RENDICONTI DEL CIRCOLO MATEMATICO DI PALERMO
Serie II, Tomo L (2001), pp. 299-311

ANALYTIC SOLUTIONS OF THE NAVIER-STOKES EQUATIONS

MARIA CARMELA LOMBARDO

We consider the time dependent incompressible Navier-Stokes equations on an half plane. For analytic initial data, existence and uniqueness of the solution are proved using the Abstract Cauchy-Kovalevskaya Theorem in Banach spaces. The time interval of existence is proved to be independent of the viscosity.

1. Introduction.

Despite of both the clear-cut physical model adopted and the simplicity of the resulting equations, up to date, there are many unanswered questions related to the unstationary Navier-Stokes problem.

In the general three dimensional case only partial regularity results have been asserted: globally in time, the existence of a weak L_2 solution is proved, but the question of uniqueness is still open. On the other hand, existence and uniqueness of a smooth solution is stated only locally in time, the size of the time interval being determined by the data of the problem.

If one considers the L_2 norm of the velocity and its derivatives, it is possible to show that, in the case of periodic boundary conditions, the following inequality holds (see [1]):

$$\frac{1}{2}\frac{d}{dt}\|\nabla \mathbf{u}\|_2^2 \le -\nu\|\Delta \mathbf{u}\|_2^2 + c\|\Delta \mathbf{u}\|_2^{3/2}\|\nabla \mathbf{u}\|_2^{3/2}$$

from which, with the aid of Hölder's inequality, one can derive:

$$\frac{d}{dt}\|\nabla \mathbf{u}\|_2^2 \le c\nu^{-3}\|\nabla \mathbf{u}\|_2^5$$

which degrades as $\upsilon \to 0$. The question that naturally arises is: does the time of existence of a regular solution tend to zero as the viscosity tends to zero?

The question of regularity is, other than a mathematical problem, also a philosophical one. A discontinuity in the solution would imply a macroscopic change over a microscopic infinitesimal interval, which, in turn, would reveal the presence of small scale structures in the flow. This would be in conflict with the assumption under which the incompressible Navier-Stokes equations are derived, namely interacting particles in a limit of infinite separation of length and time scales between the microscopic and the macroscopic phenomena.

In the rest of this paper we shall prove that, in a suitable Banach space of analytic functions, a unique regular solution of the 3 - D Navier-Stokes equations on an half space exists for a time which is independent of the viscosity. We shall use an abstract formulation of the Cauchy-Kovalevskaya Theorem which allows to estimate the nonlinear term through an iterative procedure. In order to apply the above mentioned theorem without any loss of regularity, the essential point is the use of the Cauchy estimates for the derivatives of the analytic functions.

Our analysis will strictly follow Sammartino and Caflish ([4] and [5]). In their papers the authors proved that the solution of the Navier-Stokes equations with analytic initial data can be decomposed in the form of an asymptotic series. The zero-order term is composed of the sum of an Euler solution plus a Prandtl solution exponentially decaying outside the boundary layer, while the norm of the correction term is showed to be of order $\sqrt{\upsilon}$ in a proper Banach space of analytic functions.

We generalize the results obtained therein. In fact we show that the time of existence of a regular solution does not depend on the boundary layer solution whereas in [4] and [5] the size of the domain of analyticity was shrinking at each step of the asymptotic expansion.

The paper is organized as follows: in Section 2 we shall set the notation and the functional spaces where existence and uniqueness will be proved. In Section 3 we will state the Abstract Cauchy-Kovalevskaya Theorem (ACK) in the form proposed by Safonov [3]. In Section [4] we shall introduce the Navier-Stokes operator we will need to put the Navier-Stokes equations in a suitable form for the application of the ACK theorem. Through the Navier-Stokes operator the Navier-Stokes equations will be solved in Section 5 and the iterative procedure of the ACK theorem will be shown to converge to a unique solution.

2. Statement of the problem and function spaces.

We shall deal with the incompressible Navier-Stokes equations in a 3 - D half space for the velocity field $\mathbf{u} = (u, v)$, u and v being the components of the velocity tangent and ortogonal to the boundary respectively. Namely:

$$\partial_t \mathbf{u} - \nu\Delta\mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p = 0, \tag{2.1}$$

$$\nabla\cdot\mathbf{u} = 0, \tag{2.2}$$

$$\gamma\mathbf{u} = 0, \tag{2.3}$$

$$\mathbf{u}(t=0) = \mathbf{u}_0, \tag{2.4}$$

where γ is the trace operator defined as:

$$\gamma u = (u(x, y=0, t), v(x, y=0, t))$$

and $u_0 = (u'_0, u_{0_n})$ is the initial data. Primed quantities denote the tangential components of a vector, while the subscript n denotes the normal component. In the sequel we shall denote $\varepsilon^2 = \nu$ and $Y = y/\varepsilon$.

