NUMERICAL COMPUTATION OF CHARACTERISTIC MULTIPLIERS FOR LINEAR TIME PERIODIC COEFFICIENTS DELAY DIFFERENTIAL EQUATIONS

Dimitri Breda [*,1] **Stefano Maset** [**]
Rossana Vermiglio [*]

[*] *Dipartimento di Matematica e Informatica*
Università degli Studi di Udine
via delle Scienze 208 I-33100 Udine, Italy
{dbreda,vermiglio}@dimi.uniud.it
[**] *Dipartimento di Matematica e Informatica*
Università degli Studi di Trieste
via Valerio 12 I-34127 Trieste, Italy
maset@univ.trieste.it

Abstract: in this work we address the question of asymptotic stability of linear delay differential equations (DDEs) with time periodic coefficients, a class which is recognized to be fundamental in machining tool.
Since the dynamics of such a class of delay systems is governed by the dominant eigenvalues (multipliers) of the *monodromy operator* associated to the system of DDEs, i.e. the solution operator over the period of the coefficients, we discretize it by using pseudospectral differencing techniques based on collocation and approximate the dominant multipliers by the eigenvalues of the resulting matrix. The use of pseudospectral methods has already been proposed in the context of simpler DDEs. Here we fully generalize the method to the class of linear time periodic coefficients DDEs with arbitrary period and multiple discrete and distributed delays.
The scheme is shown to have spectral accuracy by means of several numerical examples.

Keywords: delay differential equations, periodic coefficients, characteristic multipliers, monodromy operator, pseudospectral methods

1. INTRODUCTION

Nowadays many phenomena arising from engineering as well as from physical and biological sciences are modeled with systems of differential equations involving delay terms. The presence of the delay makes models better suit real dynamics and permits a deeper understanding of phenomena behavior. In spite of this, the introduction of past-dependence in the evolution of a system increases its complexity since, opposite to ordinary differential equations, delay models are infinite dimensional systems and their integration and/or study of their stability properties require much more effort. Moreover, delay terms are important since they may change the system dynamics drastically, inducing dangerous instability and lost of

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performance as well as improving stability. The asymptotic stability of constant coefficients delay systems has been treated extensively in several monographs (Niculescu, 2001; Stepan, 1989; Bellen and Zennaro, 2003) either from the analytical and from the numerical point of view, giving raise to a collection of techniques for the computation of the stability boundaries.

With respect to the constant coefficients case, the situation is even more complicated when time periodic coefficients occur as system parameters. Such cases are fundamental for instance in machine tool vibrations (Butcher *et al*, 2004), but also in other engineering fields such as parametric control of robotic systems (Insperger and Stepan, 2000), neural networks (Hagan *et al*, 1996) and optimal control (Deshmukh *et al*, 2004). The question of asymptotic stability for this class of systems is more challenging than for time periodic systems without delay, for which the well-established Floquet theory allows one to determine stability from the eigenvalues of the Floquet transition matrix. An extension of this theory has been shown (Hale and Verduyn Lunel, 1993) to be suitable for delay systems with periodically varying parameters, where a *monodromy operator* can be defined playing the role of an infinite dimensional Floquet transition matrix.

In the last decade, a few numerical methods have been proposed to address the study of the asymptotic stability of time periodic delay systems including time finite element analysis (Bayly *et al*, 2001), numerical simulation (Zhao and Balachandran, 2001), statistical signal variance (Schmitz *et al*, 2002), harmonic balance and infinite determinants (Budak and Altintas, 1998), approximation of the delay by weighted integrals (Insperger and Stepan, 2000) and semi-discretization method (Insperger and Stepan, 2004). Other interesting papers are (Szalai *et al*, 2006; Verheyden *et al*, 2005).

More recently, pseudospectral differencing techniques (Trefethen, 2000) have been applied in (Breda *et al*, 2005) for the computation of the characteristic roots of systems of linear DDEs with constant coefficients by discretizing the infinitesimal generator of the semigroup of solution operators associated to the system. The discretization of the solution operator with similar techniques has been presented first in (Breda, 2004) for the constant coefficients case with multiple discrete and distributed delays. This work represents a natural extension of the above techniques to the discretization of the monodromy operator, i.e. the idea is therefore to compute the eigenvalues of the resulting discretization matrix as approximations to the exact Floquet multipliers, i.e. the eigenvalues of the monodromy operator. Also Beuler (2004) proposed to compute approximations to the monodromy operator by means of pseudospectral differencing methods for systems with integer delays. Here the method is presented for the most general class of linear time periodic DDEs with no restriction between period and maximum delay or between the delays themselves.

