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Distributionally Robust Facility Location with Bimodal Random Demand

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Abstract

In this paper, we study a facility location problem in which customer demand is bimodal, i.e., display, or belong to, two spatially distinct distributions. We assume that these two distributions are ambiguous (unknown), and only their mean values and ranges are known. Therefore, we propose a _distributionally robust facility location_ (DRFL) problem that seeks to find a subset of locations from a given set of candidate sites to open facilities to minimize the fixed cost of opening facilities, and worst-case (maximum) expected costs of transportation and unmet demand over a family of distributions characterized through the known means and support of these distributions. We propose a decomposition-based algorithm to solve DRFL, which include valid lower bound inequalities to accelerate the convergence of the algorithm. In a series of numerical experiments, we demonstrate the superior computational and operational performance of our approach as compared with the stochastic programming approach and a DR approach that does not consider bimodality of the demand. Our results draw attention to the need to consider the impact of uncertainty of customer demand when it does not follow one distinct and known distribution in many strategic real-world problems.

Keywords: Facility location; Distributionally robust optimization; Bimodal Demand; Mixed-Integer programming; Cutting plane

1. Introduction

In this paper, we consider a decision-maker who wants to determine a subset of locations from a given set of candidate sites to open facilities and accordingly assign customer demand to these open facilities. Different than classical facility location settings, we focus on the case in which customer demand is bimodal. We use the term "*bimodal*" in a slightly informal way to refer to the tendency of a random demand to display, or belong to, two spatially distinct distributions. For example, depending on the occurrence of a random event, the demand may follow two distinct distributions, one before the occurrence of the event and one after it takes place. We assume that these two distributions are ambiguous (unknown), and only their mean values and ranges are known. The

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quality of facility location decisions is a function of the fixed cost of opening facilities and a measure of the costs of transportation and unmet demand.

Determining facility locations is a fundamental managerial problem and has many applications such as transportation, logistics, healthcare, to name a few (Ahmadi-Javid et al., 2017; Melo et al., 2009; Owen and Daskin, 1998; Turkoglu and Genevois, 2019). Customer demand often derives facility location decisions. Unfortunately, the precise volume of customer demand is not known at that time when making facility location decisions. Even in a perfect world where we can forecast, estimate, or obtain an approximation of the expected demand, many random events can change/shift customer demand (from low to high, for example), and the probability of such shift is hard to predict in advance. For different scenarios, the support of the demand could be different. At the same time, conditioning on the event realization, the expectation of the demand and thus its distribution may also differ.

In the carsharing industry such as Zipcar, for example, customers choose vehicles to rent for a short time from the most convenient rental location. A competitive company that offers cheaper rental options, more modern cars, or more convenient locations in a certain service region may reduce customer demand for Zipcar. And the probability of observing such an event is not known at the time when decision-makers locate their Zipcars. As pointed out by Hao et al. (2019), customer demand for Zipcar or last-mile transportation services such as Taxi and Uber may be higher on rainy/snowy days than other days. The future weather information at the demand location is uncertain at the point when companies decide where to locate their vehicles (Hao et al., 2019).

Indeed a deterministic approach that relies on estimated demand values, which may be easy to solve from a computational perspective, can produce sub-optimal facility location decisions as it does not capture the bimodality and variability of the demand. By incorporating uncertainty, classical two-stage stochastic programming (SP) models seek to find facility location decisions that minimize the first-stage fixed costs of locating facilities and the expected cost of transportation and unmet demand. Here, the expectation is taken with respect to known probability distributions of random demand. In reality, it is often challenging, if not impossible, to estimate the probability distributions of the demand accurately (Basciftci et al., 2019; Lei et al., 2016; Liu et al., 2019). If we locate facilities according to the (optimistically) biased SP solutions, then we may fail to satisfy the demand of a large number of customers.

In this paper, we address the distributional ambiguity and bimodality of customer demand via scenario-wise distributionally robust (DR) optimization. Specifically, we consider the case in which the demand has two scenarios with two distinct distributions (e.g., one before and the other after the occurrence of a random event). Accordingly, we construct a scenario-wise ambiguity set (i.e., a family of distributions) with two scenarios that correspond to two distinct distributions. We characterize this ambiguity set by the known means and ranges of the unknown distributions of the

demand. Our ambiguity set is a special case of the general scenario-wise or multi-modal ambiguity set of Chen et al. (2019) and resemble the bimodal ambiguty set of Shehadeh et al. (2020). Then, we formulate a *distributionally robust facility location* (DRFL) model that seeks to find a subset of locations from a given set of candidate sites to open facilities to minimize the fixed cost of opening facilities, and worst-case expected costs of transportation and unmet demand over the ambiguity set.

We propose a cutting plane decomposition-based algorithm to solve DRFL and derive lower bound inequalities to accelerate the convergence of the algorithm. The results of our extensive numerical experiments demonstrate the superior computational and operational performance of our approach as compared with the SP approach. More broadly, our results draw attention to the need to consider the impact of uncertainty of customer demand when it does not follow one distinct and known distribution in many strategic real-world problems. Thus our results motivate the need for new approaches that consider the multi-modality and ambiguity of the distributions of random parameters in real-world optimization problems.

To the best of our knowledge, and according to the recent survey of Turkoglu and Genevois (2019), this paper is the first to consider the bimodal ambiguity of the demand distribution and accordingly propose a tractable DRFL approach. Although we use the occurrence of a random event to illustrate our ideas, present our approach, and derive useful insights, our approach is applicable in other applications of DRFL in which the distribution of the demand is bimodal, i.e., may follow two distinct and unknown distributions. The reminder of this paper is structured as follows. In Section 2, we review relevant literature. In Section 3, we formally define DRFL and its reformulation. In Section 4, we introduce our decomposition algorithm to solve DRFL. In Section 5, we test various instances to demonstrate the computational efficacy and solution performance of our DR model as compared to the SP model. We draw conclusions and discuss future directions in 6.

2. Relevant Literature

Service and facility location problem. Service and facility location have been extensively studied in the literature for a wide range of private (e.g., industrial plans, warehouses, distribution centers, etc.) and public sectors (e.g., emergency medical services, fire station, etc.). Various operations research techniques have been developed to handle these problems (Chan, 2001). We refer to ReVelle and Eiselt (2005), Turkoglu and Genevois (2019), and Owen and Daskin (1998) for a comprehensive and comparative survey of service facility location problems. Given the uncertain world that we live in, facility location under uncertainty has received a significant attention. We refer to Snyder (2006) for a comprehensive review of facility location problems under random demand, characteristics, and cost parameters. We refer to Ahmadi-Javid et al. (2017) for a thorough

review of deterministic and stochastic healthcare facility location problems and future directions. Most of this literature assumes that customers demand following a fully known one probability distribution.

Stochastic optimization. As pointed out by Chen et al. (2019), there are three frameworks for optimization under uncertainty; stochastic programming (SP), robust optimization (RO), and, more recently, distributionally robust (DR) optimization. Classical SP extends the linear optimization framework to minimize the total expected cost associated with the optimal *here-and-now* (i.e., first-stage) and *wait-and-see* (i.e., second-stage recourse) decisions under a known probability distributions of random parameters. We refer to Birge and Louveaux (2011) and Shapiro et al. (2014) for a thorough discussion on SP. While SP is a powerful and mature modeling approach, its applicability is limited to the cases in which the distribution of the underlying uncertainty is fully known. If we calibrate an SP to a data sample from a biased distribution, then the resulting (*optimistically biased*) decisions will have a disappointing out-of-sample performance. This phenomenon is well-known as the optimizers' curse (Smith and Winkler, 2006). Furthermore, SP approaches are often computationally expensive and intractable.

Classical RO models assume that uncertain parameters reside in a so-called *uncertainty set* of possible outcomes, and optimization is based on the worst-case scenario occurring within the uncertainty set (Bertsimas and Sim, 2004; Ben-Tal et al., 2015; Soyster, 1973). As argued by Chen et al. (2019), Delage and Ye (2010), and Thiele (2010), sometimes classical RO models can yield overly-conservative (pessimistically biased) solutions and poor expected performances because it cannot capture the distributional information of uncertainty.

DR optimization has been developed in recent years and becomes an attractive approach for addressing optimization problems contaminated with uncertain data. DR optimization bridge the gap between the conservatism of RO and the requirement of known and exact distributions in SP. In DR optimization, one assumes that the distribution of uncertain parameters resides in a so-called "*ambiguity set*" and optimization is based on the worst-case distribution within the ambiguity set. The ambiguity set is a family of distributions characterized through certain known properties of the unknown distributions (Esfahani and Kuhn, 2018). Maybe surprisingly, it turns out that DR models, where the distribution of uncertain parameters is a decision variable, are often more tractable than their SP counterparts in many real-world applications (Delage and Ye, 2010). One can use information that is easy to compute such as the mean and range of random parameters to construct the ambiguity sets and build DR models that better mimic reality and less conservative than RO.

Our research belongs to a new class of DR optimization with scenario–wise and multi-modal ambiguity to address the ambiguity and bimodality of random demand in facility location. Here, we use the term "*multi-modal*" in a slightly informal way to refer to the tendency of a random

parameter to display several spatially distinct distributions (the terms bimodal and bimodality are to be interpreted analogously). For different scenarios (e.g., before the occurrence of an event vs. after the event takes place), the random parameter could be different (e.g., typical vs. high demand), while conditioning on the scenario realization, the expectation and distribution of random parameter can also be different (Chen et al., 2019).

Despite the advantages of DR optimization in producing robust and efficient solutions and its success in many real-world applications, it has been less used to address facility location problems. To date, the concept of demand bimodality has not been studied yet. Basciftci et al. (2019), Luo and Mehrotra (2018), Santiváñez and Carlo (2018), and Liu et al. (2019) are some of the pioneering work that uses DR optimization to address the uncertainty of the distribution of customer demand and optimally locate facilities. However, these studies and references therein assumed that the demand follows one unknown distribution and thus did not consider the possibility that the demand could be bimodal. Many random events could affect the volume of customer demand. For example, Hao et al. (2019) point out that the possibility of rain can affect customer demand for last-mile transportation services. The future weather information at the demand location is uncertain at the point when companies decide where to locate their vehicles.

