A Quantum-Bayesian Route to Quantum-State Space

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Abstract In the quantum-Bayesian approach to quantum foundations, a quantum state is viewed as an expression of an agent's personalist Bayesian degrees of belief, or probabilities, concerning the results of measurements. These probabilities obey the usual probability rules as required by Dutch-book coherence, but quantum mechanics imposes additional constraints upon them. In this paper, we explore the question of deriving the structure of quantum-state space from a set of assumptions in the spirit of quantum Bayesianism. The starting point is the representation of quantum states induced by a symmetric informationally complete measurement or SIC. In this representation, the Born rule takes the form of a particularly simple modification of the law of total probability. We show how to derive key features of quantum-state space from (i) the requirement that the Born rule arises as a simple modification of the law of total probability and (ii) a limited number of additional assumptions of a strong Bayesian flavor.

1 Introduction

In the standard formulation of (finite-dimensional) quantum mechanics, a quantum state is a density operator, ρ, on a d-dimensional Hilbert space. A measurement with

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m outcomes is described by a POVM, $\{E_1, \ldots, E_m\}$, a collection of positive semi-definite operators that sum to the identity. The probability, $p(i)$, of the i-th measurement outcome is given by the Born rule,

$$p(i) = \text{tr}(\rho E_i). \tag{1}$$

If the POVM $\{E_i\}$ is informationally complete [1], the state ρ is fully determined by the outcome probabilities $\{p(i)\}$. With respect to some fiducial informationally complete POVM, the vector of probabilities $p(i)$ is thus an alternative description of the quantum state. This means that quantum-state space can be viewed as a subset of the probability simplex.

According to the quantum-Bayesian approach to quantum foundations [1–12], the probabilities $p(i)$ represent an agent's Bayesian degrees of belief, or personalist probabilities [13–18]. They are numbers expressing the agent's uncertainty about which measurement outcome will occur and acquire an operational meaning through decision theory [17]. Quantum-Bayesian state assignments are personalist in the sense that they are functions of the agent alone, not functions of the world external to the agent [8]. In other words, there are—in principle—potentially as many quantum states for a given quantum system as there are agents who care to take note of it. Nonetheless, despite not being specified by agent-independent facts, personalist probability assignments are far from arbitrary. Dutch-book coherence [13, 16, 18] as a normative principle requires that an agent's degrees of belief conform to the usual rules of the probability calculus, and this is a surprisingly powerful constraint when coupled with the agent's overall belief system [19].

In addition to the rules required by Dutch-book coherence, the Born rule (1) puts further constraints on the probabilities used in quantum mechanics. From this arises two questions which are of central importance for the quantum-Bayesian program. One question, on which there has been much progress recently [12, 20, 21], is that of the mathematical structure of the set of probabilities resulting from (1). The second question concerns the origin of the quantum-mechanical constraints on the agent's probabilities, i.e., the origin of the Born rule. The authors' present view on this question is that the Born rule should be seen as an empirical addition to Dutch-book coherence [12].

What we mean by this is the following. Dutch-book coherence, though a normative rule, is of a purely logical character [22]. The way Bernardo and Smith [17] put its significance is this:

> Bayesian Statistics offers a rationalist theory of personalistic beliefs in contexts of uncertainty, with the central aim of characterising how an individual should act in order to avoid certain kinds of undesirable behavioural inconsistencies. … The goal, in effect, is to establish rules and procedures for individuals concerned with disciplined uncertainty accounting. The theory is not descriptive, in the sense of claiming to model actual behaviour. Rather, it is prescriptive, in the sense of saying 'if you wish to avoid the possibility of these undesirable consequences you must act in the following way.'

On the other hand, to a quantum Bayesian it is crucial that there is no such thing as a "right and true" quantum state [8]. But if so, what is one to make of the Born

rule in (1)? What are these things ρ and E_i that the probabilities are being calculated from? The meaning of the rule calls for an explanation in our terms. Our solution is to think of the Born rule in a normative way, rather than as a strict law of nature. It is something along the lines, but not identical to, Dutch-book coherence: The Born rule should be viewed as a normative principle for relating one's various degrees of belief about the outcomes of various measurements. The idea is that if one does not make sure his probability assignments are related according to the dictum of the Born rule, nature is liable to give "undesirable consequences" for his decisions. In contrast to usual Dutch-book coherence, though, the origin of the normative rule is not of a purely logical character. It should rather be seen as dependent upon contingent features of the particular physical world we happen to live in.

