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A parsimonious model for generating arbitrage-free scenario trees

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Simulation models of economic, financial and business risk factors are widely used to assess risks and support decision-making. Extensive literature on scenario generation methods aims at describing some underlying stochastic processes with the least number of scenarios to overcome the 'curse of dimensionality'. There is, however, an important requirement that is usually overlooked when one departs from the application domain of security pricing: the no-arbitrage condition. We formulate a moment matching model to generate multi-factor scenario trees for stochastic optimization satisfying no-arbitrage restrictions with a minimal number of scenarios and without any distributional assumptions. The resulting global optimization problem is quite general. However, it is non-convex and can grow significantly with the number of risk factors, and we develop convex lower bounding techniques for its solution exploiting the special structure of the problem. Applications to some standard problems from the literature show that this is a robust approach for tree generation. We use it to price a European basket option in complete and incomplete markets.

Keywords: Scenario trees; Global optimization; Convex lower bounding; Stochastic programming; Pricing in incomplete markets

1. Introduction

Simulation models are widely used to assess risk exposures and support financial decision-making. Risk management, in particular, is often based on simulations of the risk factors (or assets) of the balance sheet (see, e.g. Jamshidian and Zhu 1997, Rebonato *et al.* 2005). Scenario trees are widely used in multistage stochastic programming, where the time dimension and non-anticipativity of future events are key features of the model (Mulvey and Vladimirou 1992, Carinõ and Ziemba 1998, Zenios *et al.* 1998, Consiglio *et al.* 2006, Consigli *et al.* 2010). The trend for risk management at an enterprise-wide level, Dembo, Aziz, *et al.* (2000), broadens the risk factors to include not only financial but also economic and business risks and their inter-dependencies. For a review of literature on scenario methods for risk management and portfolio optimization see Dupačová *et al.* (2000), Kaut and Wallace (2007) and chap. 9 in Zenios (2007).

In synthesis, we identify three approaches¶:

(1) The *moment matching approach* describes the joint distribution of scenarios in terms of moments, including cross-moments to take into account inter-dependencies. It solves a set of non-convex equations to match the mathematical expressions of the factor moments to exogenously given values. The main idea is found in the seminal paper by Høyland and Wallace (2001). Refinements suggested by Date *et al.* 2008, Høyland *et al.* (2003) reduce the computational complexity of the underlying optimization problem. Chen and Xu (2014), Xu *et al.* (2012) use K-means clustering of historical observations and linear programming to approximate moments, allowing also subjective estimates of future scenarios to be included.
(2) The *copula approach* postulates the distribution function of the marginals and then, by imposing an associative structure, determines the multivariate joint distribution. This method became popular as the 'copula approach' since the copula function is used to model dependencies among the variables, Cherubini *et al.* (2004).
(3) The *distance minimization approach* approximates the true distribution (continuous or discrete) with

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¶We leave out the literature on simulations for security pricing, see, e.g. Glasserman (2004), which focuses on a specific problem and hence may take advantage of specific stochastic process structures; we take up this issue in the application section.

a few mass points that minimize the (Kantorovich) distance between an original stochastic optimization model and the approximated one. The algorithmic implementation usually starts from a set of points generated by a discrete reference process or by discretizing a continuous process and then, using specific metrics, partitions the points at each stage into disjoint subsets that reduce the total number of scenarios and shape the tree structure (Hochreiter and Pflug 2007, Dupačová *et al.* 2003).

A common aim of these methods is to approximate the underlying stochastic process or probability distribution with the least number of scenarios. However, an important requirement is usually overlooked when one departs from the security pricing literature: the scenario approximation should not present arbitrage opportunities. The generation of arbitrage-free scenarios is complicated by the need to use two probability measures—the objective and the martingale—and returns compatible with both, to match the approximated process to the original.

The significance of no-arbitrage scenarios is well understood in the pricing literature. The problem has resurfaced in more complex forms in recent works where pricing options in the context of multiperiod stochastic models for risk management (Consiglio and De Giovanni 2008, Topaloglou *et al.* 2008). Even with simple asset classes, the absence of arbitrage is a key property, and Geyer *et al.* (2010) show that scenario trees with arbitrage opportunities can produce spurious results when used in portfolio optimization.

In a commentary to Høyland and Wallace (2001), Klaassen (2002) suggests two alternatives to handle arbitrage opportunities. One is to re-apply the scenario generation method from a different starting point, and/or increase the number of scenarios, in the hope that the newly generated set of scenarios is free of arbitrage. The other is to explicitly add no-arbitrage constraints to the original set of equations. Adding a set of equations to generate arbitrage-free scenarios is a viable approach, but it leads to a system of non-convex equations whose solution is prohibitive. To solve real-world applications, Høyland *et al.* (2003) propose a heuristic that does not guarantee the arbitrage-free property (although it works well for their applications).

Arbitrage opportunities are usually eliminated by re-sampling and/or increasing the number of scenarios. Such an approach is not free of faults or limitations. As shown in Geyer *et al.* (2014a), increasing the number of scenarios does not necessarily produce arbitrage-free scenarios, as that depends on the structure of the expected returns and the covariance matrix, and they characterize three regions for the existence of no-arbitrage scenarios using Mahalanobis distances. Moreover, increasing the number of scenarios may not be viable for multistage financial planning models, as the dimensionality of the model grows exponentially with the size of the tree. In a follow-up paper Geyer *et al.* (2014b) show how to generate arbitrage-free random samples by rotating a simplex, thus avoiding the need for re-sampling, but this is done only for matching the first two moments.

Our paper resolves the limitations of existing literature providing a general methodology that applies to higher moments. We formulate the moment matching scenario generation model with no-arbitrage constraints as an optimization problem whose global minimal value is zero, if a solution exists, using the method of Maranas and Floudas (1995). The resulting optimization problem is non-convex and local search algorithms can be trapped in local minima with non-zero value, thus leading to the erroneous conclusion that no solution exists. To overcome this difficulty, we develop a global optimization approach based on convex lower bounding techniques (see Floudas and Gounaris 2009, for a review) that take advantage of the problem structure and is computationally tractable.

Our paper makes two innovations: first, it formulates a global optimization model to generate moment-matching arbitrage-free trees for an arbitrary number of risk factors (or, assets) and moments. Second, it develops an algorithm to exploit the special structure of the model, thus showing global optimization to be a robust tool for scenario generation. As a result of the model we obtain both *objective* (P) and *risk neutral* (Q) probability measures. Therefore, extensions of pricing models to distributions with general moments become straightforward, as the state price density is readily obtained by the ratio Q/P for each node of the tree (see Section 4 and Pliska (1997)).

The model satisfies the principles of parsimony, see Vandekerckhove *et al.* (2015), in the sense that it uses only available observed information (the moments) and fits them using the minimal number of scenarios. It makes no assumptions about the underlying distribution. Also, if for a given application it is known that only the first few moments are relevant, the model can fit only those. If, on the other hand, an underlying distribution is known to apply—perhaps from empirical studies or theoretical arguments—then a tree with minimal number of scenarios can approximate the distribution by matching as many of its moments as needed.

The paper is organized as follows. Sections 2 and 3 formulate the model and develop the solution method. Section 4 reports on the implementation to solve some standard models from the literature and to the pricing of a European basket option in complete and incomplete markets. Section 5 concludes.

2. Notation and model set-up

We assume that asset returns follow stochastic processes in discrete space and time. We have J assets (or risk factors) with returns labelled by index set $\mathcal{J} = \{1, 2, \ldots, J\}$ which are observed on a finite number of *time stages*, $t = 0, 1, 2, \ldots, T$:

$$R = \left(R_t^1, \ldots, R_t^J\right)_{t=0}^T. \tag{1}$$

(If some portfolio decision needs to be made at each stage t, such as in portfolio replication or portfolio optimization, these are called *decision stages*. Time stages for asset prices and decision stages for portfolios do not need to coincide, but for simplicity we assume they do.)

