbb{Z}_7^5 \oplus \mathbb{Z}_{127}^5 \oplus \mathbb{Z}_{337}^5$	$\mathbb{Z}_5^5 \oplus \mathbb{Z}_8^5 \oplus \mathbb{Z}_{17}^5 \oplus \mathbb{Z}_{41}^5 \oplus \mathbb{Z}_{193}^5$	memory limit
$(0,1,2,4,4,4)$ $(1,1,1,4,4,4)$		$\mathbb{Z}_3^6 \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_{13}^6 \oplus \mathbb{Z}_{16}^2$	$\mathbb{Z}_5^4 \oplus \mathbb{Z}_{25}^2 \oplus \mathbb{Z}_{311}^6$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_8^4 \oplus \mathbb{Z}_9^2 \oplus \mathbb{Z}_{16}^2 \oplus \mathbb{Z}_{19}^6 \oplus \mathbb{Z}_{43}^6$	$\mathbb{Z}_7^4 \oplus \mathbb{Z}_{49}^2 \oplus \mathbb{Z}_{127}^6 \oplus \mathbb{Z}_{337}^6$	memory limit	memory limit
$(0,1,3,3,4,4)$ $(0,2,2,3,4,4)$ $(0,2,2,3,4,4)_2$ $(0,2,3,3,3,4)$ $(0,2,3,3,3,4)_2$ $(0,2,3,3,3,4)_3$ $(0,3,3,3,3,3)$ $(1,1,3,3,3,4)$ $(1,1,3,3,3,4)_2$ $(1,1,3,3,3,4)_3$ $(1,1,2,3,4,4)$ $(1,1,2,3,4,4)_2$ $(1,1,2,3,4,4)_3$ $(1,1,2,3,4,4)_4$ $(1,2,2,3,3,4)$ $(1,2,2,3,3,4)_2$ $(1,2,2,3,3,4)_3$ $(1,2,2,3,3,4)_4$ $(1,2,2,3,3,4)_5$ $(1,2,2,3,3,4)_6$ $(1,2,2,3,3,4)_7$ $(1,2,2,3,3,4)_8$ $(1,2,2,3,3,4)_9$ $(1,2,2,3,3,4)_{10}$ $(1,2,2,3,3,4)_{11}$ $(1,2,2,3,3,4)_{12}$ $(1,2,2,2,4,4)$ $(1,2,2,2,4,4)_2$ $(1,2,2,2,4,4)_3$ $(1,2,3,3,3,3)$ $(1,2,3,3,3,3)_2$ $(1,2,3,3,3,3)_3$ $(1,2,3,3,3,3)_4$ $(2,2,2,3,3,3)$ $(2,2,2,3,3,3)_2$ $(2,2,2,3,3,3)_3$ $(2,2,2,3,3,3)_4$ $(2,2,2,3,3,3)_5$ $(2,2,2,2,3,4)$ $(2,2,2,2,3,4)_2$ $(2,2,2,2,3,4)_3$ $(2,2,2,2,3,4)_4$		$\mathbb{Z}_3^6 \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_{13}^6$	$\mathbb{Z}_5^6 \oplus \mathbb{Z}_{311}^6$	$\mathbb{Z}_3^6 \oplus \mathbb{Z}_8^6 \oplus \mathbb{Z}_{19}^6 \oplus \mathbb{Z}_{43}^6$	memory limit	memory limit	memory limit

Table 6.9: Digraph groups with the Mennicke relator for $4 \leqslant q \leqslant 10$

6.5 The Johnson relator with $q = 2$: groups of order 2^t and $2^t \cdot 7$

In this section, we investigate the Johnson relator with $q = 2$, that means $R(a, b) = ab^{-1}ab^{-3}$ for n-vertex tournaments with $n \leqslant 12$.

Table 6.10: All possible n-vertex tournaments for $3 \leqslant n \leqslant 6$ and some when $7 \leqslant n \leqslant 12$.

$R(a,b) = ab^{-1}ab^{-3}$				
Score Vectors	$\lvert G \rvert$	G/G'	G'/G''	DL
$(0,1,2)$	∞	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	
$(1,1,1)$	$2^{11} \cdot 7$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_7$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8^2$	2
$(0,1,2,3)$	∞	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4$	
$(0,2,2,2)$	$2^{14} \cdot 7$		$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(1,1,1,3)$	∞	$\mathbb{Z}_2^4$	$\mathbb{Z} \oplus \mathbb{Z}_4^3$	
$(1,1,2,2)$	2^{12}		$\mathbb{Z}_4^4$	2
$(0,1,2,3,4)$	∞	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^2$	
$(0,2,2,2,4)$			$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^3$	
$(0,1,3,3,3)$	$2^{17} \cdot 7$		$\mathbb{Z}_2 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(0,2,2,3,3)$	2^{15}		$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4$	
$(1,1,1,3,4)$	∞	$\mathbb{Z}_2^5$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4^3$	
$(1,1,2,2,4)$			$\mathbb{Z} \oplus \mathbb{Z}_4^4$	
$(1,1,2,3,3)$	2^{15}	$\mathbb{Z}_2^5$	$\mathbb{Z}_4^5$	2
$(1,1,2,3,3)_2$				
$(1,2,2,2,3)$				
$(1,2,2,2,3)_2$				
$(1,2,2,2,3)_3$				
$(2,2,2,2,2)$				
$(0,1,2,3,4,5)$	∞	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^3$	
$(0,2,2,2,4,5)$				
$(0,1,3,3,3,5)$			$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_4^4$	
$(0,2,2,3,3,5)$				
$(0,1,3,3,4,4)$			$\mathbb{Z} \oplus \mathbb{Z}_4^5$	
$(0,2,2,3,4,4)$				
$(0,2,2,3,4,4)_2$				
$(0,2,3,3,3,4)$				
$(0,2,3,3,3,4)_2$				

$(0, 2, 3, 3, 3, 4)_3$	∞	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$	$\mathbb{Z} \oplus \mathbb{Z}_4^5$	
$(0, 3, 3, 3, 3, 3)$				
$(1, 1, 1, 4, 4, 4)$	$2^{20} \cdot 7$	$\mathbb{Z}_2^6$	$\mathbb{Z}_4^4 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(0, 1, 2, 4, 4, 4)$		$\mathbb{Z}_2^5 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	
$(1, 1, 2, 3, 3, 5)$	∞	$\mathbb{Z}_2^6$	$\mathbb{Z} \oplus \mathbb{Z}_4^5$	
$(1, 1, 2, 3, 3, 5)_2$				
$(1, 2, 2, 2, 3, 5)$				
$(1, 2, 2, 2, 3, 5)_2$				
$(1, 2, 2, 2, 3, 5)_3$				
$(2, 2, 2, 2, 2, 5)$				
$(1, 1, 1, 3, 4, 5)$			$\mathbb{Z}^2 \oplus \mathbb{Z}_4^4$	
$(1, 1, 2, 2, 4, 5)$				

$(1,1,3,3,3,4)$	2^{18}	$\mathbb{Z}_2^6$	$\mathbb{Z}_4^6$	2
$(1,1,3,3,3,4)_2$				
$(1,1,3,3,3,4)_3$				
$(1,1,2,3,4,4)$				
$(1,1,2,3,4,4)_2$				
$(1,1,2,3,4,4)_3$				
$(1,1,2,3,4,4)_4$				
$(1,2,2,3,3,4)$				
$(1,2,2,3,3,4)_2$				
$(1,2,2,3,3,4)_3$				
$(1,2,2,3,3,4)_4$				
$(1,2,2,3,3,4)_5$				
$(1,2,2,3,3,4)_6$				
$(1,2,2,3,3,4)_7$				
$(1,2,2,3,3,4)_8$				
$(1,2,2,3,3,4)_9$				
$(1,2,2,3,3,4)_{10}$				
$(1,2,2,3,3,4)_{11}$				
$(1,2,2,3,3,4)_{12}$				
$(1,2,2,2,4,4)$				
$(1,2,2,2,4,4)_2$				
$(1,2,2,2,4,4)_3$				
$(1,2,3,3,3,3)$				
$(1,2,3,3,3,3)_2$				
$(1,2,3,3,3,3)_3$				
$(1,2,3,3,3,3)_4$				
$(2,2,2,3,3,3)$				
$(2,2,2,3,3,3)_2$				
$(2,2,2,3,3,3)_3$				
$(2,2,2,3,3,3)_4$				
$(2,2,2,3,3,3)_5$				
$(2,2,2,2,3,4)$				
$(2,2,2,2,3,4)_2$				
$(2,2,2,2,3,4)_3$				
$(2,2,2,2,3,4)_4$				
$(1,1,3,3,3,4,6)$	∞	$\mathbb{Z}_2^7$	$\mathbb{Z} \oplus \mathbb{Z}_4^6$	
$(1,1,1,3,4,5,6)$			$\mathbb{Z}^2 \oplus \mathbb{Z}_4^5$	
$(1,2,3,3,3,4,5)$	2^{21}	$\mathbb{Z}_2^7$	$\mathbb{Z}_4^7$	2
$(0,3,3,3,3,4,5)$		$\mathbb{Z}_2^6 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^6$	

$(0, 1, 2, 3, 5, 5, 5)$ $(0, 2, 2, 2, 5, 5, 5)$	$2^{23} \cdot 7$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(1, 1, 2, 2, 5, 5, 5)$ $(1, 1, 1, 3, 5, 5, 5)$	$2^{23} \cdot 7$	$\mathbb{Z}_2^7$	$\mathbb{Z}_4^5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(2, 2, 3, 3, 3, 4, 5, 6)$				
$(0, 1, 2, 3, 4, 6, 6, 6)$	$2^{26} \cdot 7$	$\mathbb{Z}_2^7 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^5 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(1, 2, 3, 4, 4, 4, 5, 6, 7)$				
$(2, 2, 2, 3, 3, 3, 7, 7, 7)$	$2^{29} \cdot 7$	$\mathbb{Z}_2^9$	$\mathbb{Z}_4^7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(0, 1, 2, 3, 4, 5, 7, 7, 7)$	$2^{29} \cdot 7$	$\mathbb{Z}_2^8 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^6 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(0, 1, 2, 3, 4, 5, 6, 8, 8, 8)$	$2^{32} \cdot 7$	$\mathbb{Z}_2^9 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(0, 1, 2, 3, 4, 5, 6, 7, 9, 9, 9)$	$2^{35} \cdot 7$	$\mathbb{Z}_2^{10} \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^7 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2
$(0, 1, 2, 3, 4, 5, 6, 7, 8, 10, 10, 10)$	$2^{38} \cdot 7$	$\mathbb{Z}_2^{11} \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4^8 \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$	2

We now state some lemmas and conjectures that were motivated by the table.

Lemma 6.5.1. *Let* $R(a, b) = ab^{-1}ab^{-3}$*. If* Γ_1, Γ_2 *are non-isomorphic* 5*-vertex tournaments which do not have sinks and do not have* 3 *vertices of out-degree* 3*, then* $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ *and* $G_{\Gamma_1}(R) \not\cong G_{\Gamma_2}(R)$*.*

Proof. There are 7 non-isomorphic tournaments with 5 vertices which do not have sinks and do not have 3 vertices of out-degree 3. By Table 6.3 on page 134 the orders of the corresponding digraph groups are all equal to 2^{15}. We will give a computational proof here that the corresponding digraph groups are pairwise non-isomorphic. We compare abelianization of low index subgroups to understand whether the digraph groups are isomorphic or not. We use the command *IdGroup(G)* in *GAP*.