In this paper, we propose the first DR approach for facility location problems with bimodal (event-wise) customer demand. We characterize our ambiguity set with the mean and range of the two unknown distributions of random demand. We propose a cutting plane decomposition-based algorithm to solve DRFL and derive lower bound inequalities to accelerate the convergence of the algorithm. The results of our extensive numerical experiments demonstrate the superior computational and operational performance of our approach as compared with the SP approach, and draw attention to the need to consider the impact of uncertainty of customer demand when it does not follow one distinct and known distribution in many strategic real-world problems.

Notation: For $a, b \in \mathbb{Z}$, we define $[a] := \{1, 2, \ldots, a\}$ and $[a, b]_{\mathbb{Z}} := \{c \in \mathbb{Z} : a \leq c \leq b\}$. The abbreviations "w.l.o.g." and "w.l.o.o." respectively represent "without loss of generality" and "without loss of optimality." Table 1 summarizes other notation.

3. DRFL Formulation and Analysis

We present a distributionally robust facility location (DRFL) problem, in which customer demand is bimodal, and we need to determine a subset of locations from a given set of candidate sites to open facilities and accordingly assign customer demand to these open facilities. The quality of facility location decisions is a function of the fixed cost of opening facilities and a measure of the costs of transportation and unmet demand. In Section 3.1, we define DRFL, introduce the ambiguity set for describing the distributional information and bimodality of demand, and present

Table 1: Notation.

Indices	
i	index of location, $i = 1, \ldots, I$
j	index of costumer, $j = 1, \ldots, J$
Parameters and sets	
I	number, or set, of locations
J	number, or, set of costumers
f_i	cost of opening a facility
$t_{i,j}$	unit transport cost from facility i to customer j
p_j	penalty of not satisfying demand at site j
C_i	capacity of facility i
d_j	demand at customer site j
$\underline{d}_j^{\mathrm{B}}/\overline{d}_j^{\mathrm{B}}$	lower/upper bound of the demand before the event at customer site j
$\underline{d}_j^{\mathrm{A}}/\overline{d}_j^{\mathrm{A}}$	lower/upper bound of the demand after the event at customer site j
First-stage decision variables	
y_i	$\begin{cases} 1, & \text{if a facility is open at location } i, \\ 0, & \text{otherwise.} \end{cases}$
Second-stage decision variables	
$x_{i,j}$	amount of demand of customer j satisfied by facility i
u_j	amount of unsatisfied demand of customer j

our min-max DRFL model accordingly. Then, in Section 3.2, we reformulate the min-max DRFL model to a solvable one.

3.1. Definitions and formulation

We consider a set of I candidate locations for building facilities and J customer sites that generate demand. customer demand is random and "bimodal," i.e., display two spatially distinct distributions. For example, depending on the occurrence of a random event, the demand may follow two distinct and unknown probability distributions, one before the occurrence of the event and one after it takes place. Note that the bimodality of the demand could be due to other reasons than the occurrence of a random event. For ease of presentation, hereafter, we use the idea that the demand bimodality is a function of a random event to present our models and derive useful insights and results.

We assume that only certain properties are known about the distribution $\mathbb{P}_j$ of the demand d_j at customer site j. In particular, we let $\mathbb{P}_j = q_j \mathbb{P}^{\mathrm{B}} + (1 - q_j)\mathbb{P}^{\mathrm{A}}$, where $\mathbb{P}^{\mathrm{B}}$ and $\mathbb{P}^{\mathrm{A}}$ are the distributions of the demand before and after the occurrence of the event, respectively. q_j is 0-1 Bernoulli random variable such that $q_j = 1$ if the event occurs and $q_j = 0$ otherwise. In other words, $q_j = 1$ if the demand follows $\mathbb{P}^{\mathrm{B}}$ (distribution 1) and $q_j = 0$ if it follows $\mathbb{P}^{\mathrm{A}}$ (distribution 2). Accordingly, the demand d_j at each customer site j is $d_j = q_j d_j^{\mathrm{B}} + (1 - q_j)d_j^{\mathrm{A}}$, where $d_j^{\mathrm{B}} \sim \mathbb{P}^{\mathrm{B}}$ and $d_j^{\mathrm{A}} \sim \mathbb{P}^{\mathrm{A}}$.

We further assume that we know the support (i.e., upper and lower bound) and the mean values of the random parameters (q, d^B, d^A). Mathematically, we consider support $S = S^q \times S^B \times S^A$, where S^q, S^B, and S^A are respectively the supports of random parameters q, d^B, and d^A defined as follows:

$$\mathcal{S}^q := \{0,1\}^J,$$

$$\mathcal{S}^B := \left\{ d^B \geq 0: \ \underline{d}_j^B \leq d_j^B \leq \overline{d}_j^B, \forall j \in [J]. \right\},$$

$$\mathcal{S}^A := \left\{ d^A \geq 0: \ \underline{d}_j^A \leq d_j^A \leq \overline{d}_j^A, \forall j \in [J]. \right\},$$

In addition, we let μ^q, μ^B, and μ^A represent the mean values of q, d^B, and d^A, respectively. We denote $\xi := [q, d^B, d^A]^\top$ and $\mu := \mathbb{E}_{\mathbb{P}}[\xi] = [\mu^q, \mu^B, \mu^A]^\top$ for notational brevity. Then, we consider the following mean-support ambiguity set $\mathcal{F}(S, \mu)$:

$$\mathcal{F}(S, \mu) := \left\{ \mathbb{P} \in \mathcal{P}(S): \ \begin{array}{c} \int_S d\mathbb{P} = 1 \\ \mathbb{E}_{\mathbb{P}}[\xi] = \mu \end{array} \right\} \tag{1}$$

where $\mathcal{P}(S)$ in $\mathcal{F}(S, \mu)$ represents the set of probability distributions supported on S and each distribution matches the mean values of q, d^B, and d^A. For all $i \in [I]$, let binary variable y_i represents the location decision such that $y_i = 1$ if a facility is open at location i and $y_i = 0$ otherwise. For all $i \in [I]$ and $j \in [J]$, we let decision variable $x_{i,j}$ represents the amount of satisfied demand at customer site j by facility i. Let decision variable u_j represents the amount of unsatisfied demand at each customer site $j \in [J]$. Finally, we let parameter C_i represents facility capacity in location i and parameters f_i, $t_{i,j}$, and p_j represent the cost of opening a facility at location i, unit transportation cost from location i to site j, and penalty of each unit of unsatisfied demand at site j, respectively. Using this notation and ambiguity set $\mathcal{F}(S, \mu)$, we formulate DRFL as

$$\min_{y \in \mathcal{Y} \subseteq \{0,1\}^I} \left\{ \sum_{i \in I} f_i y_i + \sup_{\mathbb{P} \in \mathcal{F}(S, \mu)} \mathbb{E}_{\mathbb{P}}[Q(y, \xi)] \right\} \tag{2a}$$

where for a given $y \in \mathcal{Y}$ and a joint realization of uncertain parameters $\xi := [q, d^B, d^A]^\top$

$$Q(y, \xi) := \min_{x,u} \left(\sum_{j \in J} \sum_{i \in I} t_{i,j} x_{i,j} + \sum_{j \in J} p_j u_j \right) \tag{3a}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{i,j} + u_j = q_j d_j^B + (1 - q_j) d_j^A, \qquad \forall j \in [J] \tag{3b}$$

$$\sum_{j \in J} x_{i,j} \leq C_i y_i, \qquad \forall i \in [I] \tag{3c}$$

$$u_j, x_{i,j} \geq 0, \qquad \forall i \in [I], \ j \in [J] \tag{3d}$$

Formulation DRFL searches for facility location decisions that minimizes the total cost of locating facilities and the maximum worst-case of transportation and unmet demand over a family of distributions characterized by the ambiguity set $\mathcal{F}(S, \mu)$. Constraints (3b) ensures that demand

7

at each customer site is either satisfied by other locations or penalized, and constraints (3c) respect the capacity of each open facility. Polyhedron $\mathcal{Y}$ can include any constraints related to the facility location decision y.

3.2. *Reformulation*

In this section, we use duality theory and follow a standard approach in DR optimization to reformulate the min-max DRFL model in (2) to a one that is solvable. We first consider the inner maximization problem $\sup_{\mathbb{P}\in\mathcal{F}(S,\mu)} \mathbb{E}_{\mathbb{P}}[Q(x,\xi)]$ for a fixed facility location decision $y \in \mathcal{Y}$, where $\mathbb{P}$ is the decision variable, i.e., we are choosing the distribution that maximizes the expected value of $Q(y,\xi)$. For a fixed $y \in \mathcal{Y}$, we formulate $\sup_{\mathbb{P}\in\mathcal{F}(S,\mu)} \mathbb{E}_{\mathbb{P}}[Q(y,\xi)]$ as the following linear functional optimization problem.

$$\max \ \mathbb{E}_{\mathbb{P}}[Q(y,\xi)] \tag{4a}$$

$$\text{s.t.} \ \mathbb{E}_{\mathbb{P}}[\xi] = \mu, \tag{4b}$$

$$\mathbb{E}_{\mathbb{P}}[\mathbb{1}_S(\xi)] = 1 \tag{4c}$$

where $\mathbb{1}_S(\xi) = 1$ if $\xi \in S$ and $\mathbb{1}_S(\xi) = 0$ if $\xi \notin S$. In Proposition 1, we show that problem (4) is equivalent to problem (5) (see Appendix A for detailed proof).

Proposition 1. *For any $y \in \mathcal{Y}$, problem* (4) *is equivalent to*

$$\min_{\rho,\alpha,\lambda} \left\{ \sum_{j\in J} \mu_j^B \rho_j + \mu_j^A \alpha_j + \mu_j^q \lambda_j + \max_{(q,d^B,d^A)\in S} \left\{ Q(y,q,d^B,d^A) + \sum_{j\in J} -(d_j^B\rho_j + d_j^A\alpha_j + q_j\lambda_j) \right\} \right\} \tag{5}$$

Note that $Q(y,q,d^B,d^A)$ is a minimization problem, and thus in (5) we have an inner max-min problem. We next analyze the structure of $Q(y,q,d^B,d^A)$ for a fixed y and a realized value of (q,d^B,d^A). Taking the dual of $Q(y,q,d^B,d^A)$ lead to the following proposition (see Appendix B for a detailed proof).

