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Fixed Point Results for Nonexpansive Mappings on Metric Spaces

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Abstract. In this paper we obtain some fixed point results for a class of nonexpansive single-valued mappings and a class of nonexpansive multi-valued mappings in the setting of a metric space. The contraction mappings in Banach sense belong to the class of nonexpansive single-valued mappings considered herein. These results are generalizations of the analogous ones in Khojasteh et al. [Abstr. Appl. Anal. 2014 (2014), Article ID 325840].

1. Introduction

Let (X, d) be a metric space and $f : X \to X$ be a single-valued mapping on X. Then, f is a k-Lipschitz mapping if $d(fx, fy) \leq kd(x, y)$ for all $x, y \in X$, where $k \geq 0$. In particular, if $k \in [0, 1[$, f is called contraction mapping, and if $k = 1$, f is called nonexpansive mapping. Now, it is well known that the contraction mapping principle in [1], is one of the most important theorems in classical functional analysis and is widely considered as the source of metric fixed point theory, where a point $z \in X$ is a fixed point of f if $fz = z$. In fact, the study of fixed points of mappings satisfying a certain metrical contractive condition attracted many researchers, see for example [2, 3, 6, 11–13]. Also, the notion of nonexpansive mapping has a crucial role in fixed point theory. In fact, various researchers investigated the theory of nonexpansive mappings for establishing the existence of fixed points [4, 7, 10, 14]. In almost all papers authors used some iteration techniques for obtaining theoretical results; in particular, we refer to the study of the Picard sequence of initial point x_0, say $\{x_n\}$ with $x_n = f^n x_0 = fx_{n-1}$ for all $n \in \mathbb{N}$. Also, for a lecture on nonexpansive mappings and their properties, we refer to [5].

In this paper, inspired and motivated by Khojasteh et al. in [8], we give sufficient conditions for establishing the existence of fixed points for single-valued and multi-valued nonexpansive mappings. We point out that the class of nonexpansive mappings considered herein contains the class of Banach contraction mappings. Also, some auxiliary facts on the convergence of Picard sequences and distance between fixed points of single-valued mappings are proved, by using a binary relation. Clearly, our theorems are generalizations of the results in [8] and many others. Some examples are given to support the new theory.

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2. Preliminaries

Here we recall some notions and results on the theory of multi-valued mappings. Also we give some auxiliary results useful for the proof of theorems in next sections.

Let (X, d) be a metric space and let $CB(X)$ be the collection of all non-empty closed bounded subsets of X. For $A, B \in CB(X)$, define

$$H(A, B) = \max\{\delta(A, B), \delta(B, A)\},$$

where

$$\delta(A, B) = \sup\{d(a, B) : a \in A\}, \qquad \delta(B, A) = \sup\{d(b, A) : b \in B\}$$

with

$$d(a, C) = \inf\{d(a, x) : x \in C\}.$$

The function $H : CB(X) \times CB(X) \to [0, +\infty[$ is called the Pompeiu-Hausdorff metric induced by the metric d.

We recall the following properties.

Lemma 2.1. *Let (X, d) be a metric space. For any $A, B, C \in CB(X)$ and any $x, y \in X$, we have the following:*

(i) $d(x, B) \le d(x, b)$, *for any* $b \in B$;
(ii) $\delta(A, B) \le H(A, B)$;
(iii) $d(x, B) \le H(A, B)$, *for any* $x \in A$;
(iv) $H(A, A) = 0$;
(v) $H(A, B) = H(B, A)$;
(vi) $H(A, C) \le H(A, B) + H(B, C)$;
(vii) $d(x, A) \le d(x, y) + d(y, A)$.

Definition 2.2. *Let (X, d) be a metric space and let $T : X \to CB(X)$ be a multi-valued mapping. A point $z \in X$ is a fixed point of T if $z \in Tz$.*

The following theorem, proved by Nadler [9] is a generalization of contraction mapping principle, in the case of a multi-valued mapping.

Theorem 2.3. *Let (X, d) be a complete metric space and $T : X \to CB(X)$ be a multi-valued mapping such that*

$$H(Tx, Ty) \le kd(x, y),$$

for all $x, y \in X$, where $k \in [0, 1[$. Then T has a fixed point $z \in X$.

