Arch. Rational Mech. Anal. 222 (2016) 15–45
Digital Object Identifier (DOI) 10.1007/s00205-016-0995-x

Zero Viscosity Limit for Analytic Solutions of the Primitive Equations

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Communicated by P. Constantin

Abstract

The aim of this paper is to prove that the solutions of the primitive equations converge, in the zero viscosity limit, to the solutions of the hydrostatic Euler equations. We construct the solution of the primitive equations through a matched asymptotic expansion involving the solution of the hydrostatic Euler equation and boundary layer correctors as the first order term, and an error that we show to be $O(\sqrt{\nu})$. The main assumption is spatial analyticity of the initial datum.

1. Introduction

In this paper, we address the vanishing viscosity limit and analyze the structure of the boundary layer for solutions of the primitive equations with the non-slip boundary condition on the bottom of the domain.

The primitive equations provide a fundamental model in the study of the global circulation dynamics and weather prediction. They are obtained from the full Boussinesq system under the hydrostatic approximation, where the equation for the motion of the third component of the velocity is replaced by the hydrostatic equation for the pressure. This approximation is natural when comparing the sizes of different terms in the momentum equation for physical data. The resulting equations are challenging mathematically due to the loss of a derivative compared either to the original Boussinesq model or to the Navier–Stokes equations.

The study of the primitive equations was initiated by Lions, Wang, and Temam [31–33], who provided the mathematical setting for the equations and established results pertaining to the existence of weak and strong solutions. This theory was advanced by Temam and Ziane [51], who proved the existence of a global weak and the local strong solution (cf. also [44,53]). The global existence of strong solutions was proved by Cao and Titi in [11] (see also [29,30] for non-slip boundary conditions addressed in the present paper). For other works on the regularity of solutions for the primitive equations cf. [14,26].

On the other hand, the hydrostatic Euler equations (or simply hydrostatic equations) were derived by Lions in [34] as a thin domain approximation of the Euler equation (cf. also [5,20]). However, the equations seem to be unstable, due to the aforementioned derivative loss in the equations, and the local existence is currently only known for convex data [4,20,40], analytic data [27], or a combination thereof [25]. Due to instability of the spectrum, it is believed that for Sobolev data the equations are locally ill-posed [20,46]; currently a finite time blow-up may occur as shown in [10].

Since the primitive equations model large scale motion of the ocean, while the viscosity acts on small scales, it seems natural to expect that the viscosity does not play a role in the dynamics and can be neglected. This leads to the question of whether the primitive equations converge to the hydrostatic Euler equations in the limit of a vanishing viscosity. While a lot is known about this singular limit in the case of the Navier–Stokes equations, the case of the primitive equations is open for any class of initial data.

The main goal of this paper is to prove that the vanishing viscosity theorem holds for the primitive equations with non-slip boundary conditions given analytic initial data (cf. Theorem 8.1 below). For the Neumann (no-flow) boundary conditions, we can extend the solutions by reflection and periodicity to the case of a domain with no boundary [19,45]. It is then possible to obtain the vanishing viscosity limit theorem for analytic data by an adaptation of the techniques in the present paper. When considering the non-slip boundary condition we are faced, in addition, with different boundary conditions at the limit, the hydrostatic equations allowing a tangential slip.

Here we briefly summarize some of the work on the vanishing viscosity limit for the Navier–Stokes equations. First, the case of the domain without boundary was completely resolved in [13,21,39,43]. On the other hand, the convergence of the Navier–Stokes solution toward the Euler solution is still an open problem in the case of non-slip (Dirichlet) boundary condition. An important necessary and sufficient condition was found by Kato [22], who showed that the vanishing limit theorem holds if the total dissipation in a small neighborhood of size ν of the boundary vanishes in the limit $\nu \to 0$. Other sufficient conditions were found in [12,23,50,52] (cf. also a review paper [2]). On the other hand, for analytic data, the vanishing viscosity limit was established in [48,49] for the case of the half space. In addition, an asymptotic expansion of the Navier–Stokes solution was provided as a sum of the Euler equation and the Prandtl corrector in the physical boundary layer. The approach was extended to the case of the exterior of a disk [6] (see also [9] for non-compatible initial data). For non-analytic data, the vanishing limit only holds in certain situations in the case when the Navier–Stokes nonlinearity decouples, in the presence of certain symmetries, if the vertical viscosity is smaller compared to the horizontal one, or in the case of compactly supported vorticity [3,24,37–39,42].

Compared to the Navier–Stokes counterpart, we are faced with several difficulties. The first is the loss of the derivative in the vertical velocity component. Due to this fact, the hydrostatic Euler equation is not solvable except in analytic spaces or with convex data. The second difficulty we are faced with is that the physics requires non-slip boundary conditions on the bottom, but Neumann conditions on the top, representing the ocean surface. This is overcome by a reflection principle, which allows us to set the problem so we have non-slip boundary conditions on both sides of the boundary. However, we then need to provide asymptotic expansions on both sides of the boundary. Since the expansions on both sides interfere with each other, we need to add additional correction terms. Finally, the non-slip boundary conditions lead to an additional boundary term when inverting the pressure equation, and this term has a derivative loss as well.

In order to solve the vanishing viscosity problem for the primitive equations, we develop the solution in an asymptotic expansion near the boundary, which in turn requires the analysis of the boundary layer equations. The main result of this paper states that the solution $\mathbf{u}$ of the primitive equations (PE), in the limit of small viscosity $\nu \to 0$, has the asymptotic structure

$$\mathbf{u} = \mathbf{u}^{(0)} + \sqrt{\nu}\mathbf{u}^{(1)} + \sqrt{\nu}\mathbf{e}, \tag{1.1}$$

where

$$\mathbf{u}^{(0)} = \mathbf{u}^{H} + \tilde{\mathbf{u}}^{BL}$$

with $\mathbf{u}^H$ the solution of the hydrostatic equations and $\mathbf{u}^{BL}$ a boundary layer (BL) corrector that varies rapidly close to the boundaries and decays exponentially small away from them. This boundary layer (BL) corrector is needed to compensate for the tangential slip generated by the hydrostatic solution. The first order term in the above expansion $\mathbf{u}^{(1)}$ is given by

$$\mathbf{u}^{(1)} = \mathbf{u}^{H(1)} + \mathbf{u}^{BL(1)}$$

where $\mathbf{u}^{H(1)}$ is a small correction to the hydrostatic flow, which solves Equations (5.3a)–(5.3d), and $\mathbf{u}^{BL(1)}$ is a small correction to the boundary layer flow which is assumed to solve a linear heat equation with the term $\mathbf{e}$ an overall correction solving Equations (5.7a)–(5.7e) below.

The paper is organized as follows. In Section 2, we recall the primitive and the hydrostatic equations. We introduce the expansion of the solution in terms of the hydrostatic equation, the boundary layer corrections $\tilde{\mathbf{u}}^{+}$, $\tilde{\mathbf{u}}^{-}$, and the error $\mathbf{w}$. Section 3 contains the local existence result for the hydrostatic equation with analytic data, proved using the abstract Cauchy–Kowalevski (ACK) theorem. In Section 4, we provide a construction of the boundary layer correctors. The reminder $\mathbf{w}$ is decomposed further in Section 5, where we introduce the refined error $\mathbf{e}$. In the remainder of the paper, we finally prove that the error $\mathbf{e}$ is uniformly bounded on a local time interval. In Section 8 the main result of the paper is formally stated.

2. The Primitive and the Hydrostatic Equations

2.1. The Primitive Equations

The primitive equations (PE) for a velocity field $\mathbf{u}(\mathbf{x}, z) = (\mathbf{u}, u_3)$, where $\mathbf{x} \in \mathbb{T}^2$ and $z \in [-h, h]$ read

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + u_3 \partial_z \mathbf{u} + \nabla p = \nu \Delta \mathbf{u} + \nu \partial_{zz} \mathbf{u} \tag{2.1a}$$
$$\nabla \cdot \mathbf{u} + \partial_z u_3 = 0 \tag{2.1b}$$
$$\gamma^+ \mathbf{u} = \gamma^+ u_3 = 0 \tag{2.1c}$$
$$\gamma^- \mathbf{u} = \gamma^- u_3 = 0 \tag{2.1d}$$
$$\mathbf{u}\,(\mathbf{x}, z, t = 0) = \mathbf{u}_0\,(\mathbf{x}, z)\,. \tag{2.1e}$$

In the above equations $\nabla = (\partial_x, \partial_y)$ and $\Delta = \partial_{xx} + \partial_{yy}$ denote the horizontal gradient and the Laplacian, while

$$\gamma^+ f(x, y, z) = f(x, y, z = h) \tag{2.2}$$

and

$$\gamma^- f(x, y, z) = f(x, y, z = -h) \tag{2.3}$$

represent the traces on the upper and the lower boundary respectively. The normal velocity u_3 is computed from

$$u_3 = -\int_{-h}^{z} \nabla \cdot \mathbf{u}\, dz'.$$

Note that the boundary conditions for u_3 at the top of the domain are equivalent to the condition

$$\int_{-h}^{h} \nabla \cdot \mathbf{u}\, dz' = 0.$$

We shall assume that the initial datum is compatible with the boundary condition

$$\gamma^{\pm} \mathbf{u}_0 = 0, \tag{2.4}$$

and satisfies

$$\int_{-h}^{h} \nabla \cdot \mathbf{u}_0\, dz' = 0. \tag{2.5}$$

Taking the two dimensional divergence of (2.1a) and then averaging in the vertical direction give an equation for the pressure

$$-\Delta p = \frac{1}{2h}\left(\int_{-h}^{h} \nabla \cdot (\nabla \cdot (\mathbf{u}\mathbf{u}))\; dz' - \nu \left[\partial_z \nabla \cdot \mathbf{u}\right]_{z=-h}^{z=h}\right). \tag{2.6}$$

Equation (2.6) ensures that the zero-average condition for the divergence of the initial datum $\mathbf{u}_0$ is maintained by the solution.

2.2. The Hydrostatic Euler Equations

Next we recall the hydrostatic Euler equations

$$\partial_t \mathbf{u}^H + \mathbf{u}^H \cdot \nabla \mathbf{u}^H + u_3^H \partial_z \mathbf{u}^H + \nabla p^H = 0 \tag{2.7a}$$
$$\nabla \cdot \mathbf{u}^H + \partial_z u_3^H = 0 \tag{2.7b}$$
$$\gamma^+ u_3^H = 0 \tag{2.7c}$$
$$\gamma^- u_3^H = 0 \tag{2.7d}$$
$$\mathbf{u}^H(\mathbf{x}, z, t = 0) = \mathbf{u}_0(\mathbf{x}, z). \tag{2.7e}$$

Also in this case the normal velocity u_3 is computed by

$$u_3^H = -\int_{-h}^{z} \nabla \cdot \mathbf{u}^H \, dz'. \tag{2.8}$$

Moreover, the boundary condition for u_3^H at the top of the domain is equivalent to the condition

$$\int_{-h}^{h} \nabla \cdot \mathbf{u}^H \, dz' = 0,$$

which by taking the two dimensional divergence of (2.7a) and averaging in the z direction leads to

$$-\Delta p^H = \frac{1}{2h} \int_{-h}^{h} \nabla \cdot \left(\nabla \cdot \left(\mathbf{u}^H \mathbf{u}^H \right) \right) dz'. \tag{2.9}$$

The above equation for the pressure ensures that the zero-average condition for the divergence of the initial datum $\mathbf{u}_0$ is maintained by the solution.

2.3. The Expansion

First, we expand the PE solution as

$$\mathbf{u} = \mathbf{u}^H + \tilde{\mathbf{u}}^+ + \tilde{\mathbf{u}}^- + \varepsilon \mathbf{w} \tag{2.10a}$$
$$u_3 = u_3^H + \varepsilon \tilde{u}_3^+ + \varepsilon \tilde{u}_3^- + \varepsilon w_3, \tag{2.10b}$$

where $\varepsilon = \sqrt{\nu}$. The terms $(\tilde{\mathbf{u}}^\pm, \varepsilon \tilde{u}_3^\pm)$ specified in the next section are boundary layer (BL) correctors at the top and bottom boundaries. They correct (cancel) the tangential slip introduced by the hydrostatic solution. We show below that these correctors decay exponentially fast away from the boundary layer. Also, we show that

$$\begin{aligned} \tilde{\mathbf{u}}^- &= \mathbf{u}^{BL-} - \gamma^- \mathbf{u}^H \\ \tilde{u}_3^- &= \int_Z^{h/\varepsilon} \nabla \cdot \mathbf{u}^{BL-} \, dZ' \end{aligned} \tag{2.11}$$

where $\mathbf{u}^{BL-}$ is a BL solution which approximates the primitive equation solution at the bottom boundary. The variable Z is a rescaled normal variable, the meaning of which is explained in the next subsection.

