UNIVERSITÀ DEGLI STUDI DI PALERMO
Dottorato in *Fisica applicata*
Indirizzo *Storia e didattica delle matematiche, della fisica e della chimica*
Dipartimento di Fisica e Chimica
SSD: MAT/04

TEACHING REAL NUMBERS IN THE HIGH SCHOOL: AN ONTO-SEMIOTIC APPROACH TO THE INVESTIGATION AND EVALUATION OF THE TEACHERS' DECLARED CHOICES

DOTTORANDO
LAURA BRANCHETTI

COORDINATORE/REFERENTE
Prof. AURELIO AGLIOLO GALLITTO

TUTOR
Prof. CLAUDIO FAZIO

CO TUTOR
Prof. GIORGIO BOLONDI

CICLO XXV – 2016

Contents

Appendices

Acknowledgments

This thesis would not have been possible without the collaboration and the suggestions of Giorgio Bolondi and Bruno D'Amore. I especially wish to thank them for their patience. I want to thank, as well, all the teachers who collaborated in this research and all the Professor, Researchers and PhD students who discussed with me about my work. In particular I thank Olivia Levrini and her "Quantum teachers" research group and Giulia Tasquier, she knows why.
Last, but not least, I owe special thanks and a touched and affectionate thought to Mustafa Alpaslan and Zişan Güner Alpaslan, the Angels of Mathematics education young researchers.

Abstract

This thesis addresses the topics of investigating teachers' declared choices of practices concerning real numbers and the continuum in the high school in Italy, evaluating their didactical suitability and the impact of a deep reflection about some historical and didactical issues on the teachers' decision-making process. Our research hypothesis was that teachers' choices of teaching sequences concerning real numbers, with particular attention to the representations of real numbers, could be very relevant in order to interpret some of the well-known students' difficulties. After a pilot study in form of a teaching experiment and a literature review concerning students' and teachers' difficulties with real numbers and the continuum, we observed that some causes of potential difficulties could be situated indeed in the very beginning of the teaching-learning process, even before entering the classrooms: the phase in which a teacher choose the practices and objects by means of whom introducing and work with real numbers and the continuum. In particular the choice of the objects involved in the practice seemed to be relevant, since every object emerge from previous practices and its meaning is identified by the practices in which it emerged. Thus we got interested in investigating the personal factors that affect the process of selection of practices: personal meaning, goals and orientations, as it was stressed by Schoenfeld in his goal-oriented decision-making approach to the analysis of teachers choices. Furthermore we decided to explore the teachers' choices of sequences of practices and of representation of the mathematical objects and then to evaluate their suitability in relation to the literature review concerning students' difficulties with real numbers and to the complexity of the mathematical object as it emerge from an historical analysis. After having analysed the theoretical frameworks in mathematics education that could permit us to carry out our research, we decided to use the OSA, (onto-semiotic approach) elaborated by Godino, Batanero & Font, described in their paper in 2007, and some evolutions like the CDM (*Conoscimiento didactico matematico*) model proposed by Godino in 2009. We evaluated also other frameworks, in particular the ATD (Chevallard, 2014), but we found the OSA better for the analysis we would like to carry out. In particular the operationalization of the methodologies of analysis of the teachers' personal meaning of mathematical objects and the construct of didactical suitability were more effective for our porpouses. Our main results are the following: mny teachers' personal meanings of real numbers are far from the epistemic one; many of the teachers who studied real numbers at a formal level at school and at the University and perceived them as difficult and not useful try to avoid to deepen the issues concerning real numbers with their student, thinking they would not understand; in general the experiences as students affect the teachers' choices; the teachers usually refer to real numbers also when the meaning is partial and doesn't coincide with one of the most general epistemic meanings of real numbers; very few teachers are aware of the complexity of the real numbers and are as aware of it to be able to control the relations between their many facets; also the teachers with a PhD in Mathematics operate choices that we can evaluate as unsuitable standing on the literature review and our framework; the teacher consider very hard to work with discrete and dense sets and prefer the intuitive approach to continuous sets rather than deepen the relation between dense and continuous sets, different degrees of infinity and so on; some teachers reasoning during the interviews changed their mind, getting aware of the complexity and admitting that simplifying too much can constitute a further cause of difficulty; the teachers refer to the students difficulties to justify their choice of simplifying, but when they face some crucial issues, often they admit to consider them not useful or too difficult; nevertheless no teachers declare that would renounce to introduce the field of real numbers, at least intuitively; the most of the teachers declare that nothing more is introduced about real numbers in the last years and that the partial meanings introduced in the first years are used to face the Calculus problems (intuitive approach to the Calculus); all the teachers consider necessary to introduce R or adequate subsets of R as domains of the functions expressed analytically because of their continuous graphic

Introduction

The thesis address the topic of evaluating the didactical suitability of teachers' choices of practices concerning real numbers and the continuum in the high school in Italy. Our research hypothesis was that teachers' choices of teaching sequences concerning real numbers and of representations of real numbers could be very relevant in order to interpret the students' difficulties. After a pilot study in form of a teaching experiment and a literature review concerning students' and teachers' difficulties with real numbers and the continuum, we observed that some causes of potential difficulties could be situated in the very beginning of the teaching-learning process: the choice of practices and objects by means of whom introducing and work with real numbers and the continuum. In particular the choice of the objects involved in the practice seemed to be relevant, since every object emerge from previous practices and its meaning is identified by the practices in which it emerged. Thus we decided to explore the teachers' choices of sequences of practices and of representation of the mathematical objects and then to evaluate their suitability in relation to the literature review concerning students' difficulties with real numbers and to the complexity of the mathematical object as it emerge from an historical analysis.

After having analyzed the theoretical frameworks in mathematics education that could permit us to carry out the analysis of practices and to differentiate the meaning of objects involved in the different didactical practices at school, as long as the different interpretations of the signs produced and used by teachers and students ay school, we decided to use the OSA, onto-semiotic approach (Godino, Batanero & Font, 2007). We evaluated also other frameworks, in particular the ATD (Chevallard, 2014), but we found the OSA better for the analysis we would like to carry out. In particular the attention to the personal meaning of mathematical objects and the construct of didactical suitability were more effective for our purposes. Also we choose the Schoenfeld's goal-oriented decision-making approach to the study of teachers' choices (2010). We present briefly our framework and then we resume some results as example of the analysis we carried out. Theoretical framework As theoretical framework for teachers' choice we chose the goal-oriented decision-making theory by Schoenfeld (2010).

Drawing on this theoretical framework we will consider teachers as decision-makers, whose choices are determined by "their resources (their knowledge, in the context of available material and other resources), goals (the conscious or unconscious aims they are trying to achieve), and orientations (their beliefs, values, biases, dispositions, etc.) [...] at both macro and micro levels." (Schoenfeld, 2010, p. 14). The choices are embedded in the institutional context, as explained above. In order to rebuild the knowledge to make it teachable, meaningful and useful making transpositional choices teachers, more or less consciously, decompose mathematical objects' structure and select aspects to omit and to diffuse, how to transform and finally how to rebuild the aspects of the mathematical object they choose to teach. Chevallard (1999) proposed the quatern (task, tecnique, technologies, theories) but in this paper we opt for an onto-semiotic approach (OSA) to mathematics teaching and learning that let as stress the personal dimension of knowledge, not necessary identifiable with the institutional one (D'Amore & Godino, 2006).

We will frame the intuitive concept of "lack of knowledge" presented in the introduction in the OSA defining a lack of knowledge a gap between teachers' personal meaning of O and the epistemic meaning. OSA also include the notion of didactical suitability (Godino, Wilhelmi & Bencomo, 2005), a complex construct that we will use only partially.

We formulated two general research questions:

GQ - 1 How can we describe the complexity of the teaching-learning processes involving real numbers and continuous sets from an epistemic and cognitive point of view?

GQ - 2 Are the teachers' choices epistemically and cognitively suitable ?

We carried out parallel researches: we explored the practices concerning the continuum and real numbers in the history of Mathematics in order to delineate the complex meaning of the continuum and the real numbers; we analysed some textbooks and manuals; we analysed the literature in mathematics education concerning the students' and teachers' conceptions and difficulties with the continuum and the real numbers; finally we interviewed 116 in-service teachers by means of a questionnaire, interviews in focus groups and individual interviews. The product of the first analysis is the epistemic meaning of real numbers; the results of the second analysis is a list of "standard" practices and traditional teaching sequences in the high school; the third analysis allowed us both to anticipate some possible teachers' wrong conceptions and to construct a picture of possible difficulties to put in correspondence, if it is possible, to some of the teachers' declared choices; the last analysis allowed us to make emerge the way the teachers approach the teaching-learning processes in this case and to identify categories of teachers. Then we evaluated the epistemic and cognitive suitability of teachers' choices and put it in relation with the teachers' profiles constructed in the previous analysis. In the end we followed in particular 11 high school teacher. In our first pilot study analysis we found out that one of the most relevant object used in the high school practices leading to real numbers was the interval, both of rational and real numbers. Wondering how it is possible to have a wrong conception of the infinity of the points included in a limited segment and also of its continuity (or better completeness, in the topological sense) and, at the same time, work correctly with intervals represented by segments, that are expected to be continuous sets, we faced the complexity of the definition of continuum itself: what is continuum? Has it a unique characterization or are there many conception of continuum that can't be resumed in a only, more abstract, object that include all the others? This research lead us to explore the history of the continuum in the history of mathematics and its relation with real numbers, too often identified at school with the more complex and articulated meaning of the continuum. We ended up in the collection of many different conceptions and images of continuum in the history of mathematics, philosophy and science and also in the didactical practices at school. Are teachers aware of this complexity and the lack of equivalence between different conceptions of the continuum? Are teachers aware that "to pass through" an interval without stops and to say an interval is a subset of the complete field of real numbers is not exactly the same? Do they in general take it in account while they are planning teaching sequences? Are they aware that objects that aroused from previous practices may not have the meaning they are expected to have? In the teachers' interviews we found a very articulated panorama: some teachers' personal meaning are not correct or not complex enough because of a lack of adequate formation in the Master courses or in the teachers training courses; some teachers are partially aware of the complexity but a lack of knowledge in mathematics education leads them to choose sequences without taking care of students' potential difficulties highlighted by researchers in Mathematics education; some teachers are aware of the complexity but, trying to help students, give too much attention to cognitive suitability and compromise the epistemic suitability, generating new causes of potential difficulties, like for instance using concrete examples from real life to represent irrational numbers (what is a rope with a length of $\sqrt{2}$ km?). In fact as we observed in our pilot study and Bagni highlighted in his researches concerning continuous and dense sets, a lack of complexity may cause other problems to students while learning real numbers. Conclusions and future works In our analysis we found out some profiles of teachers concerning the choice of teaching sequences concerning real numbers. Also studying the epistemic meaning of real numbers we discovered a complexity even greater than it was already stressed in the other researches about the teaching-learning processes concerning real numbers. The evaluation of the didactical suitability from the point of view of epistemic and cognitive suitability of teachers' choices made emerge a heterogeneous panorama that lead us to hypothesize some possible implications of our research in teachers' training: some teachers should be made aware of the possible meanings of the continuum and real numbers, that are some of the most quoted objects in the didactical practices in the high school but are underestimated in their complexity; some teachers that have a good knowledge concerning the continuum and the real numbers may choose potentially wrong teaching sequences because of lack of knowledge in Mathematics education or because of a lack of reflection at an epistemological level; some teachers that have a very good knowledge about the continuum and the real numbers, also from an historical-epistemological point of view may choose to avoid to take in account the complexity of real numbers at school and make it too simple because of a greater to attention to the cognitive suitability than to the epistemic one, creating this way other potential sources of difficulties to their students.

1. Research problem and literature review

1.1 **The complex relation between *continuum* and real numbers**

<< [...] lo studio dell’infinito in quanto tale è tutto sommato recente, e non si afferma che alla fine del secolo scorso. Nei duemila anni che separano la nascita della geometria greca dalle profonde intuizioni di Cantor, la trattazione matematica dell’infinitamente grande non fa registrare che progressi modesti, quasi che l’immensità dell’oggetto valga a precludere ogni sua analisi approfondita. Così chi voglia studiare la storia dell’infinito matematico dovrà rivolgersi piuttosto alla sua immagine speculare, ed indagare l’evoluzione dei temi e delle teorie legate all’infinitamente piccolo.

Tra esse, un posto particolare spetta alle dottrine del continuo, soprattutto a causa del ruolo centrale di quest’ultimo, quasi un ponte gettato tra la geometria, scienza del continuo per definizione, e la filosofia naturale, che nella composizione del continuo trova uno dei temi più dibattuti. [...]

"Il rapporto tra teorie geometriche e struttura del continuo va in senso contrario alla successione logica: la discussione delle proprietà del continuo non precede, come sarebbe logico e lecito attendersi, la formulazione e lo sviluppo delle teorie geometriche delle quali esso costituisce per così dire la materia. Al contrario, il continuo è piuttosto un risultato finale, un sottoprodotto, della geometria; un risultato peraltro che non è quasi mai esplicito, e che è piuttosto suggerito che enunciato, meno che mai dimostrato.

In altre parole, quella del continuo non è una scienza, una teoria, sulla quale si possa fondare la geometria; ma piuttosto un’immagine che si forma nella mente del geometra alla fine delle sue elucubrazioni; immagine costruita pezzo a pezzo mediante le proprietà che al continuo si sono attribuite *nel corso delle dimostrazioni, e che vengono via via a modificare immagini precedenti. La geometria genera immagini del continuo; e così ai cambiamenti di punti di vista in geometria corrisponderanno analoghe revisioni della nozione di continuità, in modo che i periodi di grande attività creatrice come il XVII secolo, sono anche caratterizzati da una forte instabilità fondazionale; periodi in cui nuove immagini del continuo sono create, modificate, e infine rimpiazzate da altre immagini, non più fondate queste ultime, o meno arbitrarie, di quelle che le hanno precedute."* (Giusti, 2000).

In these words of Giusti (2000) some crucial questions that inspired our research in the beginning are presented.

The first important remark concerns the particular nature of the *continuum* as a mathematical objects. "Magnitudes that may assume all the possible intermediate stages", "set of infinite points without dimensions", "continuum as a particular determination of the contiguous", "what we can trace without interruptions or breaks" and the most of the characterizations of continuity emerged from an historical review are far from being rigorous definitions. If we consider the conceptions of continuity in the whole history of mathematics, the most of them have a deep link with intuition, perception and, in the end, subjectivity. The numerical characterization of continuity came late in the history of mathematics and the construction of connections with the physical and geometrical conceptions of continuity have not been trivial and free from consequences. Weyl, for instance, spent a lot of time investigating this topic. He changed many times perspective and came to the conclusion that we can not transform completely the intuition of continuity as a product of our consciousness in a formal definition without loosing our global sense of continuity. The attempts to formalize the intuition of continuity, motivated by the search for higher precision, more effective representations, manipulation techniques and by the claims for rigor in Calculus' methods, obliged mathematicians and philosophers of the XIX century to transform the nature of continuous objects and processes. Continuous magnitudes (segments, figures, varieties) and variations (motion, time, increasing or decreasing values of variables) wasn't considered as wholes but as a system of parts related each other (points, numbers, instants of time etc), whose totality could form the whole. John L. Bell, one of the contemporary most experts of infinitesimals and continuity in the history of mathematics, named this process the reduction of *continuum* to discrete. This change cause the abandon of the space-temporal based concept of variation and the consideration of continuous mathematical objects in favor of static and local definitions. It was not the first period in which the composition and decomposition of continuum was investigated, since this topic has been present quite in all the centuries from Anaxagoras to our days. But this was the first time that the intuitive dimension of continuity gave way for the numerical, discrete attempts to determine the continuum, being usually numbers considered useful for calculation and applications but not suitable enough to be the base of mathematical theories (Giusti, 2000).

This operation was nor neutral nor immune from a philosophical point of view. On the contrary this reduction implied a revolution in the way the *continuum* was considered and managed. For instance standing on the famous Aristotle's definition of *continuum* magnitudes the continuum ends to be continuous if it's composed by discrete indivisible parts.

This issue opens the path to other themes that are crucial for the issue of formalization of intuituive continua, that we can synthesize in three questions:

1. Is a continuum a set of points, or small increments, or infinitesimal variations, or indivisible things? Is our goal to describe a natural "real" motion, supposed continuum? What does it mean that a function is continuous in one point ?
2. What do we consider "formal enough" and why do we need to be formal talking about the *continua*? Are there practical issues? Do all mathematicians agree in considering rigorous a definition of the continuum rather than another?
3. If formalizing we change the nature of the object, can we consider the formal and the previous object the same mathematical object? Can we re-frame and re-interpret the difference between concept image and concept definition used in Tall & Vinner (1981) and the cognitive conflicts observing that, effectively, the formalization changes at all the mathematical object? May the incommensurability of discourses identified by Nunez (2000) be an alert in this sense?

The first question addresses the ontological issue of continuity in mathematics, both in an absolute sense and in relation to the characterization of discrete magnitudes.

The second question addresses the following anthropological and sociocultural matter: what were and what are the mathematical *continua* used in the concrete life of mathematicians in different epochs, with different sensations, problems, aims, environments? How the mathematical *continua* was transformed by different philosophers, mathematicians, physicians and scientist facing new problems and being animated by different "spirits" (<< immagine costruita pezzo a pezzo mediante le proprietà che al continuo si sono attribuite nel corso delle dimostrazioni, e che vengono via via a modificare immagini precedenti >>, Giusti, 2000)? A paradigmatic example in this sense is Leibniz, as Giusti (2012) stressed using the plural *continua* in the title of the paragraph dedicated to the conceptions of continuum by the German mathematician and philosopher. The attention to personal transformation of *continua* in the mathematicians' practices should not mislead the readers, driving them to think that a particular choice is able to solve the ontological problem of continuum or allows to avoid the problem of definition of *continua,* that was proved to be an epistemological obstacle. What emerges from historical researches is the existence of "local coherences" in the works of philosophers and mathematicians related to different domains, goals and practices, but none of them can be considered the winner in the everlasting debate on discrete and continuum nor the best definition of continuity.

The processes of using and defining and re-defining the continuum are in a dialectic relation and owe particular importance since, for instance, some mathematicians used the word continuum with very different meanings, sometimes opposite. Furthermore even if each mathematician addressed his specific goals and modeled its definition on his necessities, the issue of coherence was faced by the most of them. The mathematicians at least declared their position in relation to the other definitions of *continua* used by the Ancients or by their contemporaries.

The third question is crucial in a didactical sense, since it reframes the previous questions in the problem of didactical transposition. May a teacher plan a vertical didactical sequence that ends with one of the formalizations of the concept of continuity as the top elements of a generalization chain or do we lose or change something significant in the modern formalizations of continuity, that is impossible to put in the same image of continuum?

To answers the three questions we had to explore deeply the literature review concerning the evolution of the ideas concerning the mathematical *continua* in the history of mathematics, from an epistemological - and, more generally, philosophical - point of view, but also from a semiotic point of view. The analyses that we had to carry out to answer these questions drove us to choose a theoretical framework for mathematical objects that include the personal meaning of mathematical objects, the deep relation between objects and practices, an allowed to take care of the evolution and generation of objects in the practices of mathematicians community.

A good metaphor for the issues that emerges from the three questions can be Bohr's complementary principle for couples of magnitudes in quantum physics:

"Within the scope of classical physics, all characteristic properties of a given object can in principle be ascertained by a single experimental arrangement, although in practice various arrangements are often convenient for the study of different aspects of the phenomena. In fact, data obtained in such a way simply supplement each other and can be combined into a consistent picture of the behaviour of the object under investigation. In quantum mechanics, however, evidence about atomic objects obtained by different experimental arrangements exhibits a novel kind of complementary relationship. Indeed, it must be recognized that such evidence which appears contradictory when combination into a single picture is attempted, exhaust all conceivable knowledge about the object. Far from restricting our efforts to put questions to nature in the form of experiments, the notion of complementarity simply characterizes the answers we can receive by such inquiry, whenever the interaction between the measuring instruments and the objects form an integral part of the phenomena (Bohr 1962, p.4)"

We may reformulate an analog principle for the continuum, considering the puntual, or infinitesimal, and the intuitive whole continuum as two aspects hard to see together. Every time we want to see one aspect or the other we will see it through adequate assumption and limitations, but trying to consider the two aspects together we can get lost.

This argument was supported by eloquent mathematicians and modern and contemporary philosophers like René Thom and Hermann Weyl and is highlighted also in Tall (??), Tall & Vinner (1981), even if this dualities are presented like tricky problems to overcome in order to create a "concept definition image", supposed to be one and all-embracing.

This awareness may suggest to use the complementarity of the two classes of images of o continuum (whole/intuitive-systemic/formal) as a potential resource for didactics instead of trying to construct a vertical way to modern formalizations.

A first hypothesis that should be evaluated is to avoid both the strategies that aim at forcing the didactical actions to construct one unique path and a global image and the mixed and confusing didactical transposition that use the two classes of images as equivalent. In this two complementary images of *continuum* the field of real numbers considered as an object of pure mathematics () would find his own position on the side of numerical, decomposed (systemic) continuum and the description of the trace of the whole continuos processes in the intuitive sense in terms of discrete entities like points, "epsilons" and numbers would rightly be presented as a Icarus flight, so fascinanting in his infinity as somewhat intrinsecally impossible with our wings.

This sentence may seem to forgot, or deny, a very important period of mathematics and its wonderful products, like Cantorian continuous or Hilbert's axiomatization or Cauchy's, Dedekind's, Weiestrass' efforts to trasform the vague conceptions of continuous magnitudes always expressed by processes in mathematical "static" objects that could make rigorous the infinitesimal analysis.

This is not true, since the constructions or axiomatizations themselves are intrinsecally characterized both from the tension to transform the intuition of continuity into a set of numbers algebrically structured and the awareness of the limits that vanify the complete mission (necessity of an infinity of existence axioms, need of a postulate of continuity, lack of constructive procedures for Aleph1 irrational numbers, ...).

The eminent mathematicians that dealt with the construction or the axiomatization of the field of real numbers had clear in their mind the complexity of this issue and pursued their aims aware of the loss of the intuitive dimension of continuity they will have caused through formalization.

The fields of real numbers (the plural is necessary, since there are at least two logically not equivalent constructions) are endpoints of social processes that involved the mathematicians' community, asked to clarify and make rigourous the foundations of Analysis even if the criteria to determine what is rigourous in Mathematics were not unique and clear at that time as now.

The posterity didn't accept completely this kind of "formal solution" and this is the reason why the critics to the "cantorian paradise" (Hilbert, 1926) were not lacking. This reply by Wittengstein to Hilbert resumes perfectly the spirit of this reaction:

< If one person can see it as a paradise of mathematics, why should not another see it as a joke?>

Proofs of the controversy that Cantor's set theory provoked in mathematicians' debates are easy to find: see for instance Poincaré's rejection of actual infinity (Heinzmann and Stump, 2014); the sharp opposition to Cantor's ideas that gave rise to the Brouwerian intuitionism and in general to all the costructivist thread in mathematics research (Iemhoff, 2015); Hermann Weyl's dissertations about the continuum, in which he criticizes harshly the attempts to transform *tout-court* the continuous space and time variations in assumptions by a variable of values in an interval of real numbers (Bell, 2014). Also the Cantorian axiom of infinity was judged scarsely self-evident at all (Mayberry, 1983) and not adeguately justified by analogy to finite sets (Bell, 2014).

Standing on the historical review, that we will present in the following, we can state that there is not a "solution" to the incommensurability between intuitive and numerical continuum, but rather there have been choices oriented to specific goals, both practical and theoretical, not often well explicit, representations and problem-situations have been very important in the development of different nuances and potentialities of the mathematical continuum. Understanding how the traces of these different paths are intertwined and how they have wandered in didactical practices time after time until nowadays is the main interest for the investigation of didactical transposition and thus is one of the aims of our work. In particular we will look for the relations between the search for intuitive examples and representations of continuity, to wish in the high school.The core of our a-priori historical and epistemological review is thus to look for different *continua* in the history of Italian secondary school. This configurations of objects will be compared first of all with the objects emerging from didactical practices declared by the teachers involved in our investigation, in order to find out the potential cognitive conflicts (Tall & Vinner, 1981; Godino et a., 2007) that can be forecasted through the epistemological analysis. Then we will evaluate the didactical suitability of the teachers' choices from the epistemic, cognitive and interactional points of view.

We will avoid to look for a unique general mathematical object including the field of real numbers and the continuity, as is usually done in EOS, but we will rather take care of preserving a certain degree of necessary and fruitful complexity.

This would not have been impossible at all, but we don't consider this attempt useful, nor for the historical analysis (as Giusti explains very well), nor for the analyses of declared pratices.

1.1.1 Different continua in the history of Mathematics

a. **Aristotle's continuum**

Aristotle's continuum have been framed in a wider philosophical dissertation.

<<Aristotle distinguishes discrete quantity (poson) from continuous (suneches) quantity. He includes lines, surfaces, bodies, time, and place in the latter category. Aristotle also distinguishes quantity that is continuous from that which is non-continuous. A magnitude, he says, is quantity that is measurable (as opposed to numerable or countable), and a magnitude is divisible into parts that are continuous. Among magnitudes, “that which is continuous in one [dimension] is length, that in two breadth, and that in three depth. I think that is fair to say that Aristotle’s conception of continu- ous quantity or magnitude is a geometrical conception. I shall postpone discussion of the continuous “quantities” of time and place, concentrating for the moment on magnitudes of the three spatial dimensions. There are both what might be called “structural” (geometrical or topological) and metaphysical features to be found in Aristotle’s analysis of *megethos* (magnitudes).

Some of the former, structural features correspond to properties central to the developing geometry of the fourth century BCE. Aristotle’s basic structural property, however, is continuity (*sunecheia*); and it does not have an explicit role in Euclidean geometry (where it appears principally in the notion of a “continuous proportion” of three or more terms). A principle of continuity of geometrical magnitude is assumed, however, in many Euclidean constructions: it guarantees the existence of points at the intersection of two lines, the existence of lines at the intersection of two surfaces or planes, etc. (Heath 1956). of. Having stipulated that “something is contiguous (*echomenon*) [to something] that is successive to and touches it” (227a6), he proceed as follows:

<I say that something is continuous (suneches), which is a kind of being contiguous, whenever the limit of both things at which they touch becomes one and the same and, as the word implies, they are "stuck together" (sunechêtai). But this is not possible if the extremities are two. It is clear from this definition that continuity pertains to those things from which there naturally results a sort of unity in virtue of their contact.>

In Phys VI.1 Aristotle argues that his conception of continuity implies that "it is impossible that what is continuous be composed of indivisibles, e.g., a line from points". >> (White, 1980)

In fact if we can distinguish the parts that compose the continuum and they are indivisible, the limits of the these parts don't exist and it doesn't make sense to think they are in contact. Aristotle defines the continuum as a determination of *contiguo,* i.e what is consecutive and in contact (Giusti, 2000) so it's impossible for a continuous to be composed of parts. The Aristostle's complete distinction between magnitude is presented by Giusti (2000):

"Aristotele distingue tre tipi di grandezze, a seconda dell'accoppiamento

tra le loro parti. In primo luogo la grandezza discreta, le cui parti si susseguo-

no consecutivamente senza che tra di esse vi sia alcunché di simile, pur non

escludendo la possibilità che tra esse siano intercalati altri oggetti eterogenei.

Così ad esempio tra due linee consecutive potremo trovare uno spazio, ma non

una linea; e tra due case consecutive un prato, ma non una casa:

Il consecutivo… è ciò che non presenta alcun intermedio dello stesso suo

genere tra sé stesso e quello di cui è consecutivo (dico ad esempio, che non vi

siano una linea o più linee dopo la linea, una unità o più unità dopo l'unità,

ovvero una casa dopo una casa; nulla però impedisce che vi sia in mezzo qual-

cosa di altro genere).

Contiguo è ciò che, oltre ad essere consecutivo, è anche in contatto.

Il continuo [suneches] è una determinazione del contiguo, ed io dico che c'è conti-

nuità quando i limiti di due cose, mediante i quali l'una e l'altra si toccano, *diventano uno solo e il medesimo e, come dice la parola stessa, si tengono insieme. Questo però non può verificarsi quando gli estremi sono due. Tenendo conto di questa precisazione, risulta chiaro che il continuo è in quelle cose da cui per natura vien fuori qualcosa di unico in virtù del contatto."*

<< Magnitudes of null dimension (points), of one dimension (lines), and of two dimensions (surfaces) are ontologically dependent on physical (changeable and sensible) body. Although this account of mathematical objects is difficult, Aristotle appears to hold that they are not, in reality, "separable" (*chôrista*) from sensible, physical reality. Rather, the geometer considers physical bodies *qua* geometrical, abstracting (*aphairein*) their spatial/geometrical characteristics and considering such properties separately from other kinds of property of such bodies>> (White,1980)

While Aristotle's conception of magnitude (*megethos*) is strongly geometrical, his concept of place (*topos*) finds more direct employment in his physics: motion is indeed motion with respect to a place. Place is classified as a kind of continuous quantity ("how much" – poson). (White,)

Between infinite, space, and time in Aristotle's thought, the last has surely received the most attention. In his "physical" analysis of time: "this, then, is [a] time: the number of motion with respect to the earlier [or 'prior' – proteron] and the later [or 'posterior' – husteron]". Another important feature of time, << which borders on the ineffable, is the transitory, evanescent, or "flowing" character that attaches to our experience of time>>. Just as a line is not composed of points according to Aristotle, so time is not composed of nows or instants. Any interval or stretch of time is continuous and "infinitely divisible" (in Aristotle's "potential infinity" sense) into smaller sub-intervals. Temporal "points" or instants can be demarcated or "constructed" (as, for example, the boundaries of processes and, perhaps, as instantaneous events).

Resuming the previous nuances of Aristotelic continuum, we can find these main features:

1. infinitely divisible but not composed of infinite elementar parts
2. a particular case of contiguous, whose parts are kept together in order to form a whole
3. allows a potential infinite process of division
4. is coherent with the Eudosso's theory of proportions
5. is strictly connected to physical continuum, since mathematical objects are inseparable from reality
6. the numeric dimension of continuum is allowed just for comparisons (earlier and later, previous or sequent)

Aristotle's vision is partially influencial still now, in particular to the mathematicians who deny the use of actual infinite, considered too "artificial", a sort of catch, a joke and - paradoxically - a play with the unreal.

<<Whatever the relation between Aristotle's doctrine of the potential infinite and contemporary ancient mathematical practice, the Aristotelian conception became, in the long run, the orthodox view, particularly in physical and mathematical contexts. Despite the difficulty in working out the foundations of the calculus in terms of "Aristotelian" potential infinity (a problem that was finally solved by B. Bolzano, A. Cauchy, and K. Weierstrass in the nineteenth century), the basic Aristotelian view persisted in technical contexts until the work of Georg Cantor in the late nineteenth century. In contemporary mathematics, the Aristotelian influence is detectable in the so-called "intuitionist" and constructivist traditions. In words that could have been written by Aristotle himself, M. A. E. Dummett writes that "in intuitionistic mathematics, all infinity is potential infinity; there is no completed infinity. This means, simply, that to grasp an infinite structure is to grasp the process which generates it; to recognise it as infinite is to recognise that the process is such that it will not terminate" (Dummett 1974)>>

c. Archimedean continuum

Archimedes (287 a.C.-212 a.C.) have been one the most influencial thinker of Ellenics. His work was underestimated until the XVII century both because of historical happenings concerning the discovery and interpretation of his writings and because some of the significant and revolutionary ideas that underlie to them was not completely grasped by posterity, at least until '600 (Volterra,)

His conception of continuum was more complex than we can think if we only focus our attention on the most famous of its assumption, that is universally known with Archimedean postulate.

In fact from the Archimede's correspondence with Eratostene (Heiberg,) we can understand that there were at least "two Archimedes": the one who was working to find results and the one who was presenting his results. The "second" Archimedes appears to be coherent with the ban of actual infinity, only seems to work with exhaustion methods and to deal with potential infinity. Archimedes avoid thus to use infinitesimal quantities, and only says that the parts of continumm he consider can be small of your choosing.

As we can see reading this excerpt, his method is very familiar for us, since this is indeed the method that inspired Weierstrass approach to limits and the one we that influenced more actual didactics of Analysis in the secondary school and university:

"*Supponiamo che l'area del cerchio non sia metà del prodotto del raggio per la sua circonferenza; sia allora d la differenza fra la maggiore e la minore delle due quantità. Se circoscriviamo alla circonferenza un poligono di n lati, l'area di tale poligono sarà la somma delle aree degli n triangoli che lo compongono, tutti di altezza 1, e quindi l'area complessiva sarà p, essendo p il semiperimetro del poligono. Preso n sufficientemente grande, possiamo far sì che l'area del poligono differisca dall'area del cerchio meno della metà di d. Poiché il perimetro del poligono differirà dalla circonferenza per meno della metà di d, l'area del cerchio e il semiprodotto del raggio per la circonferenza differiranno meno di d, contro l'ipotesi di partenza. Quindi d deve essere zero".* (Maffini, 2001).

This approach need to include the Archimedean axiom for numbers corresponding to measures of magnitudes, i.e.

For any two segments a and b there is a positive integer $n \in N$ such that $a < n \cdot b$

The Archimedean postulate is very important also for the modern theories of real numbers: fixed by Hilbert as a postulate in its axiomatizations of real numbers, the relation with this postulate is a key element for comparing Cantor's and Dedekind's postulate of continuity.

As highlighted by Maffini, the rigid approach to infinite and infinitesimal quantities adopted by the Greeks is one of the most substantiated reasons why the infinitesimal analysis did not develop in the Ellenics, even if its seeds had already been sown at that time. An interesting suggestion to prove how deep this sentence is came us from Volterra (), who reported an anomalous behavior of the Syracusan. It seems that when Archimedes was working in practice to find results he was using not so orthodoxical methods, but rather infinitesimal methods, exactly as Eudosso did but more and more frequently, or series:

"Dalla lettera di Archimede ad Eratostene si rileva come egli facesse uso per le sue scoperte del metodo delle quantità infinitamente piccole, e, solo per esporre i risultati al pubblico, ricorresse al metodo dell'esaustione e a quello delle serie. Basta ricordare le differenti soluzioni ch'egli ha date, allo scopo di trovare l'area della parabola, per riconoscere i principî fondamentali, mediante i quali il calcolo infinitesimale si è sviluppato da quell'epoca remota fino ai nostri giorni. [...] basta infatti classificarli [i metodi, nba], come ora abbiamo fatto, in tre gruppi; quello degli infinitesimi, quello di esaustione ed infine quello delle serie, per veder delinearsi tutte le concezioni fondamentali del calcolo infinitesimale." (Volterra,)

Standing on this consideration it appears evident that the epistemological methicolousness concerning the use of infinite infinitesimal quantities is fascinating in his firmness, but is somehow unreasonable if we observe two evidences: 1) the infinitesimal method works and leads to useful results; 2) the Ellenics somehow accepted and used results coming from its unbelievable assumptions, but accepted them just if they were expressed in "acceptable terms", i.e. respecting a strict relation with sensible reality - whatever it means - and with a sort of intuitiveness.

This became more evident in Cusano and Cavalieri, Barrow and Torricelli methods and, at the same time, in all the harsch critiques moved toward them in the sequent centuries.

Standing on these observation we can hypothesize that there is at least a distance, not to say a trench, between practice with continuum and its parts, connected to infinitesimal methods, both derivative and integral, and the theorization of continuity in its various nuances (arithmetization, algebrization, axiomatization, ...) and that this cut is not easy to mend.