The return process is modelled on the probability space $(\Omega, \mathcal{F}, P)$, where the sample space Ω is assumed to be finite. Such a formulation allows for market representation through *scenario trees*, Pliska (1997). We denote by $\mathcal{N}_t$ the set of nodes at stage t. Each node $n \in \mathcal{N}_t$ corresponds one-to-one with an atom of the filtration $\mathcal{F}_t$. Every node $n \in \mathcal{N}_t$, $t = 1, \ldots, T$, has a unique ancestor node $a(n) \in \mathcal{N}_{t-1}$, and every node

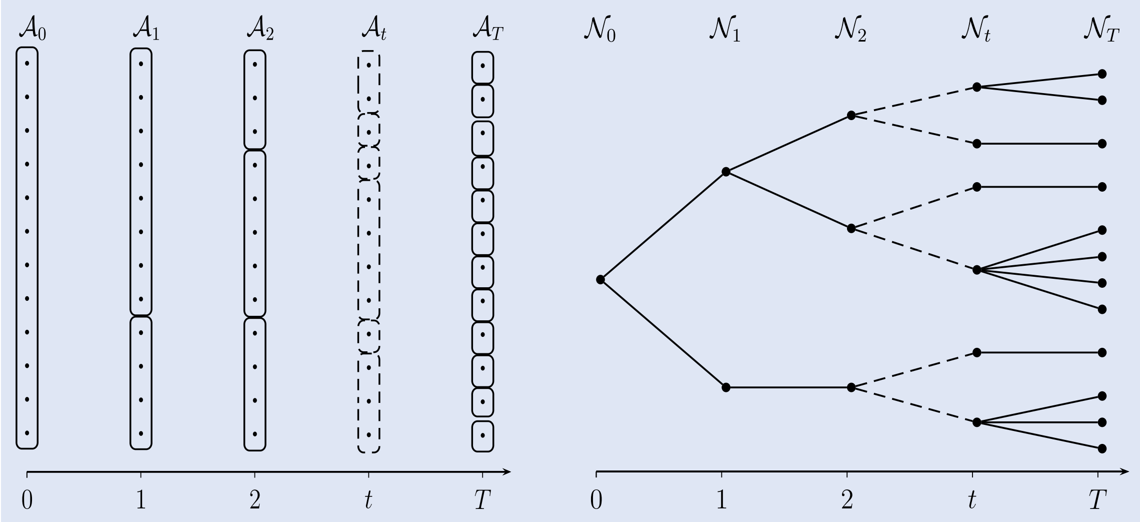

Figure 1. A finite filtration (left panel) and its associated tree (right panel).

$n \in \mathcal{N}_t$, $t = 0, \ldots, T-1$, has a non-empty set of child nodes $\mathcal{C}(n) \subset \mathcal{N}_{t+1}$. The collection of all nodes is denoted by $\mathcal{N} \equiv \bigcup_{t=0}^{T} \mathcal{N}_t$.

In the probabilistic context, if we assume that the sample space Ω is finite, every algebra $\mathcal{F}$ corresponds to a partitioning of Ω into mutually disjoint subsets (the $\mathcal{F}$-atoms). In a scenario tree, there is a one-to-one map between the nodes $n \in \mathcal{N}_t$ and the partition sets $\mathcal{A}_t$, for each $t = 1, \ldots, T$. A filtration simply corresponds to a sequence of algebras generated by successively finer partitions of Ω, see left panel of figure 1.

The tree in the right panel of figure 1 is general, allowing the branching factor to vary in each stage. To simplify notation, we work with trees having the same number of child nodes per ancestor.

To form a tree for a given set of stages, we match the moments of the sub-tree emanating from each node, and repeat the matching procedure for each non-final node. The model can be extended to match multiple sub-trees simultaneously in case there are significant inter-temporal dynamics, although it becomes large. In general, any temporal relationship between the moments, such as autocorrelation or GARCH effects, can be accounted exogenously by specifying the dynamics of the input data, Høyland and Wallace (2001).

We describe now the equations and the variables for matching a generic sub-tree. L is the number of child nodes with ancestor $a(n)$ and $\mathcal{L} = \{1, 2, \ldots, L\}$ denotes the set of indices for the scenarios in the next period. Since we focus on matching generic sub-trees, we drop the subscript t and let R_{jl} be the return of each asset $j \in \mathcal{J}$ and scenario $l \in \mathcal{L}$ for the sub-tree, and let p_l be the corresponding objective probability.

We formulate the model for most practical applications, whereby we are interested in matching up to the first four central moments of the asset return distributions and pair-wise correlations. Let μ_j, σ_j, γ_j and κ_j denote, respectively, expected return, standard deviation, skewness and kurtosis, for $j \in \mathcal{J}$, and ρ_{jk} the correlations for pairs $j, k \in \mathcal{J}$ with $j \neq k$. The arbitrage-free moment matching problem is formulated as the system of non-linear equations:

Problem 1 Arbitrage-free moment matching.

$$\sum_l p_l R_{jl} = \mu_j, \quad j \in \mathcal{J}, \tag{2}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)^2 = \sigma_j^2, \quad j \in \mathcal{J}, \tag{3}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)^3 = \gamma_j \sigma_j^3, \quad j \in \mathcal{J}, \tag{4}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)^4 = \kappa_j \sigma_j^4, \quad j \in \mathcal{J}, \tag{5}$$

$$\sum_l p_l \left(R_{jl} - \mu_j\right)\left(R_{kl} - \mu_k\right) = \rho_{jk} \sigma_j \sigma_k, \quad j, k \in \mathcal{J}, k > j, \tag{6}$$

$$\sum_l q_l R_{jl} = r, \quad j \in \mathcal{J}, \tag{7}$$

$$\sum_l p_l = 1, \tag{8}$$

$$\sum_l q_l = 1, \tag{9}$$

$$q_l > 0,\ p_l \geq 0, \quad l \in \mathcal{L}. \tag{10}$$

Problem 1 describes the matching of moments and cross-moments of the joint probability distribution to exogenously given values μ_j, σ_j, γ_j, κ_j and ρ_{jk}. Equation (7) are the no-arbitrage constraints, Pliska (1997), where, without loss of generality, we assume that r is the deterministic *risk free* rate. For stochastic risk free rate, equation (7) are modified according to Pliska (1997) as

$$\sum_l q_l \frac{R_{jl} - r_l}{1 + r_l} = 0. \tag{11}$$

This model does not make any assumptions on the probability distributions. It simply matches observed moments consistently with the no-arbitrage theory. In this sense the model is general. This is an advantage for cases where no theoretical arguments or empirical observations can justify any assumption on the underlying distributions, such as is the case for models that include both financial and economic random variables, or when business random variables are included. When something more is known, this could be incorporated in the model through additional constraints. For instance, Cochrane and Saa-Requejo (2000) argue for 'good deal' bounds for incomplete markets, and such considerations fit naturally in our model set-up. However, we point out that the computational tractability of such extensions is not necessarily the same as we demonstrate in this paper and a suitable solution method would have to be devised. An important advantage of our approach is that the model may admit more than one arbitrage-free solutions and therefore it produces a range of plausible prices instead of a point estimate. We illustrate this point in the applications section.

3. A global optimization approach

We now develop a solution method for Problem 1 based on Maranas and Floudas (1995). They employ a partitioning strategy of the interval of the variables, and convex relaxations of the non-linear terms of each equation, and we specialize this approach to the structure of Problem 1.

3.1. Variable bounds and scaling

First, we standardize the variable of the problem. In particular, denote by

$$z_{jl} = \frac{R_{jl} - \mu_j}{\sigma_j} \tag{12}$$

the standardized returns R_{jl} to obtain $R_{jl} = \mu_j + \sigma_j z_{jl}$, and substitute in equations (2)–(10).

