

# A note on strong protomodularity, actions and quotients

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## Abstract

In order to study the problems of extending an action along a quotient of the acted object and along a quotient of the acting object, we investigate some properties of the fibration of points. In fact, we obtain a characterization of protomodular categories among quasi-pointed regular ones, and, in the semi-abelian case, a characterization of strong protomodular categories. Eventually, we return to the initial questions by stating the results in terms of internal actions.

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## 1. Introduction

The present work originates from the investigation of the categorical properties related to two well-known features of group actions.

### Actions on quotients

Suppose we are given a pair  $(\xi, g)$ :

$$A \times Y \overset{\xi}{\rightsquigarrow} Y \xrightarrow{g} Z,$$

where  $\xi$  is a left-action of groups, and  $g$  is a surjective homomorphism<sup>1</sup>. We discuss the following problem: under what conditions does the action  $\xi$  induces an action on the quotient  $Z$ ?

Indeed, it is not difficult to see that  $\xi$  is well-defined on the cosets of  $Y \text{ mod } X = \text{Ker}(g)$ , precisely when it is well-defined on the 0-coset  $X$ , i.e. when it restricts to  $X$ . We shall state this property as follows:

(KC) An action passes to the quotient if, and only if, it restricts to the kernel.

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<sup>1</sup>The arrow labelled  $\xi$  in the diagram is wavy in order to emphasize that it is not a group homomorphism, i.e. that it is not *internal* to the category of groups.

### Action of quotients

Suppose now that we are given a group action  $\xi$  as before, and a surjective group homomorphism  $q: A \rightarrow Q$ . A natural question arises: when does the given  $A$ -action induce a  $Q$ -action? In this case, the restriction of the action  $\xi$  to the kernel  $K$  of  $q$  always exists, and the condition under which the action of the quotient is well defined, amounts to the fact that the kernel of  $q$  acts trivially.

These issues can be addressed in any category where a notion of internal object action is available, e.g. in any semi-abelian category (see [10]). Indeed, we will show that the property (KC) characterizes strongly protomodular categories among semi-abelian categories, and that, in such contexts, actions of quotients behave substantially in the same way as in the case of groups.

On the other hand these issues can be dealt with also in more general contexts. Indeed, when an object  $A$  acts on object  $X$ , just like in the case of group, one can consider the split epimorphism  $X \rtimes A \rightarrow A$  given by the semidirect product projection together with its canonical section. Vice-versa, any split epimorphism with codomain  $A$  gives rise to the *conjugation*  $A$ -action on the kernel of the split epimorphism.

This allows to formulate our issues in terms of split epimorphisms, or *points*, even in contexts where the machinery of internal actions is not at all available. This line of investigation will lead us to the study of some new classifying aspects of the fibration of points. In particular, with Proposition 3.3, we will give a characterization of protomodular categories among quasi-pointed regular ones as those with kernel functors that reflect short exact sequences. Then, we will show that the problem of extending actions along quotients translates (in term of points) in a property closely connected with strong protomodularity, i.e. the fact that kernel functors reflect kernels. In fact, this property coincides with strong protomodularity in the semi-abelian case (Proposition 3.6). On the other hand, the property of extending an action along a quotient of the acting object has a counterpart in terms of points in a property of change of base functors, as described in Proposition 4.1. This observation eventually provides an exhaustive description of change of base functors of the fibration of points along a regular epimorphism.

Our work confirms that strongly protomodular categories are a convenient setting for working with internal actions, and related constructions. Indeed, in the (strongly semi-abelian) varietal case, not only internal actions can be described externally, i.e. with suitable set-theoretical maps, but also, they behave *nicely* with respect to quotients. This fact allows to apply varietal techniques to the intrinsic setting.

Many varieties of universal algebra are strongly protomodular: the categories of groups, Lie algebras, rings and, more generally, all distributive  $\Omega_2$ -groups, i.e. distributive  $\Omega$ -groups with only unary and binary operations (see [11]), as for instance the categories of interest in the sense of G. Orzech [12].

The paper is organized as follows.

In Section 2 we recall the basic notions and fix the notation. Section 3 and Section 4 are quite independent to each other. Section 3 is devoted to the study

of the exactness properties of kernel functors. We prove that in quasi-pointed regular categories, protomodularity is equivalent to the fact that kernel functors reflect short exact sequences. Then we give a characterization of strongly semi-abelian categories among semi-abelian ones (Theorem 5.5). In Section 4 the context is assumed to be strongly semi-abelian. Here we approach the problem of determining the conditions that make it possible to factor the change of base functor of the fibration of points along a regular epimorphism as an equivalence of categories followed by a full embedding. Actions on quotients and actions of quotients are treated explicitly in Section 5, where the results obtained in the previous sections are reconsidered in terms of internal object actions.

## 2. Preliminaries

Here we recall some basic notions from [4], and fix the notation.

### 2.1. Protomodularity

Let  $\mathcal{C}$  be a category with finite limits. We denote by  $\mathbf{Pt}(\mathcal{C})$  the category with objects the quadruples  $(B, A, b, s_b)$  in  $\mathcal{C}$ , with  $b: B \rightarrow A$  and  $b \cdot s_b = 1_A$ , and with morphisms  $(f, g): (D, C, d, s_d) \rightarrow (B, A, b, s_b)$ :

$$\begin{array}{ccc} D & \xrightarrow{f} & B \\ \begin{array}{c} \uparrow \\ d \downarrow \end{array} & \begin{array}{c} s_d \\ \uparrow \end{array} & \begin{array}{c} \uparrow \\ b \downarrow \end{array} \\ C & \xrightarrow{g} & A \end{array} \quad (1)$$

such that both the upward and the downward directed squares commute. The codomain assignment  $(B, A, b, s_b) \mapsto A$  gives rise to a fibration, the so called *fibration of points*:

$$\mathcal{F}: \mathbf{Pt}(\mathcal{C}) \rightarrow \mathcal{C}.$$

For an object  $A$  of  $\mathcal{C}$ , we denote by  $\mathbf{Pt}_A(\mathcal{C})$  the fiber of  $\mathcal{F}$  over  $A$ . Cartesian morphism are given by commutative diagrams (1) with the downward directed square a pullback. This way, any morphism  $g: C \rightarrow A$  defines a “change of base” functor  $g^*: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathbf{Pt}_C(\mathcal{C})$ .