Proposition 2. *For fixed $y \in \mathcal{Y}$ and (q,d^B,d^A), it holds that*

$$\max_{(q,d^B,d^A)\in S} \left\{ Q(y,q,d^B,d^A) + \sum_{j\in J} -(d_j^B\rho_j + d_j^A\alpha_j + q_j\lambda_j) \right\}$$

$$\equiv \max_{(\beta,v)\in\Omega} \left\{ \sum_{j\in J} \max\{\overline{d}_j^B, \overline{d}_j^A\}\beta_j + \sum_{i\in I} C_i y_i v_i \right.$$

$$\left. \sum_{j\in J} -\left(\overline{d}_j^B\rho_j + (\underline{d}_j^B - \overline{d}_j^B)(\rho_j)^+\right) - \left(\overline{d}_j^A\alpha_j + (\underline{d}_j^A - \overline{d}_j^A)(\alpha_j)^+\right) + (-\lambda_j)^+ \right\} \tag{6}$$

In view of Equation 6, formulation (5) is equivalent to:

$$\min_{\rho,\alpha,\lambda} \left\{ \sum_{j\in J} \mu_j^B \rho_j + \mu_j^A \alpha_j + \mu_j^q \lambda_j - \left(\overline{d}_j^B\rho_j + (\underline{d}_j^B - \overline{d}_j^B)(\rho_j)^+\right) - \left(\overline{d}_j^A\alpha_j + (\underline{d}_j^A - \overline{d}_j^A)(\alpha_j)^+\right) + (-\lambda_j)^+ \right.$$

$$\left. + F(y) \right\} \tag{7}$$

where $F(y) = \max\limits_{(\beta,v)\in\Omega} \{\sum\limits_{j\in J} \max\{\overline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{A}}\}\beta_j + \sum\limits_{i\in I} C_i y_i v_i\}$. Combining the inner problem in the form of (7) with the outer minimization problem in (2), we derive a reformulation of the DR model in (2) as

$$\min\limits_{\substack{y\in\mathcal{Y}\rho,\alpha,\lambda \\ w,z,r,\delta}} \Big\{ \sum\limits_{i\in I} f_i y_i + \sum\limits_{j\in J} \big(\mu_j^{\mathrm{B}}\rho_j + \mu_j^{\mathrm{A}}\alpha_j + \mu_j^{\mathrm{q}}\lambda_j\big)$$

$$\sum\limits_{j\in J} -\big(\overline{d}_j^{\mathrm{B}}\rho_j + (\underline{d}_j^{\mathrm{B}} - \overline{d}_j^{\mathrm{B}})w_j\big) - \big(\overline{d}_j^{\mathrm{A}}\alpha_j + (\underline{d}_j^{\mathrm{A}} - \overline{d}_j^{\mathrm{A}})z_j\big) + r_j + \delta \Big\} \tag{8a}$$

$$\text{s.t.} \quad w_j \geq \rho_j, \ w_j \geq 0, \ z_j \geq \alpha_j, \ z_j \geq 0, \ r_j \geq -\lambda_j, \ r_j \geq 0, \quad \forall j \in [J] \tag{8b}$$

$$\delta \geq F(y) \tag{8c}$$

Next, we analyze structural properties of function $F(y) = \max\limits_{(\beta,v)\in\Omega} \{\sum\limits_{j\in J} \max\{\overline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{A}}\}\beta_j + \sum\limits_{i\in I} C_i y_i v_i\}$ as a function of variables $y \in Y$ in Proposition (3) (see Appendix C for a detailed proof)

Proposition 3. *For any fixed values of variables y, $F(y) < \infty$. Furthermore, $F(y)$ is convex and piecewise linear in y with finite number of pieces.*

4. Solution Approaches

Given the two-stage characteristics of the problem and Proposition (3), it is natural to attempt to solve formulation (8) (or equivalently, the DR model in (2)) with a decomposition algorithm. In Section 4.1, we present our decomposition (cutting–plane) algorithm to solve the DRFL model in (8). Then, in Sections 4.2, we derive valid lower bound inequalities for the master problem.

4.1. Decomposition Algorithm

Proposition 3 suggests that constraint (8c) describes the epigraph of a convex and piecewise linear function of decision variables in formulation (8). This observation facilitates us applying a separation-based decomposition algorithm to solve model (8) as in Jiang et al. (2017), Thiele et al. (2009), and Lei et al. (2016). Algorithm 1 presents DRFL–decomposition algorithm. Algorithm 1 is finite because we identify a new piece of the function $F(y)$ each time when we augment the set $\{L(y,\delta) \geq 0\}$ in step 4, and the function $F(y)$ has a finite number of pieces (Proposition 3).

4.2. Lower bound inequalities

Since the master problem is a relaxation of the DR problem (i.e., provide a lower bound), the tightness of the lower bound is the key to convergence efficiency. In this section, we aim to incorporate more second-stage information into the master problem without adding optimality cuts into the master problem by exploiting the specific characteristics of the second-stage (recourse) problem. In Proposition 4 and Proposition 5, we derive lower bound inequalities for the master problem, which exploits the structure of the recourse problem (see Appendix D and Appendix E for a detailed proof).

Algorithm 1: DRFL–decomposition algorithm.

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1. Input. Feasible regions $\mathcal{Y}$ and Ω; Set of cuts $\{L(y, \delta) \geq 0\} = \emptyset$; $LB = -\infty$ and $UB = \infty$.

2. Master Problem. Solve the following master problem

$$
Z = \min_{\substack{y \in \mathcal{Y}, \rho, \alpha, \lambda \\ w, z, r, \delta}} \Bigg\{ \sum_{i \in I} f_i y_i + \sum_{j \in J} (\mu_j^{\mathrm{B}} \rho_j + \mu_j^{\mathrm{A}} \alpha_j + \mu_j^{\mathrm{q}} \lambda_j)
$$
$$
+ \sum_{j \in J} -\big(\overline{d}_j^{\mathrm{B}} \rho_j + (\underline{d}_j^{\mathrm{B}} - \overline{d}_j^{\mathrm{B}}) w_j\big) - \big(\overline{d}_j^{\mathrm{A}} \alpha_j + (\underline{d}_j^{\mathrm{A}} - \overline{d}_j^{\mathrm{A}}) z_j\big) + r_j
$$
$$
+ \delta \Bigg\} \tag{9a}
$$

s.t. $\quad w_j \geq \rho_j, \; w_j \geq 0, \; z_j \geq \alpha_j, \; z_j \geq 0, \; r_j \geq -\lambda_j, \; r_j \geq 0, \quad \forall j \in [J]$ \hfill (9b)

$$
L(y, \delta) \geq 0 \tag{9c}
$$

and record an optimal solution (y^*, δ^*) and set $LB = Z^*$.

3. Sub-problem. With y fixed to y^*, solve the following problem

$$
W = \max_{(\beta, v) \in \Omega} \Bigg\{ \sum_{j \in J} \max\{\overline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{A}}\} \beta_j + \sum_{i \in I} C_i y_i v_i \Bigg\} \tag{10a}
$$

and record optimal solution (β^*, v^*) and set $UB = \min\{UB, \; W^* + (LB - \delta^*)\}$

4. if $\delta^* \geq \sum_{j \in J} \max\{\overline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{A}}\} \beta_j^* + \sum_{i \in I} C_i y_i^* v_i^*$ **then**

      stop and return y^* as the optimal solution to the DR formulation (2)

else

      add the cut $\delta \geq \sum_{j \in J} \max\{\overline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{A}}\} \beta_j^* + \sum_{i \in I} C_i y_i v_i^*$ and go to step 2.

end if

Proposition 4. *Inequality* (11) *is a valid lower bound inequality for DRFL.*

$$
\delta \geq \sum_{j \in J} \min\{p_j, \min_{i \in I}\{t_{i,j}\}\} \min\{\underline{d}_j^B, \underline{d}_j^A\} \tag{11}
$$

Proposition 5. *Inequality* (12) *is a valid lower bound inequality for DRFL.*

$$
\delta \geq \min_{j \in J} p_j \Bigg\{ \sum_{j \in J} \min\{\underline{d}_j^B, \underline{d}_j^A\} - \sum_{i \in I} C_i y_i \Bigg\} \tag{12}
$$

5. Computational Experiments

In this section, we generate random instances of DRFL and compare our DR approach with the SP approach and draw several insights. Specifically, we compare the optimal solutions of the DR-bimodal in (8) with those yielded by: (1) SP-bimodal, a SP approach that considers bimodality of demand (see Appendix F for the formulation), (2) DR-plain, a DR model that ignores the bimodality of the demand, and (3) SP-plain, a SP approach that ignores the bimodalitiy of demand. In the

Table 2: DFRLC instances. Notation: I is # of locations, J is # of customers.

Inst	J	I	Inst	J	I
1	10	5	7	40	20
2	10	10	8	40	40
3	20	10	9	50	25
4	20	20	10	50	50
5	30	15	11	100	50
6	30	30	12	100	100

plain models, we ignore the bimodality of demand and assume that it follow a single probability distribution. However, SP-bimodal and SP-plain only differ in how the demand is sampled; from two known distributions and from a single distribution, respectively. The SP models minimize the fixed cost of opening facilities plus the expected transportation and unmet demand costs via the the sample average approximation (SAA) approach (see, e.g., Kim et al. (2015); Kleywegt et al. (2002) for a detailed discussion on SAA).

We summarize our computational study as follows. We first follow a distributional belief to generate N independent and identically distributed (i.i.d) samples of each random parameter. Second, we compute the upper and lower bounds information from the generated samples and use them to obtain the (in-sample) optimal solutions of the DR model. Third, we solve the SP model using the generated sample and compare (1) solution times of DR and SP, (2) optimal facility locations of DR and SP, and (3) the in-sample and out-of-sample performance of the optimal solutions of DR and SP. Section 5.1 presents the details of data generation and experimental design. In Section 5.2, we compare solution times of DR and SP models. In Section 5.3, we compare the optimal solutions of the DR and SP. In Section 5.4, we compare the out-of-sample performance of these solutions. In Section 5.5, we conduct sensitivity analysis, and derive insights into DRFL.

5.1. *Experimental Design*

We construct 12 DRFL instances based on the same parameter settings and assumptions made in the literature. We summarize our test instances in Table 2. Each of the 12 DRFL instances is characterized by the number of customers J and number of candidate facilities I.