Second, we specify bounds on each variable. This is important as the solution search proceeds through successively finer partitions of the hyper-rectangle specified by the variable bounds, and the smaller this initial range, the faster the convergence. Natural bounds are available for the variables p_l and q_l since they are probabilities and $p_l, q_l \in (0, 1]$.† Less obvious are the bounds on z_{jl} since $z_{jl} \in (-\infty, \infty)$. However, as z_{jl} denotes standardized returns we set $R_{jl} > -1$ to rule out negative prices, and the lower bound for the standardized variable is

$$\underline{z}_j = \frac{-1 - \mu_j}{\sigma_j}. \tag{13}$$

Further restrictions of the range of z_{jl} can be imposed by analysis of historical price series. In general, we bound the variable to stay within 3 to 5 standard deviations from the mean, with larger bounds being appropriate for higher kurtosis, although algorithmic efficiency deteriorates.

With these transformations, we can now develop a solution procedure using convex relaxation of *posynomial* functions (see appendix 1 for a formal definition). Such functions are characterized by strictly positive variables. The positivity of p_l and q_l is rooted in their meaning as probabilities but for z_{jl} a suitable transformation is needed. Furthermore, to enhance numerical stability of the algorithm, we scale all variables to have the same range with p_l and q_l.

†According to theory, risk neutral probabilities have to be strictly greater than zero.

The transformation of z_{jl} is given by

$$\begin{aligned} z_{jl} &= \underline{z}_j + t_{jl}(\overline{z}_j - \underline{z}_j) \\ &= \underline{z}_j + t_{jl}\Delta_j, \end{aligned} \tag{14}$$

where $0 < \underline{t}_j \leq t_{jl} \leq \overline{t}_j = 1$ and $\Delta_j = \overline{z}_j - \underline{z}_j$.‡ Substituting equation (14) in the standardized equations (2)–(7), and after some algebra, reported in appendix 2, we obtain the following model:

Problem 2 Arbitrage-free moment matching with scaled variables.

$$\sum_l p_l t_{jl} = A_j, \quad j \in \mathcal{J}, \tag{15}$$

$$\sum_l p_l t_{jl}^2 = B_j, \quad j \in \mathcal{J}, \tag{16}$$

$$\sum_l p_l t_{jl}^3 = C_j, \quad j \in \mathcal{J}, \tag{17}$$

$$\sum_l p_l t_{jl}^4 = D_j, \quad j \in \mathcal{J}, \tag{18}$$

$$\sum_l p_l t_{jl} t_{kl} = F_{jk}, \quad j, k \in \mathcal{J}, k > j, \tag{19}$$

$$\sum_l q_l t_{jl} = H_j, \quad j \in \mathcal{J}, \tag{20}$$

$$\sum_l p_l = 1, \tag{21}$$

$$\sum_l q_l = 1, \tag{22}$$

$$0 < \underline{p}_l \leq p_l \leq \overline{p}_l = 1, \quad l \in \mathcal{L}, \tag{23}$$

$$0 < \underline{q}_l \leq q_l \leq \overline{q}_l = 1, \quad l \in \mathcal{L}, \tag{24}$$

$$0 < \underline{t}_j \leq t_{jl} \leq \overline{t}_j = 1, \quad j \in \mathcal{J}, l \in \mathcal{L}. \tag{25}$$

3.2. The branch and bound algorithm

Finding all the solutions of Problem 2 is now re-formulated as a global optimization problem. Following Maranas and Floudas (1995), we index by $m \in \mathcal{M}$ the equations of the model, i.e. $\mathcal{M} = \{1, 2, \ldots, M\}$ is the index set of equations (15)–(22). We denote by $\boldsymbol{\beta}$ a vector stacking the variables p_l, q_l, t_{jl}, and by $\underline{\boldsymbol{\beta}}$, $\overline{\boldsymbol{\beta}}$, respectively, their lower and upper bounds. We also denote by $e_m(\boldsymbol{\beta})$ the difference between the value of the equation at $\boldsymbol{\beta}$ and its right-hand-side term.

Let s be a scalar slack variable. Then the following inequality constrained problem solves Problem 2 if an optimal solution $(\boldsymbol{\beta}^*, s^*)$ satisfies $s^* = 0$:

Problem 3 Inequality-constrained minimization

$$\min_{\boldsymbol{\beta}, s} s \tag{26}$$

$$\text{s.t.} \tag{27}$$

$$e_m(\boldsymbol{\beta}) - s \leq 0, \quad m \in \mathcal{M}, \tag{28}$$

$$-e_m(\boldsymbol{\beta}) - s \leq 0, \quad m \in \mathcal{M}, \tag{29}$$

$$\underline{\boldsymbol{\beta}} \leq \boldsymbol{\beta} \leq \overline{\boldsymbol{\beta}}. \tag{30}$$

‡Note that lower bounds of t_{jl}—and also of p_l, q_l—are strictly greater than zero. There is no rule to determine them and we use 1E−04.

Table 1. Percentage of no-arbitrage scenarios generated using the heuristic of Høyland *et al.* (2003). *NA* indicates that arbitrage-free scenarios are not possible because the number of scenarios does not exceed the number of assets.

Problem	Number of Scenarios 10	15	20	30	50	100
HKW-8	0%	6%	25%	63%	93%	100%
HKW-12	NA	0%	0%	11%	56%	100%
HKW-20	NA	NA	NA	0%	0%	0%
FINLIB-15	NA	NA	0%	0%	0%	12%
FINLIB-20	NA	NA	NA	0%	0%	6%

Note that a non-zero s^* means that Problem 2 is infeasible. However, since the equations involved are, in general, non-convex, a local optimization algorithm could lead to solutions which are locally optimal thus missing the global optima. Even worse, if a local minimum has a non-zero objective value, we erroneously conclude that no feasible solutions exist to the original system. Hence we need a solution method that can identify all solutions. In practice we may terminate once a zero solution is found.

Global optimization algorithms to solve non-convex problems have been widely studied. They are mainly subdivided in three classes: deterministic, stochastic and meta-heuristic. Deterministic global optimization algorithms are usually based on a branch and bound search strategy, where the *bound phase* is implemented by minimizing a *convex relaxation* of Problem 3. On each sub-rectangle $[\underline{\boldsymbol{\beta}}', \overline{\boldsymbol{\beta}}'] \subset [\underline{\boldsymbol{\beta}}, \overline{\boldsymbol{\beta}}]$, obtained in the *branching phase*, the constrained global minimum of the convex relaxed problem can be routinely found with any local optimization algorithm. Note that, since the convex relaxation is obtained by a convex underestimation of each non-convex term of the model equations, the minimum of the relaxed problem will be an underestimation of the global minimum. This implies that if the relaxed global minimum is positive, then the relative partition can be fathomed, as the slack variable s cannot be driven to zero, and therefore the moment matching model has no solution in this specific partition. On the other hand, if the relaxed global minimum is negative, then no conclusion can be drawn and the interval is further partitioned. The algorithm terminates when all the hyper-rectangles with a negative lower bound cannot be further partitioned, or, in practice, when their norm is within a given tolerance, i.e. $\|\overline{\boldsymbol{\beta}}' - \underline{\boldsymbol{\beta}}'\| \leq \epsilon_d$.

In appendix 1, we give a convex reformulation of the model that exploits its special structure to allow for efficient solutions. The reformulation is essential for the solution of the model, but not for the model itself.

4. Applications

In this section, we apply the method to generate scenarios for some problems from the literature and to price a basket option in complete and incomplete markets. We carry out experiments to assess the performance of our approach, compare to available software for scenario generation and assess the quality of the scenarios when used for security pricing.

The data-sets are taken from real problem instances and they contain asset classes ranging from cash to stock. In particular, we perform the experiments on data-sets from Høyland *et al.* (2003) and the FINLIB library of Consiglio *et al.* (2009), and label them, respectively, `HKW-X` and `FINLIB-Y`, where `X = 8,12,20` and `Y = 15,20` denote the number of assets. All experiments are carried out on a Linux machine with 2.00 GHz Xeon Quad-Core. The convexified problems are solved with GAMS/CONOPT. (Some problems were also solved with GAMS/SNOPT, and in general we found the GAMS solvers for convex optimisation robust for our test problems).

