Efficient and Convergent Federated Learning

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Abstract—Federated learning has shown its advances over the last few years but is facing many challenges, such as how algorithms save communication resources, how they reduce computational costs, and whether they converge. To address these issues, this paper proposes a new federated learning algorithm (FedGiA) that combines the gradient descent and the inexact alternating direction method of multipliers. It is shown that FedGiA is computation and communication-efficient and convergent linearly under mild conditions.

Index Terms—GD and inexact ADMM-based federated learning, communication-efficient, communication-efficient, global convergence, linear convergence rate

I. INTRODUCTION

C Federated learning (FL), as an effective machine learning technique, gains popularity in recent years due to its ability to deal with various issues like data privacy, data security, and data access to heterogeneous data. Typical applications include vehicular communications [1], [2], [3], [4], digital health [5], and smart manufacturing [6], just to name a few. The earliest work for FL can be traced back to [7] in 2015 and [8] in 2016. It is still undergoing development and also facing many challenges [9], [10], [11].

A. Related work

Gradient descent-based learning. In recent years, there is an impressive body of work on developing FL algorithms. One of the most popular approaches benefits from the stochastic gradient descent (SGD). The general framework is to run certain steps of SGD in parallel by clients/devices and then average the resulting parameters from clients by a central server once in a while. Representatives of SGD family consists of the famous Federated averaging (FedAvg [12]) and Local SGD (LocalSGD [13], [14]). Other state-of-the-art ones can be seen in [15], [16], [17], [18]. These algorithms execute global averaging/aggregation periodically and thus can reduce the communication rounds (CR), thereby saving resources (e.g., transmission power and bandwidth in wireless communication) for real-world applications.

However, to establish the convergence theory, most SGD algorithms assume that the local data is identically and independently distributed (i.i.d.), which is unrealistic for FL applications where data is usually heterogeneous. More details can be referred to the LocalSGD [13], K-step averaging SGD [19], and Cooperative SGD [18].

A parallel line of research aims to investigate gradient descent (GD) based-FL algorithms. Since full data is used to construct the gradient, these algorithms do not impose assumptions on distributions of the involved data [20], [21], [22], [23], [24]. Nevertheless, strong conditions for the objective functions of the learning optimization problems are still required to guarantee the convergence. Typical assumptions are gradient Lipschitz continuity (also known as L-smoothness), strong smoothness, convexity, or strong convexity.

ADMM-based learning. The alternating direction method of multipliers (ADMM) has shown its advances both in theoretical and numerical aspects over the last few decades, with extensive applications into various disciplines. In particular, there is a success of implementation ADMM in distributed learning [25], [26], [27], [28], [29]. Fairly recently, ADMM-based FL draws much attention due to its simple structure and easy implementation. They can be categorized into two classes: exact and inexact ADMM. The former aims at updating local parameters through solving sub-problems exactly, which hence brings more computational burdens for local clients [30], [31], [32], [33], [34].

Therefore, inexact ADMM is an alternative to reduce the computational complexity for clients [35], [36], [37], [38], where clients update their parameters via solving sub-problems approximately, thereby alleviating the computational burdens and accelerating the learning speed. Again, we shall emphasize that those algorithms that have been established for convergence properties still impose some restrictive assumptions. Fairly recently, an algorithm from the primal-dual optimization perspective was developed in [38] and turned out to be a member of inexact ADMM-based FL. It is shown that the algorithm converges under weaker assumptions. Finally, it is worth mentioning that ADMM is very useful in FL for the purpose of data privacy [30], [32], [34], [35], [33], [37].

B. Our contributions

The main contribution of this paper is to develop a new FL algorithm that is capable of saving communication resources, reducing computational burdens, and converging under relatively weak assumptions.

I) The proposed algorithm, FedGiA in Algorithm 1, has a novel framework. After each round of communication (i.e., iteration k is a multiple of a given integer k_0), all clients are split into two groups randomly. One group adopts the scheme of the inexact ADMM to update their parameters k_0 times. While the second group exploits the GD approach to update their parameters just once. Therefore, FedGiA possesses three advantages as follows.

- It is communication-efficient since CR can be controlled by setting k_0 . Our numerical experiments have shown that CR decline when k_0 increases, see Figure 2.
- It is computation-efficient due to the nature of inexact updates for all local clients. The communication efficiency

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has been demonstrated by our numerical comparisons with two state-of-the-art algorithms, see Table III.

• It is possible to cope with scenarios where a portion of clients are in bad conditions. The sever could select them for the second group where less effort is required to update their parameters.

II) The assumptions to guarantee the convergence are mild. We prove that FedGiA converges to a stationary point (see Definition II.1) of the learning optimization problem in (4) with a linear rate $O(k_0/k)$ only under two conditions: gradient Lipschitz continuity (also known for the L-smoothness in many publications) and the boundedness of a level set, as shown in Theorem IV.2. These conditions do not impose convexity or strong convexity. Hence, they are weaker than those used to establish convergence for most current distributed learning and FL algorithms. If we further assume the convexity, then FedGiA achieves the optimal solution, as shown in Corollary IV.1.

C. Organization and notation

This paper is organized as follows. In the next section, we introduce FL and the framework of ADMM. In Section III, we present FedGiA and highlight its advantages, followed by the establishment of its global convergence and convergence rate in Section IV. We then conduct some numerical experiments and comparisons with two popular algorithms to demonstrate the performance of FedGiA in Section V. Concluding remarks are given in the last section.

We end this section with summarizing the notation that will be employed throughout this paper. We use plain, bold, and capital letters to present scalars, vectors, and matrices, respectively, e.g., m, r, and σ are scalars, \mathbf{x}, \mathbf{x}_i and \mathbf{x}_i^k are vectors, X and X^k are matrices. Let $\lfloor t \rfloor$ represent the largest integer strictly smaller than t + 1 and $[m] := \{1, 2, \ldots, m\}$ with ':=' meaning define. In this paper, \mathbb{R}^n denotes the *n*dimensional Euclidean space equipped with the inner product $\langle \cdot, \cdot \rangle$ defined by $\langle \mathbf{x}, \mathbf{y} \rangle := \sum_i x_i y_i$. Let $\| \cdot \|$ be the Euclidean norm for vectors (i.e., $\| \mathbf{x} \|^2 = \langle \mathbf{x}, \mathbf{x} \rangle$) and Spectral norm for matrices, and $\| \cdot \|_H$ be the weighted norm defined by $\| \mathbf{x} \|_H^2 := \langle H \mathbf{x}, \mathbf{x} \rangle$. Write the identity matrix as I and a positive semidefinite matrix A as $A \succeq 0$. In particular, $A \succeq B$ represents $A - B \succeq 0$. A function, f, is said to be gradient Lipschitz continuous with a constant r > 0 if

$$\|\nabla f(\mathbf{x}) - \nabla f(\mathbf{z})\| \le r \|\mathbf{x} - \mathbf{z}\|.$$
(1)

for any x and z, where $\nabla f(\mathbf{x})$ represents the gradient of f with respect to x.

II. GD AND INEXACT ADMM-BASED FL

Suppose we have *m* local clients/edge nodes with datasets $\{\mathcal{D}_1, \mathcal{D}_2, \ldots, \mathcal{D}_m\}$. Each client has the total loss $f_i(\mathbf{x}) := \frac{1}{d_i} \sum_{(\mathbf{a}, b) \in \mathcal{D}_i} \ell_i(\mathbf{x}; (\mathbf{a}, b))$, where $\ell_i(\cdot; (\mathbf{a}, b)) : \mathbb{R}^n \to \mathbb{R}$ is a continuous loss function and bounded from below, d_i is the cardinality of \mathcal{D}_i , and $\mathbf{x} \in \mathbb{R}^n$ is the parameter to be learned. Below are two examples used for our numerical experiments.

Example II.1 (Least square loss). Suppose the *i*th client has data $\mathcal{D}_i = \{(\mathbf{a}_1^i, b_1^i), \dots, (\mathbf{a}_{d_i}^i, b_{d_i}^i)\}$, where $\mathbf{a}_j^i \in \mathbb{R}^n$, $b_j^i \in \mathbb{R}$. Then the least square loss is

$$f_i(\mathbf{x}) = \sum_{j=1}^{d_i} \frac{1}{2d_i} (\langle \mathbf{a}_j^i, \mathbf{x} \rangle - b_j^i)^2.$$
⁽²⁾

Example II.2 (ℓ_2 norm regularized logistic loss). Similarly, the *i*th client has data \mathcal{D}_i but with $b_j^i \in \{0, 1\}$. The ℓ_2 norm regularized logistic loss is given by

$$f_i(\mathbf{x}) = \frac{1}{d_i} \sum_{j=1}^{d_i} [\ln(1 + e^{\langle \mathbf{a}_j^i, \mathbf{x} \rangle}) - b_j^i \langle \mathbf{a}_j^i, \mathbf{x} \rangle + \frac{\mu}{2} \|\mathbf{x}\|^2], \quad (3)$$

where $\mu > 0$ is a penalty parameter.

The overall loss function can be defined by

$$f(\mathbf{x}) := \frac{1}{m} \sum_{i=1}^{m} f_i(\mathbf{x}),$$

Federated learning aims to learn a best parameter \mathbf{x}^* that reaches the minimal overall loss, namely,

$$\mathbf{x}^* := \operatorname{argmin}_{\mathbf{x} \in \mathbb{R}^n} f(\mathbf{x}). \tag{4}$$

Since f_i is bounded from below, we have

$$f^* := f(\mathbf{x}^*) > -\infty. \tag{5}$$

By introducing auxiliary variables, $\mathbf{x}_i = \mathbf{x}$, problem (4) can be equivalently rewritten as

$$\min_{\mathbf{x}, \mathbf{x}_1, \dots, \mathbf{x}_m \in \mathbb{R}^n} \quad \frac{1}{m} \sum_{i=1}^m f_i(\mathbf{x}_i), \quad \text{s.t.} \quad \mathbf{x}_i = \mathbf{x}, \ i \in [m].$$
(6)

Throughout the paper, we shall place our interest on the above optimization problem. For simplicity, we also denote $X := (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m)$ and

$$F(X) := \frac{1}{m} \sum_{i=1}^{m} f_i(\mathbf{x}_i).$$
 (7)

It is easy to see that $f(\mathbf{x}) = F(X)$ if $X = (\mathbf{x}, \mathbf{x}, \dots, \mathbf{x})$.