After a brief recall of the extended Floquet theory in Section 2, the discretization scheme is presented in detail in Section 3 and 4 and spectral accuracy (Trefethen, 2000) of the method is shown by means of numerical examples in Section 5.

2. MODEL AND ASYMPTOTIC STABILITY

In this section we briefly recall from (Hale and Verduyn Lunel, 1993, §8.1) the extended Floquet theory for linear time periodic DDEs. We consider the most general class of linear DDEs described by

$$y'(t) = \int_{-\tau}^{0} [d_\theta \eta(t,\theta)] y(t+\theta),\ t \in \mathbb{R}, \tag{1}$$

where τ is the maximum delay and $\eta(t,\theta)$ is an $m \times m$ matrix function of bounded variation in $\theta \in [-\tau, 0]$ for each $t \in \mathbb{R}$ and periodic in the argument t with period $\omega > 0$, i.e. $\eta(t+\omega,\theta) = \eta(t,\theta)$.

Let X be the Banach space $\mathcal{C}([-\tau,0],\mathbb{C}^m)$ equipped with the supremum norm. For any $s \geq 0$ and $\varphi \in X$ there is a solution $y = y(\cdot; s, \varphi)$ of (1) defined on $[s, +\infty)$ with initial function $\varphi \in X$. The solution operator is defined as the linear bounded operator on X given by

$$T(t,s)\varphi = y_t(s,\varphi),\ \varphi \in X, \tag{2}$$

for all $t \geq s$ where $y_t(\theta; s, \varphi) = y(t+\theta; s, \varphi)$, $\theta \in [-\tau, 0]$, is the state of (1) at time t. Now, let the *monodromy operator* $U : X \to X$ be defined by

$$U\varphi = T(\omega, 0)\varphi = y_\omega.$$

Since $\omega > 0$, there exists an integer $q > 0$ such that $q\omega \geq \tau$ and hence $U^q = T(q\omega, 0)$ is compact. By the polynomial spectral theorem and the theory of compact operators it can be shown that the spectrum $\sigma(U)$ of U is an at most countable compact set of $\mathbb{C}$ with the only possible accumulation point being zero. Moreover any element μ in $\sigma(U) \setminus \{0\}$ is an eigenvalue of U and is called a *characteristic* or *Floquet multiplier* of (1).

It can be shown that the zero solution of (1) is uniformly asymptotically stable if and only if all characteristic multipliers of (1) have moduli less than 1. Therefore the idea is to get approximations to the dominant multipliers by computing the eigenvalues of a matrix discretization of the infinite dimensional operator U which is obtained by a collocation technique based on pseudospectral differencing methods (Trefethen, 2000).

3. NUMERICAL DISCRETIZATION OF THE SOLUTION OPERATOR

Goal of this section is to find out a matrix discretizing the solution operator $T(s+\Delta, s)$ defined in (2) and this can be scored by building discrete approximations to the initial state $\varphi \in X$ and to the state $y_{s+\Delta}(s, \varphi) \in X$. We assume that $\Delta := \frac{\tau}{k}$ where k is a positive integer.

For fixed N, N positive integer, let θ_i, $i = 0, \ldots, N$, be the Chebyshev nodes on the interval $[-1, 0]$ given by

$$\theta_i = \frac{1}{2}\left(\cos\left(\frac{i\pi}{N}\right) - 1\right)$$

and $t_{n,i}$, $n = 0, \ldots, k$, $i = 0, \ldots, N$, the points in $[s-\tau, s+\Delta]$ defined as (Figure 1)

$$t_{n,i} = s + \Delta(1 - n + \theta_i). \tag{3}$$

Observe that $t_{n,0} = t_{n-1,N}$ for all $n = 1, \ldots, k$. Among these points, select those in the delay interval $[s-\tau, s]$ which define the mesh

$$\Omega_N = \{t_{n,i},\ n = 1, \ldots, k,\ i = 0, \ldots, N\}.$$

and replace the continuous space X by the discrete space $X_N = (\mathbb{C}^m)^{\Omega_N} \cong \mathbb{C}^{m(1+kN)}$.

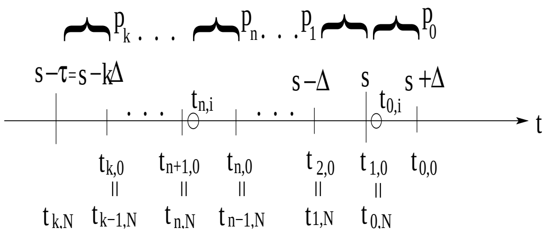

Figure 1. Discretization points in $[s-\tau, s+\Delta]$.