Let $(I), (II), (III), (IV), (V), (VI), (VII)$ denote the tournaments with their score vectors as $(1, 2, 2, 2, 3)_1, (1, 2, 2, 2, 3)_2, (1, 2, 2, 2, 3)_3, (1, 1, 2, 3, 3)_1$, $(1, 1, 2, 3, 3)_2$, $(2, 2, 2, 2, 2), (0, 2, 2, 3, 3)$, respectively.

By studying their index 2 subgroups we see that $G_{(I)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 192]$ whereas $G_{(II)}(R)$, $G_{(III)}(R)$, $G_{(IV)}(R)$, $G_{(V)}(R)$, $G_{(VI)}(R)$ do not and $G_{(VI)}(R)$ has $[64, 246]$ whereas $G_{(I)}(R)$ does not. Thus, $G_{(I)}(R)$ is not isomorphic to any of $G_{(II)}(R)$, $G_{(III)}(R)$, $G_{(IV)}(R)$, $G_{(V)}(R)$, $G_{(VI)}(R)$, $G_{(VII)}(R)$.

By studying their index 2 subgroups we see that $G_{(II)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 246]$ whereas $G_{(VI)}(R)$ do not. $G_{(IV)}(R), G_{(V)}(R), G_{(VII)}(R)$ have an index 2 subgroup whose abelianization is the group $[128, 2301]$, $[128, 1601]$ and $[128, 2150]$, respectively, whereas $G_{(II)}(R)$ does not. Thus, $G_{(II)}(R)$ is not isomorphic to any of $G_{(IV)}(R)$, $G_{(V)}(R)$, $G_{(VI)}(R)$, $G_{(VII)}(R)$. By studying their index 4 subgroups we see that $G_{(II)}(R)$ has an index 4 subgroup whose abelianization is the group $[64, 55]$ whereas $G_{(III)}(R)$ does not. Thus, $G_{(II)}(R)$ is not isomorphic to any of $G_{(III)}(R)$, $G_{(IV)}(R)$, $G_{(V)}(R)$, $G_{(VI)}(R)$, $G_{(VII)}(R)$.

By studying their index 2 subgroups we see that $G_{(III)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 246]$ whereas $G_{(VI)}(R)$ does not. $G_{(IV)}(R), G_{(V)}(R), G_{(VII)}(R)$ have an index 2 subgroup whose abelianization is the group $[128, 2301]$, $[128, 1601]$ and $[128, 2150]$, respectively, whereas $G_{(III)}(R)$ does not. Thus, $G_{(III)}(R)$ is not isomorphic to any of $G_{(IV)}(R)$, $G_{(V)}(R)$, $G_{(VI)}(R)$, $G_{(VII)}(R)$.

By studying their index 2 subgroups we see that $G_{(IV)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 2301]$ whereas $G_{(V)}(R)$, $G_{(VI)}(R)$, $G_{(VII)}(R)$ do not. Thus, $G_{(IV)}(R)$ is not isomorphic to any of $G_{(V)}(R)$, $G_{(VI)}(R)$, $G_{(VII)}(R)$.

By studying their index 2 subgroups we see that $G_{(V)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 1601]$ whereas $G_{(VI)}(R)$, $G_{(VII)}(R)$ do not. Thus, $G_{(V)}(R)$ is not isomorphic to any of $G_{(VI)}(R)$, $G_{(VII)}(R)$.

By studying their index 2 subgroups we see that $G_{(VII)}(R)$ has an index 2 subgroup whose abelianization is the group $[64, 260]$ whereas $G_{(VI)}(R)$ does not. Thus, $G_{(VI)}(R)$ is not isomorphic to $G_{(VII)}(R)$.

Thus, corresponding digraph groups of these 7 tournaments are non-isomorphic though the corresponding digraph groups have same order.

□

Lemma 6.5.2. *Let $R(a, b) = ab^{-1}ab^{-3}$. If Γ_1, Γ_2 are non-isomorphic 6-vertex tournaments which have 3 vertices of out-degree 4, then $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ and $G_{\Gamma_1}(R) \ncong G_{\Gamma_2}(R)$.*

Proof. There are 2 non-isomorphic tournaments with 6 vertices which have no sinks and have 3 vertices of out-degree 4. By Table 6.3 on page 134 the orders of the corresponding digraph groups are all equal to $2^{20} \cdot 7$. We will give a computational proof here that the corresponding digraph groups are pairwise non-isomorphic. We compare abelianization of low index subgroups to understand whether the digraph groups are isomorphic or not. We use the command *IdGroup(G)* in *GAP*.

Let $(I), (II)$ denote the tournaments with score vectors $(1, 1, 1, 4, 4, 4)$, $(0, 1, 2, 4, 4, 4)$, respectively.

By studying their index 2 subgroups we see that $G_{(II)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 2150]$ whereas $G_{(I)}(R)$ does not. Thus, $G_{(I)}(R)$ is not isomorphic to $G_{(II)}(R)$. It is also seen that G/G' are different which means the $G_\Gamma(R)$ is not isomorphic to each other. □

Lemma 6.5.3. *Let $R(a, b) = ab^{-1}ab^{-3}$. If Γ_1, Γ_2 are non-isomorphic 6-vertex tournaments which do not have sinks, do not have sources and do not have 3 vertices of out-degree 4, then $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ and $G_{\Gamma_1}(R) \ncong G_{\Gamma_2}(R)$.*

Proof. There are 35 non-isomorphic tournaments with 6 vertices which do not have sinks, do not have sources and do not have 3 vertices of out-degree 4. By Table 6.10 on page 149 the orders of the corresponding digraph groups are all equal to 2^{18}.

Let $(1), (2), \ldots, (35)$ denote the tournaments with score vectors in the order of the Table 6.11 on page 156 as $(1, 1, 3, 3, 3, 4), (1, 1, 3, 3, 3, 4)_2, \ldots, (2, 2, 2, 2, 3, 4)_4$, respectively.

By Table 6.11 (abelianization of index subgroups 4), we have that

$(5), (8), (20)$ are identical to each other.

$(9), (16)$ are identical to each other.

$(10), (12)$ are identical to each other.

$(11), (34)$ are identical to each other.

$(13), (26)$ are identical to each other.

Therefore it is not possible to distinguish the corresponding groups. However all other tournaments unique and so groups corresponding to these tournaments are pairwise not isomorphic. Note that we use identical if the rows in the Table 6.11 are exactly same to each other and if a row in the Table 6.11 does not same with others, then we say it is unique. Also note that if the rows are identical, then we cannot decide whether it is isomorphic or not but if it is unique then we say that it is not isomorphic to others (to decide the rows are identical with some of the other rows such as $(13), (26)$ or unique such as 1, see the Table 6.11). Now we investigate those identical above for other index.

By studying their index 2 subgroups we see that $G_{(5)}(R)$ has an index 2 subgroup whose abelianization is the group $[256, 56069]$ whereas $G_{(8)}(R)$, $G_{(20)}(R)$ do not. Also, $G_{(20)}(R)$has an index 2 subgroup whose abelianization is the group $[128, 2301]$ whereas $G_{(8)}(R)$ does not. Thus, $G_{(5)}(R)$, $G_{(8)}(R)$ and $G_{(20)}(R)$ are not isomorphic to each other.

By studying their index 2 subgroups we see that $G_{(16)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 2301]$ whereas $G_{(9)}(R)$ does not. Thus, $G_{(9)}(R)$ is not isomorphic to $G_{(16)}(R)$.

By studying their index 8 subgroups we see that $G_{(12)}(R)$ has an index 8 subgroup whose abelianization is the group $[256, 56059]$ whereas $G_{(10)}(R)$ does not. Thus, $G_{(10)}(R)$ is not isomorphic to $G_{(12)}(R)$.

By studying their index 2 subgroups we see that $G_{(34)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 2150]$ whereas $G_{(11)}(R)$ does not. Thus, $G_{(11)}(R)$ is not isomorphic to $G_{(34)}(R)$.

By studying their index 2 subgroups we see that $G_{(13)}(R)$ has an index 2 subgroup whose abelianization is the group $[128, 2301]$ whereas $G_{(26)}(R)$ does not. Thus, $G_{(13)}(R)$ is not isomorphic to $G_{(26)}(R)$.

Thus, corresponding digraph groups of these 35 tournaments are non-isomorphic though the corresponding digraph groups have same order.

□

We can say that the derived length of $G_\Gamma(R)$, when this is finite, is equal to 2 for all n-vertex tournaments up to $n = 6$. Thus, $G_\Gamma(R)$ is solvable for all n-vertex tournaments up to $n = 6$ if $G_\Gamma(R)$ is finite. $G_\Gamma(R)$ for all n-vertex tournaments up to $n = 6$ is also not abelian since derived length is not 1. It is well known that if the group is not abelian, then it is not cyclic. Thus,

By Index 2	[16,14]	[32,45]	[32,51]	[64,192]	[64,246]	[64,260]	[64,267]	[128,997]	[128,1601]	[128,2150]	[128,2301]	[128,2319]	[128,2328]	[256,13313]	[256,26973]	[256,53038]	[256,55626]	[256,56059]	[256,10298]
(1)=(1,1,3,3,3,4)	✗	✗	✓	✓	✓	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✓	✓	✓	✗
(2)=(1,1,3,3,3,4)$_2$	✗	✓	✓	✗	✓	✓	✓	✗	✓	✓	✗	✓	✗	✗	✗	✗	✓	✓	✗
(3)=(1,1,3,3,3,4)$_3$	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗	✓	✓	✓	✓	✗
(4)=(1,1,2,3,4,4)	✗	✗	✓	✗	✗	✓	✓	✗	✗	✓	✗	✓	✗	✗	✗	✗	✓	✗	✗
(5)=(1,1,2,3,4,4)$_2$	✗	✗	✓	✗	✗	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✗	✓	✗	✗
(6)=(1,1,2,3,4,4)$_3$	✗	✗	✓	✗	✗	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✓	✓	✗	✗
(7)=(1,1,2,3,4,4)$_4$	✗	✗	✓	✗	✗	✓	✓	✗	✗	✓	✗	✓	✗	✗	✗	✓	✓	✗	✗
(8)=(1,2,2,3,3,4)	✗	✗	✓	✗	✗	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✗	✓	✗	✗
(9)=(1,2,2,3,3,4)$_2$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
(10)=(1,2,2,3,3,4)$_3$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓	✗	✗
(11)=(1,2,2,3,3,4)$_4$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✗
(12)=(1,2,2,3,3,4)$_5$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓	✗	✗
(13)=(1,2,2,3,3,4)$_6$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓	✓	✗
(14)=(1,2,2,3,3,4)$_7$	✗	✗	✓	✓	✗	✓	✓	✗	✓	✓	✗	✓	✗	✗	✓	✓	✓	✗	✗
(15)=(1,2,2,3,3,4)$_8$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✗	✓	✗
(16)=(1,2,2,3,3,4)$_9$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓
(17)=(1,2,2,3,3,4)$_1$0	✗	✓	✓	✓	✗	✓	✓	✗	✓	✓	✗	✓	✗	✗	✗	✗	✗	✗	✗
(18)=(1,2,2,3,3,4)$_1$1	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✗
(19)=(1,2,2,3,3,4)$_1$2	✗	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗
(20)=(1,2,2,2,4,4)	✗	✗	✓	✗	✗	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✗	✓	✗	✗
(21)=(1,2,2,2,4,4)$_2$	✗	✗	✓	✓	✗	✓	✓	✗	✗	✗	✓	✓	✗	✗	✗	✗	✗	✓	✗
(22)=(1,2,2,2,4,4)$_3$	✗	✗	✓	✓	✗	✓	✓	✗	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗
(23)=(1,2,3,3,3,3)	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✗	✓	✓	✗	✗
(24)=(1,2,3,3,3,3)$_2$	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✗	✓	✗	✓	✓	✓	✗
(25)=(1,2,3,3,3,3)$_3$	✓	✓	✓	✗	✓	✓	✓	✗	✓	✓	✓	✓	✗	✗	✗	✗	✗	✗	✗
(26)=(1,2,3,3,3,3)$_4$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓	✓	✗
(27)=(2,2,2,3,3,3)	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✗	✓	✓
(28)=(2,2,2,3,3,3)$_2$	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✗	✓	✗	✗	✗
(29)=(2,2,2,3,3,3)$_3$	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✗	✓	✗	✓	✗	✗	✓
(30)=(2,2,2,3,3,3)$_4$	✗	✗	✓	✓	✗	✓	✓	✗	✗	✓	✓	✓	✗	✗	✓	✗	✗	✗	✗
(31)=(2,2,2,3,3,3)$_5$	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓	✓	✗	✗	✗	✗	✗	✗
(32)=(2,2,2,2,3,4)	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✓
(33)=(2,2,2,2,3,4)$_2$	✗	✗	✓	✗	✗	✓	✓	✗	✓	✓	✗	✓	✗	✗	✗	✗	✗	✗	✗
(34)=(2,2,2,2,3,4)$_3$	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓	✗	✓	✗	✓	✗
(35)=(2,2,2,2,3,4)$_4$	✓	✓	✓	✓	✓	✓	✓	✗	✓	✓	✓	✓	✓	✗	✗	✓	✗	✗	✗