The term $(\mathbf{w}, w_3)$ is a remainder term which is expanded further in Section 5.

2.4. The Boundary Layer Equation: the Bottom Boundary

We introduce the rescaled variable

$$Z = \frac{z}{\varepsilon}.$$

Assuming that close to the bottom boundary the solution of the PE depends on Z, denoting the boundary layer approximation of the PE solution with $(\mathbf{u}^{BL-}, \varepsilon u_3^{BL-})$, and imposing the usual boundary layer asymptotic hypotheses, we obtain the boundary layer equations

$$\partial_t \mathbf{u}^{BL-} + \mathbf{u}^{BL-} \cdot \nabla \mathbf{u}^{BL-} + u_3^{BL-} \partial_Z \mathbf{u}^{BL-} + \nabla p^H = \partial_{ZZ} \mathbf{u}^{BL-} \tag{2.12a}$$

$$u_3^{BL-} = -\int_{-h/\varepsilon}^{Z} \nabla \cdot \mathbf{u}^{BL-}(\mathbf{x}, Z', t)\, dZ' \tag{2.12b}$$

$$\mathbf{u}^{BL-}(\mathbf{x}, Z = -h/\varepsilon, t) = 0 \tag{2.12c}$$

$$\mathbf{u}^{BL-}(\mathbf{x}, Z = h/\varepsilon, t) = \gamma^- \mathbf{u}^H \tag{2.12d}$$

$$\mathbf{u}^{BL-}(\mathbf{x}, Z, t = 0) = 0. \tag{2.12e}$$

Introducing the new function $\tilde{\mathbf{u}}^-$ defined by

$$\tilde{\mathbf{u}}^- = \mathbf{u}^{BL-} - \gamma^- \mathbf{u}^H \tag{2.13}$$

and using

$$\partial_t \gamma^- \mathbf{u}^H + \gamma^- \mathbf{u}^H \cdot \nabla \gamma^- \mathbf{u}^H + \nabla p^H = 0,$$

which are derived by taking the trace of (2.1a) at the bottom boundary, we may write the equation for $\tilde{\mathbf{u}}^-$ as

$$\partial_t \tilde{\mathbf{u}}^- + \tilde{\mathbf{u}}^- \cdot \nabla \tilde{\mathbf{u}}^- + \tilde{\mathbf{u}}^- \cdot \nabla(\gamma^- \mathbf{u}^H) + (\gamma^- \mathbf{u}^H) \cdot \nabla \tilde{\mathbf{u}}^- + u_3^{BL-} \partial_Z \tilde{\mathbf{u}}^- = \partial_{ZZ} \tilde{\mathbf{u}}^-, \tag{2.14a}$$

$$u_3^{BL-} = -\int_{-h/\varepsilon}^{Z} \nabla \cdot \tilde{\mathbf{u}}^-\, dZ' - (Z + h/\varepsilon)\, \nabla \cdot (\gamma^- \mathbf{u}^H), \tag{2.14b}$$

$$\tilde{\mathbf{u}}^-(\mathbf{x}, Z = -h/\varepsilon, t) = -\gamma^- \mathbf{u}^H, \tag{2.14c}$$

$$\tilde{\mathbf{u}}^-(\mathbf{x}, Z = h/\varepsilon, t) = 0, \tag{2.14d}$$

$$\tilde{\mathbf{u}}^-(\mathbf{x}, Z, t = 0) = 0. \tag{2.14e}$$

Note that the initial condition and the boundary datum for $\tilde{\mathbf{u}}^-$ are compatible thanks to the fact that the initial datum for the PE satisfies the compatibility condition (2.4). We define the normal velocity at the bottom boundary layer as

$$\tilde{u}_3^- = \int_Z^{h/\varepsilon} \nabla \cdot \tilde{\mathbf{u}}^-\, dZ'.$$

It is identically zero at the top boundary and decays exponentially fast away from the bottom boundary. Therefore, the vector $(\tilde{\mathbf{u}}^-, \varepsilon \tilde{u}_3^-)$ is the appropriate bottom BL corrector to the solution of the hydrostatic equation $(\mathbf{u}^H, u_3^H)$. In fact $(\tilde{\mathbf{u}}^-, \varepsilon \tilde{u}_3^-)$ has the following properties:

- it is divergence free;
- it corrects the tangential slip generated by $\mathbf{u}^H$;
- it decays away from the bottom boundary layer;
- it generates normal flux at the bottom boundary which, however, is $O(\varepsilon)$.

The normal flux (recall that the solution of the PE has zero normal flux at the bottom boundary) will be corrected by the higher order terms in the asymptotic expansion.

2.5. The Boundary Layer Equation: the Top Boundary

For the boundary layer at the top we write analogous equations

$$\partial_t \mathbf{u}^{BL+} + \mathbf{u}^{BL+} \cdot \nabla \mathbf{u}^{BL+} + u_3^{BL+} \partial_Z \mathbf{u}^{BL+} + \nabla p^H = \partial_{ZZ} \mathbf{u}^{BL+} \tag{2.15a}$$
$$u_3^{BL+} = -\int_{h/\varepsilon}^{Z} \nabla \cdot \mathbf{u}^{BL+}(\mathbf{x}, Z', t)\, dZ' \tag{2.15b}$$
$$\mathbf{u}^{BL+}(\mathbf{x}, Z = h/\varepsilon, t) = 0 \tag{2.15c}$$
$$\mathbf{u}^{BL+}(\mathbf{x}, Z = -h/\varepsilon, t) = \gamma^+ \mathbf{u}^H \tag{2.15d}$$
$$\mathbf{u}^{BL+}(\mathbf{x}, Z, t = 0) = 0. \tag{2.15e}$$

Introducing the new function

$$\tilde{\mathbf{u}}^+ = \mathbf{u}^{BL+} - \gamma^+ \mathbf{u}^H \tag{2.16}$$

and using

$$\partial_t \gamma^+ \mathbf{u}^H + \gamma^+ \mathbf{u}^H \cdot \nabla \gamma^+ \mathbf{u}^H + \nabla p^H = 0,$$

we write the equations for $\tilde{\mathbf{u}}^+$ as

$$\begin{aligned} \partial_t \tilde{\mathbf{u}}^+ + \tilde{\mathbf{u}}^+ \cdot \nabla \tilde{\mathbf{u}}^+ + \tilde{\mathbf{u}}^+ \cdot \nabla(\gamma^+ \mathbf{u}^H) + (\gamma^+ \mathbf{u}^H) \cdot \nabla \tilde{\mathbf{u}}^+ + u_3^{BL+} \partial_Z \tilde{\mathbf{u}}^+ \\ = \partial_{ZZ} \tilde{\mathbf{u}}^+, \end{aligned} \tag{2.17a}$$
$$u_3^{BL+} = -\int_{h/\varepsilon}^{Z} \nabla \cdot \tilde{\mathbf{u}}^+\, dZ' - (Z - h/\varepsilon)\ \nabla \cdot (\gamma^+ \mathbf{u}^H), \tag{2.17b}$$
$$\tilde{\mathbf{u}}^+(\mathbf{x}, Z = h/\varepsilon, t) = -\gamma^+ \mathbf{u}^H, \tag{2.17c}$$
$$\tilde{\mathbf{u}}^+(\mathbf{x}, Z = -h/\varepsilon, t) = 0, \tag{2.17d}$$
$$\tilde{\mathbf{u}}^+(\mathbf{x}, Z, t = 0) = 0. \tag{2.17e}$$

We also define the normal velocity at the bottom boundary layer as

$$\tilde{u}_3^+ = \int_Z^{-h/\varepsilon} \nabla \cdot \tilde{\mathbf{u}}^+\, dZ'$$

and note that the corrector at the top boundary $(\tilde{\mathbf{u}}^+, \varepsilon \tilde{u}_3^+)$ satisfies the properties analogous to the (a)–(d) for the bottom BL corrector.

2.6. The Equation for the Remainder

We define the zeroth order approximation to the PE solution as

$$(\mathbf{u}^{(0)}, u_3^{(0)}) = (\mathbf{u}^H + \tilde{\mathbf{u}}^+ + \tilde{\mathbf{u}}^-, u_3^H + \varepsilon\tilde{u}_3^+ + \varepsilon\tilde{u}_3^-).$$

This approximation satisfies the no-slip condition at the boundary, but it may have a non-zero flux. The remainder $(\mathbf{w}, w_3)$ in the expansion (2.10a)–(2.10b) must therefore cancel this normal flux. After some straightforward calculations we derive the equations for the remainder $(\mathbf{w}, w_3)$, which read

$$\begin{aligned}
&\partial_t \mathbf{w} + \mathbf{w}\cdot\nabla\mathbf{u}^{(0)} + \mathbf{u}^{(0)}\cdot\nabla\mathbf{w} + \varepsilon\mathbf{w}\cdot\nabla\mathbf{w} + w_3\partial_z\mathbf{u}^{(0)} + u_3^{(0)}\partial_z\mathbf{w}\\
&\qquad + \varepsilon w_3\partial_z\mathbf{w} + \nabla p^w = \varepsilon^2\Delta_3\mathbf{w} + \mathbf{F}_0 + \mathbf{F}_{-1} &\text{(2.18a)}\\
&\nabla\cdot\mathbf{w} + \partial_z w_3 = 0 &\text{(2.18b)}\\
&\gamma^+\mathbf{w} = 0, \qquad \gamma^+ w_3 = \nabla\cdot\mathbf{G}^+ &\text{(2.18c)}\\
&\gamma^-\mathbf{w} = 0, \qquad \gamma^- w_3 = \nabla\cdot\mathbf{G}^- &\text{(2.18d)}\\
&\mathbf{w}(\mathbf{x}, Z, t=0) = 0 &\text{(2.18e)}
\end{aligned}$$

where

$$\mathbf{G}^- = -\int_{-h/\varepsilon}^{h/\varepsilon} \tilde{\mathbf{u}}^-\, dZ', \qquad \mathbf{G}^+ = \int_{-h/\varepsilon}^{h/\varepsilon} \tilde{\mathbf{u}}^+\, dZ';$$

the source term has been split, for clarity's sake, in a part which formally is of zeroth order in ε

$$\begin{aligned}
\mathbf{F}_0 = {} & \varepsilon\Delta_3\mathbf{u}^H + \varepsilon(\Delta\tilde{\mathbf{u}}^+ + \Delta\tilde{\mathbf{u}}^-) - \varepsilon^{-1}[\tilde{\mathbf{u}}^+\cdot\nabla\tilde{\mathbf{u}}^- + \tilde{\mathbf{u}}^-\cdot\nabla\tilde{\mathbf{u}}^+]\\
& - \varepsilon^{-1}[(\mathbf{u}^H - \gamma^+\mathbf{u}^H)\cdot\nabla\tilde{\mathbf{u}}^+ + \tilde{\mathbf{u}}^+\cdot\nabla(\mathbf{u}^H - \gamma^+\mathbf{u}^H)\\
& + (\mathbf{u}^H - \gamma^-\mathbf{u}^H)\cdot\nabla\tilde{\mathbf{u}}^- + \tilde{\mathbf{u}}^-\cdot\nabla(\mathbf{u}^H - \gamma^-\mathbf{u}^H)]\\
& - \varepsilon^{-1}[(u_3^H + (z+h)\gamma^-\nabla\cdot\mathbf{u}^H)\partial_z\tilde{\mathbf{u}}^- + (u_3^H + (z-h)\gamma^+\nabla\cdot\mathbf{u}^H)\partial_z\tilde{\mathbf{u}}^+]\\
& - \tilde{u}_3^+\partial_z\mathbf{u}^H - \tilde{u}_3^-\partial_z\mathbf{u}^H - \tilde{u}_3^+\partial_z\tilde{\mathbf{u}}^- - \tilde{u}_3^-\partial_z\tilde{\mathbf{u}}^+
\end{aligned}$$

(actually the first term is $O(\varepsilon)$, while the second, third, and the last two terms are exponentially small) and a part which is $O(\varepsilon^{-1})$

$$\mathbf{F}_{-1} = \nabla\cdot\mathbf{G}^+\,\partial_z\tilde{\mathbf{u}}^+ + \nabla\cdot\mathbf{G}^-\,\partial_z\tilde{\mathbf{u}}^-.$$

3. Results: the Construction of the Hydrostatic Solution

In this section, we provide a well-posedness result for the hydrostatic equations (2.7a)–(2.7e) in the analytic setting. In the literature, several results of this type have appeared, treating analytic data with uniform analyticity radius in the whole domain. Here we shall state a result in the analytic function spaces more appropriate for the subsequent analysis and addressing data whose analyticity radius decays linearly when approaching the boundary.