Given the vector in X_N

$$x^{(0)} = \left(x_{1,0}^T, x_{1,1}^T, \ldots, x_{1,N}^T, x_{2,1}^T, \ldots, x_{k,N}^T\right)^T,$$

where $x_{n,i} = x^{(0)}(t_{n,i})$, $n = 1, \ldots, k$, $i = 0, \ldots, N$, consider the piecewise polynomial p defined on $[s-\tau, s+\Delta]$ as

$$p(t) = p_n(t),\ t \in [t_{n,0}, t_{n,N}],\ n = 0, \ldots, k,$$

where p_n, $n = 1, \ldots, k$, is the unique N-degree polynomial interpolating the values $x_{n,i}$ on the nodes $t_{n,i}$, $i = 0, \ldots, N$, while, for $n = 0$, p_0 is the N-degree polynomial obtained by collocation of (1) on the nodes $t_{0,i}$, $i = 0, \ldots, N-1$, and by continuity with p_1 in $t_{0,N} = t_{1,0} = s$:

$$\begin{cases} p_0'(t_{0,i}) = \displaystyle\int_{-\tau}^{0} [d_\theta \eta(t_{0,i}, \theta)] p(t_{0,i} + \theta) \\ p_0(s) = p_1(s) \end{cases}$$

or

$$\begin{cases} p_0'(t_{0,i}) = \displaystyle\int_{-\tau+t_{0,i}}^{t_{k,0}} [d_\sigma \eta(t_{0,i}, \sigma - t_{0,i})] p_k(\sigma) + \\ \quad + \displaystyle\sum_{n=1}^{k-1} \int_{t_{n+1,0}}^{t_{n,0}} [d_\sigma \eta(t_{0,i}, \sigma - t_{0,i})] p_n(\sigma) + \\ \quad + \displaystyle\int_{s}^{t_{0,i}} [d_\sigma \eta(t_{0,i}, \sigma - t_{0,i})] p_0(\sigma) \\ p_0(s) = p_1(s). \end{cases}$$

Now, by setting

$$x_{0,i} = p(t_{0,i}),\ i = 0, \ldots, N,$$

and by using the Lagrange representation of p_n:

$$p_n(t) = \sum_{j=0}^{N} l_{n,j}(t) x_{n,j}$$

for $t \in [t_{n,N}, t_{n,0}]$, $n = 0, \ldots, k$, where

$$l_{n,j}(t) = \prod_{i=0,\ i \neq j}^{N} \frac{t - t_{n,i}}{t_{n,j} - t_{n,i}}$$

are the Lagrange basis polynomials relevant to the nodes (3), one obtains:

$$\begin{cases} \displaystyle\sum_{j=0}^{N} \Bigg\{ l_{0,j}'(t_{0,i}) I_m - \\ - \displaystyle\int_{s}^{t_{0,i}} [d_\sigma \eta(t_{0,i}, \sigma - t_{0,i})] l_{0,j}(\sigma) \Bigg\} x_{0,j} = \\ = \displaystyle\sum_{j=0}^{N} \Bigg\{ \int_{-\tau+t_{0,i}}^{t_{k,0}} [d_\sigma \eta(t_{0,i}, \sigma - t_{0,i})] l_{k,j}(\sigma) x_{k,j} + \\ + \displaystyle\sum_{n=1}^{k-1} \int_{t_{n+1,0}}^{t_{n,0}} [d_\sigma \eta(t_{0,i}, \sigma - t_{0,i})] l_{n,j}(\sigma) x_{n,j} \Bigg\}, \\ x_{0,N} = x_{1,0}, \end{cases} \tag{4}$$

where $i = 0, \ldots, N-1$ and I_m is the $m \times m$ identity matrix.