Table 6.11: Abelianization of index 4 subgroups

we can say that $G_\Gamma(R)$ for all n-vertex tournaments up to $n = 6$ is also non cyclic. It seems that it also holds for $7 \leqslant n \leqslant 12$ in our limited examples (see the Table 6.10 on page 149).

Conjecture 6.5.4. *Let* $R(a, b) = ab^{-1}ab^{-3}$ *and let* Γ *be an* n*-vertex tournament* $(n \geqslant 3)$ *which has 3 vertices of out-degree* $n-2$*. Then* $|G_\Gamma(R)| = 2^{3n+2}\cdot 7$*, and if there is also a sink, then*

$G/G' \cong \mathbb{Z}_2^{n-1} \oplus \mathbb{Z}_4$ *and* $G'/G'' \cong \mathbb{Z}_2 \oplus \mathbb{Z}_4^{n-3} \oplus \mathbb{Z}_7 \oplus \mathbb{Z}_8^2$.

We confirmed that it is true for $3 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament $(7 \leqslant n \leqslant 12)$ by using *GAP*.

Conjecture 6.5.5. *Let* $R(a, b) = ab^{-1}ab^{-3}$ *and let* Γ *be an* n*-vertex tournament* $(n \geqslant 3)$ *which does not have a source and does not have a sink and does not have* 3 *vertices of out-degree* $n - 2$*. Then* $|G_\Gamma(R)| = 2^{3n}$, $G/G' \cong \mathbb{Z}_2^n$ *and* $G'/G'' \cong \mathbb{Z}_4^n$.

We confirmed that it is true for $3 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament $(7 \leqslant n \leqslant 12)$ by using *GAP*.

6.6 The relator $R(a,b) = abab^3$: groups of order $2^t \cdot 3^s$

In this section, we investigate a new fixed word $R(a,b) = abab^3$ for n-vertex tournaments with $n \leqslant 12$.

Table 6.12: All possible n-vertex tournaments for $3 \leqslant n \leqslant 6$ and some when $7 \leqslant n \leqslant 12$

$R(a,b) = abab^3$				
Score Vectors	$\|G\|$	G/G'	G'/G''	DL
$(0,1,2)$	∞	$\mathbb{Z}_2^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2$	
$(1,1,1)$	$2^{11} \cdot 3^2$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_9$	$\mathbb{Z}_4 \oplus \mathbb{Z}_8^2$	2
$(0,1,2,3)$	∞	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	
$(1,1,1,3)$		$\mathbb{Z}_2^4 \oplus \mathbb{Z}_3$	$\mathbb{Z} \oplus \mathbb{Z}_4^3$	
$(0,2,2,2)$	$2^{14} \cdot 3^3$	$\mathbb{Z}_2^3 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_4 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$	2
$(1,1,2,2)$	$2^{12} \cdot 3$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4^4$	
$(0,1,2,3,4)$	∞	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_4^2$	
$(0,2,2,2,4)$			$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^3$	
$(1,1,1,3,4)$		$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_4^3$	
$(1,1,2,2,4)$			$\mathbb{Z} \oplus \mathbb{Z}_4^4$	
$(0,1,3,3,3)$	$2^{17} \cdot 3^4$	$\mathbb{Z}_2^4 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^2 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$	2
$(0,2,2,3,3)$	$2^{15} \cdot 3^2$		$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^4$	
$(1,1,2,3,3)$ $(1,1,2,3,3)_2$ $(1,2,2,2,3)$ $(1,2,2,2,3)_2$ $(1,2,2,2,3)_3$ $(2,2,2,2,2)$	$2^{15} \cdot 3$	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4^5$	2
$(0,1,2,3,4,5)$	∞	$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_4^3$	
$(0,1,3,3,3,5)$			$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_4^4$	
$(0,2,2,2,4,5)$			$\mathbb{Z}^2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^3$	
$(0,2,2,3,3,5)$			$\mathbb{Z} \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^4$	
$(1,1,1,3,4,5)$		$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^4$	
$(1,1,2,2,4,5)$			$\mathbb{Z}^2 \oplus \mathbb{Z}_4^4$	

<table>
<tr><td>$(1,1,2,3,3,5)$
$(1,1,2,3,3,5)_2$
$(1,2,2,2,3,5)$
$(1,2,2,2,3,5)_2$
$(1,2,2,2,3,5)_3$
$(2,2,2,2,2,5)$</td><td>∞</td><td>$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3$</td><td>$\mathbb{Z} \oplus \mathbb{Z}_4^5$</td><td></td></tr>
<tr><td>$(0,1,2,4,4,4)$</td><td>$2^{20} \cdot 3^5$</td><td rowspan="2">$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$</td><td>$\mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_4^3 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$</td><td rowspan="2">2</td></tr>
<tr><td>$(0,1,3,3,4,4)$</td><td>$2^{18} \cdot 3^3$</td><td>$\mathbb{Z}_2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_4^5$</td></tr>
<tr><td>$(0,2,2,3,4,4)$
$(0,2,2,3,4,4)_2$
$(0,2,3,3,3,4)$
$(0,2,3,3,3,4)_2$
$(0,2,3,3,3,4)_3$
$(0,3,3,3,3,3)$</td><td>$2^{18} \cdot 3^2$</td><td>$\mathbb{Z}_2^5 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$</td><td>$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^5$</td><td>2</td></tr>
<tr><td>$(1,1,1,4,4,4)$</td><td>$2^{20} \cdot 3^3$</td><td>$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3$</td><td>$\mathbb{Z}_4^4 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$</td><td>2</td></tr>
</table>

$(1,1,3,3,3,4)$ $(1,1,3,3,3,4)_2$ $(1,1,3,3,3,4)_3$ $(1,1,2,3,4,4)$ $(1,1,2,3,4,4)_2$ $(1,1,2,3,4,4)_3$ $(1,1,2,3,4,4)_4$ $(1,2,2,3,3,4)$ $(1,2,2,3,3,4)_2$ $(1,2,2,3,3,4)_3$ $(1,2,2,3,3,4)_4$ $(1,2,2,3,3,4)_5$ $(1,2,2,3,3,4)_6$ $(1,2,2,3,3,4)_7$ $(1,2,2,3,3,4)_8$ $(1,2,2,3,3,4)_9$ $(1,2,2,3,3,4)_{10}$ $(1,2,2,3,3,4)_{11}$ $(1,2,2,3,3,4)_{12}$ $(1,2,2,2,4,4)$ $(1,2,2,2,4,4)_2$ $(1,2,2,2,4,4)_3$ $(1,2,3,3,3,3)$ $(1,2,3,3,3,3)_2$ $(1,2,3,3,3,3)_3$ $(1,2,3,3,3,3)_4$ $(2,2,2,3,3,3)$ $(2,2,2,3,3,3)_2$ $(2,2,2,3,3,3)_3$ $(2,2,2,3,3,3)_4$ $(2,2,2,3,3,3)_5$ $(2,2,2,2,3,4)$ $(2,2,2,2,3,4)_2$ $(2,2,2,2,3,4)_3$ $(2,2,2,2,3,4)_4$	$2^{18} \cdot 3$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4^6$	2
$(1,2,3,3,3,4,5)$	$2^{21} \cdot 3$	$\mathbb{Z}_2^7 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4^7$	2
$(0,3,3,3,3,4,5)$	$2^{21} \cdot 3^2$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^6$	2
$(1,1,1,4,4,5,5)$	$2^{21} \cdot 3^2$	$\mathbb{Z}_2^7 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_4^7$	2
$(0,1,2,4,4,5,5)$	$2^{21} \cdot 3^4$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_4^6$	2

$(0,2,2,2,5,5,5)$	$2^{23} \cdot 3^4$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$	2
$(0,1,2,3,5,5,5)$	$2^{23} \cdot 3^6$	$\mathbb{Z}_2^6 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_4^4 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$	2
$(1,1,1,3,4,5,6)$	∞	$\mathbb{Z}_2^7 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3^2 \oplus \mathbb{Z}_4^5$	
$(1,1,1,3,5,5,5)$	$2^{23} \cdot 3^4$	$\mathbb{Z}_2^7 \oplus \mathbb{Z}_3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_4^5 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$	2
$(2,2,3,3,3,4,5,6)$	$2^{24} \cdot 3$	$\mathbb{Z}_2^8 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4^8$	2
$(0,1,2,3,4,6,6,6)$	$2^{26} \cdot 3^7$	$\mathbb{Z}_2^7 \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$	$\mathbb{Z}_2 \oplus \mathbb{Z}_3^4 \oplus \mathbb{Z}_4^5 \oplus \mathbb{Z}_8^2 \oplus \mathbb{Z}_9$	2
$(1,1,1,3,4,5,6,7)$	∞	$\mathbb{Z}_2^8 \oplus \mathbb{Z}_3$	$\mathbb{Z}^2 \oplus \mathbb{Z}_3^3 \oplus \mathbb{Z}_4^6$	
$(1,2,3,4,4,4,5,6,7)$	$2^{27} \cdot 3$	$\mathbb{Z}_2^9 \oplus \mathbb{Z}_3$	$\mathbb{Z}_4^9$	2

We now state some lemmas and conjectures that were motivated by the table.