We shall also make use of the Fourier transform. Namely, given $f(x)$ we define its Fourier transform as:

$$\hat{f}(\xi') = \frac{1}{(2\pi)^{1/2}} \int dx f(x) e^{-ix\xi'},$$

where the above integral is on the whole real line. We shall adopt the convention of using ξ' as the dual of x. Moreover if $\sigma(T)(\xi')$ is a function of ξ' such that

$$\widehat{Tf}(\xi') = \sigma(T)(\xi')\hat{f}(\xi')$$

where T is an operator acting on functions of one variable, then $\sigma(T)$ is called the symbol of the pseudodifferential operator T. If T acts on functions of two (or more) variables,the definition is analogous.

As far as this chapter is concerned we shall always be dealing with functions that are analytic in the two complex variables x and y.

We first introduce the "strip" and the "conoid" in the complex plane.

$$D(\rho) = \mathbb{R} \times (-\rho, \rho) = \{x \in \mathbb{C} : \Im x \in (-\rho, \rho)\} \tag{2.5}$$

$$\begin{aligned} \Sigma(\theta, a) = &\{y \in \mathbb{C} : 0 \le \Re y \le a \text{ and } |\Im y| \le \Re y \tan\theta\} \\ &\cup \{y \in \mathbb{C} : \Re y \ge a \text{ and } |\Im y| \le a \tan\theta\}. \end{aligned} \tag{2.6}$$

The functions will be L^2 in both the tangential and normal variable. Hence we introduce the paths along which the L^2 integration is performed:

$$\Gamma(b) = \{x \in \mathbb{C} : \Im x = b\} \tag{2.7}$$

$$\begin{aligned} \Gamma(\theta', a) = &\{y \in \mathbb{C} : 0 \le \Re y \le a \text{ and } \Im y = \Re y \tan\theta'\} \\ &\cup \{y \in \mathbb{C} : \Re y \ge a \text{ and } \Im y = a \tan\theta'\} \end{aligned} \tag{2.8}$$

In what follows the values of the angle θ and the parameter l counting the number of derivatives will always be restricted to

$$0 < \theta < \pi/4$$
$$4 \le l.$$

Let us now introduce the Banach spaces: we begin with the space of functions which are analytic only in the x variable.

DEFINITION 2.1. *$H^{\prime l, \rho}$ is the set of all complex functions $f(x)$ such that*

1) f is analytic in $D(\rho)$

2) $\partial_x^\alpha f \in L^2(\Gamma(\Im x))$ for $\Im x \in (-\rho, \rho)$, $\alpha \le l$; i.e. if $\Im x$ is inside $(-\rho, \rho)$, then $\partial_x^\alpha f(\Re x + i\Im x)$ is a square integrable function of $\Re x$

3) $|f|_{l,\rho} = \sum_{\alpha \le l} \sup_{\Im x \in (-\rho, \rho)} \|\partial_x^\alpha f(\cdot + i\Im x)\|_{L^2(\Gamma(\Im x))} < \infty.$

In the above defined norm one has first to compute the usual L^2 norm, performing the integration with respect to the real part of x and then one takes the $\sup_{\Im x \in (-\rho, \rho)}$ with respect to the imaginary part of x.

Notice that the above definition of the norm is equivalent to:

$$|f|_{l,\rho} = \sum_{k \le l} \left\{ \int d\xi' \left| e^{\rho|\xi'|} |\xi'|^k \hat{f}(\xi') \right|^2 \right\}^{1/2}. \tag{2.9}$$

We now introduce the dependence on the time variable.

DEFINITION 2.2. $H'^{l,\rho}_{\beta,T}$ *is the set of all complex functions* $f(x,t)$ *such that*

1) $\partial_t^j f(x,t) \in L^\infty([0,T], H'^{l-j,\rho}$ *for* $j \leq l$;

2) $|f|_{l,\rho,\beta,T} = \sum_{j=0}^{l} \sup_{0\leq t\leq T} |\partial_t^j f(\cdot,\cdot,t)|_{l-j,\rho-\beta t} < \infty.$

Let us now introduce the dependence on the normal variable. Since we want the time of existence of the solution to be independent of the viscosity, we define a rescaled normal variable $Y = 1/\varepsilon$.