By considering the vector $x^{(0)}$ as a discrete version of the initial state φ in (2), the vector in X_N

$$x^{(\Delta)} = \left(x_{0,0}^T, x_{0,1}^T, \ldots, x_{0,N}^T, x_{1,1}^T, \ldots, x_{k-1,N}^T\right)^T$$

turns out to be a discrete version of the state $y_t(s, \varphi)$. We can write

$$x^{(\Delta)} = T_N(s+\Delta, s) x^{(0)} \tag{5}$$

and the matrix $T_N(s+\Delta, s)$ has the form

$$T_N(s+\Delta, s) = \begin{pmatrix} A_N \\ B_N \end{pmatrix},$$

$$B_N = \begin{pmatrix} 0 & I_{mN} & 0 & \cdots & \cdots & 0 & 0 \\ 0 & 0 & I_{mN} & 0 & \cdots & \cdots & 0 \\ & & & \ddots & & & \\ & & & & \ddots & & \\ 0 & 0 & \cdots & \cdots & 0 & I_{mN} & 0 \end{pmatrix}$$

where the zeros in B_N are matrices, A_N can be obtained by (4) and B_N corresponds to the part of $x^{(0)}$ shifted to $x^{(\Delta)}$.

It is now evident that (5) is the discrete counterpart of (2). Note that when $\Delta = \tau$, i.e. $k = 1$, $T_N(s + \Delta, s)$ is made only of the matrix A_N.

4. NUMERICAL DISCRETIZATION OF THE MONODROMY OPERATOR

The cases $\omega \leq \tau$ and $\omega > \tau$ are treated separately.

In the case $\omega \leq \tau$, let us set

$$\tau' = k\omega, \quad k = \left\lceil \frac{\tau}{\omega} \right\rceil,$$

where $\lceil z \rceil$ is the smallest integer $\geq z$ and consider $\tau' \geq \tau$ as the new maximum delay for (1). As a discrete approximation of the monodromy operator $U = T(\omega, 0)$ we take $U_N = T_N(\omega, 0)$. Note that if $\omega = \tau$, then $k = 1$ and $\tau' = \tau$.

In the case $\omega > \tau$, let us set

$$\tau' = \frac{\omega}{h}, \quad h = \left\lfloor \frac{\omega}{\tau} \right\rfloor,$$

where $\lfloor z \rfloor$ is the greatest integer $\leq z$ and use $\tau' \geq \tau$ as the new maximum delay for (1). Since

$$U = T(\omega, 0) = T(h\tau', (h-1)\tau') \cdots T(\tau', 0),$$

the operator U is then approximated by

$$U_N = T_N(\omega, 0) = T_N(h\tau', (h-1)\tau') \cdots T_N(\tau', 0).$$

The eigenvalues of U_N approximate (a finite number of) the characteristic multipliers of (1) as it will be shown in Section 5.

5. NUMERICAL RESULTS

We first investigate the so-called damped delayed Mathieu equation (Insperger and Stepan, 2004):

$$y''(t) + \kappa y'(t) + \left(\delta + \varepsilon \cos\left(\frac{2\pi t}{\omega}\right)\right) y(t) = by(t - \tau). \tag{6}$$

In Figure 2 we show the first dominant eigenvalues of (6) approximated with the proposed technique using $N = 100$. It can be seen that the dominant multiplier is outside the unit disk, hence the system is asymptotically unstable. Moreover it can be noticed how the multipliers with smaller modulus accumulate at zero. As to show the spectral accuracy behavior (Trefethen, 2000) of the numerical scheme, i.e. the error between an exact multiplier and its approximation is $O\left(N^{-N}\right)$, we compute approximations to the dominant multiplier for $N = 1, \ldots, 50$ and compare the results with a much more accurate value (more than 13 digits) obtained with a larger number of discretization points ($N = 100$). Results are depicted in Figure

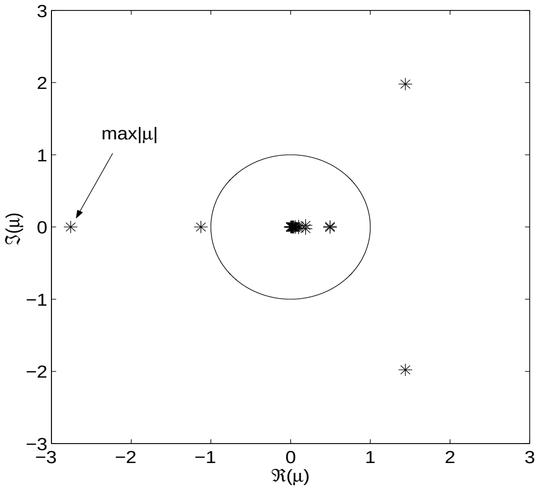

Figure 2. Approximated eigenvalues for (6) with $b = -1.5$, $\kappa = 0.2$, $\varepsilon = 2$, $\tau = \omega = 2\pi$ and $\delta = 1$.