Lemma 6.6.1. *Let* $R(a,b) = abab^3$*. If* Γ_1, Γ_2 *are non-isomorphic* 5*-vertex tournaments which have no sources and no sinks, then* $|G_{\Gamma_1}(R)| = |G_{\Gamma_2}(R)|$ *and* $G_{\Gamma_1}(R) \ncong G_{\Gamma_2}(R)$*.*

Proof. There are 6 non-isomorphic tournaments with 5 vertices which do not have have sources and sinks. By Table 6.3 on page 134. the orders of the corresponding digraph groups are all equal to $2^{15} \cdot 3$. We will give a computational proof here that the corresponding digraph groups are pairwise non-isomorphic. We compare abelianization of low index subgroups to understand whether the digraph groups are isomorphic or not. We use the command *IdGroup(G)* in *GAP*.

Let $(1),(2),(3),(4),(5),(6)$ denote the tournaments with score vectors

$(1,2,2,2,3),(1,2,2,2,3)_2,(1,2,2,2,3)_3,(1,1,2,3,3),(1,1,2,3,3)_2,$
$(2,2,2,2,2)$, respectively. Note that we use identical if the columns in the Table 6.13 on page 161 are exactly same to each other and if a column in the Table 6.13 does not same with others, then we say it is unique. Also note that if the related columns are identical, then we cannot decide whether it is isomorphic or not but if it is unique then we say that it is not isomorphic to others (to decide the columns are identical or unique, check the Table 6.13)

By Table 6.13 (abelianization of index subgroups 2), we have that $(2),(3)$ are identical to each other. Others are not isomorphic to each other since they are unique.

Table 6.13: Abelianization of index 2 subgroups

By Index 2	(1)	(2)	(3)	(4)	(5)	(6)
[48,52]	✓	✓	✓	✓	✓	✓
[96,220]	✓	✓	✓	✓	✓	✓
[96,231]	✓	✓	✓	✓	✓	✓
[192,1400]	✓	✗	✗	✗	✗	✗
[192,1454]	✗	✓	✓	✓	✓	✗
[192,1530]	✓	✓	✓	✓	✓	✗
[384,17309]	✗	✗	✗	✗	✓	✗
[384,20029]	✗	✗	✗	✓	✗	✗

By studying their index 5 subgroups we see that $G_{(2)}(R)$ has an index 5 subgroup whose abelianization is the group $[192, 807]$ whereas $G_{(3)}(R)$ does not. Thus, $G_{(2)}(R)$ is not isomorphic to $G_{(3)}(R)$.

Thus, corresponding digraph groups of these 6 tournaments are non-isomorphic though the corresponding digraph groups have same order.

□

Conjecture 6.6.2. *Let $R(a,b) = abab^3$ and let Γ be an n-vertex tournament $(n \geqslant 3)$ which has a source. Then $G_\Gamma(R)$ is infinite.*

We confirmed that it is true for $3 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament $(7 \leqslant n \leqslant 12)$ by using *GAP*.

Conjecture 6.6.3. *Let $R(a,b) = abab^3$, suppose that Γ be an n-vertex tournament $(n \geqslant 3)$ which does not have a source and let $G = G_\Gamma(R)$, then G is finite and the derived length of G is 2 (and so G is solvable, not abelian and not cyclic).*

We confirmed that it is true for $3 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament $(7 \leqslant n \leqslant 12)$ by using *GAP*.

Conjecture 6.6.4. *Let $R(a,b) = abab^3$ and let Γ be an n-vertex tournaments $(n \geqslant 3)$.*

If Γ has a sink, then $G/G' \cong \mathbb{Z}_2^{n-1} \oplus \mathbb{Z}_3 \oplus \mathbb{Z}_4$.

If Γ does not have a sink, then $G/G' \cong \mathbb{Z}_2^n \oplus \mathbb{Z}_3$.

We confirmed that it is true for $4 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament ($7 \leqslant n \leqslant 12$) by using *GAP*.

Question 6.6.5. *Let $R(a,b) = abab^3$. There are 6-vertex tournaments such that the corresponding group has order $2^{18} \cdot 3^2$. Are each pair of these groups isomorphic ?*

There are also 35 possible 6-vertex tournaments such that the corresponding group has order $2^{18} \cdot 3$. Are each pair of these groups isomorphic ?

We used low index subgroups to prove that in previous sections but we are unable to confirm it here since this technique does not work with this word efficiently.

6.7 The relator $R(a,b) = abab^{-2}$: perfect groups

In this section, we investigate a new fixed word $R(a,b) = abab^{-2}$ for n-vertex tournaments with $n \leqslant 12$. We find out perfect groups with this new fixed word. Some of these groups will be a group called the *double cover of the alternating group* A_n which is denoted $2 \cdot A_n$. The definition of this group (which is unimportant for our purposes) is given in [32], but we note that $2 \cdot A_4 \cong SL(2,3), 2 \cdot A_5 \cong SL(2,5), 2 \cdot A_6 \cong SL(2,9)$. The number of perfect groups of a given order may be found using the *NumberPerfectGroups* command in *GAP*. There is only one perfect group of order 720, namely $SL(2,9)$ or *Double cover of A6* and there is only one perfect group of order 5040, namely *Double cover of A7*. We have this perfect group with 4-vertex tournaments and 5-vertex tournaments respectively. There are 4 perfect groups with order 40320 and one of these arises as a digraph group where the digraph is a 6-vertex tournament as the group $2 \cdot A8$. There are 6 perfect groups with order 362880 and one of these arises as a digraph group where the digraph is a 7-vertex tournaments as the group $2 \cdot A9$. We will also give a computational proof if Γ is a 6-vertex tournament without a source then $G_\Gamma(R) \cong 2 \cdot A8$, which is a perfect group in Lemma 6.7.1.

Table 6.14: All possible n-vertex tournaments for $3 \leqslant n \leqslant 6$ and some when $7 \leqslant n \leqslant 12$

$R(a,b) = abab^{-2}$					
Score Vectors	$\|G\|$	Perfect	G/G'	G'/G''	G
$(0,1,2)$	∞	No	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$G''/G''' = \mathbb{Z}_2^4$ $G'''/G'''' = \mathbb{Z}^9 \oplus \mathbb{Z}_2$
$(1,1,1)$	840	No	$\mathbb{Z}_7$	1	$\mathbb{Z}_7 \times SL(2,5)$
$(0,1,2,3)$ $(1,1,1,3)$	?	No	$\mathbb{Z}_2$	1	
$(0,2,2,2)$ $(1,1,2,2)$	6!	Yes	1		$2 \cdot A6 = SL(2,9)$
$(0,1,2,3,4)$ $(0,2,2,2,4)$ $(1,1,1,3,4)$ $(1,1,2,2,4)$	?	No	$\mathbb{Z}_2$	1	
$(0,1,3,3,3)$ $(0,2,2,3,3)$ $(1,1,2,3,3)$ $(1,1,2,3,3)_2$ $(1,2,2,2,3)$ $(1,2,2,2,3)_2$ $(1,2,2,2,3)_3$ $(2,2,2,2,2)$	7!	Yes	1		$2 \cdot A7$
$(0,1,2,3,4,5)$ $(0,1,3,3,3,5)$ $(0,2,2,2,4,5)$ $(0,2,2,3,3,5)$ $(1,1,1,3,4,5)$ $(1,1,2,2,4,5)$ $(1,1,2,3,3,5)$ $(1,1,2,3,3,5)_2$ $(1,2,2,2,3,5)$ $(1,2,2,2,3,5)_2$ $(1,2,2,2,3,5)_3$ $(2,2,2,2,2,5)$	?	No	$\mathbb{Z}_2$	1	

$(0,1,3,3,4,4)$ $(0,1,2,4,4,4)$ $(0,2,2,3,4,4)$ $(0,2,2,3,4,4)_2$ $(0,2,3,3,3,4)$ $(0,2,3,3,3,4)_2$ $(0,2,3,3,3,4)_3$ $(0,3,3,3,3,3)$ $(1,1,1,4,4,4)$ $(1,1,3,3,3,4)$ $(1,1,3,3,3,4)_2$ $(1,1,3,3,3,4)_3$ $(1,1,2,3,4,4)$ $(1,1,2,3,4,4)_2$ $(1,1,2,3,4,4)_3$ $(1,1,2,3,4,4)_4$ $(1,2,2,3,3,4)$ $(1,2,2,3,3,4)_2$ $(1,2,2,3,3,4)_3$ $(1,2,2,3,3,4)_4$ $(1,2,2,3,3,4)_5$ $(1,2,2,3,3,4)_6$ $(1,2,2,3,3,4)_7$ $(1,2,2,3,3,4)_8$ $(1,2,2,3,3,4)_9$ $(1,2,2,3,3,4)_{10}$ $(1,2,2,3,3,4)_{11}$ $(1,2,2,3,3,4)_{12}$ $(1,2,2,2,4,4)$ $(1,2,2,2,4,4)_2$ $(1,2,2,2,4,4)_3$ $(1,2,3,3,3,3)$ $(1,2,3,3,3,3)_2$ $(1,2,3,3,3,3)_3$ $(1,2,3,3,3,3)_4$ $(2,2,2,3,3,3)$ $(2,2,2,3,3,3)_2$ $(2,2,2,3,3,3)_3$ $(2,2,2,3,3,3)_4$ $(2,2,2,3,3,3)_5$	$8!$	Yes	1		$2 \cdot A8$

$(2,2,2,2,3,4)$ $(2,2,2,2,3,4)_2$ $(2,2,2,2,3,4)_3$ $(2,2,2,2,3,4)_4$	8!	Yes	1		$2 \cdot A8$
$(1,2,3,3,3,4,5)$	9!	Yes	1		$2 \cdot A9$
$(0,3,3,3,3,4,5)$	9!	Yes	1		$2 \cdot A9$
$(2,2,3,3,3,4,5,6)$	10!	Yes	1		$2 \cdot A10$
$(1,1,1,4,4,4,6,7)$	10!	Yes	1		$2 \cdot A10$
$(1,1,1,3,4,5,6,7)$	10!	Yes	1		$2 \cdot A10$
$(1,2,3,4,4,4,5,6,7)$	11!	Yes	1		$2 \cdot A11$
$(1,2,3,4,4,5,5,6,7,8)$	12!	Yes	1		$2 \cdot A12$

We now state some lemmas and conjectures that were motivated by the table.

Lemma 6.7.1. *Let $R(a,b) = abab^{-2}$. If Γ is a $6-$vertex tournament without a source, then $G_\Gamma(R) \cong 2 \cdot A8$, which is a perfect group.*

Proof. There are 44 non-isomorphic tournaments with 6 vertices which do not have sources. By Table 6.14 on page 163 , the orders of the corresponding digraph groups are all equal to 8!.

The *GAP* code works by counting the number of subgroups of the possible groups up to given index. Let G be our group and L be a list of Low index subgroups which mean an algorithm for finding all subgroups of up to a given index in a finitely presented group G. Firstly, we see our group has index 8 subgroup by typing *L:=LowIndexSubgroupsFpGroup(G,8)* into *GAP* and then *Index(G,L[2])* into *GAP*. By this code, G has a proper subgroup of index 8, but G does not have proper subgroups in any smaller index. We know there are 4 perfect groups with the order of 40320 by typing *Number-PerfectGroups(40320)* into *GAP*. Now, we called *PerfectGroup(40320,1)* as $G1$, *PerfectGroup(40320,2)* as $G2$, *PerfectGroup(40320,3)* as $G3$ and *Perfect-Group(40320,4)* as $G4$, namely $G1 = 272\ L3(2)$, $G2 = 2^4\ A7$, $G3 = 2\ A8$ and $G4 = 2\ L3(4)$.