If the category  $\mathcal{C}$  is finitely complete, also the fibers  $\mathbf{Pt}_A(\mathcal{C})$  are, and every change of base functor is left exact. In the present work,  $\mathcal{C}$  will be always finitely complete.

A category  $\mathcal{C}$  is called *protomodular* when every change of base of the fibration of points is conservative, i.e. when it reflects isomorphisms (see [4]).

When  $\mathcal{C}$  admits an initial object  $0$ , for any object  $A$  of  $\mathcal{C}$ , one can consider the change of base along the initial arrow  $!_A: 0 \rightarrow A$ . This defines a *kernel functor*  $\mathcal{K}_A$ , for every object  $A$ . In the presence of an initial object, the protomodularity condition can be simplified by requiring that just kernel functors are conservative.

The category  $\mathcal{C}$  is called *quasi-pointed* when the unique arrow  $0 \rightarrow 1$  is a monomorphism. Considering this being the case, the domain functor  $\mathbf{Pt}_0(\mathcal{C}) \rightarrow$

$\mathcal{C}$  defines an embedding of categories. Its isomorphic image is the subcategory  $\mathcal{C}_0$  spanned by objects with null support (i.e. objects  $A$  equipped with a necessarily unique arrow  $\omega_A: A \rightarrow 0$ ) so that we can factor

$$\mathcal{K}_A: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathcal{C}_0 \hookrightarrow \mathcal{C}.$$

When  $0 \rightarrow 1$  is an isomorphism, we say that  $\mathcal{C}$  is pointed; if this is the case, clearly  $\mathcal{C}_0 = \mathcal{C}$ .

Let  $\mathcal{C}$  be a quasi-pointed finitely complete category. We shall call *kernel map* any  $f: X \rightarrow Y$ , pullback of an initial arrow, i.e. when  $f$  fits into a pullback diagram as it is shown below:

$$\begin{array}{ccc} X & \xrightarrow{\omega_X} & 0 \\ f \downarrow & & \downarrow !_Z \\ Y & \xrightarrow{g} & Z \end{array} \quad (2)$$

In this case, we write  $f = \ker(g)$  or  $f = k_g$ . We denote by  $\mathcal{K}$  the class of kernel maps of a given category  $\mathcal{C}$ .

Following [5], we say that  $g$  is the cokernel of  $f$ , and we write  $g = \text{coker}(f)$ , when (2) is a pushout. Let us notice that this definition of cokernel is not dual to that of kernel given above, unless the category is pointed. When both conditions above are satisfied, i.e. when (2) is at the same time both a pullback and a pushout, we call the pair  $(f, g)$  *short exact sequence* (see [5]), and we describe it by the diagram:

$$X \xrightarrow{f} Y \xrightarrow{g} Z$$

We recall from [8] that, if  $\mathcal{C}$  is quasi-pointed and protomodular, every regular epimorphism is the cokernel of its kernel, so that the pair  $(f, g)$  is a short exact sequence precisely when  $g$  is a regular epimorphism, and  $f$  is its kernel.

Recall that an (internal) equivalence relation is called *effective* when it is the kernel pair of a map. A category  $\mathcal{C}$  is *regular*, if it is finitely complete, it has pullback-stable regular epimorphisms, and all effective equivalence relations admit coequalizers. A regular category  $\mathcal{C}$  is *Barr exact* when all equivalence relations are effective (see [2]).

Quasi-pointed protomodular regular categories are called *sequestiable*. If they are in fact pointed, they are called *homological*, and they are termed *semi-abelian* when they are also Barr exact and with finite coproducts (see [4]).

An important feature of sequestiable categories is that, in such contexts, intrinsic versions of some classical lemmas of homological algebra hold. This is the case of the  $3 \times 3$  lemma (see [8]), that will be a basic tool in the development of the present work.

## 2.2. Strong protomodularity

In [7], Bourn introduces a more general notion of *normal monomorphism* that objectifies an equivalence class of an internal equivalence relations.

In a category  $\mathcal{C}$  with finite limits, a morphism  $f: X \rightarrow Y$  is *normal* to an equivalence relation  $(R, r_1, r_2)$  on the object  $X$  when the following two diagrams are pullbacks:

$$\begin{array}{ccc} X \times X & \longrightarrow & R \\ \parallel & \lrcorner & \downarrow \langle r_1, r_2 \rangle \\ X \times X & \xrightarrow{f \times f} & Y \times Y \end{array} \qquad \begin{array}{ccc} X \times X & \longrightarrow & R \\ p_1 \downarrow & \lrcorner & \downarrow r_1 \\ X & \xrightarrow{f} & Y \end{array}$$

When the category  $\mathcal{C}$  is protomodular, normality becomes a property: if  $f$  is normal to a relation  $R$ , then  $R$  is unique. We denote by  $\mathcal{N}$  the class of normal monomorphisms.

Indeed, in quasi-pointed protomodular categories, any kernel is normal to its associated kernel relation. On the other hand, not every normal monomorphism is a kernel, i.e.  $\mathcal{K} \subseteq \mathcal{N}$ , and the inclusion may be strict, in general.

Let us recall from [7] that if  $\mathcal{C}$  is finitely complete, pointed and protomodular, then the class  $\mathcal{K}$  coincides with the class  $\mathcal{N}$  precisely when every equivalence relation is effective.