For each DRFL instance, we randomly generate a set of potential facility locations and customer sites as uniformly distributed random numbers on a 100 by 100 plane (as in Basciftci et al. (2019), Lei et al. (2014), Lei et al. (2016), and references therein). We compute the distance, $t_{i,j}$, between each candidate location $i \in [I]$ and customer location $j \in [J]$ in Euclidean sense (Basciftci et al., 2019; Lei et al., 2014). We generate fixed opening cost f from uniform distribution as $f_i \in U[2000, 5000]$. We set the capacity $C_i = 150$, for all $i \in [I]$. For most of the experiments, we set the unit penalty cost of unmet demand $p_j > \max\{t_{i,j}\}$ at each $j \in [J]$ (as in Lei et al. (2014), Lei et al. (2016), and references therein). We conduct a sensitivity analysis of these parameters in Section 5.5.

We follow the same procedures in the DR applications literature (see, e.g., Jiang et al. (2017) Mak et al. (2014), Shehadeh et al. (2020)) to generate random parameters as follows. We randomly sample the mean value of the demand before the event as $\mu^{\mathrm{B}} \in U[20, 40]$ and after the event $\mu^{\mathrm{A}} \in U[30, 60]$. We set the standard deviation of the demand before and after the event as $\sigma^{\mathrm{B}} = 0.5\mu^{\mathrm{B}}$ and $\sigma^{\mathrm{A}} = 0.5\mu^{\mathrm{A}}$. To approximate the lower $(\underline{d}^{\mathrm{B}}, \underline{d}^{\mathrm{A}})$ and upper bound $(\overline{d}^{\mathrm{B}}, \overline{d}^{\mathrm{A}})$ values of $(d^{\mathrm{B}}, d^{\mathrm{A}})$, we respectively use the 20% and 80% of the N in-sample data. We generate the in-sample data of d^{B} and d^{A} by following lognormal (LogN) distributions. Specifically, we sample $N = 1000$ realizations of d_j^{B}, for all $j \in [J]$, from LogN with $(\mu^{\mathrm{B}}, \sigma^{\mathrm{B}})$, and $N = 1000$ realizations of d_j^{A}, for all $j \in [J]$, from LogN with $(\mu^{\mathrm{A}}, \sigma^{\mathrm{A}})$. We generate $(q_1^n, \ldots, q_j^n)$, $n \in [N]$, from Bernoulli distribution with $\mu^{\mathrm{q}} = 0.8$. We generate data for the plain models by following the same steps.

For each instance of DRFL, we optimize the SP model with the generated N scenarios and the DR model with the generated mean and support of random parameters. We implemented the SP, DR, and DRFL–decomposition algorithm using AMPL2016 programming language calling CPLEX V12.6.2 as a solver with default settings and relative MIP gap of 1-2%. We ran all experiments on a computer with an Intel Core i7 processor, 2.5 GHz CPU, and 16 GB (1600MHz DDR3) of memory. We imposed a solver time limit of 1 hour.

5.2. *CPU Time*

In this section, we compare solution times of SP and DR models. In addition to the default capacity of $C = 150$, we study solution times of SP and DR with $C = 100$ and a tight capacity of $C \in U[20, 50]$ (i.e., uniformly generated as in Basciftci et al. (2019)). For each of the 12 DRFL instances in Table 2 and choice of C, we randomly generate 5 instances as described in Section 5.1 for a total of 180 SAAs and DR instances. We solve each instance using the SAA formulation of the SP model (see Appendix F for the formulation) and our DR model via the DRFL–decomposition algorithm.

In Table 3, we compare the minimum (Min), average (Avg), and maximum (Max) SP and DR solution times (in seconds) of the 180 instances. From Table 3, we first observe that solution times increase as the number of customer and locations increase under all values of C. Second, we observe that the SP takes a significantly longer time to solve each of the 180 instances than the DR model. The SP can solve all of the 165 SAAs corresponding to instances 1–11 with solution times ranging from 1 to 2787 seconds. The SP terminates with no optimal solution ($\sim 10\%$ MIP relative gap) for all of the 15 SAAs instances corresponding to instance 12.

In contrast, solution times of the DR ranges from 0.1 to 187 seconds for instances 1–10. The DR can quickly solve all of the 90 DRFL instances corresponding to instance 11 and 12 with a relaxed tolerance level $\epsilon = 0.01$ in Algorithm 1 (i.e., the algorithm terminates with near-optimal solutions). Note that when solving these instances with $\epsilon := \frac{UB-LB}{UB} = 0.01$, the gap remained at 0.01% for several hours. This could indicate that CPLEX finds good integer solutions early, but

Table 3: Solution times (in seconds) using the SP and DR formulations. Instances marked with $*$ are solved with $\epsilon := \frac{UB-LB}{UB} = 0.01$

Inst	Model	$C = 150$			$C = 100$			$C \in [20, 50]$		
		Min	Avg	Max	Min	Avg	Max	Min	Avg	Max
1 (10, 5)	DR	0.2	0.2	0.3	0.1	0.1	0.1	0.1	0.1	0.1
	SP	3.6	4.5	6.8	4	6	7	1.0	1.0	1
2 (10, 10)	DR	0.3	0.4	0.6	0.4	0.5	6	0.1	0.1	0.1
	SP	8	8	9	17	32	51	2	3	4
3 (20,10)	DR	0.2	0.2	0.3	0.1	0.1	0.1	0.1	0.1	0.1
	SP	89	92	94	5	15	31	4	4	5
4 (20, 20)	DR	2.0	37	96	1.0	1.6	2.1	0.1	0.1	0.2
	SP	657	763	974	208	573	1079	11	14	24
5 (30,15)	DR	0.3	0.7	0.9	0.1	0.1	0.2	0.1	0.1	0.1
	SP	60	75	126	13	41	77	9	10	11
6 (30, 30)	DR	0.3	0.6	0.9	1.8	2.8	5.2	0.1	0.1	0.2
	SP	215	273	427	97	172	285	34.0	45.0	51
7 (40, 20)	DR	0.6	1.6	2.1	0.1	0.1	15	0.1	0.1	0.10
	SP	185.9	291	417	69	171	259	19	22	24
8 (40, 40)	DR	4.2	56	138	1.9	4.3	6.4	0.2	0.2	0.2
	SP	2164	2467	2690	291	499	783	77	97	136
9 (50, 25)	DR	1.0	2.3	5.5	0.1	0.1	0.1	0.1	0.1	0.1
	SP	357.2	481	721	70	331	781	32	37	41
10 (50, 50)	DR	3.7	8.9	16.7	17	104	187	0.1	0.1	0.1
	SP	1183	1183	1889	464	1110	2787	124	145	220
11 (100, 50)$*$	DR	0.4	0.5	0.6	0.12	0.13	0.14	0.1	0.1	0.07
	SP	211	319	673	242	874	1931	206	230	255
12 (100, 100)$*$	DR	32	65	114	39.0	63.0	86.0	0.1	0.2	0.2
	SP	-	-	-	-	-	-	-	-	-

examine many additional nodes to prove optimality.

The results in Table 3 also suggest that it is easier and faster to solve DRFL instances with a tighter capacity $C \in [20, 50]$ than with a relaxed capacity of $C = 100, 150$. Consider instance 8 (40, 40), for example. Increasing C from $U[20, 50]$ to 100 increased the average solution times of the DR and SP respectively from 0.2 and 97 to 4.30 and 499 seconds. One possible explanation for the increase in solution times could be that the relaxed capacity allows for satisfying a larger amount of demand, which may cause both the SP and DR models to search for various alternative combinations of facility locations than under the tighter capacity.

Finally, it is worthy of mentioning that C is the only parameter affecting the solution times significantly. Using other values of the fixed cost, f, and unmet penalty, p, we obtain similar solution times.

5.3. *Optimal Open Facility Locations*

In this section, we compare the optimal facility location decisions of the DR and SP models. For presentation brevity and illustrative purposes, we consider instances 3 (20, 10), 8 (40, 40), and 12 (100, 100) as examples of relatively small, medium, and large instances, respectively. Table 4 presents the optimal decisions of the SP and DR models under the default parameter settings.

From Table 4 we first observe that to mitigate the ambiguity of the demand, the DR models tends to open more facilities than the SP models. Consider instance 8, for example, the DR-bimodal and DR-plains respectively open 15 and 10 facilities as compared to 9 and 8 facilities opened by the SP-bimodal and SP-plain, respectively. DR-bimodal and SP-bimodal open more facilities than their plain counterparts.

Second, we observe that the DR-bimodal mitigates the bimodality of the demand by opening more facilities than the DR-plain. By opening more facilities, the DR-bimodal satisfies a larger amount of customer demand than the SP models and the DR-plain (reflected by the zero unsatisfied demand in Tables 5–6 presented later in Section 5.4). Although this comes at a higher (one-time) fixed cost associated with opening facilities, as we show in the next section, it results in a lower total cost and better service quality (in terms of satisfying customers demand) in the long run.

5.4. *Out-of-Sample Simulation Performance*

In this section, we compare the simulation performance of the optimal solutions of the DR and SP models. Considering that the results obtained for different instances consistently share some common features, for presentation brevity and illustrative purposes, we again present results with 3 (20, 10), 8 (40, 40), and 12 (100, 100) as examples of relatively small, medium, and large instances, respectively.

We evaluate the out-of-sample performance of the optimal DR and SP solutions to these instances (see Table 4) under both perfect (in-sample) and misspecified (out-of-sample) distributional

14

Table 4: Optimal Facility Location of SP and DR models.