To shed further light on the similarities and differences between Dutch book coherence and the Born rule, here we renew the question of deriving the structure of quantum-state space from a set of assumptions formulated and motivated fully in terms of the probability assignments of a Bayesian agent. In Sect. 3 we show one way to derive several key features of quantum-state space from the assumption that the Born rule arises as a simple modification of the law of total probability, complemented by a few further assumptions of a strong Bayesian character. Many details of the derivation are left out since a full account can be found in [12]. Indeed the role of the present paper should be viewed as a supplement to the (very long) [12]. Here we try to be as logically crisp as possible, deleting extended examples and motivations, and very carefully labeling every assumption.

The set of assumptions in Sect. 3 was inspired by the observation that the mapping between the Hilbert space and probability simplex formulations of quantum mechanics becomes very simple if the fiducial POVM is a symmetric informationally complete POVM, or SIC [23–28]. In particular, in this representation the Born rule takes the form of an extremely simple modification of the law of total probability [12], thus motivating our main assumption in Sect. 3. This motivation will be explained in Sect. 2, where we prepare the ground for the rest of the paper by reviewing SICs briefly and showing how to rewrite the Born rule in terms of them.

2 SICs and the Born Rule

Consider a set of d^2 one-dimensional projection operators, Π_i, in d-dimensional Hilbert space such that

$$\operatorname{tr} \Pi_i \Pi_j = \frac{d\delta_{ij} + 1}{d + 1}. \tag{2}$$

The informationally complete POVM $\{E_i\}$ defined by $E_i = \frac{1}{d}\Pi_i$ is called a SIC [24]. SICs have been explicitly proven to exist in dimensions $d = 2$–15, 19, 24, 35, and 48 (see references in [29]). Furthermore, they have been observed by computational means, to a numerical precision of 10^{-38}, in dimensions $d = 2$–67 [29]. For this paper, we will assume that SICs exist in all dimensions.

With respect to a SIC, a density operator can be recovered easily from the outcome probabilities given by the Born rule (1) by using the beautiful formula [24, 26]

$$\rho = \sum_{i=1}^{d^2} \left((d+1)p(i) - \frac{1}{d} \right) \Pi_i. \tag{3}$$

Using this equation one can show that, in the representation induced by a fiducial SIC, the set of all quantum states can be characterized very elegantly. According to one such characterization, a probability vector $p(i)$ is a pure quantum state if and only if it satisfies the constraints [28]

$$\sum_i p(i)^2 = \frac{2}{d(d+1)} \tag{4}$$

and

$$\sum_{ijk} c_{ijk} p(i) p(j) p(k) = \frac{d+7}{(d+1)^3}, \tag{5}$$

where the coefficients c_{ijk} are defined by

$$c_{ijk} = \operatorname{Re} \operatorname{tr}(\Pi_i \Pi_j \Pi_k). \tag{6}$$

All other quantum states, which means all mixed states, are constructed by taking convex combinations of the states given by (4) and (5). A key question for the quantum Bayesian program is how to understand and motivate the structure of quantum-state space expressed in these equations as restrictions on an agent's personalist probability assignments.

A strong hint as to where to look for an answer is given by the surprising form the Born rule takes when written in SIC language. Consider the scenario in Fig. 1, where one measurement (which we call the "measurement on the ground") is analyzed in terms of another measurement (the "measurement in the sky"), and assume for the time being that the measurement in the sky is a SIC, implying that it has $n = d^2$ outcomes. The probabilities $p(i)$ are thus a representation of the agent's prior state assignment. The conditional probabilities $r(j|i)$ are the agent's probabilities to obtain outcome j on the ground assuming that the measurement in the sky was actually performed and resulted in outcome i.

Let us now denote by $s(j)$ the agent's probabilities for the outcomes of the measurement on the ground in this situation, i.e., when the measurement in the sky was actually carried out before the measurement on the ground. Dutch-book coherence requires that the agent computes $s(j)$ from $p(i)$ and $r(j|i)$ by using the law of total probability,