Now, we need to check whether the groups have a proper subgroup of index less than 8 or not. If we have it then G is not the group that we investigate. To see for the first perfect group $G1$, we type *L1:=LowIndexSubgroupsFpGroup(G1,7)* into *GAP* and we see $G1$ has a proper subgroup of index less than 8. Thus, $G \neq G1$. In that way, it can be seen $G \neq G2$ and $G \neq G4$. Hence G is $G3$ which is $2 \cdot A8$. By this techniques, we see all 44 tournaments are giving the same perfect groups $2 \cdot A8$. See the related *GAP* codes in Appendix A.4.

□

Conjecture 6.7.2. *Let $R(a, b) = abab^{-2}$, suppose that Γ does not have a source and if*

Γ is an 3-vertex tournament, then $|G_\Gamma(R)| = 840$ and $G = \mathbb{Z}_7 \times SL(2, 5)$

Γ is an n-vertex tournament, $n \geqslant 4$, then $|G_\Gamma(R)| = (n + 2)!$ and $G = 2.\ A(n + 2)$ and these tournaments give the same perfect group although the tournaments are non-isomorphic.

We confirmed that it is true for all n-vertex tournaments when $3 \leqslant n \leqslant 6$ and some when $7 \leqslant n \leqslant 12$ by using *GAP*.

Question 6.7.3. *Let $R(a, b) = abab^{-2}$ be and suppose that Γ has a source, then is $G_\Gamma(R)$ infinite ?*

It is true for only when $n = 3$ but for $n \geqslant 4$ the computations in *GAP* do not complete. I believe this is because the corresponding groups are infinite.

6.8 The relator $R(a, b) = ab^2a^2b^{-2}$: 3-groups

In this section, we investigate a new fixed word $R(a, b) = ab^2a^2b^{-2}$ for n-vertex tournaments with $n \leqslant 8$.

If there is a sink in the graph and $\beta = 0$, then the corresponding digraph groups are infinite by Lemma 6.2.1. Since $\beta = 2 + (-2) = 0$, we have an infinite group whenever we have a sink, which is showed as 0 in the score vector.

Table 6.15: All possible n-vertex tournaments for $3 \leqslant n \leqslant 6$ and some when $n = 7, 8$.

$R(a,b)=ab^2a^2b^{-2}$					
Score Vectors	$\lvert G\rvert$	G/G'	G'/G''	G''/G'''	DL
$(0,1,2)$	∞				
$(1,1,1)$	3^9	$\mathbb{Z}_3^3$	$\mathbb{Z}_3 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3$	3
$(0,1,2,3)$	∞				
$(0,2,2,2)$					
$(1,1,1,3)$	3^{11}	$\mathbb{Z}_3^4$	$\mathbb{Z}_3^2 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3$	3
$(1,1,2,2)$	3^8	$\mathbb{Z}_3^4$	$\mathbb{Z}_3^4$	-	2
$(0,1,2,3,4)$	∞				
$(0,2,2,2,4)$					
$(0,1,3,3,3)$					
$(0,2,2,3,3)$					
$(1,1,1,3,4)$	3^{13}	$\mathbb{Z}_3^5$	$\mathbb{Z}_3^3 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3$	3
$(1,1,2,2,4)$	3^{10}	$\mathbb{Z}_3^5$	$\mathbb{Z}_3^5$	-	2
$(1,1,2,3,3)$					
$(1,1,2,3,3)_2$					
$(1,2,2,2,3)$					
$(1,2,2,2,3)_2$					
$(1,2,2,2,3)_3$					
$(2,2,2,2,2)$					
$(0,1,2,3,4,5)$	∞				
$(0,1,3,3,3,5)$					
$(0,2,2,2,4,5)$					
$(0,2,2,3,3,5)$					
$(0,1,3,3,4,4)$					
$(0,1,2,4,4,4)$					
$(0,2,2,3,4,4)$					
$(0,2,2,3,4,4)_2$					
$(0,2,3,3,3,4)$					
$(0,2,3,3,3,4)_2$					
$(0,2,3,3,3,4)_3$					
$(0,3,3,3,3,3)$					
$(1,1,1,3,4,5)$	3^{15}	$\mathbb{Z}_3^6$	$\mathbb{Z}_3^4 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3$	3
$(1,1,1,4,4,4)$					

$(1,1,2,2,4,5)$ $(1,1,2,3,3,5)$ $(1,1,2,3,3,5)_2$ $(1,2,2,2,3,5)$ $(1,2,2,2,3,5)_2$ $(1,2,2,2,3,5)_3$ $(2,2,2,2,2,5)$ $(1,1,3,3,3,4)$ $(1,1,3,3,3,4)_2$ $(1,1,3,3,3,4)_3$ $(1,1,2,3,4,4)$ $(1,1,2,3,4,4)_2$ $(1,1,2,3,4,4)_3$ $(1,1,2,3,4,4)_4$ $(1,2,2,3,3,4)$ $(1,2,2,3,3,4)_2$ $(1,2,2,3,3,4)_3$ $(1,2,2,3,3,4)_4$ $(1,2,2,3,3,4)_5$ $(1,2,2,3,3,4)_6$ $(1,2,2,3,3,4)_7$ $(1,2,2,3,3,4)_8$ $(1,2,2,3,3,4)_9$ $(1,2,2,3,3,4)_{10}$ $(1,2,2,3,3,4)_{11}$ $(1,2,2,3,3,4)_{12}$ $(1,2,2,2,4,4)$ $(1,2,2,2,4,4)_2$ $(1,2,2,2,4,4)_3$ $(1,2,3,3,3,3)$ $(1,2,3,3,3,3)_2$ $(1,2,3,3,3,3)_3$ $(1,2,3,3,3,3)_4$ $(2,2,2,3,3,3)$ $(2,2,2,3,3,3)_2$ $(2,2,2,3,3,3)_3$ $(2,2,2,3,3,3)_4$ $(2,2,2,3,3,3)_5$ $(2,2,2,2,3,4)$ $(2,2,2,2,3,4)_2$ $(2,2,2,2,3,4)_3$ $(2,2,2,2,3,4)_4$	3^{12}	$\mathbb{Z}_3^6$	$\mathbb{Z}_3^6$	-	2

$(1,2,3,3,3,4,5)$	3^{14}	$\mathbb{Z}_3^7$	$\mathbb{Z}_3^7$	-	2
$(0,3,3,3,3,4,5)$	∞				
$(1,1,1,3,4,5,6)$	3^{17}	$\mathbb{Z}_3^7$	$\mathbb{Z}_3^5 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3$	3
$(1,1,2,3,4,5,6,6)$	3^{16}	$\mathbb{Z}_3^8$	$\mathbb{Z}_3^8$		2
$(1,1,1,3,4,5,6,7)$	3^{19}	$\mathbb{Z}_3^8$	$\mathbb{Z}_3^6 \oplus \mathbb{Z}_9^2$	$\mathbb{Z}_3$	3

We now state some lemmas and conjectures that were motivated by the table. However, we are not able to give lemmas as previous sections in this section since low index subgroups technique does not work with this word efficiently.

Lemma 6.8.1. *Let $R(a,b) = ab^2a^2b^{-2}$ and let Γ be an n-vertex tournament ($n \geqslant 3$) which has a sink. Then $G/G' \cong \mathbb{Z} \oplus \mathbb{Z}_3^{n-1}$,*

This can be seen by similar proof of Theorem 6.4.1.

Conjecture 6.8.2. *Let $R(a,b) = ab^2a^2b^{-2}$, suppose that Γ be an n-vertex tournament ($n \geqslant 3$) which does not have a sink and let $G = G_\Gamma(R)$, then the derived length of G is 2 or 3 (and so G is solvable, not abelian and not cyclic).*

Conjecture 6.8.3. *Let $R(a,b) = ab^2a^2b^{-2}$, and let Γ be an n-vertex tournaments ($n \geqslant 3$) which has no sinks and does not have 3 vertices of out-degree 1. Then, $|G_\Gamma(R)| = 3^{2n}$, $G/G' \cong \mathbb{Z}_3^n$ and $G'/G'' \cong \mathbb{Z}_3^n$.*

Conjecture 6.8.4. *Let $R(a,b) = ab^2a^2b^{-2}$ and let Γ be an n-vertex tournament ($n \geqslant 3$) which has no sinks and has 3 vertices of out-degree 1. Then $|G_\Gamma(R)| = 3^{2n+3}$,$G/G' \cong \mathbb{Z}_3^n$, $G'/G'' \cong \mathbb{Z}_3^{n-2} \oplus \mathbb{Z}_9^2$ and $G''/G''' \cong \mathbb{Z}_3$.*

Each of the three conjectures above has been verified for $3 \leqslant n \leqslant 6$ for all tournaments and for some n-vertex tournament when $n = 7, 8$ by using *GAP*.

Question 6.8.5. *Let $R(a,b) = ab^2a^2b^{-2}$. There are 7 possible 5-vertex tournaments such that the corresponding group has order 3^{10}. There are 2 possible 6-vertex tournaments such that the corresponding group has order 3^{15}. There are 35 possible 6-vertex tournaments such that the corresponding group has order 3^{12}.*

Are each pair of the groups with the same order isomorphic ?

We used low index subgroups to prove that in previous sections but we are unable to confirm it here since this technique does not work with this word efficiently.

APPENDIX A

Appendix: *GAP* Codes

We will provide the related *GAP* codes here.

A.1 Functions

We are giving the functions for all the groups that we used throughout Chapter 6 here. Therefore, readers can easily repeat any experiment from the thesis if they want to.