To sum up, in Archimedes we find two conceptions of continuum:

1. practical continuum: actually infinite, composed by infinite infinitesimal parts (infinitesiaml segments and rectanguls)
2. theoretical continuum: potentially infinite, consistent with Eudosso's exhaustion method and with the ban of actual infinite and infinitesimal quantities

d. Ockham's continuum

William of Ockham (c. 1280–1349) the principal difficulty presented by the continuous is the infinite divisibility of space, and in general, that of any continuum. The treatment of continuity rests on the idea that between any two points on a line there is a third—perhaps the first explicit formulation of the property of density—and on the distinction between a continuum "whose parts form a unity" from a contiguum of juxtaposed things. Ockham recognizes that it follows from the property of density that on arbitrarily small stretches of a line infinitely many points must lie, but resists the conclusion that lines, or indeed any continuum, consists of points. Concerned, rather, to determine "the sense in which the line may be said to consist or to be made up of anything.", Ockham claims that "no part of the line is indivisible, nor is any part of a continuum indivisible." While Ockham does not assert that a line is actually "composed" of points, he had the insight, startling in its prescience, that a punctate and yet continuous line becomes a possibility when conceived as a dense array of points, rather than as an assemblage of points in contiguous succession. (Bell,)

Ockham's continuum is so:

1. punctate and dense, but not actually composed of points nor of indivisibles
2. whose parts form a unity
3. not set of points in contiguous succession

e. **Cusanus' continua**

Nicolaus Cusanus (1401–64) pointed out an interesting distinction. He asserts that any continuum, be it geometric, perceptual, or physical, is divisible in two senses, the one ideal, the other actual. Ideal division "progresses to infinity"; actual division terminates in atoms after finitely many steps.

So Cusanus' possible features of continuum are two:

1. Ideal continuum: infinitely divisible (actual infinity)

2. Actual continuum: atomic and only finitely divisible

It's not surprising that precisely in Nicolaus Cusanus we find two somewhat opposite characterizations of the continuum, since it's theological approach to the science attributed to Mathematics the role and the potentiality to realize in the infinite the *coincidence of the opposites (coincidentia oppositorum).* Finite human mind can only investigate finite things, not the infinite (the famous *docta ignorantia).* Knowing is an approximation to the truth, so as an approximation to infinite. Truth is to knowledge what a circle is to a poligonal. In Cusano's infinite - an actual infinite, explicitly in contradiction with all the Greek mathematics' assumptions - the coincidence of opposite reveals itself and finite and infinite are not antithetical but, on the contrary, finite things have a sort of symbolical relation with infinite. The potential infinite is necessary to human mind, that only can think finitely, but this is not a proof of the inexistence of actual infinite, that is included in everything as God is "compacted" in every creature.

<< "....Più cose non sono, dunque, in una qualsiasi cosa in atto, ma tutte sono senza pluralità questa cosa stessa. L'universo è nelle cose in modo contratto, e ogni cosa che esiste in atto contrae i suoi universi, affinché essi siano in atto ciò che essa è. Tutto ciò che esiste in atto è in Dio, perché Dio è l'atto di tutto. Ma

l'atto è la perfezione e il fine della potenza. Ed essendo l'universo contratto in qualsiasi cosa esistente in atto, è chiaro che Dio, che è nell'universo, è in qualsiasi cosa e che qualsiasi cosa che esiste in atto è, come universo, immediatamente in Dio."

Ciò che emerge quindi è che l'uomo pensa all'infinito in termini potenziali, non potendo fare diversamente la finitezza della sua mente, ma ciò non toglie che esista un infinito in atto che Cusano identifica con Dio. >> (Maffini, 2001)

His actual infinite conception of continuum revelead to be not only philosophical speculation, but also very useful tool: for instance lead him to consider *infinilateral* regular polygon - a regular polygon with an infinite number of (infinitesimally short) sides. By dividing it up into a correspondingly infinite number of triangles, its area, as for any regular polygon, can be computed as half the product of the apothem (in this case identical with the radius of the circle), and the perimeter. The idea of considering a curve as an infinilateral polygon was employed by a number of later thinkers, for instance, Kepler, Galileo and Leibniz.

g. Galileo Galilei's continuum

Galileo Galilei faced the issue of the composition of the continuum when he was working on physical problems, in particular motion and dynamics. His idea of decomposing motion in infinitesimal parts was a revolutionary step for physics, since it allowed to consider locally constant the velocity and to use for general situations laws that had been enounced in simplier cases. This way he brought infinitesimal methods into physics, opening the path to fruitful exchanges between geometry and physics that produced very important developments in both the sciences (Volterra,). But Galileo's fame linked to the interpretation of the continuum is not due to this intuition, although to the rediscussion of the Aristotle's dogma, ancient but still influencial at that time, and to the creation of the corrispondence between naturals and their squares. Aristotle had stressed that the continuum can't be composed by discrete elements without loosing its continuous characterization, while Galileo's continuum is fragmented, atomic, composed by infinite elements that are results of the infinitive process of division. An important feature of the Galileian continuum is strictly linked to its infinite cardinality. Let's imagine to compare two segments using the number of its elements. Since they are both composed by infinite indivisible elements there is no way to establish an order relation between them. An excerpt from the *Dialogue* clarify Galileo's position, represented as usually by Salviati: (from http://www.mathedpage.org/infinity/galileo.html)

Simplicio*: Here a difficulty presents itself which appears to me insoluble. Since it is clear that we may have one line segment longer than another, each containing an infinite number of points, we are forced to admit that, within one and the same class, we may have something greater than infinity, because the infinity of points in the long line segment is greater than the infinity of points in the short line segment. This assigning to an infinite quantity a value greater than infinity is quite beyond my comprehension.*

Salviati*: This is one of the difficulties which arise when we attempt, with our finite minds, to discuss the infinite, assigning to it those properties which we give to the finite and limited; but this I think is wrong, for we cannot speak of infinite quantities as being the one greater or less than or equal to another. To prove this I have in mind an argument, which, for the sake of clearness, I shall put in the form of questions to Simplicio who raised this difficulty. [...]*

Salviati*: If I should ask further how many squares there are, one might reply truly that there are as many as the corresponding number of roots, since every square has its own root and every root its own square, while no square has more than one root and no root more than one square.*

Simplicio*: Precisely so.*

Salviati*: But if I inquire how many roots there are, it cannot be denied that there are as many as there are numbers because every number is a root of some square. This being granted we must say that there are as many squares as there are numbers because they are just as numerous as their roots, and all the numbers are roots. Yet at the outset we said there are many more numbers than squares, since the larger portion of them are not squares. Not only so, but the proportionate number of squares diminishes as we pass to larger numbers. Thus up to 100 we have 10 squares, that is, the squares constitute 1/10 part of all the numbers; up to 10,000 wefind only 1/100th part to be squares; and up to a million only l/1000th part; on the other hand in an infinite number, if one could conceive of such a thing, he would be forced to admit that there are as many squares as there are numbers all taken together.*

Sagredo*: What then must one conclude under these circumstances?*

Salviati*: So far as I see we can only infer that the totality of all numbers is infinite, that the number of squares is infinite, and that the number of their roots is infinite; neither is the number of squares less than the totality of all numbers, nor the latter greater than the former; and finally the attributes "equal," "greater," and "less" are not applicable to infinite, but only to finite, quantities. When, therefore, Simplicio introduces several lines of different lengths and asks me how it is possible that the longer ones do not contain more points than the shorter, I answer him that one line does not contain more or less or just as many points as another, but that each line contains an infinite number. Or if I had replied to him that the points in one line segment were equal in number to the squares; in another, greater than the totality of numbers; and in the little one, as many as the number of cubes, might I not, indeed, have satisfied him by thus placing more points in one line than in another and yet maintaining an infinite number in each? So much for the first difficulty.*

Sagredo*: Pray stop a moment and let me add to what has already been said an idea which just occurs to me. If the preceding be true, it seems to me impossible to say that one infinite number is greater than another. ...*

In summary Galileo's continuum had the following features:

1. composed of infinite elementary and invisible parts
2. different continuous segments couldn't be compared since they are composed of infinite elements and all the infinite quantities have the same cardinality

h. Bonaventura Cavalieri's continuum

Bonaventura Cavalieri was a pupil of Galileo Galilei and, to a great extent, he continued the works of his master. This obliged Cavalieri to follow a very curvy path when he had to deal with the composition of the continuum. In fact the assumptions he needed to adopt in order to go on with his quantitative methods for calculating areas and volumes clashed with a fundamental ban introduced by his master: infinitive quantities can't be compared, it's denied to define ratios between infinitive quantities since it's senseless. Getting out of thorny ancient and modern philosophical implications Cavalieri succeeded in "tailoring" a continuum suitable for his goals and not inconsistent with his master's view, but he had to accept an important restriction: renounce to a global vision of continuum in mathematics and philosophy, in particular natural philosophy, avoiding to consider its *indivisibles* as *atomic elementar parts of the matter.* This renounce can be considered as a crucial point in the history of continua since it sanctioned the separation between the mathematical (Cavalieri's one) and the physical material continuum (Galilei's one).

The first acquired the freedom of being composed of elementar constituents that were pure abstract objects created by the mathematicians imagination, with no needs to represent atoms or monads, pure frutiful inventions useful time after time in different practices involving the infinitesimal quantities.

The second is more complex to analyse, because of its many facets. For a clarification about the issues of continuity in Physics we refer to a specific chapter of this work, but we anticipate here just some excerpts in order to clarify the meaning of the distinction between mathematical andd physical continua operated by Cavalieri. Two big categories of continuity can be individuated in physics: the continuous evolution of processes (in space and time, which in turn can be continuous) and the continuity of matter (opposite to the discretization of matter in atoms).

The freedom acquired by mathematical continua involved the first kind of continuity, when processes, space and time have been modeled through mathematical tools (laws, functions, systems composed by infinite parts). The analysis of the composition of matter (and of the space, if considered full of matter like *ether*) followed instead its own path towards the current quantistic theory of matter.

Cavalieri was interested in Eudosso's exhaustion method in order to simplify it and make it more effective. He needed to conjugate the theory of ratios, on whom Eudosso's methods relied on, and his *Geometry of indivisibles*, that uses n-1 dimensional elements, i.e. segments for calculating areas, rectanguls for calculating volumes. Eudosso's theory of ratios was coherent with the ban of actual infinite and took care of the problem of incommensurability already studied by the Pythagoric school. The problem of incommensurability was strictly linked to hypothesis that underlies the atomic composition of the Pythagoric continuum, based on the concept of monad and number, so it was denied to consider a segment composed of indivisible points-monads. In order to be consistent with this constraint, Cavalieri could not consider segments as composed by points-monads. This was fortunately not necessary since in his method Cavalieri was considering both the segments and the figures themselves as wholes. This "stratagem" also preserves partially Cavalieri from the inconsistency with Galileian ban concerning quantifications and comparisons of infinite quantities of atomic elements, that in his approach were really the elementar parts of the continuum and had the same cardinality in two segments of different lenght. Cavalieri had to do more than considering *indivisibles* as wholes, but also he had to justify the core operation of his methods: the comparison between indivisibles. This comparison could not be realized standing on the number of their elements, because, as his master has proved, the number of elements is infinite and so is the same. Cavalieri thus opted for a continuum:

1) composed by *indivisibles* considered as wholes that can be compared through methods not involving infinite quantities;

2) if the figure was considered in his totality, it was considered as a whole and not as a sum of indivisible elements;

3) the *indivisibles* were not elementar parts of the continuum

4) obliquous segment were not comparable with horizontals (in order to avoid the incommensurability diagonal-edge)

i. **Viete's and Descartes' continua: a break-point with tradition**

Until now we have presented merely geometrical approaches to continuum, to which numbers were connected only by a hierarchical relation, in particular after the premature collapse of the numerical theory of continuum, dreamt and desidered by Pytagora and his followers, due to the discovery of incommensurable quantities. This didn't implicate, of course, the abandon of a numerical mathematics, as for instances all the Arabic tradition and the fascinating history of equations testify, but scarred the numerical continuum as a low quality product, afflicted by dangerous bugs. What happened in France by means of Viéte and Descartes is really more than a crucial turning point in the history of continuum, but rather a milestone, a revolution in mathematics, since it reopenend by force the doors between two rooms that hadn't anything to do eachother for many centuries. In Classical geometry the numbers had assumed a subordinate position in respect of magnitudes and proportions. Viéte and Déscartes mainteined a strong interest in Geometry, but two main innovations were suggesting a possible improvement of mathematics substained by numbers: the overflow of numerical methods in practical problems of real life - encouraged by decimal system's spread that accelerated calculations and improve approximations; the impressive development in the algebraic field - let's think for instance at Bombelli's *Algebra* - lead them to prefer the generality of Algebra to the rigour of Classical geometry. The Algebraic geometry that they were building up was founded on numbers, in spite of all the ancient recommendations, and were upsetting the hierarchy between numbers and magnitudes (Giusti, 2012). To the ancient limitations of numbers, the new Algebra had already replied with the creative power of signs in at least two ways: the symbol Rx to indicate not rational solutions to rational equations (called *numeri surdi*, absurd numbers) and the imaginary unity *i*.

The sentences by Leibniz concerning *i* let half-view the hybrid and mysterious atmosphere wrapping the Algebra at that time, useful but somewhat incomprehensible in its deep meaning:

"Ex irrationalibus oriuntur quantitates impossibiles seu imaginariae, quarum mira est natura, et tamen non contemnenda utilitas."

"The imaginary number is a fine and wonderful recourse of the divine spirit, almost an amphibian between being and not being."

The terms "real number" and "imaginary number" have been invented precisely by Descartes, who was addressing the topic to manage visually two curves' intersections but trying to merge algebraic and geometrical methods realized that some algebraic solution corresponded to invisible points.

As Giusti (2000) stresses, algebraic questions demanded not only practical solutions, as it was frequent before, but also theoretical answers, usually reserved to geometry. Viéte did more than inventing the syncopated language that made him famous all over the world, but operated a real change of perspective transforming even geometrical problems in algebraic problems through the appoinment of a numerical value to magnitudes. This way he enlarged the concept of number itself: a number is solution to a geometrical problem, even if the solution is unknown. Viéte replaced the numeric algebra (*algebra numerosa*) with a new symbolic algebra (*algebra speciosa*) in which objects were not labeled with numbers but with letters: consonants for known numbers, vocal for the unknown ones. The new algebra dealt with forms, not only of with letters. This is the crucial point: to substitute numbers with letters wasn't revolutionary in itself, since it would have been sufficient to provide suitable example in order to express a rule, no need to use the verbal register. The true innovation *algebra speciosa* was the new unity established between algebra and geometry through the algebraic manipulation of forms, whose value was even unknown, in which the introduction of symbols was not the cause but the consequence (ibid.). Nevertheless it's undeniable that the semiotic choice to represent and manipulate letters instead of numbers was determinant for the development of the algebraic geometry: the semantic neutrality of letters, sometimes numbers, sometimes magnitudes made possible the union between algebraic procedures and geometrical methods, towards a new geometry that was founded on the ambiguity of what the letters actually represent. Geometrical magnitudes could thus be represented by a letter avoiding the mediation of numbers. This way a new, very complex relation is established between magnitudes and numbers, so powerful to seem to wipe away the past fears linked to the use of numbers in geometry and the problem of incommensurability. This acquired flexibility allowed, for instance, to express geometrical constructions through algebraic procedures, like compass-and-straightened procedures that were containing square roots - for which a symbol already existed - but also to create new numbers that were arising from new geometrical constructions - like cubic roots. Descartes accepted completely Viéte innovations, indeed accentuated more the numeric aspects of magnitude introducing the measurement unity, towards an algebraic theory of continuum, more and more oriented to abandon geometry and its strict axiomatic constraints towards an algebraic structure able to take the place of the axiomatic structure. This is also revolutionary, and characterize strongly both the Viéte's and Descartes' conception of continuum and the future mathematical approaches to this topic.

Summing up, the "amphibium" Viéte's and Descartes' continuum (ibid.) was characterized by these features:

1. accepted the classical continuum of magnitudes represented by the Euclidean straight and approximation methods
2. substituted to the axiomatic structure of Eudosso's theory of proportions the algebraic structure of the numerical continuum, whose rules were established by operations and descriptions of procedures
3. had no contradictions with the use infinitesimal quantities since it renounced to the precision and the rigour of Eudosso's axiomatization

For this last reason in particular the algebraic continuum is a truly new continuum, on which growed up the new mathematical analysis. It is thus contradictory to frame mathematical analysis in the classical continuum, ignoring this crucial transistion. It's not surprising to recognize similarities with Archimedes infinitesimal procedures, that would have find room in this new continuum, and the fact that, after less of one century, actual infinitesimal quantities and infinitesimal procedures, directly descending from Archimedes', entered mathematics, leading to the impressive steps forward of infinitesimal analysis. Given the reach of this revolutionary change of perspective on continuum it's also not surprising that infinitesimal analysis had to face urgent and frequent critiques of vagueness, lack of exactness and rigour, lack of strong bases, and so on. The abandon of the safe port of classical continuum to the new not-axiomatic conceptions leaved a trace in the newborn analysis. This Achilles' heel took after around two centuries to the born of theories of real numbers.

l. Leibniz's continua

Lebniz is one of the most influencial mathematicians in the history of Calculus. Also his philosophical speculations concerning the world (The monadology) had been intertwined with mathematics, enriching his theories with frequent comparisons with the real world, with whom mathematical theories had to be harmonized. Well-known is the Leibniz's principle *Natura non facit saltus*, become a discussion topic of the modern Quantistic Physics thanks to Niels Bohr.

On the topic of continuum Leibniz's position was very articulated, as Giusti () stresses. Leibniz never reached a unique theory continuum but rather different images of different continua, changing time after time depending on the case he was analyizing. In the end we will individuate at least three Leibnizian continua, each of whom coexists with the other without hierarchy and fulfills a particular task. In Giusti (2000) the Leibnizian continua are labeled with the following names:

1. Continuum with infinitesimals

2. Formalized classical continuum

3. Hyperdense continuum

Only the third one is suitable for the Calculus, since the others have no sufficient complexity to accomodate "*i valori della variabile, le differenze, le differenze delle differenze, e via sminuzzando, [che] si muovono sempre all'interno del punto strutturato ma inesteso del continuo iperdenso leibniziano. [...] Non sono necessarie infatti solo delle grandezze infinitamente piccole; occorre anche e soprattutto che esse siano paragonabili tra loro in modo che, come dice Leibniz, uno zero sia più grande di un altro. Inoltre si deve poter operare su di esse con le regole formali delle operazioni aritmetiche: sommarle, sottrarle, moltiplicarle tra loro e con altre grandezze finite."* (Giusti, 2000)

The first time in mathematics the term "function" appeared it was written by Leibniz, even if the concept was already "in the air" since many centuries (Volterra, 1920). The conception of Leibnizian function is founded on the third continuum.

I. Continuum with infinitesimals for motion and qualitative analysis of processes

The continuum with infinitesimals is analyzed in order to make room to natural discontinuities and describe qualitative changes in a suitable image of continuous process. His bumps were the discontinuity between life and death and between near and far. The natural reference for this kind of dissertation were the Aristotelic characterizations of continuous and contiguos, that Leibniz take as starting point to abandon them in the end of its analysis. Qualititative differences can not coexist, so one can be alive or died but not the both of them. So life and death are contiguous but their extremes can not coincide since it would implicate that a moment exists in which one is alive and dead at the same time.

1. Contiguous entities are produced by cuts of continuous entities (reversing Aristotle's determination of continuous as a particular case of contiguous)
2. The mental act of separation produces two points, distinct but not distant. The instant-point is splitted in two.
3. The process of split can not be repeated generating consecutive, infinitely near points (that would lead to the ancient contradictions), since after the first cut there are no segments to divide since the two points, let them be, B and C, are not distant and AB=AC. So a straight is not a set of points, nor of indivisibles, but infinitesimal are created by every single cut.

4) The motion doesn't happen in an instant or in a point, but is a whole process; points are results of a cut.

II. Formalized classical continuum for geometrical problems

The formalized classical continuum has a strong relation with the Euclidean Book I since its construction deals with intersections between geometrical entities. As a reader can imagine the problem to face will be partially similar to the one presented before, since the local composition of continuum in proximity of the intersection points concerns cuts and near points. Nevertheless this continuum has a peculiarity that makes it different from the first.

Let's start from Euclids. Lebniz objects to the statements concerning intersections between for instance lines and circumferences, in which it's not explicited that the existence of the point of intersection was not obvious but postulated. Lebniz fills the gap expliciting that there is always at least a point in common between curves that intersect each other. In this context the continuum assumes nuances that make it more similar the Aristotelic one.

To compare Leibniz's and Aristotle's continuum it's necessary to point out that the concept of part was different: Aristotle's parts were intervals resulting from a division, while Lebniz's concept of part was more similar to set (not necessarily ordered). This explains some apparent incongruences but merely justify the - although implicit - use of superpositions between cointegrant parts of a continuum by Lebniz; superpositions that would have been impossible in Aristotle's continuum.

A section is thus an intersection between not superposed parts. To avoid empty intersections Lebniz implies that subsets are closed, as all the parts of the continuum and all the figures.

Leibniz's geometry is geometry of closed sets.

The characterization of the formalized classical continuum is thus this one:

1. closed and connected set
2. every intersection is not empty, so the point now belongs to every part (contrary to the case of motion explored before)
3. no need infinitely near points since the split is not only not necessary, but instead not desiderable.
4. every point belong both to the internal and the external part of an intersection

III. Hyperdense continuum

As we anticipated the continuum with infinitesimals couldn't satisfy all the necessities of the Calculus, since the infinitesimals E had not only to be manipulated algebrically so to keep equal to a segment x the segment x + E, with E the distance between two distinct and not distant points, but every point needed to have a cloud of infinite infinitely close points. Robinson identified with the concept of monad this differential structure of point without extension but composed of infinite infinitely close points; so the point isn't what is not divisible in parts. Distances had to be at the same time zeros but a zero had to be larger than another zero. The operations had to be confined into the infinitely populated cloud around the point, in which we find the arbitrary but assigned sequence of infinitely close values that the Leibnizian variables assumed (Bos, 1974).

Differentials become themselves variables, may be differentiated and may assume values inside the cloud. Everything happens *intra-extensionem,* the microworld of the cloud have the same structure of the macroworld of the line.

To sum up the hyperdense continuum has the following features:

1. each point has around a cloud of infinite infinitely close points, whose distances are comparable each other
2. variables assume assigned but arbitrary sequences of values *intra-extensionem*
3. differences of any order are all inside the cloud, so as results of every kind of operation involving differentials
4. the whole continuum is not sum of its points

This may be considered a critical point of the hyperdense continuum if we consider integration, as it was for Cavalieri, since if the operation cannot exit the cloud, how can a sum of infinitesimals form a figure? Leibniz never explicit this point but coherently with his model never sums effectively quantities but always traits integration as anti-derivation, being derivation the local operator for whom the model had been tailored. As Cavalieri did indeed in practice, Leibniz considered always the figures as wholes to study locally and to re-obtained inverting the local operation of derivation, but never as sums of elements.

On the contrary further approaches reversed the hierarchy between integration and derivation (f.i. Volterra, 1920). So it's proved that there are no right approaches but only goal-oriented practical choices that delimit a space of action in which mathematicians stay until they do not need new freedoms.

m. Newtonian continuum

Newton had come to treat continuity facing both the issue of abstract description of Space and Time as absolute entities, pursued through mathematical tools, and the analysis of magnitudes variations and motion. The long-term debate concerning the born of the Calculus that involved Leibniz's and Newton's scholars, that has been fascinating students for all the following centuries, is very well-known but it's usually narrated merely from a social point of view, i.e. that of the attribution of the paternity of Calculus to Leibniz or Newton. The intriguing happenings obscured partially the presence a very deep difference between the two approaches to the Calculus, that determined the affirmation of the Leibnizian formalism and, in general, of the Leibnizian method for differential analysis. A key element to grasp the difference between the two methods is precisely the conception of continuity. The dynamic approach to variations proposed and carried out by Newton and its scholars implied a dynamic conception of continuity, framed in a more general mechanical conception of Geometry. In Newton we can read:

< *Quantitates Mathematicas, non ut ex partibus quam minimis costantes,*

sed ut motu continuo descriptas hie considero. Lineae describuntur ... per

motum continuum Punctorum; Superficies per motum Linearum; Solida per

motum Superficierum; Anguli per rotationem Laterum; Tempora per fluxum

continuum & sic in ceteris. Hae Geneses in rerum naturae locum vere habent,

& in Motu Corporum quotidie cernuntur > (Giusti, 1988)

The term themselves used by Newton, like fluxions and fluents, lead to think at the intuitive, perceptual dimension of continuity as a flow, something that pass through a points from an endpoint to another, a process of becoming. The deny of space-temporal intuitions of continuity that will characterize the arithmetization of Analysis indeed after a century in favour of puntual, static characterization of continuity (also for the infinitesimal variations) has a strong, maybe causal, relation with the affirmation of the Leibnizian methods on those created by Netwon. The Newtonian method obliged him to borrow and then transformate deeply and make more effective the Mercator's series, since the approximation is the key of the method. The famous mathematicians MacLaurin, belonging to the Newtonian school, readfirms the preference for a dynamic approach to continuity of his school, in particular talking about limits:

< *Quando la certezza di una parte qualunque della geometria è messa in discussione, la maniera più efficace per ristabilire la verità sulla sua piena luce e prevenire dispute, è di dedurla da assiomi o principi primi di evidenza indiscutibile, con dimostrazioni del tipo più rigoroso alla maniera degli antichi geometri. Questo è il nostro intento nel trattato che segue; nel quale non proponiamo di cambiare la nozione di flussione di Sir Isaac Newton, 35 ma di spiegare e dimostrare il suo metodo deducendolo per disteso da poche verità autoevidenti, in maniera rigorosa: e, nel trattare ciò, di fare astrazione da tutti i principi e i*

postulati che possono richiedere di immaginare altre quantità che non possano essere facilmente concepite come realmente esistenti. Non considereremo una parte qualunque dello spazio o del tempo come indivisibile o infinitesima; ma considereremo un punto come il termine o il limite di una linea, e un momento come un termine o un limite del tempo.> (MacLaurin, in Cinti, 2013)

The conception of continuum in Netwon was thus:

1. dynamic
2. associated to functions and variations
3. related to physical *phenomena*, like motion, magnitudes variation, time lapsing.

n. Euler's continuum

R may be regarded as the space of ratios of microquantities. This was essentially the view of Euler, who regarded (real) numbers as representing the possible results of calculating the ratio 0/0. For this reason Lawvere has suggested that R be called the space of Euler reals. In Euler the continuity law was traduced in the expressibility of a function through an analytical expression in his whole domain.

o. **D'Alembert's continuum**

D'Alembert studies about continuity was merely connected to the mathematical description of physical laws and so the central concepts are functions and variations. The French scholar in particular was interested in functions that describe waves in a chord, so not only continuous but at least derivable two times (for calculating local speed and acceleration).

In D'Alembert, similarly to Euler, the continuity law was traduced in the expressibility of a function through an analytical expression in his whole domain.

p. Arbogast's continuity

In 1787 in St. Petersburg a curious concours was announced: find a way to determinate the nature of continuous functions. In 1791 the competition was won by Luis-Francois Arbogast, that propose the following definition that broaden the D'Alemberts' one, but also the definition proposed by Euler:

"The law of continuity consists in that a quantity cannot pass from one state [value] to another [value] without passing through all the intermediate states [values] ...."

This insight was made rigorous in an 1817 pamphlet by Bernhard Bolzano (1781--1848) and is now know as the Intermediate Value Theorem (Thomas and Weir,).

Two cases of not continuous functions were taken in account: *discontinuous*, i.e. the law, so the function, change "completely", or *discontiguous* i.e. different parts of a curve are unconnected. From this characterization also emerges the conception of function as a curve.

The idea of "passing through" is anchored to the intuitive conception of physical motion, assumed to be continuous. To assume all the values is here used equivalently to pass through without interruptions or "saltus" (Leibniz,). Discontinguity contradict the notion of continuum as something with *no breaks*.

q. Lagrange's continuum

Lagrange faced first the problem of foundation of Calculus with the "rigour of the ancient proofs", trying to provide an algebraic solution. He tries to refer all the Calculus procedures and definitions to Algebra, getting it away from the geometrical evidences that had made room to metaphysical arguments to criticize the differential Analysis. This way Lagrange inaugurated the season of "arithmetization of Analysis" that seemed to sort out all the problems, giving the desiderated rigour to the Calculus.

r. Bolzano's continuum

Bernhard Bolzano's contribution to the debate on continuity is merely to place on the side of the question of methods and rigour in Analysis. It's very important for our research to trace the paths towards contemporary images and conceptions of continuum passing through the role of rigour and precision in the choice of representations of continuity in the mathematicians' practices. In particular we are interested in the forces that transformed the actual infinite arithmetic continuum, forbidden in the Ancient Hellad, in the solution to the issues of credibility, methodological strenght, acceptable foundation of the Analysis.

The "[...] platonist orthodoxy [of founding mathematics on set-theory, nba] is a comparatively recent phenomenon, dating back only a century or so. Before then, it was more common to view infinity as potential infinity. It is illuminating to look at how and why the change-over to actual infinity occurred. The transition arose out of the needs of nineteenth-century mathematics, particularly the arithmetisation of analysis. Four reasons can be traced" (Fletcher, 2007).

One of these reasons is related to Bolzano's methodological issues: the rejection of spatial and temporal intuition.

"Newton based his ideas of limits and differentiation on intuitions of motion; other mathematicians based their ideas of continuity on spatial intuition. These kinematic and geometric conceptions fell into disfavour in the nineteenth century, as they had failed to provide satisfactory theories of negative numbers, irrational numbers, imaginary numbers, power series, and differential and integral calculus (Bolzano, 1810, preface). Dedekind pointed out that simple irrational equations such as $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ lacked rigorous proofs (1872, §6). Even the legitimacy of the negative numbers was a matter of controversy in the eighteenth and nineteenth centuries (Ewald, 1996, vol. 1, pp.314–8, 336). Moreover, Bolzano, Dedekind, Cantor, Frege and Russell all believed that spatial and temporal considerations were extraneous to arithmetic, which ought to be built on its own intrinsic foundations" (Fletcher, 2007).

Bolzano didn't spare of course the intuitive ideas of continuity, keeping continuity at a safe distance from motion and time:

"He criticized demonstrations of Kaestner, Clairaut, Lacroix, Metternich, Rösling, Klügel, and Lagrange for the involvement of geometrical and physical images (time and movement, transfer) and the lack of analyticity in their reasoning, i.e. lack of understanding of the continuity as a mathematical notion" (Sinkevic, 2015). Bolzano wrote:

"*As a matter of fact, if we take into account that a proof in science must not at all be just words but argumentation, i.e. be the exposition of objective cause for the true being proved, then it goes without saying, that if an affirmation is correct only for the values in the space, it may not be correct for all variables, whether or not they are in the space. The most common kind of proof depends on a truth borrowed from geometry, namely, that every continuous line of simple curvature of which the ordinates are first positive and then negative (or conversely) must necessarily intersect the x-axis somewhere at a point that lies in between those ordinates. There is certainly no question concerning the correctness, nor indeed the obviousness, of this geometrical proposition. But it is clear that it is an intolerable offense against correct method to derive truths of pure (or general) mathematics (i.e., arithmetic, algebra, analysis) from considerations which belong to a merely applied (or special) part, namely, geometry. No one will deny that the concepts of time and motion are just as foreign to general mathematics as the concept of space. We strictly require only this: that examples never be put forward instead of proofs and that the essence of a deduction never be based on the merely metaphorical use of phrases or on their related ideas, so that the deduction itself would become void as soon as these were changed*» (Bell, , translation of Bolzano, 1817).

"These considerations led them to reject the potential-infinity notion of a quantity capable of augmentation without end, with its connotations of time and change, and to replace it with the static notion of the infinite set of all possible values of the quantity." (Fletcher, 2007)

The paper published by Bolzano in 1817 represents an important stage in the rigorous foundation of analysis and is one of the earliest occasions when the continuity of a function and the convergence of an infinite series are both defined and used correctly (Russ, 1980).

The change of images of continuum from Geometry to Algebra and Arithmetics that we can observe in the following century may be directly connected to Bolzano's statements. Let's finally see in depth which was the conception of the continuum emerging from Bolzano's works.

Bolzano, like Galileo, considered that an actual infinity could arise from aggregation of infinte points (Bell, 2014):

"If we try to form a clear idea of what we call a 'continuous extension' or 'continuum', we are forced to declare that a continuum is present when, and only when, we have an aggregate of simple entities (instants

or points or substances) so arranged that each individual member of the aggregate has, at each individual and sufficiently small distance from itself, at least one other member of the aggregate for a neighbour".

Bolzano's continuum is thus:

1. dense (convinced that it was sufficient to make a set continuum)
2. actual infinite and composed, through aggregration, by infinite points
3. admitted the existence of infinitesimal quantities as the reciprocals of infinitely great quantities (algebrical)
4. infinitesimal quantities cannot be considered as zeros (contrary of Euler's position), but also cannot be considered as "infinitely small geometrical entities". This was true only for isolated values.

s. Cauchy's continuum

Le Cours D'Analyse, published by Cauchy in 1821, can be considered the first didactical manual for Analysis published ever, and published "for a major utility for students". Cauchy's work was determinant in development of the Analysis, also thanks to this book in which he reorganized all the previous knowledge refounding it in order to give it rigorous foundations, as Lagrange and Bolzano had already tried to do. It may surprise - and it should deal us to deep didactical reflections - that the concept of rigour that guided Cauchy in his mission was exactly the opposite of the two eminent mathematicians that came before him. Cauchy always thought indeed that the maximum rigour was given by Geometry instead of the more generic algebra, whose formulas may be valid only under particular conditions (Cinti,), but was anyway convinced the intuitive and evident concepts were not enough yet. Their success in geometry and physics couldn't satisfy the compelling foundational necessities. So Cauchy aimed to a rigour "like that of Geometry" - but not geometrical - avoiding to base his methods on the algebraic general procedures, that had been supposed and pretended to be more general than it was really. The semiotic dimension of this careful procedure of refoundation of Calculus deserves a special attention. As Leibniz did, Cauchy needed new symbols to concile the rigour with the simplicity - desiderable for students' manuals - that came from intuitive infinitesimals. The new concept, with its relative coherent and goal-oriented representations, on which Cauchy refounded the Analysis was the limit (Cinti,). His symbols are not the most used today - invented by Weierstrass - but it was anyway already clear to Cauchy what they had to permti: to suggest the process of getting closer and closer without using intuitive and personalizable intuitive ideas, but, on the contrary, delimiting precisely the kind of "approach" the variables' values had to follow. This is the crucial concept of Cauchy's sequence and leads to define precisely the infinitesimals, leaving aside the old intuitive conceptions: infinitesimals were the differences between two terms – closer and closer - of a sequence of elements that defined a limit.

The first example of limits is precisely the real numbers, that, if are not rational, coincided with the sequence of rational numbers that approached it, following the rules Cauchy had established for getting closer to a fixed point.

Cauchy was aware of the importance of a suitable concept of continuity in the complex structure that the Calculus need to come out on top.