In [8] Bourn calls *normal*, a left exact functor that is conservative and reflects normal monomorphisms. A relevant application of this definition is related to the fibration of points. When all the change of base functors are normal, the category is called *strongly-protomodular* (see [6, 4]). In the presence of initial object, it suffices to consider the kernel functors  $\mathcal{K}_A$ , for every object  $A$ . A strongly protomodular semi-abelian category is termed *strongly semi-abelian*.

Bourn, in [6], gives a characterization of normal subobjects in  $\mathbf{Pt}_A(\mathcal{C})$ . When  $\mathcal{C}$  is quasi-pointed protomodular,  $\varphi: (B, b, s_b) \rightarrow (C, c, s_c)$  is normal in  $\mathbf{Pt}_A(\mathcal{C})$  if, and only if,  $\varphi \cdot k_b$  is normal in  $\mathcal{C}$ , where  $X = \text{Ker}(b)$  and  $Y = \text{Ker}(c)$ :

$$\begin{array}{ccccc} X & \xrightarrow{k_b} & B & \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{s_b} \end{array} & A \\ \downarrow f & & \downarrow \varphi & & \parallel \\ Y & \xrightarrow{k_c} & C & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s_c} \end{array} & A \end{array}$$

This, in turns, gives a criterion for strong protomodularity: it suffices to check, for every morphism of split short exact sequences as above, that if  $f$  is normal, then also  $k_c \cdot f$  is.

## 3. Exactness properties of kernel functors

In this section, we analyze some issues related to the behavior of kernel functors with respect to kernels, cokernels and short exact sequences, in the

quasi-pointed regular setting. As recalled before, in this case, the kernel functor takes values in the base category  $\mathcal{C}$ . Moreover, when  $\mathcal{C}$  is protomodular, regular (or Barr-exact), then also  $\mathbf{Pt}_A(\mathcal{C})$  is protomodular, regular (or Barr-exact) respectively (see [4], for instance). These circumstances suggest to investigate the exactness properties of kernel functors, in the sense of homological algebra.

Our main motivation rests in the observation that a notion similar to strong protomodularity, but stated in terms of kernels instead of normal monomorphisms, is connected with (actually equivalent to) the problem of extending actions along quotients. This connection will be made explicit in the next sections.

The preservation property described in the next proposition is little more than a reformulation of some arguments analyzed in [8].

**Proposition 3.1.** *Let  $\mathcal{C}$  be a quasi-pointed protomodular category with pullback stable regular epimorphisms. Then  $\mathcal{K}_A$  preserves short exact sequences, for every  $A$  in  $\mathcal{C}$ .*

*Proof.* Let us consider a short exact sequence  $(\varphi, \gamma)$ , and let  $f = \mathcal{K}_A(\varphi)$  and  $g = \mathcal{K}_A(\gamma)$ , as described by the following diagram.

$$\begin{array}{ccccc}
 \mathcal{K}_A(b) & \xrightarrow{k_b} & B & \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{s_b} \end{array} & A \\
 f \downarrow & & \varphi \downarrow & & \parallel \\
 \mathcal{K}_A(c) & \xrightarrow{k_c} & C & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s_c} \end{array} & A \\
 g \downarrow & & \gamma \downarrow & & \parallel \\
 \mathcal{K}_A(d) & \xrightarrow{k_d} & D & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{s_d} \end{array} & A
 \end{array} \tag{3}$$

Since kernel functors preserve limits,  $\ker(g) = f$ . Furthermore, by Lemma 1 in [8], the left-down square is a pullback, and since  $\gamma$  is a regular epimorphism, so is  $g$ . Finally, by Proposition 2 in [8],  $g = \text{coker}(f)$ .  $\square$

Before we can treat reflection properties of kernel functors, let us develop the necessary description of kernels in  $\mathbf{Pt}_A(\mathcal{C})$ .

We have just recalled Bourn's characterization of normal subobject in  $\mathbf{Pt}_A(\mathcal{C})$ . In the sequentiable setting, one can recover a similar characterization for kernels.

**Proposition 3.2.** *In a sequentiable category  $\mathcal{C}$ , let us consider a morphism of points  $\varphi: (B, b, s_b) \rightarrow (C, c, s_c)$ , together with its restriction to kernels, as described by the commutative diagram below:*

$$\begin{array}{ccccc}
 X & \xrightarrow{k_b} & B & \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{s_b} \end{array} & A \\
 f \downarrow & & \varphi \downarrow & & \parallel \\
 Y & \xrightarrow{k_c} & C & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s_c} \end{array} & A
 \end{array} \tag{4}$$

Then

- (1)  $\varphi$  is a kernel in  $\mathbf{Pt}_A(\mathcal{C})$  if, and only if,  $\varphi \cdot k_b$  is a kernel in  $\mathcal{C}$ ;
- (2) in this case, the cokernel of  $\varphi$  in  $\mathbf{Pt}_A(\mathcal{C})$  is given by the cokernel of  $\varphi \cdot k_b$  in  $\mathcal{C}$ .

*Proof.* Point (1). The fact that  $\varphi$  is a kernel, amounts to the existence of a morphism of points  $\gamma: (C, c, s_c) \rightarrow (D, d, s_d)$ , such that the commutative square  $\gamma \cdot \varphi = s_d \cdot b$  is a pullback in  $\mathcal{C}$ . Then, pasting it with the kernel diagram of  $(k_b, b)$ , one easily sees that  $k_c \cdot f = \varphi \cdot k_b = \ker(\gamma)$  in  $\mathcal{C}$ .