Model	3(20, 10)		8 (40, 40)		12 (100, 100)	
	# open	location	# open	location	# open	location
DR-bimodal	8	1,3,4,5,7,8,9,10	15	1,9,12,13,15,16, 19,20,22, 24,26, 37,38,39,40	35	1,2,3,4,5,6,8,10, 11,12,13,18,19,20, 21,22,23,26,27,28, 29,31,32,33,34,35, 36,40,41,43,45,46, 47,49,50
SP-bimodal	5	1,3,4,8,9	9	12,13,19,22,24, 26,37,38	22	1,2,3,4,6,12,17,18, 22,24,25,27,35,36, 37,38,41,42,43,44, 45,46
SP-plain	5	1,3,5,8,9	8	13,19,22,24,26, 37,38,40	19	1,3,4,6,12,17,18,22, 24,25,27,35,37,41, 42,44,45,46,47
DR-plain	5	1,3,4,8,9	10	9,12,13,15,19,22, 24,37,38,40	25	1,2,3,4,6,9,11,12,17, 18,22,24,25,27,32, 35,36,37,38,41,42, 43,44,45,46

information as follows. First, we fix the optimal first-stage decisions y in the SP model. Then, we solve the second-stage recourse problem in (3) using the following two sets of $N' = 10000$ out-of-sample data $(q_1^n, d_1^{B,n}, d_1^{A,n}), \ldots, (q_J^n, d_J^{B,n}, d_J^{A,n})$, for all $n \in [N']$, to compute the corresponding second stage and unmet demand costs.

1. *Perfect distributional info:* we use the same parameter settings and distributions (i.e., LogN) that we use for generating the N in-sample data to generate the N' data points.

2. *Misspecified distributional info:* we use the same mean values (μ^q, μ^B, μ^A) and standard deviations (σ^B, σ^A) of random parameters (q, d^B, d^A), but we vary distribution type of (d^B, d^A) to generate the N' data. Specifically, we follow Weibull distributions with ranges $[0, \overline{d}^B]$ and $[0, \overline{d}^A]$ to generate $(d_1^{B,n}, d_1^{A,n}), \ldots, (d_J^{B,n}, d_J^{A,n})$, for all $n \in [N']$. This is to simulate the out-of-sample performance of the DR and SP optimal solutions when the in-sample data is biased.

Table 5 presents the means and quantiles of second-stage cost (2-stage), cost of unmet demand, and total cost yielded by optimal solutions of the DR and SP in 4 under perfect (in-sample) distributional information.

Clearly, by opening more facilities, the DR models satisfy a larger amount of customer demand than their SP counterparts. The DR-bimodal has the highest fixed cost because it opens the largest number of facilities as compared to other models. However, DR-bimodal has the best performance with zero unmet demand and thus a significantly lower second-stage and total costs on average and at all quantiles than that of the SP-bimodal and the plain models. In fact, the DR-bimodal has a significantly better performance at the 0.75- and 0.95- quantiles, especially for instance 8 and 12. By opening more facilities than the SP models, DR-plain satisfy a larger amount of demand and thus has a lower second stage and total costs. The SP-plain, which open the least number of facilities and thus has the lowest fixed cost, yields the highest unmet demand and total costs.

Table 5: Out-of-sample performance of optimal schedules given by DR and SP schedules under perfect distributional information (LogN).

Inst	Model	Metric	Fixed	2-stage	unmet	TC
3	DR- bimodal	mean	26815	7416	0	34231
	SP-bimodal		14080	23912	8699	37992
	SP-plain		14080	23917	8846	37997
	DR-plain		14080	23912	8699	37992
	DR- bimodal	0.75–quantile	26815	9348	0	36163
	SP-bimodal		14080	29117	12600	43197
	SP-plain		14080	28717	12300	42797
	DR-plain		14080	29117	12600	43197
	DR- bimodal	0.95–quantile	26815	10353	0	37168
	SP-bimodal		14080	59773	41700	73853
	SP-plain		14080	61105	43035	75185
	DR-plain		14080	59773	41700	73853
8	DR- bimodal	mean	37975	12643	0	50618
	SP-bimodal		22095	44827	26168	66922
	SP-plain		19432	66287	50100	85719
	DR-plain		28282	25372	4539	53654
	DR- bimodal	0.75–quantile	37975	13353	0	51328
	SP-bimodal		22095	62558	42900	84653
	SP-plain		19432	92138	75000	111570
	DR-plain		28282	25291	4539	53573
	DR- bimodal	0.95–quantile	37975	14292	0	52267
	SP-bimodal		22095	100849	79770	122944
	SP-plain		19432	131562	113400	150994
	DR-plain		28282	57130	34500	85412
12	DR- bimodal	mean	98817	20437	0	119254
	SP-bimodal		58089	149360	120957	207449
	SP-plain		49253	262349	239700	311602
	DR-plain		77368	44519	11861	121887
	DR- bimodal	0.75–quantile	98817	21108	0	119925
	SP-bimodal		58089	185324	155700	243413
	SP-plain		49253	300582	277200	349835
	DR-plain		77368	91453	53700	168821
	DR- bimodal	0.95–quantile	98817	22123	0	120940
	SP-bimodal		58089	245994	213960	304083
	SP-plain		49253	355743	331200	404996
	DR-plain		77368	91453	53700	168821

These results imply that when the distributional information is accurate, the DR- bimodal yields near-optimal solutions that provide a better quality of service by satisfying customers demand.

Table 6 presents the out-of-sample performance of the DR and SP models under misspecified distributional information. From these results, we observe that the DR-bimodal yields zero unmet

Table 6: Out-of-sample performance of optimal schedules given by DR and SP schedules under misspecified distributional information (Weibull).

Inst	Model	Metric	Fixed	2-stage	unmet	TC
3	DR- bimodal	mean	26815	8806	0	35621
	SP-bimodal		14080	24054	8700	38134
	SP-plain		14080	26022	10800	40102
	DR-plain		14080	24054	8700	38134
	DR- bimodal	0.75–quantile	26815	9493	0	36308
	SP-bimodal		14080	29431	12900	43511
	SP-plain		14080	32912	16500	46992
	DR-plain		14080	29431	12900	43511
	DR- bimodal	0.95–quantile	26815	10427	0	37242
	SP-bimodal		14080	58501	40500	72581
	SP-plain		14080	63624	45600	77704
	DR-plain		14080	58501	40500	72581
8	DR- bimodal	mean	37975	11697	0	49672
	SP-bimodal		22095	44759	26100	66854
	SP-plain		19432	74340	58651	93772
	DR-plain		28282	29723	7854	58005
	DR- bimodal	0.75–quantile	37975	12224	0	50199
	SP-bimodal		22095	62558	42900	84653
	SP-plain		19432	95649	79200	115081
	DR-plain		28282	31150	7800	59432
	DR- bimodal	0.95–quantile	37975	12950	0	50925
	SP-bimodal		22095	100879	79800	122974
	SP-plain		19432	132329	114735	151761
	DR-plain		28282	69371	44235	97653
12	DR- bimodal	mean	98817	19084	0	117901
	SP-bimodal		58089	148006	120600	206095
	SP-plain		49253	277469	255600	326722
	DR-plain		77368	51341	16800	128709
	DR- bimodal	0.75–quantile	98817	19650	0	118467
	SP-bimodal		58089	184121	155700	242210
	SP-plain		49253	313893	291300	363146
	DR-plain		77368	61319	25200	138687
	DR- bimodal	0.95–quantile	98817	20406	0	119223
	SP-bimodal		58089	244354	213900	302443
	SP-plain		49253	367534	343800	416787
	DR-plain		77368	122469	84000	199837

demand and the lowest total cost as compared to the DR-plain and the SP. The cost reductions are reflected in all quantiles of the random second-stage cost and total cost. The DR-plain has a lower total costs than the two SP models. Moreover, the SP-bimodal satisfies a larger amount of demand and yields a lower second-stage and total costs than the SP-plain.

These simulation results show how the DR-bimodal approach can produce facility location decisions that are robust (i.e., maintain a good performance under various probability distributions of demand). Satisfying customer demand is a desirable property in many, if not all, real-world applications.

5.5. *Sensitivity Analysis*

In this section, we study the sensitivity of DR-bimodal and SP-bimodal to different parameter settings. For illustrative purposes, we consider instance 8 (40, 40) for this experiment. For each experiment, we obtain the DR and SP optimal solutions and then simulate their performance under a sample of 10000 scenarios of the demand generated from LogN distribution.

Impact of event likelihood, μ^{q}

First, we analyze the impact of μ^{q} on the optimal number of open facilities, average total cost, average cost of unmet demand, and average transportation cost. We fix $f = 5000$ and vary $\mu^{\mathsf{q}} \in \{0.3, 0.5, 0.8, 1\}$. Note that $\mu^{\mathsf{q}} = 0.3$ and 0.8 indicate that the event occurs 70% and 20% of the time on average, respectively. We keep all other parameter settings as described in Section 5.1. Figures 1a–1d compares the results under different μ^{q}. It is apparent from Figure 1a that DR tends to open more facilities to mitigate the ambiguity and bimodality of the demand as the event is more likely to occur (i.e., μ^q decreases). Figure 1b indicates that the average total costs of the optimal DR and SP solutions are approximately equal when μ^q approaches to the lower and upper limits of the specified set. The SP solution results in a slightly better average total cost when $\mu^q = 0.3$ and 1. On the other hand, when $\mu^q = 0.5$, the average total cost reaches to the peak point for the optimal SP solution, which is significantly higher than the average total cost of the optimal DR solution. These results indicate that the average total cost is quite sensitive to μ^q for the SP approach. The SP approach shows poor performance compared to the DR approach, especially when the event's probability is similar to the probability that the event does not occur. Figure 1c and Figure 1d also show that the optimal DR solution provides a better service quality with 0 unmet demand and less or approximately the same transportation cost.

Impact of capacity

Second, we analyze the optimal number of open facilities as a function of the capacity parameter C. Specifically, we fix $f = 5000$ and vary $C \in \{100, 150, 200, 250\}$. As C increases from 100 to 250, both models open fewer facilities. When $C \in \{100, 150, 200, 250\}$, the numbers of open facilities by DR are $\{22, 15, 11, 11\}$, and by the SP are $\{12, 8, 7, 5\}$. It makes sense to open fewer facilities when the capacity increases as each facility can satisfy higher demand. The DR solutions result in 0 unmet demand and significantly less transportation cost than the SP model (see Figures 1e–1f), thus providing better service quality.