A. ADMM

The backgrounds of ADMM can be referred to the earliest work [39] and a nice book [25]. To apply ADMM for problem (6), by letting $X := (\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_m)$ ad $\Pi := (\pi_1, \pi_2, \dots, \pi_m)$, we introduce the augmented Lagrange function defined by,

$$\mathcal{L}(\mathbf{x}, X, \Pi) = \sum_{i=1}^{m} (\underbrace{f_i(\mathbf{x}_i)/m + \langle \mathbf{x}_i - \mathbf{x}, \pi_i \rangle + (\sigma/2) \|\mathbf{x}_i - \mathbf{x}\|^2}_{=:L(\mathbf{x}, \mathbf{x}_i, \pi_i)}).$$
(8)

Here, $\pi_i \in \mathbb{R}^n, i \in [m]$ are the Lagrange multipliers and $\sigma > 0$. The framework of ADMM for problem (6) is given as follows: for an initialized point $(\mathbf{x}^0, X^0, \Pi^0)$, perform the following updates iteratively for every $k \ge 0$,

$$\begin{cases} \mathbf{x}^{k+1} = \underset{\mathbf{x} \in \mathbb{R}^{n}}{\operatorname{argmin}} \mathcal{L}(\mathbf{x}, X^{k+1}, \Pi^{k+1}) \\ = \frac{1}{m} \sum_{i=1}^{m} (\mathbf{x}_{i}^{k+1} + \frac{1}{\sigma} \boldsymbol{\pi}_{i}^{k+1}), \\ \mathbf{x}_{i}^{k+1} = \underset{\mathbf{x}_{i} \in \mathbb{R}^{n}}{\operatorname{argmin}} L(\mathbf{x}^{k}, \mathbf{x}_{i}, \boldsymbol{\pi}_{i}^{k}), \quad i \in [m], \\ \boldsymbol{\pi}_{i}^{k+1} = \boldsymbol{\pi}_{i}^{k} + \sigma(\mathbf{x}_{i}^{k+1} - \mathbf{x}^{k}), \quad i \in [m]. \end{cases}$$
(9)

B. Stationary points

To end this section, we present the optimality conditions of problems (6) and (4).

Definition II.1. A point $(\mathbf{x}^*, X^*, \Pi^*)$ is a stationary point of problem (6) if it satisfies

$$\begin{cases} \frac{1}{m} \nabla f_i(\mathbf{x}_i^*) + \boldsymbol{\pi}_i^* = 0, & i \in [m], \\ \mathbf{x}_i^* - \mathbf{x}^* = 0, & i \in [m], \\ \sum_{i=1}^m \boldsymbol{\pi}_i^* = 0. \end{cases}$$
(10)

A point \mathbf{x}^* is a stationary point of problem (4) if it satisfies

$$\nabla f(\mathbf{x}^*) = 0. \tag{11}$$

Note that any locally optimal solution to problem (6) (resp. (4)) must satisfy (10) (resp. (11)). If f_i is convex for every $i \in$ [m], then a point is a globally optimal solution to problem (6) (resp.(4)) if and only if it satisfies condition (10) (resp. (11)). Moreover, it is easy to see that a stationary point $(\mathbf{x}^*, X^*, \Pi^*)$ of problem (6) indicates

$$\nabla f(\mathbf{x}^*) = \frac{1}{m} \sum_{i=1}^m \nabla f_i(\mathbf{x}^*) = -\frac{1}{m} \sum_{i=1}^m \pi_i^* = 0.$$

That is, \mathbf{x}^* is also a stationary point of the problem (4).

III. ALGORITHMIC DESIGN

The framework of ADMM in (9) encounters three drawbacks in reality. (i) It repeats the three updates in every step, leading to communication inefficiency. In FL, the framework manifests that local clients and the central server have to communicate at every step. However, frequent communications would come at a huge price, such as a long learning time and large amounts of resources. (ii) Solving the second subproblem in (9) would incur expensive computational cost as it generally does not admit a closed-form solution. (iii) In real applications, some clients may suffer from bad conditions, which leads to computational difficulties. It is necessary to leave them more time to update their parameters. Therefore, to overcome the above mentioned drawbacks, we cast a new algorithm in Algorithm 1, where

$$\overline{\boldsymbol{g}}_i^{k+1} := \frac{1}{m} \nabla f_i(\mathbf{x}^{\tau_{k+1}}).$$

The merits of Algorithm 1 are highlighted as follows.

(i) Communication efficiency: Algorithm 1 shows that communications only occur when $k \in \mathcal{K} = \{0, k_0, 2k_0, \ldots\},\$ where k_0 is a predefined positive integer. Therefore, CR can be reduced if setting a big k_0 , thereby saving the cost vastly. In fact, such an idea has been extensively used in literature [15], [16], [13], [17], [14], [18].

(ii) Fast computation using inexact updates: We update \mathbf{x}_{i}^{k+1} by (13) instead of solving the second sub-problem in (9). It can accelerate the computation for local clients significantly, as the computation is relatively cheap if H_i is chosen properly (e.g, diagonal matrices). We point out that (13) is a result of

$$\mathbf{x}_{i}^{k+1} = \operatorname{argmin}_{\mathbf{x}_{i}} (1/m) h_{i}(\mathbf{x}_{i}; \mathbf{x}^{\tau_{k+1}}) + \langle \mathbf{x}_{i} - \mathbf{x}^{\tau_{k+1}}, \boldsymbol{\pi}_{i}^{k} \rangle + \frac{\sigma}{2} \| \mathbf{x}_{i} - \mathbf{x}^{\tau_{k+1}} \|^{2}$$
(19)
$$= \mathbf{x}^{\tau_{k+1}} - (H_{i}/m + \sigma I)^{-1} (\overline{\boldsymbol{g}}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k}),$$

Algorithm 1: FL via GD and inexact ADMM (FedGiA)

Given an integer $k_0 > 0$ and a constant $\sigma > 0$, every client *i* initializes $H_i \succeq 0, \mathbf{x}_i^0, \pi_i^0$ and $\mathbf{z}_i^0 = \mathbf{x}_i^0 + \pi_i^0 / \sigma, i \in [m]$. Let τ_k be a function of k as $\tau_k := \lfloor k/k_0 \rfloor$. for $k = 0, 1, 2, 3, \dots$ do if $k \in \mathcal{K} := \{0, k_0, 2k_0, 3k_0, \ldots\}$ then Weights upload: (Communication occurs) All clients upload $\{\mathbf{z}_1^k, \dots, \mathbf{z}_m^k\}$ to the server. Global aggregation:

The server calculates average parameter $\mathbf{x}^{\tau_{k+1}}$ by

$$\tau_{k+1} = \frac{1}{m} \sum_{i=1}^{m} \mathbf{z}_{i}^{k}.$$
 (12)

Weights broadcast: (Communication occurs) The server broadcasts $\mathbf{x}^{\tau_{k+1}}$ to all clients.

Clients selection:

x

The server randomly selects a new set $\mathcal{C}^{\tau_{k+1}} \subseteq [m]$ of clients for training in the next round.

end for every $i \in C^{\tau_{k+1}}$ do

Local update: Client i updates its parameters by

$$\mathbf{x}_{i}^{k+1} = \mathbf{x}^{\tau_{k+1}} - (H_{i}/m + \sigma I)^{-1} (\overline{g}_{i}^{k+1} + \pi_{i}^{k}), \quad (13)$$

$$\boldsymbol{\pi}_{i}^{k+1} = \boldsymbol{\pi}_{i}^{k} + \sigma(\mathbf{x}_{i}^{k+1} - \mathbf{x}^{\tau_{k+1}}), \tag{14}$$

$$\mathbf{z}_{i}^{k+1} = \mathbf{x}_{i}^{k+1} + \pi_{i}^{k+1} / \sigma.$$
 (15)

end

for every $i \notin C^{\tau_{k+1}}$ do *Local invariance:* Client *i* keeps parameters by

$$\mathbf{x}_i^{\kappa+1} \equiv \mathbf{x}^{\tau_{k+1}},\tag{16}$$

$$\boldsymbol{\pi}_{i}^{k+1} \equiv -\overline{\boldsymbol{g}}_{i}^{k+1},\tag{17}$$

$$\mathbf{z}_{i}^{k+1} \equiv \mathbf{x}^{\tau_{k+1}} - \overline{g}_{i}^{k+1} / \sigma.$$
(18)

end

where $h_i(\mathbf{x}_i; \mathbf{z})$ is an approximation of $f_i(\mathbf{x}_i)$, namely,

$$f_i(\mathbf{x}_i) \approx \underbrace{f_i(\mathbf{z}) + \langle \nabla f_i(\mathbf{z}), \mathbf{x}_i - \mathbf{z} \rangle + (1/2) \|\mathbf{x}_i - \mathbf{z}\|_{H_i}^2}_{=:h_i(\mathbf{x}_i;\mathbf{z})}.$$
 (20)

(iii) Mixed updates: At every $k \in \mathcal{K}$ in Algorithm 1, all clients are divided into two groups. For clients in $C^{\tau_{k+1}}$, they update their parameters k_0 times based on the inexact ADMM, while for clients not in $C^{\tau_{k+1}}$, they update their parameters just once based on the GD. This suggests that, in real applications, the sever should try to select clients with good devices to form $\mathcal{C}^{\tau_{k+1}}$ and put the rest as the second group. This would leave clients in the second group having more time to update their parameters since they only need to update them once.

We would like to point out that FedAvg [12], [40] and FedProx [41] select partial devices to join in the training in each communication round. That is, if $k \in \mathcal{K}$, they randomly select a subset $C^{\tau_{k+1}}$ of clients and only clients in $C^{\tau_{k+1}}$ update their parameters and the rest clients remain unchanged.

end

IV. CONVERGENCE ANALYSIS

To establish the convergence, we need one assumption.

Assumption IV.1. Every $f_i, i \in [m]$ is gradient Lipschitz continuous with a constant $r_i > 0$.

Assumption IV.1 implies that there is always a Θ_i satisfying $r_i I \succeq \Theta_i \succeq 0$ such that

$$f_i(\mathbf{x}_i) \le f_i(\mathbf{z}_i) + \langle \nabla f_i(\mathbf{z}_i), \mathbf{x}_i - \mathbf{z}_i \rangle + \frac{1}{2} \| \mathbf{x}_i - \mathbf{z}_i \|_{\Theta_i}^2, \quad (21)$$

for any $\mathbf{x}_i, \mathbf{z}_i \in \mathbb{R}^n$. Apparently, many Θ_i s satisfy the above condition (e.g., $\Theta_i = r_i I$). In the subsequent convergence analysis, we suppose that every client *i* chooses $H_i = \Theta_i$.

A. Global convergence

For notational convenience, hereafter we let $\mathbf{a}^k \to \mathbf{a}$ stand for $\lim_{k\to\infty} \mathbf{a}^k = \mathbf{a}$ and denote

$$\mathcal{L}^k := \mathcal{L}(\mathbf{x}^{\tau_k}, X^k, \Pi^k), \qquad r := \max_{i \in [m]} r_i.$$
(22)

With the help of Assumption IV.1, our first result shows that whole sequences of $\{\mathcal{L}^k\}$, $\{F(X^k)\}$, and $\{f(\mathbf{x}^{\tau_k})\}$ converge.