DEFINITION 2.3. $L^{l,\rho,\theta}$ *is the set of all functions* $f(x,Y)$ *such that*

1) f *is analytic inside* $D(\rho) \times \sum(\theta, a/\varepsilon)$

2) $\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(x,Y) \in L^2(\Gamma(\theta', a/\varepsilon); H'^{0,\rho})$ *with* $|\theta'| \leq \theta, \alpha_1 \leq 2, \alpha_1 + \alpha_2 \leq l$ *and* $\alpha_2 \leq l-2$ *when* $\alpha_1 > 0$

3) $$|f|_{l,\rho,\theta} = \sum_{\alpha\leq l} \sup_{|\theta'|\leq\theta} \| |\partial_x^\alpha f(\cdot,Y)|_{0,\rho} \|_{L^2(\Gamma(\theta',a/\varepsilon))}$$
$$+ \sum_{0<\alpha_1\leq 2} \sum_{0\leq\alpha_2\leq l-2} \sup_{|\theta'|\leq\theta} \| |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,Y)|_{0,\rho} \|_{L^2(\Gamma(\theta',a/\varepsilon))} < \infty.$$

DEFINITION 2.4. *The space* $L^{l,\rho,\theta}_{\beta,T}$ *is the set of functions* $f(x,Y,t)$ *such that*

1) $f \in C^0([0,T], L^{l,\rho,\theta})$ *and* $\partial_t\partial_x^\alpha f \in C^0([0,T], L^{0,\rho,\theta})$ *with* $\alpha \leq l-2$

2) $$|f|_{l,\rho,\theta,\beta,T} = \sum_{0\leq j\leq 1} \sum_{\alpha\leq l-2j} \sup_{0\leq t\leq T} |\partial_t^j \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}$$
$$+ \sum_{0<\alpha_1\leq 2} \sum_{\alpha_2\leq l-2} \sup_{0\leq t\leq T} |\partial_Y^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} < \infty.$$

Finally we recall the following Sobolev inequality, which will be useful in the sequel.

PROPOSITION 2.1. *Let* $f, g \in L^{l,\rho,\theta}_{\beta,T}$ *and* $l \geq 4$. *Then* $f\cdot g \in L^{l,\rho,\theta}_{\beta,T}$, *and*

$$|f\cdot g|_{l,\rho,\theta,\beta,T} \leq c|f|_{l,\rho,\theta,\beta,T}|g|_{l,\rho,\theta,\beta,T}.$$

The main result of this paper is the following theorem.

THEOREM 2.1. *Suppose that* $\mathbf{u}_0 \in L^{l,\rho_0,\theta_0}$, $l \geq 4$, *with* $\nabla \cdot \mathbf{u}_0 = 0$ *and* $\gamma u_0 = 0$. *Then there exist* ρ, β, θ, T *such that* $0 < \rho < \rho_0$, $0 < \theta < \theta_0$, $0 < \beta$, $0 < T$, *such that the Navier-Stokes equations Eqs. (2.1) - (2.4) admit a unique solution* $\mathbf{u} \in L^{l,\rho,\theta}_{\beta,T}$. *This solution satisfies the following bound in* $L^{l,\rho,\theta}_{\beta,T}$:

$$|\mathbf{u}|_{l,\rho,\theta,\beta,T} < c|\mathbf{u}_0|_{l,\rho_0,\theta_0}.$$

3. The Abstract Cauchy-Kovalevskaya Theorem.

In order to prove existence and uniqueness for the Navier-Stokes equations we are going to use an abstract version of the Cauchy-Kovalevskaya Theorem formulated in Banach spaces. We shall refer to the version proposed by Safonov [3].

We first define a Banach scale $\{\mathbb{X}_\rho : 0 < \rho \leq \rho_0\}$ with norms $|\ |_\rho$ as a family of Banach spaces such that $\mathbb{X}_{\rho'} \subset \mathbb{X}_{\rho''}$ and $|\ |_{\rho''} \leq |\ |_{\rho'}$ when $\rho'' \leq \rho' \leq \rho_0$.