3 for the case $\omega = \tau = 2\pi$ and can be compared with the results in (Insperger and Stepan, 2004). Moreover, to show the correctness of the method also for the cases $\omega \neq \tau$, we present some results in Figure 4 for $\tau = 2\pi$ and $\omega = \sqrt{2}\pi = \tau/\sqrt{2}$ (top row) and $\omega = 2\sqrt{2}\pi = \sqrt{2}\tau$ (bottom row).

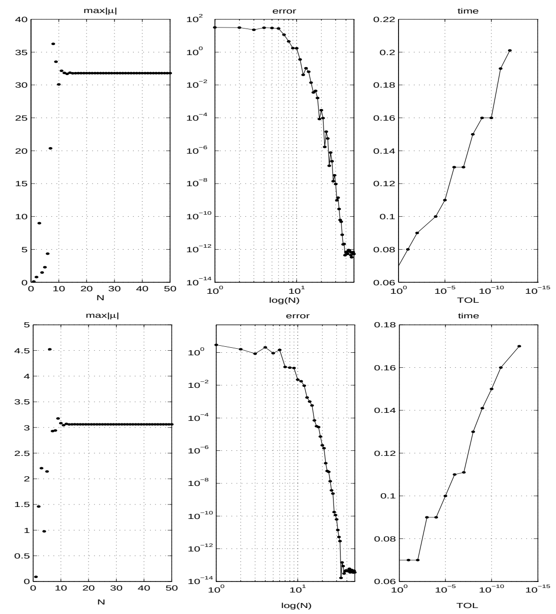

Figure 3. Absolute value (left column), error (center column) and CPU time (seconds, right column) of dominant eigenvalue for (6) with $b = -1.5$, $\kappa = 0.2$, $\varepsilon = 2$, $\tau = \omega = 2\pi$ and $\delta = 0$ (top row), $\delta = 1$ (bottom row).

Given the rapid convergence, the method is suitable to compute stability charts, i.e. the set of stable/unstable regions in the plane of two uncertain system parameters. This is done using the algorithm presented in (Breda *et al*, 2005). In particular, the autonomous case $\varepsilon = 0$ for equation (6), the so-called Hsu-Bhatt-Vyshnegradskii stability chart, is shown in Figure 5 (green regions

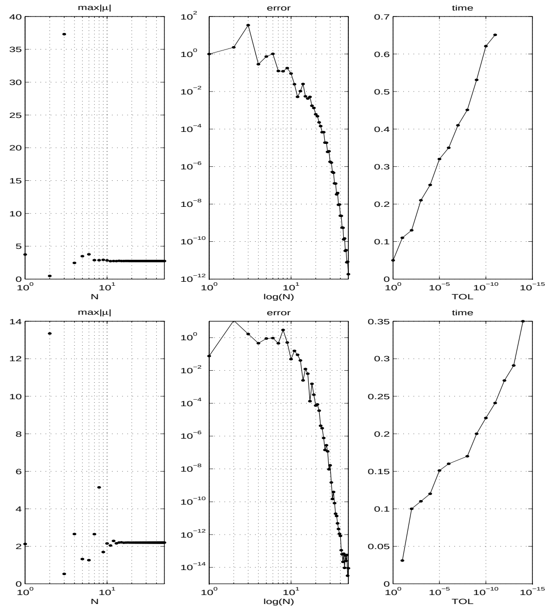

Figure 4. Absolute value (left column), error (center column) and CPU time (seconds, right column) of dominant eigenvalue for (6) with $b = -1.5$, $\kappa = 0.2$, $\varepsilon = 1$, $\tau = 2\pi$ and $\delta = 1$, $\omega = \sqrt{2}\pi = \tau/\sqrt{2}$ (top row) and $\delta = 2$, $\omega = 2\sqrt{2}\pi = \sqrt{2}\tau$ (bottom row).

are stable). Other stability charts for the general equation (6) with $\varepsilon = 1$ are presented in Figure 6 for different choices of the ratio $\frac{\omega}{\tau}$. Compare all the results with those in (Insperger and Stepan, 2004).

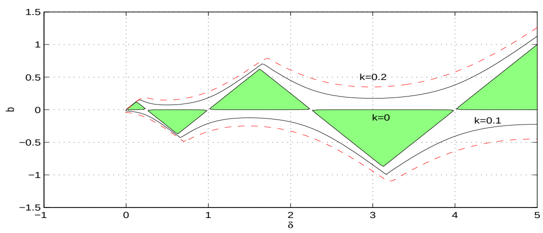

Figure 5. Hsu-Bhatt-Vyshnegradskii stability chart of equation (6) with $\varepsilon = 0$ and $\tau = 2\pi$.