Impact of variability in demand/demand ranges

18

(a) No. of open facilities vs μ^{q}

(b) Average total cost vs μ^{q}

(c) Average cost of unmet demand vs μ^{q}

(d) Average transportation cost vs μ^{q}

(e) Average cost of unmet demand vs C
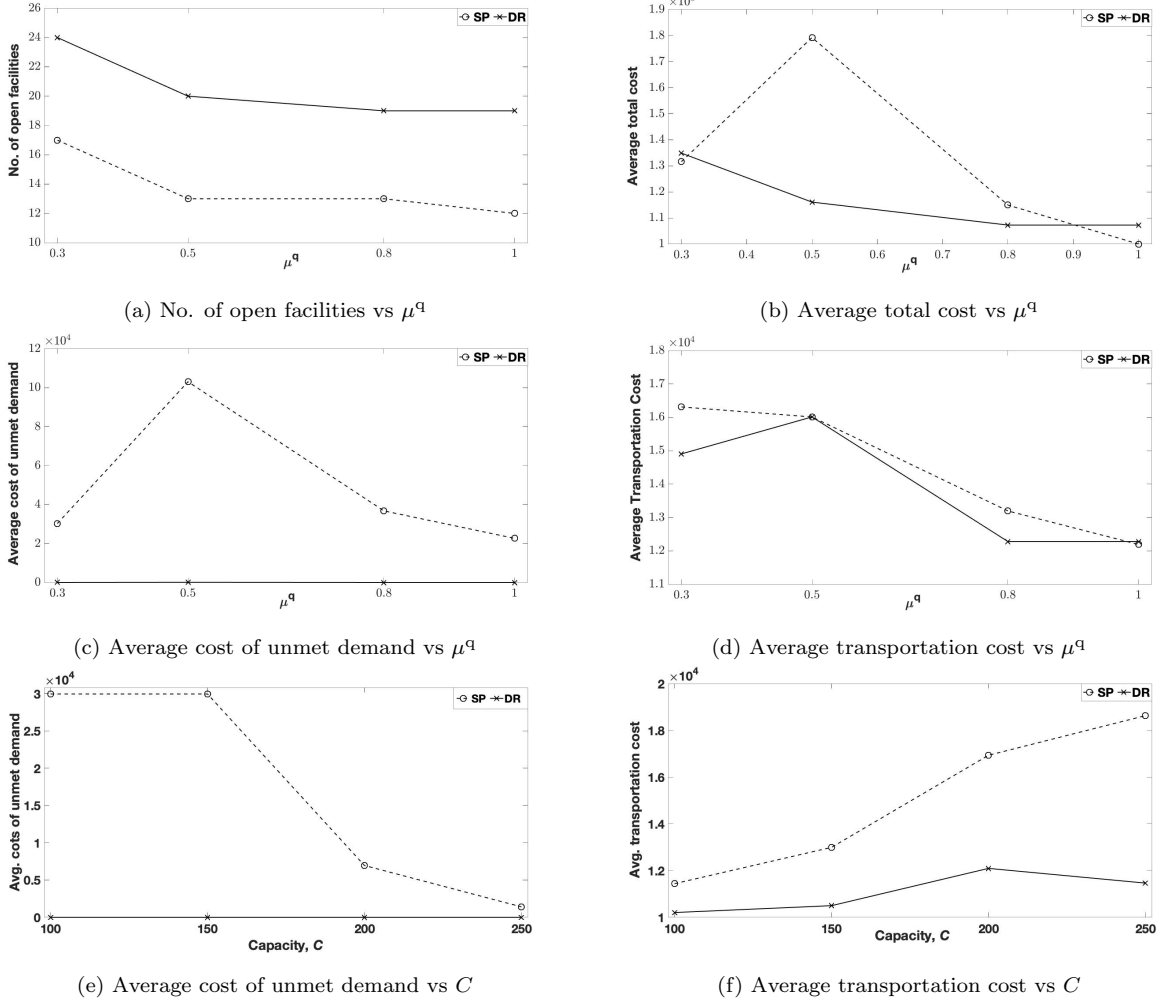
(f) Average transportation cost vs C

Figure 1: Comparison of the results under different parameter settings

Finally, we analyze the sensitivity of the DR and SP solutions to the variability and volume of the demand range. In addition to the base range (Range 1: $\mu^{\mathrm{B}} \in U[20, 40]$ and $\mu^{\mathrm{A}} \in U[30, 60]$), we consider two additional ranges. In Range 2, we increase the variability (difference between the lower and upper bounds) of μ^{B} and μ^{A} to $\mu^{\mathrm{B}} \in [10, 50]$ and $\mu^{\mathrm{A}} \in [20, 70]$. In Range 3, we keep the difference between the upper and lower bounds of μ^{B} and μ^{A} as in Range 1 (20 and 30, respectively) and increase the demand volume (lower and upper bounds) to $\mu^{\mathrm{B}} \in [30, 50]$ and $\mu^{\mathrm{A}} \in [40, 70]$.

Figure 2 presents the optimal number of open facilities, average cost of unmet demand, and average transportation cost under Range 1–Range 3. It is quite apparent from Figure 2a that both models tend to open more facilities under a higher variability and volume of the demand. By opening more facilities than SP, the DR mitigates the increase in the variability and amount of the demand better by maintaining a zero unmet demand (see Figure 2b) and significantly lower transportation cost (Figure 2c). The average second stage cost of the SP under Range 1, Range 2, and Range 3 are respectively 2.75, 1.5, and 1.5 times higher than that of the DR.

19

(a) No. of open facilities

(b) Average cost of unmet demand
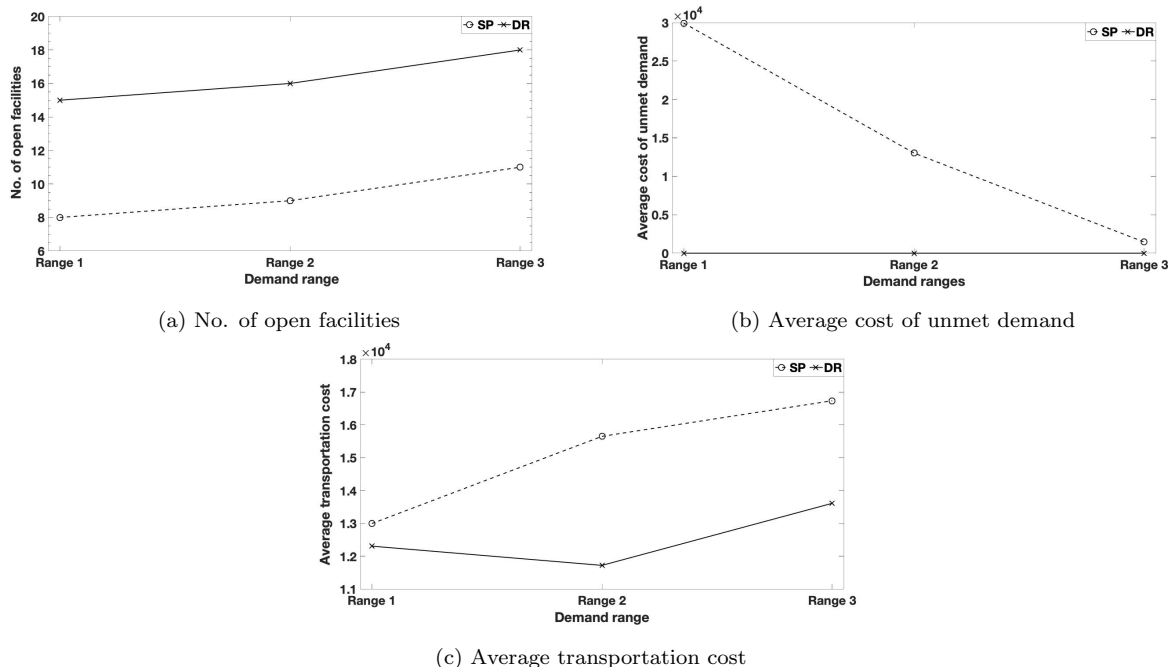

(c) Average transportation cost

Figure 2: Effect of demand range

Our experiments in this section provide an example of how our computationally efficient DR approach can be used to generate robust facility locations decisions under different parameter settings. Practitioners should thus decide whether satisfying customer demand (by adopting DR solutions) is more important for their business and reputation or not (by selecting the SP solution which performs poorly in terms of demand satisfaction, transportation cost, and often total cost).

6. Conclusion

In this paper, we consider a facility location problem, recognizing the bimodality of random demand. That is, customer demand tends to display two spatially distinct distributions. We assume that these two distributions are ambiguous, and only their mean values and ranges are known. Therefore, we propose a _distributionally robust facility location_ (DRFL) problem that seeks to find a subset of locations from a given set of candidate sites to open facilities to minimize the fixed cost of opening facilities, and worst-case expected costs of transportation and unmet demand over a family of distributions characterized through the known means and support of these distributions. We propose a decomposition-based algorithm to solve DRFL, which include valid lower bound inequalities in the master problem.

Using a set of DRFL instances of various sizes constructed based on prior studies, we conduct a series of numerical experiments to draw insights into DRFL. Specifically, we demonstrate that: (1) our DR-bimodal approach has a superior computational performance as compared to the SP

approach, (2) DR-bimodal can produce facility locations decisions that can satisfy customer demand (providing a better quality of service) and maintain lower unmet demand and transportation costs than the SP-bimodal, SP-plain, and DR-plain models (which fail to satisfy customer demand and have higher transportation costs), under various probability distributions (and extreme scenarios) of the random parameters. Although we use the occurrence of a random event to illustrate our ideas, to present our model and derive useful insights, our approach is applicable in other applications of DRFL in which the distribution of the demand is bimodal, i.e., tends to display two spatially distinct and unknown distributions.

We suggest the following areas for future research. First, due to the lack of data, our results are based on assumptions and parameter settings made in prior studies, and we assume that we know the capacity and the number of potential facility locations. We aim to extend our model to optimize the capacity and location of the facilities jointly. Second, we want to extend our approach by incorporating multi-modal probability distributions and higher moments of the demand in a data-driven DR approach. Third, it would be theoretically interesting to extend our approach by considering locating facilities in a country-wide setting (i.e., a larger network) and other non-classical settings such as mobile facility. This may require us to investigate efficient exact methods to solve larger instances of these problems under the general case of multi-modal distribution. Fourth, we aim to incorporate other sources of uncertainty (e.g., random capacity, product usability and lifetime, etc.) and objectives (e.g., holding cost).