4.1. *Checking for arbitrage*

As noted earlier, neither re-sampling nor increasing the number of scenarios are foolproof methods for generating arbitrage-free scenarios. Table 1 summarizes results with the generation of no-arbitrage scenarios using the heuristic of Høyland *et al.* (2003).† For a given number of scenarios, we re-sample 100 different instances and check for arbitrage using the model of King (2002). An unbounded solution signals arbitrage. The table summarizes the success rate of producing arbitrage-free scenarios by re-sampling. When matching only the first two moments, the percentage of no-arbitrage scenarios depends on the structure of the covariance matrix; Geyer *et al.* (2014a, 2014b). Our test problems match higher moments and are therefore more complex.

Increasing the number of scenarios, as suggested by Klaassen, improves the success rate (although for problem `HKW_20` it was not possible to generate arbitrage-free scenarios). The success rate is lower for problems with more assets and the number of scenarios needed are on average more than double the number of assets. This is crucial for practical applications. As we will see in the next section, to price an option in complete or incomplete markets, we need to build scenario trees that grow exponentially with the number of time steps and scenarios. Therefore, a desirable property of arbitrage-free scenarios is to match the moments of the distribution with the minimum number of scenarios. According to theory, Pliska (1997), for complete markets the number of scenarios should be equal to the number of assets plus one, $L = J + 1$. Note that if it is possible to generate trees such that each sub-tree has a number of scenarios equal to $J + 1$, then the option price can be determined by simply discounting the final payoff under the risk neutral measure, given in our model by the probabilities q_l for $l \in \mathcal{L}$.

†The error tolerances for the heuristic ε_X and ε_Y are set to 1E−03 (default value) and to 5E−02. We choose these values after an exploratory phase, where we tried different tolerance values and picked values such that the heuristic converges for all test problems.

Table 2. Mahalanobis distances and arbitrage bounds delimiting the no-arbitrage and arbitrage regions for the case of complete markets.

Problem	Mahalanobis distance	No arbitrage region	Arbitrage region
HKW-8	1.62E+00	1.25E−01	8
HKW-12	1.73E+00	8.33E−02	12
HKW-20	1.20E+01	5.00E−02	20
FINLIB-15	1.29E+01	6.67E−02	15
FINLIB-20	1.86E+01	5.00E−02	20

Table 3. Average maximum error over 10 different trees.

Problem	μ	σ	γ	κ	ρ	r
HKW-8	1.08E−06	4.20E−06	2.84E−05	1.40E−04	4.88E−06	3.87E−07
HKW-12	1.38E−06	9.14E−06	6.45E−05	2.51E−04	5.71E−06	2.70E−07
HKW-20	7.22E−06	8.91E−06	7.86E−05	3.34E−04	1.50E−05	2.60E−07
FINLIB-15	1.41E−06	3.83E−06	5.26E−05	2.49E−04	5.56E−06	1.12E−07
FINLIB-20	2.13E−06	7.38E−06	6.00E−05	1.58E−04	8.03E−06	3.66E−07

Table 4. Standard deviation of the maximum error of 10 different trees.

Problem	μ	σ	γ	κ	ρ	r
HKW-8	8.67E−07	3.67E−06	2.34E−05	1.30E−04	7.83E−06	5.89E−07
HKW-12	1.52E−06	8.11E−06	3.06E−05	1.70E−04	9.43E−06	3.94E−07
HKW-20	9.69E−06	6.62E−06	1.71E−05	1.07E−04	1.40E−05	2.43E−07
FINLIB-15	2.49E−06	5.38E−06	3.38E−05	2.06E−04	7.04E−06	1.34E−07
FINLIB-20	2.46E−06	5.40E−06	3.37E−05	1.13E−04	6.76E−06	7.64E−07

To establish the difficulty of the test problems, and to provide a link of our work with that of Geyer *et al*. (2014a), we report in table 2 the Mahalanobis distances and the arbitrage bounds in case of market completeness ($L = J + 1$). We set $r = 0$ to obtain a conservative value of the Mahalanobis distance. Geyer *et al*. prove that, for a given variance-covariance matrix, the space of the expected excess returns can be partitioned in three region: (i) a no-arbitrage region, whose bound is given by $\frac{1}{L-1}$; (ii) a region where to rule out arbitrage opportunities, the scenario set obtained has to be checked via linear programming; (iii) the arbitrage region, whose bound is given by $L - 1$.

From table 2, we observe that all test sets have Mahalanobis distance inside region (ii), where arbitrage opportunities depend on the set of generated scenarios. Hence, all test problems are feasible, but not trivial. For test HKW-20, we see that even if the Mahalanobis distance is within region (ii), all scenarios generated by the heuristic have arbitrage. We point out that the general results of characterizing three regions is valid when matching the first two moments. Matching of higher order moments could introduce incompatibilities with the arbitrage-free constraints and that's where our model is uniquely applicable.

4.2. *Accuracy of the solution*

In this section, we show that the global optimization approach is a feasible alternative to re-sampling procedures, that, as seen above, perform poorly in terms of success rate and in terms of number of scenarios needed to guarantee absence of arbitrage. Our objective is to generate sets of scenarios with the minimum number of branches, possibly satisfying the *completeness hypothesis* whereby the number of scenarios equals the number of assets plus one. We point out that the global optimization approach is able to locate all the global minima of the problem. If the system of equations is consistent, the moment matching problem could have infinite solutions and in our experiments we terminate after 10 solutions.

In table 3, we display the average maximum error for each set of equations of the moment matching problem. That is, for each set of equations corresponding to the moment to be matched, we record the maximum error obtained over the set of assets, where error is the difference between the value of the expression on the left-hand-side of the equation and the parameter on the right-hand-side.† This value is then averaged over the 10 generated trees. For example, column ρ displays the maximum mismatch over all the equations describing correlations, averaged over the 10 solutions. The average maximum error is negligible and the solutions found match closely the empirical moments. Moreover, the trees generated have the minimum number of scenarios required to exclude arbitrage.

To confirm robustness of the solution algorithm, we display in table 4 the standard deviations of the maximum error for

†The symbols of the columns correspond to the equations of Problem 1. For readability, we omit the errors for the normalization constraints for p_l and q_l, which are in the range 3.26E−6 to 9.33E−7.

Table 5. Solution times for one (T_1) and ten (T_{10}) solutions in h:min:s.

	T_1	T_{10}
HKW-8	0:00:09	0:01:41
HKW-12	0:02:00	0:15:02
HKW-20	0:43:41	4:50:14
FINLIB-15	0:00:53	1:08:57
FINLIB-20	0:04:58	2:51:31