$$s(j) = \sum_{i=1}^{d^2} p(i) r(j|i). \tag{7}$$

If, however, the measurement in the sky is not performed and remains counterfactual, Dutch-book coherence places no constraints on the agent's probabilities, denoted
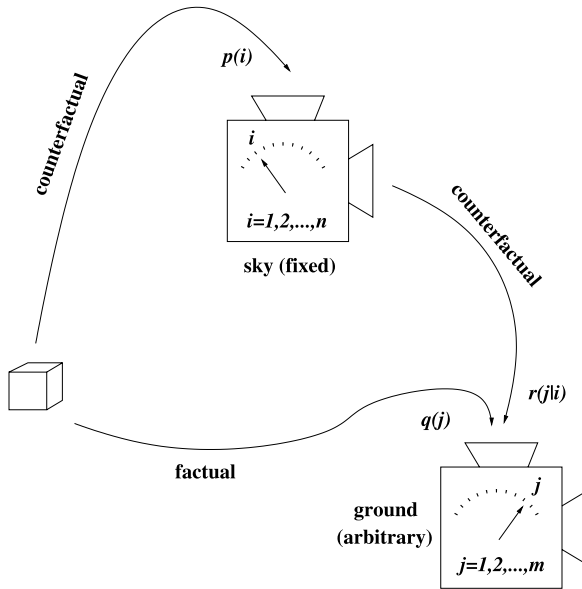
Fig. 1 This diagram shows the conceptual framework of this paper. A system is imagined to be measurable in two ways. The measurement on the ground, with outcomes $j = 1, \ldots, m$, is an arbitrary measurement that could be performed in the laboratory. The measurement in the sky, with outcomes $i = 1, \ldots, n$, is a fixed fiducial measurement introduced to analyze the measurement on the ground. The probability distributions $p(i)$ and $r(j|i)$ represent an agent's probabilities assuming the measurement in the sky is actually performed. The probability distribution $q(j)$ represents instead the agent's probabilities under the assumption that the measurement in the sky is *not* performed. If the measurement in the sky is a SIC with $n = d^2$ outcomes, $q(j)$ is related to $p(i)$ and $r(j|i)$ by the simple relation (8)

by $q(j)$, for the outcomes on the ground. In this case, it is the quantum formalism through the Born rule that restricts the agent's distribution $q(j)$. This follows easily by setting $q(j) = \operatorname{tr} \rho F_j$ and $r(j|i) = \operatorname{tr} \Pi_i F_j$ for some general POVM $\{F_j\}$ with m outcomes and using (3). One obtains

$$q(j) = \sum_{i=1}^{d^2} \left((d+1)p(i) - \frac{1}{d} \right) r(j|i). \tag{8}$$

This is the Born rule $q(j) = \operatorname{tr} \rho F_j$ expressed in the SIC representation. As in [12] we stress the central importance of this equation by calling it the *Urgleichung*. The similarity with the law of total probability is striking—it is basically the simplest modification of that law that it could possibly be [30]. To get from the law of total probability to the Urgleichung, one merely makes the replacement

$$\sum_i p(i)r(j|i) \longrightarrow \sum_i f\big(p(i)\big)r(j|i), \tag{9}$$

with f simply an affine mapping, $f(x) = (d+1)x - \frac{1}{d}$. The functional form of the Born rule expressed by the Urgleichung will be the pivot for the development in the next section.

3 Deriving the Structure of Quantum-State Space

In this section, the main section of the paper, we formulate a series of assumptions from which a number of key features of the structure of quantum-state space can be derived. As stated previously we omit many of the mechanical details of the proofs for greater clarity. The omitted details can be found in [12]. The basic situation we consider is as in Fig. 1. We imagine a fiducial, counterfactual n-outcome measurement "in the sky" in terms of which we analyze an arbitrary m-outcome measurement "on the ground." Initially there are no restrictions on the numbers n and m. As before, $p(i)$, the *prior in the sky*, represents the probabilities in the sky, and $q(j)$ represents the probabilities on the ground. We write $r(j|i)$ to represent the conditional probability for obtaining j on the ground, given that i was found in the sky. When we want to suppress components, we will write vectors $\|p\rangle\rangle$ and $\|q\rangle\rangle$, and write R for the matrix with entries $r(j|i)$—by definition, R is a stochastic matrix.

We start by postulating a generalized Urgleichung where we take the mapping f in (9) to be a general affine mapping $f(x) = \alpha x - \beta$.

Assumption 1 (Generalized Urgleichung) *For any measurement on the ground, $q(j)$ should be calculated according to*

$$q(j) = \sum_{i=1}^{n} \big(\alpha p(i) - \beta\big) r(j|i), \tag{10}$$

where α and β are fixed nonnegative real numbers.