1.4 Teachers' choices and students' learning: two *case - studies*

1.4.1 Discrete, dense and continuous sets after Dedekind's cuts and the Calculus: a teaching experiment results analysed by Bagni (2000)

Bagni (2000) carried out a research in order to analyse the importance of discrete, dense and continuous sets in the secondary school (16-19 years-old students). In particular in the first part the idea of connected set referred to the set of real numbers have been studied. Two classes of high school (Liceo Scientifico) have been tested with the same questionnaire and the results have been compared. Then the students' knowledge have been explored by means of interviews.
The research was carried out with:
a. a class of 26 students, 16-17 years-old (11-degree), with a previous knowledge of sets and cardinality, who had studied real numbers as Dedekind's cuts and had never studied infinite sets (definition, transfinite numbers).
b. a class of 22 students, 18-19 years-old (13-degree), with a previous knowledge of limits and continuous functions and who had never studied infinite sets (definition, transfinite numbers).
Comparing the test 1 and 2 (before and after the introduction of the main concept of the Calculus in the tradition curriculum of Liceo Scientifico) the authors aim at analysing the role of these concepts in the learning process of the notions of dense and continuous set.

We will resume the results that were presented by Bagni (2000):

Test 1
"All the students of the first group (11-degree) learnt in a satisfying manner the use of the set theory's symbology: the first item was indeed answered correctly by all the students.
The density of the set Q was apprehended by the most of the students as the property of existence of an element of Q that lies of a couple of given elements, whatever the elements are chosen.
The infinite cardinality of Q (third item, 73 %) and R (fifth item, 85 %) were correctly learnt by the most of the students, even if it's necessary to analyse with attention the answers by means of interviews. Difficulties emerged concerning the concept of irrationality (fourth item): only the 58 % of students state that $\sqrt{26} \notin Q$. This will be explored through interviews.
The answers to the sixth item were even more uncertain, so interviews were carried out also to interpret them.

Interviews concerning test 1
The author invited the students to reflect about the answers and to justify them by means of interviews; in particular he focused on the sixth item.
7 on 11 students who answered that the cardinality of R is bigger than that of Q:
$N \subset Q \subset R \Rightarrow$ the cardinality of R is bigger than Q and N
"The cardinality is that of R: the other sets are included in their subsets [proper]" (Dara)
The students who answered that the bigger cardinality is that of R and Q justified their answers stated that two infinite sets have the same cardinality. For instance an interesting statement is the following:

"I_N only occupies the whole points and I_Q occupies all the fractions; I_R occupies all the spaces between the fractions.
But since I_Q and I_R are infinite and their cardinality is the same. With the rationals between 3 and 10 I can cover the whole segment because I can draw points as close as I want." (Marco,)

Analysis of interviews and conclusions (test 1)
For what concerns the misconceptions linked to the concepts of density and continuity and infinite sets, there is an interesting partition of the group in two parts:

- some students found their reasonings on the consideration : $N \subset Q \subset R \Rightarrow$ the cardinality of R is bigger than Q and N; i.e. reason in the same way with finite and infinite sets and don't consider the fact that an infinite set can be put in correspondence with one of its subsets (the definition itself);
- other students think that to compare two infinite quantity is impossible and that a segment of a real line contains the same number of rational and real numbers.

The first result was predictable since the notion of cardinality had not been introduced before and the aim of the research was to highlight the lack of a right introduction of infinite.

Marco's answer is very interesting since highlights the role of the graphic representations (Duval, 1993). Some students may thus be tempted to approach the concepts of density and continuity in terms of graphic representations, probably because of the habit of using graphic representations in the high school in Italy (Bagni, 2007). While the difference between a discrete and continuous set is visualizable (as the student did representing N), but it's not true for the difference between dense and continuous sets (the student indeed talked about the subsets of Q and R but was not able to represent them). The graphic approach was thus misleading.

Test 2 (13th degree)
Confirming the results of the previous test the students showed good knowledge about the symbology of set theory and the density of Q and the infinite cardinality of the sets Q and R. By means of interviews the answers have been then checked.
Some difficulty emerged concerning the irrationality (72%) and in the sixth item as in the previous case.

Interviews (test 2)
As the author observed in the previous interviews the most of the students who stated that the cardinality of R is bigger than the others (9 on 11) use the argument: $N \subset Q \subset R \Rightarrow$ the cardinality of R is bigger than Q and N.

The following justification is interesting: "For sure the cardinality of I_N is lower than the other since it's finite and the other infinite. But in the other case can an infinite be bigger than another?"

The students who answered that R and Q have the biggest cardinality state that two infinite have the same cardinality (5 students, 23 %) and the same happened with the students that stated that all the sets have the same cardinality.

The infinite sets are considered to have the same cardinality.

The issue of irrationality was faced in this ways by the students who answered correctly (16):

- $\sqrt{26} \notin Q$ since $\sqrt{26} = \sqrt{2} \cdot \sqrt{13} \wedge \sqrt{2} \notin Q \wedge \sqrt{26} \notin Q \Rightarrow \sqrt{26} \notin Q$ (8 on 16)

$\sqrt{26}$ is not rational but I can't explain the reason why. It's hard to imagine to calculate the power 2 of a decimal number with an infinity of digits. Maybe the digits ends at any point but I can know it.
(operational difficulties didn't end up in argumentations; for similar results see Romero i Chesa & Azcarte Gimenes, 1994; Fischbein, Jehiam & Cohen, 1995).

Interviews analysis and conclusions (test 2)

Apart from small quantitative differences the 13th degree students show analogies with the 11th degree students we analysed before. In particular the argument concerning the inclusion $N \subset Q \subset R$ is significant and no differences have been detected between the arguments concerning finite and infinite sets.

The case of the student who justifies the irrationality of $\sqrt{26}$ with the irrationality of $\sqrt{2}$ and $\sqrt{13}$ is interesting since the students could present the argumentation concerning the irrationality of $\sqrt{13}$ for $\sqrt{26}$ itself. The reference to $\sqrt{2}$ may be interpreted as a matter of assurance linked to the didactical contract (Brousseau, 1987)

In the high school the presentation of the continuity of the number line appeared to be weak and incomplete. The introduction had been realized in the traditional way it's presented in the Scientific high school in Italy. In particular the difference between density and continuity didn't seem to be understood by the students.
In particular after the presentation of the Dedekind's cuts in the 10th degree the students of the both of the classroom never referred to them to evaluate the irrationality of the numbers and the sets' cardinality. Also even after studying the Calculus the situation was not so much better.
Some students tried to use graphic representations to distinguish dense and continuous sets but this attempt of course failed: as Duval highlighted "learning by means of graphic representation makes necessary a particular work and it's not possible to use spontaneous interpretations of figures and images" (Duval, 1994, translated by us).
Indeed the "double nature" of mathematical objects, being both abstract and real (figural concepts, Fischbein, 1993) didn't seem to be very useful in this case since to advantage of this double nature with continuous sets is hard requires to make choices carefully and opens the way to misleading ambiguities (Bagni, 1997); the coordination of different registers of representation in general is useful (Duval, 1993; Kaldrimidou, 1987; Vinner, 1992) but in the case should be analysed in depth the way to carry it out.
A specific teaching sequence concerning infinite sets and transfinite numbers is thus necessary to carry out a theoretical approach to the study of real numbers.

1.4.2 A teaching experiment concerning real numbers as Dedekind's cuts: emergent relations between students' difficulties and teachers' choices

In the Master thesis discussed in 2011(Branchetti, 2011) a teaching experiment concerning real numbers in the high school was described and partially analysed within an heterogeneus framework composed merely by the Theory of obstacles (Brousseau, 1986) and the first conception of Didactical transposition (Chevallard, 1985). The aim of the teaching experiment was trying to implement a didactical proposal for introducing real numbers in the high school based on some lessons presented in an academic course of Foundations of Geometry (Coen, 2010). The core element of the lessons concerning real numbers was the analogy (and the possibility of defining an isomorphism) between real numbers as Dedekind's cuts and "rational segments" introduced by Euclid in the
, based on Eudosso's theory of ratios of homogeneus magnitudes. This analogy had been presented as a possible didactical resource since the Euclid's construction was somehow more intuitive and elementar. We decided to investigate wether and how a didactical sequence in which the Euclid's construction was introducted before the Dedekind's cuts could have been useful in order to introduce Dedekind's cuts as a modern version of the same theory. The modern theory would have been presented as just more formal and expressed through more effective mathematical tools that simplify the rigorous but complicated Eudosso's definition (functions, minimum and maximum, algebraic operations, set theory).

The complexity of real numbers from an historical point of view at that time had not been considered very much since it was quite unknown at all to the teacher and to the researcher to. The only aspects of the complexity of real numbers taken in account had been:

1. the lack of equivalence between different theories of real numbers and, in particular, the exemplar case of the asymmetry between the postulate fixed by Cantor and Dedekind for the continuity of the line;
2. the analogies and the differences between the definition of ratios of homogeneus magnitudes proposed by Eudosso, reported in Euclid (
3. ,) and the Dedekind's field of real numbers as "rational cuts";
4. the relation between space and time (supposed continuous and corrisponding through a bijection to real numbers) and numbers as presented in many textbooks' problems is mediated by the practice of approximation, that changes the properties of the set of numbers really used and the manipulation rules.

The teaching experiment has been planned in order to:

1. respect some constraints decided after a literature review concerning students' difficulties in the learning process;
2. pursue some intermediate disciplinar objectives necessary to reach the construction of real numbers as Dedekind's cuts, spending also time in verifying the students' previous knowledge concerning the objects involved in the teaching experiment;
3. use the history of mathematics in order to motivate students on one hand, and to present the cultural problem that characterize continuous and discrete sets since the very origins of Western mathematics;
4. avoid to use the concept of limit and infinitesimal and the language of the Calculus, since the teaching experiment was addressed to students that hadn't studied them yet.

The choice of Dedekind's cuts as a way to construct the field of real numbers was due to some features of this construction that permit to address the most of the previous objectives:

1. discussing with the teacher before the teaching experiment, the objects involved in the construction should have been known quite well by the students;
2. the time the teacher considered available for the teaching experiment was 12 hours, including the final questionnaire and the discussion. To respect this constraint the Dedekind's construction was considered good because it was planned to be implemented in 4 hours and this feature allowed to re-analyse with the students in classroom discussions the previous knowledge (rational numbers, representation of numbers on the line, approximation, ...);
3. from an historical point of view the Dedekind's cuts are inspired by Eudosso's theory of ratios and,
4.
5. the two construction have a similar structure (Coen,);
6. in the whole process of construction of the field of real numbers as Dedekind's cuts the concept of limit or infinitesimal are never involved explicitely.

In order to make students as much as possibile aware of the epistemological status of the construction of a set of real numbers and to avoid to make them confused about the necessity of constructing the set of real numbers, the teaching experiment was planned to start with tasks involving computers and discrete sets of finite numbers, showing what it was possible to do with finite numbers. Then the line was used to represent the "set of all the numbers" and the set of finite numbers that a computer can use (fixed the limitations for the mantissa and the exponents of their representations) have been represented on the number line, showing its distribution on the line. This decision was inspired by the book
(Villani, 2000) in which different sets of numbers that may be involved in high schools' didactical practices, included finite, rational and real numbers, were presented. The introduction of the set of finite numbers was planned in order to highlight some of its properties starting from its decimal and graphic representation:

1. the set of finite numbers doesn't complete the whole line;

2. there is a finite quantity of finite numbers;
3. the distribution on the line is not homogeneus but there is a different density near the number 0 and near the upper and lower endpoints;
4. the set is not closed for any operation, i.e. for any arithmetic operation between finite numbers there is always at least couple of numbers whose addition or multiplication (and inverse operations, when possible) is not a representable finite number;
5. there is an effect of numerical elimination of terms of the mantissa that transform a result in a truncated or approximated number.

Introducing these properties in relation to the graphic representations we aimed at creating the opportunity to involve the classroom in a discourse concerning the properties of this set, in order to compare it with other number sets. In particular we aimed at driving the students to reflect about the differences and the relation between concrete problems concerning numbers and their approximations and theoretical problems concerning the definitions, the local and the global structure and the properties that can characterize a set of numbers as an algebraic and topological structure with operations.

Secondly we decided to plan a phase in which the students could recall to their memory the previous knowledge concerning rational numbers in different representations (verbal definitions, fractions, graphic representation onto the number line, ...). Then the already known elements of the set would have been characterized by properties and organized in a structure in relation to the set of the finite numbers.

In a further phase some problems concerning the limitations of rational numbers when trying to describe physical phenomena and geometrical constructions with numbers would have been presented in order to justify the introduction of new numbers that could complete the line. First of all we planned to present the problem of the incommensurability of $\sqrt{2}$ in a modern form, more familiar to the students, using the fractions to represent ratios and the decomposition in prime factors of natural numbers to prove the irrationality of $\sqrt{2}$ by means of the theorem of arithmetics.

Then we planned to propose two other problems to solve in group works: the problem of the bottle and the definition of Cantorian set (in the Appendices). These two problems had been chosen because they permit to stress the difference between an object or a process supposed to be continuous and the numerical description of it.
In the problem of the bottle the instant of time
in which the bottle is empty isn't finite nor rational. The instant exists because it's identified by a process - the bottle's quantity of water decrease - but the rational numbers are not sufficient to capture that instant of time.
The Cantorian set is possible to define through a process of division but its description requires more than the rational numbers.
Once introduced three problems to show cases that rational numbers don't permit to describe, we planned to propose a definition of all the possible numbers, rational and not, as all the points of the line and to propose also a formal definition. In order to define formally the real numbers we opted for a sequencial and gradual introduction of collateral definitions, images, procedures, starting from Euclid and reaching in the end the Dedekind's cuts. In the end we planned to reframe the rational numbers as Dedekind's cuts and to promote the use of the formal definition and algebraic considerations rather than graphical procedures in order to establish if a couple of rational sections was a Dedekind's cut or not and to identify eventually the cut with a rational or irrational number.
In this passage we didn't stress (since we were not completely aware of it) that the new numbers are created and that the correspondence between points, obtained by means of rational cuts, and real numbers is not provable but it's rather postulated. Also we used the line as an image both for representing rational and real numbers, distinguishing the two only as figural concepts, whose properties had been introduced in practices involving other representations of the line.
In the following Paragraph we report the designs' description that we wrote before the experience.
1. Computer

A support to face this issue is the computer, that students use in the everyday life and at school. While usually the computer is used in order to motivate students or to take advantage of its potentialities during the lessons, in this case the computer is the very object of the analysis. Its way of representing and using number is so explored showing its rules and its limitations in comparison with the pure mathematics.

Numbers in the Greek ancient mathematics

One of the objectives of this sequence is to make students' models of numbers emerge, keeping in our mind that the images correlated to this models origin in the first years of school. As we showed in the literature review the first use of numbers is related to quantification and is deeply linked to concrete objects (fingers, little stones, ...). The conception of number as tool for indicate and compare quantities may be used in order to associate numbers and segments by means of the conception of a segment as an aggregation of points, or better, atomic elements. This is precisely the Pythagoric idea of aggregating a given quantity of monads in order to form a segment. We can associate the measure of a segment and the practice of computation of elements in order to introduce the crisis of this model, that may be an intuitive model also formed by the didactical practice in the primary school (Sbaragli, 2006), and to present the concept of commensurable magnitudes and the problem of incommensurability from an historical point of view.

3. The set of finite numbers

The set of finite numbers is not one of the sets of numbers usually defined at school. Usually teachers talks about natural, whole, rational, real numbers, even if they very often use numbers that arise from operations with computers or approximations. The set of finite numbers that we can generate with a computer has not a unique definition because of the arbitrary number of elements of its decimal representation, depending time after time on the particular computer. Also if we don't take in account the limitation the set of finite numbers with any possible lengths we can't define on it an algebraic structure and it's poor of properties. For instance it's not closed for operations and it has more than one neutral element for addition (Villani,). Nevertheless its use in the didactical practice is so widespread that these numbers are very often used by the students and they are used indeed to represent and manipulate the most of the numbers by means of approximation of every rational or irrational number.

There are important consequences of this fact on students' conception of numbers:

1. in this set, once fixed the maximum length, it's possible to identify the consecutive number of every element;
2. all the numbers, rational or irrational, are reduced to finite numbers with the same methods of approximation, implying implicitly the lack of importance of the so stressed difference between rational or irrational.

We could use an a-didactical situation including computers in order to make emerge the students' conceptions of numbers, with high probability linked in the definitions to rational or irrational numbers but in the practice to finite numbers.

A possible way to make the limitations of the set of finite numbers emerge is to modify a non-vectorial image coded, for instance, in jpg. The image reconstruction after a modification is often different from the initial one and this is because This way the students may be motivated by graphical operations and, by means of suitable transformations of graphical operations into arithmetical ones, students may see "in action" the limitations of a set of numbers that doesn't own the property of continuity, but instead is:

1. discrete
2. not equally distributed.

The set of finite numbers may be introduced through attempts to define it and to construct its elements. Since the computers' codes are binary, we could present first numbers in the binary representation and then explicite the similar structure of decimal numbers. The scientific notation adapted to computers may be introduced in order to define a particular set of finite numbers, i.e. that of the computer the students are using, that can be determined once the computer's characteristics are known. After writing the definitions, the numbers are placed on a line in order to show their distribution.

4. Decimal numbers

The set of finite numbers is a subset of the set of decimal numbers, that should already have been introduced following the planning we are presenting. Now, in order to permit the students to recall their conceptions about decimal numbers and so make them aware of some properties of the numbers they are used to manipulate, the set of decimal numbers may be explored better and compared with the already known sets:

natural, integers, rational numbers. Also, since the students should already know some irrational numbers, the properties and definitions of this decimal numbers are explored in order to distinguish them from the rational ones. Now the challenge is the creation of a bridge between the set of all the decimal numbers and all the points of a line, that will be created after an exploration of the representation of rational numbers on the number line.

5. Rational numbers as ratios

The practice of placing rational numbers onto the line is linked to the conception of rational numbers as ratios between two segments, one of whom is the unit. This is the originary definition of ratios, proposed by Eudosso when numbers where still considered segments, and in modern terms, it deals to the representation of rational numbers as fractions. The adjective 'rational' comes itself from 'ratio'. Thus we could introduce the Eudosso's definition of ratios of homogeneus magnitued for segments in order to connect rational numbers and segments of the line.

As we have already pointed out, the relation between points and line (i.e., at an abstract level, between an object and its elementary parts) expressed through graphical representations of the both of them, may lead students with high probability to the "chain model" (Sbaragli, 2006).

Starting from the Pytagoric conception of segment as result of an aggregation of monads, we could propose an analysis of what is the meaning of a number that represent a "rational segment". How can the Pytagoric conception of number/segment generate a wider conception as that of rational number? Why do we indicate points with rational numbers? The strategies of placement of rational numbers onto the line could be put in correspodence with the Eudosso's definition.

This approach may allow the teacher to rethink with the students in a classroom discussion a critical point concerning the structure of the line: are the points small pearls? How far can we divide them? How many "pearls-points" there should be in the unit? Reflecting on the possible fractions we can identify in the interval [0,1] it will emerge a property of rational numbers that also is useful to change the image of the line as an aggregate of small pearls: there is an infinity of numbers in the interval [0,1] and also in the segment we used as a unit. In other words: the line is dense as long as the set of rational numbers. The property of the set of all the possible ratios included in the interval [0,1] may this way be assumed by the line itself. This could be a useful step to introduce the density of real numbers using the line as a model of them, since in the graphic register it's hard to represent the property of density (Bagni, 2000)

6. Different representations of rational numbers

The representations of rational numbers as decimals has a particular feature: there is not a bijective mapping between representations and numbers. In fact there are special cases in which two different decimal representations identify indeed the same number. This is the case of any decimal with 9 as period and the decimal whose antiperiod has the last digit equal to the consecutive of the digit of the first at the same position and the other digits equal to 0, like f.i. 0.(9) and 1.(0), 1,2(9) and 1.3(0). To show this anomaly is interesting in order to highlight the difference between a number and its representation. Also this could be used in order to discuss the possibility of introducing or not a limitation, like avoiding the representations which 9 as a period, to construct the map. This meta-discourses may help to shift the discussion from practical to theoretical issues.

7. Irrationality of $\sqrt{2}$

Once analysed in depth the set Q, it's possible to go forward and construct R starting from Q. This process is coherent with a famous sentence pronounced by Dedekind himself:

Here comes the issue of the previous knowledge and the way the constrution can be realized. Being R an algebraic structure we have to operate an enlargement taking in account the comparison between the two structure from the algebraic point of view.

Students are expected to know:

1. the standard order of Q;
2. some irrational numbers

Since the students already know some irrational numbers, both algebraic and not, like $\sqrt{2}$, e and π, the enlargement can't ignore them. A problem that may justify the enlargement coherently with the whole teaching experiment is the prove of the existence of numbers that are not rational using a geometrical example. The classical construction of sqrt 2 as diagonal of a square with edge's lenght equal to a unit may be a good example in this sense since the number is represented by a segment. The segment can be projected onto the line by means of a compass showing graphically that the segment does belong to the line but proving arithmetically, at the same time, that there are no rational numbers that can describe it. So rational numbers are not enough to describe all the segments that we can individuate onto a line. This example is still framed in a geometrical construction in which numbers and lenghts of segment are identified but we have to stress we are providing an example that only may be used for positive numbers. Once proved the existence of one segment that can't be described in terms of rational numbers we may go on looking for other irrational numbers as points of a line. Other example can be presented that the students already know and then formulate the crucial questions concerning irrational numbers and segments: how can I describe all the "irrational points"? To answer this question we will introduce the Dedekind's cuts.

8. Dedekind's cuts

In order to present the set of rational and irrational numbers and to describe the line in terms of properties, like we did before with the density of rational numbers, we need to define a complete set and to do this also we need the notion of extreme, minimum and maximum of a set.

An example of not complete set is Q and the previous introduction of $\sqrt{2}$ may be used to explain that Q is not complete using the representation of the segment $\sqrt{2}$ ad a set of points that are indicated with numbers whose square is less than 2. This transformation may be proposed both in the algebraic and graphic register coordinating the two semiotic registers. After this example other examples, algebraic and trascendental, may be presented, like $\sqrt{5}$, ln 2, e, and so on. Since we need a procedure to enlarge Q to create R numbers we use to define the segments are rational. The endpoint of the segment may be rational or not. We can define the point using the Dedekind's rational cuts , i.e. two subsets of Q whose intersection is empty, whose union is Q and with the lower set without the maximum. To do this we can work in the graphical register. After definining the Dedekind's rational cuts we can define all the numbers, rational or irrational, through a couple of sections.

The implementation of the planned teaching experiment have been realized in a high school classroom of twentyfive 17-18 years-old students (Liceo scientifico, grade 12). The planning has been discussed with the teacher. The students had not studied the Calculus yet and they had never studied limits and the properties of infinite sets. The teacher had declared that in his in mind the most important practices in which students should use a not elementary knowledge about real numbers is the extension of the exponential function from Q to R, since the lack of a procedure to go closer and closer to a point with rational numbers it's impossible to make sense of expressions like e^√2. The teacher is expert since he have been teaching for 20 years more or less and has got a Master degree in Physics. The teacher accepted our request to experiment our sequence in one of his classroom for 6 lessons, including the assessment and a final discussion. The teacher has always been present during the teaching experiment but it has been conducted by us.

The first lesson took place in the Informatics Lab. Students didn't know the content of the teaching experiment. The first assigned task was to open a folder containing images and to modify the image using a sofware, with particular attention to make it bigger or smaller and to observe how it changed. Their observation have been collected and shared with the classmates. The categories of answers were:

1. the images are blurry;
2. if we try to change the colour filling a part of the image, we only colour one pixel;
3. if we enlarge and then we try to return the image to its original status we fail since the quality of the image is worse than it was in the beginning
4. every is similar to the whole.

The last observation was referred to the particular image the student were modifying, since it contained a fractal. Even if the observation didn't concern the activity as it was planned it have been written down on the blackboard since we decided not to filter the observations since the very beginning.

Then the students were asked to formulate hypothesis about the way a computer colour an image in order to go a step towards the numerical intepretation of the manual procedures of enlargement and restriction and to start a discourse concerning the quantity of numbers a computer can use to carry out its tasks.

The proposals have been again collected onto the blackboard. The proposals have been synthesized in a unique proposal after a brief discussion, reaching in a few minutes an agreement in the classroom about the method but not about the quantity of colours a computer can use.

The method they decided to propose was the costruction of a scale from black to white, deciding a quantity of colours to use and assigning 0 to black and the higher number to white.
The quantity of numbers to use divided the classroom in groups:

- 30, as in the interface of the software Paint for Windows.
- 50, since it can be enough
- infinite
- a finite number since in the reality they are infinite but in the computer it's impossible
- a lot.

As we were wishing before starting the lesson, the keywords came from the students in a classroom discussion. After that we decided to institutionalize the quantity of number as finite.

In the first step we aroused curiosity about the way a computer colour, construct and reconstruct an image. In the second phase we presented the RGB (red, green, blue) methods used by many softwares: to every pixel a colour is assigned through a number, but the number is not a casual number on a linear scale but a 3D vector with a number assigned to any component. Students had studied and used vectors in the Physics lessons so we could stand on their previous knowledge about vectors. The third phase of the first lesson was the transformation of the vector into a number using the sum of powers of 16 with the three components as coefficients.
In the last phase of the first lesson we came up to highlight that the computer modify the image by means of numerical operations: the colour of every pixel must be computed. Is this the reason why we can't use with the "inverse operation" of enlargement of reduction coming back to the strarting point?
To fix this point ax example concerning the reduction by 1/4 was presented, showing the difference between the "pure operation", the expected invertibility of the procedure and the observed image that contradict the invertibility. This issue has been used for introducing the set of finite numbers that a computer can represent, using the notions of mantissa and exponent and the usual limitations imposed to the quantity of digits (Basic single - 32 bit, 1 for the sign, 8 for the exponent, 23 for the mantissa, Basic Double - 64 bit, 1 for the sign, 11 for the exponent, 52 for the mantissa)

The lessons end with a question: how are the finite numbers distributed onto the number line? Which is the relation between them and the numbers the students' already know and use in the didactical practices? How many numbers we can represent with a computer? Are there holes in the line if we only use finite numbers?
The students' answers to the questions posed in the end of the first lesson have been collected in the beginning of the second one. These are the proposed answers:

- All the numbers can be represented by means of a computer
- Numbers with too many digits can't be represented

Some further questions were posed in order to make the classroom discuss about the two different positions. How many numbers can we represent with these computers? Does any number that we can't represent exist? A group of students convinced all the others that the second thesis was correct, proposing argumentations like this: the numbers with a quantity of significant digits bigger than the quantity of bits available for the mantissa can't be represented. An interesting fact is that the examples presented were all about irrational numbers, to whom all the students associated the infinity of digits, but no examples concerning rational numbers were analysed. In particular they analyzed the case of $\sqrt{3}$. Also no students faced the problem of underflow or overflow for the exponent but only for the mantissa. Other examples were proposed by the teacher and by us and in the end we came to a definition of finite number, before referring to computable finite numbers and then a more general one:

A number
is a finite number if exists a number
so that a
-truncation of
is equal to
.
For the distribution of computable finite numbers an image was proposed by us, showing how many numbers we can't represent.

The question concerning the cardinality of the general set of finite numbers was answered by the students without institutionalization. A temporary agreement was found in the classroom, fixing that the numbers were infinite but less than all the possible numbers. The discussion have been interrupted since we didn't feel it was the right moment to profundize this aspect. Instead we decided to focus their attention on the distinction between rational and irrational numbers starting from a student's previous sentence.

"One of you said that we can't represent √3 with a computer since it has an infinite quantity of digits, like all the irrational numbers. What about rational numbers? Do all of them have a finite quantity of digits?". Once again there was a lack of agreement in the classroom and this made emerge a new debate: "Considering all the finite numbers do we take in account all the rational numbers?". The teacher proposed a reformulation of the problem: "All the finite numbers are rational. Is the reverse statement true?". All the students that were discussing answered "Yes". A not so convinced student posed the issue of the definition of rational numbers and the students tried to formulate definitions comparing them in a classroom discussion. The main question was "What is a rational number? How can we define it?". A student while discussing posed a crucial question: "Are the period numbers rational or irrational?", driving the group to focus the very problem. The two possibilities have been taken in account seriously by the classroom and two groups aroused that weren't finding any agreement. Reasoning in the decimal register the students' were discussing and proposing arbitrary argumentations based on the memory of previous teaching-learning happenings, stressing only the teleogical and communicative rationality but avoiding the epistemic level (Boero & Morselli,). The choice of the decimal register was a natural implication of the previous activities. To restart the discussion we proposed a question. "Is 5 a rational number?". All the students answered "Yes.". When asking for explanations we collected some argumentations and a student proposed to answer using the algorithm of trasformation af a fraction in a decimal number and the inverse one in order to decide if a decimal number was rational or not. By means of examples proposed by the teacher the group lead to the statement that not only the finite numbers, but also the periodic ones are rational. This statement was instituttionalized as a definition of rational number.

is a rational number if
is a finite decimal number or a periodic decimal number.
The ambiguity of the representation of decimal periodic numbers with 9 as period is presented in the case of 0.(9) and 1. We proposed the following argumentation:

1/3=0,(3)

So:

0.(3) · 3 = 0.(9)

1/3 · 3 = 1

Thus:

0.(9)=1.

One student tried to confute the argumentation using his calculator and we took advantage of this happening to recall the main concept of the previous lesson: a calculator, like every computer, substitute the most of the numbers with its approximations, in particular the number with an infinity of digits. In this moment we observed that the most of the students were reminding successfully the previous lesson. The issue of periodic numbers with period 9 was proposed as a matter of choice, giving to the students the freedom of deciding if:

1. exclude the periodic numbers by means of an explicit deny in the definition;
2. accept that some rational numbers may have two different decimal representations.

This open question obliged the students to reconsider the definition of rational numbers.

is a rational number if
is a finite decimal number or a periodic decimal number with period different from 9.

is a rational number if
is a finite decimal number or a periodic decimal number. If a number has 9 as period it's equal to a number that has the same digits of the anteperiod but the last, that's incremented by 1.
In the end we proposed a further definition:

is a rational number if we can express it as ratio between two whole numbers, i.e. if exist
in Z so that q=m/n.
The second lesson ended with a question for the next day: "Does any not rational numbers exist"
The third lesson was planned to be frontal and dedicated to the History of Mathematics. The theme was the conception of numbers in the Ancient Greece, from the Pytagorics to Archimedes and Euclid, with a special attention to the
The lesson was supported by a multimedial presentation. The main contents have been the concept of numbers for the Pytagorics, the notion of commensurability between homogeneus magnitudes by Eudosso, the issue of irrationality of √2 as it was presented by Plato in the
and the solution proposed by Euclid in the Elements
The conceptions of numbers emerging from this historical path were: number as quantity (natural number), number as ratio between homogeneus magnitudes (rational numbers). Furthermore the Ellenics already were aware of the necessity of introducing a new kind of numbers. Using the definition of rational number formulated during the previous lesson the irrationality of √2 have been proved by means of an arithmetic prove, i.e. the scomposition in prime numbers of 2 and the impossibility of finding a rational number whose square was equal to 2. So an irrational number was defined as a number that couldn't be expressed as a ratio of whole numbers.

The fourth lesson was the last in terms of introduction of new concepts. The 4 hours left had already been planned to be dedicated to assessment and final discussion. The aim of the last lesson was the introduction of real numbers as Dedekind's cuts in order to finish the teaching experiment with a cosntruction of a field of real numbers. The prevailing lesson methodology was frontal as the previous one but the lesson started with a group problem solving. The problems concerned irrational numbers and the numerical solution of problems contextualized once in the geometry and once in the real-life. The problems were the two we presented before in the planning description: the problem of the bottle and the description of a generic element of the Cantorian set. The class was splitted in two groups and to one group only one problem was assigned.
Both the problems concerns the logarithmic and exponential functions since the teacher had to work on them the next week. The problems were proposed in order to pose the problem of the relation between real numbers and continuity. In fact the solution of the problem of the bottle is irrational but we measure time using a discrete system (seconds, minutes, hours, ...). The numerical solution represent an instant in temporal continuity, while we usually represent the instants by means of a discrete set. A student had an interesting reaction: once understood the problem she asked "Does that instant exist anyway?". We answered that the istant exists even if our usual measures don't permit to represent it. After this problem solving session we introduced some propedeutical concepts, like maximum, minimum and endpoint, and the definition of rational section in the sense of Dedekind. We expressed the definition of Dedekind's sections in the language

of set theory but also representing them as half lines in the graphical register. Some slides are reported. Also we used the definition of maximum to justify the third part of the definition, showing examples of couples of subsets of Q without maximum in the first section. In particular the example of √2 defined as a couple (A,B), A,B, subsects of Q, so that: A={x in Q | x < 0 v x^2 <2}, B= {x in Q | x > 0 & x^2 >2}. Also we presented a transcendental example defining ln 3. In the end we showed how the same procedure may be used for rational numbers. We defined R as the set of all the possible sections of Q, precising that if the number is rational we say it a separator element for (A,B), if it's not rational we create the irrational number. It was stressed the analogy between this definition and that presented by Euclid in
, highlighting differences and analogies.

The fifth day of the teaching experiment was used for a formative assessment in written form (whose complete version is reported in the Appendices). In the test the notion of consecutive number, never explicitly introduced before, have been briefly introduced and then used in order to explore the students' knowledge avoiding to use the same language we used during the lessons. This choice aimed at investigating their deep understanding of the concept of density.

The sixth and last meeting with the classroom have been divided in two phases. In the first part the students have been asked to read a resume of the last lesson concerning real numbers and to answer two questions, similar to one of the most critical questions of the test (Exercise 6). In the last part a discussion concerning the test results have been carried out in order to close the teaching experiment with a final collective reflection. The 6th question had been re-proposed to the students since the answers were significantly worse than the others and, merely, they were difficult to interpret.

A mass phenomenon had emerged from the tests analysis and we tried to better investigate it by means of a further brief test similar to the first but with more precise questions. Then we guided a discussion concerning all the questions so to make order in the different conception emerged from the answers and to institutionalize the concepts that any question was planned to investigate.

- all periodics decimals (1)
- all irrationals (1)
- all but irrationals (1)
- all whole numbers (1)
- all natural numbers (1)
- all but natural numbers (1)
- naturals and rationals (1)
- irrational and rational, ut not whole numbers (2)
- all the kind of numbers (2)
- natural and irrational numbers (2)
- all kind of numbers but all positive (3)

The first question concerns the enlargement of the set Q. We wished to verify if the students had correctly conceptualized the enlargement or the considered real numbers as complementary set of rationals. The answers showed many different students' profiles but none but one listed only irrational numbers.

- Every animal is a figure, so quantity of figures = quantity of animals (1)
- infinite, but in a finite square the infinite can only be imagined (1)
- A very big quantity, but not infinite, since in the reality the infinite doesn't exist because of spatial limitations (1)
- Infinite, since I can divide infinite times the figures (9)
- Theoretically infinite, but finite in the practice (4)
- Potentially infinite, but finite in the practice (1)
- A very big quantity, but not infinite, since the infinite isn't realizable (1)
- I can't determine it since they are so much and so small (1)
- Infinite (2)
- Theoretically infinite (1)
- Infinite, but there is an optical effect (1)
- Infinite, because they are so small that to count them is impossible (1)
- They have the order 10^2, since there should be about 50 figures in every circle (1).

18 students answered that the figures are infinite (72%), 4 answered that they couldn't be infinite (16%), 1 answered that we can't determine it, 1 provided a finite estimation and 1 didn't answer (the last three are together 12 %).

- it's not periodic but it's unlimited and not periodic (1)
- we can express it as a ratio between whole numbers (3)
- it has a finite quantity of decimal digits or we know all its decimal digits (1)
- we can express it completely by means of digits, I don't need other symbols (1)
- a finite quantity of digits (5)
- it's possible to transform its decimal representation into a fraction representation (1)
- we can represent it as a fraction (5)
- it's represented in base 10 and it has a finite quantity of digits or it's periodic (1)
- we can represent it as a ratio between rational numbers, in particular whole numbers (2)
- decimal finite and not periodic
- we can express it as a ratio between natural numbers (1)
- no answer (2)

The mathematically correct answers are 15 (60%), one of which is referred only to naturals (Eudosso's conception of rational number); 8 answers identify rational numbers with decimals with a finite quantity of digits; 2 students didn't answer.