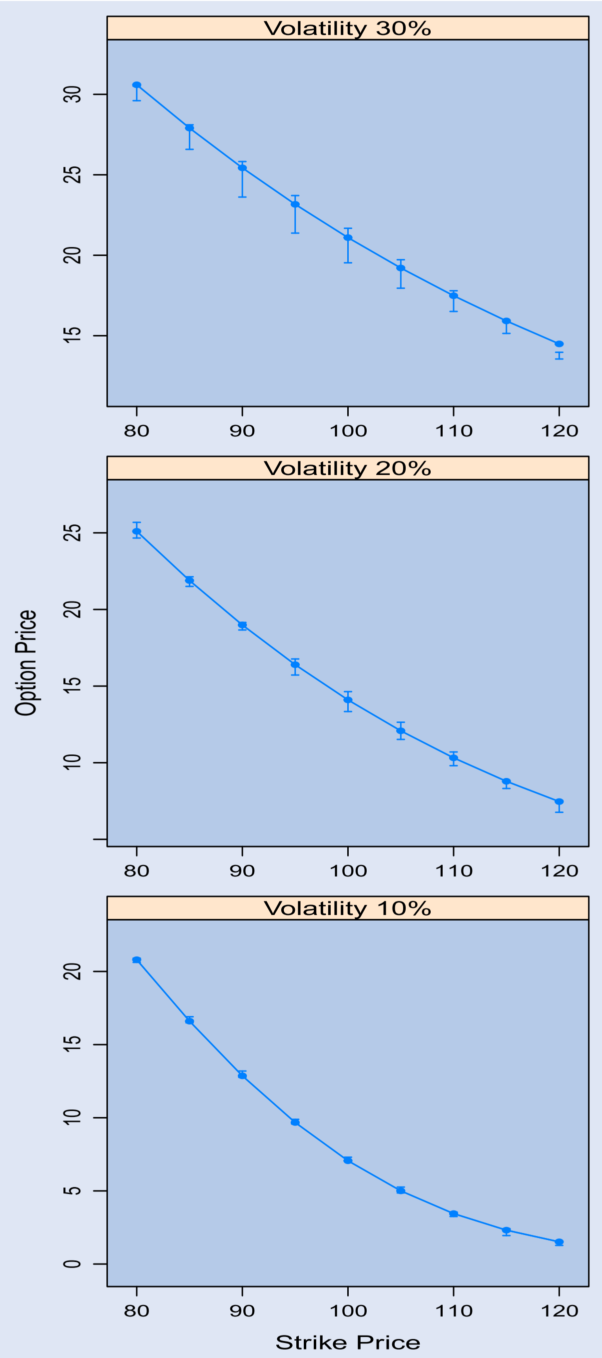

Figure 2. The range of prices obtained from 20 calibrated trees for different strike prices and volatilities; the Montecarlo estimate under Gaussian assumptions is indicated by a bullet.

each moment matched over the 10 different trees. In table 5, we report the computational times to find one and ten solutions. We observe that these models are solved with modest computational resources. Solution times increase with the number of assets due to the exponential nature of the branch and bound algorithm, and solving multiple instances of a model may require substantially different computational times due to the complexity of global optimization models.

4.3. Options pricing applications

We showed that it is possible to build trees with a small number of scenarios to match with high accuracy a given set of moments and obtain the corresponding objective and risk neutral measures. Our work was motivated by applications in risk management, especially when using multiperiod optimization models. Now we illustrate two additional applications with options pricing in complete and incomplete markets. We will see that high-quality solutions are obtained even with the very small number of scenarios we use.

4.3.1. Complete markets. We start by assessing the quality of our trees with respect to a financial problem whose solution, under some assumptions, can be obtained with other methods. We consider the pricing of a European basket option, written on $J = 4$ equally weighted assets with maturity $T = 5$ years and correlation among the underlying assets 0.5. The objective of this experiment is to assess the quality of our scenario trees. For evaluation of basket options under normality assumptions, more efficient methods are available; things become more complex if we need to take into account stylized facts, such as fat-tails or skewness of the distributions of the underlying assets. We compute the price of a basket option assuming normality using Monte Carlo simulation with 1,000,000 scenarios drawn from the Gaussian risk-neutral distribution.

For our method, the Black–Scholes (BS) hypotheses is satisfied by generating scenarios with drift $\mu = r$, (without loss of generality we set the risk free $r = 0$), standard deviations $\sigma = 10\%, 20\%, 30\%$ per year, skewness $\gamma = 0$, kurtosis $\kappa = 3$, and correlation among the four assets $\rho = 0.5$. Note that $\gamma = 0$ and $\kappa = 3$ satisfy the Gaussian hypothesis of the BS model. Matching only these two moments does not ensure that the distributions of the asset returns is normal (in theory, infinite moments of the Gaussian should be matched). For market completeness we set $L = J + 1$.

The five-year tree is constructed sequentially, that is, at each branching node a one-year subtree is generated using our optimization procedure. For consistency with BS, we assume that the input moments of the conditional distributions are time independent. Therefore, the five-year tree is obtained by generating a single one-year subtree and replicating it to each branching node.

Table 6. Montecarlo price MC of the basket option obtained by drawing 1 000 000 scenarios from the risk neutral distribution, average price 'Opt.' using twenty 5-year trees, and mean absolute error between the two values.

	$\sigma = 10\%$			$\sigma = 20\%$			$\sigma = 30\%$		
	MC	Opt.	Error (%)	MC	Opt.	Error (%)	MC	Opt.	Error (%)
80	20.80	20.81	0.49	25.10	25.17	1.69	30.59	30.36	0.76
85	16.60	16.74	1.04	21.89	21.84	0.97	27.90	27.77	0.79
90	12.87	12.98	1.38	18.99	18.89	0.89	25.43	25.51	1.74
95	9.68	9.72	0.91	16.40	16.43	1.93	23.17	23.45	2.79
100	7.06	7.21	2.16	14.10	14.20	3.06	21.10	21.45	3.17
105	5.00	5.12	2.83	12.08	12.15	3.62	19.21	19.49	2.75
110	3.45	3.37	3.40	10.32	10.29	2.81	17.49	17.55	1.46
115	2.32	2.15	7.99	8.79	8.59	2.24	15.92	15.62	1.84
120	1.52	1.43	7.50	7.46	7.07	5.61	14.49	13.73	5.29

Table 7. Descriptive statistics of the price of a basket option in incomplete markets over a sample of 20 trees.

Strike Price	80	90	100	110	120
Buyer side					
Mean	16.44	9.60	4.87	2.25	0.80
St. dev.	1.085	0.815	0.698	0.447	0.218
Min	14.78	8.27	3.95	1.39	0.50
Max	19.16	11.45	6.53	3.13	1.30
Writer side					
Mean	24.48	15.97	9.29	4.96	2.03
St. dev.	0.800	0.746	0.528	0.461	0.189
Min	22.82	14.43	8.15	4.04	1.67
Max	24.93	16.46	9.75	5.38	2.34

Following the notation in section 2, we denote by $\mathcal{N}_T$ the set of nodes at the final period that coincides with the maturity of the option. There is a unique path from the root node to the final nodes, and we denote by $\mathcal{H}(n)$ the index set of nodes which belong to the path leading from the root of the tree to the final nodes $n \in \mathcal{N}_T$.

For each final node n, we compute the price of each asset $j \in \mathcal{J}$ as follows:

$$P_n^j = P_0^j \prod_{m \in \mathcal{H}(n)} \left(1 + R_m^j\right), \tag{31}$$

where R_m^j is the return of the asset j at node m (with $R_0^j = 0$), and P_0^j is the price of the asset j at the root node $m = 0$ (we set $P_0^j = 100$ for each $j \in \mathcal{J}$). We compute in a similar way the risk neutral probabilities of each final node n,

$$q_n^* = \prod_{m \in \mathcal{H}(n)} q_m, \tag{32}$$

where q_m is the risk neutral probability of each node $m \in \mathcal{H}(n)$ and $q_0 = 1$.

The price of the option is now obtained as the present value of the expected value of the final payoff under risk-neutral probabilities q_n^*, $n \in \mathcal{N}_T$,

$$C = e^{-r} \sum_{n \in \mathcal{N}_T} q_n^* \max(P_n^B - K, 0), \tag{33}$$

where $P_n^B = \sum_{j \in \mathcal{J}} w_j P_n^j$. We remark that due to the assumed market completeness, it is possible to price the option by simple discounting of expected value under q^*.

We fit scenario trees for volatilities $\sigma = 10\%, 20\%, 30\%$ and price the basket option for strike prices ranging from 80 to 120 (recall that the initial price of each asset $P_0^j = 100$). We generate 20 solutions of the non-convex optimization problem, and also use Monte Carlo simulation with 1,000,000 scenarios to estimate the 'true' price under normality assumption. The results are illustrated in figure 2 where we show the range of prices obtained by the calibrated trees and the Monte Carlo estimate. In all cases—except for deep in-the-money and deep out-of-the-money options under the extreme volatility scenarios—the range of prices obtained from the scenario tree brackets the Monte Carlo estimate, and the range is small.

In table 6, we compute the average price over the 20 trees obtained by equation (33), and report the mean absolute percentage difference between the average price from the trees and the Montecarlo price estimate.