Since the $q(j)$ are probabilities, they satisfy the double inequality $0 \le q(j) \le 1$ or

$$0 \le \sum_{i=1}^{n} \big(\alpha p(i) - \beta\big) r(j|i) \le 1. \tag{11}$$

We call this double inequality the *Urungleichung*. It puts immediate restrictions on the distributions $p(i)$ and $r(j|i)$, i.e., on the vector $\|p\rangle\rangle$ and the matrix R. For an agent to accept quantum mechanics it means, at least in part, he commits to these restrictions on his Bayesian probability assignments. Our ultimate goal—which in this paper we will achieve only partially—is the precise characterization of these restrictions in Bayesian terms. We will denote by $\mathcal{P}_0$ the set of all priors for the sky permitted by quantum mechanics, and by $\mathcal{R}_0$ the set of all permitted conditional distributions R. Our next assumption is about the sets $\mathcal{P}_0$ and $\mathcal{R}_0$. To formulate it, we need a definition.

Definition 1 Let $\mathcal{P}$ be a set of priors in the sky and let $\mathcal{R}$ be a set of stochastic matrices. We say that $\mathcal{P}$ and $\mathcal{R}$ are *consistent* if all pairs $(\| p \rangle\rangle, R) \in \mathcal{P} \times \mathcal{R}$ obey the Urungleichung (11). Furthermore, we say $\mathcal{P}$ and $\mathcal{R}$ are *maximal* whenever $\mathcal{P}' \supseteq \mathcal{P}$ and $\mathcal{R}' \supseteq \mathcal{R}$ imply $\mathcal{P}' = \mathcal{P}$ and $\mathcal{R}' = \mathcal{R}$ for any consistent $\mathcal{P}'$ and $\mathcal{R}'$.

Assumption 2 (Maximality) *The sets $\mathcal{P}_0$ and $\mathcal{R}_0$ of all valid priors for the sky and all valid conditionals R are taken to be consistent and maximal.*

In other words, we assume that quantum mechanics restricts the set of probabilities available to the agent as little as possible given the universal validity of the generalized Urgleichung (10).

Unfortunately, Assumption 2 does not fix the sets $\mathcal{P}_0$ and $\mathcal{R}_0$ uniquely. There are many consistent and maximal pairs $(\mathcal{P}, \mathcal{R})$. The assumptions below constitute one way to proceed toward the goal of a complete characterization of $\mathcal{P}_0$ and $\mathcal{R}_0$. There is little doubt that there exist simpler and more compelling sets of assumptions to achieve this goal. Finding these is work in progress.

Assumption 3 (Possibility of complete ignorance) *The constant vector*

$$\| p \rangle\rangle = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)^{\mathrm{T}} \tag{12}$$

is in the set $\mathcal{P}_0$.

This assumption makes sure that the agent can be in a state of complete ignorance about the outcome of the measurement in the sky.

Assumption 4 (Priors span the simplex) *The elements of $\mathcal{P}$ span the probability simplex in n dimensions.*

If this assumption were not satisfied, one could use a smaller simplex for all considerations.

Assumption 5 (Principle of Reciprocity) *For any $R \in \mathcal{R}_0$ and any outcome j on the ground, the vector $\| p \rangle\rangle$ with components*

$$p(i) = \frac{r(j|i)}{\sum_k r(j|k)} \tag{13}$$

is in the set $\mathcal{P}_0$ of valid priors for the sky. Conversely, all valid priors $\| p \rangle\rangle \in \mathcal{P}_0$ can be written in this way.

To motivate this assumption and its name, imagine that both the measurement in the sky and the measurement on the ground are performed and the agent learns the outcome j on the ground while remaining ignorant of the outcome in the sky. Imagine further that his prior in the sky before the measurement is given by the state (12) of complete ignorance. The expression (13) is then the agent's posterior probability for

the outcome i in the sky, given the outcome j on the ground, as computed by Bayes's rule. The content of the Principle of Reciprocity is that the set of priors in the sky is equal to the set of posteriors upon learning the outcome on the ground.

The assumptions so far are very natural and already lead to a number of interesting consequences [12]. For instance, it follows immediately from Assumption 1 that the relation

$$\alpha = n\beta + 1 \tag{14}$$

holds between the three constants of the generalized Urgleichung (10). Assumption 2 implies that the sets $\mathcal{P}_0$ and $\mathcal{R}_0$ are both convex, and even compact [21], so that they necessarily have well-defined extreme points. And Assumption 5 implies the existence of an important class of special priors:

Definition 2 Let the measurement on the ground be identical to the measurement in the sky. Denote the components of the matrix R by $r_s(j|i)$ in this case. By the Principle of Reciprocity (Assumption 5), the distributions $\|e_k\rangle\rangle$, $k = 1, \ldots, n$, with components

$$e_k(i) = \frac{r_s(k|i)}{\sum_l r_s(k|l)} \tag{15}$$

are in the set $\mathcal{P}_0$. They are called *basis states*.