In 8 definitions the correct conception of consecutive number has been defined (32%), even if the language was still natural. Some definitions were good for N and Z (48%) even if there was a lack of generality. 4 students (16%) identified consecutive with bigger. 1 students wrote down a tautology "consecutive the consecutive". All the students answered correctly the 2nd and the 3rd questions, but only 10 students (40%) answered that it's impossible to find the consecutive number of a rational number because of the density. Other 4 students didn't write down the consecutive number but didn't comment; maybe they understood or not, so the percentage of correct answers is between 40% and 56%. 4 (16%) students showed a partial understanding: the first stated that the consecutive exists but we couldn't write it because of the density; the second stated that there are infinite numbers between -1 and 0 but then wrote down the consecutive numbers adding one to the last digit to the decimal representation; the third stated that there are consecutives in the et of rational numbers because of the density, but answered that there was always the consecutive in the set of real numbers and it was the minimum of the set of the numbers bigger than a given number; the last stated that the consecutive element didn't exist for algebraic irrational numbers expressed in form of radicals and rational numbers expressed in form of fraction but wrote down the consecutive number of rational numbers expressed in form of finite decimals. Furthermore there were 3 (12%) answers completely wrong and 2 (8%) missing answers.

All the students said that the first couple of subsets of Q was a Dedekind's cut. 24 students (96%) answered correctly that the number was rational, but 4 of them didn't indentify the correct number. In the second case just 3 student answered correctly that the couple was a Dedekind's cut. There have been other interesting answers, even if not correct, in which the errors didn't concern the new concepts. The most of the students made mistakes solving the inequalities that were defining the subsets; many students also didn't interpret correctly the connectives. The representation we chose for the couple of sets revealed to be very unsuitable for the students. In particular the use of polynomial 2-degree inequalities affected in a very negative way their strategies since they recognized the representation as a 2-degree inequality they were used to solve and used the usual procedures, ignoring that the numbers were rational. The students who transformed the inequality in the graphical register in particular came to contrasting results and in the end they got lost in the procedures because of a loss of sense. At a closer view this undesired happening let us reason about a crucial point: students are used to solve inequalities without taking in account the domain and use the number line in ways that veil the properties of R as a complete and the structure of the set Q as it's represented onto the line.

Also an important remark concerns the algebraic numbers. The students are not used to distinguish between rational numbers and rational numbers representation onto the number line since they act in the same way when they have to put the two kind of numbers on to number line, i.e. transforming any other possible representation into the decimal one and then approximate by truncation, at least arroundment. This may cause serious problems when trying to teach them real numbers since properties of the sets of numbers N, Z, Q , R vanish using the line, realm of finite numbers instead of being a privilidged representation of the set R. The link between the line and the set R is its continuity, properties that we can only partially associate the completeness of the numerical set R but rather belongs to intuitive, space-time representation of motion and trajectory.
The answers to the fifth item lead to an interesting phenomenon: all the students but one didn't recognize in the couple:

$C = \{x \in Q \mid x < 0 \vee x^2 < 5\}$, $D=\{x \in Q \mid x > 0 \vee x^2 > 5\}$
the irrational number $\sqrt{5}$. This was the only item to cause so many problems to the students and the quantity of right answers isn't comparable at all with all the others. In particular it has exactly an opposite profile of answers in respect of 5.a, that was formulated in the same way but changing the inequalities that represented the set and the nature of the number, once rational, once irrational.
There are significant differences between the strategies and the argumentations that lead to wrong answers, but a common point is the attempt to solve the inequality instead of intepreting the expression ' $x \in \mathbb{Q} : x^2 < 5$ ' as ' all the rational numbers whose square is less than 5 '. The strategies used by the students to solve the inequality in the domain Q showed habits in her previous didactical practices that we hypothesize to be in a causal relation with their errors.

First of all the students used the same procedure without considering the domain: if the domain had been Q or R or another they should have used the same procedure. This emerge clearly from the answers.

Also the loss of connection between the teaching experiment and the strategies is an interesting phenomenon to investigate. In fact we just modified the way to present the question and the students' behaviors changed in the respect of what we observed during the classroom discussion. In particular the changes we made affected the semiotic dimension of the presentatio.
The slides we used to introduce the Dedekind's cuts always presented at the same time two representations of subsets of Q, one in the graphic register, the other in the algebraic register. In the test we omitted the graphic representation but rather we stressed more the algebraic dimension, also using connectives and other logical symbols.
The answers to the two item were the following:
11 students stated that the second couple of sets identifies an irrational numbers (44%) but only 3 of them (15%) answer that this is a Dedekind's cut. This implies that 8 students on 11 were accepting that an irrational number can be not representable by Dedekind's cuts. We coulnd't infere certain conclusions since the students didn't state it explicitely but we deduced it by their brief answers. Since the phenomenon was very interesting we categorized the answers trying to figure out possible students' profiles to investigate better.

11 students on the 22 that answered solved the inequality on the margin of the text using the methodoly that defines the set R4. We didn't know which procedures or argumentations they used the other students to answer, but we hypothesized that in the most of the cases also the students that didn't explain their procedures could have found problems in the conversion between one representation and the other of the sets C and D, that lead them to change the practices and indeed to face another problem. For instance the students who transformed the inequality used to represent the set of the rational numbers whose square is less of 5 in a task that we can resume in "Solve the inequality" changed the nature of the problem ignoring the domain Q and applying a usual procedure. In the end some of them abandoned the attempt and didn't go on, the others tried to match the solution with the problem, but the core of the problem was indeed in the difference between the two procedures.

The previous practices carried out to solve inequalities ignored the domain and, even if the set of real numbers had not been studied, the solution had always been represented with segments and expressed in the verbal register as "all the numbers between
and
". The procedure can be summarized in two steps:

1. solve the equation obtained by changing the sign < or ≤ with =
2. represent the solutions, if they exists in the real numbers, on the number line and study the intervals that make the function positive or negative

The intervals endpoints may be rational or irrational, algebraic or trascendental. What numbers lie in the middle of the endpoints doesn't seem to matter. In this procedure the algorithm is more important than the concept and following the steps of solving the equation before solving the inequality it makes sense to use respectively < or ≤ to exclude or include the endpoints. In fact, if we can solve the equation, the numbers to use as endpoints are known and we can include or exclude them. In fact there are no doubts about the existence of a number to put in the place of the variable to make the expression equal to 0 since it is exactly the solution of the equation solved at the first step.
This way the line is already manipulated as if it was complete, even if the numbers associated by teachers and students to the points are only a subset of real numbers (rational and algebraic numbers).
This makes very hard to use this procedure to introduce new numbers that make it possible to find numbers in any cut, since students may already believe it's possible with the numbers the know.
(D'Amore, Fandino Pinilla, Santi, Sbaragli, 2011)

The categories, not incompatible, are:
In order to investigate better the phenomenon we decided to use a part of the last lesson to readminister a similar version of the question after giving the students a written note in which the two representations, graphic and algebraic, were used at the same time like we had done during the lessons. The relation between the two representations had been explained again. Only 19 on 25 students answered the new questionary.

The new questionary was composed by 2 questions.

For every couple of sets say if it is a Dedekind's cut and, if so, say if it's rational or irrational.

1) A = {x∈Q | x < 0∨x^2 ≤ 5},   B={x∈Q | x > 0∨x^2 > 5}

2) C = {x∈Q | x < 0∨x^2 ≤ 5},   D={x∈Q | x > 0∨x^2 > 5}

The first couple is very similar to the same with presented before, but the sign used in the first case was ≤ rather than <.

The second couple was defined by polynomial inequalities but of a different grade. The first subset was defined by means of the sign < , while for the second we used ≤, i.e. the contrary respect of the previous one.

We made two clear variation: once we differentiated between < and ≤, in the other case we changed the degree of the inequality in order to analyse the case of inequalities the students were not used to solve at school. We tried to figure out which criteria the students were using to decide if a couple was or not a Dedekind's cut.
The results were the following:
1)
a. Right answer (8)
b. No, because ±√5 are not in Q (2)
c. No, because A∩B=√5 (4)
d. No (3)
e. No answers (2)

5 students reported the solution of the procedure to solve the inequality on a margin of the text. We will use the letters to indicate a category we attributed to a student who answered the first question in order to see if the student remained in the same category in the second ot not.
2) To answer the second question some students changed their position.

One student belonging to 'a' changed strategy in the second case. In the first case he didn't solve the inequality but answered immediately. In the second case he solved the inequality and interpreted the sign $\leq$ as inclusion of the element and turned to the wrong strategy.
Two other students belonging to 'a' wrote only moved to 'd', without explanations.

Two students belonging to 'c' and one student belonging to 'd' changed their position in the second case stating that it's a section because $C \cap D = \emptyset$. In this case probably the sign $<$ instead of $\leq$ may have lead the students to think that the intersection was empty.

This changes confirmed that the algebraic representation is deeply linked to equations and to the practice we named R4 and that this representation changes indeed the nature of the problem from which the real numbers as Dedekind cuts emerge as a solution: to complete the set of numbers we put in correspondence with the line.

2. Research framework

2.1 Onto-semiotic approach (OSA) to Mathematics teaching-learning processes

The onto-semiotic approach (EOS, using the Spanish achronymus; OSA, using the English one) to mathematics teaching and learning stresses the personal dimension of knowledge, not necessary identifiable with the institutional one (D'Amore & Godino, 2006). This is very important in our analysis since the italian context is very eterogeneus from the point of view of teachers' formation because mathematics teacher can have a Master Degree in Mathematics but also in Physics, Statistics and Engineering.

2.1.1 Mathematical "objects" and meaning in OSA: system of practices and configurations of objects

Since this research aims to analyze teachers choices of representation of mathematical objects in the didactical transposition of real numbers, first of all we will clarify what we mean with the term *mathematical object* .

In mathematics education, as in epistemology and philosophy of language, both the terms mathematical concepts and mathematical objects are used frequently. These terms are often used intuitively as synonyms, but the distinction hides deep reasons. The different meanings of these terms depends first of all on different epistemological perspectives, that can be gathered in the main threads of pragmatism and realism (D'Amore and Godino, 2006). Also different nuances of theories, so pragmatic as realist, make sense of these terms in different ways according to his uses (Font, Godino and Gallardo, 2013). Both the interpretation of the terms *concept* and *object* are too complex to be explained fully in this work; for a complete dissertation see (D'Amore, 2001). In this paper we will draw upon terms of the ontosemiotic theory (Godino and Batanero, 2003; Godino, 2006; Godino and Font, 2007) and merely we will use the term *object.* We clarify the meaning of the term *object* and other terms used in this context in order to make it consistent with the onto-semiotic perspective we rely on.

Since the onto-semiotic approach to mathematics education is pragmatic, it is based on the definition of practice (P). Godino and Batanero (1998) consider "mathematical practice any action or manifestation (linguistic or otherwise) carried out by somebody to solve mathematical problems, to communicate the solution to other people, so as to validate and generalize that solution to other contexts and problems". The practices can be idiosyncratic of a person or institutional. We are merely interested in systems of practices (SP) that a teacher associate to the problems-situations concerning the domain of her didactical transposition. In this case we are interested in systems of practice involving single real numbers, subset of real numbers and the set R as a whole.

We opted for the definition of object and meaning by Godino and Batanero (1998), according to which we use the expression:

* *mathematical object O* indicating any entity or thing to which we refer, or talk about it, be it real or imaginary and that intervenes in some way in mathematical activity;

* *meaning of the object O* indicating the system of practices that a person carries on (personal meaning), or are shared within an institution (institutional meaning), related to the object O.

In the following part of this paragraph we will resume the elements of the onto-semiotic approach that will be used in this paper. All of these were presented in the synthesis paper by Godino, Batanero and Font (2007), that can be used as reference for the complete description of the onto-semiotic approach (that we will call *EOS* drawing on the Spanish achronymus).

Meaning can be characterized with different nuances.

Institutional meanings of the object O are linked to these types of practices of teachers:

- *Implemented*: system of practices effectively implemented related to O;

- *Assessed*: system of practices related to O used to assess students' learning;

- *Intended*: system of practices related to O planned;

- *Referential*: system of practices related to O used as reference to elaborate the intended meaning after an historical and epistemological study of the object O.

Personal meaning of the object O is characterized by these types of practices:

- *Global:* set of personal practices related to a specific mathematical object.

- *Declared*: the personal practices effectively shown in assessment tasks and questionnaires (institutionally correct or wrong)

- *Achieved*: personal practices that fit the institutional meaning fixed by an institution.

Personal practices are usually referred to students but in this paper we will refer them also to teachers as past students since the personal practices can affect teachers' choices the didactical transposition.

In mathematical practices are always involved ostensive (symbols, graphs etc.) and non-ostensive objects (objects brought to mind, unperceivable).

The six types of primary mathematical objects are defined in Godino, Batanero and Font (2007) this way:

• *Language* (terms, expressions, notations, graphics);

• *Situations* (problems, applications, exercises, ...);

• *Concepts*, given by their definitions or descriptions (number, point, straight line, mean, function, etc.);

• *Propositions*, properties or attributes;

• *Procedures* (operations, algorithms, techniques, ...);

• *Arguments* used to validate and explain the propositions and procedures (deductive, inductive, etc.).

New objects can emerge from the system of practices, with an organization and a structure (types of problems, procedures, definitions, properties, arguments). These objects will form configurations (CO), i.e. network of objects characterized by relationships established between them.

Objects and relationships (CO) emerge by mean of sequences of practices, called processes. The primary objects emerges throughout the respective primary mathematical processes of:

* *communication;*

* *problem posing;*

** definition;*

** enunciation;*

** elaboration of procedures (algorithms, routines, ...);*

** argumentation.*

2.1.2 Dualities

Objects can have different facets, that will be coupled in pairs since they're complementary:

- *Personal – institutional*: personal objects emerge from practices carried on by a person; institutional objects emerge from systems of practices shared within an institution
- *Ostensive – non ostensive*: mathematical objects (both at personal or institutional levels) are non perceptible. However their associated ostensives (notations, symbols, graphs, etc.) allow to use them in practices. The distinction between ostensive and on-ostensive depends on the language game in which they take part. Ostensive objects can also be thought, imagined by a subject or be implicit.
- *Extensive – intensive:* this duality deals with the use of generic elements in mathematics pratices and merely let us distinguishing the particular and the general.
- *Unitary – systemic: "*in some circumstances mathematical objects are used as unitary entities (they are supposed to be previously known), while in other circumstances they are seen as systems that could be decomposed to be studied. For example, in teaching, addition and subtraction, algorithms, the decimal number system (tens, hundreds, …) is considered as something known, or as unitary entities. These same objects in other grades are to consider as complex objects to be learned." (Godino, Batanero & Font, 2007)

- *Expression – content*: expression and content are the antecedent and consequent of semiotic functions. All kind of objects can play both the roles for a subject that establish the relation through a semiotic function.

2.1.3 Strategies to connect old and new configurations of objects

To explain the nature of the relationship between objects, the definition of semiotic function is crucial, since it allows to address one goal of EOS we refer to: capturing the intrinsically relational nature of mathematics.

Semiotic function is "the correspondences (relations of dependence or function) between an antecedent (expression, signifier) and a consequent (content, signified or meaning), established by a subject (person or institution) according to certain criteria or a corresponding code." (Godino, Batanero and Font, 2007). All the six different types of objects can be expression or content of the semiotic functions. The relations of dependence between antecedent and consequent can be:

- representational (one object put in place of another),

- instrumental (an object uses another as an instrument),
- structural (two or more objects are parts of a system from which new objects emerge).

2.1.4 Didactical suitability

Didactical suitability concerns <<objective criteria that serve to improve the teaching and learning and guide the evaluation of the teaching/learning processes. [...] *Epistemic suitability* measures the extent to which the implemented meaning represents adequately the intended meaning (the curricular guidelines for this course or classroom). *Cognitive suitability* is the degree to which [...] the implemented meaning is included in the students' zone of proximal development, and whether the students' learning (personal meaning achieved) is close to the intended meaning. *Interactive suitability* is the extent to which the organisation of the teaching and the classroom discourse serve to identify and solve possible conflicts and difficulties that appear during the instructional process>> (Godino, Ortiz, Roa & Wilhemi, 2011).

The criteria of didactical suitability we planned to investigate in our a priori analysis are those used in other OSA's prototypical investigations: epistemic representativeness, declared cognitive proximity and declared strategies of negotiation of meanings in the didactical interaction. In Ordoñez y Wilhelmi (2010) a typical methodology to investigate teachers' reflection on their practices is presented; in particular in relation to the epistemic meaning the teacher should be able to explicit the objects and processes involved in the didactical practice and to propose changes to make it as best as it can be (Pino-Fan, Godino & Font, 2014).

2.1.5 CDM (*Conocimiento Didactico Matematico*): a proposal for the analysis of the teachers' knowledge

In the model CDM, proposed by Godino (2009), the teachers' knowledge's dimensions that may affect the design and realization of teaching sequences are three: 1) mathematical; 2) didactical; 3) meta didactical-mathematical.

The mathematical dimension concerns the teachers' knowledge of the mathematical configurations of objects and processes, categorized in:

- common knowledge (CK)
- advanced knowledge (AK)

The didactical dimension concern the knowledge about the six aspects of the teaching-learning processes: epistemic, cognitive, affective, interactional, mediational, ecologic.

The meta-didactical-mathematical dimension regards the knowledge the teacher need to reflect on and evaluate the suitability of the teaching-learning environments and actions they design and realize.

The notion of configuration of objects and processes is a tool that allows to describe the mathematical practices' complexity and to use that complexity to interpret the cognitive conflicts (Tall & Vinner, 1981) emerging in the teaching-learning processes.

2.1.6.1 Didactical transpositions in ATD: past and current definitions

The concept of didactical transposition in mathematics education was introduced by Chevallard in 1985. As it happened for many other constructs, it was declined in many forms by the author itself and other researchers, also belonging to other research fields. In this paper we will refer to the review article by Chevallard and Bosch (2014), according to whom didactical transposition is an analytical instrument to avoid the <illusion of “authenticity” of the knowledge taught at school>. Didactical transposition consists in rebuilding <an appropriate environment with activities aimed at making this knowledge “teachable,” meaningful, and useful>.

The institutional dimension is the most stressed in this framework, so much that the concept of didactical transposition was generalized to institutional transposition: the knowledge is transposed from one social institution to another (citare). In this change it “is thus important to understand the choices made in the designation of the knowledge to be taught and the construction of the taught knowledge to analyze what is transposed and why and what mechanisms explain its final organization and to understand what aspects are omitted and will therefore not be diffused.” Teachers, as long as producers of knowledge and curriculum designers, contribute in this decision process. The different weights of teachers choices depend on the autonomy of teachers in the school system.

The previous conceptions of didactical transposition, proposed by Chevallard in 1985 and then in 1999, stressed more the importance of the transformation of the knowledge in three main phases and three related “identities” of the mathematical knowledge at school: 1) Savoir savant 2) Knowledge to teach; 3) Taught knowledge. The passage from the 1st to the 2nd step was called Didactical transposition, while the second was called Didactical Engineering (Chavallard, ; Sarrazy;).

2.1.6.2 Institutional and personal relation with the mathematical objects in ATD and OSA

In order to rebuild the knowledge to make it teachable, meaningful and useful making transpositional choices the teachers need to decompose *mathematical objects’s* structure and explore their nature, so as to select aspects to omit and to diffuse and to better understand what to change, how to transform the and finally how to rebuild the aspects of the mathematical object we chose to teach.

While in the first formulation the concept of mathematical object was more static and somehow universal, the second version, developed thanks to the new conception of *praxeology* (Chevallard,), took in account more the pragmatic dimension of mathematical objects and its relation with the social human development in communities and institutions. *Praxeologies* – composed by praxis and logos – include <the study, not only of what people do, and how they do it, but also of what they think, and how they do so> (Chevallard, 2005). Chevallard proposed the quatern (task, tecnique, technologies, theories). Mathematics turned to be considered as a product of human communities of practices, more and more shared, widespread and consolidated through reflexive practices that lead progressively to theories. These processes were named institutionalizations of knowledge, whose products were indeed a set of cultural objects, that we call Mathematics.

This conception of Mathematics inspired a true revolution in Mathematics education, describing in a complex manner the different natures of the mathematical practices and permitting to rethink the design of didactical activities and curricula.

In the framework called ATD, Anthropology Theory of Didactics (Chevallard,), it was also stressed the role of the personal relationship between the subject and the mathematical objects, but this aspect was explored more by the authors of another framework, inspired to ATD and many other previous contributions and theories of Mathematics education: the onto-semiotic approach to Mathematics teaching and learning investigation (OSA).

2.2 Schoenfeld's theory of goal-oriented teachers' decision-making

As theoretical framework for teachers' choice we chose the goal-oriented decision-making theory by Schoenfeld (2010). Drawing on this theoretical framework we will consider teachers as decision-makers, whose choices are determined by "their *resources* (their knowledge, in the context of available material and other resources), *goals* (the conscious or unconscious aims they are trying to achieve), and *orientations* (their beliefs, values, biases, dispositions, etc.) [...] at both macro and micro levels." (Schoenfeld, 2010, p. 14). The choices are embedded in the institutional context, as explained above.

3. Research design

3.1 Research problems and research questions

1) In order to evaluate the instructional processes' epistemic suitability we need to analyse deeply the complexity of real numbers as a mathematical object (in the sense of OSA).

GQ - 1 How can we describe the complexity of the teaching-learning processes involving real numbers and continuous sets from an epistemic and cognitive point of view?

PQ - 1.1 What is the epistemic meaning of real numbers?

PQ - 1.2 What are the teachers' personal concepts of real numbers?

PQ - 1.3 What relations are there between teachers' formation and their personal conception of real numbers (personal objects)?

PQ - 1.4 What practices concerning real numbers do the teachers declare to prefer and to choose?

PQ - 1.5 What are the relations between teachers' personal object, teachers' orientation and goals and choices?

PQ - 1.6 Is any categorization of teachers' profiles possible?

2) If the systems of practices from whom an object emerge are not rich enough in terms of epistemic representativeness, when the teachers tries to involve students' in discourses concerning the object maybe the students will not be able to participate in the discourses

GQ - 2 Are the teachers' choices epistemically and cognitively suitable ?

PQ - 2.1 How the practices declared by the teacher connected?

PQ - 2.2 Are there systems of practices in which are involved objects whose meaning is not rich enough?

PQ - 2.3 May teachers' search for cognitive suitability cause the lack of epistemic suitability in the case of real numbers?

3.2 Methodologies of qualitative research

Our methodology was inspired by those presented by Neuman and Pirie in the monograph on qualitative research in mathematics education edited by Teppo (1998), even if the subject involved in their researches were students. As Neuman we opted for a phenomenographic observation technique and for the construction of descriptive categories. While in this framework the results depend strictly on the observative model, in our research analyzing data other questions emerged that we didn't consider in the beginning. This is contemplated by Pirie's methodology, that allow to let emerge new questions from data, going on with cyclic analyses and making more and more precise the categorization.

3.3 Research methodology

We interviewed 116 mathematics teachers in the high school in Italy. Teachers were asked to answered a questionnaire designed to investigate:
- teachers' formation (master degree, training courses attended)
- teachers' knowledge (configurations of objects they associated to R set)
- the practices chosen by teachers involving elements or subsets of R or objects traditionally used in the constructions of real numbers like inequalities, Q etc.
- the semiotic representations of subsets of rational and real numbers they consider best in order to address a goal.

In the first part teachers were asked to answer direct open questions (the question intent corresponded to the request) about knowledge and goals, while in the second part, more maieutical, the questions aimed at making come into light preferences and orientations.
Using Schoenfeld's model's categories (2011) some of the questions about knowledge give information about resources, the other questions concerning knowledge concern the goals of didactical transposition. Questions about practices and semiotic representations were posed in indirect form. Teachers were asked to comment videos about a construction of using a graphical method, (http://www.youtube.com/watch?v=jk08WkwqT_Q), an applet titled "Bijection between real numbers and

points of a line" (http://www.youtube.com/watch?v=kuKTyp_b8WI) and a video-recorded lesson carried out by a teacher about different ways to solve and to represent the solution of inequalities (http://www.youtube.com/watch?v=UEBK5DfPxvk). Teachers were asked to indicate preferences about didactical materials and students' answers - in order to investigate orientations.
After answering the questionnaire the teachers were interviewed in focus groups (3 or 4 members) in which we guided a discussion on questionnaire answers in order to make them explicit their personal choices and the reasons of their choices and to investigate their general orientations concerning the didactical transposition of real numbers and students' difficulties.
We defined some *a priori* categories standing on research results in the fields of goal-oriented teachers' decision-making and didactical suitability, presented in the framework. We established as a criterion for collocating a teacher in a category the presence of sentences that unequivocally let us deciding if she belonged or not to the category. Some further categories emerged after the first data analysis and other categories were created. Only significant categories are reported.

3.4 CDM analysis of the methodology

In this Paragraph, inspired by the analysis proposed in Pino-Fan, Godino & Font (2014), we present the analysis of the Didactical-mathematical knowledge (CDM, to use the Spanish acronym) of the in-service teachers, proposed by Godino (2009) and framed in the OSA (Godino, Batanero & Font, 2007). CDM was elaborated standing on the models proposed by other authors about teachers' mathematical, didactical and pedagogical knowledge in a wide sense .

There could be significant differences between teachers' declared practices and teachers' real practices and also there are other dimensions of suitability of the didactical actions but in this particular case we conjecture that unsuitable choices in the designing of teaching sequences vanish all the attempts to make effective the didactical actions and furthermore that deficiencies or unawareness in this phase imply unavoidably bad choices concerning the other dimensions.

We used OSA's methodologies and analytical tools to analyse the practices the teachers report from their didactical past experience concerning real numbers and continuous sets or declare to do in their classrooms usually, depending on the kind of students they are working with. In particular we will analyse the configurations emerging from the different practices they declare to plan or to have already experienced and we will compare the configurations they would need in the teaching sequences they propose with the ones we can hypothesize they use standing on their statements and reports.
We interviewed 116 mathematics teachers in the high school in Italy. Teachers were asked to answered a questionnaire designed to investigate:
- teachers' formation (master degree, training courses attended)
- teachers' knowledge (configurations of objects they associated to R set)
- the practices chosen by teachers involving elements or subsets of R or objects traditionally used in the constructions of real numbers like inequalities, Q etc.
- the semiotic representations of subsets of rational and real numbers they consider best in order to address a goal.
In the first part teachers were asked to answer direct open questions (the question intent corresponded to the request) about knowledge and goals, while in the second part, more maieutical, the questions aimed at

making come into light preferences and orientations.
Using Schoenfeld's model's categories (2011) some of the questions about knowledge give information about resources, the other questions concerning knowledge concern the goals of didactical transposition. Questions about practices and semiotic representations were posed in indirect form. Teachers were asked to comment videos about a construction of using a graphical method, (http://www.youtube.com/watch?v=jk08WkwqT_Q), an applet titled "Bijection between real numbers and points of a line" (http://www.youtube.com/watch?v=kuKTyp_b8WI) and a video-recorded lesson carried out by a teacher about different ways to solve and to represent the solution of inequalities (http://www.youtube.com/watch?v=UEBK5DfPxvk). Teachers were asked to indicate preferences about didactical materials and students' answers - in order to investigate orientations.
After answering the questionnaire the teachers were interviewed in focus groups (3 or 4 members) in which we guided a discussion on questionnaire answers in order to make them explicit their personal choices and the reasons of their choices and to investigate their general orientations concerning the didactical transposition of real numbers and students' difficulties.
We defined some *a priori* categories standing on research results in the fields of goal-oriented teachers' decision-making and didactical suitability, presented in the framework. We established as a criterion for collocating a teacher in a category the presence of sentences that unequivocally let us deciding if she belonged or not to the category. Some further categories emerged after the first data analysis and other categories were created. Only significant categories are reported.

The questionnaire

The version of the questionnaire we used to investigate the teachers' didactical-mathematical knowledge (CDM, Godino, 2009) is the result of an elaboration of a previous pilot version we tested with 20 in-service teachers and that we asked researchers and teachers to evaluate.
We asked to answer online the definitive version of our questionnaire to 107 in-service teachers. The teachers could use books and other resources, answer the questionnaire using a lot of time and re-opening the questionnaire at any time after the conclusion of the questionnaire. This way we gave the teachers the time to rethink at the mathematical object and at their teaching-learning experiences.

Not all the teachers answered the whole questionnaire but anyway we took in account their answers for quantitative and qualitative analyses, even if the *case-studies* we used for our conjectures and conclusions are based on completed questionnaires and interviews.

The 116 in-service teachers were all Italian but had very different backgrounds from the point of view of formation (PhD in mathematics, Master Degree in Mathematics, Teachers training courses, Master Degree in STEM disciplines different from Mathematics, like Physics, Engineering, Statistics or combination of these like PhD in Mathematics who attended teacher training courses and so on). Also they had many different features like different ages, different teaching experience, different regional provenience, different socio-economics contexts in which they had taught, different kind of high school (technical, humanistic, scientific, classical, professional high schools).

The reasons why in the first part of the investigation we decided to ask different teachers' to answer the questionnaire were merely three:

- in Italy there are many possible paths that allow to teach Mathematics in the high school and, being the mathematical object we are analysing very complex, the teachers' mathematical knowledge concerning this object could be very different. In order to investigate teachers' choices and to propose possible teachers' training courses concerning this topic we needed to have a wide panoramic on different possibilities.
- Since we were going to correlate teachers' CDM (Godino, 2009) and their declared choices, a research problem we consider interesting in this case is to understand how far teachers knowledge influence teachers' choices. We conjectured in the very beginning that, given the nature of this mathematical object, in some aspects intuitive and formally very articulated, the choices proposed by the teachers about some of the partial meanings could be very similar, in spite of their personal knowledge.
- Teachers with different background or teachers who work in very different context may have different goals and so they may plan differently the teaching sequences or have different opinion about the effectiveness of teaching practices.

The questionnaire was composed by 18 questions. The first questions concerning teachers' knowledge concern the mathematical aspect.

In the first question we asked the teachers informations about their background: which Master degree they obtained, which certificated teachers' training courses they attended, in which kind of high school they teach actually and since how many years they are teaching.
The aim of this first question was to identify the teachers and so to interpret better their answers.

In the second question we asked the teachers informations about their learning experiences concerning real numbers, giving the opportunity to choose more than one possibility between: at school, at the University in a Calculus course, at the University in other courses, in a teachers' training course, in divulgative books and in the original books as a self-taught.

In the third question we asked the teachers what are the main features of the set of real numbers in an open format in which the teachers could answer freely.

In the fourth question we posed a crucial question concerning the way the set of real number can be constructed starting from Q as an enlargement. The formulation of the question was explicitly elaborated in order to recall a very widespread practice of introduction of real numbers when the teachers introduce quadratic equations that have no rational solutions or geometric magnitudes whose lengths, in respect of a given unity, needs to be expressed with a not rational root square. Many textbooks and teachers we had talked with reported the usual discourse concerning real numbers as the set in which it's possible to find the root squares of numbers - not always expliciting positive numbers. The answer to this questions allowed us to discriminate very soon between teachers who had studied and understood at least one real numbers set's construction. Indeed, as we explained before, the large heterogeneity of the paths that permit to the people to

become mathematics teachers in the high didn't guarantee that the teacher had neither a good common knowledge concerning real numbers. If the teachers as students learnt that the set of real numbers could be constructed by means of an enlargement of Q with roots (algebraic extension of Q) and some transcendental numbers, maybe as a teacher she would re-propose the same scheme, being this partial meaning her meaning of real numbers.
Even if a written answer doesn't allow to figure out complete profiles of the teachers, the way we expressed the question after our pilot study permitted to us to differentiate the teachers in categories by means of their written answers.

In the fifth question we explored more in depth the teachers' knowledge about the differential and topological structure of R and its relation with the structure of Q, asking if it was possible to define a limit point in Q or if we need R. The aims of these question were indeed more than one. This question permitted us also to classify teachers in relation to their didactical goals and to understand the role they give to real numbers in the Calculus, that is the field in which, depending on the planned didactical sequences, the properties of real numbers could be really essential. In fact we asked to the teachers if the limit point, usually introduced in the high school in order to talk about the limits of functions in the boundaries of their domain, need the properties of real numbers to be introduced or is an independent object, that we can define in a field with less properties. The teachers we interviewed in the pilot study and the textbooks we analysed introduced indeed real numbers a lot of time before the limit points. We consider this point crucial. In fact the didactical sequences in which the introduction of real numbers arrives very early, indeed the whole meaning comes to be identified with a partial, very poor, meaning. This can deal to the following kind of conflicts or deviations from the original meaning:

1. the students may not be able to fit the old meanings of real numbers with the concept of limit point, crucial to pose the problem of the construction of a complete set, without a specific discourse in which the different branches in the epistemic meaning tree;
2. to quote R as the set who contains all the limit points of sequences of Q, R with high probability R must appear as a unitary entity, maybe represented with a line or simply indicated with the symbol R; in both the cases the choice is quite problematic since as we observed before, standing on many results (Bagni, Sbaragli, Azcarate i Gimenez, D'Amore & Arrigo, Tall,) the line is not usually interpreted as dense by the students, nor truly composed by infinite elements and the visual representation is very often the only one associated to R as a unitary object. Since we need topological and differential local configurations to talk about dense sets, necessary to introduce limit points, there is a concrete risk to use partial meanings of real numbers that are not complex enough to address this topic. This can cause cognitive conflicts.

The two following questions concern the teacher's relation with the national curricula and possible functional goals of the introduction of real numbers. This last point is crucial since, while the real numbers should be part of the curricula, often they are introduced to solve other problems, of different nature, or to define other concepts rather than being the direct goal of a learning sequence. Since this could be very relevant for analysing teachers' practices concerning real numbers but maybe the teacher would have take it for granted, we listed explicitly possible contents in which the properties of real numbers could be considered necessary.

Once established which was the teachers' aware knowledge concerning real numbers and what were the teachers' goals related to real numbers, we went on showing the teachers some videos reporting very frequent teaching practices concerning real numbers. Two of these (D8 and D9) were explicitly aimed at making emerge partial meanings of real numbers, the last one should use and represent subsets of real numbers, at least in the the original intention. We know from the literature that students', and sometimes teachers', interpretation of the properties of subsets of real numbers in the "algebraic" and graphic register are quite different from those we can expect standing on the institutional meaning. Also through this question we could explore the way the teachers perceive the relation between the "algebraic" and the graphic representations of the intervals, two different partial meanings of real numbers historically connected in a very complex and interesting way.

The last questions concerned tasks assigned to the students and the degree of acceptance of the correctness of the solutions that the students proposed. Each of the solutions could provide us relevant informations concerning the representation of intervals of numbers and the association of personal meanings to them by the teachers. In this questions teachers were expected to choose the solution they considered right and to say why the other were wrong. Asking the teachers to comment on the correctness we ask indeed them to interpret the signs and so to associate their personal meanings to them. Furthermore we ask them to put in the clothes of a student who answered that way, in order to see how the teachers correct the students, i.e. if they take in account their personal meaning or correct the students referring to their own meanings.