Note that for in-the-money and at-the-money options (strike prices $\leq$ 105) the option price error is fairly small. The quality of the solution deteriorates somewhat for out-of-the-money and low volatility. On average, the option price is fairly close to the Monte Carlo price. However, the coarse granularity of the tree (only $5^5 = 3125$ scenarios vs. 1,000,000 of the Monte Carlo) yields option prices very close to zero in some tree instances.

As already pointed out, there are more accurate methods to price basket options under the BS hypotheses, and the value of our method is not so much the accuracy of the average price it computes, but the fact that it generates a range of prices that bracket the Monte Carlo price. That is, if normality holds, then our method approximates accurately the Monte Carlo estimate even with very few (3125) scenarios. But in the absence of any distributional information and with only the empirically observed moments to go by, the range of prices is

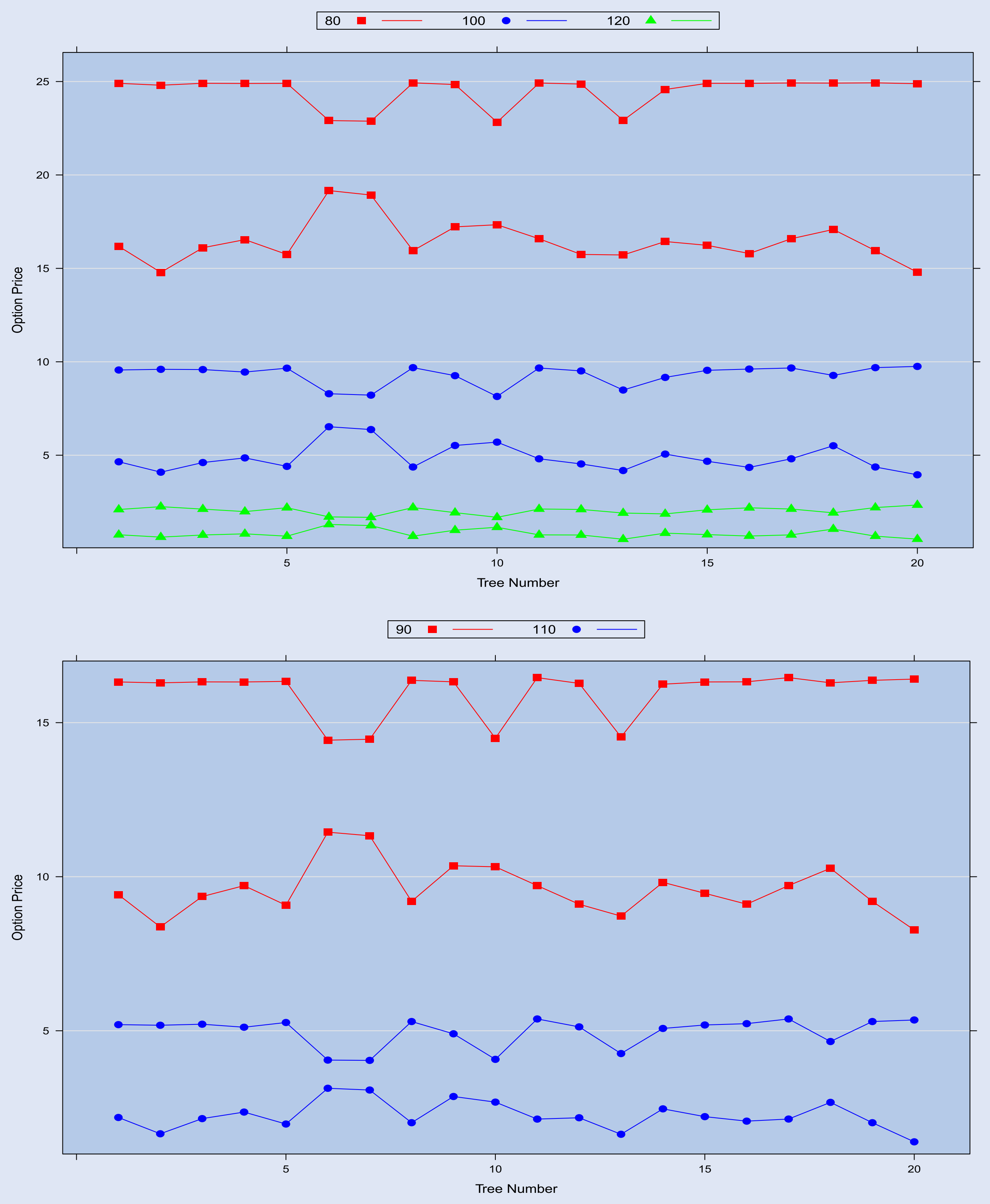

Figure 3. Bid/ask prices of the basket option in an incomplete market for different strike prices and trees and volatility of assets $\sigma = 20\%$

a better indicator. Furthermore, the methodology can be applied to pricing more complex instruments, path-dependent options and in incomplete markets, which we consider next.

4.3.2. Incomplete markets. The representation through trees of the underlying stochastic process is particularly suitable to price option in incomplete markets, see, e.g. Dembo, Rosen, *et al.* (2000). Market incompleteness arises when the number of risky factors is greater than the available securities to hedge them. This is simulated in our experiments by assuming a non-traded underlying asset.

In case of incompleteness, the martingale measures are infinite, and, therefore, there are infinite prices of the option under scrutiny, lying between a lower (bid price) and an upper (ask price) bound. Such bounds can be determined by appropriately modelling of the hedging process. In our experiment, we adopt a super-replication strategy, where at each node $n \in \mathcal{N}_t$, $t = 0, 1, \ldots, T-1$, the portfolio of assets is self-financing, and at each final node, $n \in \mathcal{N}_T$, the hedging portfolio super-replicates the option payoff. Such a strategy is equivalent to solving a linear stochastic programming model (one for the buyer and one for the writer), where the value of the portfolio at the root node is the option price, King (2002). In this context, a reference value to serve as the 'true' value of the option is not available, and we assess the quality of the scenario trees through sensitivity analysis of the option price.

We calibrated multi-period scenario trees using our method and formulated the linear stochastic programming model of King (2002) on the calibrated tree. In figure 3 we display the ask and bid prices of the basket option, with volatility $\sigma = 20\%$, and strike prices ranging from 80 to 120. More detailed statistics are reported in table 7. The low standard deviation over the sample of 20 trees is evidence that the scenario generation method we propose is stable in incomplete markets too; the trees yield very similar option prices.

5. Conclusions

The generation of arbitrage-free scenario trees that match the moments of a set of risk factors is a prevalent problem in risk management and in pricing financial instruments, especially in incomplete markets and for enterprise-wide risk management. We proposed a model that casts this problem as a global optimization problem whose solution is zero if a solution to the original problem exists. Exploiting the special structure of the model with linear relaxations of a convex reformulation, we have shown that the method is robust and computationally tractable. Experiments highlighted the efficacy of the methodology in pricing synthetic options in complete and incomplete markets. The result is a general purpose parsimonious methodology that can generate theoretically correct and accurate scenario trees with a minimal number of scenarios and no more input requirements than the available moments. It makes no distributional assumptions, but if the underlying distribution is known then it can be approximated by matching an arbitrary number of moments.

Disclosure statement

No potential conflict of interest was reported by the authors.

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References

Carinõ, D.R. and Ziemba, W.T., Formulation of the Russel–Yasuda Kasai financial planning model. *Oper. Res.*, 1998, **46**(4), 433–449.

Chen, Z. and Xu, D., Knowledge-based scenario tree generation methods and application in multiperiod portfolio selection problem. *Appl. Stoch. Models Bus. Ind.*, 2014, **30**, 240–257.

Cherubini, U., Luciano, E. and Vecchiato, W., *Copula Methods in Finance*, 2004 (Wiley: Chichester).