Theorem IV.1. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$ and $\sigma > 6r/m$. The following results hold under Assumption IV.1.

i) Three sequences $\{\mathcal{L}^k\}$, $\{F(X^k)\}$, and $\{f(\mathbf{x}^{\tau_k})\}$ converge to the same value, namely,

$$\lim_{k \to \infty} \mathcal{L}^k = \lim_{k \to \infty} F(X^k) = \lim_{k \to \infty} f(\mathbf{x}^{\tau_k}).$$
(23)

ii) $\nabla F(X^k)$ and $\nabla f(\mathbf{x}^{\tau_k})$ eventually vanish, namely,

$$\lim_{k \to \infty} \nabla F(X^k) = \lim_{k \to \infty} \nabla f(\mathbf{x}^{\tau_k}) = 0.$$
 (24)

Theorem IV.1 states that the objective function values converge, and its establishment does not relied on the random selection of $C^{\tau_{k+1}}$. In the below theorem, we would like to see the convergence performance of sequence $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ itself. To proceed with that, we need the assumption on the boundedness of the following level set

$$\mathcal{S}(\alpha) := \{ \mathbf{x} \in \mathbb{R}^n : f(\mathbf{x}) \le \alpha \}$$
(25)

for a given $\alpha > 0$. We point out that the boundedness of the level set is frequently used in establishing the convergence properties of optimization algorithms. There are many functions satisfying this condition, such as the coercive functions¹.

Theorem IV.2. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$ and $\sigma > 6r/m$. The following results hold under Assumption IV.1 and the boundedness of $S(\mathcal{L}^0)$.

i) Then sequence {(x^{τk}, X^k, Π^k)} is bounded, and any its accumulating point, (x[∞], X[∞], Π[∞]), is a stationary point of (6), where x[∞] is a stationary point of (4).

¹A continuous function $f : \mathbb{R}^n \mapsto \mathbb{R}$ is coercive if $f(\mathbf{x}) \to +\infty$ when $\|\mathbf{x}\| \to +\infty$.

ii) If further assume that \mathbf{x}^{∞} is isolated, then the whole sequence, $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$, converges to $(\mathbf{x}^{\infty}, X^{\infty}, \Pi^{\infty})$.

It is noted that if f is locally strongly convex at \mathbf{x}^{∞} , then \mathbf{x}^{∞} is unique and hence is isolated. However, being isolated is a weaker assumption than locally strong convexity. It is worth mentioning that the establishment of Theorem IV.2 does not require the convexity of f_i or f, because of this, the sequence is guaranteed to converge to the stationary point of problems (6) and (4). In this regard, if we further assume the convexity of f, then the sequence is capable of converging to the optimal solution to problems (6) and (4), which is stated by the following corollary.

Corollary IV.1. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$ and $\sigma > 6r/m$. The following results hold under Assumption IV.1, the boundedness of $S(\mathcal{L}^0)$, and the convexity of f.

i) Three sequences $\{\mathcal{L}^k\}$, $\{F(X^k)\}$, and $\{f(\mathbf{x}^{\tau_k})\}$ converge to the optimal function value of (4), namely

$$\lim_{k \to \infty} \mathcal{L}^k = \lim_{k \to \infty} F(X^k) = \lim_{k \to \infty} f(\mathbf{x}^{\tau_k}) = f^*.$$
 (26)

- ii) Any accumulating point $(\mathbf{x}^{\infty}, X^{\infty}, \Pi^{\infty})$ of sequence $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ is an optimal solution to (6), where \mathbf{x}^{∞} is an optimal solution to (4).
- iii) If further assume f is strongly convex. Then whole sequence {(x^{τk}, X^k, Π^k)} converges the unique optimal solution, (x*, X*, Π*), to (6), where x* is the unique optimal solution to (4).

Remark IV.1. Regarding the assumption in Corollary IV.1, we note that f being strongly convex does not require that every $f_i, i \in [m]$ is strongly convex. If one of f_i s is strongly convex and the remaining is convex, then $f = \sum_{i=1}^{m} w_i f_i$ is strongly convex. Moreover, the strongly convexity suffices to the boundedness of level set $S(\alpha)$ for any α . Therefore, under the strongly convexity, the assumption on the boundedness of $S(\mathcal{L}^0)$ can be exempted.

B. Complexity analysis

Finally, we investigate the convergence speed of the proposed Algorithm 1. The following result states that the minimal value among $\|\nabla f(\mathbf{x}^{\tau_j})\|^2, j \in [k]$ vanishes with a linear rate $O(rk_0/k)$.

Theorem IV.3. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$ and $\sigma > 6r/m$. If Assumption IV.1 holds, then it follows

$$\min_{j \in [k]} \|\nabla f(\mathbf{x}^{\tau_j})\|^2 \le \frac{\rho k_0}{k} (\mathcal{L}^0 - f^*).$$

where $\rho := 5m\sigma^2/\eta$ with η given by (39).

We would like to point out that the establishment of such a convergence rate only requires the assumption of gradient Lipschitz continuity, namely, Assumption IV.1. Moreover, if we take $\sigma = tr/m$ with t > 6, then

$$\frac{\rho k_0}{k} = \frac{10t^3}{t^2 - 5t - 6} \cdot \frac{rk_0}{k} = O(\frac{rk_0}{k}).$$

This is what we expected. The larger k_0 is, the more iterations is required to converge stated as below.

Remark IV.2. Theorem IV.3 hints that Algorithm 1 should be terminated if

$$\|\nabla f(\mathbf{x}^{\tau_k})\|^2 \le \epsilon,\tag{27}$$

where ϵ is a given tolerance. Therefore, after

$$k = \left\lfloor \frac{\rho k_0(\mathcal{L}^0 - f^*)}{\epsilon} \right\rfloor = O(\frac{rk_0}{\epsilon})$$
(28)

iterations, Algorithm 1 meets (27) and the total CR are

$$CR := \left\lfloor \frac{2k}{k_0} \right\rfloor = \left\lfloor \frac{2\rho(\mathcal{L}^0 - f^*)}{\epsilon} \right\rfloor = O(\frac{r}{\epsilon}).$$
(29)

V. NUMERICAL EXPERIMENTS

This section conducts some numerical experiments to demonstrate the performance of FedGiA in Algorithm 1. All numerical experiments are implemented through MATLAB (R2019a) on a laptop with 32GB memory and 2.3Ghz CPU.

A. Testing example

We use Example II.1 with synthetic data and Example II.2 with real data to conduct the numerical experiments.

Example V.1 (Linear regression with non-i.i.d. data). For this problem, local clients have their objective functions as (2). We randomly generate d samples $(\mathbf{a}_1^i, b_1^i), i \in [d]$ from three distributions: the standard normal distribution, the Student's t distribution with degree 5, and the uniform distribution in [-5,5]. Then we shuffle all samples and divided them into m parts (A_i, \mathbf{b}_i) for m clients, where $A_i = (\mathbf{a}_1, \dots, \mathbf{a}_{d_i})^{\top}$ and $\mathbf{b}_i = (b_1, \dots, b_{d_i})^{\top}$. Therefore, $d = d_1 \dots + d_m$. The data size of each part, d_i , is randomly chosen from [50, 150]. For simplicity, we fix n = 100, but choose $m \in \{64, 96, 128, 196, 256\}$. In the reagrd, each client has non-*i.i.d.* data (A_i, \mathbf{b}_i) .

Example V.2 (Logistic regression). For this problem, local clients have their objective functions as (3), where $\mu = 0.001$ in our numerical experiments. We use two real datasets described in Table I to generate \mathbf{a}_j^i and b_j^i . We randomly split d samples into m groups corresponding to m clients.

TABLE I: Descriptions of two real datasets.

Data	Datasets	Source	n	d
qot	Qsar oral toxicity	uci	1024	8992
sct	Santander customer transaction	kaggle	200	200000

B. Implementations

As mentioned in Remark IV.2, we terminate FedGiA if $k \ge 10^4$ or solution \mathbf{x}^{τ_k} satisfies

Error :=
$$\|\nabla f(\mathbf{x}^{\tau_k})\|^2 \le n10^{-9}$$
. (30)

and initialize $\mathbf{x}_i^0 = \boldsymbol{\pi}_i^0 = 0$. For every $k \in \mathcal{K}$, we randomly select sm clients to form $\mathcal{C}^{\tau_{k+1}}$, namely, $|\mathcal{C}^{\tau_{k+1}}| = sm$ and $s \in (0, 1]$. Here, s = 1 means all clients are chosen. Parameters

 σ , H_i and $C^{\tau_{k+1}}$ are set as follows. Theorem IV.1 suggests that σ should be chosen to satisfy $\sigma = tr/m$, where t is given in Table II. Finally, H_i is chosen as Table II, where FedGiAg and FedGiA_D represent FedGiA under H_i opted as a Gram and Diagonal matrix, respectively.

TABLE II: Choices of t and H_i .

		FedGiA _G	FedGiA _D
	t	H_i	H_i
Example V.1	0.15	$\frac{1}{d_i}A_i^{\top}A_i$	$\frac{1}{d_i} \ A_i^\top A_i\ $
Example V.2	$\max\{0.1, \frac{8}{n}\ln(d)\}$	$\frac{1}{4d_i}A_i^{\top}A_i$	$\frac{1}{4d_i} \ A_i^\top A_i\ $

C. Numerical performance

In this part, we conduct some simulation to demonstrate the performance of FedGiA including global convergence, convergence rate, and effect of k_0 and $C^{\tau_{k+1}}$. To measure the performance, we report the following factors: $f(\mathbf{x}^{\tau_k})$, error $\|\nabla f(\mathbf{x}^{\tau_k})\|^2$, CR, and computational time (in second). We only report results of FedGiA solving Example V.1 and omit ones for Example V.2 as they show the similar observations.

1) Global convergence with rate $O(k_0/k)$: We fix m = 64, s = 0.5, and $k_0 \in \{1, 5, 10, 15, 20\}$ and present the results in Figure 1. From the left sub-figure, as expected, all lines eventually tend to the same objective function value, well testifying Theorem IV.1. It is clear that the bigger values of $k_0 > 1$ (i.e., the wider gap between two global aggregations) are, the more iterations are required to reach the optimal function value. From the right sub-figure, the trends show that all errors vanish gradually along with the iterations rising, and the big values of k_0 , the more iterations required to converge, which perfectly justifies Theorems IV.3 that the convergence rate $O(k_0/k)$ relies on k_0 .



Fig. 1: Objective function values and errors v.s. iterations. FedGiA_G (solid lines) and FedGiA_D (dashed lines) solve Example V.1 with m = 64 and s = 0.5.

2) Effect of k_0 : Next, we would like to see how the choices of k_0 impact the performance of FedGiA. To proceed with that, for each dimension (m, d_1, \ldots, d_m) of the dataset, we generate 20 instances of Example V.1 solved by FedGiA with fixing s = 0.5 and $k_0 \in [20]$ and report the average results in Figure 2. With the increasing of k_0 , CR decreases first and then stabilize at a certain level. To this end, it is efficient to save communication costs if we set a proper k_0 . However, it is unnecessary to set a big value of k_0 as it results in longer computational time. In comparison with FedGiA_G, FedGiA_D needs more CR for small k_0 and fewer CR for large k_0 but always rans faster.