3.1. Function Spaces for the Hydrostatic Equations

For functions defined on the boundary, we introduce

$$D(\rho) = \{x \in \mathbb{C}^2 : \Re \mathbf{x} \in \mathbb{T}^2, \ \Im \mathbf{x} \in (-\rho, \rho)^2\}, \tag{3.1}$$

which expresses the domain of analyticity with respect to the tangential variable $\mathbf{x}$. Then we define the space of the functions analytic with respect to the tangential variable. Analyticity is expressed in terms of decay properties of the Fourier spectrum.

Definition 3.1. The space $H^{l,\rho}$ is the space of functions $f(x)$ such that

- $\partial_x^j f$, with $0 \leqq j \leqq l$, is analytic in $D(\rho)$;
- $|f|_{l,\rho} = \sum_{0 \leqq j \leqq l} \left(\sum_{k \in \mathbb{Z}^2} e^{2\rho|k|}(1+k^2)^j |f_k|^2\right)^{1/2} < \infty.$

The following space is the space of functions depending on (x, t). The analyticity radius decreases linearly in time at the rate β. The trace of the solution of the hydrostatic equations belongs to the space below and therefore also the boundary data for the equations of the BL corrector.

Definition 3.2. The space $H^{l,\rho}_{\beta,T}$ is the set of functions $f(x, t)$ for which

- $\partial_t^i \partial_x^j f(x, t) \in H^{0,\rho-\beta t}$ for all $0 \leqq t \leqq T$ with $0 \leqq i + j \leqq l$ and $0 \leqq i \leqq 1$.

We now define the domain of analyticity with respect to the normal variable z

$$\begin{aligned}\Sigma(\theta) = \{&z \in \mathbb{C} : \Re z \in [-h, 0], |\Im z| < (\Re z + h)\tan\theta\}\\ &\cup \{z \in \mathbb{C} : \Re z \in [0, h], |\Im z| < (h - \Re z)\tan\theta\}, \qquad \theta < \pi/4\end{aligned}$$

and the path of integration in the complex plane

$$\begin{aligned}\Gamma(\theta') = \{&z \in \mathbb{C} : \Re z \in [-h, 0], |\Im z| = (\Re z + h)\tan\theta'\} \cup\\ &\{z \in \mathbb{C} : \Re z \in [0, h], \Im z = (h - \Re z)\tan\theta'\}, \qquad \theta' < \theta.\end{aligned}$$

Definition 3.3. The space $H^{l,\rho,\theta}$, with $\theta < \pi/4$, is the set of the functions $f(x, z)$ such that

- f is analytic in $D(\rho) \times \Sigma(\theta)$;
- $\partial_x^{\alpha_1}\partial_z^{\alpha_2} f \in L^2(\Gamma(\theta'); H^{0,\rho})$ with $|\theta'| \leqq \theta$, $\alpha_1 + \alpha_2 \leqq l$;
- $|f|_{l,\rho,\theta} = \sum_{\alpha_1+\alpha_2 \leqq l} \sup_{|\theta'| \leqq \theta} \| |\partial_x^{\alpha_1}\partial_z^{\alpha_2} f(\cdot, z)|_{0,\rho} \|_{L^2(\Gamma(\theta'))} < \infty.$

Definition 3.4. The space $H^{l,\rho,\theta}_{\beta,T}$ is defined as the set of all functions $f(x, z, t)$ such that

- $f(x, z, t) \in H^{l,\rho-\beta t,\theta-\beta t}$ and $\partial_t f(x, z, t) \in H^{l-1,\rho-\beta t,\theta-\beta t}$ for all $t \in [0, T]$,

Moreover,

$$\begin{aligned}|f|_{l,\rho,\theta,\beta,T} \equiv &\sum_{\alpha_1+\alpha_2 \leqq l} \sup_{0 \leqq t \leqq T} |\partial_x^{\alpha_1}\partial_z^{\alpha_2} f(\cdot, \cdot, t)|_{0,\rho-\beta t,\theta-\beta t}\\ &+ \sum_{\alpha_1+\alpha_2 \leqq l-1} \sup_{0 \leqq t \leqq T} |\partial_t \partial_x^{\alpha_1}\partial_z^{\alpha_2} f(\cdot, \cdot, t)|_{0,\rho-\beta t,\theta-\beta t} < \infty.\end{aligned}$$

Remark 1. Here and in the rest of the paper we suppose that the index l is large enough to give to all the function spaces the structure of an algebra. For example we have that if $f, g \in H^{l,\rho,\theta}$, then $fg \in H^{l,\rho,\theta}$ with the estimate

$$|fg|_{l,\rho,\theta} < c|f|_{l,\rho,\theta}|g|_{l,\rho,\theta}.$$

We shall make the same assumption for the function spaces K (that we shall introduce for the construction of the boundary layer solutions), M (introduced in Section 6.1 in the analysis of the first order hydrostatic Euler equations), and L (introduced in Section 7 and used in the construction of the correction $\mathbf{e}$).

3.2. Well-Posedness Result for the Hydrostatic Equations

The well-posedness of the hydrostatic equations is expressed by the following theorem.

Theorem 3.1. *Suppose the initial datum of the hydrostatic equations* $\mathbf{u}_0$ *belongs to* H^{l,ρ_0,θ_0} *and satisfies the compatibility condition* (2.5). *Then there exist* $\beta_0 > 0$ *and* $T_0 > 0$ *such that Equations* (2.7a)–(2.7e) *admit a unique solution* $\mathbf{u}^H \in H^{l,\rho_0,\theta_0}_{\beta_0,T_0}$.

Note that, for the above theorem to hold, one only needs to require the compatibility condition between the initial datum and the incompressibility condition (2.5). It is not necessary to impose condition (2.4).

The proof is based on the ACK theorem given in Appendix E. The crucial step in recasting the hydrostatic equation in the operator form suitable for application of the ACK theorem is to eliminate the pressure from the equation using (2.9) which gives the expression

$$\nabla p^H = -\nabla\Delta^{-1}\frac{1}{2h}\int_{-h}^{h}\nabla\cdot(\nabla\cdot(\mathbf{u}^H\mathbf{u}^H))dz'. \tag{3.2}$$

Moreover, one has the following lemma which gives an estimate of the gradient of the pressure in terms of the velocity.

Lemma 3.1. *Let* $\mathbf{u} \in H^{l,\rho,\theta}$, *and let* p *be expressed by*

$$p = -\Delta^{-1}\left(\frac{1}{2h}\int_{-h}^{h}\nabla\cdot(\nabla\cdot(\mathbf{uu}))\,dz'\right).$$

Then, if $0 < \rho' < \rho$, *we have* $\nabla p \in H^{l,\rho'}$ *and the estimate*

$$|\nabla p|_{l,\rho'} \leqq c\frac{|\mathbf{u}|_{l,\rho,\theta}}{\rho-\rho'}$$

holds.

The proof is achieved by expressing the operators ∇ and Δ^{-1} in terms of their Fourier symbol, using Remark 1, and the Cauchy estimate for the derivative of an analytic function.

We then rewrite the hydrostatic equations in the integrated form

$$\mathbf{u}^H + F(\mathbf{u}^H, t) = 0$$

where

$$F(\mathbf{u}^H, t) = \mathbf{u}_0^H + \int_0^t (\mathbf{u}^H \cdot \nabla \mathbf{u}^H + u_3^H \partial_z \mathbf{u}^H + \nabla p^H)\, ds,$$

where the pressure gradient is given by (3.2), while u_3^H is given by (2.8).

With the help of the lemma above, and of the Cauchy estimate to bound the operators ∇ and ∂_z, we obtain that F is bounded and quasi-contractive and thus satisfies the hypotheses of the ACK theorem. This concludes the proof Theorem 3.1.

4. Results: the Construction of the Layer Correctors

4.1. Function Spaces for the Boundary Layer Equations

We first introduce the domain of analyticity

$$\Xi(\theta) = \{Z \in \mathbb{C} : \Re Z \in [-h/\varepsilon, 0], |\Im z| < (\Re Z + h/\varepsilon)\tan\theta\}$$
$$\cup \{Z \in \mathbb{C} : \Re Z \in [0, h/\varepsilon], |\Im Z| < (h/\varepsilon - \Re Z)\tan\theta\}, \qquad \theta < \pi/4$$

of the normal variable for the BL correctors.

Definition 4.1. The space $K^{-,l,\rho,\theta,\mu}$ is the space of functions $f(x, Z)$ such that, for $i + j \leqq l$ and $j \leqq 2$, $e^{\mu(\Re Z + h/\varepsilon)} \partial_x^i \partial_Z^j f \in L^\infty(\Xi(\theta), K^{0,\rho})$. The norm in $K^{-,l,\rho,\theta,\mu}$ is defined as

$$|f|_{l,\rho,\theta,\mu} \equiv \sum_{j \leqq 2} \sum_{i \leqq l-j} \sup_{Z \in \Sigma^-(\theta)} e^{\mu(\Re Z + h/\varepsilon)} \, |\partial_Z^j \partial_x^i f(\cdot, Z)|_{0,\rho}.$$

Definition 4.2. The space $K_{\beta,T}^{-,l,\rho,\theta,\mu}$, with $\theta < \pi/4$, is the space of the functions $f(x, Z, t)$ such that

- $f \in K^{-,l,\rho-\beta t,\theta-\beta t,\mu-\beta t}$ and $\partial_t \partial_x^i f \in K^{-,0,\rho-\beta t,\theta-\beta t,\mu-\beta t}$ for all $0 \leqq t \leqq T$, where $0 \leqq i \leqq l-2$.

Moreover,

$$|f|_{l,\rho,\theta,\mu,\beta,T} \equiv \sum_{0 \leqq j \leqq 2} \sum_{i \leqq l-j} \sup_{0 \leqq t \leqq T} |\partial_Z^j \partial_x^i f(\cdot, \cdot, t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t}$$
$$+ \sum_{i \leqq l-2} \sup_{0 \leqq t \leqq T} |\partial_t \partial_x^i f(\cdot, \cdot, t)|_{0,\rho-\beta t,\theta-\beta t,\mu-\beta t} < \infty.$$

The spaces $K^{+,l,\rho,\theta,\mu}$ and $K_{\beta,T}^{+,l,\rho,\theta,\mu}$ are defined analogously, imposing the exponential decay away from the top boundary.

4.2. Well-Posedness Result for the Boundary Layer Equations

We now state the existence and uniqueness result concerning system (2.14a)–(2.14e). Given that we have proven that $\mathbf{u}^H \in H^{l,\rho_0,\theta_0}_{\beta_0,T_0}$, we may assume that the gradients of $\mathbf{u}^H$, which are present in (2.14a), have the same regularity with a smaller analyticity radius. This means that we can assume that $\nabla \mathbf{u}^H$, $\partial_Z \mathbf{u}^H \in H^{l,\rho_1,\theta_1}_{\beta_0,T_0'}$ with $\rho_1 < \rho_0$, $\theta_1 < \theta_0$ and $T_0' < T_0$. The same reasoning is applied to the subsequent terms in the asymptotic expansion and is thus omitted.

With the same techniques used in [8,28,35,48] one can prove the following result.

Theorem 4.1. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *and satisfies the compatibility conditions* (2.4)-(2.5). *Then there exist* $\beta_1 < \beta_0$ *and* $T_1 < T_0$ *and* $\mu_1 > 0$ *such that the system* (2.14a)–(2.14e) *admits a unique solution* $\tilde{\mathbf{u}}^- \in K^{-,l,\rho_1,\theta_1,\mu_1}_{\beta_1,T_1}$ *and* $\tilde{u}_3^- \in K^{-,l-1,\rho_1,\theta_1,\mu_1}_{\beta_1,T_1}$.

An analogous statement holds for the system (2.17a)–(2.17e).

Theorem 4.2. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *satisfies the compatibility conditions* (2.4) *and* (2.5). *Then there exist* $\beta_1 > \beta_0$ *and* $T_1 < T_0$ *such that the system* (2.17a)–(2.17e) *admits a unique solution* $\tilde{\mathbf{u}}^+ \in K^{+,l,\rho_1,\theta_1,\mu_1}_{\beta_1,T_1}$ *and* $\tilde{u}_3^+ \in K^{+,l-1,\rho_1,\theta_1,\mu_1}_{\beta_1,T_1}$.