A.1.1 Tournaments

We give digraph codes for all tournaments up to 6-vertex tournaments and for some from 7 to 12-vertex tournaments here.

```
LoadPackage("digraphs");

T012 :=DigraphByEdges([ [2,1],[3,1],[3,2] ]);
T111 :=DigraphByEdges([ [1,2],[2,3],[3,1] ]);

T0123:=DigraphByEdges([ [1,2],[1,3],[1,4], [2,3], [2,4], [3,4] ]);
```

T1113:=DigraphByEdges([[1,2],[2,3],[3,1], [4,1], [4,2], [4,3]]);
T0222:=DigraphByEdges([[1,2],[1,4],[2,3], [2,4], [3,1], [3,4]]);
T1122:=DigraphByEdges([[1,2],[1,4],[2,3], [3,1], [4,2], [4,3]]);

T01234:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [3,4], [3,5], [4,5]]);
T02224:=DigraphByEdges([[1,2],[1,5],[2,3], [2,5], [3,1], [3,5], [4,1], [4,2], [4,3], [4,5]]);
T11134:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [3,4], [4,5], [5,3]]);
T11224:=DigraphByEdges([[1,2],[2,3],[2,5], [3,1], [4,1], [4,2], [4,3], [4,5], [5,1], [5,3]]);
T01333:=DigraphByEdges([[1,4],[2,1],[2,4], [2,5], [3,1], [3,2], [3,4], [5,1], [5,3], [5,4]]);
T02233:=DigraphByEdges([[1,2],[1,4],[2,3], [2,4], [2,5], [3,1], [3,4], [5,1], [5,3], [5,4]]);
T11233:=DigraphByEdges([[1,2],[1,3],[1,4], [2,3], [2,4], [2,5], [3,4], [3,5], [4,5], [5,1]]);
T11233_2:=DigraphByEdges([[1,2],[1,3],[1,4], [2,3], [2,4], [2,5], [3,5], [4,3], [5,1], [5,4]]);
T12223:=DigraphByEdges([[1,2],[1,3],[1,4], [2,3], [2,4], [3,4], [3,5], [4,5], [5,1], [5,2]]);
T12223_2:=DigraphByEdges([[1,3],[1,4],[2,1], [2,4], [3,2], [3,4], [3,5], [4,5], [5,1], [5,2]]);
T12223_3:=DigraphByEdges([[1,2],[1,3],[1,4], [2,3], [2,5], [3,4], [3,5], [4,2], [4,5], [5,1]]);
T22222:=DigraphByEdges([[1,2],[1,4],[2,4], [2,5], [3,1], [3,2], [4,3], [4,5], [5,1], [5,3]]);

T012345:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,4], [3,5], [3,6], [4,5], [4,6], [5,6]]);
T012444:=DigraphByEdges([[1,3],[1,4],[1,5], [1,6], [2,1], [2,4], [2,5], [2,6], [3,2], [3,4], [3,5], [3,6],[4,5], [4,6], [5,6]]);

T013335:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,5], [2,6], [3,4], [3,5], [3,6], [4,2], [4,5], [4,6], [5,6]]);
T013344:=DigraphByEdges([[1,3],[1,4],[1,5], [1,6], [2,1], [2,3], [2,5], [2,6], [3,4], [3,5], [3,6],[4,2],[4,5], [4,6], [5,6]]);
T022245:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6]]);
T022335:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,4], [2,5], [2,6], [3,2], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6]]);
T022344:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,3], [2,4], [2,5], [2,6], [3,4], [3,5], [3,6], [4,5], [4,6], [5,1], [5,6]]);
T022344_2:=DigraphByEdges([[1,3],[1,4],[1,5], [1,6], [2,1], [2,4], [2,5], [2,6], [3,2], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6]]);
T023334:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,3], [2,4], [2,6], [3,4], [3,5], [3,6], [4,5], [4,6],[5,1], [5,2], [5,6]]);
T023334_2:=DigraphByEdges([[1,3],[1,4],[1,6], [2,1], [2,4], [2,6], [3,2], [3,4], [3,5], [3,6], [4,5], [4,6], [5,1], [5,2], [5,6]]);
T023334_3:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,3], [2,5], [2,6], [3,4], [3,5], [3,6], [4,2], [4,5], [4,6], [5,1], [5,6]]);
T033333:=DigraphByEdges([[1,2],[1,4],[1,6], [2,4], [2,5], [2,6], [3,1], [3,2], [3,6], [4,3], [4,5], [4,6], [5,1], [5,3], [5,6]]);
T111345:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,4], [3,5], [3,6], [4,5], [5,6], [6,4]]);
T111444:=DigraphByEdges([[1,2],[1,4],[1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,1], [3,4], [3,5], [3,6], [4,5], [5,6], [6,4]]);
T112245:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,5], [3,6], [4,3], [4,5], [5,6], [6,4]]);
T112335:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [3,4], [3,5], [3,6], [4,5], [4,6], [5,6], [6,2]]);
T112335_2:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [3,4], [3,5], [3,6],[4,6], [5,4], [6,2], [6,5]]);
T112344:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [2,6], [3,4], [3,5], [3,6], [4,6], [5,4], [6,1], [6,5]]);
T112344_2:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5],

[2,6], [3,4], [3,5], [3,6], [4,5], [4,6], [5,6], [6,1]]);
T112344_3:=DigraphByEdges([[1,2],[1,3],[1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,5], [3,6], [4,1], [4,3], [4,5], [5,6], [6,4]]);
T112344_4:=DigraphByEdges([[1,2],[1,4],[1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,1], [3,5], [3,6], [4,3], [4,5], [5,6], [6,4]]);
T113334:=DigraphByEdges([[1,3],[1,4],[1,5], [1,6], [2,1], [2,3], [2,5], [3,4], [3,5], [3,6], [4,2], [4,5], [4,6], [5,6], [6,2]]);
T113334_2:=DigraphByEdges([[1,3],[1,4],[1,5], [2,1], [2,3], [2,5], [2,6], [3,4], [3,5], [3,6], [4,2], [4,5], [4,6], [5,6], [6,1]]);
T113334_3:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,5], [2,6], [3,4], [3,5], [3,6], [4,2], [4,5], [4,6], [5,6], [6,1]]);
T122235:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [3,4], [3,5], [3,6], [4,5], [4,6], [5,2], [5,6], [6,2]]);
T122235_2:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [3,4], [3,5], [3,6], [4,5], [5,2], [5,6], [6,2], [6,4]]);
T122235_3:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [3,4],[3,6], [4,5], [4,6], [5,3], [5,6], [6,2]]);
T122244:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [2,6], [3,5], [3,6], [4,3], [4,5], [5,6], [6,1], [6,4]]);
T122244_2:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,3], [2,4], [2,5], [2,6], [3,5],[3,6], [4,3], [4,5], [5,1], [5,6], [6,4]]);
T122244_3:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [2,6], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6], [6,1]]);
T122334:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [3,4], [3,5], [3,6], [4,5], [4,6], [5,6], [6,1], [6,2]]);
T122334_2:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,3], [2,4], [2,5], [3,4], [3,5], [3,6], [4,5], [4,6], [5,1], [5,6], [6,2]]);
T122334_3:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,6], [3,4],[3,5], [3,6], [4,5], [5,2], [5,6], [6,1], [6,4]]);
T122334_4:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,3], [2,4], [2,5], [3,4], [3,5], [3,6], [4,5], [5,1], [5,6], [6,2], [6,4]]);
T122334_5:=DigraphByEdges([[1,2],[1,4],[1,5], [1,6], [2,3], [2,4], [2,5], [3,1], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6], [6,2]]);

T122334_6:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,4], [2,5], [2,6], [3,2], [3,4],[3,6], [4,5], [4,6], [5,3], [5,6], [6,1]]);
T122334_7:=DigraphByEdges([[1,2],[1,4],[1,5], [1,6], [2,4], [2,5], [2,6], [3,1],[3,2], [3,4], [4,5], [4,6], [5,3], [5,6], [6,3]]);
T122334_8:=DigraphByEdges([[1,3],[1,4],[1,5], [1,6], [2,1], [2,4], [2,5], [3,2], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6], [6,2]]);
T122334_9:=DigraphByEdges([[1,2],[1,3],[1,5], [1,6], [2,4], [2,5], [2,6], [3,2], [3,4], [4,1], [4,5], [4,6], [5,3], [5,6], [6,3]]);
T122334_10:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,4], [2,5], [3,2], [3,4], [3,6], [4,5], [4,6], [5,1], [5,3], [5,6], [6,2]]);
T122334_11:=DigraphByEdges([[1,2],[1,3],[1,5], [1,6], [2,4], [2,5], [3,2], [3,4],[3,6], [4,1], [4,5], [4,6], [5,3], [5,6], [6,2]]);
T122334_12:=DigraphByEdges([[1,2],[1,3],[1,4], [1,6], [2,4], [2,5], [2,6], [3,2], [3,4], [4,5], [4,6], [5,1], [5,3], [5,6], [6,3]]);
T123333:=DigraphByEdges([[1,2],[1,3],[1,5], [2,4], [2,5], [2,6], [3,2], [3,4], [3,6], [4,1], [4,5], [4,6], [5,3], [5,6], [6,1]]);
T123333_2:=DigraphByEdges([[1,2],[1,3],[1,4], [2,4], [2,5], [2,6], [3,2], [3,4], [3,6], [4,5], [4,6], [5,1], [5,3], [5,6], [6,1]]);
T123333_3:=DigraphByEdges([[1,2],[1,3],[1,4], [2,3], [2,5], [2,6], [3,4], [3,5], [3,6], [4,2], [4,5], [4,6], [5,1], [5,6], [6,1]]);
T123333_4:=DigraphByEdges([[1,2],[1,4],[2,4], [2,5], [2,6], [3,1], [3,2], [3,6], [4,3], [4,5], [4,6], [5,1], [5,3], [5,6], [6,1]]);
T222234:=DigraphByEdges([[1,3],[1,4],[1,5], [1,6], [2,1], [2,3], [2,5], [3,5], [3,6], [4,2], [4,3], [5,4], [5,6], [6,2], [6,4]]);
T222234_2:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,3], [2,4], [2,5], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6], [6,1], [6,2]]);
T222234_3:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,4], [2,5], [2,6], [3,2], [3,4], [4,5], [4,6], [5,3], [5,6], [6,1], [6,3]]);
T222234_4:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [2,4], [2,5], [3,2], [3,4], [3,6], [4,5], [4,6], [5,3], [5,6], [6,1], [6,2]]);
T222225:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,5], [3,5], [3,6], [4,2], [4,3], [5,4], [5,6], [6,2], [6,4]]);
T222333:=DigraphByEdges([[1,2], [1,4],[1,6], [2,4], [2,5], [2,6], [3,1],

[3,2], [3,6], [4,3], [4,5], [5,1], [5,3], [6,4], [6,5]]);
T222333_2:=DigraphByEdges([[1,3],[1,5],[1,6], [2,1], [2,3], [2,6], [3,4], [3,5], [4,1], [4,2], [5,2], [5,4], [5,6], [6,3], [6,4]]);
T222333_3:=DigraphByEdges([[1,2], [1,5], [1,6], [2,3], [2,5], [3,1], [3,5], [3,6], [4,1], [4,2], [4,3], [5,4], [5,6], [6,2], [6,4]]);
T222333_4:=DigraphByEdges([[1,3], [1,6], [2,1], [2,3], [2,6], [3,4], [3,5], [4,1], [4,2], [5,1], [5,2], [5,4], [6,3], [6,4], [6,5]]);
T222333_5:=DigraphByEdges([[1,3], [1,6], [2,1], [2,3], [2,5], [3,4], [3,5], [4,1], [4,2], [5,1], [5,4], [5,6], [6,2], [6,3], [6,4]]);

T1133346:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [2,3], [2,4], [2,5], [2,6],[3,4], [3,5], [3,6], [4,5], [4,6], [4,7], [5,6], [6,7], [7,2], [7,3], [7,5]]);
T1233345:=DigraphByEdges([[1,2], [1,3], [1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [3,4], [3,5], [3,6], [4,5], [4,6], [4,7], [7,1], [7,2], [7,3], [5,6], [5,7], [6,7]]);
T0333345:=DigraphByEdges([[1,2], [1,3], [1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,7], [3,4], [3,5], [3,6], [4,5], [4,6], [4,7], [7,1], [7,3], [7,5], [6,2], [6,5], [6,7]]);
T0123555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [3,6], [4,5], [4,6], [5,6],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T0222555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [4,5], [4,6], [6,3], [6,5],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T1122555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [4,5], [4,6], [5,6], [6,3],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T1113555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [3,6], [4,5], [5,6], [6,4],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T1114455:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [3,6], [3,7], [4,5], [5,6], [6,4], [7,1], [7,4], [7,5], [7,6]]);
T1113456:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [2,3], [2,4], [2,5], [2,6],[2,7], [3,4], [3,5], [3,6], [3,7], [4,5], [4,6], [4,7], [5,6], [6,7], [7,5]]);
T1133346:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [2,3], [2,4], [2,5], [2,6], [3,4],[3,5], [3,6], [4,5], [4,6], [4,7], [5,6], [6,7], [7,2], [7,3], [7,5]]);
T1233345:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5],