Lemma 2.4. *If $\{a_n\}$ is a nonincreasing sequence of nonnegative real numbers, then the sequence*

$$\left\{\frac{a_n + a_{n+1}}{a_n + a_{n+1} + 1}\right\}$$

is nonincreasing too.

Proof. We note that

$$\frac{a_n + a_{n+1}}{a_n + a_{n+1} + 1} \ge \frac{a_{n+1} + a_{n+2}}{a_{n+1} + a_{n+2} + 1}$$

if and only if

$$(a_n + a_{n+1})(a_{n+1} + a_{n+2} + 1) \ge (a_{n+1} + a_{n+2})(a_n + a_{n+1} + 1).$$

Clearly, this holds since $a_n \ge a_{n+2}$. □

Corollary 2.5. *Let* (X, d) *be a metric space,* $f : X \to X$ *be a nonexpansive mapping and* $x_0 \in X$*. If* $\{x_n\}$ *is a Picard sequence of initial point* x_0*, then the sequence*

$$\left\{ \frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} \right\}$$

is nonincreasing.

Proof. Since f is nonexpansive, we have that $d(x_{n-1}, x_n) \geq d(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Thus the statement holds by Lemma 2.4. □

3. Fixed Points for Single-Valued Mappings

We prove some results for single-valued mappings defined on a metric space endowed with an arbitrary binary relation.
Let $f : X \to X$ be a mapping and $\mathcal{M}$ be a binary relation on X, that is, $\mathcal{M}$ is a subset of $X \times X$. Then, $\mathcal{M}$ is Banach f-invariant if $(fx, f^2x) \in \mathcal{M}$ whenever $(x, fx) \in \mathcal{M}$. Also, a subset Y of X is well ordered with respect to $\mathcal{M}$ if for all $x, y \in Y$ we have $(x, y) \in \mathcal{M}$ or $(y, x) \in \mathcal{M}$.
Let $F^{ix}(f) = \{x \in X : x = fx\}$ denote the set of all fixed points of f on X.

Theorem 3.1. *Let* (X, d) *be a complete metric space endowed with a binary relation* $\mathcal{M}$ *on* X *and* $f : X \to X$ *be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + k \right) d(x, y), \tag{1}$$

for all $(x, y) \in \mathcal{M}$*, where* $k \in [0, 1[$*. Also assume that*

- *(a)* $\mathcal{M}$ *is Banach* f*-invariant;*
- *(b) if* $\{x_n\}$ *is a sequence in* X *such that* $(x_{n-1}, x_n) \in \mathcal{M}$ *for all* $n \in \mathbb{N}$ *and* $x_n \to z \in X$ *as* $n \to +\infty$*, then* $(x_{n-1}, z) \in \mathcal{M}$ *for all* $n \in \mathbb{N}$*;*
- *(c)* $F^{ix}(f)$ *is well ordered with respect to* $\mathcal{M}$*.*

If there exists $x_0 \in X$ *such that* $(x_0, fx_0) \in \mathcal{M}$ *and*

$$\frac{d(x_0, fx_0) + d(fx_0, f^2x_0)}{d(x_0, fx_0) + d(fx_0, f^2x_0) + 1} + k < 1, \tag{2}$$

then

- *(i)* f *has at least one fixed point* $z \in X$*;*
- *(ii) the Picard sequence of initial point* $x_0 \in X$ *converges to a fixed point of* f*;*
- *(iii) if* $z, w \in X$ *are two distinct fixed points of* f*, then* $d(z, w) \geq (1 - k)/2$*.*

Proof. Let $x_0 \in X$ be such that $(x_0, fx_0) \in \mathcal{M}$ and (2) holds, and let $\{x_n\}$ be a Picard sequence of initial point x_0. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From $(x_0, x_1) = (x_0, fx_0) \in \mathcal{M}$, since $\mathcal{M}$ is Banach f-invariant, we deduce $(x_1, x_2) = (fx_0, f^2x_0) \in \mathcal{M}$. This implies

$$(x_{n-1}, x_n) = (f^{n-1}x_0, f^n x_0) \in \mathcal{M} \quad \text{for all } n \in \mathbb{N}.$$

By using the contractive condition (1) with $x = x_{n-1}$ and $y = x_n$, we get

$$d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) \leq \left(\frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} + k \right) d(x_{n-1}, x_n),$$
$$\leq \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} + k \right) d(x_{n-1}, x_n), \tag{3}$$

for all $n \in \mathbb{N}$.