5. Asymptotic Expansion of the Remainder

In this section, we decompose the remainder as

$$\mathbf{w} = \mathbf{u}^{H(1)} + \tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)} + \mathbf{e} \tag{5.1a}$$

$$w_3 = u_3^{H(1)} + \varepsilon \tilde{u}_3^{+(1)} + \varepsilon \tilde{u}_3^{-(1)} + e_3 \tag{5.1b}$$

$$p^w = p^{H(1)} + p^e. \tag{5.1c}$$

In the rest of the paper we shall use the following notation:

$$\mathbf{u}^{(1)} = \mathbf{u}^{H(1)} + \tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)} \tag{5.2a}$$

$$u_3^{(1)} = u_3^{H(1)} + \varepsilon \tilde{u}_3^{+(1)} + \varepsilon \tilde{u}_3^{-(1)}. \tag{5.2b}$$

The term $(\mathbf{u}^{H(1)}, u_3^{H(1)})$ is a first order correction to the hydrostatic solution $(\mathbf{u}^H, u_3^H)$. It is introduced to correct the vertical inflow introduced at the bottom boundary by the zeroth order BL correction $(\tilde{\mathbf{u}}^-, \varepsilon\tilde{u}_3^-)$, as well the vertical inflow introduced at the top boundary by the zeroth order BL correction $(\tilde{\mathbf{u}}^+, \varepsilon\tilde{u}_3^+)$. However, $\mathbf{u}^{H(1)}$ introduces tangential slip at both boundaries.

The term $(\tilde{\mathbf{u}}^{+(1)}, \varepsilon\tilde{u}_3^{+(1)})$ is a first order BL corrector at the top boundary, with a similar meaning for $(\tilde{\mathbf{u}}^{-(1)}, \varepsilon\tilde{u}_3^{-(1)})$. They are introduced to correct the tangential slip introduced by $\mathbf{u}^{H(1)}$. However, these first order BL correctors introduce normal inflow at the boundaries (in the same fashion as the zeroth order BL correctors $(\tilde{\mathbf{u}}^+, \varepsilon\tilde{u}_3^+)$ and $(\tilde{\mathbf{u}}^-, \varepsilon\tilde{u}_3^-)$ did in the previous step of the asymptotic expansion).

The term $\mathbf{e}$ is an error term and is introduced to correct these normal inflows, see the normal boundary conditions (5.7c) below.

In the rest of this section, we write the equations satisfied by the above terms.

5.1. The First Order Hydrostatic Equations

The term $(\mathbf{u}^{H(1)}, u_3^{H(1)})$ satisfies the system

$$\begin{aligned}&\partial_t \mathbf{u}^{H(1)} + \mathbf{u}^H \cdot \nabla \mathbf{u}^{H(1)} + \mathbf{u}^{H(1)} \cdot \nabla \mathbf{u}^H + \varepsilon \mathbf{u}^{H(1)} \cdot \nabla \mathbf{u}^{H(1)} \\ &\quad + u_3^H \partial_z \mathbf{u}^{H(1)} + u_3^{H(1)} \partial_z \mathbf{u}^H + \varepsilon u_3^{H(1)} \partial_z \mathbf{u}^{H(1)} + \nabla p^{H(1)} = 0 \end{aligned} \tag{5.3a}$$

$$\nabla \cdot \mathbf{u}^{H(1)} + \partial_z u_3^{H(1)} = 0 \tag{5.3b}$$

$$\gamma^+ u_3^{H(1)} = \nabla \cdot \mathbf{G}^+ \tag{5.3c}$$

$$\gamma^- u_3^{H(1)} = \nabla \cdot \mathbf{G}^- \tag{5.3d}$$

$$\mathbf{u}^{H(1)}(\mathbf{x}, z, t = 0) = 0. \tag{5.3e}$$

Differently from the zeroth order hydrostatic equations the above equations have a non-zero flux at the boundary. Therefore, we find it more convenient to solve the above equation introducing the new unknown $(\mathbf{v}, v_3)$ defined as

$$\mathbf{v} = \mathbf{u}^{H(1)} - \left(R \frac{\cosh[|k|(z+h)]}{\sinh[2|k|]} \nabla \cdot \mathbf{G}^+ + R \frac{\cosh[|k|(h-z)]}{\sinh[2|k|]} \nabla \cdot \mathbf{G}^- \right) \tag{5.4a}$$

$$v_3 = u_3^{H(1)} - \left(\frac{\sinh[|k|(z+h)]}{\sinh[2|k|]} \nabla \cdot \mathbf{G}^+ - \frac{\sinh[|k|(h-z)]}{\sinh[2|k|]} \nabla \cdot \mathbf{G}^- \right) \tag{5.4b}$$

where R is the operator

$$R = \frac{ik}{|k|}.$$

It is easy to verify that $\gamma^{\pm} v_3 = 0$ and that $\nabla \cdot \mathbf{v} + \partial_z v_3 = 0$. Thus $(\mathbf{v}, v_3)$ satisfies a system analogous to (5.3a)–(5.3e). The boundary conditions are homogeneous and, given that $\nabla \cdot \mathbf{G}^{\pm}(t = 0) = 0$, we can also say that the initial data are homogeneous: the well-posedness can therefore be proved analogously to what has been done for $(\mathbf{u}^H, u_3^H)$. The formal statement is given in Theorem 6.1.

5.2. The First Order Boundary Layer Corrector

We shall impose that the equation satisfied by $\tilde{\mathbf{u}}^{-(1)}$ is a linear heat equation with the appropriate boundary conditions so that the tangential slip generated by $\mathbf{u}^{H(1)}$ at the bottom boundary is cancelled:

$$\partial_t \tilde{\mathbf{u}}^{-(1)} - \partial_{ZZ}\tilde{\mathbf{u}}^{-(1)} = 0 \tag{5.5a}$$

$$\tilde{u}_3^{-(1)} = \int_Z^{h/\varepsilon} \nabla \cdot \tilde{\mathbf{u}}^{-(1)}\, dZ' \tag{5.5b}$$

$$\gamma^- \tilde{\mathbf{u}}^{-(1)} = -\gamma^- \mathbf{u}^{H(1)} \tag{5.5c}$$

$$\gamma^+ \tilde{\mathbf{u}}^{-(1)} = 0 \tag{5.5d}$$

$$\tilde{\mathbf{u}}^{-(1)}(\mathbf{x}, Z, t = 0) = 0. \tag{5.5e}$$

An analogous system is satisfied by $(\tilde{\mathbf{u}}^{+(1)}, \varepsilon\tilde{u}_3^{+(1)})$,

$$\partial_t \tilde{\mathbf{u}}^{+(1)} - \partial_{ZZ}\tilde{\mathbf{u}}^{+(1)} = 0 \tag{5.6a}$$

$$\tilde{u}_3^{+(1)} = -\int_{-h/\varepsilon}^{Z} \nabla \cdot \tilde{\mathbf{u}}^{+(1)}\, dZ' \tag{5.6b}$$

$$\gamma^+ \tilde{\mathbf{u}}^{+(1)} = -\gamma^+ \mathbf{u}^{H(1)} \tag{5.6c}$$

$$\gamma^- \tilde{\mathbf{u}}^{+(1)} = 0 \tag{5.6d}$$

$$\tilde{\mathbf{u}}^{+(1)}(\mathbf{x}, Z, t = 0) = 0. \tag{5.6e}$$

For the above systems one can recognize that the initial and the boundary conditions are compatible. Note also that these first order BL correctors introduce normal influx at the boundaries. In particular, the normal influx at the bottom boundary is

$$\gamma^- \varepsilon \tilde{u}_3^{-(1)} = -\varepsilon \nabla \cdot \mathbf{G}^{-(1)} \qquad \text{where} \quad \mathbf{G}^{-(1)} \equiv -\int_{-h/\varepsilon}^{h/\varepsilon} \tilde{\mathbf{u}}^{-(1)}\, dZ',$$

while the influx at the top boundary is

$$\gamma^+ \varepsilon \tilde{u}_3^{+(1)} = -\varepsilon \nabla \cdot \mathbf{G}^{+(1)} \qquad \text{where} \quad \mathbf{G}^{+(1)} \equiv \int_{-h/\varepsilon}^{h/\varepsilon} \tilde{\mathbf{u}}^{+(1)}\, dZ'.$$

These normal fluxes will be cancelled by the normal fluxes of the error term $(\mathbf{e}, e_3)$ with appropriate boundary conditions, cf. (5.7d) below.

5.3. The Error Equation

Finally one can write the equation satisfied by the error as

$$\begin{aligned}(\partial_t - \varepsilon^2 \Delta)\mathbf{e} + \mathbf{e} \cdot \nabla[\mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)}] + [\mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)}] \cdot \nabla \mathbf{e} + \varepsilon \mathbf{e} \cdot \nabla \mathbf{e} \\ + [u_3^{(0)} + \varepsilon u_3^{(1)}]\partial_z \mathbf{e} + e_3 \partial_z [\mathbf{u}^{(0)} + \varepsilon \mathbf{u}^{(1)}] + \varepsilon e_3 \partial_z \mathbf{e} + \nabla p^e = \mathbf{\Psi}\end{aligned} \tag{5.7a}$$

$$\nabla \cdot \mathbf{e} + \varepsilon^{-1} \partial_Z e_3 = 0 \tag{5.7b}$$

$$\gamma^- \mathbf{e} = 0, \qquad \gamma^+ \mathbf{e} = 0 \tag{5.7c}$$

$$\gamma^- e_3 = \varepsilon \nabla \cdot \mathbf{G}^{-(1)}, \qquad \gamma^+ e_3 = \varepsilon \nabla \cdot \mathbf{G}^{+(1)} \tag{5.7d}$$

$$\mathbf{e}|_{t=0} = 0 \tag{5.7e}$$

being the expression for the source term specified in Appendix D, where it is shown to be $O(1)$.

Note the crucial fact that the influxes at the boundaries are $O(\varepsilon)$ quantities, and are in the form of a derivative of a function.

6. Results: Construction of the First Order Hydrostatic Solution and of the First Order BL Corrector

6.1. Function Spaces

First we define the domain of analyticity with respect to the normal variable z

$$\Sigma(\theta) = \{z \in \mathbb{C} : \Re z \in [-h, 0], \Im z < (\Re z + h)\tan\theta\} \\ \cup \{z \in \mathbb{C} : \Re z \in [0, h], \Im z < (h - \Re z)\tan\theta\}, \qquad \theta < \pi/4$$

and the path of integration in the complex plane

$$\Gamma(\theta') = \{z \in \mathbb{C} : \Re z \in [-h, 0], \Im z = (\Re z + h)\tan\theta'\} \\ \cup\{z \in \mathbb{C} : \Re z \in [0, h], \Im z = (h - \Re z)\tan\theta'\}, \qquad \theta' < \theta.$$

Definition 6.1. The space $M^{l,\rho,\theta}$ is the set of all functions $f(x, z)$ such that

- f is analytic inside $D(\rho) \times \Sigma(\theta)$;
- $\partial_z^{\alpha_1}\partial_x^{\alpha_2} f(x, y) \in L^2(\Gamma(\theta'); H^{0,\rho})$ with $|\theta'| \leqq \theta$, $\alpha_1 + \alpha_2 \leqq l$ and $\alpha_2 \leqq l - 2$ when $\alpha_1 > 0$;
- $|f|_{l,\rho,\theta} = \sum_{\alpha\leqq l} \sup_{|\theta'|\leqq\theta} \||\partial_x^{\alpha} f(\cdot, z)|_{0,\rho}\|_{L^2(\Gamma(\theta'))}$
$+ \sum_{0<\alpha_1\leqq 2}\sum_{\alpha_2\leqq l-2} \sup_{|\theta'|\leqq\theta} \||\partial_z^{\alpha_1}\partial_x^{\alpha_2} f(\cdot, z)|_{0,\rho}\|_{L^2(\Gamma(\theta'))} < \infty.$

Definition 6.2. The space $M^{l,\rho,\theta}_{\beta,T}$ is defined as the set of all functions $f(x, z, t)$ such that

- For each $t \in [0, T]$, we have $f(x, z, t) \in M^{l,\rho-\beta t,\theta-\beta t}$ and $\partial_t\partial_x^j f(x, z, t) \in M^{0,\rho-\beta t,\theta-\beta t}$ with $j \leqq l - 2$;
- $|f|_{l,\rho,\theta,\beta,T} = \sum_{0\leqq j\leqq 1}\sum_{\alpha\leqq l-2j} \sup_{0\leqq t\leqq T} |\partial_t^j\partial_x^{\alpha} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}$
$+ \sum_{0<\alpha_1\leqq 2}\sum_{\alpha_2\leqq l-2} \sup_{0\leqq t\leqq T} |\partial_z^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} < \infty$

in which the norms on the right are those defined in $M^{l,\rho,\theta}$.