[2,6], [3,4], [3,5], [3,6], [4,5], [4,6], [4,7],[7,1], [7,2], [7,3], [5,6], [5,7],[6,7]]);
T0333345:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,7], [3,4], [3,5], [3,6], [4,5], [4,6], [4,7],[7,1], [7,3], [7,5], [6,2], [6,5],[6,7]]);
T0123555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7],[3,4], [3,5], [3,6], [4,5], [4,6], [5,6],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T0222555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7],[3,4], [3,5], [4,5], [4,6], [6,3], [6,5],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T1122555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [4,5], [4,6], [5,6], [6,3],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T1113555:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [2,3], [2,4], [2,5], [2,6], [2,7],[3,4], [3,5], [3,6], [4,5], [5,6], [6,4],[7,1], [7,3], [7,4], [7,5], [7,6]]);
T1113456:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [2,3], [2,4], [2,5], [2,6], [2,7], [3,4], [3,5], [3,6], [3,7], [4,5], [4,6], [4,7], [5,6], [6,7], [7,5]]);

T01234666:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [3,4], [3,5], [3,6], [3,7], [4,5], [4,6], [4,7], [5,6], [5,7], [6,7], [8,1], [8,3], [8,4], [8,5], [8,6], [8,7]]);

T012345777:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [1,8], [2,3], [2,4], [2,5], [2,6], [2,7], [2,8],[2,9], [3,4], [3,5], [3,6], [3,7], [3,8], [4,5], [4,6], [4,7], [4,8], [5,6], [5,7], [5,8], [6,7], [6,8], [7,8], [9,1], [9,3], [9,4], [9,5], [9,6], [9,7], [9,8]]);
T222333777:=DigraphByEdges([[1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [1,8], [2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [2,9], [3,4], [3,5], [3,6], [4,5], [4,6], [4,7], [5,6], [5,7], [5,8], [6,7], [6,8], [7,8], [7,3], [8,3], [8,4], [9,1], [9,3], [9,4], [9,5], [9,6], [9,7], [9,8]]);

T0123456888:=DigraphByEdges([[1,2], [1,3], [1,4], [1,5], [1,6], [1,7], [1,8], [1,9], [2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [2,9], [2,10], [3,4], [3,5], [3,6], [3,7], [3,8], [3,9], [4,5], [4,6], [4,7], [4,8],[4,9], [5,6], [5,7], [5,8], [5,9], [6,7], [6,8], [6,9], [7,8],[7,9],[8,9], [10,1], [10,3], [10,4], [10,5], [10,6], [10,7], [10,8], [10,9]]);

```
T01234567999:=DigraphByEdges([ [1,2], [1,3], [1,4], [1,5], [1,6], [1,7],
[1,8], [1,9], [1,10], [2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [2,9], [2,10], [2,11],
[3,4], [3,5], [3,6], [3,7], [3,8], [3,9], [3,10], [4,5], [4,6], [4,7], [4,8], [4,9],
[4,10], [5,6], [5,7], [5,8], [5,9], [5,10], [6,7], [6,8], [6,9], [6,10], [7,8],
[7,9], [7,10], [8,9],[8,10], [9,10], [11,1], [11,3], [11,4], [11,5], [11,6], [11,7],
[11,8], [11,9], [11,10]]);

T12:=DigraphByEdges([ [1,2],[1,3],[1,4], [1,5], [1,6], [1,7], [1,8], [1,9],
[1,10], [1,11], [2,3], [2,4], [2,5], [2,6], [2,7], [2,8], [2,9], [2,10], [2,11],
[2,12], [3,4], [3,5], [3,6], [3,7], [3,8], [3,9], [3,10], [3,11], [4,5], [4,6], [4,7],
[4,8], [4,9], [4,10], [4,11], [5,6], [5,7], [5,8], [5,9], [5,10], [5,11], [6,7], [6,8],
[6,9], [6,10], [6,11], [7,8],[7,9], [7,10], [7,11], [8,9], [8,10], [8,11], [9,10],
[9,11], [10,11], [12,1], [12,3], [12,4], [12,5], [12,6], [12,7], [12,8], [12,9],
[12,10], [12,11]]);
```

A.1.2 Johnson Digraph Group when $q = 2$

We will provide Johnson relation here and reader can change the relators and hence they can obtain Mennicke group and the other groups that we have in this thesis.

```
LoadPackage("digraphs");
JohnsonDigraphGroup:=function(gr,q)
local F,EdgeSet,n,e,i,j,rels;
n:=Size(DigraphVertices(gr));
F:=FreeGroup(n);
EdgeSet:=DigraphEdges(gr);
rels:=[];
for e in EdgeSet do
        i:=e[1];j:=e[2];
        AddSet(rels,F.(i)*F.(j)^(1-q)*F.(i)*F.(j)^(-q-1));
od;#e
return(F/rels);
```

```
end;

*Put the code related tournaments defined in Section A.1.1.
Then enter the code, below

J012:=JohnsonDigraphGroup(T012,2);
J111:=JohnsonDigraphGroup(T111,2);

J0123:=JohnsonDigraphGroup(T0123,2);
J1113:=JohnsonDigraphGroup(T1113,2);
J0222:=JohnsonDigraphGroup(T0222,2);
J1122:=JohnsonDigraphGroup(T1122,2);
J01234:=JohnsonDigraphGroup(T01234,2);
J02224:=JohnsonDigraphGroup(T02224,2);
J11134:=JohnsonDigraphGroup(T11134,2);
J11224:=JohnsonDigraphGroup(T11224,2);
J01333:=JohnsonDigraphGroup(T01333,2);
J02233:=JohnsonDigraphGroup(T02233,2);
J11233:=JohnsonDigraphGroup(T11233,2);
J11233_2:=JohnsonDigraphGroup(T11233_2,2);
J12223:=JohnsonDigraphGroup(T12223,2);
J12223_2:=JohnsonDigraphGroup(T12223_2,2);
J12223_3:=JohnsonDigraphGroup(T12223_3,2);
J22222:=JohnsonDigraphGroup(T22222,2);

J012345:=JohnsonDigraphGroup(T012345,2);
J012444:=JohnsonDigraphGroup(T012444,2);
J013335:=JohnsonDigraphGroup(T013335,2);
J013344:=JohnsonDigraphGroup(T013344,2);
J022245:=JohnsonDigraphGroup(T022245,2);
J022335:=JohnsonDigraphGroup(T022335,2);
J022344:=JohnsonDigraphGroup(T022344,2);
J022344_2:=JohnsonDigraphGroup(T022344_2,2);
```

```
J023334:=JohnsonDigraphGroup(T023334,2);
J023334_2:=JohnsonDigraphGroup(T023334_2,2);
J023334_3:=JohnsonDigraphGroup(T023334_3,2);
J033333:=JohnsonDigraphGroup(T033333,2);
J111345:=JohnsonDigraphGroup(T111345,2);
J111444:=JohnsonDigraphGroup(T111444,2);
J112245:=JohnsonDigraphGroup(T112245,2);
J112335:=JohnsonDigraphGroup(T112335,2);
J112335_2:=JohnsonDigraphGroup(T112335_2,2);
J112344:=JohnsonDigraphGroup(T112344,2);
J112344_2:=JohnsonDigraphGroup(T112344_2,2);
J112344_3:=JohnsonDigraphGroup(T112344_3,2);
J112344_4:=JohnsonDigraphGroup(T112344_4,2);
J113334:=JohnsonDigraphGroup(T113334,2);
J113334_2:=JohnsonDigraphGroup(T113334_2,2);
J113334_3:=JohnsonDigraphGroup(T113334_3,2);
J122235:=JohnsonDigraphGroup(T122235,2);
J122235_2:=JohnsonDigraphGroup(T122235_2,2);
J122235_3:=JohnsonDigraphGroup(T122235_3,2);
J122244:=JohnsonDigraphGroup(T122244,2);
J122244_2:=JohnsonDigraphGroup(T122244_2,2);
J122244_3:=JohnsonDigraphGroup(T122244_3,2);
J122334:=JohnsonDigraphGroup(T122334,2);
J122334_2:=JohnsonDigraphGroup(T122334_2,2);
J122334_3:=JohnsonDigraphGroup(T122334_3,2);
J122334_4:=JohnsonDigraphGroup(T122334_4,2);
J122334_5:=JohnsonDigraphGroup(T122334_5,2);
J122334_6:=JohnsonDigraphGroup(T122334_6,2);
J122334_7:=JohnsonDigraphGroup(T122334_7,2);
J122334_8:=JohnsonDigraphGroup(T122334_8,2);
J122334_9:=JohnsonDigraphGroup(T122334_9,2);
J122334_10:=JohnsonDigraphGroup(T122334_10,2);
J122334_11:=JohnsonDigraphGroup(T122334_11,2);
```

```
J122334_12:=JohnsonDigraphGroup(T122334_12,2);
J123333:=JohnsonDigraphGroup(T123333,2);
J123333_2:=JohnsonDigraphGroup(T123333_2,2);
J123333_3:=JohnsonDigraphGroup(T123333_3,2);
J123333_4:=JohnsonDigraphGroup(T123333_4,2);
J222234:=JohnsonDigraphGroup(T222234,2);
J222234_2:=JohnsonDigraphGroup(T222234_2,2);
J222234_3:=JohnsonDigraphGroup(T222234_3,2);
J222234_4:=JohnsonDigraphGroup(T222234_4,2);
J222225:=JohnsonDigraphGroup(T222225,2);
J222333:=JohnsonDigraphGroup(T222333,2);
J222333_2:=JohnsonDigraphGroup(T222333_2,2);
J222333_3:=JohnsonDigraphGroup(T222333_3,2);
J222333_4:=JohnsonDigraphGroup(T222333_4,2);
J222333_5:=JohnsonDigraphGroup(T222333_5,2);
```

A.2 Size and Derived Series

We will give an example how to specify size and derived series such as $G/G', G'/G''$ and derived length of the corresponding group of a tournament here, the reader can check the *GAP* code to see the results.