Conversely, let us assume that  $k_c \cdot f = \varphi \cdot k_b$  is a kernel in  $\mathcal{C}$ , and let  $\gamma: C \rightarrow D$  be its cokernel (always in  $\mathcal{C}$ ). Then  $\gamma$  underlies a morphism of points. Indeed, since  $c \cdot \varphi \cdot k_b$  factors through 0, we get a unique  $d: D \rightarrow A$  such that  $d \cdot \gamma = c$ . In fact,  $d$  is a split epimorphism with section  $s_d = \gamma \cdot s_c$ , and  $\gamma: (C, c, s_c) \rightarrow (D, d, s_d)$  is a morphism of points. We are to prove that  $\varphi$  is the kernel of  $\gamma$  in  $\mathbf{Pt}_A(\mathcal{C})$ . To this end, let us consider the commutative diagram

$$\begin{array}{ccccc}
 X & \xrightarrow{k_b} & B & \xrightarrow{\varphi} & C \\
 \downarrow & & \uparrow s_b & \downarrow b & \downarrow \gamma \\
 0 & \longrightarrow & A & \xrightarrow{s_d} & D
 \end{array}$$

The whole diagram and the square on the left are pullbacks, so that by the pullback cancellation property of protomodular categories (see [5]) also the square on the right is a pullback, thus showing that  $\varphi$  is the kernel of  $\gamma$  in  $\mathbf{Pt}_A(\mathcal{C})$ .

Point (2). Clearly  $\mathbf{Pt}_A(\mathcal{C})$  is pointed, moreover it is protomodular and regular (see [4]) as  $\mathcal{C}$  is. Actually, as showed in the proof of point 1,  $\gamma = \text{coker}(\varphi \cdot k_c)$  in  $\mathcal{C}$  underlies a regular epimorphism. In order to conclude the proof, it suffices to recall that in homological categories a regular epimorphism is always the cokernel of its kernel.  $\square$

We are now ready to show how, in the sequentiable setting, kernel functors also reflect short exact sequences. Moreover, this property characterizes sequentiable categories among quasi-pointed regular ones.

**Proposition 3.3.** *Let  $\mathcal{C}$  be a quasi-pointed regular category. The following statements are equivalent:*

- (1)  $\mathcal{C}$  is protomodular,
- (2)  $\mathcal{K}_A$  reflects short exact sequences, for every  $A$  in  $\mathcal{C}$ .

*Proof.* In a protomodular category  $\mathcal{C}$ , let us consider a pair  $(\varphi, \gamma)$  of morphisms of points over  $A$ , such that applying the kernel functor  $\mathcal{K}_A$  one obtains a short exact sequence  $(f, g)$ , see diagram (3). Since  $g \cdot f = 0$ ,  $\gamma \cdot \varphi$  factors through  $A$ . More precisely,  $\gamma \cdot \varphi = s_d \cdot b$ , as one can prove by pre-composing this equality

with the jointly epic pair  $(k_b, s_b)$ . Then we consider the diagram below:

$$\begin{array}{ccccc}
\mathcal{K}_A(b) & \xlongequal{\quad} & \mathcal{K}_A(b) & \longrightarrow & 0 \\
f \downarrow & & \downarrow \varphi \cdot k_b & & \downarrow \\
\mathcal{K}_A(c) & \xrightarrow{k_c} & C & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s_c} \end{array} & A \\
g \downarrow & & \downarrow \gamma & & \parallel \\
\mathcal{K}_A(d) & \xrightarrow{k_d} & D & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{s_d} \end{array} & A
\end{array} \tag{5}$$

We can apply the  $3 \times 3$  lemma: the three rows are short exact, and so are the leftmost and the rightmost columns. The middle column is zero, hence we can conclude that it is short exact. By Proposition 3.2, the pair  $(\varphi, \gamma)$  is a short exact sequence in  $\mathbf{Pt}_A(\mathcal{C})$ .

Conversely, we have to prove that, for any object  $A$ , the kernel functor  $\mathcal{K}_A$  reflects isomorphisms. To this end, we consider a map  $\varphi$  in  $\mathbf{Pt}_A(\mathcal{C})$  such that its restriction to kernels is an isomorphism  $f$ . Then, since kernels have null support, the cokernel of  $f$  exists, and of course it is trivial. Thus one can consider the following diagram:

$$\begin{array}{ccccc}
\mathcal{K}_A(b) & \xrightarrow{k_b} & B & \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{s_b} \end{array} & A \\
f \downarrow \simeq & & \downarrow \varphi & & \parallel \\
\mathcal{K}_A(c) & \xrightarrow{k_c} & C & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s_c} \end{array} & A \\
\downarrow & & \downarrow c & & \parallel \\
0 & \longrightarrow & A & \begin{array}{c} \xrightarrow{1_A} \\ \xleftarrow{1_A} \end{array} & A
\end{array}$$

By applying the hypothesis, we obtain that the sequence  $(\varphi, c)$  is short exact in  $\mathbf{Pt}_A(\mathcal{C})$ , so that  $\varphi$  is the pullback of  $1_A$  along  $c$ , hence an isomorphism.  $\square$

Proposition 3.1 and Proposition 3.3 together, imply immediately the following corollary.

**Corollary 3.4.** *Let  $\mathcal{C}$  be sequentiable. Then for any map  $e: E \rightarrow A$ , the change of base  $e^*: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathbf{Pt}_E(\mathcal{C})$  preserves and reflects short exact sequences.*

In the last part of this section we would like to examine the behavior of the kernel functors with respect to kernels and (some specific class of) cokernels. We start by considering a distinguished class of morphisms of points, i.e. those maps  $\varphi$  in  $\mathbf{Pt}_A(\mathcal{C})$  such that their restriction to the kernel functor  $\mathcal{K}_A$  is a kernel map in  $\mathcal{C}$ .

Of course, if  $\varphi$  is a kernel in  $\mathbf{Pt}_A(\mathcal{C})$ , then  $\mathcal{K}_A(\varphi)$  is a kernel in  $\mathcal{C}$ . On the other hand, we wish to investigate when the other implication holds. This is done in the next proposition.