A priori analysis of the tasks proposed to the teachers concerning the mathematical aspect of CDM
Standing on the assumptions of the model CDM the questions posed to the teacher must have two main features: they must provide information about the degree of correctness of the teachers' personal meaning in respect of the global meaning of the mathematical object ((Pino-Fan, Godino & Font, 2011; Pino, Godino & Font, 2014); they must allow the researcher in the phase of data analysis to investigate the different representations, the partial meanings and the possible configurations that the teachers know and may use in their practice. The choice of the representations was inspired by the works: Font (2000) in the case of derivatives, reported in Pino, Godino & Font (2014); Pino-Fan, Godino y Font (2013).
This last point is very important in our case since, as we highlighted before, there is not a unique final configuration but there are instead more than one meaning and the meanings are not commensurable and synthesizable in a mathematical object
Standing on this observation it's also more significant to investigate the partial meanings associated to the signs since these could be confused or mixed.

The kind of representations and meanings we explored, in line with our first characterization of the epistemic meaning of real numbers, were those implied in:

1. *common knowledge* (CK), i.e. the representations and practices involving real numbers most used in

the italian textbooks and the most recurrent in our pilot study;

2. *advanced knowledge* (AK), i.e.the representations and practices involving real numbers used in the historical steps of evolution of the concept and in the more advanced mathematical practices and in the interdisciplinary significant connected meanings;
3. *knowledge of partial meanings* (PEM) of the epistemic meanings involved in different practices that could be the meaning the students associate to the representations or the partial meaning required in a specific practice or in an historical version of a practice (for instance introducing the root square of 2 using the Pythagoras' meaning of incommensurability requires the partial meaning of numbers as linear magnitudes).

We will use before a common set of a-priori categories to label the teachers cognitive configurations, the we will go on inverting the direction and creating new a-posteriori categories.

Then in a second phase of the research we will associate these categories to the teachers' declared choice and reported experiences in order to evaluate the weight of the personal cognitive configurations in the teaching sequences planning and in the use of the concepts in the practices.

The questions we posed in the different fields were also of a different kind. In particular we investigated:

1. CK and PEM using videos reporting didactical proposals on partial meanings of real numbers and reports of students' answers to tasks we proposed in the pilot study concerning intervals of rational and real numbers.
2. AK using direct, both open and cloze questions, and asking comments on representations of intervals that could be used also in other mathematical contexts. The aim of the cloze questions was to be sure to have information about crucial nodes.

Questions

There are more than one reason why the teachers may have answered that R is necessary. We synthesized them in some possible a-priori categories:

1. The teachers know that Q is dense and that the density is the needed feature but thinks that not all the limit point of rational sequences are rational numbers so introduce before the set that contain all these points before introducing limit points;
2. The teachers know that Q is dense and that the density is the needed feature but are used to use limit points in the particular case of limits of rational sequences and recreate the ambient in which they studied limit points without making new choices;
3. The teachers don't know the density is the needed feature and are used to use limit points in the "scholastic Calculus" that concerns real function in the real domain.

Analyzing the teachers belonging to the category K4_B_a we detected great differences in their profiles. This is interesting since it's a counterexample to the assumption that a different background necessarily leads to different choices. In the Chapter 3 we will discuss this topic about all the categories of teachers and choices we created.

In this Chapter we comment specifically on the choice a., that is particularly interesting if it's compared to the choice of introducing real numbers by means of a particular irrational number, usually the root square of 2.
Some teachers who were interviewed while they were answering the questionnaire and other teachers who commented on the answer in a written form explained this choice saying that not all the limit points are rational so they need R to introduce this concept.
To put light on this complementary choices, we will define:

1. "whole to single" choices, i.e. choices that put the whole before the single element, constructing before the "new world" in which an element find a place thanks to common properties we define before and then an example of possible element;
2. "single to whole" choices i.e. choices that put the single element before the whole, using the single element, critical and intrinsically characterized by properties that are then extended to create a "new world", as a particular element in which recognize properties that will be common to the future elements of the whole set.

An aspect of this distinction may be reframed in the OSA's duality extensive-intensive but there are other possible nuances in the case of real numbers.
In this context we are interested in comparing the different teachers orientations in relation to the practices:

1. constructing real numbers starting from Q
2. defining limit points

We first of all show prototypical examples of "single to whole" and of "whole to single" choices and then we will comment on the particular case of teachers who make inconsistent choices interpreting a priori a possible cause for the inconsistency.

Before starting we have to remark that inconsistencies at this level are not surprising, in particular in the case of real numbers and continuity, as Tall & Vinner explained in depth yet in 1981. As the authors interpreted using the constructs concept image and concept definition and the cognitive conflict, a person can perceive an inconsistency if a conflict is presented to her in a form in which she can become aware of it.
We are now re-describing and re-interpreting the apparent inconsistency between the choices using a more complex set of tools that allows us to go beyond the difference between concept image and concept image definition and to extend adequately these categories in order to make the analysis a tool for training the teachers and to provide solutions to the "solve" the inconsistency that we can previse to lead the teachers to unsuitable choices.

Root square of 2

1. "Whole to single": *Let's consider the set of real numbers. Some of its elements are rational while some other elements are not rational. We can provide an example of this kind of not rational elements: the square of 2. This number represent the diagonal of a 1-edged square.*
2. "Single to whole": *Let's construct a 1-edged square and let's draw its diagonal. We can project this segment on a line putting the left endpoint of the diagonal in 0. We can prove that there are no rational numbers equal to the root square of 2, so a not rational number exists. Let's now consider the set obtained joining Q the set of this new kind of numbers, that we call irrational numbers.*

Limit point

1. "Whole to single": *Let's consider the set of real numbers. Its rational elements are limits of rational Cauchy's sequences, but not all the rational Cauchy's sequences converge to rational points. Let's define a limit point as the point whose circular neighbourhoods minus the point itself always contain at least a point, whatever their radius is. Some limit points of Q are not elements of Q. The set that contains all the limit points of Q is R.*
2. "Single to whole": *Let's define a limit point as the point whose circular neighbourhoods minus the point itself always contain at least a point, whatever their radius is. Q is dense so all its elements are limit points in Q, but some limit points of Q are not elements of Q. Let's consider the rational Cauchy's sequence defined by the condition: "The square of the element is lower than 2". It converges but not to a rational point and this point has all the features we require. We can construct a set that contains all the limit points of Q: the set of real numbers R.*

In the first kind of introduction R is given, it appears in the discourses in an extensive form; in the second case we use the discourse to define and sometimes construct the set of real numbers.

What can be problematic in the first case is that R is a very complex mathematical object and that different kind of practices may lead to quite different meanings.

In other words if we recall a partial meaning of real numbers by means of the sign R the teachers should first of all check the students' personal meaning since they are not sure it is the needed meaning. Also they should take care of the relations the students are able to establish between objects emerged in previous different practices.

For instance we could introduce the set of real numbers as the set containing all the rational and irrational numbers and then prove that the root square of 2 is irrational ("whole to single") as an example of irrational numbers, taking care of clarifying the existence of a semiotic function that connects fractions, used in the proof of the irrationality of the root square of 2, to decimal numbers.
This new configurations is possible to construct since the partial meaning used are coherent and sufficient,

but this is again a partial meaning of real numbers and the required meanings are very simple: a student must know that a rational number is a number that we can represent by means of ratios between whole numbers and must know what a prime number is.
With these “poor materials” we can obtained a quite poor meaning of real numbers: a set composed by numbers that we can represent in a decimal form, that we can compare and that we can’t always represent by means of fractions. Nothing more.

Let’s now consider the limit points. In this case if we wish to introduce the concept of limit points basing our definitions on a previous knowledge of real numbers we need much more. As we anticipated the crucial property of a set in which we can define a limit point is the density. The property of density is not easy to represent and the objects emerging from the teachers’ practices all along the school years may have properties different in respect of the desired one (Bagni, 1999). Also the students need to know that a not rational point on the line can be around by rational numbers whatever the radio of the neighborhood we consider is, so they need to know the differential structure of the points of the line. Furthermore, even if we explain to them what a rational Cauchy’s sequence is, it’s not trivial to prove that an irrational number can be thought as limit of a rational sequence. Historically the considerations that lead to construct the set of real numbers was in the reverse order, i.e. the real numbers was thought by Cauchy as limit points of the sequences that didn’t converge and then he defined a the property of Cauchy-completeness, that is the discriminant property to distinguish Q and R.
Two main remarks are so necessary:

1. the teachers can’t count easily on previous knowledge of real numbers when they introduce the limit points of Q because the student would need a very complex partial meaning of the set of real numbers that include its differential structure; for instance partial meanings like that presented before for the root square of 2 are not enough.
2. if the teacher introduce the set of real numbers as Dedekind cuts, or use dynamic representations for the limits, we already showed that the partial meanings don’t include the differential nor the topological properties of R, so they’re not suitable even if they are complex configurations. This observation is crucial since yet in the textbooks’ analysis we found attempts to introduce the real numbers only focused on the features of their single elements, in particular in the first years. This is easy to understand since the students’ own poor partial meanings and this introductions aim at finding a new place to numbers emerging from algebraic or geometrical problems, but it’s important to make the teachers aware that this partial meanings are not sufficient to introduce limit points.

There are many reasons to consider the approach “whole to single” more suitable in the case of the root square of 2 rather than in the case of the limit point, nevertheless we saw more rarely this approach in the answers concerning the first rather than the other.

Let’s now consider the other approach (“single to whole”). In this case the introduction of square of 2 is more critical and potentially problematic than the other.

When a teachers introduce real numbers using the root square of 2 he is indeed using a particular example that may lead to a different general elements of real numbers. First of all this generalization from an algebraic number to a general irrational number don't include the distinction between algebraic and transcendental numbers.
Then other differences lies in the kind of process that, stemming from the particular element, drive to the whole set.
First of all the generalization of the properties of the root square of 2 can be carried out in different ways depending, time after time, on the practices and the partial meanings that emerged from the practices the teacher choose to introduce it. For instance possible paths are:

1. the root square of 2 is introduced as a case of incommensurability between segments;
2. the root square of 2 is introduced as the solution of a quadratic equation;
3. the root square of 2 is presented as an irrational numbers and then is approximated more and more by means of couple of rational numbers, one greater and one lower;
4. concrete examples of problem-situation involving diagonals or circles are described, an "irrational problem" is posed and then the root square of 2 is used to indicate a concrete object or a measure of a concrete object.

Starting from these four starting point we stress different features of the elements of the new set we are creating. This observation are important also in the analysis of the group of questions Q8-9.

There are many reasons to consider the approach "single to whole" more suitable in the case of in the case of the limit point rather than in the case of the root square of 2, nevertheless we saw more rarely this approach in the answers concerning the first rather than the other.

Furthermore in our pilot study we didn't observe a coherence in the teachers answers in the two cases we analyzed, so the preference for one approach or the other doesn't seem to be a strong orientation but rather a context-dependent contingency, maybe based on didactical traditions. Of course both the two directions are necessary to construct the meaning of real numbers and a teacher can't be considered inconsistent if once decide to move from single to whole and once from the whole to the single element. What we are discussing here is the possibility of realizing such a "generalization" or on the contrary "exemplification" when the mathematical object is the set of real numbers, why the teachers sometimes consider necessary to know real numbers in advance and to use it in the extensive meaning and sometimes not and how far the teachers are aware of the fact the some paths are possible and other are unsuitable.

In the case of real numbers the semiotic functions must be treated with particular attention since the general real numbers is a very complex object, whose nature has been debated for long time and is epistemologically and logically hard to grasp.

Also we didn't observe a strict relation between the richness of the teachers' mathematical knowledge and a reasoned approach to this matters in the pilot study. In the a-posteriori analysis we present a detailed analysis of the interviews in which we detected interesting dynamics reported by the teachers that permit us to understand better the reasons of this apparent inconsistency.

3.4. Cognitive configurations in the question Q7

Thinking more closely to the complementarity we analyzed before, we understood that that was indeed only a particular case of a more general complementarity in the teachers' choices about real numbers and the continuum, that we can synthesize as it follows:

1. to introduce R before introducing other objects, using R as extensive in the discourses about the objects quoted before;
2. to introduce the objects necessary to make a configuration of R from a sequence of practices emerge and then construct R as a configuration of objects emerged before.

Teachers may be consistent or not in reference to this choice, as we saw in the analysis of the different approach to the introduction of the root square of 2 and of the limit points.

3.4 Research hypothesis

Our research hypothesis is that often the configurations of objects and processes used to introduce other situations-problems and to construct other configurations are not complex and rich enough to support the new attempt or constitute different meanings in respect of the needed ones (see for instance the conception of R as union of rational and irrational numbers and the definition of R as set of the limit points of Q; it's not the same to use one or the other to introduce limits of real functions in the real domain).

The relation between the institutional and the teachers' cognitive meanings is crucial since indeed the teachers are "institutional messengers" and the students are given a knowledge filtered by their teachers' personal meanings and choices.

4.Data analysis

4.1 **Analysis of the epistemic meaning of real numbers by means of the OSA's tools**

In the following paragraph we will present the holistic meaning of real numbers. This scheme was created following an economic principle: to use the minimum objects and processes that were necessary to create a new configuration. This is the reason why this scheme doesn't represent exactly the historical evolution, or better the historical systems of practices, and doesn't follow a strictly chronological order, even if the chronomological evolution is obviously important. Making our choice we were aware that the evolution of mathemtical concepts is not linear and that the socio-cultural contexts influenced deeply every mathematician's work. For instance let's consider the cases of the famous epistolar relations between mathematicians: Eratostene and Archimedes, Cantor and Dedekind. When we will not connect two configurations or systems of practices attributed to mathematicians who collaborated in their life we focus only on the configurations of objects and processes really necessary to them to develop their partial meaning of real numbers.

To explicit our choices we report an intermediate scheme that resumes the main steps of the historical evolution of real numbers we took in account (attached in the end of the book)

4.1.1 Systems of practices and configurations in the history of real numbers: the *scheme* of the real numbers' holistic meaning

1. **First level: Primitive problem and emerging configurations of objects and processes (1st level's partial meaning)**

P1: *To measure and compare homogeneous magnitudes*
CE1: [Eudosso's theory of magnitudes] & [Measures are ratios whose second term is the unit] & [Pythagoric problem of incommensurability of linear magnitudes]

P2: *To approximate magnitudes*
CE2: [Archimedes' approximation methods] & [Eudosso's exhaustion method]

P3: *To characterize the continuum in terms of atomic discrete entities*
CE3: [Anaxagoras' composition of segment with different finite units] & [Unlimited divisibility of a segment] & [Points are results of cuts of the segments] & [In a cut a point can lie on one of the two parts or the cut may generate two endpoints, distinct but not far] & [Points are atomic] & [The line is composed by points] & [Euclidean geometry's relations between points and lines]

P4: *To describe trajectories as result of motion (no breaks)*
CE4:= Line as trajectory
[Segments are traces of mobile points] & [Points are extremes of extendable segments] & [No stops in the motion implies no breaks in the segments and no gaps]

P5: *To solve equations and inequalities*
CE5: [Assyrs and Babiloneses tables] & [Diofanto's equation] & [Methods for 2nd order equations or more]

P6: *To operate with quantities in contextualized problems of real life (addition, subtraction, multiplication, division, power, exponential, logarithms)*
CE6: [Operations algorithms] & [Modelization of real life problems with operations] & [Procedures and methods to simplify computation with great numbers] & [Positional systems with Indo-arabic digits and decimal representation]

P7: *To linearize geometrical constructions*
CE7: [Cube's duplication] & [Angles trisection] & [Circles' rectification] & [Line and compass constructions] & [Regular polygons' construction] & [Aureum section] & [Representation of new magnitudes on a line to compare them using ratios and unit]

P8: *To find a relation between finite and infinite*
CE8: [Infinite as unlimited and unbounded] & [Zeno's paradox and infinite paths that generate a convergent process] & [Infinite's definition] & [Cusanus' "infinite finite", curves as limits of polygonals] & [Galileo's composition of curves of infinite infinitely small linear elements] & [Cavalieri's indivisible element that don't recreate the whole figure] & [Leibniz's integration as anti-differentiation]

P9: *To describe qualitative changes in phenomena*
CE9: [Characterization and relation between contiguous and continuous] & [Infinitely small distances between

contiguous parts of a segment]

P10: T*o describe physical dependent variations in space and time*
CE10: [Time-dependence of the space] & [Graphic representation of phenomena in time and space]

2. **Second level: New problems and configurations of objects (2nd level's partial meaning)**

P2.1: *To model physical phenomena in terms of variable's variation and dependence using mathematical models*
C2.1 := CE2 & CE4 & CE8 : [D'Alembert Wave's function] & [Arbogast's definition of continuous variation] & [Study of bounded or asymptotic variations corresponding to bounded variations of an independent variable] & [Interpolation of data through continuous function associated to phenomena tha are considered continuous (volume's variations, pressure's variations, ...)]
P2.2: *To describe motion, segments, 2D figures, volumes using infinitesimal quantities*
C2.2 := CE3 & CE8 & CE9 : [Galileo's cinematic problem of describing curves by means of infinitely small linear variation] & [Cavalieri's infinitesimal method to calculate areas] & [First integration methods] & [Torricelli-Barrow theorem] & [Continuum is a limit of "sums" of infinite, infinitely small, discrete elements]

P2.3: *To associate points of the line to geometrical constructions and algorithmic procedures*
C2.3 := CE1 & CE3 & CE7 : [Segments' endpoints represent ratios with the unity] & [The line is composed by endpoints of segments that are multiples of the unit or obtained by divisions or linearized geometrical constructions]

P2.4: *To describe incommensurable magnitudes by means of ratios of commensurable magnitudes*
C2.4 := CE1 & CE2 & CE9 : [Archimedean axiom] & [Incommensurable magnitudes can be approximated with couples of commensurable ones] & [Measures are limits of more and more precise measure's processes] & [Rational sequences' limits]

P2.5: *To express operations in a general form using symbols (Viète)*
C2.5 := CE5 & CE6 : [Modern algebra] & [Generalization of numerical operations] & [Manipulations of symbols to simplify computations] & [Manipulations procedures] & [Generic expressions]

P2.6: *To describe curves in terms of functions*
C2.6 := CE3 & CE4 & CE10 : [Functions represented by curves] & [Curves' discontinuities as interruptions/breaks correspond to holes in the function's domain] & [Flat neighborhoods] & [Intersection of trajectories and tangents] & [Continuous functions are represented by graphics that we can divide in contiguous parts everywhere]

P2.7: *To construct a bijection between numbers that results from operations and their representations*
C2.7 := CE1 & CE6 : [Equivalence relations between procedures: equivalent ratios and proportions, equivalent segment as results of different steps of addition or subtraction of two segments] & [Set of magnitudes that are "quotient" of equivalent procedures]

3. Third level: New problems and macroconfigurations (3rd level's partial meaning)

P3.1: *To create a bridge between numerical and geometrical generic procedures*

C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like $\pi, \sqrt{2}, e$] & [Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Distinction between algebraic and transcendental numbers] & [Cartesian geometry] & [R contains all the numbers generated algebrically and geometrically]

P3.2: *To characterize set of numbers in terms of properties in respect of operations*

C3.2 := C2.5 & C2.7 : Algebraic structures and numerical sets

[Generic procedures may have some properties] & [A subset of all the possible number may satisfy operations properties] & [Different kind of algebraic structures] & [Bombelli's complex numbers] & [Distinction between N, Z, Q] & [Real vs Imaginary numbers] &[Z, Q and C as "quotients" in respect of equivalence relations defined by means of one operation] & [Operations with roots] & [Impossibility to define R as quotient defined by generic numerical operations]

P3.3: *To estimate the rate of change of a flowing variable known the flowing variable and viceversa (Newton, Taylor, MacLaurin)*

C3.3 := C2.1 & C2.6 & C3.1 : [Varying flowing quantities and their rate of variations are represented by motion] & [Fluxions and fluent] & [Netwon's differential Calculus] & [Taylor's and MacLaurin's series development of analytical functions]

P3.4: *To treat analytically in the Cartesian geometry infinitesimal variations of curves, in particular for the problem of tangents and subtangentes (Leibniz)*

C3.4 := C2.2 & C2.6 & C3.1 : [Infinitesimals are variables] & [Comparison between infinitesimal quantities is possible] & [High-order infinitesimals exist] & [Infinitesimal differences' ratios make sense] & [Leibniz's point as a "differential cloud"] & [Operations with infinitesimal quantities] & [Products of infinitesimals are not absolute zeros] & [Infinitesimals can be neglected when infinitely small with respect to other quantities]

4. Fourth level: Configurations of sets of real numbers (meanings of real numbers)

P4.1: *To construct the set of real numbers and to put it in relation with the line*

C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes [Cauchy's iterative static description of limit processes] & [Cauchy's infinitesimal sequences and Cauchy-convergence of rational series] & [Some Cauchy's rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to

0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Cantor's postulate of continuity of the line]

P4.2: *To construct the set of real numbers and to put it in relation with the line*
C4.2 := C2.3 & C2.7&C3.1 & C3.2 : R is the set of Dedekind's rational cuts
[Some numbers are not rational, necessity of creating the new numbers starting from Q like the other equivalence relations lead to Z and Q] & [Dedekind's completeness i.e. existence of the supreme of every subset in the subset itself] & [Q is not complete] & [Points on the line are results of cuts that leave the dividing element in the left part] & [The point may correspond to a rational number or not; if it doesn't the cut creates an irrational number] & [Dedekind's equivalence relation between rational sections, numbers are equivalence classes] & [R is the complete field of Dedekind's rational cuts classes] & [Operations are derived by Q-operations] & [Dedekind's postulate of continuity of the line]

P4.3: *To mathematize physical variation using the functions and the Calculus*
C4.3 := C3.3 & C3.4 Physical integral-differential Calculus
[Irrational numbers may represent positions or instants] & [Irrational numbers are finite sums of smaller and smaller increments i.e. series] & [Continuous trajectories are limits of infinitesimal polygonals] & [Curves-trajectories can be "mathematized" locally in small neighborhoods of space-time by means of series and linear differentials] & [Continuous variations are limit of discontinuous infinitesimal linear variations] & [The length of a curve is an integral] & [Continuous functions describe continuous processes but they need to be more than continuous since a variation has a speed and acceleration at least, so the derivatives also need to be continuous]

5. Fifth level: Further configurations

P5.1 := *To compare different kind of infinite quantities*
C5.1:= CE8 & C4.1: Cantor's transfinite numbers , set theory and Cohen-Godel's continuum hypothesis
[Cantor's diagonal theorem] & [Different infinite cardinalities between N-Z-Q and R-C] & [Cantor's set theory and numbers as sets' cardinalities] & [Transfinite numbers] & [Continuum hypothesis]

P5.2:= *To create a system of axioms for the Geometry*
C5.2 := C3.2 & C4.2: Hilbert's axiomatization of real numbers' set
[Axiomatization of the properties of order; field with the usual operations; Archimedean axiom; Completeness axiom] & [System respecting the Euclidean axioms not extendable] & [R is the only ordered complete field] & [Dedekind's postulate of continuity implies Cantor's one but not viceversa] & [Hilbert's completeness axiom leads indirectly to the introduction of limit points, and, hence, renders it possible to establish a one-to-one correspondence between the points of a segment and the system of real numbers] & [The existence of limit points is derivable but not postulated]

P5.3 := *To treat in a precise manner the Leibnizian infinitesimal quantities and to operate with them consistently with the real numbers' properties*
C5.3 := C2.2 & C4.1 : Non-standard Analysis (Robinson)

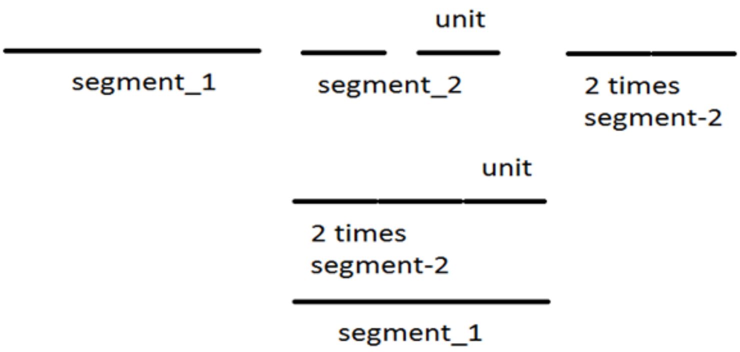
[Debates about the infinitesimals and the non-equivalence between intuitive and formal conceptions of the continuum and limits] & [Brouwer's intuitionism] & [Weyl's classification of the different conceptions of the continuum] & [Non-standard analysis reformulates the Calculus using a logically rigorous notion of infinitesimal numbers] & [Reconsideration of Leibniz's infinitely small or infinitely large with the same properties of real numbers] & [Necessity of a set that contains infinitesimal and infinite elements i.e. is non-Archimedean] & [Hyperreal numbers or other non standard model of real numbers]

Hybrid continuum and dualities

The extensive-intensive duality is crucial in the construction of the hybrid configuration of the continuum by Viète and Descartes, since the problematic aspect of this configuration, that is also its major feature, is the relation between particular and general objects. As Giusti (2012) stressed indeed the innovation that changed radically the conception of the continuum introduced by Viète and Descartes is the double-faced nature of the symbols they used as general objects. The symbol x can represent both the numbers and the (linearized) magnitudes. The general objects is thus the bridge between two independent configurations and the possibility to use algebraic expressions as extensive in relations to different possible intensive configurations is the decisive step towards an hybrid continuum, free from the constraints imposed by the ratios' theory by Eudosso. In the "algebraic discourses" the extensive can represent both numbers and magnitudes.
Let's consider the example proposed by Godino, Batanero & Font (2007) in order to explain the intensive-extensive duality.

"*Extensivo – intensivo (ejemplar - tipo)*. Un objeto que interviene en un juego de lenguaje como un caso particular (un ejemplo específico, p.e., la función y = 2x + 1) y una clase más general (p.e., la familia de funciones y = mx + n). La dualidad extensivo-intensivo se utiliza para explicar una de las características básicas de la actividad matemática: el uso de elementos genéricos (Contreras y cols, 2005). Esta dualidad permite centrar la atención en la dialéctica entre lo particular y lo general, que sin duda es una cuestión clave en la construcción y aplicación del conocimiento matemático. "La generalización es esencial porque este es el proceso que distingue la creatividad matemática de la conducta mecanizable o algorítmica (Otte, 2003, p. 187)."
The particular example y = 2x + 1 can also be interpreted as a general expression, in which the variables y and x can both represent numbers and magnitudes:

1. y=3, x=1: 3=2+1
2. y=segment_1, y=segment_2, 1=the unit, 2=scalar factor



4.2 **Analysis of the institutional meaning of real numbers by OSA tools**

4.2.1 Systems of practices concerning real numbers in the national curricula

(First 2 years on 5)

a. The student will acquire an intuitive knowledge of real numbers, with particular attention to their geometrical representation on a line.

EPI:

P2.3: To associate points of the line to geometrical constructions and algorithmic procedures
C2.3 := CE1 & CE3 & CE7 : [Segments' endpoints represent ratios with the unity] & [The line is composed by endpoints of segments that are multiples of the unit or obtained by divisions or linearized geometrical constructions]
COG:
Tall, D'Amore&Arrigo, Sbaragli, Bagni: Models of pearls (features from partial meanings in researches)
Approximations and constructions: the calculator and finite numbers
INT:
Representations: decimals, fractions and segments

b. The proof of the irrationality of $\sqrt{2}$and of other numbers will be an important occasion of conceptual detailed study.

EPI:

P3.1: To create a bridge between numerical and geometrical generic procedures
C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)
[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like 2, , e] & [Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Algebraic and transcendental numbers] & [Cartesian geometry]

http://amslaurea.unibo.it/1821/1/amerise_patrizia_tesi.pdf
COG:
Necessity of the absurd proof
INT:

Coordination of different representations

c. The study of irrational numbers and of their possible expressions will provide a significant example of algebraic computation's application and an occasion to face the issue of approximation.

EPI:
P3.2: *To characterize set of numbers in terms of properties in respect of operations*
C3.2 := C2.5 & C2.7 : Algebraic structures and numerical sets
[Generic procedures may have some properties] & [A subset of all the possible number may satisfy operations properties] & [Different kind of algebraic structures] & [Bombelli's complex numbers] & [Distinction between N, Z, Q] & [Z, Q and C as "quotients" in respect of equivalence relations defined by means of one operation] & [Operations with roots] & [Impossibility to define R as quotient defined by generic numerical operations]

P2.4: *To describe incommensurable magnitudes by means of ratios of commensurable magnitudes*
C2.4 := CE1 & CE2 & CE9 : [Archimedean axiom] & [Incommensurable magnitudes can be approximated with couples of commensurable ones] & [Measures are limits of more and more precise measure's processes] & [Rational sequences' limits]

COG:
Non-whole numbers
Exact numbers vs approximations (mia tesi)
INT:
Decimal representation of irrational numbers
Finite numbers

d. Lo studente acquisirà la capacità di eseguire calcoli con le espressioni letterali sia per rappresentare un problema (mediante un'equazione, disequazioni o sistemi) e risolverlo, sia per dimostrare risultati generali, in particolare in aritmetica.

EPI:
P2.5: *To express operations in a general form using symbols (Viète)*
C2.5 := CE5 & CE6 : [Modern algebra] & [Generalization of numerical operations] & [Manipulations of symbols to simplify computations] & [Manipulations procedures] & [Generic expressions]
COG:
Difficulties to give sense to symbolic expressions
INT:
Letters, difficulties in conversion from verbal expressions to symbolic expressions (conversion, Duval, 1993; Ferrari,) but also in the symbolic register itself .

e. Particular attention will be given to the Pythagoras' theorem so that its geometrical aspects and its implications in Number theory (introduction of irrational numbers) are understood, stressing merely the conceptual aspects.

EPI:
P3.1: To create a bridge between numerical and geometrical generic procedures
C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)
[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure

represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like $\sqrt{2}$, π , e] & [Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Algebraic and transcendental numbers] & [Cartesian geometry]
http://amslaurea.unibo.it/1821/1/amerise_patrizia_tesi.pdf

COG:
Conversion between numerical and geometrical representation of numbers
Approximations of irrational numbers used instead of the number

INT:
Conversion between numerical and geometrical representation of numbers

f. The student will study some kinds of functions: f(x) = ax + b, f(x) = |x|, f(x) = a/x, f(x) =x^2both in terms strictly mathematical and to describe and solve applicative problems.

EPI:
P2.6: *To describe curves in terms of functions*

C2.6 := CE3 & CE4 & CE10 : [Functions represented by curves] & [Curves' discontinuities as interruptions/breaks correspond to holes in the function's domain] & [Flat neighborhoods] & [Intersection of trajectories and tangents] & [Continuous functions are represented by graphics that we can divide in contiguous parts everywhere]

COG:
The domain is not explicited
The continuous graph is immediately associated to the analytical function
To find some points and then to trace a line

INT:
Graphic representations to coordinate with tabular and numerical ones

(3^{rd} and 4^{th} years on 5)

g. The study of the circumference and of the circle, of the number π, of the contexts in which exponential growths involving the number e appear will allow to examine in depth the knowledge of real numbers, regarding in particular the transcendental numbers.

EPI:
P6: *To operate with quantities in contextualized problems of real life (addition, subtraction, multiplication, division, power, exponential, logarithms)*

CE6: [Operations algorithms] & [Modelization of real life problems with operations] & [Procedures and methods to simplify computation with great numbers] & [Positional systems with Indo-arabic digits and decimal representation]

P3.1: *To create a bridge between numerical and geometrical generic procedures*

C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like 2, , e] &

[Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Algebraic and transcendental numbers] & [Cartesian geometry]

COG:

Approximation of irrational numbers may vanish the attempt to conceptualize transcendental numbers
Confusion between angles and magnitudes (Radiants or meters)

INT:

Irrational numbers are represented by means of non-decimal symbols or using approximations obtained with the calculator

c. Through a first knowledge of the problem of the formalization of real numbers the student will be introduced to the issue of mathematical infinity and its connections with the philosophical thought.

EPI:

P8: *To find a relation between finite and infinite*

CE8: [Infinite as unlimited and unbounded] & [Zeno's paradox and infinite paths that generate a convergent process] & [Infinite's definition] & [Cusanus' "infinite finite", curves as limits of polygonals] & [Galileo's composition of curves of infinite infinitely small linear elements] & [Cavalieri's indivisible element that don't recreate the whole figure] & [Leibniz's integration as anti-differentiation]

P4.1: *To construct the set of real numbers and to put it in relation with the line*

C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes [Cauchy's iterative static description of limit processes] & [Cauchy's infinitesimal sequences and Cauchy-convergence] & [Some Cauchy's rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to 0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Cantor's postulate of continuity of the line]

P4.2: *To construct the set of real numbers and to put it in relation with the line*

C4.2 := C2.3 & C2.7&C3.1 & C3.2 : R is the set of Dedekind's rational cuts

[Some numbers are not rational, necessity of creating the new numbers starting from Q like the other equivalence relations lead to Z and Q] & [Dedekind's completeness i.e. existence of the supreme of every subset in the subset itself] & [Q is not complete] & [Points on the line are results of cuts that leave the dividing element is the left part] & [The point may correspond to a rational number or not; if it doesn't the cut creates an irrational number] & [Dedekind's equivalence relation between rational sections, numbers are equivalence classes] & [R is the complete field of Dedekind's rational cuts classes] & [Operations are derived by Q-operations] & [Dedekind's postulate of continuity of the line]

P5.1 := *To compare different kind of infinite quantities*

C5.1:= CE8 & C4.1: Cantor's transfinite numbers , set theory and Cohen-Godel's continuum hypothesis [Cantor's diagonal theorem] & [Different infinite cardinalities between N-Z-Q and R-C] & [Cantor's set theory and numbers as sets' cardinalities] & [Transfinite numbers] & [Continuum hypothesis]

COG:

Finite quantities are assimilated to big finite quantities

Potential infinite vs actual infinite

The line is usually perceived as composed by a finite number of small points-balls

INT:

Graphic representations are considered intuitive, while the other no, but the intuition doesn't lead immediately to infinite points nor to the problem of incompleteness of Q and to the relation between completeness and continuity

d. The student will learn to study quadratic functions, to solve quadratic equations and inequalities and to represent and to solve problems involving them.

EPI:

P5: *To solve equations and inequalities*

CE5: [Assyrs and Babiloneses tables] & [Diofanto's equation] & [Methods for 2nd order equations or more]

P6: *To operate with quantities in contextualized problems of real life (addition, subtraction, multiplication, division, power, exponential, logarithms)*

CE6: [Operations algorithms] & [Modelization of real life problems with operations] & [Procedures and methods to simplify computation with great numbers] & [Positional systems with Indo-arabic digits and decimal representation]

COG:

Methods for linear equations and inequalities adapted to the 2^{nd} degree in a wrong way
The lack of the domain in the previous stidy of functions may lead the students to wrong solutions for applicative problems (negative numbers for measures, numbers belonging to Q\Z to represent non divisible quantities, irrational numbers to represent measures in Physics)
Procedures are not supported by conceptualization; this implies problems in the modelization using quaratic equations and inequalities (Boero, Bazzini, Garuti, 2000)

INT:

The representatons of intervals are a lot and depends on the specific set considered
The treatment and the conversion between representations may be difficult if the representations arised as objects from not connected practices and the students don't know any semiotic function that can create a new, more general, configuration

e. The student will learn to construct simple growth models using the exponentials, in addition to periodic behaviors, also in relation with other disciplines; this will be carried out both in continuous and discrete contexts .