From (3), by Corollary 2.5, we deduce

$$\begin{aligned} d(x_n, x_{n+1}) &\leq \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} + k \right) d(x_{n-1}, x_n) \\ &\leq \left(\frac{d(x_0, x_1) + d(x_1, x_2)}{d(x_0, x_1) + d(x_1, x_2) + 1} + k \right) d(x_{n-1}, x_n) \\ &= \lambda\, d(x_{n-1}, x_n) \end{aligned} \tag{4}$$

for all $n \in \mathbb{N}$, where

$$\lambda = \frac{d(x_0, x_1) + d(x_1, x_2)}{d(x_0, x_1) + d(x_1, x_2) + 1} + k < 1.$$

Then, by (4), $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point for f. By hypothesis (b), we deduce that $(x_n, z) \in \mathcal{M}$. Then, using the contractive condition (1) with $x = x_n$ and $y = z$, we get

$$\begin{aligned} d(x_{n+1}, fz) = d(fx_n, fz) &\leq \left(\frac{d(x_n, fz) + d(z, fx_n)}{d(x_n, fx_n) + d(z, fz) + 1} + k \right) d(x_n, z) \\ &= \left(\frac{d(x_n, fz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, fz) + 1} + k \right) d(x_n, z) \end{aligned} \tag{5}$$

for all $n \in \mathbb{N}$.

On taking limit as $n \to +\infty$ on both sides of (5), we get $d(z, fz) \leq 0$. This implies that $d(z, fz) = 0$, that is, $z = fz$ and hence z is a fixed point of f. Thus (i) and (ii) hold.

If $w \in X$, with $z \neq w$, is another fixed point of f, then by hypothesis (c) we can assume that $(z, w) \in \mathcal{M}$ and hence, using (1) with $x = z$ and $y = w$, we get

$$d(z, w) = d(fz, fw) \leq (d(z, fw) + d(w, fz) + k)d(z, w).$$

This implies that $d(z, w) \geq (1 - k)/2$, that is, (iii) holds. □

In the following result we consider a weak contractive condition.

Theorem 3.2. *Let (X, d) be a complete metric space endowed with a binary relation $\mathcal{M}$ on X and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + k \right) d(x, y) + Ld(y, fx), \tag{6}$$

for all $(x, y) \in \mathcal{M}$, where $k \in [0, 1[$ and L is a nonnegative real number. Also assume that

(a) $\mathcal{M}$ is Banach f-invariant;

(b) if $\{x_n\}$ is a sequence in X such that $(x_{n-1}, x_n) \in \mathcal{M}$ for all $n \in \mathbb{N}$ and $x_n \to z \in X$ as $n \to +\infty$, then $(x_{n-1}, z) \in \mathcal{M}$ for all $n \in \mathbb{N}$;

(c) $F^{ix}(f)$ *is well ordered with respect to* $\mathcal{M}$.

If there exists $x_0 \in X$ *such that* $(x_0, fx_0) \in \mathcal{M}$ *and* (2) *holds, then*

(i) f *has at least one fixed point* $z \in X$;

(ii) the Picard sequence of initial point $x_0 \in X$ *converges to a fixed point of* f;

(iii) if $z, w \in X$ *are two distinct fixed points of* f, *then* $d(z, w) \geq \max\{\frac{1-k-L}{2}, 0\}$.