**Proposition 3.5.** *Let  $\mathcal{C}$  be sequentiable, and let  $\mathcal{K}_A: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathcal{C}$  be the kernel functor relative to an object  $A$  in  $\mathcal{C}$ . Then the following statements are equivalent:*

- (1) *the kernel functor  $\mathcal{K}_A$  reflects kernel maps,*
- (2) *the kernel functor  $\mathcal{K}_A$  lifts the cokernels of the maps  $\varphi$  such that  $\mathcal{K}_A(\varphi)$  is a kernel, i.e.:*
- (C\*) *for every  $\varphi$  in  $\mathbf{Pt}_A(\mathcal{C})$  such that  $\mathcal{K}_A(\varphi) = f$  is a kernel map in  $\mathcal{C}$ , there exists a unique  $\gamma$  in  $\mathbf{Pt}_A(\mathcal{C})$  such that  $\gamma = \text{coker}(\varphi)$ , and  $\mathcal{K}_A(\gamma) = \text{coker}(f)$ .*

For the notion of (co)limit lifting functor, the reader can refer to Definition 13.17 of [1].

*Proof.* (1)  $\Rightarrow$  (2). Let us consider a morphism  $\varphi: (B, b, s_b) \rightarrow (C, c, s_c)$  such that  $f = \mathcal{K}_A(\varphi)$  is a kernel, and let  $g = \text{coker}(f)$ . By (1)  $\varphi$  is a kernel in  $\mathbf{Pt}_A(\mathcal{C})$ , hence by Proposition 3.2, also  $k_c \cdot f$  is.

Let  $(D, \gamma)$  be the cokernel of  $k_c \cdot f$ , so that the pair  $(k_c \cdot f, \gamma)$  is a short exact sequence. We can consider the following commutative diagram

$$\begin{array}{ccccc}
 \mathcal{K}_A(b) & \xlongequal{\quad} & \mathcal{K}_A(b) & \longrightarrow & 0 \\
 \downarrow f & & \downarrow k_c \cdot f & & \downarrow \\
 \mathcal{K}_A(c) & \xrightarrow{k_c} & C & \xrightleftharpoons[s_c]{c} & A \\
 \downarrow g & & \downarrow \gamma & & \parallel \\
 Z & \xrightarrow{\alpha} & D & \xrightarrow{\beta} & A
 \end{array}$$

where  $\alpha$  and  $\beta$  are obtained by the universal properties of the cokernels  $(D, \gamma)$  and  $(Z, g)$ . Since  $\mathcal{C}$  is sequentiable, we can apply the  $3 \times 3$  lemma, and conclude that the sequence  $(\alpha, \beta)$  is short exact. Moreover,  $\beta$  is split by  $\gamma \cdot s_c$ , so that  $\gamma$  induces a morphism of points  $(C, c, s_c) \rightarrow (E, \beta, \gamma \cdot s_c)$ ; in fact, by Proposition 3.2,  $\gamma = \text{coker}(\varphi)$  in  $\mathbf{Pt}_A(\mathcal{C})$ .

Now, universality of kernels implies the existence of a unique isomorphism  $\tau: Z \rightarrow \mathcal{K}_A(\beta)$  such that  $\alpha = k_\beta \cdot \tau$ . Of course, as  $g$  is a cokernel of  $f$  also  $\tau \cdot g$  is, moreover  $\mathcal{K}_A(\gamma) = \tau \cdot g$ , so that the existence part of (C\*) is granted. Uniqueness comes from the fact that, since  $\mathcal{C}$  is protomodular, the pair  $(k_c, s_c)$  is jointly strongly epic.

(2)  $\Rightarrow$  (1). Let us assume that  $f = \mathcal{K}_A(\varphi)$  is a kernel. By (C\*) then, there is a  $\gamma = \text{coker}(\varphi)$  in  $\mathbf{Pt}_A(\mathcal{C})$  such that  $\mathcal{K}_A(\gamma) = \text{coker}f$ . Now apply Proposition 3.3 and get  $(\varphi, \gamma)$  is short exact. In particular,  $\varphi$  is a kernel.  $\square$

Whenever kernel maps and normal monomorphisms coincide, condition (1) of Proposition 3.5 above, expresses precisely the strong protomodularity axiom. This proves the following proposition.

**Proposition 3.6.** *Let  $\mathcal{C}$  be a semi-abelian category. The following statements are equivalent*

- (1)  $\mathcal{C}$  is strongly semi-abelian,
- (2) for every object  $A$  of  $\mathcal{C}$ , the kernel functor  $\mathcal{K}_A$  lifts the cokernels of  $\mathcal{K}_A$ -kernels, i.e. condition  $(C^*)$  of Proposition 3.5 is satisfied.

#### 4. Change of base: the other direction

As we have recalled in Section 2, for any map  $f: E \rightarrow A$ , the change of base functor  $f^*: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathbf{Pt}_E(\mathcal{C})$ , is defined by pulling back along  $f$ . In other terms,  $f$  defines a functor between the fibers that moves *backward*, with respect to the direction of  $f$ . A quite natural question to ask is whether there are conditions allowing to *push forward* along a map. More precisely, given a map  $q: A \rightarrow Q$ , we aim to define a functor  $q_*: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathbf{Pt}_Q(\mathcal{C})$ .

In the present work, we restrict our attention to the case when  $q$  is a regular epimorphism. The following result shows that such a push forward can be performed if, and only if, the pullback along  $k = \ker(q)$  trivializes the pointed object we started with:

**Proposition 4.1.** *In a strongly semi-abelian category  $\mathcal{C}$ , we consider a pointed object  $(C, c, s_c, A)$ , and a regular epimorphism  $q: A \rightarrow Q$ . Then, if we denote by  $(K, k)$  and  $(Y, k_c)$  the kernels of  $q$  and of  $c$  respectively, the following statements are equivalent:*

- (1) the pullback along  $k$  of  $(C, c, s_c, A)$  is the pointed object

$$(Y \times K, \pi_2, \langle 0, 1 \rangle, K),$$

- (2) there exist a pointed object  $(D, d, s_d, Q)$  and a cartesian morphism

$$(\gamma, q): (C, c, s_c, A) \rightarrow (D, d, s_d, Q).$$

The situation is described by the following diagram.