6.2. Well-Posedness Result for the First Order Hydrostatic Equations

In Section 5.1 we found that, through the change of variable defined by (5.4a)–(5.4b), one can write a system with homogeneous normal boundary conditions. Moreover note that, given that the boundary influx, as expressed by (5.3c)–(5.3d), is a derivative, the change of variable (5.4a)–(5.4b) is regular (no issue arises at $k = 0$). One can therefore state the following well-posedness result.

Theorem 6.1. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *and satisfies the compatibility conditions* (2.4) *and* (2.5)*. Then there exist* $\beta_2 > \beta_1$*,* $T_2 < T_1$ $\rho_2 < \rho_1$*,* $\theta_2 < \theta_1$*, such that the system* (5.3a)–(5.3e) *admits a unique solution* $\mathbf{u}^{H(1)} \in M^{l,\rho_2,\theta_2}_{\beta_2,T_2}$.

6.3. Well-Posedness Result for the First Order Boundary Layer Corrector

The equations satisfied by the first order BL corrector (the Equations (5.5a)–(5.5e) for the bottom boundary) are linear heat equations. The following theorem is therefore easily proved.

Theorem 6.2. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *and satisfies the compatibility conditions* (2.4) *and* (2.5)*. There exist* $\beta_3 > \beta_2$*,* $T_3 < T_2$ $\rho_3 < \rho_2$*,* $\theta_3 < \theta_2$*,* $\mu_3 < \mu_1$*, such that Equations* (5.5a)–(5.5e) *admit a unique solution* $\tilde{\mathbf{u}}^{-(1)} \in K^{-,l,\rho_3,\theta_3,\mu_3}_{\beta_3,T_3}$ *and* $\tilde{u}_3^{-(1)} \in K^{-,l-1,\rho_3,\theta_3,\mu_3}_{\beta_3,T_3}$.

Obviously, an analogous result holds for the top BL corrector.

Theorem 6.3. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *and satisfies the compatibility conditions* (2.4) *and* (2.5)*. There exist* $\beta_3 > \beta_2$*,* $T_3 < T_2$ $\rho_3 < \rho_2$*,* $\theta_3 < \theta_2$*,* $\mu_3 < \mu_1$*, such that Equations* (5.6a)–(5.6e) *admit a unique solution* $\tilde{\mathbf{u}}^{+(1)} \in K^{+,l,\rho_3,\theta_3,\mu_3}_{\beta_3,T_3}$ *and* $\tilde{u}_3^{+(1)} \in K^{+,l-1,\rho_3,\theta_3,\mu_3}_{\beta_3,T_3}$.

The following lemma, where we express the regularity of the boundary data for the error Equations (5.7a)–(5.7d), is an immediate consequence of the above theorems:

Lemma 6.2. *The traces of the boundary layer correctors are such that* $\mathbf{G}^{+(1)}$*,* $\mathbf{G}^{-(1)} \in H^{l+1,\rho_3}_{\beta_3,T_3}$.

Note that we have raised the regularity of the boundary data $\mathbf{G}^{+(1)}$ and $\mathbf{G}^{-(1)}$ stating that they belong to an $l+1$ space rather than l. This can be assumed by shrinking the size of the domain of analyticity from ρ_3 to ρ_3' and, for notational convenience, renaming ρ_3'.

7. Results: Analysis of the Error Equation

7.1. Function Spaces

First introduce, in the complex plane, the trapezoid $\Sigma(\theta, a)$ and a family of paths $\Gamma(\theta', a)$, where $\theta' < \theta < \pi/4$ and $0 < a < h$. These concepts will be useful to define the domain of analyticity, with respect to the normal variable, of the solution of the error equations and the path of integration in the complex plane. Namely, let

$$\begin{aligned}\Sigma(\theta, a) &= \{z \in \mathbb{C} : h - a < |\Re z| < h, |\Im z| < (h - |\Re z|)\tan\theta\} \\ &\quad \cup \{z \in \mathbb{C} : |\Re z| < a, |\Im z| < a\tan\theta\} \\ \Gamma(\theta', a) &= \{z \in \mathbb{C} : h - a < |\Re z| < h\ , |\Im z| = (h - |\Re z|)\tan\theta'\} \\ &\quad \cup \{z \in \mathbb{C} : |\Re z| < a, |\Im z| = a\tan\theta'\}\end{aligned}$$

for $0 < a < h$, $0 < \theta < \pi/4$, and $0 < \theta' < \theta$.

Definition 7.1. The space $L^{l,\rho,\theta}$ is the set of all functions $f(x, Z)$ such that

- f is analytic inside $D(\rho) \times \Sigma(\theta, a/\varepsilon)$
- $\partial_Z^{\alpha_1}\partial_x^{\alpha_2} f(x, Z) \in L^2(\Gamma(\theta', a/\varepsilon); H^{0,\rho})$ with $|\theta'| \leqq \theta$, $\alpha_1 \leqq 2$, $\alpha_1 + \alpha_2 \leqq l$ and $\alpha_2 \leqq l-2$ when $\alpha_1 > 0$
- $|f|_{l,\rho,\theta} = \sum_{\alpha \leqq l} \sup_{|\theta'|\leqq\theta} \| \, |\partial_x^\alpha f(\cdot, Z)|_{0,\rho} \|_{L^2(\Gamma(\theta',a/\varepsilon))}$ $+ \sum_{0<\alpha_1\leqq 2} \sum_{0\leqq\alpha_2\leqq l-2} \sup_{|\theta'|\leqq\theta} \| \, |\partial_Z^{\alpha_1}\partial_x^{\alpha_2} f(\cdot, Z)|_{0,\rho} \|_{L^2(\Gamma(\theta',a/\varepsilon))}$ $< \infty$.

Definition 7.2. The space $L^{l,\rho,\theta}_{\beta,T}$ is the set of functions $f(x, Z, t)$ such that

- $f(x, Z, t) \in L^{l,\rho-\beta t,\theta-\beta t}$ and $\partial_t\partial_x^\alpha f(x, Z, t) \in L^{0,\rho-\beta t,\theta-\beta t}$ for all $\alpha \leqq l-2$; and $t \in [0, T]$;
- $|f|_{l,\rho,\theta,\beta,T} = \sum_{0\leqq j\leqq 1} \sum_{\alpha\leqq l-2j} \sup_{0\leqq t\leqq T} |\partial_t^j \partial_x^\alpha f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t}$ $+ \sum_{0<\alpha_1\leqq 2} \sum_{\alpha_2\leqq l-2} \sup_{0\leqq t\leqq T} |\partial_Z^{\alpha_1}\partial_x^{\alpha_2} f(\cdot,\cdot,t)|_{0,\rho-\beta t,\theta-\beta t} < \infty$ where the norms on the right are those defined in $L^{l,\rho,\theta}$.

The following theorem is the main result of this section.

Theorem 7.1. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *and satisfies the compatibility conditions* (2.4) *and* (2.5). *Then there exist* $\beta_4 > \beta_3$, $T_4 < T_3$, $\rho_4 < \rho_3$, $\theta_4 < \theta_3$, *such that Equations* (5.7a)–(5.7e) *admit, for* ε *sufficiently small, a unique solution* $\mathbf{e} \in L^{l,\rho_4,\theta_4}_{\beta_4,T_4}$.

7.2. The Equation of the Error in the Operator Form

We now express the error $(\mathbf{e}, e_3)$ as the sum of two terms:

$$\mathbf{e} = \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}) + \boldsymbol{\sigma} \qquad \text{and} \qquad e_3 = \mathcal{S}_3^s(\boldsymbol{\eta}) + \sigma_3. \tag{7.1}$$

The term $(\boldsymbol{\sigma}, \sigma_3)$ takes care of the boundary data. It solves the following Stokes problem with boundary data, whose solution is constructed in Appendix B:

$$(\partial_t - \varepsilon^2\Delta_2 - \partial_{ZZ})\boldsymbol{\sigma} + \nabla p^\sigma = 0 \tag{7.2a}$$
$$\nabla \cdot \boldsymbol{\sigma} + \partial_Z\sigma_3 = 0 \tag{7.2b}$$
$$\gamma^\pm\boldsymbol{\sigma} = 0 \tag{7.2c}$$
$$\gamma^\pm\sigma_3 = \varepsilon\nabla\cdot\mathbf{G}^{\pm(1)} \tag{7.2d}$$
$$\boldsymbol{\sigma}|_{t=0} = 0. \tag{7.2e}$$

Note that the boundary data for the above problem, being the normal flux $O(\varepsilon)$ and of divergence type, have the same form of the boundary data as the system (B.4a)–(B.4e); the above problem can therefore be solved using the operator $\left(\boldsymbol{\mathcal{S}}^b, \mathcal{S}_3^b\right)$ introduced in Appendix B.1. Using Theorem B.1, and also due to Lemma 6.2, one can therefore see that $\boldsymbol{\sigma} \in L^{l,\rho_3,\theta_3}_{\beta_3,T_3}$ while $\sigma_3 \in L^{l-1,\rho_3,\theta_3}_{\beta_3,T_3}$. Shrinking the domain of the analyticity of σ_3 from ρ_3 to ρ_3' and renaming ρ_3', one can state the following proposition:

Proposition 7.1. *The system Equations* (7.2a)–(7.2e) *admits, for ε sufficiently small, a unique solution $\boldsymbol{\sigma}$ which belongs to $L^{l,\rho_3,\theta_3}_{\beta_3,T_3}$.*

We may therefore write the equation for $\boldsymbol{\eta}$ which is

$$\boldsymbol{\eta} + \boldsymbol{\mathcal{G}}(\boldsymbol{\eta}) = \mathbf{f}; \tag{7.3}$$

where the operator $\boldsymbol{\mathcal{G}}$ is defined as

$$\begin{aligned}\boldsymbol{\mathcal{G}}(\boldsymbol{\eta}) = {} & \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}) \cdot \nabla[\mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)} + \varepsilon\boldsymbol{\sigma}] \\ & + [\mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)} + \varepsilon\boldsymbol{\sigma}] \cdot \nabla\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}) + \varepsilon\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}) \cdot \nabla\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}) \\ & + \mathcal{S}^s_3(\boldsymbol{\eta})\partial_z[\mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)} + \varepsilon\boldsymbol{\sigma}] + [u_3^{(0)} + \varepsilon u_3^{(1)} + \varepsilon\sigma_3]\partial_z\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}) \\ & + \varepsilon\mathcal{S}^s_3(\boldsymbol{\eta})\partial_z\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta})\end{aligned} \tag{7.4}$$

and where the operator $\boldsymbol{\mathcal{S}}^s$ solves the Stokes equation with source term and is introduced in Appendix A, while $\mathbf{f}$ is a source term whose expression is

$$\begin{aligned}\mathbf{f} = {} & \boldsymbol{\Psi} - [\mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)}] \cdot \nabla\boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot \nabla[\mathbf{u}^{(0)} - \varepsilon\mathbf{u}^{(1)}] + \varepsilon\boldsymbol{\sigma} \cdot \nabla\boldsymbol{\sigma} \\ & - [u_3^{(0)} + \varepsilon u_3^{(1)}]\partial_z\boldsymbol{\sigma} - \sigma_3\partial_z[\mathbf{u}^{(0)} + \varepsilon\mathbf{u}^{(1)}] - \varepsilon\sigma_3\partial_z\boldsymbol{\sigma}.\end{aligned} \tag{7.5}$$

In order to prove that the error $\mathbf{e}$ exists and is bounded, and given the boundedness of $\boldsymbol{\sigma}$, it is sufficient to estimate $\boldsymbol{\eta}$ as solution Equation (7.3). We shall prove the existence and uniqueness of Equation (7.3) using the ACK theorem stated in Appendix E, verifying that the operator $\boldsymbol{\mathcal{G}}(\boldsymbol{\eta}) - \mathbf{f}$ satisfies the hypotheses (i)–(iii).

7.3. Estimate on the Source Term f

In order to verify the hypothesis (i) (namely the boundedness of the source term), one has to inspect expression (7.5). In Appendix D we have shown the boundedness of $\boldsymbol{\Psi}$. The other terms, all containing $\boldsymbol{\sigma}$, are all easily estimated, the more problematic terms are those involving $\partial_z\boldsymbol{\sigma}$. However, one immediately recognizes that the coefficients of $\partial_z\boldsymbol{\sigma}$ are, after some rearrangement, u_3^H and $\varepsilon[\tilde{u}_3^+ + \tilde{u}_3^- + u_3^{(1)} + \sigma_3]$; using the fact that both these coefficients are zero at the boundaries, and with Cauchy estimates on $\partial_z\boldsymbol{\sigma}$, one gets the desired estimate.