Using Assumption 4, one can show that these components take the form

$$e_k(i) = \frac{1}{\alpha}(\delta_{ki} + \beta) \tag{16}$$

and satisfy the relation

$$\sum_i e_k(i)^2 = \frac{1}{\alpha^2}\left(1 + 2\beta + n\beta^2\right). \tag{17}$$

However, to pin down the sets $\mathcal{P}_0$ and $\mathcal{R}_0$ further, and in particular to fix the parameterized form of the constants n, α, and β—i.e., that $n = d^2$, $\alpha = d + 1$, and $\beta = \frac{1}{d}$—we need two additional postulates. First here is another definition.

Definition 3 A measurement on the ground is said to have the property of *in-step unpredictability* (ISU) if a uniform prior in the sky implies a uniform probability assignment for the probabilities on the ground, i.e., for an ISU measurement, whenever $\|p\rangle\rangle$ is the uniform distribution (12), then $\|q\rangle\rangle$ is given by the uniform distribution $(\frac{1}{m}, \frac{1}{m}, \ldots, \frac{1}{m})^{\mathrm{T}}$.

The existence of ISU measurements, which will be postulated in Assumption 6 below, means that an agent may be totally ignorant about both the (counterfactual) outcome in the sky and the outcome on the ground.

Let us now denote by $r_{\mathrm{ISU}}(j|i)$ the components of the matrix R for a measurement on the ground with m outcomes, $m \neq n$, and in-step unpredictability. It can be shown

[12] that one must have

$$\sum_i r_{\text{ISU}}(j|i) = \frac{n}{m}. \tag{18}$$

By the Principle of Reciprocity, this ISU measurement gives rise to a class of priors which we denote by $\| p_k \rangle\rangle$, $k = 1, \ldots, m$. Their components are given by

$$p_k(i) = \frac{m}{n} r_{\text{ISU}}(k|i); \tag{19}$$

each vector $\| p_k \rangle\rangle$ represents a valid prior in the sky. This leads to our next definition.

Definition 4 We say that a measurement with in-step unpredictability *achieves the ideal of certainty* if $\| p \rangle\rangle = \| p_k \rangle\rangle$ implies that $q(j) = \delta_{jk}$, i.e., for such a measurement and a prior in the sky given by $\| p_k \rangle\rangle$, the agent is certain that the outcome on the ground will be k.

This is a very specific definition. It is motivated by the following consideration. Consider a setup with an ISU measurement on the ground, i.e., a measurement with total ignorance for both ground and sky, and imagine the measurement in the sky is actually performed. Observing k on the ground while remaining ignorant about the sky then gives rise to the posterior $\| p_k \rangle\rangle$ for the sky (see the discussion following Assumption 5). Now go back to the usual situation in which the measurement in the sky remains counterfactual, and assume the agent's prior for the sky is $\| p_k \rangle\rangle$. If the measurement achieves the ideal of certainty, the agent will be certain that the measurement on the ground results in the very outcome k.

Assumption 6 (Availability of Certainty) *For any system, there is a measurement with in-step unpredictability of some number $m_0 \geq 2$ of outcomes that (i) achieves the ideal of certainty and (ii) for which one of the priors $\| p_k \rangle\rangle$ defined in (19) has the form of a basis distribution (15).*

For a measurement of this type, we have that [12]

$$\langle\langle p_j \| p_k \rangle\rangle = \frac{1}{\alpha}\left(\frac{m_0}{n}\delta_{jk} + \beta\right), \quad j, k = 1, \ldots, m_0, \tag{20}$$

where $\langle\langle \cdot \| \cdot \rangle\rangle$ denotes the inner product. Using condition (ii) of the above assumption, it follows that the squared norm $\langle\langle p_k \| p_k \rangle\rangle$ of any of the vectors $\| p_k \rangle\rangle$ is equal to the squared norm of the basis vectors given by (17). This, together with (14) now implies the equality

$$\frac{m_0}{n}\alpha - \beta = 1 \tag{21}$$

for any measurement satisfying Assumption 6.