$$\begin{array}{ccccc}
 Y & \xrightarrow{k\pi_2} & Y \times K & \xrightleftharpoons[\langle 0, 1 \rangle]{\pi_2} & K \\
 \parallel & & \downarrow \varphi & \downarrow (\diamond) & \downarrow k \\
 Y & \xrightarrow{k_c} & C & \xrightleftharpoons[s_c]{c} & A \\
 \parallel & & \downarrow \gamma & \downarrow (\diamond\diamond) & \downarrow q \\
 Y & \dashrightarrow & D & \dashrightarrow & Q \\
 & & k_d & & s_d
 \end{array} \tag{6}$$

*Proof.* (1)  $\Rightarrow$  (2). By the assumption in (1),  $\varphi$  is a kernel, since it is the pullback of a kernel. Now, if we focus on the square  $(\diamond)$  above, we can consider the kernels of the horizontal split epimorphisms and the cokernels of the vertical monomorphisms. Since the base category is homological, not only can we say that such kernels are isomorphic, but also the cokernels of  $k$  and  $\varphi$  are. This last claim is proved by applying the  $3 \times 3$  lemma to the diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{\langle 1,0 \rangle} & Y \times K & \xrightleftharpoons[\langle 0,1 \rangle]{\pi_2} & K \\
\parallel & & \downarrow \varphi & & \downarrow k \\
Y & \xrightarrow{k_c} & C & \xrightleftharpoons[s_c]{c} & A \\
\downarrow & & \downarrow q \cdot c & & \downarrow q \\
0 & \longrightarrow & Q & \xlongequal{\quad} & Q
\end{array}$$

Now, let us consider the following morphism of short exact sequences:

$$\begin{array}{ccccc}
K & \xrightarrow{k} & A & \xrightarrow{q} & Q \\
\langle 0,1 \rangle \downarrow & & \downarrow s_c & & \parallel \\
Y \times K & \xrightarrow{\varphi} & C & \xrightarrow{q \cdot c} & Q
\end{array}$$

Since  $\langle 0,1 \rangle$  is a kernel, applying Axiom M1.2 of [14] (which holds in every strongly semi-abelian category) we deduce that also  $s_c \cdot k = \varphi \cdot \langle 0,1 \rangle$  is a kernel. Let us compute the cokernel  $\gamma = \text{coker}(s_c \cdot k)$ , and arrange our data in the diagram below:

$$\begin{array}{ccccc}
K & \xrightarrow{\langle 0,1 \rangle} & Y \times K & \xrightarrow{\pi_1} & Y \\
\parallel & & \downarrow \varphi & & \downarrow \alpha \\
K & \xrightarrow{s_c \cdot k} & C & \xrightarrow{\gamma} & D \\
\downarrow & & \downarrow q \cdot c & & \downarrow \beta \\
0 & \longrightarrow & Q & \xlongequal{\quad} & Q
\end{array}$$

where  $\alpha$  and  $\beta$  are obtained by the universal property of the cokernels involved. By the  $3 \times 3$  lemma, we deduce that the sequence  $(\alpha, \beta)$  is short exact. Finally, the square  $\beta \cdot \gamma = q \cdot c$  is a pullback, since  $\text{Ker}(\beta) = \text{Ker}(c)$ :

$$\begin{array}{ccccc}
& & K & \xlongequal{\quad} & K \\
& & \downarrow s_c \cdot k & & \downarrow k \\
Y & \xrightarrow{k_c} & C & \xrightleftharpoons[s_c]{c} & A \\
\parallel & & \downarrow \gamma & & \downarrow q \\
Y & \xrightarrow{\alpha} & D & \xrightarrow{\beta} & Q
\end{array}$$

Then also  $\text{Ker}(\gamma) = \text{Ker}(q)$ . Moreover  $\beta$  is a split epimorphism. In order to prove this assertion, we notice that  $(\gamma \cdot s_c) \cdot k = \gamma \cdot (s_c \cdot k) = 0$ , and by the universal property of the cokernel  $q$ , there exists a (unique) map  $\sigma: Q \rightarrow D$  such that  $\sigma \cdot q = \gamma \cdot s_c$ . Hence  $\beta \cdot \sigma \cdot q = \beta \cdot \gamma \cdot s_c = q \cdot c \cdot s_c = q$ , and, since  $q$  is epic, we get  $\beta \cdot \sigma = 1_Q$ .

(2)  $\Rightarrow$  (1). Assume we are in the situation as described by the diagram below

$$\begin{array}{ccc}
 B & \begin{array}{c} \xrightarrow{b} \\ \xleftarrow{s_b} \end{array} & K \\
 \varphi \downarrow & (i) & \downarrow k \\
 C & \begin{array}{c} \xrightarrow{c} \\ \xleftarrow{s_c} \end{array} & A \\
 \gamma \downarrow & (ii) & \downarrow q \\
 D & \begin{array}{c} \xrightarrow{d} \\ \xleftarrow{s_d} \end{array} & Q
 \end{array}$$

with (i) and (ii) pullbacks, and  $(k, q)$  short exact. Let us denote by  $Y$  the kernel of  $c$ . Then (i) + (ii) is a pullback, and since  $q \cdot k$  factors through 0, the pointed object  $(B, b, s_b, K)$  is isomorphic to the product projection  $(Y \times K, \pi_2, \langle 0, 1 \rangle, K)$ .  $\square$

For a map  $k$  with codomain  $A$ , we denote by  $\mathbf{Pt}_A(\mathcal{C})|_k$  the full subcategory of  $\mathbf{Pt}_A(\mathcal{C})$ , with objects those split epimorphisms such that the change of base along  $k$  gives a product projection.

Then it is easy to prove that Proposition 4.1 above can be used in order to give a description of the change of base  $q^*$  when  $q$  is a regular epimorphism.