7.4. Quasi-Contractivity of the Operator $\boldsymbol{\mathcal{G}}$

We now verify the hypothesis (iii) of the ACK theorem, namely the quasi-contractivity of the operator $\boldsymbol{\mathcal{G}}$. Hypothesis (ii), namely the continuity of the operator $\boldsymbol{\mathcal{G}}$, can be easily estimated using the same ideas.

The linear terms in (7.4) are immediately estimated using: (a) the estimates, given in the previous sections, for the lower order terms in the asymptotic expansion; (b) the estimate on $\boldsymbol{\sigma}$ given in Proposition 7.1; (c) the estimate on the Stokes operator with source term $\boldsymbol{\mathcal{S}}^s$ given in (A.4a) and (A.4c) of Theorem A.1; (d) the Cauchy estimate to bound terms involving the z-derivative of rapidly varying functions.

Concerning the nonlinear terms, we first consider the term involving the tangential derivatives. Suppose

$$\left|\boldsymbol{\eta}^{(1)}\right|_{l,\rho_0,\theta_0,\beta,T} < c\,, \quad \text{and} \quad \left|\boldsymbol{\eta}^{(2)}\right|_{l,\rho_0,\theta_0,\beta,T} < c.$$

Then, for $\rho' < \rho < \rho_0 - \beta t$ and $\theta' < \theta < \theta_0 - \beta t$ one can write the estimate

$$\begin{aligned}
&|\varepsilon \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)}) \cdot \nabla \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)}) - \varepsilon \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)}) \cdot \nabla \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)})|_{l,\rho',\theta'} \\
&\quad \leqq |[\varepsilon \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)}) - \varepsilon \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)})] \cdot \nabla \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)})|_{l,\rho',\theta'} \\
&\qquad + |\varepsilon \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)}) \cdot \nabla [\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)}) - \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)})]|_{l,\rho',\theta'} \\
&\quad \leqq \varepsilon c \int_0^t |\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta'}\, ds \quad \frac{|\boldsymbol{\eta}^{(1)}|_{l,\rho_0,\theta'}}{\rho_0 - \rho'} \\
&\qquad + \varepsilon c \int_0^t \frac{|\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho(s),\theta'}}{\rho(s) - \rho'}\, ds \quad |\boldsymbol{\eta}^{(2)}|_{l,\rho',\theta'} \\
&\quad \leqq \varepsilon c \int_0^t |\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta'}\, ds \\
&\qquad + \varepsilon c \int_0^t \frac{|\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho(s),\theta'}}{\rho - \rho'}\, ds \\
&\quad \leqq \varepsilon c \int_0^t \frac{|\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho(s),\theta'}}{\rho(s) - \rho'}\, ds.
\end{aligned}$$

To pass from the second to the third line we have used the estimate on the operator $\boldsymbol{\mathcal{S}}^s$ and used the Cauchy estimate on the tangential derivative. In particular, note that, given the expression (7.3) for $\boldsymbol{\eta}$ and the expression (7.4) for $\boldsymbol{\mathcal{G}}$, one has $\gamma^{\pm}(\boldsymbol{\eta}^{(1)} - \boldsymbol{\eta}^{(2)}) = 0$. This has allowed us to use the estimate on $\boldsymbol{\mathcal{S}}^s$ given in Theorem A.2.

We now estimate the nonlinear terms involving the normal derivative; the procedure is the same as that used for the tangential derivative, the only difference being that here the ε factor is needed to estimate the $O(\varepsilon^{-1})$ z-derivative:

$$\begin{aligned}
&|\varepsilon \mathcal{S}_3^s(\boldsymbol{\eta}^{(1)}) \partial_z \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)}) - \varepsilon \mathcal{S}_3^s(\boldsymbol{\eta}^{(2)}) \partial_z \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)})|_{l,\rho',\theta'} \\
&\quad \leqq |[\varepsilon \mathcal{S}_3^s(\boldsymbol{\eta}^{(1)}) - \varepsilon \mathcal{S}_3^s(\boldsymbol{\eta}^{(2)})] \partial_z \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)})|_{l,\rho',\theta'} \\
&\qquad + |\varepsilon \mathcal{S}_3^s(\boldsymbol{\eta}^{(2)}) \partial_z [\boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(1)}) - \boldsymbol{\mathcal{S}}^s(\boldsymbol{\eta}^{(2)})]|_{l,\rho',\theta'} \\
&\quad \leqq c \int_0^t |\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta'}\, ds \quad \frac{|\boldsymbol{\eta}^{(1)}|_{l,\rho',\theta_0}}{\theta_0 - \theta'} \\
&\qquad + c \int_0^t \frac{|\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta(s)}}{\theta(s) - \theta'}\, ds \quad |\boldsymbol{\eta}^{(2)}|_{l,\rho',\theta'}
\end{aligned}$$

$$
\begin{aligned}
&\leqq c \int_0^t |\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta'}\, ds \\
&\quad + c \int_0^t \frac{|\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta(s)}}{\theta(s) - \theta'} \\
&\leqq c \int_0^t \frac{|\boldsymbol{\eta}^{(1)}(\cdot,\cdot,s) - \boldsymbol{\eta}^{(2)}(\cdot,\cdot,s)|_{l,\rho',\theta(s)}}{\theta(s) - \theta'}\, ds.
\end{aligned}
$$

Having shown that all the hypotheses of the ACK theorem are satisfied, we have proved the following theorem:

Theorem 7.2. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *satisfies the compatibility conditions* (2.4) *and* (2.5). *Then there exist* $\beta_4 > \beta_3$, $T_4 < T_3$, $\rho_4 < \rho_3$, $\theta_4 < \theta_3$, *such that Equation* (7.3) *admits, for* ε *sufficiently small, a unique solution* $\boldsymbol{\eta} \in L^{l,\rho_4,\theta_4}_{\beta_4,T_4}$.

Now, Theorem 7.1 follows immediately from the above theorem and Proposition 7.1.

8. The Main Result

We can now state formally the main result of the paper as a direct consequence of Theorems 3.1, 4.1, 4.2, 6.1, 6.2, 6.3 and 7.1. Defining $\bar{T}$ as the common time of existence of the hydrostatic terms and of the Boundary Layer correctors as well as of the error term, and with $\bar{\rho}$ and $\bar{\theta}$ the smallest radius of analyticity of the various terms, and with $\bar{\mu}$ the smallest exponential decay rate of the BL correctors away form the boundaries, we can expand the solution of the primitive equations as follows.

Theorem 8.1. *Suppose that* $\mathbf{u}_0 \in H^{l,\rho_0,\theta_0}$ *satisfies the compatibility conditions* (2.4) *and* (2.5). *Then, for* ν *sufficiently small, there exist* $\bar{\rho}$, $\bar{\theta}$, $\bar{\mu}$, $\bar{\beta}$, *and* $\bar{T}$ *such that the unique solution of the primitive equations* (2.1a)–(2.1e) *can be written as*

$$
\begin{aligned}
\mathbf{u} &= \mathbf{u}^H + \tilde{\mathbf{u}}^- + \tilde{\mathbf{u}}^+ + \sqrt{\nu}(\mathbf{u}^{H(1)} + \tilde{\mathbf{u}}^{-(1)} + \tilde{\mathbf{u}}^{+(1)} + \mathbf{e}) \\
u_3 &= u_3^H + \sqrt{\nu}(\tilde{u}_3^- + \tilde{u}_3^+) + \sqrt{\nu}[u_3^{H(1)} + \sqrt{\nu}(\tilde{u}_3^{-(1)} + \tilde{u}_3^{+(1)} + e_3)]
\end{aligned}
$$

where

- $\mathbf{u}^H$ *and* u_3^H *solve Equations* (2.7a)–(2.7e) *and belong to* $H^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},\bar{T}}$
- $\tilde{\mathbf{u}}^-$ *and* $\tilde{u}_3^-$ *solve Equations* (2.14a)–(2.14e) *and belong to* $K^{-,l,\bar{\rho},\bar{\theta},\bar{\mu}}_{\bar{\beta},\bar{T}}$
- $\tilde{\mathbf{u}}^+$ *and* $\tilde{u}_3^+$ *solve Equations* (2.17a)–(2.17e) *and belong to* $K^{+,l,\bar{\rho},\bar{\theta},\bar{\mu}}_{\bar{\beta},\bar{T}}$
- $\mathbf{u}^{H(1)}$ *and* $u_3^{H(1)}$ *solve Equations* (5.3a)–(5.3e) *and belong to* $M^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},\bar{T}}$
- $\tilde{\mathbf{u}}^{-(1)}$ *and* $\tilde{u}_3^{-(1)}$ *solve Equations* (5.5a)–(5.5e) *and belong to* $K^{-,l,\bar{\rho},\bar{\theta},\bar{\mu}}_{\bar{\beta},\bar{T}}$
- $\tilde{\mathbf{u}}^{+(1)}$ *and* $\tilde{u}_3^{+(1)}$ *solve Equations* (5.6a)–(5.6e) *and belong to* $K^{+,l,\bar{\rho},\bar{\theta},\bar{\mu}}_{\bar{\beta},\bar{T}}$
- $\mathbf{e}$ *and* e_3 *solve Equations* (5.7a)–(5.7e) *and belong to* $L^{l,\bar{\rho},\bar{\theta}}_{\bar{\beta},\bar{T}}$.

9. Conclusions

In this paper we have constructed the solution of the primitive equations in the limit of zero viscosity, the main assumption being the analyticity of the initial data. We have shown that the solution exists for a time that depends on the size of the initial data but does remain finite when $\nu \to 0$. The solution of the PE has been constructed with an asymptotic matching procedure involving, as dominant terms, the solution of the hydrostatic equations and BL correctors. The byproduct of our construction is that the solution of the PE, away from boundaries, is well approximated by the hydrostatic equations; in fact the BL correctors decay exponentially outside a small layer, size $O(\sqrt{\nu})$, close to the boundary. The structure of the solution (1.1) describes a laminar flow, with strong gradients confined in the boundary layer.

It is interesting to note that the equations satisfied by the BL correctors share the mathematical structure with the Prandtl equations. It is well known how Prandtl solutions develop a singularity (see [17] and references therein), and one could therefore expect that the boundary layer solution that we have constructed to show similar blow-up phenomena, and to do this also if initialized with analytic data. In the classical high Reynolds number Navier–Stokes theory the boundary layer singularities (or, to be more precise, the appearance of complex singularities that are precursor of the blow-up [18]) signal the interaction stage-when the pressure profile at the boundary is modified by the vorticity generated in the boundary layer; this interaction ultimately leads to separation of the boundary layer and to the break-up of asymptotic structures like (1.1). It would be interesting to explore if the phenomenology typical of the high Reynolds number Navier–Stokes flows is also shown by the primitive equations solutions.

Acknowledgments Part of this work has been done while the third author (MS) was visiting the Math Department of USC. He expresses gratitude for the warm and friendly hospitality received. IK was supported in part by the NSF Grant DMS-1311943. MCL was supported in part by INDAM-GNFM.