Fig. 2: Effect of k_0 for FedGiA_G (solid lines) and FedGiA_D (dashed lines) solving Example V.1 with s = 0.5.

3) Effect of $C^{\tau_{k+1}}$: Finally, we would like to see how choices of $C^{\tau_{k+1}}$ impact the performance of FedGiA. We alter $s \in (0.1, 1]$ and report the average results in Figure 3. We observe that s would not have a big influence on CR when $k_0 > 5$. As expected, the larger s the longer the computational time for most cases. In general speaking, FedGiA_D needs fewer CR when s is small and more CR when s is large but always runs faster than FedGiA_G.



Fig. 3: Effect of s. FedGiA_G (solid lines) and FedGiA_D (dashed lines) solve Example V.1 with m = 64.

D. Numerical comparison

In this part, we will compare our proposed method with FedAvg [12] and LocalSGD [13], [14]. For the former, we use its non-stochastic version. Precisely, we select all clients for the training in each round of communication and sue full local dataset to calculate the gradient. The learning rate is set as $\gamma = \gamma_k(a) := a/\log_2(k+1)$ with a = 0.01 for Example V.1 and a = 0.5d/m for Example V.2, where $d = d_1 + \ldots + d_m$. For LocalSGD, as suggested by [14] using small mini-batch size to approximate the gradient for every local client, we choose mini-batch size $0.05d_i$ for the *i*th client. Its learning rate is set as $\gamma = \gamma_k(a)$ with a = 0.005 for Example V.1 and a = 0.1d/m for Example V.2. For FedGiA, we fix the size of $C^{\tau_{k+1}}$ as 0.5m (i.e., s = 0.5). To ensure relatively fair comparisons, we terminate all methods if condition (30) is satisfied or CR are over 1000.

1) Solving Example V.1: For simplicity, we fix m = 64 and n = 100. From the left sub-figure in Figure 4, we can see that (i) the objective function values for all methods eventually tend to be the same; (ii) Basically, the larger k_0 the faster

the decline of the objective function values; (iii) $FedGiA_G$ and $FedGiA_D$ behave better than FedAvg which outperforms LocalSGD. We then run 20 independent trials and report the average results in Table III. We can conclude that FedGIA and FedGIA use the fewest CR and run the fastest.

TABLE III: Comparison for four algorithms.

		Ex. V.1			Ex. V.2 with got		Ex. V.2 with sct			
Alg.	k_0	Obj.	CR	Time	Obj.	CR	Time	Obj.	CR	Time
FedAvg	1	1.684	1000	1.17	0.260	1000	9.30	0.327	96.6	4.65
	5	1.684	1000	3.27	0.237	572	14.5	0.327	20.0	2.75
	10	1.684	1000	5.64	0.237	289	13.0	0.327	10.0	2.49
LocalSGD	1	1.686	1000	2.95	0.325	1000	13.2	0.333	1000	58.9
	5	1.685	1000	11.0	0.302	1000	44.1	0.331	1000	196
	10	1.684	1000	19.5	0.299	1000	81.5	0.330	1000	369
FedGiA _G	1	1.684	13.6	0.16	0.236	19.8	0.60	0.326	5.00	0.59
	5	1.684	7.40	0.20	0.236	19.9	1.35	0.325	5.00	0.97
	10	1.684	6.10	0.25	0.236	19.9	2.10	0.325	5.00	1.45
FedGiA _D	1	1.684	18.4	0.13	0.236	19.9	0.44	0.326	5.00	0.49
	5	1.684	10.1	0.11	0.236	19.9	0.63	0.325	4.90	0.76
	10	1.684	7.00	0.11	0.236	19.9	0.87	0.325	4.90	1.05

2) Solving Example V.2: Again, we fix m = 64 for simplicity. From the two right sub-figures in Figure 4, we can observe the objective function values obtained by FedGiA decline the fastest. In addition, FedGiA uses the fewest CR to reach the optimal solutions, followed by FedAvg, and LocalSGD needs the most. We then run 20 independent trials and report the average results in Table III, where FedGIA outperforms the other two algorithms by consuming the fewest CR and running the fastest.

VI. CONCLUSION

This paper developed a new FL algorithm and managed to address three key issues in FL, including saving communication resources, reducing computational complexity, and establishing convergence property under mild assumptions. These advantages hint that the proposed algorithm might be practical to deal with many real applications such as mobile edge computing [42], [43], over-the-air computation [44], [45], vehicular communications [1], unmanned aerial vehicle online path control [46] and so forth. Moreover, we feel that the algorithmic schemes and techniques used to build the convergence theory could be also valid for tackling decentralized FL [29], [47]. We leave these for future research.

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APPENDIX A Some Useful Properties

For notational simplicity, hereafter, we denote

$$\begin{split} & \bigtriangleup \mathbf{x}_i^{k+1} := \mathbf{x}_i^{k+1} - \mathbf{x}_i^k, \qquad \bigtriangleup \boldsymbol{\pi}_i^{k+1} := \boldsymbol{\pi}_i^{k+1} - \boldsymbol{\pi}_i^k, \\ & \bigtriangleup \mathbf{x}^{\tau_{k+1}} := \mathbf{x}^{\tau_{k+1}} - \mathbf{x}^{\tau_k}, \qquad \bigtriangleup \overline{\mathbf{x}}_i^{k+1} := \mathbf{x}_i^{k+1} - \mathbf{x}^{\tau_{k+1}} \\ & \boldsymbol{g}_i^k \qquad := \frac{1}{m} \nabla f_i(\mathbf{x}_i^k), \qquad \overline{\boldsymbol{g}}_i^{k+1} \qquad := \frac{1}{m} \nabla f_i(\mathbf{x}^{\tau_{k+1}}). \end{split}$$

For any vectors $\mathbf{a}, \mathbf{b}, \mathbf{a}_i$, matrix $H \succeq 0$, and t > 0, we have

$$-\|\mathbf{b}\|^{2} = 2\langle \mathbf{a}, \mathbf{b} \rangle + \|\mathbf{a}\|^{2} - \|\mathbf{a} + \mathbf{b}\|^{2},$$

$$\|\mathbf{a} + \mathbf{b}\|^{2} \le (1+t)\|\mathbf{a}\|^{2} + (1+1/t)\|\mathbf{b}\|^{2},$$

$$\|\sum_{i=1}^{m} \mathbf{a}_{i}\|^{2} \le m \sum_{i=1}^{m} \|\mathbf{a}_{i}\|^{2},$$

(31)

$$2\langle H\mathbf{a}, \mathbf{b} \rangle \le t \|\mathbf{a}\|_H^2 + (1/t) \|\mathbf{b}\|_H^2.$$

By the Mean Value Theorem, the gradient Lipschitz continuity indicates that for any \mathbf{x}, \mathbf{z} and $\mathbf{w} \in {\mathbf{x}, \mathbf{y}}$,

$$f(\mathbf{x}) - f(\mathbf{z}) - \langle \nabla f(\mathbf{w}), \mathbf{x} - \mathbf{z} \rangle$$

= $\int_0^1 \langle \nabla f(\mathbf{z} + t(\mathbf{x} - \mathbf{z})) - \nabla f(\mathbf{w}), \mathbf{x} - \mathbf{z} \rangle dt$
 $\leq \int_0^1 r \|\mathbf{z} + t(\mathbf{x} - \mathbf{z})) - \mathbf{w}\| \|\mathbf{x} - \mathbf{z}\| dt$
= $\frac{r}{2} \|\mathbf{x} - \mathbf{z}\|^2.$ (32)

APPENDIX B PROOFS OF ALL THEOREMS

A. Key lemmas

Lemma B.1. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$. The following results hold under Assumption IV.1.

a) $\forall k \in \mathcal{K}$,

$$\sum_{i=1}^{m} \left(\frac{\boldsymbol{\pi}_{i}^{k}}{\sigma} + \mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k+1}} \right) = 0.$$
(33)

b)
$$\forall k \ge 0, \forall i \in [m],$$

$$\overline{g}_i^{k+1} + \overline{\pi}_i^{k+1} + \frac{1}{m} H_i \triangle \overline{\mathbf{x}}_i^{k+1} = 0.$$
(34)

c)
$$\forall k \ge 0, \forall i \in [m],$$

 $\| \bigtriangleup \pi_i^{k+1} \|^2 \le \frac{3r_i^2}{m^2} \| \bigtriangleup \mathbf{x}_i^{k+1} \|^2 + \frac{6r_i^2}{m^2} \| \bigtriangleup \mathbf{x}^{\tau_{k+1}} \|^2.$ (35)

Proof. a) For any $i \notin C^{\tau_{k+1}}$, we have from (15) that

$$\mathbf{z}_i^{k+1} = \frac{\boldsymbol{\pi}_i^{k+1}}{\sigma} + \mathbf{x}_i^{k+1}.$$
(36)

For any $i \notin C^{\tau_{k+1}}$, it follows from (16)-(18) that the above relation is still valid. Hence, we have (36) for any $i \in [m]$ and for any $k \geq 0$. As a result, for any $k \in \mathcal{K}$,

$$\sum_{i=1}^{m} \left(\frac{\pi_i^k}{\sigma} + \mathbf{x}_i^k - \mathbf{x}^{\tau_{k+1}}\right) \stackrel{(36)}{=} \sum_{i=1}^{m} (\mathbf{z}_i^k - \mathbf{x}^{\tau_{k+1}}) \stackrel{(12)}{=} 0.$$

b) For $i \in C^{\tau_{k+1}}$, solution \mathbf{x}_i^{k+1} in (13) satisfies (19), thereby contributing to,

$$0 = \overline{g}_{i}^{k+1} + \pi_{i}^{k} + (\frac{1}{m}H_{i} + \sigma I) \Delta \overline{\mathbf{x}}_{i}^{k+1} \stackrel{(14)}{=} \overline{g}_{i}^{k+1} + \pi_{i}^{k+1} + \frac{1}{m}H_{i} \Delta \overline{\mathbf{x}}_{i}^{k+1}.$$
(37)

For any $i \notin C^{\tau_{k+1}}$, the second equation in (37) is still valid due to $\pi_i^{k+1} = -\overline{g}_i^{k+1}$ and $\Delta \overline{\mathbf{x}}_i^{k+1} = 0$ from (16)-(18). Hence, it is true for any $i \in [m]$ and any $k \in \mathcal{K}$.

c) It follows from (34) and $r_i I \succeq H_i = \Theta_i \succeq 0$ that

$$\begin{split} \| \triangle \boldsymbol{\pi}_{i}^{k+1} \|^{2} \\ &= \| \overline{\boldsymbol{g}}_{i}^{k+1} - \overline{\boldsymbol{g}}_{i}^{k} + \frac{1}{m} H_{i} (\triangle \mathbf{x}_{i}^{k+1} - \triangle \mathbf{x}_{i}^{\tau_{k+1}}) \|^{2} \\ &\leq \frac{3r_{i}^{2}}{m^{2}} \| \triangle \mathbf{x}_{i}^{k+1} \|^{2} + \frac{3r_{i}^{2}}{m^{2}} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2} + 3 \| \overline{\boldsymbol{g}}_{i}^{k+1} - \overline{\boldsymbol{g}}_{i}^{k} \|^{2} \\ &\leq \frac{3r_{i}^{2}}{m^{2}} \| \triangle \mathbf{x}_{i}^{k+1} \|^{2} + \frac{6r_{i}^{2}}{m^{2}} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2}, \end{split}$$

which finishes the proof.