Proof. Let $x_0 \in X$ be such that $(x_0, fx_0) \in \mathcal{M}$ and (2) holds, and let $\{x_n\}$ be a Picard sequence of initial point x_0. If $x_{n-1} = x_n$ for some $n \in \mathbb{N}$, then x_{n-1} is a fixed point of f and the existence of a fixed point is proved. Now, we suppose that $x_{n-1} \neq x_n$ for all $n \in \mathbb{N}$. From $(x_0, x_1) = (x_0, fx_0) \in \mathcal{M}$, since $\mathcal{M}$ is Banach f-invariant, we deduce that $(x_1, x_2) = (fx_0, f^2x_0) \in \mathcal{M}$. This implies

$$(x_{n-1}, x_n) = (f^{n-1}x_0, f^n x_0) \in \mathcal{M} \quad \text{for all } n \in \mathbb{N}.$$

Using the contractive condition (6) with $x = x_{n-1}$ and $y = x_n$, we get

$$\begin{aligned} d(x_n, x_{n+1}) = d(fx_{n-1}, fx_n) &\leq \left(\frac{d(x_{n-1}, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} + k\right) d(x_{n-1}, x_n) + Ld(x_n, x_n), \\ &\leq \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} + k\right) d(x_{n-1}, x_n) \end{aligned}$$

for all $n \in \mathbb{N}$.

Proceeding as in the proof of Theorem 3.1, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point for f. By hypothesis (b), we deduce that $(x_n, z) \in \mathcal{M}$. Using (6) with $x = x_n$ and $y = z$, we obtain

$$\begin{aligned} d(x_{n+1}, fz) &= d(fx_n, fz) \\ &\leq \left(\frac{d(x_n, fz) + d(z, fx_n)}{d(x_n, fx_n) + d(z, fz) + 1} + k\right) d(x_n, z) + Ld(z, fx_n) \\ &= \left(\frac{d(x_n, fz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, fz) + 1} + k\right) d(x_n, z) + Ld(z, x_{n+1}) \end{aligned} \tag{7}$$

for all $n \in \mathbb{N}$.

On taking limit as $n \to +\infty$ on both sides of (7), we get $d(z, fz) \leq 0$. This implies that $d(z, fz) = 0$, that is, $z = fz$ and hence z is a fixed point of f. Thus (i) and (ii) hold.

If $w \in X$, with $z \neq w$, is another fixed point of f, then by hypothesis (c) we can assume that $(z, w) \in \mathcal{M}$ and hence, using (6) with $x = z$ and $y = w$, we get

$$d(z, w) = d(fz, fw) \leq (d(z, fw) + d(w, fz) + k)d(z, w) + Ld(w, fz).$$

This implies that $1 \leq 2d(z, w) + k + L$, that is, (iii) holds. □

4. Further Results for Single-Valued Mappings

4.1. Consequences in metric spaces

We get the following results by putting $\mathcal{M} = X \times X$ in Theorems 3.1 and 3.2.

Theorem 4.1. *Let (X, d) be a complete metric space and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + k \right) d(x, y), \tag{8}$$

for all $x, y \in X$, where $k \in [0, 1[$. If there exists $x_0 \in X$ such that

$$\frac{d(x_0, fx_0) + d(fx_0, f^2 x_0)}{d(x_0, fx_0) + d(fx_0, f^2 x_0) + 1} + k < 1, \tag{9}$$

then

(i) f has at least one fixed point $z \in X$;

(ii) the Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;

(iii) if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geq (1 - k)/2$.

Proof. All conditions of Theorem 3.1 are satisfied with $\mathcal{M} = X \times X$ and hence Theorem 4.1 follows from Theorem 3.1. □

Theorem 4.2. *Let (X, d) be a complete metric space and $f : X \to X$ be a nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + k \right) d(x, y) + L d(y, fx),$$

for all $x, y \in X$, where $k \in [0, 1[$ and L is a nonnegative real number. If there exists $x_0 \in X$ such that (9) *holds, then*

(i) f has at least one fixed point $z \in X$;

(ii) the Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;

(iii) if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geq \max\{\frac{1-k-L}{2}, 0\}$.