For t in $[0, T]$, consider the equation

$$u(t) = u_0 + \int_0^t \mathcal{F}(\tau, u(\tau))d\tau. \tag{3.1}$$

The following Theorem holds.

THEOREM 3.1. **(ACK)**

Let us assume that $\exists R > 0$, $\rho_0 > 0$, *and* $\beta_0 > 0$ *such that if* $0 < t \leq \rho_0/\beta_0$, *the following hold:*

1) $\forall 0 < \rho' < \rho \leq \rho_0$ *the function*

$$\mathcal{F}(t, u) : \{u \in \mathbb{X}_\rho : |u|_\rho \leq R\} \times [0, \rho_0/\beta_0] \to \mathbb{X}_{\rho'}$$

is continuous.

2) $\forall 0 < \rho < \rho_0$ *and* $t \in [0, \rho_0/\beta_0]$, *the function* $\mathcal{F}(t, 0)$ *is continuous in* $[0, \rho_0/\beta_0]$ *and satisfies, with a fixed constant* K, *the following bound:*

$$|\mathcal{F}(t, 0)|\rho \leq K. \tag{3.2}$$

3) $\forall 0 < \rho' < \rho'' < \rho_0, t \in [0, \rho_0/\beta_0]$ *and* $\forall u^1, u^2 \in \mathbb{X}_{\rho''}$ *with* $|u^1|_{\rho''} < R, |u^2|_{\rho''} < R$

$$|\mathcal{F}(t, u^1) - \mathcal{F}(t, u^2)|_{\rho'} \leq C \frac{|u^1 - u^2|_{\rho''}}{\rho'' - \rho'}. \tag{3.3}$$

Then for any positive ρ_0, R, C *and* K *there is a positive constant* β_0 *such that, under the above assumptions,* $\exists \beta > \beta_0$ *such that Eq. (3.1) has a unique continuously differentiable solution* $u(t)$ *with*

$$u(t) \in \{u \in \mathbb{X}_\rho : |u(t)|_\rho < R\} \quad \textit{for all } t \in [0, (\rho_0 - \rho)/\beta).$$

4. The Pseudodifferential Operators.

In this section we shall introduce the operators we need in order to put the Navier-Stokes equations in a suitable form for the application of the ACK theorem.

4.1. *The Inverse Heat Operator.*

In order to apply the ACK theorem to Eqs. (2.1) - (2.4) one has first to invert the diffusion operator $(\partial_t - \varepsilon^2 \partial_{xx} - \partial_{YY})$ through the inverse heat operator E^*. The operator E^* solves the following system:

$$\begin{aligned} (\partial_t - \varepsilon^2 \partial_{xx} - \partial_{YY}) E^* u &= u(x, Y, t), \\ E^* u(x, Y, t = 0) &= 0, \\ \gamma E^* u &= 0, \end{aligned} \tag{4.1}$$

Using the explicit expression of $E^* u$, one can prove the following Propositions (see for examlpe [2]).

PROPOSITION 4.1. *Let* $u \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $E^* u \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|E^* u|_{l,\rho,\theta,\beta,T} \leq c|u|_{l,\rho,\theta,\beta,T}.$$

4.2. *The Projected Heat Operator.*

We now introduce the divergence-free projection operator P^∞. It is the pseudodifferential operator whose symbol is (we shall omit the distinction between the operator and its symbol):

$$P^\infty = \frac{1}{\varepsilon^2\xi'^2 + \xi_n^2}\begin{pmatrix} \xi_n^2 & -\varepsilon\xi'\xi_n \\ -\varepsilon\xi'\xi_n & \varepsilon^2\xi'^2 \end{pmatrix}, \tag{4.2}$$

where ξ' and ξ_n denote the Fourier variables corresponding to x and Y respectively. The operator P^∞ can be through as

$$P^\infty = 1 - \nabla\Delta^{-1}\nabla.$$

and has the property of being divergence-free, i.e.

$$\nabla \cdot P^\infty \mathbf{u} = 0.$$

It is possible to give an explicit expression of P^∞ which avoids Fourier transform in y. One first extends u oddly to $Y < 0$, i.e.