Cochrane, J.H. and Saa-Requejo, J., Beyond arbitrage: Good-deal asset price bounds in incomplete markets. *J. Polit. Econ.*, 2000, **108**(1), 79–119.

Consigli, G., Iaquinta, G. and Moriggia, V., Path-dependent scenario trees for multistage stochastic programs in finance. *Quant. Finance*, 2010, **12**(8), 1265–1281.

Consiglio, A. and De Giovanni, D., Evaluation of insurance products with guarantee in incomplete markets. *Insur.: Math. Econ.*, 2008, **42**(1), 332–342.

Consiglio, A., Nielsen, S. and Zenios, S.A., *Practical Financial Optimization: A Library of GAMS Models*, 2009 (Wiley: Chichester). Available online at: https://www.gams.com/finlib/libhtml/ (accessed 17 November 2015).

Consiglio, A., Saunders, D. and Zenios, S.A., Asset and liability management for insurance products with minimum guarantees: The UK case. *J. Bank. Finance*, 2006, **30**, 645–667.

Date, P., Mamon, R. and Jalen, L., A new moment matching algorithm for sampling from partially specified symmetric distributions. *Oper. Res. Lett.*, 2008, **36**, 669–672.

Dembo, R., Aziz, A., Rosen, D. and Zerbs, M., *Mark to Future—A Framework for Measuring Risk and Reward*, 2000 (Algorithmics Publications: Toronto).

Dembo, R., Rosen, D. and Saunders, D., Valuation in incomplete markets: An optimization approach. *Algo Res. Q.*, 2000, **3**(2), 29–37.

Dupačová, J., Consigli, G. and Wallace, S.W., Scenarios for multistage stochastic programs. *Ann. Oper. Res.*, 2000, **100**, 25–53.

Dupačová, J., Gröewe-Kuska, N. and Römish, W., Scenario reduction in stochastic programming. *Math. Program.*, 2003, **95**(3), 493–511.

Floudas, C. and Gounaris, C., A review of recent advances in global optimization. *J. Global Optim.*, 2009, **45**(1), 3–38.

Geyer, A., Hanke, M. and Weissensteiner, A., No-arbitrage conditions, scenario trees, and multi-asset financial optimization. *Eur. J. Oper. Res.*, 2010, **206**(3), 609–613.

Geyer, A., Hanke, M. and Weissensteiner, A., No-arbitrage bounds for financial scenarios. *Eur. J. Oper. Res.*, 2014a, **236**(2), 657–663.

Geyer, A., Hanke, M. and Weissensteiner, A., No-arbitrage ROM simulation. *J. Econ. Dyn. Control*, 2014b, **45**, 66–79.

Glasserman, P., *Monte Carlo Methods in Financial Engineering*, 2004 (Springer-Verlag: New York).

Hochreiter, R. and Pflug, G.C., Financial scenario generation for stochastic multi-stage decision processes as facility location problems. *Ann. Oper. Res.*, 2007, **152**(1), 257–272.

Høyland, K., Kaut, M. and Wallace, S.W., A heuristic for moment matching scenario generation. *Comput. Optim. Appl.*, 2003, **24**, 169–185.

Høyland, K. and Wallace, S.W., Generation scenario trees for multistage decision problems. *Manage. Sci.*, 2001, **47**(2), 295–307.

Jamshidian, F. and Zhu, Y., Scenario simulation: Theory and methodology. *Finance Stoch.*, 1997, **1**(1), 43–67.

Kaut, M. and Wallace, S.W., Evaluation of scenario generation methods for stochastic programming. *Pac. J. Optim.*, 2007, **3**, 257–271.

King, A.J., Duality and martingales: A stochastic programming perspective on contingent claims. *Math. Program., Ser. B*, 2002, **91**, 543–562.

Klaassen, P., Comment on "Generating scenario trees for multistage decision problems". *Manage. Sci.*, 2002, **48**(11), 1512–1516.

Lu, H., Li, H., Gounaris, C. and Floudas, C., Convex relaxation for solving posynomial programs. *J. Global Optim.*, 2010, **46**(1), 147–154.

Lundell, A., Westerlund, J. and Westerlund, T., Some transformation techniques with applications in global optimization. *J. Global Optim.*, 2009, **43**, 391–405.

Maranas, C. and Floudas, C., Finding all solutions of nonlinearly constrained systems of equations. *J. Global Optim.*, 1995, **7**(2), 143–182.

Maranas, C. and Floudas, C., Global optimization in generalized geometric programming. *Comput. Chem. Eng.*, 1997, **21**, 351–569.
Mulvey, J.M. and Vladimirou, H., Stochastic network programming for financial planning problems. *Manage. Sci.*, 1992, **38**(11), 1642–1664.
Pliska, S.R., *Introduction to Mathematical Finance: Discrete Time Models*, 1997 (Blackwell: Malden, MA).
Rebonato, R., Mahal, S., Joshi, M., Buchholz, L. and Nyholm, K., Evolving yield curves in the real-world measures: A semi-parametric approach. *J. Risk*, 2005, **7**(3), 29–61.
Topaloglou, N., Vladimirou, H. and Zenios, S.A., Pricing options on scenario trees. *J. Bank. Finance*, 2008, **32**(2), 283–298.
Tsai, J. and Lin, M., Finding all solutions of systems of nonlinear equations with free variables. *Eng. Optim.*, 2007, **39**(6), 649–659.
Vandekerckhove, J., Matzke, D. and Wagenmakers, E.-J., Model comparison and the principle of parsimony. In *Oxford Handbook of Computational and Mathematical Psychology*, edited by J. Busemeyer, J. Townsend, Z.J. Wang and A. Eidels, pp. 300–317, 2015 (Oxford University Press: Oxford).
Xu, D., Chen, Z. and Yang, L., Scenario tree generation approaches using k-means and lp moment matching methods. *J. Comput. Appl. Math.*, 2012, **236**, 4561–4579.
Zenios, S.A., *Practical Financial Optimization: Decision making for Financial Engineers*, 2007 (Blackwell: Malden, MA).
Zenios, S.A., Holmer, M.R., McKendall, R. and Vassiadou-Zeniou, C., Dynamic models for fixed-income portfolio management under uncertainty. *J. Econ. Dyn. Control*, 1998, **22**, 1517–1541.

Appendix 1. A convex reformulation of the model

Once Problem 2 is transformed into the equivalent constrained minimization Problem 3, we are faced with a set of non-linear inequalities of the general form

$$\sum_k \prod_i x_{ki}^{\alpha_{ki}} - C - s \leq 0, \tag{A1}$$

$$-\sum_k \prod_i x_{ki}^{\alpha_{ki}} + C - s \leq 0, \tag{A2}$$

where each term of the summation is a *posynomial* and C is a constant. In particular, a posynomial function is defined as

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i}, \tag{A3}$$

where $\alpha_i \in \mathbb{R}$ and $0 < \underline{x}_i \leq x_i \leq \overline{x}_i$, for each $i = 1, 2, \ldots, m$.

Observe that the non-linear terms in equations (15)–(25) are posynomials, and this is exploited in the convex relaxation phase of the solution algorithm. Convexification of posynomial functions is carried out through the variable transformation $x \to f(y)$, where $f(y) : \mathbb{R} \to \mathbb{R}$ is a suitable mapping carrying the one-to-one relation between the original variable x and the transformed variable y. For example, Maranas and Floudas (1997) use an exponential transformation, $x \to e^y$; Tsai and Lin (2007) use $x \to y^{-1}$, which is a special case of the power transformation $x \to y^{\beta}$. Note that, in general, not all variables need to be transformed. For example, equations (21)–(22) are linear in p and q.