Remark 4.3. *Every contraction satisfies condition* (8) *and also condition* (9) *for some $x_0 \in X$.*

Remark 4.4. *From Theorem 4.1, we obtain Theorem 1 of [8] if $k = 0$.*

Example 4.5. *Let $X = [0, 1] \cup [2, +\infty[$ and $d : X \times X \to [0, +\infty[$ defined by $d(x, y) = |x - y|$. Define $f : X \to X$ by*

$$fx = \begin{cases} \dfrac{1}{2} + \dfrac{1}{2}x & \text{if } x \in [0, 1], \\ \\ 1 + \dfrac{1}{2}x & \text{if } x \in [2, +\infty[. \end{cases}$$

Clearly, (X, d) is a complete metric space and f is a nonexpansive mapping.
Now, for all $x, y \in [0, 1]$ or $x, y \in [2, +\infty[$, we have

$$d(fx, fy) \leq \frac{1}{2} d(x, y).$$

If $x \in [0, 1]$ and $y \in [2, +\infty[$, then

$$\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} \geq 1$$

and hence

$$\begin{aligned} d(fx, fy) &= 1 + \frac{1}{2}y - \frac{1}{2} - \frac{1}{2}x \leq \left(\frac{1}{2} + \frac{1}{2}\right)(y - x) \\ &\leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + \frac{1}{2}\right) d(x, y). \end{aligned}$$

Thus all the hypotheses of Theorem 4.1 are satisfied with $k = 1/2$ and $x_0 = 1/4$. In this case f has two fixed points $x = 1$ and $x = 2$. Note that f is not a contraction, in fact $d(f1, f2) = d(1, 2)$.

Example 4.6. *Let $X = [0, 1]$ and $d : X \times X \to [0, +\infty[$ defined by $d(x, y) = |x - y|$. Define $f : X \to X$ by $fx = x$. Clearly, (X, d) is a complete metric space and f is a nonexpansive mapping. For all $x, y \in X$ with $x \neq y$, we have*

$$d(fx, fy) = |x - y| \leq \left(2|x - y| + \frac{1}{2}\right)|x - y| + \frac{1}{2}|x - y|.$$

Thus all the hypotheses of Theorem 4.2 are satisfied with $k = 1/2$, $L = 1/2$ and $x_0 \in [0, 1]$. In this case f has infinite fixed points. Clearly, f is not a contraction.

4.2. Consequences in partially ordered metric spaces

Let (X, d) be a metric space and let $(X, \preceq)$ be a partially ordered set, then $(X, d, \preceq)$ is called a partially ordered metric space. Also $x, y \in X$ are called comparable if $x \preceq y$ or $y \preceq x$ holds. Further, a self-mapping $f : X \to X$ is called nondecreasing if $fx \preceq fy$, whenever $x \preceq y$ for all $x, y \in X$. Finally, $(X, d, \preceq)$ is called regular if for every nondecreasing sequence $\{x_n\}$ in X convergent to some $x \in X$, we have $x_n \preceq x$ for all $n \in \mathbb{N}$.

Example 4.7. *Let $\preceq$ be a partial order on X such that $(X, \preceq)$ is a partially ordered set. Then*

$$\mathcal{M} = \{(x, y) \in X \times X : \ x \preceq y\}$$

is a binary relation on X. Also if $f : X \to X$ is a nondecreasing mapping, then the set $\mathcal{M}$ is Banach f-invariant.

Theorem 4.8. *Let $(X, d, \preceq)$ be a partially ordered complete metric space and $f : X \to X$ be a nondecreasing nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + k\right) d(x, y),$$

for all comparable $x, y \in X$, where $k \in [0, 1[$. Also assume that

(a) $(X, d, \preceq)$ is regular;

(b) $F^{ix}(f)$ is well ordered.

If there exists $x_0 \in X$ such that $x_0 \preceq fx_0$ and (9) holds, then

(i) f has at least one fixed point $z \in X$;

(ii) the Picard sequence of initial point $x_0 \in X$ converges to a fixed point of f;

(iii) if $z, w \in X$ are two distinct fixed points of f, then $d(z, w) \geq (1 - k)/2$.

Proof. In view of Example 4.7, the proof can be completed on the lines of the proof of Theorem 3.1. □

In the following result we consider a weak contractive condition.