Lemma B.2. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$ and $\sigma > 6r/m$. If Assumption IV.1 holds, then for any $k \ge 0$,

$$\mathcal{L}^{k+1} - \mathcal{L}^{k} \le -\eta \sum_{i=1}^{m} (\| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2} + \| \triangle \mathbf{x}_{i}^{k+1} \|^{2}), \quad (38)$$

where η is given by

$$\eta := \frac{\sigma}{2} - \frac{5r}{2m} - \frac{3r^2}{\sigma m^2}.$$
 (39)

Proof. Gap $(\mathcal{L}^{k+1} - \mathcal{L}^k)$ can be decomposed as

$$\mathcal{L}^{k+1} - \mathcal{L}^k =: e_1^k + e_2^k + e_3^k, \tag{40}$$

with

$$e_{1}^{k} := \mathcal{L}(\mathbf{x}^{\tau_{k+1}}, X^{k}, \Pi^{k}) - \mathcal{L}^{k},$$

$$e_{2}^{k} := \mathcal{L}(\mathbf{x}^{\tau_{k+1}}, X^{k+1}, \Pi^{k}) - \mathcal{L}(\mathbf{x}^{\tau_{k+1}}, X^{k}, \Pi^{k}), \qquad (41)$$

$$e_{3}^{k} := \mathcal{L}^{k+1} - \mathcal{L}(\mathbf{x}^{\tau_{k+1}}, X^{k+1}, \Pi^{k}).$$

Estimating e_1^k . If $k \notin \mathcal{K}$, then $\mathbf{x}^{\tau_{k+1}} = \mathbf{x}^{\tau_k}$, yielding

$$e_1^k = 0 = -\frac{\sigma m}{2} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^2.$$

For $k \in \mathcal{K}$, multiplying both sides of the first equation in (33) by $\Delta \mathbf{x}^{\tau_{k+1}}$ yields

$$\sum_{i=1}^{m} \langle \Delta \mathbf{x}^{\tau_{k+1}}, \boldsymbol{\pi}_{i}^{k} \rangle = \sum_{i=1}^{m} \langle \Delta \mathbf{x}^{\tau_{k+1}}, \sigma(\mathbf{x}^{\tau_{k+1}} - \mathbf{x}_{i}^{k}) \rangle.$$
(42)

The fact allows us to derive that

$$e_{1}^{k} \stackrel{(8)}{=} \sum_{i=1}^{m} (L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_{i}^{k}, \boldsymbol{\pi}_{i}^{k}) - L(\mathbf{x}^{\tau_{k}}, \mathbf{x}_{i}^{k}, \boldsymbol{\pi}_{i}^{k}))$$

$$\stackrel{(8)}{=} \sum_{i=1}^{m} (\langle \Delta \mathbf{x}^{\tau_{k+1}}, -\boldsymbol{\pi}_{i}^{k} \rangle$$

$$+ \frac{\sigma}{2} \|\mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k+1}}\|^{2} - \frac{\sigma}{2} \|\mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k}}\|^{2})$$

$$\stackrel{(42)}{=} \sum_{i=1}^{m} (\langle \Delta \mathbf{x}^{\tau_{k+1}}, \sigma(\mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k+1}}) \rangle$$

$$+ \frac{\sigma}{2} \|\mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k+1}}\|^{2} - \frac{\sigma}{2} \|\mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k}}\|^{2})$$

$$\stackrel{(31)}{=} -\frac{\sigma}{2} \sum_{i=1}^{m} \|\Delta \mathbf{x}^{\tau_{k+1}}\|^{2} = -\frac{\sigma m}{2} \|\Delta \mathbf{x}^{\tau_{k+1}}\|^{2}.$$

Overall, for both scenarios, we obtained

$$e_1^k = -\frac{\sigma m}{2} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^2.$$
(43)

Estimating e_2^k . We denote

$$p_i^k := L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_i^{k+1}, \boldsymbol{\pi}_i^k) - L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_i^k, \boldsymbol{\pi}_i^k)$$

$$\stackrel{(8)}{=} \frac{1}{m} f_i(\mathbf{x}_i^{k+1}) - \frac{1}{m} f_i(\mathbf{x}_i^k) + \langle \Delta \mathbf{x}_i^{k+1}, \boldsymbol{\pi}_i^k \rangle \qquad (44)$$

$$+ \frac{\sigma}{2} \|\Delta \overline{\mathbf{x}}_i^{k+1}\|^2 - \frac{\sigma}{2} \|\mathbf{x}_i^k - \mathbf{x}^{\tau_{k+1}}\|^2.$$

We will consider two cases: $i \notin C^{\tau_{k+1}}$ and $i \in C^{\tau_{k+1}}$. For $i \notin C^{\tau_{k+1}}$, if $k \in \mathcal{K}$, then $\mathbf{x}_i^{k+1} \equiv \mathbf{x}^{\tau_{k+1}}$ (namely $\Delta \overline{\mathbf{x}}_i^{k+1} = 0$) suffices to

$$\begin{split} p_{i}^{k} \stackrel{(44)}{=} & \frac{1}{m} f_{i}(\mathbf{x}_{i}^{k+1}) - \frac{1}{m} f_{i}(\mathbf{x}_{i}^{k}) + \langle \Delta \mathbf{x}_{i}^{k+1}, \boldsymbol{\pi}_{i}^{k} \rangle - \frac{\sigma}{2} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} \\ & \stackrel{(32)}{\leq} \langle \Delta \mathbf{x}_{i}^{k+1}, \boldsymbol{g}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k} \rangle + \frac{r_{i}}{2m} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} - \frac{\sigma}{2} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} \\ & = \langle \Delta \mathbf{x}_{i}^{k+1}, \boldsymbol{\overline{g}}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k} \rangle - \frac{\sigma m - r_{i}}{2m} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} \\ \stackrel{(17)}{=} \langle \Delta \mathbf{x}_{i}^{k+1}, -\Delta \boldsymbol{\pi}_{i}^{k+1} \rangle - \frac{\sigma m - r_{i}}{2m} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} \\ \stackrel{(31)}{\leq} \frac{3r_{i}}{2m} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} + \frac{m}{6r_{i}} \| \Delta \boldsymbol{\pi}_{i}^{k+1} \|^{2} - \frac{\sigma m - r_{i}}{2m} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2} \\ \stackrel{(35)}{\leq} \frac{r_{i}}{m} \| \Delta \mathbf{x}^{\tau_{k+1}} \|^{2} - \frac{\sigma m - 5r_{i}}{2m} \| \Delta \mathbf{x}_{i}^{k+1} \|^{2}. \end{split}$$

If $k \notin \mathcal{K}$, then (16) indicates $\mathbf{x}_i^{k+1} = \mathbf{x}^{\tau_{k+1}} = \mathbf{x}^{\tau_k} = \mathbf{x}_i^k$. This immediately results in

$$p_i^k \stackrel{(44)}{=} 0 = \frac{r_i}{m} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^2 - \frac{\sigma m - 5r_i}{2m} \| \triangle \mathbf{x}_i^{k+1} \|^2.$$

Therefore, for any $i \notin C^{\tau_{k+1}}$, we showed that

$$p_i^k \leq \frac{r_i}{m} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^2 - \frac{\sigma m - 5r_i}{2m} \| \triangle \mathbf{x}_i^{k+1} \|^2$$
$$\leq \frac{r}{m} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^2 - \frac{\sigma m - 5r}{2m} \| \triangle \mathbf{x}_i^{k+1} \|^2$$

where the last inequality is due to $r \ge r_i$ for any $i \in [m]$. For any $i \in C^{\tau_{k+1}}$, direct calculation yields that

$$\begin{split} &\langle \triangle \mathbf{x}_{i}^{k+1}, \boldsymbol{g}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k} + \sigma \triangle \overline{\mathbf{x}}_{i}^{k+1} \rangle \\ &= \langle \triangle \mathbf{x}_{i}^{k+1}, \boldsymbol{g}_{i}^{k+1} - \overline{\boldsymbol{g}}_{i}^{k+1} + \overline{\boldsymbol{g}}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k} + \sigma \triangle \overline{\mathbf{x}}_{i}^{k+1} \rangle \\ \stackrel{(37)}{=} \langle \triangle \mathbf{x}_{i}^{k+1}, \boldsymbol{g}_{i}^{k+1} - \overline{\boldsymbol{g}}_{i}^{k+1} - H_{i} \triangle \overline{\mathbf{x}}_{i}^{k+1} / m \rangle \\ &= \langle \sqrt{r_{i}/m} \triangle \mathbf{x}_{i}^{k+1}, \sqrt{m/r_{i}} (\boldsymbol{g}_{i}^{k+1} - \overline{\boldsymbol{g}}_{i}^{k+1} - H_{i} \triangle \overline{\mathbf{x}}_{i}^{k+1} / m) \rangle \\ &\leq \frac{r_{i}}{2m} \| \triangle \mathbf{x}_{i}^{k+1} \|^{2} + \frac{2r_{i}}{m} \| \triangle \overline{\mathbf{x}}_{i}^{k+1} \|^{2}, \end{split}$$

where the last inequality is from $r_i I \succeq H_i = \Theta_i \succeq 0$ and the gradient Lipschitz continuity of f_i . Moreover, it follows from (31) that

$$\begin{split} & \frac{\sigma}{2} \| \triangle \overline{\mathbf{x}}_i^{k+1} \|^2 - \frac{\sigma}{2} \| \mathbf{x}_i^k - \mathbf{x}^{\tau_{k+1}} \|^2 \\ &= \langle \triangle \mathbf{x}_i^{k+1}, \sigma \triangle \overline{\mathbf{x}}_i^{k+1} \rangle - \frac{\sigma}{2} \| \triangle \mathbf{x}_i^{k+1} \|^2 \end{split}$$