The inverse non-linear transformation $y \to f^{-1}(x)$ has to be included into the transformed problem, thus moving the non-convexities from the original constraints to the inverse equality constraints. (For example, by using the power transformation, the non-convex definitional equality $y = x^{1/\beta}$ has to be associated with each transformed variable). In this respect, Lundell *et al.* (2009) approximate the inverse linear transformation constraint through a piecewise linear function, thus turning the model to a mixed-integer non-linear programme, while Lu *et al.* (2010) adopt an *ad-hoc* linear relaxation of the bilinear equation associated to each transformed variable. Since tree generation problems are characterized by a medium to high level of dimensionality, we adopt the approach of Lu *et al.* that does not require integer variables that complicate the model. (For example, with $J = 4$ assets and $L = 5$ scenarios we obtain a non-linear programme with 30 variables and 37 constraints. In general, the number of variables are $2L + JL$, and the number of equations (excluding the box constraints) $2 + 5J + (J^2 - J)/2$.)

The starting point of the analysis in Lu *et al.* is based on the following proposition:

Proposition 1 *A twice differentiable function*

$$f(x_1, x_2, \ldots, x_n) = \prod_{i=1}^{n} x_i^{\alpha_i} \tag{A4}$$

is convex if $\alpha_i < 0$ *and* $x_i > 0$, *for each* $i = 1, 2, \ldots, n$.

Let $\mathcal{K} = \{i \mid \alpha_i < 0, i = 1, 2, \ldots, n\}$ and $\overline{\mathcal{K}} = \{i \mid \alpha_i > 0, i = 1, 2, \ldots, n\}$. A convex reformulation of the posynomial function (A3) is given by

$$f\left(\{x_i\}_{i \in \mathcal{K}}, \{y_i\}_{i \in \overline{\mathcal{K}}}\right) = \prod_{i \in \mathcal{K}} x_i^{\alpha_i} \prod_{i \in \overline{\mathcal{K}}} y_i^{-\frac{\alpha_i}{\delta_i}}, \tag{A5}$$

where, $y_i = x_i^{-\delta_i}$ and $0 < \delta_i \leq 1$, for each $i \in \overline{\mathcal{K}}$.

For those variables x_i, $i \in \overline{\mathcal{K}}$, that appears in non-convex terms, it is necessary to relax the definitional equation $y_i = x_i^{-\delta_i}$. Lu *et al.* show that such relaxation is given by the following linear inequality, for each $i \in \overline{\mathcal{K}}$:

$$y_i \leq \underline{x}_i^{-\delta_i} + \frac{\overline{x}_i^{-\delta_i} - \underline{x}_i^{-\delta_i}}{\overline{x}_i - \underline{x}_i}\left(x_i - \underline{x}_i\right). \tag{A6}$$

The parameters δ_i play an important role in the relaxation. The smaller the δ_i the tighter the convex relaxation, but it cannot be chosen arbitrarily close to zero and has to be determined as a function of the computer accuracy and bounds $\overline{x}_i, \underline{x}_i$; see Lu *et al.* (2010), section 3.

To convexify Problem 2, we need to transform each variable p_l, q_l and t_{jl} since their exponents are all positive. Accordingly, we set:

$$\pi_l = p_l^{-\xi_l} \quad \text{and} \quad 0 < \xi_l \leq 1, \tag{A7}$$

$$\chi_l = q_l^{-\upsilon_l} \quad \text{and} \quad 0 < \upsilon_l \leq 1, \tag{A8}$$

$$\tau_{jl} = t_{jl}^{-\delta_{jl}} \quad \text{and} \quad 0 < \delta_{jl} \leq 1. \tag{A9}$$

Remark 1 The variables p_l and q_l are the only ones to be relaxed, and this is because such variables appear in the linear equations (21)–(22). So we have to add to the convexified problem the following inequalities, for each $l \in \mathcal{L}$,

$$\pi_l \leq \underline{p}_l^{-\xi_l} + \frac{\overline{p}_l^{-\xi_l} - \underline{p}_l^{-\xi_l}}{\overline{p}_l - \underline{p}_l}\left(p_l - \underline{p}_l\right), \tag{A10}$$

$$\chi_l \leq \underline{q}_l^{-\upsilon_l} + \frac{\overline{q}_l^{-\upsilon_l} - \underline{q}_l^{-\upsilon_l}}{\overline{q}_l - \underline{q}_l}\left(q_l - \underline{q}_l\right). \tag{A11}$$

Remark 2 The inequality constraints of Problem 3 are of two types: (28) is the sum of positive terms, also known as *posynomial*; (29) is the sum of negative terms, also known as *signomial*. The transformation $y_i = x_i^{-\delta_i}$ turns the posynomials to a convex function, and, similarly, a signomial is turned to a concave function. We relax each concave term by underestimating it through an affine function $T(y_1, y_2, \ldots, y_n)$. For example, the concave inequalities correspondent to the transformation of equation (15) are given by

$$-\sum_l \pi_l^{-1/\xi_l} \tau_{jl}^{-1/\delta_{jl}} + A_j - s \leq 0, \quad \text{for each } j \in \mathcal{J}. \tag{A12}$$

Each concave term $G_{jl}(\pi_l, \tau_{jl}) = -\pi_l^{-1/\xi_l} \tau_{jl}^{-1/\delta_{jl}}$ is underestimated by means of the affine function $T_{jl}(\pi_l, \tau_{jl}) = a_{jl}\,\pi_l + b_{jl}\,\tau_{jl} + c_{jl}$. The coefficients a_{jl}, b_{jl}, and c_{jl} are chosen in such a way that the affine function, $T_{jl}(\pi_l, \tau_{jl})$, is equal to the concave function $G_{jl}(\pi_l, \tau_{jl})$ at the vertex of the rectangular domain identified by the upper and lower bounds of variables π_l and τ_{jl}.

Appendix 2. The model with scaled variables

The derivation of equations (15)–(20) needs some tedious algebra especially for higher moments. We derive here only equations (15)–(16) and for the remaining parameters we provide the relations without the derivation, which follows along the lines described here. By substituting the scaling relation for z_{jl} in the standardized equation (2), we obtain:

$$\sum_l p_l(\underline{z}_j + t_{jl}\Delta_j) = \underline{z}_j + \Delta_j \sum_l p_l t_{jl} = 0.$$

By isolating the summation of the last relation, we obtain:

$$\sum_l p_l t_{jl} = -\frac{\underline{z}_j}{\Delta_j} = A_j.$$

We proceed similarly for the standardized equation (3) to obtain:

$$\begin{aligned}
&\sum_l p_l(\underline{z}_j + t_{jl}\Delta_j)^2 \\
&\quad = \sum_l p_l\left[\underline{z}_j^2 + t_{jl}^2\Delta_j^2 + 2\underline{z}_j t_{jl}\Delta_j\right] \\
&\quad = \underline{z}_j^2 + \Delta_j^2 \sum_l p_l t_{jl}^2 + 2\underline{z}_j\,\Delta_j \sum_l p_l t_{jl} \\
&\quad = \underline{z}_j^2 + \Delta_j^2 \sum_l p_l t_{jl}^2 + 2\underline{z}_j\,\Delta_j \left(-\frac{\underline{z}_j}{\Delta_j}\right) \\
&\quad = \underline{z}_j^2 + \Delta_j^2 \sum_l p_l t_{jl}^2 - 2\underline{z}_j^2 \\
&\quad = \Delta_z^2 \sum_l p_l t_{jl}^2 - \underline{z}_j^2 = 1.
\end{aligned}$$

Isolating the summation of the last relation, we obtain:

$$\sum_l p_l t_{jl}^2 = \frac{1 + \underline{z}_j^2}{\Delta_j^2} = B_j.$$

It is possible to verify that:

$$C_j = \frac{\gamma_j - \underline{z}_j\left[\underline{z}_j^2 + 3\right]}{\Delta_j^3}, \quad D_j = \frac{\kappa_j - \underline{z}_j\left[4\gamma_j - \underline{z}_j^3 - 6\underline{z}_j\right]}{\Delta_j^4},$$

$$F_{jh} = \frac{\rho_{jh} + \underline{z}_j\,\underline{z}_h}{\Delta_j\,\Delta_h}, \qquad H_j = \frac{r - \mu_j - \underline{z}_j\,\sigma_j}{\sigma_j\,\Delta_j}.$$