EPI:

P2.6: *To describe curves in terms of functions*

C2.6 := CE3 & CE4 & CE10 : [Functions represented by curves] & [Curves' discontinuities as interruptions/breaks correspond to holes in the function's domain] & [Flat neighborhoods] & [Intersection of trajectories and tangents] & [Continuous functions are represented by graphics that we can divide in contiguous parts everywhere] & [Dirichlet's function as example of anomalous function]

P10: T*o describe variations depending on space and time in Physics*

CE10: [Time-dependence of the space] & [Graphic representation of phenomena in time and space]

COG:
To merge dynamic and static, continuous and discrete variations
INT:
Coordination between graphical (continuous & discrete), verbal and analytical expressions of functions to model phenomena

(5^{th} year)

a. The student will deepen the study of the fundamental analytical functions, also using examples from Physics or other disciplines; she will acquire the concept of limit, both of functions and sequences, and will learn to compute limit values in simple cases.

EPI:

P3.3: *To estimate the rate of change of a flowing variable known the flowing variable and viceversa (Newton, Taylor, MacLaurin)*
C3.3 := C2.1 & C2.6 & C3.1 : [Varying flowing quantities and their rate of variations are represented by motion] & [Fluxions and fluent] & [Netwon's differential Calculus] & [Taylor's and MacLaurin's series development of analytical functions]

P3.4: *To treat analytically in the Cartesian geometry infinitesimal variations of curves, in particular for the problem of tangents and subtangentes (Leibniz)*
C3.4 := C2.2 & C2.6 & C3.1 : [Infinitesimals are variables] & [Comparison between infinitesimal quantities is possible] & [High-order infinitesimals exist] & [Infinitesimal differences' ratios make sense] & [Leibniz's point as a "differential cloud"] & [Operations with infinitesimal quantities] & [Products of infinitesimals are not absolute zeros] & [Infinitesimals can be neglected when infinitely small with respect to other quantities]

COG:
Dynamic/intuitive vs static/formal definition of limits and continuity of real functions in the real domain (Tall & Vinner; 1981; Bagni, 1999)
To merge dynamic and static, continuous and discrete variations, using the Calculus formalization (Leibniz-Newton diatribe)
INT:
Coordination between graphical (continuous & discrete), verbal and analytical expressions of functions to model phenomena

b. The main goal will be merely to make the students understand the role of the Calculus as a fundamental conceptual tool in the modelization of Physical phenomena of of other natures.

EPI:

P4.3: *To mathematize physical variation using the functions and the Calculus*

C4.3 := C3.3 & C3.4 Physical integral-differential Calculus

[Irrational numbers may represent positions or instants] & [Irrational numbers are finite sums of smaller and smaller increments i.e. series] & [Continuous trajectories are limits of infinitesimal polygonals] & [Curves-trajectories can be "mathematized" locally in small neighborhoods of space-time by means of series and linear differentials] & [Continuous variations are limit of discontinuous infinitesimal linear variations] & [The length of a curve is an integral] & [Continuous functions describe continuous processes but they need to be more than continuous since a variation has a speed and acceleration at least, so the derivatives also need to be continuous]

Comments
The problem of infinite cardinality is not faced before, but it's not a problem for the previous practices
The systems of practices concerning the real numbers and the continuum are present largely in the national curricula.
The complex relation between Mathematics and Physics concerning the hardly debated concept of "numbers varying continuously" appears at a certain time and is not treated – as it could be – since the very beginning. If we think to the critic of Bolzano to time dependent intuitions in the Calculus, to the diatribe Newton-Leibniz, engaged indeed in this field (Giusti,) or to the Intuitionistics' reactions to the Calculus' arithmetization, we can hypothesize easily that this node will create some doubts or sudden cognitive conflicts that maybe the teachers didn't expected and may not be able to manage effectively.

Taking in account the Aristotle's approach to the continuum, its generation through motions, its characterization by means of the concept of contiguous and its relation with infinite divisibility that not implies the possibility to re-compose the continuum starting from its small parts, maybe something can be inserted before in order to prepare the students to compare and opportunely distinguish these two identities, somehow incommensurable (Nunez, 2000) and complementary (Bell,).

Also it's remarkable the precocity of the intuitive approach to the representation of the real numbers on a line in the first 2 years, since it's not trivial at all, as the History and the Didactics of Mathematics researchers stressed many times. Maybe these aspects should be discussed more in depth.

From our first epistemic analysis of the national curriculum the ambitions of this curriculum seemes to be very high in comparison with the potential poorness of the students' partial meanings in the various steps reported by an amount of researchers (Ch.) and the lack of problems and tasks that lead the students to act on the "conceptual level" in the textbooks (Gonzales-Martin, 2014;). Also the teachers are assumed to be deeply aware of the epistemic meaning of real numbers and of the interdisciplinar connections with other disciplines, while this is not obvious standing on some researches' results (Tall, Gonzales-Martin, Arrigo & D'Amore). The role of the teachers in the realization of the attended curriculum seems to be very important and this confirms to us that the investigation concerning the teachers' choice is decisive to create a bridge between the expected students' knowledge about real numbers in the end of the high school and the weakness of their knowledge highlighted by the researchers.

Furthermore the teachers we interview were asked to express their degree of acceptance and agreement with the national curricula about real numbers and the opinions were very different, in particular in respect of the problem of formalization of real number and the problem of the mathematical infinite.

The reasons of this discrepancy we took in account are mainly two:
1) the teachers thought that there systems of practices that are more important than those ones;
2) the teaching-learning processes concerning real numbers in the high school are different in respect of the epistemic ones; maybe the reasons of this difference lies precisely in the students' difficulties or in the teachers' personal meanings of real numbers.

4.3 Analysis of the questions that evaluate the teachers' mathematical knowledge

Q4) In every enlargement of numerical sets *($N \rightarrow Z$, $Z \rightarrow Q$)* there was a "critical operation" involving the elements of a set whose properties lead to construct another set (f.i. subtraction, division). How is it possible to construct R strating from the elements of Q?

EPI:

P2.7: *To construct a bijection between numbers that results from operations and their representations*

C2.7 := CE1 & CE6 : [Equivalence relations between procedures: equivalent ratios and proportions, equivalent segment as results of different steps of addition or subtraction of two segments] & [Set of magnitudes that are "quotient" of equivalent procedures]

P3.2: *To characterize set of numbers in terms of properties in respect of operations*
C3.2 := C2.5 & C2.7 : Algebraic structures and numerical sets
[Generic procedures may have some properties] & [A subset of all the possible number may satisfy operations properties] & [Different kind of algebraic structures] & [Bombelli's complex numbers] & [Distinction between N, Z, Q] & [Z, Q and C as "quotients" in respect of equivalence relations defined by means of one operation] & [Operations with roots] & [Impossibility to define R as quotient defined by generic numerical operations]

P4.1: *To construct the set of real numbers and to put it in relation with the line*
C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes
[Cauchy's iterative static description of limit processes] & [Cauchy's infinitesimal sequences and Cauchy-convergence] & [Some Cauchy's rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to 0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Cantor's postulate of continuity of the line]

P4.2: *To construct the set of real numbers and to put it in relation with the line*
C4.2 := C2.3 & C2.7&C3.1 & C3.2 : R is the set of Dedekind's rational cuts
[Some numbers are not rational, necessity of creating the new numbers starting from Q like the other equivalence relations lead to Z and Q] & [Dedekind's completeness i.e. existence of the supreme of every subset in the subset itself] & [Q is not complete] & [Points on the line are results of cuts that leave the dividing element is the left part] & [The point may correspond to a rational number or not; if it doesn't the cut creates an irrational number] & [Dedekind's equivalence relation between rational sections, numbers are equivalence classes] & [R is the complete field of Dedekind's rational cuts classes] & [Operations are derived by Q-operations] & [Dedekind's postulate of continuity of the line]

COG:

The definition of Z, Q and R requires the notion of equivalence relation, that is usually stressed only in the case of rational nummbers, or better, the set of fractions
While for Z and Q an operation is enough to define the equivalence relation, in the case of R a 2^{nd} order logic and a non-algebraic procedure is necessary.

The students are used to work with numbers on the line or with approximations so, in the moment in which the teachers propose to them practices concerning definitions, properties or argumentation about the incompleteness of Q and the necessity of enlarging the set maybe the student will not be able to understand the problem, since they're used to work with the number line and to refer to it with the symbol R, even if their meanings are partial or uncorrect (Ch. ;). Furthermore if the teachers' personal meanings don't include the more general and formal aspect of the epistemic of real numbers maybe the problems will be posed in an uncorrect way and many inconsistencies may arise; for instance to introduce R as the set generated to contains all the roots of positive numbers may generate some cognitive conflicts that would require a very careful treatise to be turned into a significant and generative practice.

Some potential cognitive conflicts we can hypothesize *a priori* are:

1) once introduced the problem of incommensurability using geometrical constructions involving square roots, the students should create a new configuration in which new numbers have always a geometrical construction; in this perspective it would not be easy to connect the decimal representation, that is a product of a further important and not trivial step, to the partial meaning because the irrationl numbers created as series of rational numbers have nothing to do with geometrical constructions;
2) once introduced the square roots algebraically and associated the irrational numbers to the roots, maybe it would not be easy to explain the difference between algebraic and transcendental numbers

INT:
The numbers are maybe the most complex mathematical objects in terms of quantity of different representations and of relations between them. Enlarging numerical numbers the textboks and the teachers sometimes use time after time different kind of representations depending on the kind of enlargement:

a) From N to Z the numbers are represented usually in the positional decimal representation

b) From Z to Q usually the register of fraction is used.

c) From Q to R sometimes using the graphic representations of numbers and *reductio ad absurdum* involving arithmetical aspects of irrational numbers, sometimes using the decimal representation to synthesize all the differences (rational/irrational, algebraic/transcendental) in one, unique, representation.

Each of the representations is linked to epistemic and institutional different systems of practices and complex historical transitions – let's f.i. think about the long-term debate about the relation between numbers and magnitudes.

Furthermore, since focusing on the point c) is the goal of our question, at the basis of the graphic and the decimal representation of numbers there are also different epistemic partial meanings:

1) Graphic representation of real numbers: geometrical procedures and numbers, potential construction of the set of real numbers adding new elements

P3.1: *To create a bridge between numerical and geometrical generic procedures*

C3.1 := C2.5 & C2.3 : Hybrid continuum (Viète and Descartes)

[Symbols represent at the same time numbers (quantities) and magnitudes] & [Every general procedure represent at the same time geometrical and numerical procedures] & [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like 2, , e] & [Distinction between rational and irrational numbers] & [Irrational numbers may be results of arithmetical or geometrical procedures] & [Some irrational numbers are not possible to construct geometrically] & [Parallelism between equations and construction of segments using line and compass] & [Algebraic and transcendental numbers] & [Cartesian geometry]

2) Real numbers are all the possible decimal numbers (series of infinite elements, some periodics and some not)

P4.1: *To construct the set of real numbers and to put it in relation with the line*

C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes

[Cauchy’s iterative static description of limit processes] & [Cauchy’s infinitesimal sequences and Cauchy-convergence of rational series] & [Some Cauchy’s rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to 0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Cantor's postulate of continuity of the line]

On one hand this observation doesn't imply anything certain in itself since the way the objects are used in the practices depends on the specific context, the teachers, the previous experiences shared by the teachers and their students and also on the role the teachers use them in the linguistic game (see the Par. 2.1.2 Dualities); on the other hand this could be very probably a source of difficulties for the students since the partial meanings that underlie to the graphical and the decimal representations are different and also positionned at different levels of generality (3 vs 4).

Q5) In your mind is it possible to define a limit point in Q or is it necessary to use real numbers?

EPI:

P4.1: *To construct the set of real numbers and to put it in relation with the line*
C4.1 := C2.3 & C2.4 & C2.2 & C3.1 & C3.2 : R is the set of Cauchy-Weierstrass-Cantor contiguous classes [Cauchy's iterative static description of limit processes] & [Cauchy's infinitesimal sequences and Cauchy-convergence of rational series] & [Some Cauchy's rational sequences are convergent but no to a rational number] & [Definition of Cauchy's complete set] & [Irrational numbers are limits of rational sequences] & [More than one rational sequence converge that the same number] & [Quotient of R in respect of infinitesimal Cauchy's sequences] & [R is the set of all the possible classes of Cauchy's sequences whose difference is infinitesimal i.e. converge to 0] & [Weierstrass' flat intervals and intervals' convergence to a point] & [Cantor's contiguous classes and creation of irrational contiguous classes] & [Operation between contiguous classes] & [R is the Cauchy-complete field of contiguous classes ordered like rational numbers] & [Q is dense in R] & [Cantor's postulate of continuity of the line]
Q8) Look at the first minute of this video: http://www.youtube.com/watch?v=jk08WkwqTQ

EPI:

P2: *To approximate magnitudes*
CE2: [Archimedes' approximation methods] & [Eudosso's exhaustion method]

P2.3: *To associate points of the line to geometrical constructions and algorithmic procedures*
C2.3 := CE1 & CE3 & CE7 : [Segments' endpoints represent ratios with the unity] & [The line is composed by endpoints of segments that are multiples of the unit or obtained by divisions or linearized geometrical constructions]

INT:

Q9) Look at this video: http://www.youtube.com/watch?v=kuKTyp_b8WIIl . Can it help the students to understand the bijection between real numbers and the line's points?

EPI:

P4: *To describe trajectories as result of motion (no breaks)*
CE4: [Segments are traces of mobile points] & [Points are extremes of extendable segments] & [No stops in the motion implies no breaks in the segments and no gaps]

P2.1: *To model physical phenomena in terms of variable's variation and dependence using mathematical models*
C2.1 := CE2 & CE4 & CE8 : [D'Alembert Wave's function] & [Arbogast's definition of continuous variation] & [Study of bounded or asymptotic variations corresponding to bounded variations of an independent variable]

INT:

Q10) Look at this video, from the minute 10:20 to the minute 12:10
http://www.youtube.com/watch?v=UEBK5DfPxvk . Do you think the distinction between the graphic and algebraic representations?

EPI:

P5: *To solve equations and inequalities*
CE5: [Assyrs and Babiloneses tables] & [Diofanto's equation] & [Methods for 2nd order equations or more]

P2.5: *To express operations in a general form using symbols (Viète)*
C2.5 := CE5 & CE6 : [Modern algebra] & [Generalization of numerical operations] & [Manipulations of symbols to simplify computations] & [Manipulations procedures] & [Generic expressions]

P2.3: *To associate points of the line to geometrical constructions and algorithmic procedures*
C2.3 := CE1 & CE3 & CE7 : [Segments' endpoints represent ratios with the unity] & [The line is composed by endpoints of segments that are multiples of the unit or obtained by divisions or linearized geometrical constructions]

COG:

The solutions are presented as two different objects emerging from different practices. This choice may lead the students to keep on considering them different in the future.

The set of all the possible numbers is presented wit the line but it's not explained how to recognize a number that is bigger or lower in other representations.

The functional aspect of inequalities is not stressed (Boero, Bazzini & Garuti, 2000). The processes involved are very poor.

INT:

The representations can be both useful, but the conversion between them is not obvious and not adeuqateky stressed.

Different aspects of measure in the didactical practice in the high school

M1) Measure as a practice in Physics

P2: *To approximate magnitudes*
CE2: [Archimedes' approximation methods] & [Eudosso's exhaustion method]

M2) Measure s a theory in Mathematics

P2.2: *To describe motion, segments, 2D figures, volumes using infinitesimal quantities*
C2.2 := CE3 & CE8 & CE9 : [Galileo's cinematic problem of describing curves by means of infinitely small linear variation] & [Cavalieri's infinitesimal method to calculate areas] & [First integration methods] & [Torricelli-Barrow theorem] & [Continuum is a limit of "sums" of infinite, infinitely small, discrete elements]

M3) Observed data and interpolation

P2.1: *To model physical phenomena in terms of variable's variation and dependence using mathematical models*
C2.1 := CE2 & CE4 & CE8 : [D'Alembert Wave's function] & [Arbogast's definition of continuous variation] & [Study of bounded or asymptotic variations corresponding to bounded variations of an independent variable] & [Interpolation of data through continuous function associated to phenomena tha are considered continuous (volume's variations, pressure's variations, ...)]

4.4 **Teachers' mathematical knowledge categorizations**

4.4.1 Teachers' categories based on the question Q3: the real numbers' properties

By means of the question Q3 we investigated the teachers' opinions about the most important features of the set R, that we labeled with:

K2: Teachers' opinion concerning the most important features of real numbers (referred to the epistemic meaning of real numbers we presented in 3.1.3)

Not all the answers were precise or correct but at this level we are interested in a teachers' classification concerning the kind of knowledge they declared to have concerning the properties of real numbers, distinguishing between common knowledge [CK] and advanced knowledge [AK]. We will identify CK with the lower levels in the epistemic meaning scheme (from 1 to 3, labeled with L := lower levels) and AK with the upper levels (4 and 5, labeled with U := upper levels)

K2_A: Topological/differential structure of R (intervals, density) [13] [U]

2, 7, 11, 15, 18, 20, 47, 65, 66, 72, 83, 101, 111

K2_B: Algebraic structure of R (field) [43] [L]

1, 2, 8, 17, 20, 22, 39, 42, 46, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 65, 66, 67, 70, 72, 73, 75, 79, 83, 84, 87, 88, 90, 93, 98, 100, 101, 104, 106, 110, 111, 116, 115,

K2_C: Relation between Q and R (operations, order, density of Q in R) [10] [U]

5, 20, 21, 39, 64, 65, 68, 88, 90, 108,

K2_D: Relation between dynamic and static conception of continuity [0] [U]

No one

K2_E: Difference between rational and irrational numbers [6] [L]

20, 72, 87, 90, 92, 97

K2_F: Difference between algebraic and transcendental numbers [2] [L]: 72, 87

K2_G: History of real numbers [1] [U]: 85

K2_H: Construction of R [3] [U]: 63, 85, 88

K2_I: Necessity of a postulate of continuity [0] [U]: No one

K2_J: can represent every magnitude [1] [L]: 97

K2_L: Archimedean axiom [11] [U]: 4, 26, 38, 58, 59, 61, 65, 68, 72, 78, 105

K2_M: Lack of equivalence between Cantor's and Dedekind's postulates [0] [U]: No one

K2_N: Completeness [47] [U]: 1, 4, 5, 8, 9, 10, 13, 14, 16, 18, 22, 25, 26, 29, 32, 36, 37, 40, 41, 42, 51, 53, 54, 55, 57, 58, 61, 62, 64, 65, 68, 70, 71, 72, 73, 74, 77, 78, 87, 88, 91, 92, 93, 95, 108, 110, 116, 115

K2_O: Order [38] [L]: 1, 2, 4, 8, 10, 11, 16, 17, 18, 26, 32, 42, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 67, 70, 72, 73, 75, 78, 79, 88, 90, 91, 93, 101, 110, 116, 115

K2_P: Correspondence between R and the points of a line [10] [L]: 2, 7, 11, 39, 63, 81, 87, 90, 92, 108

K2_R: Cardinality [27] [U]: 2, 5, 7, 10, 13, 22, 26, 32, 37, 39, 40, 41, 61, 63, 64, 65, 68, 71, 73, 75, 76, 87, 90, 91, 95, 96, 105

K2_S: Continuous [15] [L]: 4, 14, 17, 22, 38, 39, 59, 60, 63, 65, 72, 76, 77, 78, 96, 111

K2_T: Make possible more operations/all the operations [6] [L]: 7, 11, 32, 34, 53, 97

K2_U: Make possible the differential Calculus [2] [U]: 11, 65

K2_V: Make representable every number in the decimal register [5][L]: 13, 40, 60, 80, 101

K2_Z: R include N, Z, Q and/or is included in C [3] [L]: 20, 87, 89

K2_W: R is metric space [3] [U]: 64, 65, 73

The empty *a priori* categories are also very interesting, since they are crucial elements of the advanced mathematical knowledge and are missing in the teachers' answers. This was confirmed by all the other data of the questionnaires and the interviews.

It's interesting that only a few teacher named some properties that were considered essential in the following of the questionnaire.

We looked with particular attention to these properties or partial configurations are maybe the ones who emerge in the practice as somewhat unconscious and that is maybe disconnected. The teachers can also belong to more than one category.

In the answers to Q3 we identified 8 kind of cognitive configurations that we grouped in cognitive configurations:

- CC1: Topological/differential (K2A, K2N, K2U, K2W) [66;3]

2, 7, 11, 15, 18, 20, 47, 65, 66, 72, 83, 101, 111; 1, 4, 5, 8, 9, 10, 13, 14, 16, 18, 22, 25, 26, 29, 32, 36, 37, 40, 41, 42, 51, 53, 54, 55, 57, 58, 61, 62, 64, 65, 68, 70, 71, 72, 73, 74, 77, 78, 87, 88, 91, 92, 93, 95, 108, 110, 116, 115; 11, 65; 64, 65, 73

- CC2: Numeric - systemic(K2E, K2F, K2H, K2V) [17; 1]

20, 72, 87, 90, 92, 97; 72, 87; 63, 80, 85, 88; 13, 40, 60, 101

- CC3: Numeric - unitary (K2R, K2Z) [30; 1]

2, 5, 7, 10, 13, 22, 26, 32, 37, 39, 40, 41, 61, 63, 64, 65, 68, 71, 73, 75, 76, 87, 90, 91, 95, 96, 105; 20, 87, 89

- CC4: Algebraic – operations' properties (K2B, K2T) [49; 1]

1, 2, 8, 17, 20, 22, 39, 42, 46, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 65, 66, 67, 70, 72, 73, 75, 79, 83, 84, 87, 88, 90, 93, 98, 100, 101, 104, 106, 110, 111, 116, 115; 7, 11, 32, 34, 53, 97

- CC5: Axiomatic (K2N, K2O, K2L, K2M) [96; 24]

4, 26, 38, 58, 59, 61, 65, 68, 72, 78, 105; 1, 4, 5, 8, 9, 10, 13, 14, 16, 18, 22, 25, 26, 29, 32, 36, 37, 40, 41, 42, 51, 53, 54, 55, 57, 58, 61, 62, 64, 65, 68, 70, 71, 72, 73, 74, 78, 87, 88, 91, 92, 93, 95, 108, 110, 116, 115;

1, 2, 4, 8, 10, 11, 16, 17, 18, 26, 32, 42, 51, 53, 54, 55, 56, 57, 58, 59, 62, 63, 64, 67, 70, 72, 73, 75, 78, 79, 88, 90, 91, 93, 101, 110, 116, 115

- CC6: Line - systemic (K2J, K2L, K2P, K2R) [48; 0]

4, 26, 38, 58, 59, 61, 65, 68, 72, 78, 105; 97; 2, 7, 11, 39, 63, 81, 87, 90, 92, 108; 2, 5, 7, 10, 13, 22, 26, 32, 37, 39, 40, 41, 61, 63, 64, 65, 68, 71, 73, 75, 76, 87, 90, 91, 95, 96, 105

- CC7: Line - unitary (K2S) [16]

4, 14, 17, 22, 38, 39, 59, 60, 63, 65, 72, 76, 77, 78, 96, 111

- CC8: Relations between Q and R (K2C, K2H) [13,1]

5, 20, 21, 39, 64, 65, 68, 88, 90, 108; 63, 85, 88

We propose here a further quantitative and qualitative analysis of the data. The aims of the quantitative analysis are:

- to show the teachers' cognitive configurations' distribution
- to decide if a group could constitute a category or is rather composed by rare cases

The qualitative analysis concerns the comparison between the epistemic meaning and the teachers' configurations and the prevalence of practices placed on lower or higher levels in the epistemic meaning. This is particularly interesting because our previous analysis showed that the relation between the formal and intuitive dimension plays a crucial role in the field of Calculus and infinity.

- Teachers who belong only to 1 to 3 category [46]

7, 8, 9, 13, 14, 15, 16, 17, 18, 21, 25, 29, 36, 42, 46, 47, 51, 53, 54, 55, 57, 59, 60, 62, 66, 67, 70, 77, 74, 76, 79, 80, 83, 84, 85, 89, 92, 93, 96, 97, 100, 104, 106, 110, 111, 115, 116

- Teachers who belong to more than 3 categories [32]

2, 4, 5, 10, 11, 20, 22, 26, 32, 37, 39, 40, 41, 58, 61, 63, 64, 65, 68, 71, 72, 73, 75, 78, 87, 88, 90, 91, 95, 101, 105,

108

45 teachers on 79 listed properties we grouped in at least 3 categories, while 32 teachers listed more than three kind of properties.

Even if this is just a very ambiguous indication, this oriented us in the very beginning in the identification of teachers that perceived the real numbers as something complex and multifaced and teachers that only focused on a few properties. Many other markers all along the questionnaire can confirm or disconfirm this hypothesis. In order to deepen more into the kind of properties the teacher listed we looked at the level of generality of the listed properties, counting the number of L and U properties the teachers proposed

- Teacher who belong only to L categories [14]

17, 46, 56, 60, 67, 79, 80, 81, 84, 89, 97, 98, 100, 106

- Teacher who belong only to U categories [16]

5, 9, 15, 21, 25, 29, 36, 37, 47, 61, 68, 71, 74, 85, 95, 105

- Teacher who belong both to U and to L categories [12+27+11=50]

a. more U than L: 7, 10, 13, 18, 26, 40, 64, 65, 73, 78, 88, 91

b. more L than U: 2, 4, 8, 11, 20, 32, 42, 51, 53, 54, 55, 57, 58, 59, 62, 63, 70, 72, 75, 87, 90, 92, 93, 101, 110, 111, 115

c. equal: 1, 14, 16, 22, 38, 39, 66, 76, 77, 83, 96, 108, 116

12 teachers on 80 listed (15,6%) only properties at a lower level; 16 teachers on 80 (19,5%) listed only properties at an upper level; the remaining 51 teachers (65%) showed an intermediate behavior.

39 teachers (50,5%) are more oriented globally to L properties, while 27 (35%) are more oriented to U properties; the remaining 11 teachers (14,5%) showed an intermediate behavior.

A priori we could expect that:

- the teachers who list all L properties:

a. weren't aware of the U dimension concerning real numbers;

b. thought at a very simple way to introduce real numbers at school, misunderstanding the question.

Both 1a. and 1.b , given the lack of the U dimension, can be categorized as teachers whose knowledge is CK, since, even if they answered thinking at high school practices, some of the U properties should have been listed because they are in the national curricula.

- the teachers who list all U properties:

a. studied real numbers at the University and just remember the most used properties of real numbers;

b. answered following the “definition principle”, using the minimum number of features that characterize R at a formal level;

c. think at real numbers as the formalized fields constructed or axiomatized between the XIX and XX centuries, avoiding to confuse them with their intuitive versions

Even if we can't only base on this answer to conclude something certain, we can hypothesize that teachers who belong to this category have a AK about real numbers, or better, once at least studied the real numbers at the University at a formal level.

3) the teachers who listed both L and U, at different degrees depending on the quantity of properties of on and another kind:

a. studied real numbers at a formal level, but feel the necessity to distinguish between formal and more operational meanings of real numbers;

b. studied real numbers at a formal level, but feel the necessity to distinguish between formal and more intuitive meanings of real numbers;

c. studied real numbers at a formal level, but feel the necessity to list properties at different degrees of complexity, thinking both at the institutional/epistemic meaning and at the students' cognitive partial meanings;

For each of these configurations we provide prototypical examples in the Appendix B.

The teachers' personal meanings of real numbers concerning the main properties of the real numbers set are associated in this way to the epistemic partial meanings of real numbers, paying attention to the position in the generality scale highlighted in the epistemic meaning (1-3 [L]; 4-5 [U]) :

CC1 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes) [U]

CC2 ↔ C3.1 (Hybrid continuum) [L]

CC3 ↔ C3.2 (Algebraic structures and numerical sets) [L]

C5.1 (Cantor's transfinite numbers) [U]

CC4 ↔ C3.2 (Algebraic structures and numerical sets) [L]

CC5 ↔ C5.2 (Axiomatization of real numbers' set) [U]

CC6 ↔ C3.1 (Hybrid continuum) [L]

CC7 ↔ CE4 (Line as trajectory) [L]

CC8 ↔ C4.1 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes) [U]

C4.2 (R is the set of Dedekind's rational cuts) [U]

In order to give an idea of the distribution of the teachers' answers in the 8 categories we report a graphic representation of the answers' frequencies:

The most of the answers (29%) concerns CC5: **Axiomatic configuration.**

Another very represented category (19,5%) is CC1: **Topological/differential configuration.**

An intermediate value of answers belonging to CC3 (**Numeric – unitary configuration**) (9%), CC4 (**Algebraic configuration**) (15%) and CC6 (**Line - systemic configuration**) (15%) was also registered.

A minority of answers belong to CC2 (**Numeric - systemic configuration**) (5%), CC7 (**Line - unitary configuration**) (4,5%) and CC8 (**Relation between Q and R**) (4%).

Globally the 52,5% of answers were U, while the 47,5% were L.

4.4.2 Teachers' categories based on the question Q4: the real numbers' set construction

K3: Teachers' declared approach to the construction or axiomatization of a field of real numbers as an enlargement of the set Q

K3_A: Dedekind's cut [14]: 11, 26, 41, 42, 55, 56, 57, 66, 68, 71, 97, 100, 110, 116

K3_B: Cauchy's classes of convergent sequences [10]: 4, 8, 29, 55, 66, 67, 71, 95, 100, 110

K3_C: Cantor's contiguous classes [14]: 2, 9, 21, 55, 57, 58, 62, 63, 66, 71, 72, 73, 74, 77

K3_D: Weierstrass' intervals [0]: *No one*

K3_E: Hilbert's axiomatization [2]: 16, 71
K3_F: Using √2 and/or other radicals, eventually with a historical approach [43]
5, 7, 10, 11, 13, 14, 15, 17, 18, 22, 25, 26, 32, 34, 37, 38, 40, 46, 47, 53, 56, 58, 59, 60, 64, 65, 72, 74, 78, 79, 80, 83, 85, 88, 89, 90, 91, 92, 93, 98, 104, 111, 115

K3_G: Extending the decimal finite numbers to infinite numbers (axiomatic) [5]
11, 32, 41, 42, 100

K3_H: Proposing rational problems without rational solutions [3]
1, 89, 115

K3_I: Union of rational and irrational numbers [7]
10, 14, 38, 86, 87, 89, 111

K3_L: Ratio between C and r (π) [3]
17, 93, 106

K3_H2: Incommensurable magnitudes [2]
17, 105

K3_I_N: Constructing Q starting from R dividing intervals [1]
39

K3_L2: Q dense in R [1]
61

K3_M: union of algebraic and transcendental numbers [5]
64, 65, 74, 87, 93

K3_N: Correspondence with the line (real numbers are all the point of a line) [3]
75, 90, 105

K3_O: R as the set that contains all the other [1]
108

K3_Z: don't remember [5]
20, 39, 76, 84, 101

The same teachers can belong to different categories since some teachers listed more than one approach.
We grouped the teachers’ answers in categories:

IC1) Dedekind (Cuts): K3A [14]

IC2) Hilbert (Axiomatic): K3E [2]

IC3) Cantor, Cauchy, Weierstrass (Limit points): K3B, K3C, K3D, K3L2 [25]

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q): K3F, K3H, K3H2, K3L, K3O [53]

IC5) Union of different kind of numbers (Rational/irrational, Algebraic/Transcendent): K3I, K3M, K3G [11]

IC6) Correspondence with the points of a line: K3N [3]

Some teachers' answers have not been categorized because they were uncorrect (lack of the first level of epistemic suitability); also 5 teachers declared not to remember how R could be constructed starting from Q. In this second case the teachers are aware that a formal construction exists, but don't remember it. These teachers consistitute a new category, to add to those we created before:

3d. studied real numbers at a formal level, but never integrated the previous, maybe intuitive and operational knowledge, with the formal and less intuitive one (this category had already been identified by Tall & Vinner, 1981); as a result they forgot the new aspects, making the old meanings prevailing.

The IC can be put in relation with the epistemic meaning of real numbers:

IC1 ↔ C4.1 (R is the set of Dedekind's rational cuts)

IC2 ↔ C5.2 (Hilbert's axiomatization of real numbers' set)

IC3 ↔ C4.2 (R is the set of Cauchy-Weierstrass-Cantor contiguous classes)

IC4 ↔ C3.1 (Hybrid continuum → [New numbers must represent incommensurable magnitudes like the diagonal of a unitary square, that have special properties and are impossible to represent in the decimal usual representation and by means of fractions, like π,])

IC5 ↔ C3.1 (Hybrid continuum → [Distinction between rational and irrational numbers] & [Distinction between algebraic and transcendental numbers])

IC6 ↔ C3.1 (Hybryd continuum → [R contains all the numbers generated algebrically and geometrically])

Furthermore we grouped the categories in couples of related macrocategories.

Keeping on categorizing the configurations in lower and upper levels in the epistemic meaning we can see that an half of the categories is U while the other half is L.

MC1a) Historical/formal approach: K3A, K3B, K3C, K3D, K3E, K3L2 [41] [U]

MC1b) Adapted approach: K3F, K3H, K3H2, K3L, K3O, K3I, K3M, K3N [65] [L]

Looking at the frequencies we can yet observe that the most of the constructions proposed belong to the L-categories (MC1b):

Other interesting categories in our investigation are:

MC2a) Static introduction: *All*
MC2b) Dynamic introduction: *No one*

Confirming what it had emerged in Q2, no approaches outline the dynamic meaning of continuity and the analogy between the segment as trajectory and other meanings of real numbers, like completeness, bijection with the line, the link between continuous variations and intervals and so on. The dynamic configurations are not considered as possible "operations" that can generate R. This could be referred to the Weyl's and Brouwer's classification of the continuum: the intuitive time-dependent, the punctual and the free-choice acceptations of continuity are considered somehow independent at this level.

MC3a) Intensive (general) introduction: K3F, K3H, K3H2, K3L, K3O [47]

MC3b) Extensive (particular) introduction: K3A, K3B, K3C, K3D, K3E, K3L2, K3I, K3M, K3N [55]

The most of the introductions proposed by the teacher tends to go on this way:

- to provide examples of problems that create a crisis in a previous numbers' model (Q);
- these problems may be algebraic or geometrical;
- in the hybrid continuum (C3.1) every construction ends to be identified with a number/point of a line;
- to appoint these particular cases of numbers that are not rational (i.e. don't correspond to geometrical constructions associated to ratios) to representative of a general element, the irrational number;
- to say R is the set that contains all the possible numbers, the previous and the numbers "like these particular cases"

This procedure, that we analyzed in depth in 2.6, is critical from the point of view of dualities (Font, Godino and D'Amore, 2006) since when the generic element is created this way, there are no patterns that permit to create a real generic object, but there are only particular examples that reasonably will remain the only content of the expression "irrational numbers".

MC4a) Unitary introduction: K3E, K3H2, K3G, K3O, K3N [12]
MC4b) Systemic introduction: K3A, K3B, K3C, K3D, K3F, K3H, K3L2, K3I, K3M [93]

Consistently with the previous trend, but much more impressively, the most of the teachers propose constructions of real numbers that stress the features of the elements of real numbers rather than to introduce the whole set as unitary. In the idea itself of construction there is the inner concept of thinking in a systemic way; nevertheless this is a crucial, maybe the most important, issue concerning real numbers, since we have no possibilities to construct it element by element, but on the other side we have not a simple rule like those that permit to construct Z and Q. This aspect will be investigated in depth in the interviews, since an interaction is necessary to conclude something significant.

It's very impressive the high number and the large heterogeneity of teachers who choose the strategy of using the root square as a way to construct real numbers starting from Q: this introduction that we can place at a medium level in the epistemic scheme is something that the most of the teachers have in common, as we will see also in the analysis of the comments on the first video.