**Corollary 4.2.** *Given a regular epimorphism  $q: A \rightarrow Q$  together with its kernel  $k = \ker(q)$  in a strongly semi-abelian category  $\mathcal{C}$ , we have a factorization  $q^* = j \cdot e$*

$$\begin{array}{ccccc}
 & & p^* & & \\
 & \curvearrowright & & \curvearrowleft & \\
 \mathbf{Pt}_Q(\mathcal{C}) & \xrightarrow{e} & \mathbf{Pt}_A(\mathcal{C})|_k & \xrightarrow{j} & \mathbf{Pt}_A(\mathcal{C})
 \end{array}$$

where the functor  $e$  is an equivalence of categories.

*Proof.* For  $\mathcal{C}$  sequentiable, it is trivial to show that the change of base along a regular epimorphism  $q$  is fully faithful. Moreover, in the strongly semi-abelian case, point (2) of Proposition 4.1 defines precisely a quasi-inverse for the equivalence

$$\mathbf{Pt}_Q(\mathcal{C}) \rightarrow \mathbf{Pt}_A(\mathcal{C})|_k.$$

$\square$

## 5. Back to action(s)

In this section we return to the problems described in the introduction, now set in the semi-abelian context.

### 5.1. Internal actions

Semi-abelian categories are a convenient setting for working with *internal actions*. Here we briefly recall their definition from [3]. This will help in formulating internally the property (KC) of the introduction.

Let  $\mathcal{C}$  be a finitely complete, pointed category with pushouts of split monomorphisms. Then, for every object  $A$  of  $\mathcal{C}$ , the functor  $\mathcal{K}_A$  has a left adjoint  $\Sigma_A$ . This can be described as follows: for an object  $X$  of  $\mathcal{C}$ ,  $\Sigma_A(X)$  is the pointed object  $A + X \xrightleftharpoons[i_A]{[1,0]} A$ . The monad corresponding to this adjunction is denoted by  $Ab(-)$ , and for any object  $X$  of  $\mathcal{C}$  one gets a kernel diagram:

$$AbX \xrightarrow{\kappa_{A,X}} A + X \xrightarrow{[1,0]} A .$$

The  $Ab(-)$ -algebras are called internal  $A$ -actions (see [3, 9]). The category  $\mathbf{Alg}(Ab(-))$  of such algebras will be more conveniently denoted by  $\mathbf{Act}(A, -)$ .

When the kernel functor  $\mathcal{K}_A$  is monadic, then  $\mathcal{C}$  is said to be a category with semi-direct products, and the canonical comparison

$$\Xi: \mathbf{Pt}_A(\mathcal{C}) \rightarrow \mathbf{Act}(A, -), \quad (7)$$

establishes an equivalence of categories. All semi-abelian categories satisfy this condition.

**Example 5.1.** In the case of group one easily recovers the classical notion of group action. For two given groups  $A$  and  $X$ ,  $AbX$  is nothing but the subgroup of the free product  $A * X$  generated by the words  $a; x; -a$ , with  $a \in A$  and  $x \in X$ , and the homomorphism  $\xi: AbX \rightarrow X$  recovers a classical group action by letting  $a \cdot x = \xi(a; x; -a)$ .

For an action  $\xi: AbX \rightarrow X$ , the semi-direct product of  $X$  with  $A$ , with action  $\xi$  is the split epimorphism corresponding to  $\xi$  via  $\Xi$ . It can be computed explicitly (see [11]) by means of the coequalizer diagram:

$$AbX \xrightleftharpoons[i_X \cdot \xi]{\kappa_{A,X}} A + X \xrightarrow{q_\xi} A \rtimes_\xi X .$$

**Example 5.2.** For objects  $A$  and  $X$ , the *trivial action* of  $A$  on  $X$  is the composite

$$\rho_{A,X} = \rho_X: AbX \xrightarrow{\kappa_{A,X}} A + X \xrightarrow{[0,1]} X .$$

The map  $\rho_{A,X}$  is natural in the two variables  $A$  and  $X$ . The corresponding split epimorphism is given by the cartesian product with the canonical section:

$$X \times A \xrightleftharpoons[\langle 0,1 \rangle]{\pi_2} A .$$

**Example 5.3.** Every object  $X$  acts on itself by conjugation. This is given by the composite

$$\chi_X: X \mathfrak{b} X \xrightarrow{\kappa_{X,X}} X + X \xrightarrow{[1,1]} X .$$

The map  $\chi_X$  is natural in the variable  $X$ . The corresponding split epimorphism is isomorphic to the cartesian product with the diagonal section:

$$X \times X \xrightleftharpoons[\langle 1,1 \rangle]{\pi_2} X .$$

### 5.2. Property (KC) for split epimorphisms

From now on, we consider  $\mathcal{C}$  semi-abelian. In this setting, we will first formulate our property (KC) in terms of split epimorphisms, according to the equivalence (7) between actions and points. Then, in the next section, we will go back to the original formulation of the problem.

Let a short exact sequence  $X \xrightarrow{f} Y \xrightarrow{g} Z$  be given, and consider a split epimorphism  $(C, c, s_c)$ , with kernel  $Y$  and codomain  $A$ . Let  $\xi$  be the corresponding action.

$$\begin{array}{ccccc} X & \xrightarrow{k_b} & B & \xrightleftharpoons[s_b]{b} & A \\ f \downarrow & & \downarrow \varphi & \text{(\dagger)} & \parallel \\ Y & \xrightarrow{k_c} & C & \xrightleftharpoons[s_c]{c} & A \\ g \downarrow & & \downarrow \gamma & \text{(\ddagger)} & \parallel \\ Z & \xrightarrow{k_d} & D & \xrightleftharpoons[s_d]{d} & A \end{array} \quad (8)$$

With reference to the diagram above, the fact that the action  $\xi$  restricts to the kernel  $X$ , amounts to the fact that there exists a morphism  $\varphi$  of split epimorphisms  $(\dagger)$ , that restricts to  $f$ , while the fact that the action  $\xi$  passes to the quotient  $Z = Y/X$  amounts to the fact that there exists a morphism  $\gamma$  of split epimorphisms  $(\ddagger)$ , that restricts to  $g$ . In this fashion, with a little abuse of language, we can translate property (KC) in the double implication  $(\dagger) \Leftrightarrow (\ddagger)$ .