Acknowledgments

Appendices

Appendix A. Proof of Proposition 1

Proof. For a fixed y, we can formulate problem (4) as the following linear functional optimization problem.

$$\max_{\mathbb{P} \geq 0} \int_S Q(y, q, d^{\mathrm{B}}, d^{\mathrm{A}}) \, d\mathbb{P} \tag{A.1a}$$

$$\text{s.t.} \int_S d_j^{\mathrm{B}} \, d\mathbb{P} = \mu_j^{\mathrm{B}} \qquad \forall j = 1, \ldots, J \tag{A.1b}$$

$$\int_S d_j^{\mathrm{A}} \, d\mathbb{P} = \mu_j^{\mathrm{A}} \qquad \forall j = 1, \ldots, J \tag{A.1c}$$

$$\int_S q_j \, d\mathbb{P} = \mu_j^{\mathrm{q}} \qquad \forall j = 1, \ldots, J \tag{A.1d}$$

$$\int_S d\mathbb{P} = 1 \tag{A.1e}$$

Letting $\rho = [\rho_1, \ldots, \rho_J]^T$, $\alpha = [\alpha_1, \ldots, \alpha_J]^\top$, $\lambda = [\lambda_1, \ldots, \lambda_J]^\top$, and θ be the dual variable associated with constraints (A.1b), (A.1c), (A.1d), and (A.1e), respectively, we present problem (A.1) in its dual form:

$$\min_{(\rho, \alpha, \lambda) \in \mathbb{R}^J, \theta \in \mathbb{R}} \sum_{j \in J} \mu_j^{\mathrm{B}} \rho_j + \mu_j^{\mathrm{A}} \alpha_j + \mu_j^{\mathrm{q}} \lambda_j + \theta \tag{A.2a}$$

$$\text{s.t.} \sum_{j \in J} (d_j^{\mathrm{B}} \rho_j + d_j^{\mathrm{A}} \alpha_j + q_j \lambda_j) + \theta \geq Q(y, q, d^{\mathrm{B}}, d^{\mathrm{A}}) \qquad \forall (q, d^{\mathrm{B}}, d^{\mathrm{A}}) \in S \tag{A.2b}$$

where ρ, α, λ, and θ are unrestricted in sign, and constraint (A.2b) is associated with the primal variable $\mathbb{P}$. Under the standard assumptions that: (1) $\mu_j^{\mathrm{B}}(\mu_j^{\mathrm{A}})$ lies in the interior of the set $\{\int_S d_j^{\mathrm{B}}(d_j^{\mathrm{A}}) \, d\mathbb{Q} : \mathbb{Q} \text{ is a probability distribution over } S\}$, and (2) μ_j^{q} lies in the interior of the set $\{\int_S q_j \, d\mathbb{Q} : \mathbb{Q} \text{ is a probability distribution over } S\}$ for each customer site j, strong duality hold between (A.1) and (A.2) (Bertsimas and Popescu (2005); Jiang et al. (2017); Mak et al. (2014); Shehadeh et al. (2020)). Note that for fixed $(\rho, \alpha, \lambda, \theta)$, constraint (A.2b) is equivalent to $\theta \geq \max_{(q, d^{\mathrm{B}}, d^{\mathrm{A}}) \in S} \{Q(y, q, d^{\mathrm{B}}, d^{\mathrm{A}}) - \sum_{j \in J} (d_j^{\mathrm{B}} \rho_j + d_j^{\mathrm{A}} \alpha_j + q_j \lambda_j)\}$. Since we are minimizing θ in (A.2), the dual formulation of (A.1) is equivalent to:

$$\min_{\rho, \alpha, \lambda} \left\{ \sum_{j \in J} \mu_j^{\mathrm{B}} \rho_j + \mu_j^{\mathrm{A}} \alpha_j + \mu_j^{\mathrm{q}} \lambda_j + \max_{(q, d^{\mathrm{B}}, d^{\mathrm{A}}) \in S} \left\{ Q(y, q, d^{\mathrm{B}}, d^{\mathrm{A}}) + \sum_{j \in J} -(d_j^{\mathrm{B}} \rho_j + d_j^{\mathrm{A}} \alpha_j + q_j \lambda_j) \right\} \right\}$$

$\square$

Appendix B. Proof of Proposition 2

Proof. For fixed $y \in \mathcal{Y}$ and $\xi = (q, d^{\mathrm{B}}, d^{\mathrm{A}})$, the dual of formulation (3) is as follow

$$Q(y, q, d^{\mathrm{B}}, d^{\mathrm{A}}) = \max_{\beta, v} \sum_{j \in J} (q_j d_j^{\mathrm{B}} + (1 - q_j) d_j^{\mathrm{A}}) \beta_j + \sum_{i \in I} C_i y_i v_i \tag{B.1a}$$

$$\text{s.t.} \quad \beta_j + v_i \le t_{i,j}, \qquad\qquad \forall i \in [I], \forall j \in [J] \tag{B.1b}$$

$$\beta_j \le p_j, \qquad\qquad \forall j \in [J] \tag{B.1c}$$

$$v_i \le 0, \qquad\qquad \forall i \in [I] \tag{B.1d}$$

where $\beta = [\beta_1, \dots, \beta_J]^\top$ and $v = [v_1, \dots, v_I]^\top$ are the dual variables associated with constraints (3b) and (3c), respectively. Note that we can rewrite constraints (B.1b) as $v_i \le \min_{j \in J}\{t_{i,j} - \beta_j\}, \forall i \in [I]$. Given that $v_i \le 0$ and the objective of maximizing a positive number times β_j, then without loss of optimality, we can assume that $\beta_j \ge 0$ (note that if $\beta_j < 0$ for one j, then $v_i \le \min_{j' \ne j}\{t_{i,j'} - \beta_{j'}\}$ and $v_i \le t_{i,j} + |\beta_j| =$ positive number. Given that $v_i \le 0$ then condition $v_i \le t_{i,j} + |\beta_j|$ is redundant, i.e., $v_i \le \min_{j' \ne j}\{t_{i,j'} - \beta_{j'}\}$, and the first term in the objective function will be negative for j).

Next, we derive several useful algebraic expressions:

$$\max_{\substack{q_j \in \{0,1\}, \\ d_j^{\mathrm{B}} \in [\underline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{B}}] \\ d_j^{\mathrm{A}} \in [\underline{d}_j^{\mathrm{A}}, \overline{d}_j^{\mathrm{A}}]}} (q_j d_j^{\mathrm{B}} + (1 - q_j) d_j^{\mathrm{A}}) \beta_j = \left\{ \begin{array}{ll} \overline{d}_j^{\mathrm{B}} \beta_j & \textbf{if } q_j = 1 \\ \overline{d}_j^{\mathrm{A}} \beta_j & \textbf{if } q_j = 0 \end{array} \right\}$$

$$\equiv \max\{\overline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{A}}\} \beta_j. \tag{B.2}$$

$$\max_{d_j^{\mathrm{B}} \in [\underline{d}_j^{\mathrm{B}}, \overline{d}_j^{\mathrm{B}}]} -d_j^{\mathrm{B}} \rho_j = \left\{ \begin{array}{ll} -\overline{d}_j^{\mathrm{B}} \rho_j & \textbf{if } \rho_j \le 0 \\ -\underline{d}_j^{\mathrm{B}} \rho_j & \textbf{if } \rho_j > 0 \end{array} \right\}$$

$$\equiv -\left(\overline{d}_j^{\mathrm{B}} \rho_j + (\underline{d}_j^{\mathrm{B}} - \overline{d}_j^{\mathrm{B}})(\rho_j)^+ \right) \tag{B.3}$$

$$\max_{d_j^{\mathrm{A}} \in [\underline{d}_j^{\mathrm{A}}, \overline{d}_j^{\mathrm{A}}]} -d_j^{\mathrm{A}} \alpha_j = \left\{ \begin{array}{ll} -\overline{d}_j^{\mathrm{A}} \alpha_j & \textbf{if } \alpha_j \le 0 \\ -\underline{d}_j^{\mathrm{A}} \alpha_j & \textbf{if } \alpha_j > 0 \end{array} \right\}$$

$$\equiv -\left(\overline{d}_j^{\mathrm{A}} \alpha_j + (\underline{d}_j^{\mathrm{A}} - \overline{d}_j^{\mathrm{A}})(\alpha_j)^+ \right) \tag{B.4}$$

$$\max_{q_j \in \{0,1\}} -q_j \lambda_j = \left\{ \begin{array}{ll} -\lambda_j & \textbf{if } \lambda_j \le 0 \\ 0 & \textbf{if } \lambda_j > 0 \end{array} \right\}$$

$$\equiv (-\lambda_j)^+ \tag{B.5}$$

Next, we observe that the feasible region $\Omega := \{(\text{B.1b})-(\text{B.1d})\}$ of $Q(y, q, d^{\text{B}}, d^{\text{A}})$ in (B.1) is bounded polyhedral and thus the optimal solution (β^*, v^*) to (B.1) is an extreme point of Ω. In addition, the support S is bounded. It follows that we can equivalently reformulate $\max\limits_{(q, d^{\text{B}}, d^{\text{A}}) \in S} \left\{ Q(y, q, d^{\text{B}}, d^{\text{A}}) + \sum_{j \in J} -(d_j^{\text{B}} \rho_j + d_j^{\text{A}} \alpha_j + q_j \lambda_j) \right\}$ as

$$\max_{\substack{(\beta, v) \in \Omega \\ (q, d^{\text{B}}, d^{\text{A}}) \in S}} \left[\sum_{j \in J} (q_j d_j^{\text{B}} + (1 - q_j) d_j^{\text{A}}) \beta_j + \sum_{i \in I} C_i y_i v_i + \sum_{j \in J} (-d_j^{\text{B}} \rho_j - d_j^{\text{A}} \alpha_j - q_j \lambda_j) \right]$$

$$\max_{(\beta, v) \in \Omega} \sum_{j \in J} \max_{\substack{q_j \in \{0,1\}, \\ d_j^{\text{B}} \in [\underline{d}_j^{\text{B}}, \overline{d}_j^{\text{B}}] \\ d_j^{\text{A}} \in [\underline{d}_j^{\text{A}}, \overline{d}_j^{\text{A}}]}} (q_j d_j^{\text{B}} + (1 - q_j) d_j^{\text{A}}) \beta_j + \sum_{i \in I} C_i y_i v_i$$

$$+ \sum_{j \in J} \max_{d_j^{\text{B}} \in [\underline{d}_j^{\text{B}}, \overline{d}_j^{\text{B}}]} -d_j^{\text{B}} \rho_j + \max_{d_j^{\text{A}} \in [\underline{d}_j^{\text{A}}, \overline{d}_j^{\text{A}}]} -d_j^{\text{A}} \alpha_j + \max_{q_j \in \{0,1\}} -q_j \lambda_j \tag{B.6}$$