Let the tournament with score vector $(1, 1, 2, 3, 3)$ and we are looking for Johnson word when $q = 2$ which means $R(a, b) = ab^{-1}ab^{-3}$.

```
LoadPackage("digraphs");

JohnsonDigraphGroup:=function(gr,q)
local F,EdgeSet,n,e,i,j,rels;
n:=Size(DigraphVertices(gr));
F:=FreeGroup(n);
EdgeSet:=DigraphEdges(gr);
rels:=[];
for e in EdgeSet do
```

```
        i:=e[1];j:=e[2];
        AddSet(rels,F.(i)*F.(j)^(1-q)*F.(i)*F.(j)^(-q-1));
od;#e
return(F/rels);
end;
T11233:=DigraphByEdges([ [1,2],[1,3],[1,4], [2,3], [2,4], [2,5], [3,4],
[3,5], [4,5], [5,1] ]);
J11233:=JohnsonDigraphGroup(T11233,2);

Size(J11233);
AbelianInvariants(J11233);
DG:=DerivedSubgroup(J11233);
AbelianInvariants(DG);
DerivedLength(J11233);

*Here is the output

<immutable digraph with 5 vertices, 10 edges>
gap> J11233:=JohnsonDigraphGroup(T11233,2);
<fp group on the generators [ f1, f2, f3, f4, f5 ]>
gap>
gap> Size(J11233);
32768
gap> AbelianInvariants(J11233);
[ 2, 2, 2, 2, 2 ]
gap> DG:=DerivedSubgroup(J11233);
Group(<fp, no generators known>)
gap> AbelianInvariants(DG);
[ 4, 4, 4, 4, 4 ]
gap> DerivedLength(J11233);
2
```

A.3 Isomorphism

We will provide a computational technique to determine whether two digraph groups are isomorphic or not. We use low index subgroup and Id group technique to check if they are not isomorphic. We provide the code here for an example

Suppose we have two tournaments with score vectors $(1, 1, 2, 3, 3)$ and $(1, 1, 2, 3, 3)_2$ and $R(a, b) = a^{-1}bab^{-3}$ which is Mennicke's word when $q = 3$. It is classified that these tournaments are not isomorphic in [28]. We investigate whether the corresponding digraph groups are isomorphic or not.

The *GAP* code for $(1, 1, 2, 3, 3)$

```
F:=FreeGroup(5);
a:=-1; b:=1; c:=1; d:=-3;

R:=[
F.1^a*F.2^b*F.1^c*F.2^d,
F.1^a*F.4^b*F.1^c*F.4^d,
F.1^a*F.5^b*F.1^c*F.5^d,
F.2^a*F.3^b*F.2^c*F.3^d,
F.2^a*F.4^b*F.2^c*F.4^d,
F.3^a*F.1^b*F.3^c*F.1^d,
F.4^a*F.3^b*F.4^c*F.3^d,
F.5^a*F.2^b*F.5^c*F.2^d,
F.5^a*F.3^b*F.5^c*F.3^d,
F.5^a*F.4^b*F.5^c*F.4^d,
];
G:=F/R;
L:=LowIndexSubgroupsFpGroup(G,3);

for i in [1..Size(L)] do
Q:=L[i]/DerivedSubgroup(L[i]);
```

```
Print(i,",",AbelianInvariants(L[i]),",",IdGroup(Q),"\n");
od;
```

*Here is the output

```
1,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
 2,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
3,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
4,[ 2, 2, 2, 2 ],[ 16, 14 ]
5,[ 2, 2, 2, 2, 8 ],[ 128, 2301 ]
6,[ 2, 2, 2, 4 ],[ 32, 45 ]
7,[ 2, 2, 2, 4 ],[ 32, 45 ]
8,[ 2, 2, 2, 2 ],[ 16, 14 ]
9,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
10,[ 2, 2, 2, 4 ],[ 32, 45 ]
11,[ 2, 2, 2, 4 ],[ 32, 45 ]
12,[ 2, 2, 2, 8 ],[ 64, 246 ]
13,[ 2, 2, 2, 2 ],[ 16, 14 ]
14,[ 2, 2, 2, 2 ],[ 16, 14 ]
15,[ 2, 2, 2, 2 ],[ 16, 14 ]
16,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
17,[ 2, 2, 2, 2, 8 ],[ 128, 2301 ]
18,[ 2, 2, 2, 2, 8 ],[ 128, 2301 ]
19,[ 2, 2, 2, 2 ],[ 16, 14 ]
20,[ 2, 2, 2, 4 ],[ 32, 45 ]
21,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
22,[ 2, 2, 2, 4 ],[ 32, 45 ]
23,[ 2, 2, 2, 2 ],[ 16, 14 ]
24,[ 2, 2, 2, 2 ],[ 16, 14 ]
25,[ 2, 2, 2, 4 ],[ 32, 45 ]
26,[ 2, 2, 2, 2, 8 ],[ 128, 2301 ]
27,[ 2, 2, 2, 2 ],[ 16, 14 ]
28,[ 2, 2, 2, 2, 4 ],[ 64, 260 ]
```

```
29,[ 2, 2, 2, 2 ],[ 16, 14 ]
30,[ 2, 2, 2, 4 ],[ 32, 45 ]
31,[ 2, 2, 2, 2 ],[ 16, 14 ]
32,[ 2, 2, 2, 4 ],[ 32, 45 ]
```

The Gap code for $(1, 1, 2, 3, 3)_2$

```
F:=FreeGroup(5);
a:=-1; b:=1; c:=1; d:=-3;
R:=[
F.1^a*F.3^b*F.1^c*F.3^d,
F.1^a*F.4^b*F.1^c*F.4^d,
F.1^a*F.5^b*F.1^c*F.5^d,
F.2^a*F.1^b*F.2^c*F.1^d,
F.2^a*F.4^b*F.2^c*F.4^d,
F.3^a*F.2^b*F.3^c*F.2^d,
F.4^a*F.3^b*F.4^c*F.3^d,
F.5^a*F.2^b*F.5^c*F.2^d,
F.5^a*F.3^b*F.5^c*F.3^d,
F.5^a*F.4^b*F.5^c*F.4^d,
];
H:=F/R;
L:=LowIndexSubgroupsFpGroup(H,3);

for i in [1..Size(L)] do
Q:=L[i]/DerivedSubgroup(L[i]);
Print(i,",",AbelianInvariants(L[i]),",",IdGroup(Q),"\n");
od;

*Here is the output

1,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
2,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
3,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
```

```
4,[ 2, 2, 2, 2 ],[ 16, 14 ]
5,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
6,[ 2, 2, 2, 8 ],[ 64, 246 ]
7,[ 2, 2, 2, 2 ],[ 16, 14 ]
8,[ 2, 2, 2, 2 ],[ 16, 14 ]
9,[ 2, 2, 2, 2, 8 ],[ 128, 2301 ]
10,[ 2, 2, 2, 4 ],[ 32, 45 ]
11,[ 2, 2, 2, 2 ],[ 16, 14 ]
12,[ 2, 2, 2, 4 ],[ 32, 45 ]
13,[ 2, 2, 2, 2 ],[ 16, 14 ]
14,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
15,[ 2, 2, 2, 2 ],[ 16, 14 ]
16,[ 2, 2, 2, 2 ],[ 16, 14 ]
17,[ 2, 2, 2, 2, 8 ],[ 128, 2301 ]
18,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
19,[ 2, 2, 2, 4 ],[ 32, 45 ]
20,[ 2, 2, 2, 8 ],[ 64, 246 ]
21,[ 2, 2, 2, 2 ],[ 16, 14 ]
22,[ 2, 2, 2, 4 ],[ 32, 45 ]
23,[ 2, 2, 2, 2 ],[ 16, 14 ]
24,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
25,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
26,[ 2, 2, 4, 8 ],[ 128, 1601 ]
27,[ 2, 2, 2, 2 ],[ 16, 14 ]
28,[ 2, 2, 2, 2, 2 ],[ 32, 51 ]
29,[ 2, 2, 2, 2 ],[ 16, 14 ]
30,[ 2, 2, 2, 4 ],[ 32, 45 ]
31,[ 2, 2, 2, 2 ],[ 16, 14 ]
32,[ 2, 2, 2, 2 ],[ 16, 14 ]
```

As we can see by the outputs, they are not isomorphic. By index 3 subgroups, G has [64,260] but H does not.

A.3.1 How it works

Let G_1, G_2 be finite groups. We aim to show that $G_1 \ncong G_2$. If $G_1^{ab} \ncong G_2^{ab}$ then $G_1 \ncong G_2$.

How do we do this?

If $IdGroup(G_1) \neq IdGroup(G_2)$ then $G_1 \ncong G_2$. But maybe GAP can't provide $IdGroup(G_1)$. If $G_1 \cong G_2$ then the set of index 2 subgroups of G_1 is equal to the set of index 2 subgroups of G_2. Therefore if G_1 has an index 2 subgroup H_1 that is not an index 2 subgroup of G_2 then $G_1 \ncong G_2$. If H_1^{ab} is not the abelianization of any index 2 subgroup of G_2 then H_1 is not an index 2 subgroup of G_2. Thus, we produce a list of abelianization of index 2 subgroups of G_1, G_2 in GAP.

We look for an index 2 subgroup of G_1 whose abelianization is not the abelianization of any index 2 subgroup of G_2. If we find one of them then $G_1 \ncong G_2$. If not, then we are looking for bigger index.

A.4 Perfect groups

We provide the *GAP* code to find out the perfect group of order 40320 here and we already explain how the code works in Lemma 6.7.1.

```
F:=FreeGroup(6);
a:=1; b:=1; c:=1; d:=-2;
R:=[
F.1^a*F.2^b*F.1^c*F.2^d,
F.2^a*F.3^b*F.2^c*F.3^d,
F.3^a*F.1^b*F.3^c*F.1^d,
F.6^a*F.1^b*F.6^c*F.1^d,
F.6^a*F.2^b*F.6^c*F.2^d,
F.6^a*F.3^b*F.6^c*F.3^d,
F.6^a*F.4^b*F.6^c*F.4^d,
F.5^a*F.6^b*F.5^c*F.6^d,
F.1^a*F.4^b*F.1^c*F.4^d,
```

```
F.3^a*F.4^b*F.3^c*F.4^d,
F.4^a*F.2^b*F.4^c*F.2^d,
F.1^a*F.5^b*F.1^c*F.5^d,
F.2^a*F.5^b*F.2^c*F.5^d,
F.3^a*F.5^b*F.3^c*F.5^d,
F.4^a*F.5^b*F.4^c*F.5^d,
];
G:=F/R;
L:=LowIndexSubgroupsFpGroup(G,8);
Index(G,L[2]);
G1:=PerfectGroup(40320,1);
G2:=PerfectGroup(40320,2);
G3:=PerfectGroup(40320,3);
G4:=PerfectGroup(40320,4);
L1:=LowIndexSubgroupsFpGroup(G1,7);
L2:=LowIndexSubgroupsFpGroup(G2,7);
L4:=LowIndexSubgroupsFpGroup(G4,8);
```

*Here is the output

```
[ Group(<fp, no generators known>), Group(<fp, no generators known>) ]
gap> Index(G,L[2]);
8
gap> G1:=PerfectGroup(40320,1);
A5 2^1 x L3(2) 2^1
gap> G2:=PerfectGroup(40320,2);
A7 2^4
gap> G3:=PerfectGroup(40320,3);
A8 2^1
gap> G4:=PerfectGroup(40320,4);
L3(4) 2^1
gap> L1:=LowIndexSubgroupsFpGroup(G1,7);
[ Group(<fp, no generators known>),
```

```
Group(<fp, no generators known>),
Group(<fp, no generators known>),
Group(<fp, no generators known>),
Group(<fp, no generators known>) ]
gap> L2:=LowIndexSubgroupsFpGroup(G2,7);
[ Group(<fp, no generators known>),
Group(<fp, no generators known>) ]
gap> L4:=LowIndexSubgroupsFpGroup(G4,8);
[ Group(<fp, no generators known>) ]
```

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