Using the above two facts derives

$$\begin{split} p_{i}^{k} &\stackrel{(44)}{=} \frac{1}{m} f_{i}(\mathbf{x}_{i}^{k+1}) - \frac{1}{m} f_{i}(\mathbf{x}_{i}^{k}) + \langle \bigtriangleup \mathbf{x}_{i}^{k+1}, \boldsymbol{\pi}_{i}^{k} \rangle \\ &+ \langle \bigtriangleup \mathbf{x}_{i}^{k+1}, \sigma \bigtriangleup \overline{\mathbf{x}}_{i}^{k+1} \rangle - \frac{\sigma}{2} \| \bigtriangleup \mathbf{x}_{i}^{k+1} \|^{2} \\ &\stackrel{(32)}{\leq} \langle \bigtriangleup \mathbf{x}_{i}^{k+1}, \boldsymbol{g}_{i}^{k+1} + \boldsymbol{\pi}_{i}^{k} + \sigma \bigtriangleup \overline{\mathbf{x}}_{i}^{k+1} \rangle - \frac{\sigma m - r_{i}}{2m} \| \bigtriangleup \mathbf{x}_{i}^{k+1} \|^{2} \\ &\leq \frac{2r_{i}}{m} \| \bigtriangleup \overline{\mathbf{x}}_{i}^{k+1} \|^{2} - \frac{\sigma m - 2r_{i}}{2m} \| \bigtriangleup \mathbf{x}_{i}^{k+1} \|^{2} \\ &\stackrel{(14)}{\leq} \frac{2r_{i}}{m\sigma^{2}} \| \bigtriangleup \boldsymbol{\pi}_{i}^{k+1} \|^{2} - \frac{\sigma m - 2r_{i}}{2m} \| \bigtriangleup \mathbf{x}_{i}^{k+1} \|^{2} \\ &\stackrel{(35)}{\leq} \frac{12r_{i}^{3}}{m^{3}\sigma^{2}} \| \bigtriangleup \mathbf{x}^{\tau_{k+1}} \|^{2} - (\frac{\sigma m - 2r_{i}}{2m} - \frac{6r_{i}^{3}}{m^{3}\sigma^{2}}) \| \bigtriangleup \mathbf{x}_{i}^{k+1} \|^{2} \\ &\leq \frac{r}{m} \| \bigtriangleup \mathbf{x}^{\tau_{k+1}} \|^{2} - \frac{\sigma m - 5r}{2m} \| \bigtriangleup \mathbf{x}_{i}^{k+1} \|^{2}, \end{split}$$

where the last inequality is due to $\sigma > 6r/m$ and $r \ge r_i$ for any $i \in [m]$. Overall, we showed

$$e_{2}^{k} = \sum_{i=1}^{m} p_{i}^{k} = \sum_{\in \mathcal{C}^{\tau_{k+1}}} p_{i}^{k} + \sum_{i \notin \mathcal{C}^{\tau_{k+1}}} p_{i}^{k}$$

$$\leq \sum_{i=1}^{m} (\frac{r}{m} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2} - \frac{\sigma m - 5r}{2m} \| \triangle \mathbf{x}_{i}^{k+1} \|^{2}).$$
(45)

Estimating e_3^k . Again, we have two cases. For client $i \in \mathcal{C}^{\tau_{k+1}}$, it has the following inequalities,

$$\begin{aligned} q_i^k &:= L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_i^{k+1}, \boldsymbol{\pi}_i^{k+1}) - L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_i^{k+1}, \boldsymbol{\pi}_i^k) \\ &\stackrel{(8)}{=} \langle \triangle \overline{\mathbf{x}}_i^{k+1}, \triangle \boldsymbol{\pi}_i^{k+1} \rangle \stackrel{(14)}{=} \frac{1}{\sigma} \| \triangle \boldsymbol{\pi}_i^{k+1} \|^2 \\ &\stackrel{(35)}{\leq} \frac{3r_i^2}{\sigma m^2} \| \triangle \mathbf{x}_i^{k+1} \|^2 + \frac{6r_i^2}{\sigma m^2} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^2. \end{aligned}$$

For client $i \notin C^{\tau_{k+1}}$, since $\mathbf{x}_i^{k+1} = \mathbf{x}^{\tau_{k+1}}$ by (16), it follows $\triangle \overline{\mathbf{x}}_i^{k+1} = 0$ and thus

$$q_i^k = 0 \le \frac{3r_i^2}{\sigma m^2} \|\Delta \mathbf{x}_i^{k+1}\|^2 + \frac{6r_i^2}{\sigma m^2} \|\Delta \mathbf{x}^{\tau_{k+1}}\|^2.$$

The above two errors and $r \ge r_i$ bring out

$$e_{3}^{k} = \sum_{i=1}^{m} q_{i}^{k} \leq \sum_{i=1}^{m} \left(\frac{6r^{2}}{\sigma m} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2} + \frac{3r^{2}}{\sigma m^{2}} \| \triangle \mathbf{x}_{i}^{k+1} \|^{2} \right).$$
(46)

Combining (40), (43), (45), (46) shows

$$\begin{aligned} \mathcal{L}^{k+1} - \mathcal{L}^{k} &= e_{1}^{k} + e_{2}^{k} + e_{3}^{k} \\ &\leq -\left(\frac{\sigma}{2} - \frac{r}{m} - \frac{6r^{2}}{\sigma m^{2}}\right) \sum_{i=1}^{m} \| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2} \\ &- \left(\frac{\sigma}{2} - \frac{5r}{2m} - \frac{3r^{2}}{\sigma m^{2}}\right) \sum_{i=1}^{m} \| \triangle \mathbf{x}_{i}^{k+1} \|^{2} \\ &\leq -\left(\frac{\sigma}{2} - \frac{5r}{2m} - \frac{3r^{2}}{\sigma m^{2}}\right) \sum_{i=1}^{m} (\| \triangle \mathbf{x}^{\tau_{k+1}} \|^{2} + \| \triangle \mathbf{x}_{i}^{k+1} \|^{2}) \end{aligned}$$

which finishes the proof.

Lemma B.3. Let $\{(\mathbf{x}^{\tau_k}, X^k, \Pi^k)\}$ be the sequence generated by Algorithm 1 with $H_i = \Theta_i, i \in [m]$ and $\sigma > 6r/m$. The following results hold under Assumption IV.1.

- i) {L^k} is non-increasing.
 ii) L^k ≥ f(x^{τ_k}) ≥ f* > -∞ for any integer k ≥ 0.
- iii) For any $i \in [m]$,

$$\lim_{k \to \infty} (\Delta \mathbf{x}_i^{k+1}, \Delta \boldsymbol{\pi}_i^{k+1}, \Delta \mathbf{x}^{\tau_{k+1}}, \Delta \overline{\mathbf{x}}_i^{k+1}) = 0.$$
(47)

Proof. i) Since $\sigma > 6r/m$, it follows $\eta > 0$ which by (38) results in the conclusion immediately .

ii) From $r_i I \succeq H_i = \Theta_i \succeq 0$ and (21), we have

$$\frac{1}{m}f_{i}(\mathbf{x}^{\tau_{k+1}}) - \frac{1}{m}f_{i}(\mathbf{x}_{i}^{k+1})$$

$$\stackrel{(32)}{\leq} \langle \Delta \overline{\mathbf{x}}_{i}^{k+1}, -\overline{\mathbf{g}}_{i}^{k+1} \rangle + \frac{r_{i}}{2m} \| \Delta \overline{\mathbf{x}}_{i}^{k+1} \|^{2}$$

$$\stackrel{(34)}{=} \langle \Delta \overline{\mathbf{x}}_{i}^{k+1}, \pi_{i}^{k+1} + \frac{1}{m}H_{i} \Delta \overline{\mathbf{x}}_{i}^{k+1} \rangle + \frac{r_{i}}{2m} \| \Delta \overline{\mathbf{x}}_{i}^{k+1} \|^{2}$$

$$\leq \langle \Delta \overline{\mathbf{x}}_{i}^{k+1}, \pi_{i}^{k+1} \rangle + \frac{3r_{i}}{2m} \| \Delta \overline{\mathbf{x}}_{i}^{k+1} \|^{2},$$
(48)

which by $\sigma > 6r/m$ allows us to obtain

$$p_i^k := \frac{1}{m} f_i(\mathbf{x}_i^{k+1}) + \langle \triangle \overline{\mathbf{x}}_i^{k+1}, \boldsymbol{\pi}_i^{k+1} \rangle + \frac{\sigma}{2} \| \triangle \overline{\mathbf{x}}_i^{k+1} \|^2$$

$$\stackrel{(48)}{\geq} \frac{1}{m} f_i(\mathbf{x}^{\tau_{k+1}}) + \frac{\sigma m - 3r_i}{2m} \| \triangle \overline{\mathbf{x}}_i^{k+1} \|^2 \ge \frac{1}{m} f_i(\mathbf{x}^{\tau_{k+1}}).$$

Using the above condition, we obtain

$$\mathcal{L}^{k+1} = \sum_{i=1}^{m} p_i^k \ge \sum_{i=1}^{m} \frac{1}{m} f_i(\mathbf{x}^{\tau_{k+1}}) = f(\mathbf{x}^{\tau_{k+1}}) \ge f^* \stackrel{(5)}{>} -\infty.$$
(49)

iii) From (38), we conclude that

$$\sum_{k\geq 0} \sum_{i=1}^{m} \eta(\|\Delta \mathbf{x}^{\tau_{k+1}}\|^2 + \|\Delta \mathbf{x}_i^{k+1}\|^2)$$

$$\leq \sum_{k\geq 0} (\mathcal{L}^k - \mathcal{L}^{k+1}) = \mathcal{L}^0 - \lim_{k\to\infty} \mathcal{L}^{k+1} \stackrel{(49)}{<} +\infty.$$

The above condition means $\|\triangle \mathbf{x}^{\tau_{k+1}}\| \to 0$ and $\|\triangle \mathbf{x}_i^{k+1}\| \to 0$, yielding $\|\triangle \boldsymbol{\pi}_i^{k+1}\| \to 0$ by (35) for any $i \in [m]$. Finally, we note that $\triangle \overline{\mathbf{x}}_i^{k+1} = \triangle \boldsymbol{\pi}_i^{k+1} / \sigma \to 0$ from (14) if $i \in \mathcal{C}^{\tau_{k+1}}$ and $\triangle \overline{\mathbf{x}}_i^{k+1} = 0$ from (16) if $i \notin \mathcal{C}^{\tau_{k+1}}$. Overall, $\triangle \overline{\mathbf{x}}_i^{k+1} \to 0$, which completes the whole proof is finished. \Box

B. Proof of Theorem IV.1

Proof. i) It follows from Lemma B.3 that $\{\mathcal{L}^k\}$ is non-increasing and bounded from below. Therefore, the whole sequence, $\{\mathcal{L}^k\}$, converges. For $i \notin \mathcal{C}^{\tau_{k+1}}$, we have $\overline{\mathbf{x}}_i^{k+1} = 0$ from (16), thereby leading to

$$L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_i^{k+1}, \boldsymbol{\pi}_i^{k+1}) \stackrel{(8)}{=} \frac{1}{m} f_i(\mathbf{x}_i^{k+1}).$$