$$u(x, Y) = -u(x, -Y) \quad \text{when} \quad Y \le 0;$$

and restricts the result of the application of P^∞ to $Y \ge 0$ for application of the norm. The expression one gets for P^∞ are:

$$\begin{aligned} P_n^\infty \mathbf{u} =& \frac{1}{2}\varepsilon|\xi'|\left[\int_0^Y dY'\left(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')}\right)(-N'u^1 + u^2)\right. \\ &\left. + \int_Y^\infty dY'\left(e^{\varepsilon|\xi'|(Y-Y')}(N'u^1 + u^2) - e^{\varepsilon|\xi'|(-Y-Y')}(-N'u^1 + u^2)\right)\right], \end{aligned} \tag{4.3}$$

$$\begin{aligned} P^{\infty\prime}\mathbf{u} =& u^1 - \frac{1}{2}\varepsilon|\xi'|\left[\int_0^Y dY'(e^{-\varepsilon|\xi'|(Y-Y')} - e^{-\varepsilon|\xi'|(Y+Y')})(u^1 + N'u^2)\right. \\ &\left. + \int_Y^\infty dY'\left(e^{\varepsilon|\xi'|(Y-Y')}(u^1 - N'u^2) - e^{\varepsilon|\xi'|(-Y-Y')}(u^1 + N'u^2)\right)\right]. \end{aligned} \tag{4.4}$$

Using the above expressions, the following estimate is easily proved.

PROPOSITION 4.2. *Let* $\mathbf{u} \in L^{l,\rho,\theta}$ *with* $\gamma\mathbf{u} = 0$. *Then* $P^\infty\mathbf{u} \in L^{l,\rho,\theta}$ *and*

$$|P^\infty \mathbf{u}|_{l,\rho,\theta} \le c|\mathbf{u}|_{l,\rho,\theta}.$$

We finally introduce the projected heat operator $\mathcal{N}_0$, acting on vectorial functions, defined as

$$\mathcal{N}_0 = P^\infty E^*. \tag{4.5}$$

P^∞ commutes with the heat operator $(\partial_t - \partial_{YY} - \varepsilon^2\partial_{xx})$. It then follows that for each u such that $\gamma u = 0$

$$\nabla \cdot \mathcal{N}_0 \mathbf{u} = 0, \tag{4.6}$$

$$(\partial_t - \partial_{YY} - \varepsilon^2\partial_{xx})\mathcal{N}_0\mathbf{u} = P^\infty \mathbf{u}. \tag{4.7}$$

Using the estimates given for P^∞ and E^*, one can prove the following:

PROPOSITION 4.3. *Suppose* $\mathbf{u} \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $\mathcal{N}_0\mathbf{u} \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\mathcal{N}_0\mathbf{u}|_{l,\rho,\theta,\beta,T} \le c|\mathbf{u}|_{l,\rho,\theta\beta,T}.$$

4.3. *The Navier-Stokes Operator.*

In order to invert the Navier-Stokes equations we first introduce the Stokes operator $\mathcal{S}$. It solves the linearized Navier-Stokes equations in the half space $y \ge 0$ with initial condition u_0 and boundary data g, namely:

$$(\partial_t - \nu\Delta)\mathcal{S}(\mathbf{g}, \mathbf{u}_0) + \nabla p = 0, \tag{4.8}$$

$$\nabla \cdot \mathcal{S}(\mathbf{g}, \mathbf{u}_0) = 0, \tag{4.9}$$

$$\gamma \mathcal{S}(\mathbf{g}, \mathbf{u}_0) = \mathbf{g}(x, t), \tag{4.10}$$

$$\mathcal{S}(\mathbf{g}, \mathbf{u}_0)(x, y, t = 0) = \mathbf{u}_0. \tag{4.11}$$

The Stokes problem Eqs. (4.8)-(4.11) can be explicitly solved following the same line as in [6], where the case of non-zero initial data is also considered. Using such an explicit form of the operator $\mathcal{S}$, one can prove the following estimate.