Using (B.2)–(B.5), we can rewrite (B.6) as

$$\max_{(q, d^{\text{B}}, d^{\text{A}}) \in S} \left\{ Q(y, q, d^{\text{B}}, d^{\text{A}}) - \sum_{j \in J} (d_j^{\text{B}} \rho_j + d_j^{\text{A}} \alpha_j + q_j \lambda_j) \right\}$$

$$\equiv \max_{\beta, v} \left\{ \sum_{j \in J} \max\{\overline{d}_j^{\text{B}}, \overline{d}_j^{\text{A}}\} \beta_j - \left(\overline{d}_j^{\text{B}} \rho_j + (\underline{d}_j^{\text{B}} - \overline{d}_j^{\text{B}})(\rho_j)^+ \right) - \left(\overline{d}_j^{\text{A}} \alpha_j + (\underline{d}_j^{\text{A}} - \overline{d}_j^{\text{A}})(\alpha_j)^+ \right) + (-\lambda_j)^+ \right.$$

$$\left. + \sum_{i \in I} C_i y_i v_i \right\} \tag{B.7}$$

This complete the proof.

$\square$

Appendix C. Proof of Proposition 3

Proof. First, note that feasible region $\Omega := \{(\text{B.1b}) - (\text{B.1d})\}$ is bounded and independent of y. Therefore, $F(y) = \max\limits_{(\beta, v) \in \Omega} \left\{ \sum_{j \in J} \max\{\overline{d}_j^{\text{B}}, \overline{d}_j^{\text{A}}\} \beta_j + \sum_{i \in I} C_i y_i v_i \right\} < \infty$. Second, for any fixed β and v, $\sum_{j \in J} \max\{\overline{d}_j^{\text{B}}, \overline{d}_j^{\text{A}}\} \beta_j + \sum_{i \in I} C_i y_i v_i$ is a linear function of y. It follows that $F(y)$ is a maximum of a set of linear functions of y and hence convex and piecewise linear. Third, it is clear that each linear piece of $F(y)$ is associated with one distinct extreme point of polyhedra Ω. Hence, $F(y)$ is finite because the bounded polyhedra Ω has a finite number of extreme points. This complete the Proof. $\square$

Appendix D. Proof of Proposition 4

Proof. Recall from the definition of the ambiguity set that the lowest demand of each customer j equals to $\min\{\underline{d}_j^{\text{B}}, \underline{d}_j^{\text{A}}\}$. Now if we assume that the facilities are uncapacitated (i.e., we relax the capacity restriction), then we will be able to satisfy the demand of each customers $j \in [J]$ at

the lowest transportation cost from the nearest open facility $i \in I' := \{i : y_i = 1\}$. Given that $I' \subseteq I$, then the lowest transportation cost from customer i to the nearest facility $i \in I'$ must be at least equal to or larger than $\min_{i \in I}\{t_{i,j}\} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\}$. If $p_j \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\} > \min_{i \in I}\{t_{i,j}\} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\}$, then the second-stage recourse objective cannot be less than $\sum_{j \in J} \min_{i \in I}\{t_{i,j}\} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\}$. Otherwise, $\delta \geq \sum_{j \in J} p_j \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\}$. It follows that (D.1) is a valid lower bound.

$$\delta \geq \sum_{j \in J} \min\{p_j, \min_{i \in I}\{t_{i,j}\}\} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\} \tag{D.1}$$

$\square$

Appendix E. Proof of Proposition 5

Proof. Note that the lowest demand for all customers is $\sum_{j \in J} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\}$ whereas the total capacity available over all facilities is $\sum_{i \in I} C_i y_i$. This means that at least $\sum_{j \in J} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\} - \sum_{i \in I} C_i y_i$ demands are not satisfied whenever the lowest demand exceeds the total capacity. Since the minimum unit penalty for unsatisfied demand is $\min_{j \in J} p_j$, (E.1) is a valid lower bound.

$$\delta \geq \min_{j \in J} p_j \left\{ \sum_{j \in J} \min\{\underline{d}_j^{\mathrm{B}}, \underline{d}_j^{\mathrm{A}}\} - \sum_{i \in I} C_i y_i \right\} \tag{E.1}$$

$\square$

Appendix F. SP Formulation

$$\min_{y \in \mathcal{Y} \subseteq \{0,1\}^I x, u} \left\{ \sum_{i \in I} f_i y_i + \frac{1}{N} \sum_{n=1}^{N} \left[\sum_{j \in J} \sum_{i \in I} t_{i,j} x_{i,j}^n + \sum_{j \in J} p_j u_j^n \right] \right\} \tag{F.1a}$$

$$\text{s.t.} \quad \sum_{i \in I} x_{i,j}^n + u_j^n = q_j^n d_j^{\mathrm{B,n}} + (1 - q_j^n) d_j^{\mathrm{A,n}}, \qquad \forall j \in [J] \ \forall n \in [N] \tag{F.1b}$$

$$\sum_{j \in J} x_{i,j}^n \leq C_i y_i, \qquad \forall i \in [I], \ \forall n \in [N] \tag{F.1c}$$

$$u_j^n, x_{i,j}^n \geq 0, \qquad \forall i \in [I], \ j \in [J], \ \forall n \in [N] \tag{F.1d}$$

References

References

Ahmadi-Javid, A., Seyedi, P., Syam, S.S., 2017. A survey of healthcare facility location. Computers & Operations Research 79, 223–263.

Basciftci, B., Ahmed, S., Shen, S., 2019. Distributionally robust facility location problem under decision-dependent stochastic demand. arXiv preprint arXiv:1912.05577 .

Ben-Tal, A., Den Hertog, D., Vial, J.P., 2015. Deriving robust counterparts of nonlinear uncertain inequalities. Mathematical programming 149, 265–299.

Bertsimas, D., Popescu, I., 2005. Optimal inequalities in probability theory: A convex optimization approach. SIAM Journal on Optimization 15, 780–804.

Bertsimas, D., Sim, M., 2004. The price of robustness. Operations research 52, 35–53.

Birge, J.R., Louveaux, F., 2011. Introduction to stochastic programming. Springer Science & Business Media.

Chan, Y., 2001. Location theory and decision analysis. Springer.

Chen, Z., Sim, M., Xiong, P., 2019. Robust stochastic optimization made easy with rsome .

Delage, E., Ye, Y., 2010. Distributionally robust optimization under moment uncertainty with application to data-driven problems. Operations research 58, 595–612.

Esfahani, P.M., Kuhn, D., 2018. Data-driven distributionally robust optimization using the wasserstein metric: Performance guarantees and tractable reformulations. Mathematical Programming 171, 115–166.

Hao, Z., He, L., Hu, Z., Jiang, J., 2019. Robust vehicle pre-allocation with uncertain covariates. Production and Operations Management .

Jiang, R., Shen, S., Zhang, Y., 2017. Integer programming approaches for appointment scheduling with random no-shows and service durations. Operations Research 65, 1638–1656.

Kim, S., Pasupathy, R., Henderson, S.G., 2015. A guide to sample average approximation, in: Handbook of simulation optimization. Springer, pp. 207–243.

Kleywegt, A.J., Shapiro, A., Homem-de Mello, T., 2002. The sample average approximation method for stochastic discrete optimization. SIAM Journal on Optimization 12, 479–502.

Lei, C., Lin, W.H., Miao, L., 2014. A multicut l-shaped based algorithm to solve a stochastic programming model for the mobile facility routing and scheduling problem. European Journal of operational research 238, 699–710.

Lei, C., Lin, W.H., Miao, L., 2016. A two-stage robust optimization approach for the mobile facility fleet sizing and routing problem under uncertainty. Computers & Operations Research 67, 75–89.

Liu, K., Li, Q., Zhang, Z.H., 2019. Distributionally robust optimization of an emergency medical service station location and sizing problem with joint chance constraints. Transportation research part B: methodological 119, 79–101.

Luo, F., Mehrotra, S., 2018. Distributionally robust optimization with decision dependent ambiguity sets. arXiv preprint arXiv:1806.09215 .

Mak, H.Y., Rong, Y., Zhang, J., 2014. Appointment scheduling with limited distributional information. Management Science 61, 316–334.

Melo, M.T., Nickel, S., Saldanha-Da-Gama, F., 2009. Facility location and supply chain management–a review. European journal of operational research 196, 401–412.

Owen, S.H., Daskin, M.S., 1998. Strategic facility location: A review. European journal of operational research 111, 423–447.

ReVelle, C.S., Eiselt, H.A., 2005. Location analysis: A synthesis and survey. European journal of operational research 165, 1–19.

Santiváñez, J.A., Carlo, H.J., 2018. Reliable capacitated facility location problem with service levels. EURO Journal on Transportation and Logistics 7, 315–341.

Shapiro, A., Dentcheva, D., Ruszczyński, A., 2014. Lectures on stochastic programming: modeling and theory. SIAM.

Shehadeh, K.S., Cohn, A.E., Jiang, R., 2020. A distributionally robust optimization approach for outpatient colonoscopy scheduling. European Journal of Operational Research 283, 549–561.

Smith, J.E., Winkler, R.L., 2006. The optimizer's curse: Skepticism and postdecision surprise in decision analysis. Management Science 52, 311–322.

Snyder, L.V., 2006. Facility location under uncertainty: a review. IIE transactions 38, 547–564.

Soyster, A.L., 1973. Convex programming with set-inclusive constraints and applications to inexact linear programming. Operations research 21, 1154–1157.

Thiele, A., 2010. A note on issues of over-conservatism in robust optimization with cost uncertainty. Optimization 59, 1033–1040.

Thiele, A., Terry, T., Epelman, M., 2009. Robust linear optimization with recourse. Rapport technique , 4–37.

Turkoglu, D.C., Genevois, M.E., 2019. A comparative survey of service facility location problems. *Annals of Operations Research* , 1–70.