For $i \in C^{\tau_{k+1}}$, it follows

$$\begin{split} & L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_{i}^{k+1}, \boldsymbol{\pi}_{i}^{k+1}) - \frac{1}{m} f_{i}(\mathbf{x}_{i}^{k+1}) \\ \stackrel{(8)}{=} \langle \triangle \overline{\mathbf{x}}_{i}^{k+1}, \boldsymbol{\pi}_{i}^{k+1} \rangle + \frac{\sigma}{2} \| \triangle \overline{\mathbf{x}}_{i}^{k+1} \|^{2} \\ \stackrel{(14)}{=} \frac{1}{\sigma} \langle \triangle \boldsymbol{\pi}_{i}^{k+1}, \boldsymbol{\pi}_{i}^{k+1} \rangle + \frac{1}{2\sigma} \| \triangle \boldsymbol{\pi}_{i}^{k+1} \|^{2} \\ &= \frac{1}{2\sigma} \| \boldsymbol{\pi}_{i}^{k+1} \|^{2} - \frac{1}{2\sigma} \| \boldsymbol{\pi}_{i}^{k} \|^{2} + \frac{1}{\sigma} \| \triangle \boldsymbol{\pi}_{i}^{k+1} \|^{2}. \end{split}$$

Using the above two conditions, we can conclude that

$$\begin{aligned} |\mathcal{L}^{k+1} - F(X^{k+1})| \\ &= |\sum_{i=1}^{m} L(\mathbf{x}^{\tau_{k+1}}, \mathbf{x}_{i}^{k+1}, \boldsymbol{\pi}_{i}^{k+1}) - \frac{1}{m} f_{i}(\mathbf{x}_{i}^{k+1})| \\ &= |\sum_{i \in \mathcal{C}^{\tau_{k+1}}} \frac{1}{2\sigma} \|\boldsymbol{\pi}_{i}^{k+1}\|^{2} - \frac{1}{2\sigma} \|\boldsymbol{\pi}_{i}^{k}\|^{2} + \frac{1}{\sigma} \| \bigtriangleup \boldsymbol{\pi}_{i}^{k+1} \|^{2}| \\ &\leq \sum_{i=1}^{m} |\frac{1}{2\sigma} \| \boldsymbol{\pi}_{i}^{k+1} \|^{2} - \frac{1}{2\sigma} \| \boldsymbol{\pi}_{i}^{k} \|^{2}| + \frac{1}{\sigma} \| \bigtriangleup \boldsymbol{\pi}_{i}^{k+1} \|^{2} \xrightarrow{(47)} 0. \end{aligned}$$

In addition, same reasoning to show (48) enables to derive

$$\begin{aligned} \frac{1}{m}f_i(\mathbf{x}_i^{k+1}) &- \frac{1}{m}f_i(\mathbf{x}^{\tau_{k+1}}) \\ &\leq \langle \triangle \overline{\mathbf{x}}_i^{k+1}, -\boldsymbol{\pi}_i^{k+1} \rangle + \frac{3r_i}{2m} \| \triangle \overline{\mathbf{x}}_i^{k+1} \|^2, \end{aligned}$$

which by (48) yields that $|q_i^k| \leq \frac{3r_i}{2m} \| \triangle \overline{\mathbf{x}}_i^{k+1} \|^2$, where

$$q_i^k := \frac{1}{m} f_i(\mathbf{x}_i^{k+1}) - \frac{1}{m} f_i(\mathbf{x}^{\tau_{k+1}}) - \langle \triangle \overline{\mathbf{x}}_i^{k+1}, \boldsymbol{\pi}_i^{k+1} \rangle.$$

Therefore, the above fact brings out

$$\begin{aligned} |\mathcal{L}^{k+1} - f(\mathbf{x}^{\tau_{k+1}})| &= |\sum_{i=1}^{m} q_i^k + \frac{\sigma}{2} || \triangle \overline{\mathbf{x}}_i^{k+1} ||^2 |\\ &\leq \sum_{i=1}^{m} (\frac{\sigma}{2} + \frac{3r_i}{2m}) || \triangle \overline{\mathbf{x}}_i^{k+1} ||^2 \to 0. \end{aligned}$$

ii) By (33), we derive that, for any $\forall k \in \mathcal{K}$,

$$0 = \sum_{i=1}^{m} (\boldsymbol{\pi}_{i}^{k} + \sigma(\mathbf{x}_{i}^{k} - \mathbf{x}^{\tau_{k+1}}))$$

$$\stackrel{(16)}{=} \sum_{i \in \mathcal{C}^{\tau_{k+1}}} (\boldsymbol{\pi}_{i}^{k} + \sigma \Delta \overline{\mathbf{x}}_{i}^{k+1} - \sigma \Delta \mathbf{x}_{i}^{k+1})$$

$$+ \sum_{i \notin \mathcal{C}^{\tau_{k+1}}} (\boldsymbol{\pi}_{i}^{k+1} - \sigma \Delta \mathbf{x}_{i}^{k+1} - \Delta \boldsymbol{\pi}_{i}^{k+1})$$

$$\stackrel{(14)}{=} \sum_{i=1}^{m} (\boldsymbol{\pi}_{i}^{k+1} - \sigma \Delta \mathbf{x}_{i}^{k+1}) - \sum_{i \notin \mathcal{C}^{\tau_{k+1}}} \Delta \boldsymbol{\pi}_{i}^{k+1},$$
(50)

which together with (47) implies $\lim_{k \in \mathcal{K} \to \infty} \sum_{i=1}^{m} \pi_i^{k+1} = 0$. Let $s := (\tau_{k+1} - 1)k_0 \in \mathcal{K}$. Then

$$\lim_{s(\in\mathcal{K})\to\infty}\sum_{i=1}^{m}\pi_{i}^{s+1}=0.$$
(51)

Moreover, for any k, it is easy to show that

$$s + 1 = (\tau_{k+1} - 1)k_0 + 1 \le k + 1 \le \tau_{k+1}k_0,$$

$$\tau_{s+1} = \lfloor (s+1)/k_0 \rfloor = \lfloor \tau_{k+1} - 1 - 1/k_0 \rfloor = \tau_{k+1}.$$
(52)

Based on this, we now estimate $\pi_i^{k+1} - \pi_i^{s+1}$ for any k. For any $i \in C^{\tau_{k+1}}$, we can show that and hence

$$\begin{split} & \|\boldsymbol{\pi}_{i}^{k+1} - \boldsymbol{\pi}_{i}^{s+1}\| \\ \stackrel{(34)}{=} \|\overline{\boldsymbol{g}}_{i}^{k+1} - \overline{\boldsymbol{g}}_{i}^{s+1} + \frac{1}{m}H_{i}(\bigtriangleup \overline{\mathbf{x}}_{i}^{k+1} - \bigtriangleup \overline{\mathbf{x}}_{i}^{s+1})\| \\ & \leq \frac{r_{i}}{m}(\|\overline{\mathbf{x}}^{\tau_{k+1}} - \overline{\mathbf{x}}^{\tau_{s+1}}\| + \|\bigtriangleup \overline{\mathbf{x}}_{i}^{k+1}\| + \|\bigtriangleup \overline{\mathbf{x}}_{i}^{s+1}\|) \\ \stackrel{(52)}{=} \frac{r_{i}}{m}(\|\bigtriangleup \overline{\mathbf{x}}_{i}^{k+1}\| + \|\bigtriangleup \overline{\mathbf{x}}_{i}^{s+1}\|) \end{split}$$

For any $i \notin C^{\tau_{k+1}}$, $\pi_i^{k+1} = \pi_i^{s+1} = -\overline{g}_i^{s+1}$ by (17). So, the above condition is still valid. Overall, we show that

$$\|\boldsymbol{\pi}_{i}^{k+1} - \boldsymbol{\pi}_{i}^{s+1}\| \leq \frac{r_{i}}{m} (\|\triangle \overline{\mathbf{x}}_{i}^{k+1}\| + \|\triangle \overline{\mathbf{x}}_{i}^{s+1}\|), \qquad (53)$$

for any $i \in [m]$ and $k \ge 0$, which by (47) allows us to show $\pi_i^{k+1} - \pi_i^{s+1} \to 0$, thereby recalling (51) suffices to

$$\lim_{k \to \infty} \sum_{i=1}^{m} \boldsymbol{\pi}_{i}^{k+1} = 0.$$
(54)

This together with (34) and (47) immediately gives us

$$\lim_{k \to \infty} \nabla f(\mathbf{x}^{\tau_{k+1}}) = \lim_{k \to \infty} \sum_{i=1}^{m} \overline{g}_i^{k+1} = 0.$$
 (55)

Finally, the above condition together with $\triangle \overline{\mathbf{x}}_i^{k+1} = (\mathbf{x}_i^{k+1} - \mathbf{x}_i^{\tau_{k+1}}) \rightarrow 0$ and the gradient Lipschitz continuity yields that

$$\lim_{k \to \infty} \sum_{i=1}^m w_i \nabla f_i(\mathbf{x}^{\tau_{k+1}}) = 0,$$

completing the whole proof.

C. Proof of Theorem IV.2

Proof. i) It follows from Lemma B.3 i) and (49) that

$$\mathcal{L}^{0} \ge \mathcal{L}^{k+1} \ge \sum_{i=1}^{m} w_{i} f_{i}(\mathbf{x}^{\tau_{k+1}}) = f(\mathbf{x}^{\tau_{k+1}}), \qquad (56)$$

which implies $\mathbf{x}^{\tau_{k+1}} \in \mathcal{S}(\mathcal{L}^0)$ and hence $\{\mathbf{x}^{\tau_{k+1}}\}$ is bounded due to the boundedness of $\mathcal{S}(\mathcal{L}^0)$. This calls forth the boundedness of $\{\mathbf{x}_i^{k+1}\}$ as $\triangle \overline{\mathbf{x}}_i^{k+1} \rightarrow 0$ from (47). The boundedness of $\{\pi_i^{k+1}\}$ can be ensured because of

$$\begin{aligned} \|\boldsymbol{\pi}_{i}^{k+1}\| \stackrel{(34)}{=} \| \overline{\boldsymbol{g}}_{i}^{k+1} + \frac{1}{m} H_{i} \triangle \overline{\mathbf{x}}_{i}^{k+1} \| \\ & \leq \| \overline{\boldsymbol{g}}_{i}^{k+1} - \boldsymbol{g}_{i}^{0} \| + \| \boldsymbol{g}_{i}^{0} \| + \frac{r_{i}}{m} \| \triangle \overline{\mathbf{x}}_{i}^{k+1} \| \\ & \stackrel{(1)}{\leq} \frac{r_{i}}{m} \| \mathbf{x}^{\tau_{k+1}} - \mathbf{x}_{i}^{0} \| + \| \boldsymbol{g}_{i}^{0} \| + \frac{r_{i}}{m} \| \triangle \overline{\mathbf{x}}_{i}^{k+1} \| < +\infty, \end{aligned}$$

where '<' is from the boundedness of $\{\mathbf{x}^{\tau_{k+1}}\}$. Overall, $\{(\mathbf{x}^{\tau_{k+1}}, X^{k+1}, \Pi^{k+1})\}$ is bounded. Let $(\mathbf{x}^{\infty}, X^{\infty}, \Pi^{\infty})$ be any accumulating point of the sequence, it follows from (34) and $\Delta \overline{\mathbf{x}}_{i}^{k+1} \rightarrow 0$ that

$$0 = \overline{g}_i^{k+1} + \pi_i^{k+1} + \frac{1}{m} H_i \triangle \overline{\mathbf{x}}_i^{k+1}$$

= $g_i^{k+1} + \pi_i^{k+1} + \overline{g}_i^{k+1} - g_i^{k+1} + \frac{1}{m} H_i \triangle \overline{\mathbf{x}}_i^{k+1}$
 $\rightarrow \frac{1}{m} \nabla f_i(\mathbf{x}_i^\infty) + \pi_i^\infty.$

Moreover, (54) and (47) suffice to $\sum_{i=1}^{m} \pi_i^{\infty} = 0$ and $\mathbf{x}_i^{\infty} - \mathbf{x}^{\infty} = 0$. By recalling (10), $(\mathbf{x}^{\infty}, X^{\infty}, \Pi^{\infty})$ is a stationary point of (6) and \mathbf{x}^{∞} is a stationary point of (4).