Theorem 4.9. *Let* $(X, d, \preceq)$ *be a partially ordered complete metric space and* $f : X \to X$ *be a nondecreasing nonexpansive mapping such that*

$$d(fx, fy) \leq \left(\frac{d(x, fy) + d(y, fx)}{d(x, fx) + d(y, fy) + 1} + k\right) d(x, y) + Ld(y, fx),$$

for all comparable $x, y \in X$, *where* $k \in [0, 1[$ *and* L *is a nonnegative real number. Also assume that*

(a) $(X, d, \preceq)$ *is regular;*

(b) $F^{ix}(f)$ *is well ordered.*

If there exists $x_0 \in X$ *such that* $x_0 \preceq fx_0$ *and* (9) *holds, then*

(i) f *has at least one fixed point* $z \in X$;

(ii) the Picard sequence of initial point $x_0 \in X$ *converges to a fixed point of* f;

(iii) if $z, w \in X$ *are two distinct fixed points of* f, *then* $d(z, w) \geq \max\{\frac{1-k-L}{2}, 0\}$.

Proof. Again, in view of Example 4.7, the proof can be completed proceeding as in the proofs of Theorems 3.1 and 3.2. □

5. Fixed Points for Multi-Valued Mappings

We give a result of existence of fixed point for a class of multi-valued mappings. Let $K(X)$ be the collection of all non-empty compact subsets of X.

Theorem 5.1. *Let (X,d) be a complete metric space and let* $T : X \to K(X)$ *be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leq \left(\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} + k\right) d(x, y), \tag{10}$$

for all $x, y \in X$, *where* $k \in [0, 1[$. *If there exists* $x_0 \in X$ *such that*

$$\frac{d(x_0, Tx_0) + d(x_1, Tx_1)}{d(x_0, Tx_0) + d(x_1, Tx_1) + 1} + k < 1 \tag{11}$$

holds, where $x_1 \in Tx_0$ *is such that* $d(x_0, x_1) = d(x_0, Tx_0)$, *then* T *has at least one fixed point* $z \in X$.

Proof. Let $x_0 \in X$, since Tx_0 is compact there exists $x_1 \in Tx_0$ such that $d(x_0, x_1) = d(x_0, Tx_0)$. Clearly, if $x_0 = x_1$ or $x_1 \in Tx_1$, we deduce that x_1 is a fixed point of T and so we can conclude the proof. Now, we assume that $x_0 \neq x_1$, $x_1 \notin Tx_1$ and (11) holds. From $d(x_1, Tx_1) > 0$, it follows that $H(Tx_0, Tx_1) > 0$. Next, since Tx_1 is compact, there exists $x_2 \in Tx_1$ such that

$$d(x_1, x_2) = d(x_1, Tx_1).$$

By (11), then we get

$$\begin{aligned} d(x_1, x_2) = d(x_1, Tx_1) \leq H(Tx_0, Tx_1) & \\ &\leq \left(\frac{d(x_0, Tx_1)}{d(x_0, Tx_0) + d(x_1, Tx_1) + 1} + k\right) d(x_0, x_1) \\ &\leq \left(\frac{d(x_0, x_1) + d(x_1, x_2)}{d(x_0, x_1) + d(x_1, x_2) + 1} + k\right) d(x_0, x_1). \end{aligned}$$

Now, we suppose to have chosen $x_1, \ldots, x_n \in X$ such that $x_{i+1} \in Tx_i$, $x_i \notin Tx_i$ and

$$d(x_i, x_{i+1}) = d(x_i, Tx_i) \leq \left(\frac{d(x_{i-1}, x_i) + d(x_i, x_{i+1})}{d(x_{i-1}, x_i) + d(x_i, x_{i+1}) + 1} + k \right) d(x_{i-1}, x_i)$$

for all $i = 1, \ldots, n-1$. Next, we choose $x_{n+1} \in Tx_n$ such that $d(x_n, x_{n+1}) = d(x_n, Tx_n)$, this is possible since Tx_n is compact.