4.4.3 Teachers' categories based on the question Q5: the limit points

Q5 completes the set of three questions concerning the teachers' mathematical knowledge. This question also open a new window, the teachers' goals, and is connected with the following one. We will use in this phase the teachers' questions in order to precise better the category concerning knowledge to which the teachers belong. Then we will also use this question and the previous category to go on categorizing teachers' goals, as we have already explained in 2.6

K4: Teachers' conception of accumulation point in Q and R

K4_A: Q is sufficient [40]

1, 2, 4, 7, 9, 10, 15, 16, 17, 20, 21, 22, 32, 37, 38, 39, 40, 47, 53, 55, 56, 57, 58, 61, 62, 63, 67, 72, 74, 76, 77, 78, 80, 84, 89, 93, 98, 104, 106, 110, 111

K4_B: R is necessary [32]

5, 8, 11, 14, 18, 26, 41, 42, 46, 59, 60, 64, 65, 66, 68, 73, 75, 79, 83, 85, 87, 88, 90, 91, 95, 100, 101, 105, 108, 115,

K4_C: Other [5]

13, 25, 71, 92, 116

4.4.4. Comparison between the mathematical knowledge categories

We matched the categories emerged in the first two steps, in order to better place the teachers in categories and to start to delineate the teachers' profiles.

IC1) Dedekind (Cuts): K3A [14] [U]

Q2.3 56, 97, 100

Q2.4 41, 68, 71

Q2.5a

b 11, 42, 55, 57, 110

c 66, 116

IC2) Hilbert (Axiomatic): K3E [2] [U]

Q2.3

Q2.4 71

Q2.5a

b 55,

c 16,

IC3) Cantor, Cauchy, Weierstrass (Limit points): K3B, K3C, K3D, K3L2 [25] [U]

Q2.3 67, 100,

Q2.4 9, 21, 29, 71, 61, 95,

Q2.5a 73,

b 2, 4, 8, 55, 57, 58, 62, 63, 72, 110,

c 66, 77,

IC4) Root square and π (Example of irrational numbers, R is an enlargement of Q): K3F, K3H, K3H2, K3L, K3O [54] [L]

Q2.3 17, 46, 56, 60, 79, 89, 98, 106,

Q2.4 5, 15, 25, 37, 47, 74, 85, 105

Q2.5a 7, 10, 13, 18, 26, 40, 64, 65, 78, 88, 91,

b 11, 32, 53, 58, 59, 72, 90, 92, 93, 111, 115

c 1, 14, 22, 38, 83

IC5) Union of different kind of numbers (Rational/irrational, Algebraic/Transcendent): K3I, K3M, K3G [12] [L]

Q2.3 89, 100,

Q2.4 41, 74,

Q2.5a 64, 65

b 11, 32, 42, 87, 93, 111,

c 38

IC6) Correspondence with the points of a line: K3N [3] [L]

Q2.3

Q2.4 105

Q2.5a

b 75, 90,

c

K3_Z: don't remember [5]
20 (Q2_5a), 39 (Q2_5c), 76 (Q2_5c) , 84(Q2_3), 101 (Q2_5c)

After two answers we distinguish teachers in terms of stability or ambiguity of their answers. The stability is not necessary a marker for good knowledge and, viceversa, the ambiguity doesn't imply poor knowledge.

In this phase we want to distinguish for instance the teachers who listed U properties and report U properties to construct it from the teachers that listed L properties but report U properties; in the first case the teacher confirm a U knowledge of real numbers, or better shows to have studied al least once formally real numbers and think that real numbers are a formal construction; in the second case a teacher could have studied real numbers formally but think that the important properties of real numbers are the most operational (2) or intuitive (3) one. The number of teachers we decided to analyze in this phase, standing on the first criteria of epistemic suitability, is 76.

Teachers L / L (teacher L who chooses only L constructions) [7] : COMMON

17, 46, 60, 80, 79, 89, 98, 104, 106

Teacher L / U (teacher L who chooses also U constructions) [4] : NOT FORMAL

56, 97, 100, 67

Teacher U / U (teacher U who chooses at least one U construction) [10] : ADVANCED

68, 71, 9, 21, 29, 61, 95

Teacher U / L (teacher U who chooses only L constructions) [9]: SIMPLIFIER

5, 15, 25, 37, 47, 74, 85, 105; 41

Teacher ML / L (teacher ML or intermediate who chooses the most of L constructions) [17]:

1, 14, 22, 34, 38, 83, 32, 75, 90, 87, 93, 111; 53, 59, 90, 92, 93, 108, 115

Teacher ML / U (teacher ML or intermediate who chooses the most of U constructions) [6]:

16, 55, 57, 66, 110, 116

Teacher ML / I (teacher ML or intermediate who chooses the same number of L and U constructions) [10]:

2, 4, 8, 11, 42, 58, 62, 63, 72, 77

Teacher MU / U (teacher MU who chooses the most of U constructions) [1] :

73

Teacher MU / L (teacher MU who chooses the most of L constructions) [11]:

7, 10, 13, 18, 26, 40, 64, 65, 78, 88, 91

The teachers who declared not to remember who R can be constructed starting from Q are all L or ML teachers.

6 L-teachers only quoted L-practices: we hypothesize that with high probability these teachers are limited to a common knowledge (CK) of real numbers; instead 4 L-teachers are more oriented to U-practices

concerning the introduction of real numbers. We hypothesize that these teachers' knowledge is partially AK but they consider the main properties of real numbers at school the L ones.

There are no intersections between teachers U/U and teachers U/L, so the U teachers result divided in 2 disjointed sets; this permit us to create a first couple of complementary categories.

Only one teacher MU quoted U introductions; maybe the teachers MU have studied the properties of real numbers and remember the properties but their personal meanings are L; this is not rare if we look t the interviews analyses (that we will present later), since the formal properties are traditionnaly quite often associated to partial meanings that are considered more useful and intuitve: the terms the mathematicians use to define very formal properties like the density, the completeness, the infinite cardinality are used without reaching a proper level in the epistemic meaning.

The students ML are distributed in the three categories with a higher frequency for L: L (16), U (6) and intermediate (10); 10 teachers on 63 keep on balancing U and L meanings.

Now we will put together the information from Q2 and Q3 and this answers.

	LL	LU	UU	UL	MLU	MLL	MLI	MUU	MUL	DON'T
K4_A	17, 80, 89, 98, 104, 106	56,67	9, 21, 61	15, 37, 47, 74,	16, 55, 57, 110,	1, 22, 32, 38, 53, 93, 111	2, 4, 58, 62, 63, 72, 77		7, 10, 40, 78	20, 39, 76, 84
K4_B	46, 60, 79	100	68,95	5,41, 85, 105	64, 65, 66, 88	14, 83, 90, 108, 115	8, 11, 42	73	18, 26, 87, 91	101
K4_C			71	25	116	92			13	

Tab. 1

We consider LL/K4_B teachers that are used to contextualize everything in the field of real numbers (intended in its personal partial meaning) rather than decide time after time; indeed, to state that real numbers are limit points of R so to define a limit point R is necessary as a container, the teachers should have an AK that didn't show before. Even if they were only referring to classroom practices and were adapting them to the students' personal meanings, the profile that contains these teachers is characterized by an **unaware low awareness of the complexity of the epistemic meaning of real numbers**. This will be put in relation with the category R NECESSARY and R BEFORE in the following Paragraph.

For the teachers who showed a relevant presence of answers U we can hypothesize, standing on the teachers' comments and also on our further interviews, we hypothesize that the choice B is reasoned and is interpretable using the dichotomy “whole-single” we introduced in the commentary in Par. 2.6.

4.4.5 Comparison between the teachers’ evaluated knowledge and the formation’s categories

We compare now the teachers' categories concerning real numbers and their formation, recalling here the categorization by formation:

K1: Teachers' formation

Only PhD in Mathematics: (58, 72), (64)

PhD & Teachers' training: (4),(8), (57, 110)

Only Master in Mathematics : (1, 32, 111) , (2, 62), (5,41), (7), (11, 42), (13), (14, 83, 108), (15, 37, 37), (21, 61), (25), (26), (55, 110) (73), (79), (83, 108); (87); (89); (97)

Master in Mathematics and Teachers' training : (9), (16), (17), (20, 39, 84); (41, 105); (42); (65); (71); (74); (88); (91); (104); (115), (116)

Only Scientific Master not in Mathematics (Physics and engineering): (18), (63), (76), (77), (78), (81)

Scientific Master not in Mathematics & Teachers' training : (10, 40), (46), (53), (92)

Other with teachers' training: (22), (66), (90), (91), (93), (106)

Other: (56), (60), (80), (85), (95), (98), (100), (101), (102)

These result show that the teachers we interview is very heterogeneous, as we hypothesize in the beginning. What we didn't expect was the high complexity of this scheme, that don't divide the teachers in profiles depending on the teachers' formations, as we could expect, but on the contrary is much more articulated.

To find out regularities we must regress one step before:

Only PhD in Mathematics:(58, 72), (64)

PhD & Teachers' training: (4,8), (57, 110)

Only Master in Mathematics : (1, 14, 32, 83, 108, 111), (2, 62), (5, 15, 37, 41), (7, 13, 26), (11, 42), (21, 61), (55, 110), (25), (73), (79), (87); (89); (97)

Master in Mathematics and Teachers' training : (9), (16, 116), (17, 104), (20, 39, 84); (41, 74, 105); (42); (65, 88, 91); (71); (115)

Only Scientific Master not in Mathematics (Physics and engineering): (18, 78), (63, 77), (76) Scientific Master not in Mathematics & Teachers' training : (10, 40), (46), (53, 92)

Other with teachers' training: (22, 90, 93), (66), (91), (106)

Other: (56, 100), (60, 80, 98), (85), (95), (101), (102)

Even if there are more recurrences, it doesn't seem that the formation is a good marker in order to create teachers' categories. In fact all the teachers belonging to the same category related to formation are distributed in the mathematical knowledge categories quite equally. This is an interesting result because *a priori* we considered probable that the mathematical knowledge could be quite homogeneous in teachers who had similar formation while this is very different from what we observed: it seems that the mathematical knowledge of real numbers escapes from this kind of categorization.

4.5 Teachers' didactical goals categorizations

4.5.1 Teachers' goals and real numbers: categories based on the question Q7

G1: *Introduce R to work with:*
G1_A: Exponential function
5, 7, 10, 11, 14, 15, 16, 18, 21, 22, 25, 26, 32, 34, 38, 40, 41, 42, 47, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 89, 90, 91, 92, 93, 100, 104, 105, 106, 116

G1_B: Logarithmic function
5, 7, 10, 11, 15, 16, 18, 22, 25, 26, 32, 34, 37, 40, 41, 42, 47, 58, 59, 61, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 90, 93, 95, 105, 116

G1_C: Differential calculus
4, 5, 11, 14, 16, 17, 22, 26, 32, 34, 37, 39, 40, 42, 46, 47, 53, 55, 56, 58, 59, 63, 64, 65, 66, 67, 72, 74, 75, 76, 77, 80, 83, 84, 85, 87, 88, 89, 90, 91, 105, 108, 110, 111, 116, 115,

G1_D: Integral calculus
4, 8, 10, 11, 13, 14, 16, 22, 26, 32, 34, 37, 39, 40, 42, 46, 53, 55, 58, 59, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 79, 80, 83, 84, 85, 87, 88, 89, 90, 91, 104, 105, 108, 110, 111, 116, 115

G1_E: Sequences and series
2, 5, 8, 13, 14, 25, 26, 38, 42, 46, 55, 56, 59, 61, 64, 65, 68, 71, 72, 74, 76, 77, 80, 88, 91, 93, 95, 100, 110, 116,

G1_F: Absolute value
13, 76, 79, 116

G1_G: R - intervals

5, 7, 8, 10, 11, 13, 15, 16, 17, 18, 20, 21, 22, 32, 34, 38, 39, 40, 41, 42, 47, 53, 55, 58, 60, 61, 63, 64, 65, 67, 68, 71, 75, 76, 77, 79, 80, 83, 84, 85, 87, 88, 89, 92, 93, 95, 100, 101, 104, 105, 106, 108, 111, 116, 115

G1_H: Limits
2, 5, 8, 10, 11, 13, 14, 16, 17, 21, 22, 25, 26, 32, 34, 37, 38, 40, 41, 42, 47, 53, 55, 56, 59, 60, 61, 63, 64, 65, 66, 67, 68, 71, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 95, 104, 100, 101, 105, 106, 108, 111, 116, 115

G1_I: Equation systems
5, 34, 55, 60, 61, 75, 76, 79, 84, 85, 87, 89, 90, 100, 104, 106, 116, 115

G1_L: Other
1, 29, 40, 57, 93, 115

Particular attention is posed to Limits and Sequences and series, since it's another information to compare with the question concerning the necessity of using R for introducing the limit points.

Categories

GO_R_1) R is a prerequisite for quite all the mathematical objects and practices / once introduced R, it is obvious the numerical domain is R if there are no other indications / R is necessary for the graphics [41][L/U] : **R FOR ALL**

5, 10, 11, 14, 16, 22, 26, 32, 34, 37, 40, 42, 47, 58, 59, 62, 63, 64, 65, 66, 67, 68, 72, 73, 74, 75, 76, 77, 79, 80, 83, 84, 85, 87, 89, 90, 91, 104, 105, 116

GO_R_1') R is not necessary at all [2] [U] : **NO R**

1, 20, 57

GO_R_2) R is necessary only for advanced mathematics (Calculus) [7] [U] : **R FOR U**

4, 13, 88, 108, 110, 111, 115

GO_R_2) R is necessary only for common mathematics/functions' graphics [12] [L]: **R FOR L**

7, 15, 18, 21, 25, 38, 41, 61, 92, 93, 95, 100

The most of the teachers consider R as necessary for introducing quite all the first 4 mathematical objects and systems of practices we listed, both advanced and not.

Two very different aspects of R are necessary in the two cases: the advanced practices need the U properties of R, while to introduce functions this property are not necessary, even if R is considered as the natural domain for the functions since they are supposed to have a continuous graph.

Following the synthetic principle we used to analyse the epistemic meaning of real numbers, the upper meanings of R are not necessary to introduce the functions with a continuous graph, because standing on the usual practices exponential and logarithmic functions are assumed to be intituively continuous because there are no reasons to consider a discontinuous behavior rather than proved to be continuous analytically.

An interesting issue concerns the teacher in the first category, both from the point of view of epistemic and cognitive suitability: are they aware of the two different meanings and of their problematic relation? Do they take care of the U properties when they use R in the Calculus' practices? Do they on the contrary try to use other partial meanings, maybe not related to infinitesimals, to the differential and topological structure of R? How do they use intervals and points, the two different kind of “elements” that compose the continuum?

Also it's interesting to investigate better the category of teachers who consider R necessary only in the L-practices, but not for the U-practices. In particular it would be interesting to compare this category with the previous ones in order to characterize better the teachers' goals and the properties they consider important at school and their knowledge of real numbers.

This help us in distinguishing the teachers that are aware of the U-practices but that are oriented to:

1) reconstruction (re-create situations and propose problems through which mathematical objects emerge)

2) simplification (to use objects as cultural static entities to take at school in a simplified version)

For instance to use R as domain always, even when it's not necessary, can be considered a marker of the second kind of approach: the teachers may know that the issue of continuity is anything but intuitive but prefer to simplify; what is particular is that these teachers don't renounce to say that the domain is R, even if they know that the students think at R as a Q enlarged with some known irrational numbers (Hybrid continuum 3.1). Maybe the same teachers avoid to deepen the questions concerning the differential/topological meaning of R when they work with Calculus.

Comparing these categories with the previous:

1) **R FOR L 's distribution [12]**

(7, 18)[MU/L]; (15, 25, 41)[U/L]; (21, 61, 95)[U/U]; (92, 93) [ML/L], (100)[L/U]

The most of these teachers of these category (8/11) indeed are teachers belonging to the categories U or MU; 5 of these showed an inclination to practices also in the previous practices.

These teachers are teacher with a U-background that decide to simplify and reduce R to its L-properties at school.

2) **R FOR U's distribution [7]**

(4) [ML/I], (13, 88) [MU/L], (108, 111, 115) [ML/L], (110) [ML/U]

In this case the most the teachers are ML; also the teachers that were MU were the ones also oriented to L-practices.

3) **R FOR ALL's distribution []**

(5, 37, 47, 85, 105)[U/L], (10, 26, 40, 64, 65, 91)[MU/L] , (11, 42, 58, 62, 63, 72, 77) [ML/I], (14, 22, 59, 75, 83, 32, 87, 90) [ML/L] , (16, 66, 116) [ML/U]; (79, 80, 104)[L/L], (67), (68), (73), (74), (76, 84) [DON'T]

Quite all the teachers proposed a L-construction of real numbers, and with this choice they confirm their choice.

Nevertheless we have to operate distinctions between the teachers who selected quite all the possibilities for a matter of “comfort”, as we stressed before, and teachers who feel sure if they use R: they got convinced that it is important to use always R, maybe because their teachers did the same and they are not completely aware of the reason why.

The first kind of teachers, once introduced intuitively (CE4) or in a hybrid form (C3.1), for habit always introduce new concepts in R; maybe they identify R both with the L-partial and with the U-partial epistemic meanings and maybe time after time refer to a different partial meaning without explaining it to the students.

In other words we are here paying attention to differentiate naive approaches due to a lack of AK and multifaced approaches, due to AK, even if maybe not related to complex and well organized meanings.

Since we hypothesized, standing on our pilot study, that it's not a suitable choice to use R dealing with L-partial meanings and then with U-partial meaning without a deep reflection about the relation between the line and the numbers, dense and complete, discontinuous, discrete and continuous sets, formal and intuitive conceptions, we think it's important to introduce further categories that substitute R FOR ALL and that permit to understand better the reason why a teacher is oriented to this choice.

For instance it's interesting to notice that only a little part of the teachers talked about construction of R as the set of all the points of the line or a set generated by continuous variations; being this the partial meanings at the basis of the intuitive approach to the functions with a continuous graph associated to the numbers, we can hypothesize that this are used at school in order to simplify but the connections between R and the continuous graphs are not clarified explicitely.

- **R HABIT (AK): Teachers who show a multifaced knowledge of R for functions and Calculus**
- **R NECESSARY (CK): Teachers who showed a weak knowledge of R for functions and Calculus**

We hypothesized that the previous categorization could help us to distiguish between these two kind of teachers, but we realized that the informations were not enough and ther were more variables to take in account. However we decided to investigate the existence of this two categories in further interviews.

What we can see with our questionnaire is instead: do the teachers see R as a starting point or as a point to reach in the end, or maybe never, in the high school? This issue is linked to the possibilities Sequences and series and Limits and the categories concerning the possibility to define the limit point in Q or in R. In fact these are all tools to construct R as long as mathematical objects that has properties that we can't always satisfy in Q. We put together here the two groups of data in order to create two categories:

L: 10(A), 11(B), 16(A), 17(A), 21(A), 22(A), 32(A), 34(A), 37(A), 40(A), 41(B), 47(A), 53(A), 60(B), 63(A), 66(B), 67(A), 73(B), 75(A), 79(B), 83(B), 84(A), 85(B), 87(B), 89(A), 90(B), 92(C), 104(A), 101(B), 105(B), 106(A), 108(B), 111(A), 115(B) [34]

SS: 46 (B), 110 (A) [2]

SS&L: 2 (A), 5(B), 8(B), 13(C), 14(B), 25(C), 26(B), 38(A), 42(B), 55(A), 56(A), 59(), 61(A), 64(B), 65(B), 68(B), 71(C), 72(A), 74(A), 76(A), 77(A), 80(A), 88(B), 91(B), 93(A), 95(B), 100(B), 116(C) [28]

64 teacher on 76 choose at least one between Sequences and series (SS) or Limits (L). In this group we look for the teacher who answered consistently with the question concerning the limit points (K4_B) and denote this category with R BEFORE ALL; the other teachers showed an apparent inconsistency, in particular the teachers SS&L who chose K2_A before, because they stated that it's possible to define a limit in Q, while they state know that the properties of real numbers are necessary to introduce limits and sequences, basic element of the limit point configuration (CE2, C2.4, C3.4). It would be interesting and not inconsistent a teacher who knows that it's possible to restrict to Q, but however decide to use R before at school, since this will be a “pure didactical” choice.

- **R BEFORE ALL [26]**

L: 11(B), 41(B), 60(B), 66(B), 73(B), 79(B), 83(B), 85(B), 87(B), 101(B), 105(B), 108(B), 115(B)

SS: 46 (B)

SS&L: 5(B), 8(B), 14(B), 26(B), 42(B), 64(B), 65(B), 68(B), 88(B), 91(B), 95(B), 100(B)

- **Q SUFFICIENT / R NECESSARY [32]**

L: 10(A), 16(A), 17(A), 21(A), 22(A), 32(A), 34(A), 37(A), 40(A), 47(A), 53(A), 63(A), 67(A), 75(A), 84(A), 89(A), 104(A),106(A), 111(A) [19]

SS: 110 (A) [1]

SS&L: 2 (A), 38(A), 55(A), 56(A), 61(A), 71(C), 72(A), 74(A), 76(A), 77(A), 80(A), 93(A [12]

4.6 Teachers' didactical orientations categorization

4.6.1 Teachers' categories based on the question Q8-9-10: the "reality" of irrational numbers

O16: Consider the first video as a good tool [L] [3.1]

O15_A: 1, 5, 7, 8, 10, 11, 13, 17, 18, 20, 21, 26, 32, 38, 41, 46, 47, 55, 59, 60, 61, 63, 67, 72, 73, 77, 78, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 95, 100, 104, 106, 108, 116, 115

O15_C: 1, 2, 4, 7, 8, 11, 13, 14, 15, 16, 17, 18, 20, 21, 22, 25, 26, 32, 38, 40, 47, 55, 57, 58, 59, 60, 61, 62, 63, 64, 66, 67, 68, 71, 72, 73, 74, 75, 76, 78, 79, 80, 83, 84, 87, 88, 89, 91, 92, 93, 95, 100, 101, 104, 105, 106, 108, 110, 111, 116, 115

O15_D: 56, 63, 65, 85, 87, 101
O15_E_: Other 32, 115
O16: Consider the first video as a bad or unuseful tool [U]
O16_B: 36, 42, 53, 65
O16_E: 4, 20, 34, 36, 37, 57

This video is rich of many interesting elements that concern some crucial debates about real numbers; for a complete analysis see Par 2.6.3.6. the video presents a geometrical problem, an arithmetical proof and a projection of a segment onto a line, called "line of the real numbers", so here the author put together the hybrid approach (3.1) and the Line-systemic configuration (CC6).

As we stressed before the most of the teachers didn't mention the CC6 configuration before. Other elements are interesting: the problem presented is concrete (a cord that bounds a garden); the line is anticipated to be the "line of real numbers"; then an approximation method is proposed and also irrational numbers are used to represent physical measures with km as unit.

We think that there are too many critical elements and that this concrete approach, maybe useful to catch the students' attention, may vanify in the future the teachers' attempts to create a linguistic game in which the students have to discuss about real numbers, since this is not a concrete problem and in the so-called applied problems of real life, irrational numbers are really not necessary at all and are substituted by their approximations. The same happens in Physics dealing with measures and errors. Irrational numbers emerge analyzing the Nature (golden ratio, π, e) or the musical scales () but this are far from being concrete problems in the sense of this video. A very interesting approach to the issue of irrationality, faithful to the historical resources and to the partial meaning to which this problem belongs, is presented in Recchiuti (2015), a Master thesis concerning the role of the History of mathematics in the introduction of irrationality. The author reports a very good reaction in the classroom and stresses the potentiality of carrying out argumentation practices. This seems much more interesting since it's consistent with the theoretical exigences and the request of rigour that followed through the evolutions of real numbers.

Nevertheless the most of the teachers seems not to notice these elements, not to consider them as important and misleading, or on the contrary, consider this elements good.

The large agreement concerning it lead us to analyze in depth the reasons of this appreciation and the kind of teachers that disregarded it.

First of all we notice that both the teachers who listed only L-practices or only U-practices indifferently think this video helps the students to understand real numbers. Only one teacher R FOR U show some doubts and critical comments. This is a bit surprising for the R FOR U and for the */U and suggests to us that at school concrete and intuitive approaches are considered good by quite all the teachers' categories; this makes emerge two different interpretations:

- the teachers, looking at the teaching materials, only “see” what they want the students to learn, defoliating respect of the other elements (interesting for interactional suitability);
- the teachers, whatever is their knowledge, at school tend to choose concrete and intuitive approaches (interesting for the intertwining between epistemic and cognitive suitability)

1) **R FOR L 's distribution [12]** (O15)

(7, 18)[MU/L]; (15, 25, 41)[U/L]; (21, 61, 95)[U/U]; (92, 93) [ML/L], (100)[L/U] (O15)

2) **R FOR U's distribution [7]** (O15)

(4) [ML/I](O15&O16), (13 (O15), 88(O15)) [MU/L], (108(O15), 111(O15), 115(O15)) [ML/L], (110)(O15) [ML/U]

The teachers who are critical are:

42(RFA), 53(RFU), 65 (RFA), 20 (NOR), 34(RFLIMITS), 37(RFA), 57(NOR)

This group of teachers is very interesting for our analysis since maybe renounce to traditional intuitive approaches and look for other paths, recognizing the different nature of the practices that lead to R as a cultural object, both to the partial meaning C3.1 or to the upper C4.1 and C4.2.

Looking better to the categories to which the teachers belong we observe that all these teachers quoted at least some U-properties. Also it's interesting that some of these teachers proposed an introduction using incommensurable magnitudes and root squares but don't agree with this approach. This is exactly what we expected from a group of teachers since the problem itself may be used treat important topics concerning about real numbers but avoiding concretizations.

NO CONCRETE R: 42(RFA/MLI), 53(RFU/MLL), 65 (RFA/MUL), 20 (NOR/LU-DONT), 34(RFU/MLL), 37(RFA/UL), 57(NOR/MLU)

4.6.2 Teachers' categories based on the question Q10: the "flowing correspondence" between the real numbers and the points of a line

The video show without audio-comments a point flowing on a line and some rational finite numbers appear time after time while the point is moving forward. The title of the video is quite ambitious: correspondence between real numbers and points of the line. This topic may be faced at many levels of generality and the continuous flow is one of these most primitive (CE4, C2.1, C2.2). Indeed even if also Newton based his Calculus on flowing magnitudes this was not a base for a U-definition of real numbers and its relations with the line, in particular for propertie like density and completeness.

Also we have to remind that no teachers mentionned the dynamical aspects of continuity talking about the properties of real numbers. Looking at the teachers' answers we can observe that quite an half of the teachers think that this video could be useful:

Some teachers highlighted positive and negative aspects.

Thus it's interesting to explore the reasons why so many teachers said this would be a good material to introduce the correspondence between line and real numbers and why some teachers don't agree.

Q13: Video 2: Yes, because...

Video 2: *Yes, because*...(No explanation)
21, 56, 61, 74, 76, 80, 101, 116

O17_A: Makes evident the association between numbers and segments / visualization numbers and points
7, 20, 38, 62, 87, 92, 100, 115

O17_B: The flow of the endpoint is very effective to show the order of real numbers: 8, 10, 108

O17_C: The students have a graphic vision of the problem: 13, 58, 60

O17_D: It's intuitive: 25, 67, 90

O17_E: Makes evident, avoiding not useful words, the completeness/density of R, we could not do the same with the other sets: 71, 83, 22

O17_F: To propose examples is always good: 74, 116

O17_G: Thinks that renforcing conceptions of real numbers through visual images of continuity is a good choice: 32, 67

Q13: Video 2: No, because...

O23: Thinks that density and continuity can't be distinguished by means of graphic representations (other problems are necessary): 1

O24: The second video may confuse the students since the numbers represented are only rational and positive numbers: 2, 10, 16, 34, 40, 41, 42, 47, 55, 79, 64, 65, 68, 72, 91, 93, 110, 111

O25: The flow of the slider of the second video would represent the correspondence with real number if there were no numbers on the line that may suggest a partition in steps

O25_A: 4, 10, 16, 66, 64, 65, 68, 72, 91, 93, 115

O25_B: 4, 5, 10, 22, 90, 93, 66

O26: A zoom of a small part of the line that become longer may suggest the correspondence between real number and points of a line: 4, 115

O27: The second video is not enough, an explanation is necessary: 5, 17, 26, 53, 63, 73, 95

O28: The scope is not clear: 14, 37, 46, 77, 78

O29: It's impossible to see the correspondence in this way: 15, 22, 57, 84, 85

O30: The dinamicity may confuse the students: 18, 22

O31: The necessity of sampling should be motivated : 40

O32: The visive intuition is not enough to make the student understand the deep questions concerning the correspondence: 53

O33: The graphic is not effective: 59, 101

O34: The movement should be continuous instead of going on step by step since the line seems to have empty spaces: 66

O35: Two representation at the same time may confuse the students: 101

O36: It seems that length and numbers are the same thing / lack of unit: 105, 110

O37: The infinity of real numbers between 1 and 2 doesn't appear:106

Macro-orientations about flow and real numbers

OF) The visualization of a flow helps the students to have intuitions about real numbers' properties

OF_2_A) It's possible to visualize density and completeness by means of a flow: O17E

OF_2_B) It's possible to visualize the order of real numbers by means of a flow : O17B, O34

OF_2_C) It's possible to visualize the correspondence between real numbers and points of a line by means of a flow: O17, O17A, O25, O36

OF') Visualization of a flow don't help the students to understand real numbers properties : O30, O36

OF_2'_A) It's impossible visualize infinite points in a segment: O37, O31

OF_2'_B) It's impossible to visualize density and completeness by means of a flow: O23

OF_2'_D) It's impossible to visualize the correspondence between real numbers and points of a line by means of a flow : O29, O32

Categories concerning the identification between dynamic and static representations of real numbers

OF) The visualization of a flow helps the students to have intuitions about real numbers' properties O17E, O17B, O34, O17, O17A, O25, O36

OF') Visualization of a flow don't help the students to understand real numbers properties

O30, O36, O37, O31, O23, O29, O32

Comparison between previous categories and the categories emerged in Q13

Yes, because (no explanation)

(21,61) [UU], (56) [LU], (74)[UL], (76, 101), (80)[LL], 116[MLU]

O17_A: Makes evident the association between numbers and segments / visualization numbers and points: (7)[MUL], (20), (38, 87, 92, 115)[MLL], (62)[MLI], (100)[LU]

O17_B: The flow of the endpoint is very effective to show the order of real numbers
(8)[MLI], (10)[MUL], (108)[MLL]

O17_C: The students have a graphic vision of the problem: 13[MUL], 58[MLI], 60[LL]

O17_D: It's intuitive: 25[U/L], 67[LU], 90[MLL]

O17_E: Makes evident, avoiding unuseful words, the completeness/density of R, we could not do the same with the other sets: 71[UU], (22, 83)[MLL]

O17_F: To propose examples is always good: 74[UL], 116 [MLU]

O17_G: Thinks that renforcing conceptions of real numbers through visual images of continuity is a good choice: 32[MLL], 67[LU]

Quite all the teachers in this category are teachers with profiles containing U-properties; the most of them showed to be oriented towards L-practices in the first categorization; looking at the reasons why they would choose this video we confirm that these teachers are oriented to try to simplify the approaches to real numbers; in particular it emerge that the way all of them try to do it is by means of visualization, also of properties like density and completeness.

Through the flowing point they try to avoid to formalize at any cost.

This category is for us: **INTUITIVE SIMPLIFIERS** and is associated to OF. This teachers are convinced that very formal properties have an intuitive dimension that can substitute every formalization.

To comment on this we can quote again Fletcher (Par. 1.1.1).

"Newton based his ideas of limits and differentiation on intuitions of motion; other mathematicians based their ideas of continuity on spatial intuition. These kinematic and geometric conceptions fell into disfavour in the nineteenth century, as they had failed to provide satisfactory theories of negative numbers, irrational numbers, imaginary numbers, power series, and differential and integral calculus (Bolzano, 1810, preface). Dedekind pointed out that simple irrational equations such as $\sqrt{2} \cdot \sqrt{3} = \sqrt{6}$ lacked rigorous proofs (1872, §6). Even the legitimacy of the negative numbers was a matter of controversy in the eighteenth and nineteenth centuries (Ewald, 1996, vol. 1, pp.314–8, 336). Moreover, Bolzano, Dedekind, Cantor, Frege and Russell all believed that spatial and temporal considerations were extraneous to arithmetic, which ought to be built on its own intrinsic foundations" (Fletcher, 2007).

The real numbers as complex results of geometrical, algebraic and arithmetical processes belong to a somehow counterintuitive thread in the mathematics, as it was confirmed by the critiques we reported briefly in Par 1.1.1 and inserted as a further development in Par 3.1.3.

Focusing on the other teachers we can observe that the main reasons why the video is considered unsuitable are exactly the contrary of those proposed by the INTUITIVE SIMPLIFIERS. This is particularly interesting.

In fact the two main teachers of objections are:

- **semiotic-representative**: the only visualization is not enough to introduce such a complex issue; the coordination with other registers and in particular with representations in the verbal register would be necessary;
- **goal-oriented:** the scope is not connected with the video and is not clear; a mediation and a discourse that pose the problem is necessary.

We name this group SEMIOTIC COMPLEXIFIERS.

SEMIOTIC COMPLEXIFIERS : 5, 17,26, 53, 63, 73, 95, 14, 37, 46, 77, 78, 15, 85, 22, 57, 84, 18, 22, 40, 59, 101, 66, 105, 110, 106

INTUITIVE SIMPLIFIERS: (21,61) [UU], (56) [LU], (74)[UL], (76, 101), (80)[LL], 116[MLU], (7)[MUL], (20), (38, 87, 92, 115)[MLL], (62)[MLI], (100)[LU], (8)[MLI], (10)[MUL], (108)[MLL], 13[MUL], 58[MLI], 60[LL], 25[U/L], 67[LU], 90[MLL], 71[UU], (22, 83)[MLL], 74[UL], 116 [MLU], 32[MLL],

	INTUITIVE/SIMPLIFIERS	SEMIOTIC/COMPLEXIFIERS
BEFORE	100 108 60 83	5 26 73 95 14 37 46 85 101 66 105

Q SUFF/R NEC	21 10 67 22 32	17 53 63 22 84 40 106

The most of the INTUITIVE SIMPLIFIER doesn't belong to the categories R before all or Q sufficient/ R necessary; maybe these two categories are in different ways “structured” while the other teachers follows the principle of lower formality and maximum degree of intuitiveness, escaping from the epistemic structures.

For all the four crossed categories it's interesting to deepen the ways they decide to approach the different problems-situations concerning the real numbers. First of all we explore their relation with the categories concerning the first video, in particular with the teachers who didn't declare that the video was a good tool.

NO CONCRETE R/SEMCOMP: 53(RFU/MLL), 37(RFA/UL)

No teachers are NO CONCRETE and INTUITIVE.

1) R FOR L 's distribution (O15)

(21)[U/U][QI]; (95) [U/U][BS], (100)[L/U][BI]

2) R FOR U's distribution (O15)

108(O15)[MLL][BI]

There are no significant relations between the categories concerning the first and the second video.

We go on analyzing the last questions, all concerning the representation of intervals. The first question concern the third video, while the other the teachers' preferences and comments about some students' answers.