A Stokes Equation with a Source Term

In this section, we solve the Stokes equations with source term and homogeneous boundary data

$$(\partial_t - \varepsilon^2 \Delta_2 - \partial_{ZZ})\mathbf{u} + \nabla p = \boldsymbol{\omega} \tag{A.1a}$$

$$\nabla \cdot \mathbf{u} + \varepsilon^{-1} \partial_Z u_3 = 0 \tag{A.1b}$$

$$\gamma^{\pm} \mathbf{u} = 0 \tag{A.1c}$$

$$\gamma^{\pm} u_3 = 0 \tag{A.1d}$$

$$\mathbf{u}_{t=0} = 0. \tag{A.1e}$$

In order to simplify the notation, we denote the solution of the above problem as

$$(\mathbf{u}, u_3) = (\boldsymbol{\mathcal{S}}^s(\boldsymbol{\omega}), \mathcal{S}_3^s(\boldsymbol{\omega})).$$

Without loss of generality, we may assume that

$$\int_{-h/\varepsilon}^{h/\varepsilon} \nabla \cdot \boldsymbol{\omega}\, dZ' = 0.$$

Therefore, $\mathbf{u}$ satisfies the equation

$$(\partial_t - \varepsilon^2 \Delta_2 - \partial_{ZZ})\mathbf{u} = -\frac{\varepsilon}{2h}[\nabla \Delta^{-1} \nabla \cdot \partial_Z \mathbf{u}]_{-h/\varepsilon}^{h/\varepsilon} + \boldsymbol{\omega} \tag{A.2a}$$

$$\gamma^{\pm}\mathbf{u} = 0 \tag{A.2b}$$

$$\mathbf{u}_{t=0} = 0. \tag{A.2c}$$

The normal component is given in terms of $\mathbf{u}$ by

$$u_3 = -\varepsilon \int_{-h/\varepsilon}^{Z} \nabla \cdot \mathbf{u}\, dZ'. \tag{A.3}$$

We then have the following statement:

Theorem A.1. *Let $\boldsymbol{\omega} \in L^{l,\rho,\theta}_{\beta,T}$. Then, for ε sufficiently small, $\boldsymbol{\mathcal{S}}^s(\boldsymbol{\omega}) \in L^{l,\rho,\theta}_{\beta,T}$, $\mathcal{S}^s_3(\boldsymbol{\omega}) \in L^{l-1,\rho,\theta}_{\beta,T}$, $\partial_z \mathcal{S}^s_3(\boldsymbol{\omega}) \in L^{l-1,\rho,\theta}_{\beta,T}$, and we have the estimates*

$$|\boldsymbol{\mathcal{S}}^s(\boldsymbol{\omega})|_{l,\rho,\theta,\beta,T} \leqq c|\boldsymbol{\omega}|_{l,\rho,\theta,\beta,T} \tag{A.4a}$$

$$|\mathcal{S}^s_3(\boldsymbol{\omega})|_{l-1,\rho,\theta,\beta,T} \leqq c|\boldsymbol{\omega}|_{l,\rho,\theta,\beta,T} \tag{A.4b}$$

$$|\partial_z \mathcal{S}^s_3(\boldsymbol{\omega})|_{l-1,\rho,\theta,\beta,T} \leqq c|\boldsymbol{\omega}|_{l,\rho,\theta,\beta,T}. \tag{A.4c}$$

In order to prove the above theorem, we construct the solution of Equations (A.2a)–(A.2c) as the limit of the sequence

$$\begin{aligned} \mathbf{u}_{(0)} &= 0 \\ &\cdots \\ (\partial_t - \varepsilon^2 \Delta_2 - \partial_{ZZ})\mathbf{u}_{(n+1)} &= -\frac{\varepsilon}{2h}[\nabla \Delta^{-1} \nabla \cdot \partial_Z \mathbf{u}_{(n)}]_{-h/\varepsilon}^{h/\varepsilon} + \boldsymbol{\omega} \end{aligned} \tag{A.5}$$

with $\mathbf{u}_{(n+1)}$ satisfying homogeneous initial data and boundary conditions. Therefore $\mathbf{u}_{(n+1)}$ can be expressed using the operator F_2 introduced in Appendix C.

$$\mathbf{u}_{(n+1)} = F_2(\boldsymbol{\omega}) - \frac{\varepsilon}{2h} F_2([\nabla \Delta^{-1} \nabla \cdot \partial_Z \mathbf{u}_{(n)}]_{-h/\varepsilon}^{h/\varepsilon}) \tag{A.6}$$

If one introduces the operator M

$$M\boldsymbol{\omega} = [\nabla \Delta^{-1} \nabla \cdot \partial_Z F_2 \boldsymbol{\omega}]_{-h/\varepsilon}^{h/\varepsilon},$$

from (A.6) one can easily prove, by induction, that

$$\mathbf{u}_{(n)} = F_2\left(\sum_{i=0}^{n-1} (-1)^i \left(\frac{\varepsilon}{2h}\right)^i M^i\ \boldsymbol{\omega}\right). \tag{A.7}$$

Using Proposition C.2 and Lemma C.4, it is easy to prove the following statement.

Lemma A.3. *Let* $\boldsymbol{\omega} \in L^{l,\rho,\theta}_{\beta,T}$. *Then* $M\boldsymbol{\omega} \in H^{l,\rho}_{\beta,T}$ *and*

$$|M\boldsymbol{\omega}|_{l,\rho,\beta,T} \leqq c\sqrt{T}\,|\boldsymbol{\omega}|_{l,\rho,\theta,\beta,T}$$

Therefore, if $\varepsilon < 2h/(c\sqrt{T})$, the sequence (A.7) is bounded, namely

$$|\mathbf{u}_{(n)}|_{l,\rho,\theta,\beta,T} \leqq c|\boldsymbol{\omega}|_{l,\rho,\theta,\beta,T} \tag{A.8}$$

Using (A.7) one immediately sees that

$$|\mathbf{u}_{(n+1)} - \mathbf{u}_{(n)}|_{l,\rho,\theta,\beta,T} = \left|F_2\left((-1)^n\left(\frac{\varepsilon}{2h}\right)^n M^n\,\boldsymbol{\omega}\right)\right|_{l,\rho,\theta,\beta,T}.$$

Using the bounds on F_2 and M, the above inequality shows that, under the smallness condition on ε, the sequence $\mathbf{u}_{(n)}$ is a Cauchy sequence.
Moreover the estimate (A.4a) is a consequence of the bound (A.8). The proof of the estimate (A.4b) can be achieved looking at the expression (A.3) for the normal component and using Jensen inequality. The estimate (A.4c) is an immediate consequence of the expression (A.3).
Moreover, we have the following result:

Theorem A.2. *Let* $\boldsymbol{\omega} \in L^{l,\rho,\theta}_{\beta,T}$ *with* $\gamma^{\pm}\boldsymbol{\omega} = 0$. *Then, for* ε *sufficiently small, we have*

$$|\boldsymbol{\mathcal{S}}^s(\boldsymbol{\omega})|_{l,\rho',\theta'} \leqq c\int_0^t |\boldsymbol{\omega}(\cdot,\cdot,s)|_{l,\rho',\theta'}\,ds$$

$$|\mathcal{S}^s_3(\boldsymbol{\omega})|_{l-1,\rho',\theta'} \leqq c\int_0^t |\boldsymbol{\omega}(\cdot,\cdot,s)|_{l,\rho',\theta'}\,ds$$

for $\rho' < \rho - \beta t$ *and* $\theta' < \theta - \beta t$.

The above statement is a consequence of the estimate on the operator F_2 given in Lemma C.5.

B Stokes Equations with Boundary Data

In this section, we solve the Stokes equations with boundary data. We shall assume that the normal influx is in gradient form and of size $O(\varepsilon)$. We first treat the case when the boundary condition for the normal influx is homogeneous.

B.1 Homogeneous Normal Influx

Here we construct the solution $(\boldsymbol{\sigma}, \sigma_3)$ of the system

$$(\partial_t - \varepsilon^2\Delta_2 - \partial_{ZZ})\boldsymbol{\sigma} + \nabla p^{\sigma} = 0 \tag{B.1a}$$

$$\nabla\cdot\boldsymbol{\sigma} + \varepsilon^{-1}\partial_Z\sigma_3 = 0 \tag{B.1b}$$

$$\gamma^{\pm}\boldsymbol{\sigma} = \mathbf{g}^{\pm} \tag{B.1c}$$

$$\gamma^{\pm}\sigma_3 = 0 \tag{B.1d}$$

$$\boldsymbol{\sigma}|_{t=0} = 0. \tag{B.1e}$$

Taking first the divergence and then the average of (B.1a), we get the expression for the pressure which reads:

$$p^{\sigma} = \frac{1}{2h}\varepsilon\Delta^{-1}[\partial_Z \nabla \cdot \boldsymbol{\sigma}]_{-h/\varepsilon}^{h/\varepsilon}.$$

Thus $\boldsymbol{\sigma}$ can be found by solving a heat-type equation:

$$(\partial_t - \varepsilon^2\Delta_2 - \partial_{ZZ})\boldsymbol{\sigma} = -\frac{1}{2h}\varepsilon[\nabla\Delta^{-1}\nabla \cdot \partial_Z \boldsymbol{\sigma}]_{-h/\varepsilon}^{h/\varepsilon} \tag{B.2a}$$

$$\gamma^{\pm}\boldsymbol{\sigma} = \mathbf{g}^{\pm} \tag{B.2b}$$

$$\boldsymbol{\sigma}|_{t=0} = 0. \tag{B.2c}$$

The third component σ_3 is given by:

$$\sigma_3 = -\varepsilon\int_{-h/\varepsilon}^{Z} \nabla \cdot \boldsymbol{\sigma}\, dZ'.$$

One can easily prove the following proposition:

Proposition B.1. *Let* $\mathbf{g}^{\pm} \in H^{m,\rho}_{\beta,T}$. *Then* $(\boldsymbol{\sigma}, \sigma_3)$, *the solution of the system* (B.1a)–(B.1e), *is in* $L^{m,\rho,\theta}_{\beta,T}$, *with* $\theta < \pi/4$, *and we have the estimate*

$$|\boldsymbol{\sigma}|_{m,\rho,\theta,\beta,T} \leqq c|\mathbf{g}^{\pm}|_{m,\rho,\beta,T}$$

$$|\sigma_3|_{m-1,\rho,\theta,\beta,T} \leqq c\varepsilon|\mathbf{g}^{\pm}|_{m,\rho,\beta,T}. \tag{B.3a}$$

B.2 Non-Homogeneous Normal Influx of the Divergence Type

We now treat the case where normal influx is present and solve the following system:

$$(\partial_t - \varepsilon^2\Delta_2 - \partial_{ZZ})\boldsymbol{\sigma} + \nabla p^{\sigma} = 0 \tag{B.4a}$$

$$\nabla \cdot \boldsymbol{\sigma} + \varepsilon^{-1}\partial_Z\sigma_3 = 0 \tag{B.4b}$$

$$\gamma^{\pm}\boldsymbol{\sigma} = \mathbf{g}^{\pm} \tag{B.4c}$$

$$\gamma^{\pm}\sigma_3 = \varepsilon\nabla \cdot \mathbf{g}_3^{\pm} \tag{B.4d}$$

$$\boldsymbol{\sigma}|_{t=0} = 0. \tag{B.4e}$$

We define $(\tilde{\boldsymbol{\sigma}}, \tilde{\sigma}_3)$ as

$$\tilde{\boldsymbol{\sigma}} = \boldsymbol{\sigma} - \varepsilon\left(R\frac{\cosh(|k|(z+h))}{\sinh(2|k|)}\nabla \cdot \mathbf{g}_3^{+} + R\frac{\cosh(|k|(h-z))}{\sinh(2|k|)}\nabla \cdot \mathbf{g}_3^{-}\right) \tag{B.5a}$$

$$\tilde{\sigma}_3 = \sigma_3 - \varepsilon\left(\frac{\sinh(|k|(z+h))}{\sinh(2|k|)}\nabla \cdot \mathbf{g}_3^{+} - \frac{\sinh(|k|(h-z))}{\sinh(2|k|)}\nabla \cdot \mathbf{g}_3^{-}\right) \tag{B.5b}$$

and observe that it solves a system with homogeneous normal boundary conditions of the form (B.1a)–(B.1e).

Given Proposition B.1, and denoting with $\left(\boldsymbol{\mathcal{S}}^b, \mathcal{S}_3^b\right)$ the operator that solves the system (B.4a)–(B.4e), we have immediately the following statement:

Theorem B.1. *Let* $\mathbf{g}^{\pm} \in H^{l,\rho}_{\beta,T}$ *and let* $\mathbf{g}_3^{\pm} \in H^{l+1,\rho}_{\beta,T}$. *Then* $\boldsymbol{\mathcal{S}}^b(\mathbf{g}^{\pm}, \mathbf{g}_3^{\pm}) \in L^{l,\rho,\theta}_{\beta,T}$, $\mathcal{S}_3^b(\mathbf{g}^{\pm}, \mathbf{g}_3^{\pm}) \in L^{l,\rho,\theta}_{\beta,T}$, *with* $\theta < \pi/4$, *and we have the estimates*

$$|\boldsymbol{\mathcal{S}}^b(\mathbf{g}^{\pm}, \mathbf{g}_3^{\pm})|_{l,\rho,\theta,\beta,T} \leqq c(|\mathbf{g}^{\pm}|_{l,\rho,\beta,T} + |\mathbf{g}_3^{\pm}|_{l+1,\rho,\beta,T})$$
$$|\mathcal{S}_3^b(\mathbf{g}^{\pm}, \mathbf{g}_3^{\pm})|_{l-1,\rho,\theta,\beta,T} \leqq c\varepsilon(|\mathbf{g}^{\pm}|_{l,\rho,\beta,T} + |\mathbf{g}_3^{\pm}|_{l+1,\rho,\beta,T}).$$

C The Heat Operators

In this section, we give the explicit solution to the heat equation

$$(\partial_t - \varepsilon^2\Delta_2 - \partial_{ZZ})u = f \tag{C.1a}$$
$$\gamma^{\pm}u = g^{\pm} \tag{C.1b}$$
$$u_{t=0} = 0. \tag{C.1c}$$

The explicit representation of the solution is given in terms of the inverse heat operators F_1 and F_2 introduced in [36] (here used in the case when $U = 0$), which we briefly recall here.