A more general form of the delayed damped Mathieu equation concerning a distributed delay term is the following (Insperger and Stepan, 2004):

$$y''(t) + \kappa y'(t) + c_0(t)y(t) = c_1 \int_{-1}^{0} w(\theta)y(t+\theta)d\theta \tag{7}$$

where

$$c_0(t) = c_{0\delta} + c_{0\varepsilon}\cos\left(\frac{2\pi t}{\omega}\right),\ w(\theta) = -\frac{\pi}{2}\sin(\pi\theta).$$

Results about the approximation of the dominant multiplier are shown in Figure 7 for $\omega = 1/2$, $\kappa = 0$, $c_{0\delta} = 10\pi^2$ and $c_1 = -\pi^2$ (top row) and $c_1 = \pi^2$ (bottom row) and it can be seen that spectral accuracy holds even for the distributed delay case.

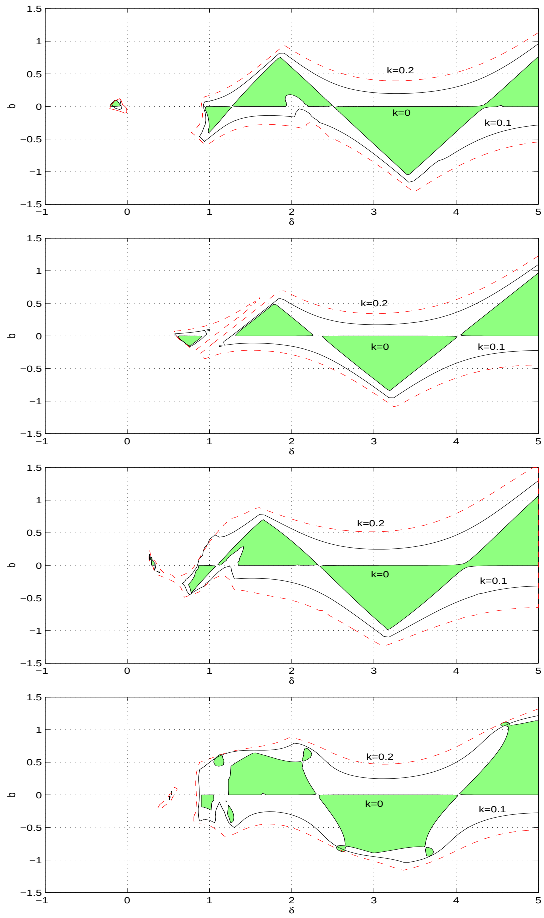

Figure 6. Stability charts of equation (6) with $\varepsilon = 1$, $\tau = 2\pi$ and $\omega = \sqrt{2}\pi = \tau/\sqrt{2}$ (top left), $\omega = 2\pi = \tau$ (top right), $\omega = 2\sqrt{2}\pi = \sqrt{2}\tau$ (bottom left) and $\omega = 4\pi = 2\tau$ (bottom right).

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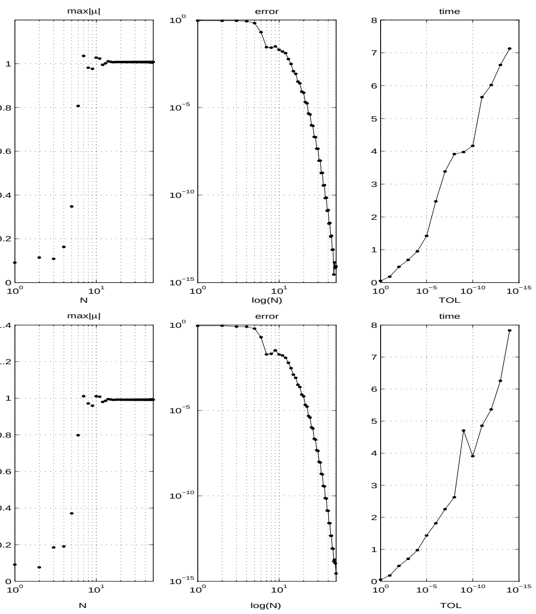

Figure 7. Absolute value (left column), error (center column) and CPU time (seconds, right column) of dominant eigenvalue for (7) with $\omega = 1/2$, $\kappa = 0$, $c_{0\delta} = 10\pi^2$ and $c_1 = -\pi^2$ (top row) and $c_1 = \pi^2$ (bottom row).

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