Then

$$\begin{aligned} d(x_n, x_{n+1}) &= d(x_n, Tx_n) \leq H(Tx_{n-1}, Tx_n) \\ &\leq \left(\frac{d(x_{n-1}, Tx_n)}{d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n) + 1} + k \right) d(x_{n-1}, x_n) \\ &\leq \left(\frac{d(x_{n-1}, x_n) + d(x_n, x_{n+1})}{d(x_{n-1}, x_n) + d(x_n, x_{n+1}) + 1} + k \right) d(x_{n-1}, x_n). \end{aligned} \tag{12}$$

If $x_{n+1} = x_n$, then x_n is a fixed point of T and the proof is finished. If $x_n \notin Tx_n$, iterating this procedure we construct a sequence $\{x_n\} \subset X$ such that $x_{n+1} \in Tx_n$, $x_n \notin Tx_n$ and (12) holds for all $n \in \mathbb{N}$. Proceeding as in the proof of Theorem 3.1, we deduce that $\{x_n\}$ is a Cauchy sequence. Since X is a complete metric space, the sequence $\{x_n\}$ converges to some $z \in X$. Now, we prove that z is a fixed point of T. Using (10) with $x = x_n$ and $y = z$, we obtain

$$\begin{aligned} d(z, Tz) &\leq d(z, x_{n+1}) + d(x_{n+1}, Tz) \\ &\leq d(z, x_{n+1}) + H(Tx_n, Tz) \\ &\leq d(z, x_{n+1}) + \left(\frac{d(x_n, Tz) + d(z, Tx_n)}{d(x_n, Tx_n) + d(z, Tz) + 1} + k \right) d(x_n, z) \\ &\leq d(z, x_{n+1}) + \left(\frac{d(x_n, z) + d(z, Tz) + d(z, x_{n+1})}{d(x_n, x_{n+1}) + d(z, Tz) + 1} + k \right) d(x_n, z) \end{aligned}$$

for all $n \in \mathbb{N}$.

On taking limit as $n \to +\infty$ on both sides, we get $d(z, Tz) = 0$. As Tz is closed,we obtain that $z \in Tz$, that is, z is a fixed point of T. □

Proceeding as in the proof of Theorem 5.1, one can prove the following theorem.

Theorem 5.2. *Let (X,d) be a complete metric space and let $T : X \to K(X)$ be a nonexpansive multi-valued mapping such that*

$$H(Tx, Ty) \leq \left(\frac{d(x, Ty) + d(y, Tx)}{d(x, Tx) + d(y, Ty) + 1} + k \right) d(x, y) + Ld(y, Tx),$$

for all $x, y \in X$, where $k \in [0, 1[$ and L is a nonnegative real number. If there exists $x_0 \in X$ such that (11) *holds, then T has at least one fixed point $z \in X$.*

Example 5.3. *Let $X = [0, 1] \cup [2, +\infty[$ and $d : X \times X \to [0, +\infty[$ defined by $d(x, y) = |x - y|$. Define $T : X \to K(X)$ by*

$$Tx = \begin{cases} [(1 + x)/2, 1] & \text{if } x \in [0, 1], \\ [2, (2 + x)/2] & \text{if } x \in [2, +\infty[. \end{cases}$$

Clearly, (X, d) is a complete metric space and T is a nonexpansive multi-valued mapping with compact values.

Now, for all $x, y \in [0,1]$ *or* $x, y \in [2,+\infty[$*, we have*

$$H(Tx,Ty) \leq \frac{1}{2}d(x,y).$$

If $x \in [0,1]$ *and* $y \in [2,+\infty[$*, then*

$$\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1} \geq 1$$

and hence

$$\begin{aligned} H(Tx,Ty) &= \max\left\{2-\frac{1+x}{2}, 1+\frac{y}{2}-1\right\} \leq \left(1+\frac{1}{2}\right)(y-x) \\ &\leq \left(\frac{d(x,Ty)+d(y,Tx)}{d(x,Tx)+d(y,Ty)+1}+\frac{1}{2}\right)d(x,y). \end{aligned}$$

Thus all the hypotheses of Theorem 5.1 are satisfied with $k = 1/2$ *and* $x_0 = 1/4$*. In this case T has two fixed points* $x = 1$ *and* $x = 2$*. Note that T is not a contraction, in fact* $H(T1,T2) = d(1,2)$*.*

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