The operator F_1 solves the above equations (C.1a)–(C.1c), with $f = 0$. Introducing the $\varphi-$function

$$\varphi(Z,t) = \sum_{n=-\infty}^{\infty} H\left(Z + 4n\frac{h}{\varepsilon}, t\right), \qquad -\infty < Z < \infty,$$

where

$$H(Z,t) = \frac{Z}{t}\frac{e^{-Z^2/(4t)}}{\sqrt{4\pi t}},$$

one can easily verify the formula

$$F_1(g^+, g^-) = \int_0^t ds\, e^{-\varepsilon^2k^2(t-s)}\varphi\left(Z + \frac{h}{\varepsilon}, t-s\right)\, g^-(k', s)$$
$$+ \int_0^t ds\, e^{-\varepsilon^2k^2(t-s)}\varphi\left(\frac{h}{\varepsilon} - Z, t-s\right)\, g^+(k', s). \tag{C.2}$$

The operator F_2 solves the equations (C.1a)–(C.1c), with $g^{\pm} = 0$ and has the explicit expression

$$F_2 f = \int_0^t dse^{-\varepsilon^2k^2(t-s)}$$
$$\times \int_{-h/\varepsilon}^{h/\varepsilon} dZ'\, \left(\theta(Z - Z', t-s) - \theta\left(Z + Z' + \frac{2h}{\varepsilon}, t-s\right)\right)\, f(k, Z', s), \tag{C.3}$$

where

$$\theta(Z,t) = \sum_{n=-\infty}^{\infty} K\left(Z + 4n\frac{h}{\varepsilon}, t\right), \qquad -\infty < Z < \infty$$

and

$$K(Z,t) = \frac{1}{\sqrt{4\pi t}} e^{-Z^2/(4t)}.$$

We now give estimates for the above heat operators, the proof following along the same lines as in [36, Section 3] (note that in [36] also a convection term is present):

Proposition C.1. *Let $g^+ \in H^{l,\rho}_{\beta,T}$, $g^- \in H^{l,\rho}_{\beta,T}$ satisfy the compatibility conditions $g^+(t=0) = g^-(t=0) = 0$. Then $F_1(g^+, g^-) \in L^{l,\rho,\theta}_{\beta,T}$ and the estimate*

$$|F_1(g^+, g^-)|_{l,\rho,\theta,\beta,T} \leqq c\big(|g^+|_{l,\rho,\beta,T} + |g^-|_{l,\rho,\beta,T}\big)$$

holds.

Proposition C.2. *Let $f \in L^{l,\rho,\theta}_{\beta,T}$. Then $F_2 f \in L^{l,\rho,\theta}_{\beta,T}$ and we have the estimate*

$$|F_2 f|_{l,\rho,\theta,\beta,T} \leqq c|f|_{l,\rho,\theta,\beta,T}.$$

Using the operators F_1 and F_2, the solution of the heat equations (C.1a)–(C.1c) may be expressed as

$$u = F_2 f + F_1(g^+, g^-) \tag{C.4}$$

and the following theorem holds.

Theorem C.1. *Let $f \in L^{l,\rho,\theta}_{\beta,T}$ and $g^+ \in H^{l,\rho}_{\beta,T}$, $g^- \in H^{l,\rho}_{\beta,T}$ satisfy the compatibility conditions $g^+(t=0) = g^-(t=0) = 0$. Then the solution of the heat equations (C.1a)–(C.1c) satisfies $u \in L^{l,\rho,\theta}_{\beta,T}$ with the estimate*

$$|u|_{l,\rho,\theta,\beta,T} \leqq c[|f|_{l,\rho,\theta,\beta,T} + |g^+|_{l,\rho,\beta,T} + |g^-|_{l,\rho,\beta,T}].$$

We now give certain additional estimates on the operator F_2 that are useful for the construction of the solution the Stokes problem.

Lemma C.4. *Let $f \in L^{l,\rho,\theta}_{\beta,T}$. Then $\gamma^\pm F_2 f \in H^{l,\rho}_{\beta,T}$ and, if $\rho' < \rho - \beta t$ and $\theta' < \theta - \beta t$, we have*

$$|\gamma^\pm \partial_Z F_2 f|_{l,\rho'} \leqq c_1 \int_0^t \frac{1}{\sqrt{t-s}} |f(\cdot,\cdot,s)|_{l,\rho',\theta'}\, ds \leqq c_2\sqrt{t}\, |f|_{l,\rho,\theta,\beta,T}.$$

Lemma C.5. *Let $f \in L^{l,\rho,\theta}_{\beta,T}$ satisfy the condition $\gamma^\pm f = 0$. Then if $\rho' < \rho - \beta t$ and $\theta' < \theta - \beta t$, we have*

$$|F_2 f|_{l,\rho'} \leqq C_1 \int_0^t |f(\cdot,\cdot,s)|_{l,\rho',\theta'}\, ds \leqq C_2 |f|_{l,\rho,\theta,\beta,T}.$$

D Estimates on the Source Terms

A simple calculation shows that the source term for the error equation (5.7a) is given by

$$\begin{aligned}
\boldsymbol{\Psi} = &-[\mathbf{u}^{(0)} \cdot \nabla(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) + \mathbf{u}^{H(1)} \cdot \nabla(\tilde{\mathbf{u}}^{+} + \tilde{\mathbf{u}}^{-} + \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})) \\
&+ (\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) \cdot \nabla\mathbf{u}^{(0)} \\
&+ (\tilde{\mathbf{u}}^{+} + \tilde{\mathbf{u}}^{-} + \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})) \cdot \nabla\mathbf{u}^{H(1)} + \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) \\
&\cdot \nabla(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) \\
&+ u_3^{(0)}\partial_z(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) + u_3^{H(1)}\partial_z(\tilde{\mathbf{u}}^{+} + \tilde{\mathbf{u}}^{-} + \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})) \\
&+ \varepsilon(\tilde{u}_3^{+(1)} + \tilde{u}_3^{-(1)})\partial_z\mathbf{u}^{(0)} \\
&+ \varepsilon(\tilde{u}_3^{+} + \tilde{u}_3^{-} + \varepsilon(\tilde{u}_3^{+(1)} + \tilde{u}_3^{-(1)}))\partial_z\mathbf{u}^{H(1)} \\
&+ \varepsilon(\tilde{u}_3^{+(1)} + \tilde{u}_3^{-(1)})\partial_z(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})] \\
&+ \varepsilon^2\Delta\mathbf{u}^{H(1)} + \varepsilon^2\Delta(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) + \mathbf{F}_0 + \mathbf{F}_{-1}.
\end{aligned}$$

In the above terms are present terms which are $O(\varepsilon^{-1})$ (being z-derivatives of the BL correctors $\tilde{\mathbf{u}}^{+}$ and $\tilde{\mathbf{u}}^{-}$); however, all these terms are multiplied by slowly varying terms that vanish at the boundaries. In fact $\boldsymbol{\Psi}$ can be rewritten as:

$$\begin{aligned}
\boldsymbol{\Psi} = &-[\mathbf{u}^{(0)} \cdot \nabla(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) + \mathbf{u}^{H(1)} \cdot \nabla(\tilde{\mathbf{u}}^{+} + \tilde{\mathbf{u}}^{-} + \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})) \\
&+ (\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) \cdot \nabla\mathbf{u}^{(0)} \\
&+ (\tilde{\mathbf{u}}^{+} + \tilde{\mathbf{u}}^{-} + \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})) \cdot \nabla\mathbf{u}^{H(1)}\} \\
&+ \varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) \cdot \nabla(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) \\
&+ u_3^{H(1)}\partial_z(\varepsilon(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})) + \varepsilon(\tilde{u}_3^{+(1)} + \tilde{u}_3^{-(1)})\partial_z\mathbf{u}^{(0)} \\
&+ \varepsilon(\tilde{u}_3^{+} + \tilde{u}_3^{-} + \varepsilon(\tilde{u}_3^{+(1)} + \tilde{u}_3^{-(1)}))\partial_z\mathbf{u}^{H(1)} \\
&+ \varepsilon(\tilde{u}_3^{+(1)} + \tilde{u}_3^{-(1)})\partial_z(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)})] \\
&+ \varepsilon^2\Delta\mathbf{u}^{H(1)} + \varepsilon^2\Delta(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) + \mathbf{F}_0 \\
&- u_3^{(0)}\partial_z(\tilde{\mathbf{u}}^{+(1)} + \tilde{\mathbf{u}}^{-(1)}) - (u_3^{H(1)} - \nabla \cdot \mathbf{G}^{+})\partial_z\tilde{\mathbf{u}}^{+} - (u_3^{H(1)} - \nabla \cdot \mathbf{G}^{-})\partial_z\tilde{\mathbf{u}}^{-}.
\end{aligned}$$

The terms in the first five lines of the above expression for $\boldsymbol{\Psi}$ are obviously $O(1)$. In the last line there are terms resulting from $\partial_z\tilde{\mathbf{u}}^{+}$ or $\partial_z\tilde{\mathbf{u}}^{-}$ (therefore, in the boundary layer are $O(\varepsilon^{-1})$) which are however multiplied by terms which are $O(\varepsilon)$ in the boundary layer, and can therefore be bounded using the Cauchy estimate.

E A Fixed Point Theorem

In order to prove the existence and uniqueness of the solution to the Prandtl equations (cf. [7,15–18,28,41]), we use the following version of the Abstract Cauchy–Kowalevski Theorem (ACK) (cf. [1,35,47] and references therein). Consider the equation

$$u + F(u, t) = 0. \tag{E.1}$$

Let $\{X_\rho : \; 0 < \rho \leqq \rho_0\}$ be a scale of Banach spaces with norms $|\cdot|_\rho$, such that $X_{\rho'} \subset X_{\rho''}$ and $|\cdot|_{\rho''} \leqq |\cdot|_{\rho'}$ when $\rho'' \leqq \rho' \leqq \rho_0$.

Theorem E.1. (ACK) *Suppose that there exist $R > 0$, $\rho_0 > 0$, and $\beta_0 > 0$ such that for $0 < \tau \leqq T \leqq \rho_0/\beta_0$ the following statements hold:*

(i) *if ρ is such that $0 < \rho \leqq \rho_0 - \beta_0\tau$, then the function $F(0,t)\colon [0,\tau] \mapsto \{u \in X_\rho : \sup_{0 \leqq t \leqq \tau} |u(t)|_\rho < \infty\}$ is continuous and*

$$|F(0,t)|_{\rho_0 - \beta_0 t} \leqq R_0 < R;$$

(ii) *if ρ', ρ are such that $0 < \rho' < \rho \leqq \rho_0 - \beta_0\tau$, then the function $F(u,t)\colon [0,\tau] \mapsto X_{\rho'}$ is continuous for all u such that $\{u \in X_\rho : \sup_{0 \leqq t \leqq T} |u(t)|_\rho \leqq R\}$;*

(iii) *if ρ' and $\rho(s)$ are such that $\rho' < \rho(s) \leqq \rho_0 - \beta_0 s$ and if u^1 and $u^2 \in \{u : u(t) \in X_{\rho_0 - \beta_0 t} : \sup_{0 \leqq t \leqq \tau} |u(t)|_{\rho_0 - \beta_0 t} \leqq R\}$, then*

$$|F(u^1,t) - F(u^2,t)|_{\rho'} \leqq C \int_0^t ds \left(\frac{|u^1 - u^2|_{\rho(s)}}{\rho(s) - \rho'} + \frac{|u^1 - u^2|_{\rho'}}{\sqrt{t-s}} \right),$$

where C is a constant independent of t, τ, u^1, u^2, ρ, ρ', $\rho(s)$.

Then there exists $\beta > \beta_0$ such that for all $0 < \rho < \rho_0$ Equation (E.1) has a unique solution $u(t) \in X_{\rho_0 - \beta t}$ with $t \in [0, \rho_0/\beta]$. Moreover, $\sup_{\rho < \rho_0 - \beta t} |u(t)|_\rho \leqq R$.

The proof of the above theorem is given in [1], where it is stated without the mild singularity in time represented by the square root singularity in the assumption (iii). This generalization is given in [35].

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(*Received July 1, 2015 / Accepted March 3, 2016*)
Published online March 16, 2016 –