Equation (20) expresses that any two of the vectors $\| p_k \rangle\rangle$ differ by the same angle, θ, defined by

$$\cos\theta = \frac{\langle\langle p_1 \| p_2 \rangle\rangle}{\langle\langle p_1 \| p_1 \rangle\rangle}. \tag{22}$$

Using the relations (14) and (21) between our four variables, α, β, n and m_0 established above, this angle can be seen to equal

$$\cos\theta = \frac{n - m_0}{(m_0 - 1)^2 + n - 1}. \tag{23}$$

We are now ready to state our last assumption.

Assumption 7 (Many Systems, Universal Angle) *The identity of a system is parameterized by its pair (n, m_0). Nonetheless for all systems, the angle θ between pairs of priors $\|p_k\rangle\rangle$ for any measurement satisfying Assumption 6 is a universal constant given by $\cos\theta = 1/2$.*

The value $\cos\theta = 1/2$ is less arbitrary than it may appear at first sight. Taken by itself, the assumption that θ is universal implies that, for any $m_0 \geq 2$, there is an integer n such that the right-hand side of (23) evaluates to the constant $\cos\theta$. It is not hard to show that this is possible only if this constant is of the form

$$\cos\theta = \frac{q}{q + 2}, \tag{24}$$

where q is a non-negative integer. The universal angle postulated above corresponds to the choice $q = 2$.

Every choice for q leads to a different relation between n and m_0. For $q = 0$, we find $n = m_0$, in which case the Urgleichung turns out to be identical to the classical law of total probability. For $q = 1$, we get the relationship $n = \frac{1}{2}m_0(m_0 + 1)$ which, although this fact plays no role in our argument, is characteristic of theories defined in real Hilbert space [31]. And for $q = 2$, we obtain

$$n = m_0^2. \tag{25}$$

Equations (21) and (25) hold for the special measurement postulated in Assumption 6. If we eliminate m_0 from these equations we find, with the help of (14), the relationships

$$n = (\alpha - 1)^2, \quad \beta = \frac{1}{\sqrt{n}}. \tag{26}$$

These equalities form the main result of this paper. They must hold for *any* measurement on the ground. If we denote the integer $\alpha - 1$ by the letter d, we recover the constants of the original Urgleichung (8).

With the Urgleichung in the form (8) as the starting point and minimal additional assumptions, a large amount of detailed information about the structure of quantum-state space can be derived. Details can be found in Ref. [12] and the paper by Appleby, Ericsson, and Fuchs [21] in this special issue.

4 Summary and Conclusion

Our main postulate, the generalized Urgleichung or Assumption 1, is an addition to standard Dutch-book coherence. It restricts an agent's probability assignments in

a situation involving a counterfactual measurement—the measurement in the sky—where Dutch-book coherence does not impose any specific constraints. The form of the generalized Urgleichung is given by a minimal modification of the law of total probability, which is the law connecting the agent's probabilities in the case the measurement in the sky is *factualized*, i.e., actually carried out. This means that the key assumption of this paper arises through a formal connection between an agent's probabilities in two complementary scenarios, one in which the measurement in the sky remains counterfactual and one in which it is factualized.

Assumption 2 guarantees that the set of probability assignments available to the agent is maximal within the constraints set by the generalized Urgleichung, i.e., Assumption 2 makes sure that the agent's probability assignments are not unduly restricted. In a similar spirit, Assumption 3 guarantees that the state of complete ignorance is among the agent's potential priors, and Assumption 4 makes sure that the set of priors available to the agent is large enough to span the probability simplex.

With Assumption 5, the Principle of Reciprocity, we return to the theme of exploring connections between the two respective scenarios of a counterfactual and a factualized measurement in the sky. The Principle of Reciprocity states that the set of priors for the sky available to the agent should be identical to the set of the agent's posteriors for a factualized measurement in the sky. The question of what motivates the particular relation between probabilities for counterfactual and factualizable measurements expressed in Assumptions 1 and 5 strikes us as a mysterious and important one.

The numerical relation between the constants α, β, and n, and in particular the fact that n is a perfect square, follows from the existence of a single special measurement defined in Assumption 6, together with the postulate of a universal angle in Assumption 7. These last two assumptions, as well as the first five, are given purely in terms of the personalist probabilities a Bayesian agent may assign to the outcomes of certain experiments. Nowhere in all this do we mention amplitudes, Hilbert space, or any other part of the usual apparatus of quantum mechanics. What has been sketched in this paper constitutes a novel approach to the quantum formalism, providing fresh insight for the foundations of quantum mechanics. Maybe even more importantly, the success of this approach provides a compelling case for quantum Bayesianism [1–12].

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