Since $\triangle \mathbf{x}^{\tau_{k+1}} \to 0$ and \mathbf{x}^{∞} being isolated, whole sequence $\{\mathbf{x}^{\tau_{k+1}}\}$ converges to \mathbf{x}^{∞} by [48, Lemma 4.10]. This together with $\triangle \overline{\mathbf{x}}_i^{k+1} \to 0$ and (34) implies that $\{X^k\}$ and $\{\Pi^k\}$ converge to X^{∞} and Π^{∞} .

D. Proof of Corollary IV.1

Proof. i) The convexity of f and the optimality of x^* lead to

$$f(\mathbf{x}^{\tau_k}) \ge f(\mathbf{x}^*) \ge f(\mathbf{x}^{\tau_k}) + \langle \nabla f(\mathbf{x}^{\tau_k}), \mathbf{x}^* - \mathbf{x}^{\tau_k} \rangle.$$
(57)

Theorem IV.1 ii) states that

$$\lim_{k \to \infty} \nabla F(X^k) = \lim_{k \to \infty} \nabla f(\mathbf{x}^{\tau_k}) = 0$$

Using this and the boundedness of $\{\mathbf{x}^{\tau_k}\}$ from Theorem IV.2, we take the limit of both sides of (57) to derive that $f(\mathbf{x}^{\tau_k}) \rightarrow f(\mathbf{x}^*)$, which recalling Theorem IV.1 i) yields (26).

ii) The conclusion follows from Theorem IV.2 ii) and the fact that the stationary points are equivalent to optimal solutions if f is convex.

iii) The strong convexity of f means that there is a positive constance ν such that

$$egin{aligned} f(\mathbf{x}^{ au_k}) - f(\mathbf{x}^*) &\geq \langle
abla f(\mathbf{x}^*), \mathbf{x}^{ au_k} - \mathbf{x}^*
angle + rac{
u}{2} \|\mathbf{x}^{ au_k} - \mathbf{x}^*\|^2 \ &= rac{
u}{2} \|\mathbf{x}^{ au_k} - \mathbf{x}^*\|^2, \end{aligned}$$

where the equality is due to (10). Taking limit of both sides of the above inequality immediately shows $\mathbf{x}^{\tau_k} \to \mathbf{x}^*$ since $f(\mathbf{x}^{\tau_k}) \to f(\mathbf{x}^*)$. This together with (47) yields $\mathbf{x}_i^k \to \mathbf{x}^*$. Finally, $\pi_i^k \to \pi_i^*$ because of

$$\begin{aligned} \|\boldsymbol{\pi}_{i}^{k} - \boldsymbol{\pi}_{i}^{*}\| \stackrel{(34),(10)}{=} \| \overline{\boldsymbol{g}}_{i}^{k} + \frac{1}{m} H_{i} \triangle \overline{\mathbf{x}}_{i}^{k} - \frac{1}{m} \nabla f_{i}(\mathbf{x}^{*}) \| \\ \stackrel{(1)}{\leq} \quad \frac{r_{i}}{m} (\| \mathbf{x}^{\tau_{k}} - \mathbf{x}^{*} \| + \| \triangle \overline{\mathbf{x}}_{i}^{k} \|) \to 0, \end{aligned}$$

displaying the desired result.

E. Proof of Theorem IV.3

Proof. Following the fact

$$\sigma > 6r/m \ge 6r_i/m, \quad \forall \ i \in [m], \tag{58}$$

for any $j \ge 1$ and (35), there is

$$\begin{aligned} \| \triangle \boldsymbol{\pi}_{i}^{j+1} \|^{2} &\leq \frac{3r_{i}^{2}}{m^{2}} \| \triangle \mathbf{x}_{i}^{j+1} \|^{2} + \frac{6r_{i}^{2}}{m^{2}} \| \triangle \mathbf{x}^{\tau_{j+1}} \|^{2} \\ &\leq \frac{\sigma^{2}}{6} (\| \triangle \mathbf{x}_{i}^{j+1} \|^{2} + \| \triangle \mathbf{x}^{\tau_{j+1}} \|^{2}). \end{aligned}$$
(59)

We note that $\triangle \overline{\mathbf{x}}_i^{k+1} = \triangle \pi_i^{k+1} / \sigma \rightarrow 0$ from (14) if $i \in C^{\tau_{k+1}}$ and $\triangle \overline{\mathbf{x}}_i^{k+1} = 0$ from (16) if $i \notin C^{\tau_{k+1}}$. Therefore,

$$\|\triangle \overline{\mathbf{x}}_i^{k+1}\| \le \|\triangle \boldsymbol{\pi}_i^{k+1} / \sigma\|, \quad \forall \ i \in [m].$$
(60)

Now we focus on $s \in \mathcal{K}$. This by (50) results in

$$\sum_{i=1}^{m} \boldsymbol{\pi}_{i}^{s+1} = \sum_{i=1}^{m} \sigma \triangle \mathbf{x}_{i}^{s+1} + \sum_{i \notin \mathcal{C}^{\tau_{s+1}}} \triangle \boldsymbol{\pi}_{i}^{s+1}$$

which leads to

$$\|\sum_{i=1}^{m} \pi_{i}^{s+1}\|^{2} \leq m \sum_{i=1}^{m} 2(\|\sigma \triangle \mathbf{x}_{i}^{s+1}\|^{2} + \|\triangle \pi_{i}^{s+1}\|^{2}).$$

Using this condition generates

$$\begin{aligned} \|\nabla f(\mathbf{x}^{\tau_{s+1}})\|^{2} &= \|\sum_{i=1}^{m} \overline{g}_{i}^{s+1}\|^{2} \\ \stackrel{(34)}{=} &\|\sum_{i=1}^{m} (\pi_{i}^{s+1} + \frac{1}{m} H_{i} \triangle \overline{\mathbf{x}}_{i}^{s+1})\|^{2} \\ &\leq & 2\|\sum_{i=1}^{m} \pi_{i}^{s+1}\|^{2} + 2m \sum_{i=1}^{m} \frac{r_{i}^{2}}{m^{2}} \|\triangle \overline{\mathbf{x}}_{i}^{s+1}\|^{2} \\ &\leq & 2\|\sum_{i=1}^{m} \pi_{i}^{s+1}\|^{2} + \frac{m}{18} \sum_{i=1}^{m} \|\triangle \pi_{i}^{s+1}\|^{2} \\ &\leq & m \sum_{i=1}^{m} (4\|\sigma \triangle \mathbf{x}_{i}^{s+1}\|^{2} + 5\|\triangle \pi_{i}^{s+1}\|^{2}) \\ &\leq & 5m\sigma^{2} \sum_{i=1}^{m} (\|\triangle \mathbf{x}_{i}^{s+1}\|^{2} + \|\triangle \mathbf{x}^{\tau_{s+1}}\|^{2}) \\ &\leq & \frac{5m\sigma^{2}}{\eta} (\mathcal{L}^{s} - \mathcal{L}^{s+1}). \end{aligned}$$
(61)

Since sequence $\{\mathcal{L}^k\}$ is non-increasing from Lemma B.3, it has $\mathcal{L}^{tk_0+1} \geq \mathcal{L}^{(t+1)k_0} \geq f^*$ by Lemma B.3 for any $t \geq 0$, thereby resulting in

$$\sum_{t=0}^{\tau_{k+1}-1} (\mathcal{L}^{tk_0} - \mathcal{L}^{tk_0+1})$$

= $\mathcal{L}^0 - \sum_{t=0}^{\tau_{k+1}-2} (\mathcal{L}^{tk_0+1} - \mathcal{L}^{(t+1)k_0}) - \mathcal{L}^{(\tau_{k+1}-1)k_0+1}$ (62)
 $\leq \mathcal{L}^0 - \mathcal{L}^{(\tau_{k+1}-1)k_0+1} \leq \mathcal{L}^0 - f^*.$

We note that for $j = 0, 1, 2, \ldots, \tau_{k+1}k_0 - 1$,

$$\tau_{j+1} = \begin{cases} 1, & j = 0, 1, \dots, k_0 - 1, \\ 2, & j = k_0, k_0 + 1, \dots, 2k_0 - 1, \\ \vdots & \vdots \\ \tau_{k+1}, & j = (\tau_{k+1} - 1)k_0, \dots, k, \dots, \tau_{k+1}k_0 - 1. \end{cases}$$

Using the above three facts and $k < \tau_{k+1}k_0 - 1$, we derive

$$\min_{j \in [k]} \|\nabla f(\mathbf{x}^{\tau_j})\|^2 \leq \frac{1}{k} \sum_{j=0}^{k-1} \|\nabla f(\mathbf{x}^{\tau_{j+1}})\|^2 \\ \leq \frac{1}{k} \sum_{j=0}^{\tau_{k+1}k_0 - 1} \|\nabla f(\mathbf{x}^{\tau_{j+1}})\|^2 \\ = \frac{k_0}{k} \sum_{t=0}^{\tau_{k+1} - 1} \|\nabla f(\mathbf{x}^{\tau_{tk_0} + 1})\|^2 \\ \stackrel{(61)}{\leq} \frac{5m\sigma^2 k_0}{\eta k} \sum_{t=0}^{\tau_{k+1} - 1} (\mathcal{L}^{tk_0} - \mathcal{L}^{tk_0 + 1}) \\ \stackrel{(62)}{\leq} \frac{5m\sigma^2 k_0}{\eta k} (\mathcal{L}^0 - f^*),$$

completing the proof.

 \square