PROPOSITION 4.4. *Suppose that* $\mathbf{u}_0 \in L^{l,\rho,\theta}$ *with* $\nabla \cdot \mathbf{u}_0 = 0$. *Moreover let* $\mathbf{g} \in H'^{l,\rho}_{\beta,T}$, *with* $\mathbf{g}(t = 0) = \gamma\mathbf{u}_0$ *and such that the normal component* *is of the form*

$$g_n = |\xi'| \int_0^{\infty} dY' f(\xi', Y', t) k(\xi', Y'), \tag{4.12}$$

with $|\xi'| \int_0^{\infty} dY' |k(\xi', Y')| \leq 1$ *and* $f \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $\mathcal{S}(\mathbf{g}, \mathbf{u}_0) \in L^{l,\rho,\theta}_{\beta,T}$, *and*

$$|\mathcal{S}(\mathbf{g}, \mathbf{u}_0)|_{l,\rho,\theta\beta,T} \leq c(|g'|_{l,\rho,\beta,T} + |f|_{l,\rho,\theta\beta,T} + |\mathbf{u}_0|_{l,\rho,\theta}.$$

Remark. Notice that it is not possible to give an estimate of the L^2 norm in Y for a general class of boundary data. Nevertheless the conditions required by Proposition (4.6) will be satisfied by the solution to the Navier-Stokes equation.

The final step of the procedure is to introduce the Navier-Stokes operator $\mathcal{N}^*$

$$\mathcal{N}^*(\mathbf{w}, \mathbf{u}_0) = \mathcal{N}_0\mathbf{w} - \mathcal{S}(\gamma\mathcal{N}_0\mathbf{w}, 0) + \mathcal{S}(0, \mathbf{u}_0), \tag{4.13}$$

which inverts the following Stokes equations:

$$(\partial_t - \varepsilon^2\partial_{xx} - \partial_{YY})\mathcal{N}^*(\mathbf{w}, \mathbf{u}_0) + \nabla p = w, \tag{4.14}$$

$$\nabla \cdot \mathcal{N}^*(\mathbf{w}, \mathbf{u}_0) = 0, \tag{4.15}$$

$$\gamma\mathcal{N}^*(\mathbf{w}, \mathbf{u}_0) = 0, \tag{4.16}$$

$$\mathcal{N}^*(\mathbf{w}, \mathbf{u}_0)(x, y, t - 0) = \mathbf{u}_0. \tag{4.17}$$

With the aid of Proposition 4.3 and 4.4 one can easily prove the following.

PROPOSITION 4.5. *Let* $\mathbf{w} \in L^{l,\rho,\theta}_{\beta,T}$ *and* $\mathbf{u}_0 \in L^{l,\rho,\theta}$ *with* $\gamma\mathbf{u}_0 = 0$ *and* $\nabla \cdot \mathbf{u}_0 = 0$. *Then* $\mathcal{N}^*(\mathbf{w}, \mathbf{u}_0) \in L^{l,\rho,\theta}_{\beta,T}$ *and*

$$|\mathcal{N}^*(\mathbf{w}, \mathbf{u}_0)|_{l,\rho,\theta\beta,T} \leq c(|\mathbf{w}|_{l,\rho,\theta\beta,T} + |\mathbf{u}_0|_{l,\rho,\theta}).$$

Proof. We only have to prove that the normal component of the boundary data $\gamma\mathcal{N}_0$ is of the form (4.12) required by Proposition 4.4.

This is easily checked by a straightforward calculation using the explicit expression of P_n^∞ given by Eq. (4.3).

5. The Main Result.

In this section we shall prove the existence and uniqueness of the solution to the Navier-Stokes equations as stated in Theorem 2.1.

With the aid of the operators introduced in the previous section, one can put the Navier-Stokes equations, Eqs. (2.1)-(2.4), in the form:

$$\mathbf{u} = \mathcal{N}^*(\mathbf{u} \cdot \nabla \mathbf{u}, \mathbf{u}_0), \tag{5.18}$$

as required by the ACK theorem. We have to verify that the hypotheses 1., 2. and 3. of the Theorem 3.1 are satisfied by the operator $\mathcal{N}^*$. The hypotheses 1. and 2. are trivial.

As far as 3. is concerned we observe that we have to deal with the estimate of the nonlinear term, which also involves the derivatives of the velocity. We hence state the following Lemmas (Cauchy estimates).

LEMMA 5.1. *Let* $f(x, Y) \in L^{l,\rho,\theta}$. *Then for* $0 < \rho' < \rho$ *and* $0 < \theta' < \theta$

$$|\partial_x f|_{l,\rho',\theta} \leq c \frac{|f|_{l,\rho,\theta}}{\rho - \rho'},$$

$$|\chi(Y) \partial_Y f|_{l,\rho,\theta'} \leq c \frac{|f|_{l,\rho,\theta}}{\theta - \theta'}.$$

In the above Proposition the Y - derivative has to be multiplied by $\chi(Y)$, which is a monotone, bounded function going to zero linearly fast near the origin, because of the conoidal shape of the domain.