**Remark 5.4.** Indeed, the implication  $(\ddagger) \Rightarrow (\dagger)$  holds in any pointed category with finite limits, with no assumption on  $g$ . In other words, this is the trivial part of our problem, and it has nothing to do with actions, etc.

On the other hand, the implication  $(\dagger) \Rightarrow (\ddagger)$  translates precisely the condition  $(C^*)$  stated in Proposition 3.5.

### 5.3. Actions on quotients

We are ready to return to our initial problem, and to formulate it in terms of internal actions.

In a pointed regular category with semi-direct products, we consider an  $A$ -action  $\xi$  on  $Y$  and a short exact sequence,  $(f, g)$ .

$$\begin{array}{ccccc}
 \text{Ab}X & \xrightarrow{1bf} & \text{Ab}Y & \xrightarrow{1bg} & \text{Ab}Z \\
 \downarrow \xi_{\downarrow} & & \downarrow \xi & & \downarrow \bar{\xi} \\
 X & \xrightarrow{f} & Y & \xrightarrow{g} & Z
 \end{array} \tag{9}$$

We can state the implications above using internal actions, as follows:

$(\ddagger) \Rightarrow (\dagger)$  If  $\xi$  induces an action on the quotient  $Z$ , then it restricts to the kernel  $X$ . In other words, if there exists an action  $\bar{\xi}$  such that the square on the right commutes, then there exists an action  $\xi_{\downarrow}$  such that the square on the left commutes.

$(\dagger) \Rightarrow (\ddagger)$  If  $\xi$  restricts to the kernel  $X$ , then it induces an action on the quotient  $Z$ . In other words, if there exists an action  $\xi_{\downarrow}$  such that the square on the left commutes, then there exists an action  $\bar{\xi}$  such that the square on the right commutes.

Of course, the implication  $(\ddagger) \Rightarrow (\dagger)$  does hold in any pointed category with semi-direct products. For what concerns property  $(\dagger) \Rightarrow (\ddagger)$ , we can translate Proposition 3.6 accordingly, in terms of internal actions. This is summarized in the following Theorem, that can be derived directly from Proposition 3.6.

**Theorem 5.5.** *Let  $\mathcal{C}$  be a semi-abelian category. The following statements are equivalent:*

- (1)  $\mathcal{C}$  is strongly protomodular,
- (2)  $(\dagger) \Rightarrow (\ddagger)$ , i.e. for any  $A$ -action  $\xi$  on an object  $Y$ , and for any normal subobject  $X$  of  $Y$  such that  $\xi$  restricts  $X$ ,  $\xi$  induces an action on the quotient  $Y/X$ .

#### 5.4. Action of quotients

So far we discussed the conditions under which an action on a given object extends to a quotient of that object. Now we change our point of view: we fix the *acted* object, and we consider when an action *of* a given object, induces an action of a quotient of that object.

More precisely, let a short exact sequence

$$K \xrightarrow{k} A \xrightarrow{q} Q$$

be given, and let us consider an action  $\xi: \text{Ab}Y \rightarrow Y$ . We pose the following question: when does the action  $\xi$  induce an action  $Q\text{b}Y \rightarrow Y$ ?

The answer, in the strongly semi-abelian context, involves the restriction to the kernel  $K$ : likewise in the case of groups,  $\xi$  induces an action  $q_*(\xi)$  of the quotient  $Q$ , precisely when  $\xi \cdot (k\text{b}1)$  is trivial.

**Proposition 5.6.** *Let  $\mathcal{C}$  be strongly semi-abelian,  $(k, q)$  a short exact sequence and  $\xi$  an action, as above. Then the following conditions are equivalent:*

- (1)  $\xi \cdot (kb1) = \rho_K$ , i.e. the trivial action on  $K$ ,
- (2) there exists an action  $q_*(\xi): QbY \rightarrow Y$  such that  $\xi = q_*(\xi) \cdot (qb1)$ .

*Proof.* This is nothing but the formulation of Proposition 4.1 in terms of internal actions.  $\square$

### Further developments

Theorem 5.5, together with Proposition 5.6, seems to suggest that strongly semi-abelian categories are a convenient setting for developing homological algebra of internal (pre)crossed modules.

Let us recall that strongly semi-abelian varieties include several classical categories of algebras. Indeed, as proved in [11], all the distributive  $\Omega_2$ -groups, also called categories of *groups with operations*, are such. Here we recall the definition for the reader's convenience.

A distributive  $\Omega_2$ -group is a variety of groups (in the sense of universal algebra) such that:  $\Omega = \Omega_0 \cup \Omega_1 \cup \Omega_2$ , with  $\Omega_0 = \{0\}$ ,  $\Omega_1 = \{-\} \cup \Omega'_1$  and  $\Omega_2 = \{+\} \cup \Omega'_2$ . where we adopted the additive notation for the (non necessarily commutative) group structure. These data must satisfy the following axioms

$$a * (b + c) = a * b + a * c, \quad (b + c) * a = b * a + c * a, \quad \text{for all } * \in \Omega'_2;$$

$$\omega(a + b) = \omega(a) + \omega(b), \quad \text{for all } \omega \in \Omega'_1;$$

$$\omega(a) * b = \omega(a * b) = a * \omega(b), \quad \text{for all } * \in \Omega'_2 \text{ and } \omega \in \Omega'_1.$$

Examples of categories of distributive  $\Omega_2$ -groups are the categories of groups, rings, Lie algebras, Leibnitz algebras among others.

Moreover, for all distributive  $\Omega_2$ -groups, it is possible to translate conditions involving internal actions, in conditions involving *external* actions as defined in [13], thus making the theory manageable in many algebraic situation of interest.

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