Using the Sobolev inequality, Proposition 2.1 we can prove the following Lemmas:

LEMMA 5.2. *Let* $f(x, Y)$ *and* $g(x, Y)$ *be in* $L^{l,\rho,\theta}$. *Then for* $0 < \rho' < \rho$

$$|g \partial_x f|_{l,\rho',\theta} \leq c |g|_{l,\rho',\theta} \frac{|f|_{l,\rho,\theta}}{\rho - \rho'}.$$

LEMMA 5.3. *Let* $f(x, Y)$ *and* $g(x, Y)$ *be in* $L^{l,\rho,\theta}$ *with* $g(x, Y = 0) = 0$. *Then for* $0 < \theta' < \theta$

$$|g\partial_Y f|_{l,\rho,\theta'} \leq c|g|_{l,\rho,\theta'} \frac{|f|_{l,\rho,\theta}}{\theta - \theta'}.$$

With the aid of the above Lemmas, the nonlinear part is easily bounded as stated by the following Proposition.

PROPOSITION 5.1. *Suppose* $\mathbf{u}^1$ *and* $\mathbf{u}^2$ *are in* $L^{l,\rho,\theta}_{\beta,T}$ *wxith* $\gamma_n \mathbf{w}^1 = \gamma_n \mathbf{w}^2 = 0$. *Then for* $0 < \rho' < \rho$ *and* $0 < \theta' < \theta$

$$|\mathbf{u}^1 \cdot \nabla \mathbf{u}^1 - \mathbf{u}^2 \cdot \nabla \mathbf{u}^2|_{l,\rho',\theta'} \leq c \left[\frac{|\mathbf{u}^1 - \mathbf{u}^2|_{l,\rho,\theta'}}{\rho - \rho'} + \frac{|\mathbf{u}^1 - \mathbf{u}^2|_{l,\rho',\theta}}{\theta - \theta'} \right]$$

where the constant c *depends only on* $|\mathbf{u}^1|_{l,\rho,\theta\beta,T}$ *and* $|\mathbf{u}^2|_{l,\rho,\theta\beta,T}$.

This concudes the proof of the Theorem 2.1.

6. Concluding Remarks.

In this paper we have considered the incompressible Navier-Stokes equations on an half space. Assuming analiticity of the initial data, we have proved that a unique regular solution exists for a time which is independent of the viscosity. Moreover, since the result has been obtained through a direct study of the Navier-Stokes equations, it clarifies how the time of existence of a regular solution does not depend on the boundary layer solution.

We stress the fact that the properties of the analytic functions (especially the Cauchy estimates), play an essential role in the proof. No analogous result, in the realm of Sobolev space, up to date is known.

Acknowledgements.

This paper is based on a part of the author's PhD thesis. The author thanks his advisor, Professor M. Sammartino, for his support and suggestion of this work. The author also thanks Professor A. Greco for the enlightning discussions and useful suggestions.

REFERENCES

[1] Doering C. R., Gibbon J. D., *Applied Analysis of the Navier-Stokes Equations,* Cambridge Texts in Applied Mathematics, 1995.

[2] Lombardo M. C., *Asymptotic Analysis and Well-Posedness Problems in the Boundary Layer Theory,* PhD Thesis, (1999).

[3] Safonov M. V., *An Abstract Cauchy-Kovalevskaya Theorem in a Weighted Banach Space,* Comm. Pure Appl. Math., **48** (1995), 629-637.

[4] Sammartino M., Caflisch R. E., *Zero Viscosity Limit for Analytic Solutions of the Navier-Stokes Equation on a Half-Space I. Existence for Euler and Prandtl Equations,* Comm. Math. Phys., **192** (1998), 433-461.

[5] Sammartino M., Caflisch R. E., *Zero Viscosity Limit for Analytic Solutions of the Navier-Stokes Equation on a Half-Space II. Construnction of the Navier-Stokes Solution,* Comm. Math. Phys., **192** (1998), 463-491.

[6] Ukai S., *A Solution Formula for the Stokes Equation in* R^n_+, Comm. Pure Appl. Math., **40** (1987), 611-621.

Pervenuto il 22 settembre 2000

Maria Carmela Lombardo
Dipartimento di Matematica
University of Palermo
Via Archirafi, 34
90123 Palermo, Italy
lombardo@dipmat.math.unipa.it