4.6.3 2 Teachers' categories based on the question Q11-12-13-14-15-16: the representations of intervals of rational and real numbers

Categories based on the third video

O38: The third video is not suitable to introduce inequalities

O38_A: The problem presented in the third video isn't useful: 1, 41, 72

O38_B: The language used in the third video isn't good: 1, 7, 88

O38_C: The solution presented is not the graphic solution (a procedure) but instead a graphic representation: 2, 4, 13, 53, 84

O39: The graphic representation allows to visualize better the solutions: 4, 5, 53, 63, 66, 72

O40_A: There are not two solutions but only two representations of the solution: 4, 7, 10, 14, 17, 18, 19, 21, 22, 26, 37, 38, 42, 57, 65, 68, 78, 84, 88, 91, 93, 95, 101, 105, 110, 115

O40_B: It's not clear that the solution of the inequalities are intervals, too many attention is posed to equations and it seems that intervals can only be represented in the graphic register: 10

O40_D: Prefer explicitely the graphic solution even if the representations are equivalent: 38, 53

O40_E: The graphic representation is intuitive: 10

O40_F: The algebraic approach helps in the "limit cases": 10

O40_G: The graphic representation makes visible the infinite quantity of solutions: 11, 34, 53, 64

O40_H: The graphic representation makes the students understand the meaning: 13, 53, 73, 87, 90

O40_I: The graphic representation is useful in the practice: 15, 47, 57, 63, 72, 73

O40_L: It's better to coordinate the registers: 16, 25, 26, 60, 65, 74, 75, 80, 83, 89, 101, 106, 107, 111, 116, 115

O40_M: the graphic representation is not a prove of the solution: 21

O40_N: The graphic solution makes understand the intervals: 32, 62, 66

O40_O: Should emphasize the starting point: 32

O40_P: It's not clear if the methods are equivalent: 40

O40_Q: The methods are not the solutions: 55

O40_R: The graphic solution is always better; it's necessary: 63, 85

O40_S: The choice of a method depends on the goals: 67

O40_T: The algebraic solution is exact, the other is approximated: 68

O40_U: The graphic solution is part of the solution procedure: 73

O40_V: The graphic solution is a "representation" of the algebraic one: 77, 87, 92

O40_Z: The graphic representation makes concrete the example: 79

O40_J: The graphic representation is not the solution and too visualization obstaculate the abstraction: 95

INTERVALS OF REAL NUMBERS

RI1: Absolute value, same register, used for limits: 55, 65, 66, 72, 80, 88, 93, 108

RI2: "Endpoints" inclusion with <: 1, 2, 4, 5, 7, 8, 10, 11, 14, 15, 16, 17, 20, 21, 22, 25, 26, 32, 34, 39, 40, 41, 42, 47, 53, 55, 57, 60, 63, 65, 67, 68, 72, 73, 75, 76, 77, 78, 79, 80, 83, 84, 85, 87, 88, 89, 90, 91, 92, 93, 95, 105, 106, 110, 111, 116, 115

RI3: < and or/and representation: 1, 20, 22, 42, 55, 57, 65, 80, 90, 93, 105, 106,

RI4: "Endpoints" inclusion with parenthesis [a,b]: 40, 61, 62, 63, 66, 68, 83, 85, 88, 95, 108, 116, 115

RI5: Graphical representation: 1, 4, 5, 14, 16, 17, 26, 32, 41, 42, 53, 57, 62, 63, 64, 65, 73, 74, 80, 87, 91, 93, 105, 106, 110, 115

INTERVALS OF RATIONAL NUMBERS

QI1: Graphical representation: 1, 13, 17, 116,

QI2: $-\sqrt{5} < x < \sqrt{5}$

1, 2, 7, 10, 11, 15, 16, 21, 22, 25, 32, 116

QI3: $[-\sqrt{5}, \sqrt{5}]$

16

QI4: $]-\sqrt{5}, \sqrt{5}[$

1, 8, 11, 15, 16, 17, 22, 32, 116,

QI5: $|x| < \sqrt{5}$

1, 4, 11, 13, 21,

O41_B: Different intervals' representations are not equivalent: 57, 90

O41_C: Some representations are more suitable than others in specific practices: 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

O41_D: Thinks that some representations of intervals are immediate/clear/explicit/intuitive or on the contrary, implicit: 4, 5, 7, 8, 10, 11, 17, 20, 26, 32, 34, 37, 42, 53, 62, 64, 66, 73, 74, 79, 91, 95, 110

O41_E: Only some of the proposed solutions are representation of intervals: 1, 2, 7, 15, 22, 34, 67, 68, 73, 75, 79, 80, 83, 84, 85, 87, 88, 89, 91, 92, 95, 105, 106, 108, 110, 111,

O41_F: Some representations are not "finished"/are the problem and not the solution: 5, 7, 10, 11, 15, 22, 32, 39, 40, 41, 53, 57, 62, 68, 77, 84, 91, 95, 115

O41_G: The synthetic representations are better: 10, 40, 63, 78, 79, 95, 115

O41_H: The representation are complementary and the meaning is a configuration of the different objects: 57, 75

O41_I: Every representation has specific features: 67, 65

O41_L: The graphic representation is not formal enough/"represents" the algebraic one: 83, 90, 95, 13

O41_M: The graphic representation of intervals can be useful to introduce upper and lower endpoints in the real numbers: 87

O42_A: The parenthesis [,] represent inclusion also in the case of rational numbers: 1, 4, 10, 11, 13, 14, 17, 21, 22, 32, 34, 40, 42, 57, 60, 62, 63, 67, 68, 72, 74, 79, 80, 85, 87, 88, 89, 91, 92, 94, 108, 116, 115

O42_B: The domain of inequalities should be indicated only in the case of rational numbers: 4, 8, 10, 20, 26, 32, 41, 57, 62, 63, 64, 65, 68, 72, 73, 78, 80, 83, 84, 85, 94, 111, 115,

O42_E: Segments "are" intervals of real numbers: 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

O42_F: The usual representations of intervals of real numbers can't represent intervals of rational numbers: 2, 5, 8, 14, 20, 22, 26, 39, 41, 53, 55, 64, 65, 73, 77, 78, 83, 84, 90, 101, 105, 110,

O42_G: All the representations are good: 7, 47, 61, 66, 76

O42_H: In the representation of intervals of rational numbers irrational numbers can't be used: 21, 87

O42_I: A draw is not a set: 62

INTERVALS REPRESENTATIONS' CATEGORIES

CIR_1) To coordinate adequately registers to represent the solutions of inequalities (intervals) is useful and important/ a lack of good coordination between verbal and other representation makes a didactical practice unsuitable: O38_B, O40_A, O40_F, O40_L, O40_P

1, 7, 88, 4, 7, 10, 14, 17, 18, 19, 21, 22, 26, 37, 38, 42, 57, 65, 68, 78, 84, 88, 91, 93, 95, 101, 105, 110, 115, 10, 16, 25, 26, 60, 65, 74, 75, 80, 83, 89, 101, 106, 107, 111, 116, 115, 40

CIR_2) There is a hierachy in the intervals' representations

a. The graphic representation is better/more intuitive, more synthetic O39, O40_D, O40_G, O40_H, O40_I, O40_N, O40_R, O40_Z, O22

4, 5, 53, 63, 66, 72, 38, 53, 11, 34, 53, 64, 13, 53, 73, 87, 90, 32, 62, 66, 15, 47, 57, 63, 72, 73, 32, 62, 66, 63, 85, 79

b. The algebraic representation is better/more precise, more formal: O40_T: 68

c. The graphic representation "represents" the algebraic one: O41_L : 83, 90, 95, 13

d. The algebraic representation "represents" the graphic one: O40_V: 77, 87, 92

CIR_3) The intervals' representations are equivalent: O40_A, O40_P, O42_G: 4, 7, 10, 14, 17, 18, 19, 21, 22, 26, 37, 38, 42, 57, 65, 68, 78, 84, 88, 91, 93, 95, 101, 105, 110, 115, 40, 7, 47, 61, 66, 76

CIR_3') The intervals' representations are not equivalent: O41_B: 57, 90

CIR_5) A segment can represent the infinite real solutions of an inequality: O39, O40_G, O42_E

11, 34, 53, 64, 4, 5, 53, 63, 66, 72, 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

CIR_5') A segment can't represent the infinite real solutions of an inequality: O40_M, O40_J, O42_I : 62, 21, 95

O7: A segment on the number line is a representation of a subset of R: 22

CIR_6: The usual representation of real intervals can represent a subset of Q: O42_A, RI5&QI1, RI4&QI3, RI1&QI5, RI2&QI2: 1, 4, 10, 11, 13, 14, 17, 21, 22, 32, 34, 40, 42, 57, 60, 62, 63, 67, 68, 72, 74, 79, 80, 85, 87, 88, 89, 91, 92, 94, 108, 116, 115, 1, 17, 16, 1, 2, 7, 10, 11, 15, 16, 21, 22, 25, 32, 116

CIR_6’’: The usual representation of real intervals can’t represent a subset of Q: O42_E, O42_F, O42_H

2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93, 2, 5, 8, 14, 20, 22, 26, 39, 41, 53, 55, 64, 65, 73, 77, 78, 83, 84, 90, 101, 105, 110

4.6.3.3 Teachers’ semiotic profiles

GENERAL SEMIOTIC CATEGORIES OF ORIENTATIONS

GSO_1) **No need of mediation:** A representation can be intuitive/immediate/clear, whoever the interpreter of the sign is: O17A, O17B, O17D, O17E, O39, O40_E, O40_G, O40_H, O40_N, O41_D, O42_E : 7, 20, 38, 62, 87, 92, 100, 115, 8, 10, 108, 13, 58, 60, 71, 83, 22, 4, 5, 53, 63, 66, 72, 11, 34, 53, 64, 10, 32, 62, 66, 4, 5, 7, 8, 10, 11, 17, 20, 26, 32, 34, 37, 42, 53, 62, 64, 66, 73, 74, 79, 91, 95, 110, 2, 4, 5, 8, 16, 22, 26, 32, 39, 42, 53, 55, 64, 65, 68, 73, 77, 78, 83, 93

GSO_1') **Mediation is necessary**: A graphic representation is not enough, language or other representations are necessary to give sense to it := O7, O23, O27, O28, O33, O40_M : 22, 1, 14, 37, 46, 77, 78, , 17, 26, 53, 63, 73, 95, 21, 59, 101

GSO_2) A **graphic representation is always useful** and more intuitive than the others (ostensive, hierarchy): O22, O17_C, O17F, O40_D, O40_H: 13, 58, 60, 74, 116, 38, 53, 13, 53, 73, 87, 90

GSO_3) The **coordination between different registers confuses the students:** O35: 101

GSO_4) S**ome representations are more suitable** than others in specific practices: O41_C, O41_M: 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

GSO_5) Some signs are not representations since they're not finished processes / **Some signs associated to intervals represent the task and not the solution:** O41_E, O41_F: 1, 2, 7, 15, 22, 34, 67, 68, 73, 75, 79, 80, 83, 84, 85, 87, 88, 89, 91, 92, 95, 105, 106, 108, 110, 111,

5, 7, 10, 11, 15, 22, 32, 39, 40, 41, 53, 57, 62, 68, 77, 84, 91, 95, 115

GSO_6) The **synthetic representations** are better: O41_G: 10, 40, 63, 78, 79, 95, 115

GSO_7) **Different representations are complementary** / the meaning is a result of configurations

O41_H, O41_I, GSO_4: 67, 65, 57, 75, 1, 2, 8, 10, 17, 33, 34, 47, 62, 72, 79, 85, 87, 88, 91, 108, 115

Intersecting these categories we create 4 profiles:

ABSOLUTE MEANING

5, 11, 32, 39, 7, 68, 92, 77, 83, 110**: GSO_1 & GSO_5**

4, 16, 42, 55, 64, 66, 20, 93, 100, 71**: GSO_1**

38, 13, 58, 60, 74**: GSO_1 & GSO_2**

65**: GSO_1 & GSO_7**

COMPLEX SEMIOTIC APPROACH

2, 62, 108**: GSO_1 & GSO_4 & GSO_5 & GSO_7**

87, 34**: GSO_1 & GSO_2 & GSO_4 & GSO_5 & GSO_7**

79, 115**: GSO_1 & GSO_2 & GSO_4 & GSO_5 & GSO_6 & GSO_7**

8, 72**: GSO_1 & GSO_4 & GSO_7**

10**: GSO_1 & GSO_4 & GSO_5 & GSO_6**

62, 91: **GSO_1 & GSO_5 & GSO_7**

<u>MEDIATION AND COORDINATION</u>

1**: GSO_1' & GSO_2 & GSO_5**

14, 46, 21, 59**: GSO_1'**

101**: GSO_1' & GSO_3**

<u>GLOBALLY INCONSISTENT</u>

22**: GSO_1 & GSO_1' & GSO_5**

53**: GSO_1 & GSO_1' & GSO_2**

63, 78**: GSO_1 & GSO_1' & GSO_6**

17**: GSO_1 & GSO_1' & GSO_4**

26, 37, 77**: GSO_1 & GSO_1'**

73**: GSO_1 & GSO_1' & GSO_2 & GSO_5**

95**: GSO_1 & GSO_1' & GSO_5 & GSO_6**

<u>OTHER</u>

90, 116**: GSO_2**

15, 41, 84, 80, 89, 105, 106, 111**: GSO_5**

57, 67, 75, **: GSO_5 & GSO_7**

40: **GSO_5 & GSO_6**

33, 47**: GSO_4 & GSO_7**

85, 88**: GSO_4 & GSO_5 & GSO_7**

PhD TEACHERS

Even if we are aware that in the process of didactical transposition teachers make, not always consciously, some choices (which practices of the system of practices propose and present to students through problem-situations) let’s suppose in a first approximation that teachers intend to represent subsets of the set of real numbers with the they listed in Q3 and Q4 as the most important properties of R, i.e. a ordered (O), complete (C) field (F) in which Q is dense (QD), *a. Visualization, graphic representation and real numbers*

Quite all the teachers affirm that visualization is important in the didactical transposition of real numbers and the graphics representation of intervals (segments) as solutions of inequalities is preferred by all the teachers. Even in one case a teacher affirm that the graphic representation let more than the other visualize the infinity of the solutions.

Even the only teacher that didn't express a preference for the graphic representation explicitly, affirmed that visualization is useful when a student has to solve inequalities, without expliciting the domain in which solutions are searched.

When teachers avoid the graphic representation "line" the reason is that graphically we can represent subsets of R and not of Q. Overstepping this last debatable sentence - how to represent for instance Dirichlet's function if not with a straight line? - we will focus the problem from the point of view of results about visualization in didactics of mathematics. According to Bagni (1998) visual techniques can be useful, but it must be controlled by teacher very carefully. Its inattentive use can cause difficulties for learners, up to generate incomplete learning and sometimes being misleading (Duval, 1994). In particular Bagni (2000, p. 4) stresses the inadequacy of graphic representations of properties like density and continuity, so important for the set of real numbers. In a research carried out in the high school in Italy he observed that some students, were "tempted" to use graphical methods to answer questions concerning density and continuity << because the habit of using graphical methods in the didactics of mathematics in the high school [...] but while the difference between discrete and continuous sets is graphically relevable, we cannot affirm the same for the difference between dense and continuous. [...] The graphical methods turn into being misleading >>. Furthermore << the attempt to apply directly graphical representations to the understanding of the difference between dense and continuous sets is doomed>>.

Many researchers showed that the students' perception of the line is anything but dense and complete set of points. Arrigo e D'Amore (1999, 2002) named *dependence* the belief that the cardinality of points of a segment get bigger when the segment become longer (Tall, 1980). Also they observe that the "visive inclusion" of a segment in a longer one, that can be associate to the Euclidean notion "The whole is bigger than one of its parts", affects negatively the students construction of meaning. We can say, using EOS terminology, that there is a practice (operation of inclusion between segments) that becomes a part of the meaning of the object 'set of real numbers' (system of practice a student associate to numbers, identified through other practices with the point of a line) that is not compatible with the desired meaning of 'set of real numbers' (infinite set). Furthermore students often don't become aware of the true issue of continuity because of an intuitive model, also visual: the "model of the necklace" (D'Amore, 1999; 2002) i.e. the image of a segment a sequence of little balls linked by a cord (Arrigo e D'Amore, 1999; 2002; Tall, 1980; Gimenez, 1990; Romero i Chesa & Azcárate Giménez, 1994). This model, incompatible with density and continuity, was showed to be a very widespread intuitive model (Fischbein, 1985?) of a line both for high school and university students and for primary school's teachers interviews (Sbaragli, 2006). Being a strong intuitive model for teachers, it is very used in the primary school and this is one of the main reason why, in absence of a reflection, this is transmitted from one generation to another. High school teachers' interpretation of the line as a continuous sets is far from being similar to those of the colleagues teaching in the primary school and maybe teachers with a PhD in Mathematics are not aware neither of the current interpretation of the line as a necklace in the primary school nor of the fact that this is the previous model onto which they try to construct the dense and complete field of real numbers. Maybe this is the reason why they consider even << too explicit >> (Teacher A, C5) the reference to R if we use a segment to represent numbers. Anyway the "expert interpretation" prevents the teacher to frame correctly the initial students' cognitive meanings, and this may .

b. Segments' endpoints and intervals of ratinal and real numbers real numbers

All the teachers expressed a preference for the representation [a,b] for an interval of real numbers (Q14). Some teachers mentioned that this a usual, traditional, conventional representation of real numbers, while some of them said that this *is* subsets of real numbers. Also teachers refers to this representation using adjectives like explicit, intuitive and clear. The same teachers affirm that this can't be a representation of an interval of rational numbers. Also teachers affirm that representation like],[are adequate for rational numbers also if the belonging to Q is precised through an intersection, f.i],[Q. On the contrary expressions like [,] are considered absolutely inadequate because extremes are included.

The representation of a set of numbers included between two numbers don't specify the kind of numbers considered and leave the interpretation open to many possibilities, neither of which is a continuous set if a students hadn't already conceptualized R as a complete set and the line as good model of R under a suitable condition fixed by a postulate of continuity. Also the distinction operated between representations of open and closed intervals only makes sense for intervals of R in this case and not for intervals of Q. The distinction between open and closed intervals of Q, if the interval is bounded by irrational number is senseless and this can create confusion. Furthermore the intersection with Q, that is used to distinguish sets of real numers and rational numbers, may not be sufficient to represent intervals of rational numbers. In fact the difficulties in the conceptualization of rational numbers are well known (Fandino Pinilla, 2006). In particular many authors, as Bagni (2000), showed that the conceptualization of density is quite hard to realize using the visual representation "line of numbers", since intervals of R and Q are indistinguishable visually. Also the teachers do not observe that in Q14 the domain is not explicited and is considered R as if it is obvious. This can drive students to answers like those observed in Fig. 2 since thy are not used to solve inequalities with sets different from R (even if they don't know R and work on the representation of the segment highlighting the segment bounded by the endpoints. This practice, used to construct R-intervals, is a personal CO different for students and teachers since teachers know R and student don't know it.

c. Movement and continuity in segments of real numbers

Some teachers affirm that a cursor scrolling along a line can represent a correspondence between the points of a line and a set of numbers. We will avoid to consider which set of numbers was considered, if a subset of R or Q, and we will analyze the possibility of a movement along a segment of representing a correspondence of its points with a dense or complete set.

Nunez (2000) showed that discourses involving the embodied practice of going through a line starting from a pint a and reaching a point b without stopping and the properties of being continuous i.e. the existence of a unique finite limit for two approximating successions, one from the left side and one coming from the right side.

e. Absolute value

The representation of intervals as sets of numbers as "disks" in one dimension is very widespread in Italian manuals, both in high school and university. Some teachers consider the representation $|x-1|<3$ not adequate for an interval since it has to be "solved", manipulated in rder to be transfromed in another representation. Maybe the same teachers use this representation to introduce limits of function of real variables, as showed for example by teacher B. It seems that in different practice the same representation assumes different meaning (D'Amore, Fandino Pinilla, Iori, 2013), so teachers change the meaning of the representation when the practice changes. This sort of incoherence may cause difficulties to students when they have to interpret this sign in the further activities, in add to known difficulties showed by Sierpinska and Douroux.

4.7 **From teachers' profiles to the teaching sequences concerning real numbers: the epistemic, institutional and personal dimensions of some teachers' choices emerging in the interviews**

A group of teachers who had participated in the previous part of the research answering the online questionnaire was interviewed in small focus groups (3-4 participants for every group), following the scheme we used in the pilot study.
The teachers' interviews were carried out following an interview protocol but in a semi-guided form, that was elaborated after the pilot study. The protocol contains general open questions concerning the system of practices the teachers implement to introduce real numbers and explain their properties, but also crucial questions that address topics that we consider core problems. These problems lie in the intersection between historical systems of practices and configurations, partial meanings of real numbers that are quoted in the national curricula (institutional meaning) and recurrent system of practices and configurations of objects emerged in previous analyses of "traditional scholastic practices" reported in the textbooks (Tall & Vinner, 1981; Bronner, 2000; Merenluoto & Lehtinen, 2002; Gonzales-Martìn, 2014) and quoted by the teachers in the pilot study (Ch 2,).

Some of the teachers we interviewed in focus group were interviewed also individually in a further session of interviews. This further step was added for two main reasons:

1. some of the teachers answers in focus groups had to be clarified more;
2. some teachers didn't participate very much in the focus group discussions;

We will report just the most important interviews analysis. Some of these were realized in focus groups while other are individual interviews.

Some of the interviews we carried out have not been analyzed because their analyses wouldn't have added something relevant in respect of the teachers' answers in the questionnaire.

Other interviews' analyses have been discarded because in this second phase of the research our analyses aimed at stressing merely the creation of significant categories concerning teachers choices of real numbers in the cases of teachers who showed a good understanding of real numbers. Since the characteristic issues regarding real numbers are very complex, as it was also shown by many authors and was confirmed by our analyses, some teachers didn't have the occasion during their formation paths to deepen the study of real numbers and sometimes their personal meanings of real numbers didn't satisfy the first condition of epistemic suitability, i.e. the correctness, and the second one, i.e. the representativeness.
On the contrary we went on analyzing teachers' interviews in the cases in which teachers' personal meaning was a configuration, or even a collection, of correct but partial meanings, and even when different systems of practices were inconsistent from an higher perspective but couldn't be discovered as inconsistent by the teachers because of a lack of critical problems that could lead them to productive cognitive conflicts (Tall & Vinner, 1981).

These choices were oriented to make the results of the analysis and of the categorization useful in order to design in the future teachers training programs and teaching-learning sequences useful for face the problem in a suitable way; in this perspective we were interested in the teachers' profiles that satisfied at least the minimum requests of mathematical knowledge and epistemic suitability.

Also we posed problems that had the potentiality to change teachers' orientations by means of cognitive conflicts and stimulated them to reflect better on the relations between historical-institutional problems, epistemic partial and general meanings, their practices, the students' behaviours and results from the literature review concerning students' difficulties. We observed interesting dynamics in the teachers' interviews and during further meetings with the teachers involved in the research only in the cases in which teachers had shown a knowledge and declared choices suitable enough in the previous part of the research. These happenings reinforced more our decision to focus our attention only on the cases in which a change or significant dynamics could be somehow prototypical for further studies on teachers' training.

The teachers’ interviews analysis consists in the following 6 steps:

1. Teachers’ interviews’ synthesis step by step
2. Declared teachers’ teaching sequences
3. Teachers’ idiosyncratic idea
4. Teachers’ reports of students’ difficulties with real numbers
5. Teachers’ didactical goals
6. Teachers’ orientations about teaching-learning sequences concerning real numbers

Every description is presented both in a narrative form, following the chronological order in which every configuration or orientation appeared and taking in account the interaction with the interviewer and the eventual groupmates, and in a synthetic form oriented at creating the teachers’ configuration to compare with the epistemic ones. In the second phase the codes assigned to every configurations and the practices, processes and objects listed to characterize every partial configuration will be used. In order to allow a better follow the analysis a detached paper reporting the epistemic meaning structures is attached in the end of the book. The entire interviews are reported in the Appendix A.

4.7.1 ***Case studies***

Teacher 1: (Real numbers are much more intuitive than rational numbers)

1. Teachers’ interviews’ synthesis step by step

Step 1 (answering to I1): Historical approach, real numbers are numbers that are not ratios of whole numbers

Step 2: Real numbers exist in the reality

Step 3: Pythagoras’ theorem oblige to speak of real numbers

Step 4: Algebraic real numbers are necessary to solve inequalities

Step 5:The solutions of inequalities are segments

Step 6:First of all it’s necessary to solve inequalities in R using the graphic representation of segments or half-lines

Step 7:R is the set of the points of the line

Step 8:Real numbers are continuous, are perceivable in a continuous way

Step 9:The segment has no holes, every point has precedent and consecutive points

Step 10:Rational numbers are approximations of real numbers

Step 11:In the reality the numbers are approximated

Step 12:In the number line passing from N to Q everything becomes “a bit jointed”.

Step 13:If a segment is result of a movement to draw it, it’s full

Step 14:Obviously it’s impossible to represent Q graphically because the segment is continuous.

Step 15 (interacting with I):Q is equipotent to N and that R is not. There in an enormous jump.

Step 16 (interacting with I):R is the mathematization of a primitive intuition.

Step 17 :Definition of limit points. Differentiation between limit of a sequence, that we calculate only to , while we can calculate the limits of real functions also in finite number: is the only limit point for the sequences

Step 18 :R is a topological space

Step 19:Calculus' main theorem are formulated in topological spaces

Step 20:Interdisciplinarity with Physics: approximations are identified with numbers and put in the line because of the necessity of using transcendental functions that they don't know and calculate using the calculator without being aware of the process.

Step 21:Difference between real numbers and their practical use through approximations, that are respectively continuous and discontinuous sets.

Step 22:Starting from R, stress the existence of N, Z, Q as discrete subsets of R.

2. Declared teachers' teaching sequences

a. In the Calculus uses the primary intuition of continuity rather than introducing more formal conceptions

3. Teachers' idiosyncratic idea

a. Real numbers should be introduced very soon
b. Real numbers are more intuitive than rational numbers
c. Operations with real numbers and real numbers should be clearly separated
d. The inequalities solved in Q are disturbing

4. Teachers' reports of students' personal objects and/or difficulties with real numbers

a. In the students' minds R is a line
b. The correspondence between numbers and points is innate, it is very natural for them
c. When a student think of a number, he thinks of a real number
d. The students perceive the numbers in a continuous way
e. Would say to the students that Q is impossible to represent graphically
f. When the students are too young they study rational numbers but they have no confidence with the decimal approximation of magnitudes
g. In the students' minds there is already a spontaneous conception of real number as a sequence of numbers
h. Only a few students understand the discourse about limit points for sequence and for real functions
i. The Calculus is so hard for the students to study that a deep discourse concerning real numbers could be useful only for some students
j. The students see the continuous, the contiguous classes
k. For the students everything is full, there is everything in the segment. If they take off something it's obvious that I find something more. The meaning of accumulation point is impossible to understand for them.
l. The students have a strong idea of continuity but they don't have a similar idea of discontinuity.

5. Teachers' didactical goals

6. Teachers' orientations about teaching-learning sequences concerning real numbers

a. It's interesting to solve inequalities in Q or in N only after introducing them in R

b. We can avoid to ask them to solve inequalities in R only because we're not able to introduce them formally
c. It's useful to work on the graphic representation as a good substitute of more formal representations of real numbers
d. It's easy to show that Q is equipotent to N and that R is not.
e. The line's topology is trivial, or better, it's not trivial but it's usually presented in the textbooks in a trivial way

7. Analysis of teachers' declared choices didactical suitability
8. Literature and teachers' orientations about students' difficulties: a comparison
9. Literature and teachers' declared teaching sequences: remarkable comparisons between results in Mathematics education research and teachers' choices

Teacher 2: (The relation between finite and infinite in the segment is a paradigm of life)

1. Teachers' interviews' synthesis step by step

Step 1: R is the set of limit points of Q

Step 2: The representation of Q has holes, R has no holes. Segments are subset of R.

Step 3 (answering I1): R is an enlargement of Q, in the chain
N ZQRC.

Step 4: Z is necessary for subtraction, Q is necessary for division. Some numbers are not included in Q.

Step 5: Proof of irrationality

Step 6: "Geometrical numbers": numbers are endpoints on a line of segments constructed through geometrical procedures

Step 7: A finite thing can contain infinite points

Step 8: Between 0 and 1 there are always points (limit points), but this is also true in Q (density of Q)

Step 9: Dividing we construct Q, Other things are in R.

Step 10: There is always a point in the middle, between two points.

Step 11: Points have no dimensions

Step 12: R is necessary for limits

Step 11: When you solve equations coming from geometrical problems like those that contain Pythagoras' theorem, you need real numbers

Step 13: is a geometrical number that can be approximated by means of geometrical procedures

Step 14: , infinitesimal quantity, is a real number

2. Declared teachers' teaching sequences

a. Problems regarding the reality that give sense to real numbers

b. In the first year introduce Q, in the second year a part of R.
c. R is necessary for limits in the last year
d. It’s important to go on with parallel discourses between Geometry and Algebra in relation with numbers and infinity
e. must be introduced geometrically, in a laboratorial way
f. infinitesimals are necessary to introduce limits, so close to x0
g. Present the passage from Q to R graphically, between rational numbers there are irrational numbers

3. Teachers’ idiosyncratic idea

a. Segments are paradigms of life: finite things contains infinite points
b. The definition of limit points in Q or R is an open problem
c. We don’t have a way to represent Q
d. I should reflect on representations

4. Teachers’ reports of students’ difficulties with real numbers

a. Only for good students the proof of irrationality of 2
b. The students pose many questions and are interested when a teacher talks about finite and infinite quantities
c. The students accept that the line is infinite, but don’t accept easily that a segment is infinite

5. Teachers’ didactical goals

a. Real numbers are necessary for the measures

6. Teachers’ orientations about teaching-learning sequences concerning real numbers
7. Analysis of teachers’ declared choices didactical suitability
8. Literature and teachers’ orientations about students’ difficulties: a comparison
9. Literature and teachers’ declared teaching sequences: remarkable comparisons between results in Mathematics education research and teachers’ choices

Teacher 3: (We have to give to the students strong bases and to present Mathematics as an history of humans’ cultural conquests)

1. Teachers’ interviews’ synthesis step by step

Step 1: R is necessary as domain of functions

Step 2: Zeno’s paradox anticipate the topology of the line

Step 3: R is a topological space

Step 4: The relation between R and the line is not trivial

Step 5: To create a 1-dimensional object using 0-dimensional objects is a problem

Step 6: The line without a point is not visualizable, but also the line, as long as the points.

Step 7: The relation between continuous (intuitive) and discrete objects is very complex

Step 8: [Irrational numbers have infinite digits without repetitions] & [Rectification of the circle through geometrical approximation procedures] are practices to treat in parallel, these are two aspects of the same discourse

Step 9: Numerical-geometrical approach to exhaustion using Archimedes' methods

Step 10: R is a field, extension using historical approaches from N to Z and to Q and then using
2.

Step 11: Algebra's fundamental theorems

Step 12: Mathematics was adapted to R and not viceversa, R is the only example we can propose of so many objects: metric spaces, complete fields, and so on.

Step 13: Algebraic symbols are representations of objects, like segments or numbers. Numbers are abstract ideas, Pythagoric.

Step 14: Q is the set of fractions; fractions are relations between magnitudes, commensurable magnitudes

Step 15: Decimal representation of Q is good for Physics

Step 16: In the Calculus we use intuitive variations/intervals rather than static representations

Step 17: Hyperreals

2. Declared teachers' teaching sequences

a. The topic maths and music is very useful to introduce the concepts of logarithms and exponentials using the musical scale: naturals, rationals, geometrical irrationals, not geometrical numbers, complex numbers
b. Task: to present x2, 2x, xx
c. Use Zeno's paradox in the third year to anticipate the definition of limit and the line's topology
d. Ask the students: "Could you assure there is a correspondence between the line and the numbers?"
e. Using Geogebra and zooming in the line we can deal students' to reflect on the fact the line is not its draw, since it has a grower thickness
f. He introduce contiguous classes once, but then he talks intuitively.
g. To approximate numbers using excess and deficiency approximations
h. Introduce R as a field using the set's chain and 2
i. Introduce the Algebra's fundamental theorem
j. Solve polynomial equations
k. Use Geogebra to create a coordination between algebraic and geometrical aspects of real numbers
l. Use x to represent segments or numbers but I always stress this is a representation.
m. Task in Geogebra: to zoom in enlarging a point, in order to show the draw is not the object
n. Represents Q by means of fractions
o. Defines limit points but work with limits without using it
p. Task: Ask the students what kind of numbers are e, , sin , ln 3

3. Teachers' idiosyncratic idea

a. A question at the University about the fact that we are not sure there are no holes in the line conditioned his way to talk about real numbers and the line to his students
b. Every time we take a point it's a conquest to say it's a real number
c. We can't visualize the line without a point
d. A draw is different from an object, it' only a representation.
e. The properties of R are shocking
f. Teachers are quite obliged to define limit points or contiguous classes but it's unuseful

g. Continuous is not the limit of discrete, it’s another thing.
h. It’s important to distinguish geometrical numbers and numbers we can’t construct by means of geometrical procedures
i. Dense and continuous may have the same representations, the continuous is a support for discrete sets
j. Real numbers, infinity and the Algebra’s fundamental theorem are conquest of the humanity.
k. To solve polynomial equations and to study polynomial functions.
l. Q is a set of relations, this is its best geometrical representation, not the points on the line
m. The existence of real numbers as results of processes doesn’t imply their geometrical construction

4. Teachers’ reports of students’ difficulties with real numbers

a. Functions like 2x, x irrational and xx are in the hyperuranium for the students
b. The students have doubt when they reflect on the the fact that 0-dimensional objects create a 1-dimensional object
c. The students create misconceptions when a teacher merges discrete and continuous.
d. When he was at school he approximated 2 with 1,4; 1,41; 1,414 reducing as much as he want the difference. He thought to have understood but then he realized he didn’t understand the deep meaning.
e. The students understand the difference between geometrical and not geometrical numbers, but they are shocked when a number is not geometrical, like 32
f. The students are not used to think in discrete sets
g. Limit points’ meaning is related to the properties of Q, examples are always in Q and never in R.

5. Teachers’ didactical goals

a. It’s very important to present to the students questions and problems that lead humanity to wonderful conquests

6. Teachers’ orientations about teaching-learning sequences concerning real numbers

a. Properties of real numbers are not necessary to introduce exponentials and logarithms but are necessary to complete the functions
b. Exponential and logarithmic functions are prototypical real functions in the real domain
c. Formalization of real numbers is not useful from a didactical point of view, even in the Scientific high schools
d. It’s important to analyse deeply the relation between the line and the numbers
e. The graphic representation is important since it’s synthetic, but it can’t ben the only one; it’ necessary to coordinate analytical aspects, limits, inequalities.
f. Avoid to talk about sequences in N and then limits of functions since this passage creates a short circuit between continuous and discrete, it can create misconceptions.
g. We have to construct strong bases and not to refer to everything to do too many things.
h. We must not introduce everything but only something that is useful for some goals.
i. To know real numbers means also to study polynomial function and to solve polynomial equations.
j. It’s important to coordinate numerical aspects, algebraic aspects and graphic/geometrical aspects
k. The axiomatic approach are not effective because students can’t understand anything by means of them; indeed the mathematics was adapted to R and not viceversa; R is the example of all this axioms and all the mathematical definitions.
l. Limits can be introduced without defining limit points
m. It’s important to defoliate concepts to give strong bases

Teacher 4: (Only a very few of real numbers can be taught in the high school)

1. Teachers’ interviews’ synthesis step by step

Step 1: Real numbers are in correspondence with the points of the line

Step 2: Irrational numbers can be approximated using rational numbers

Step 3: The problems that lead to R are geometrical

Step 4: The most important thing is to represent the total and linear order of R using the line, that is different from the axiom of the order

Step 5: R has an algebraic structure that must be consistent with that of Q in terms of properties of the operations

Step 6: R is approximate with Q

Step 7: R is a set whose elements are identified by their properties and not by their decimal representation

Step 8: Q is dense in R, ordered a consistent way in respect of Q, complete

Step 9: Actual infinity of R vs potential infinity of the limits' approach

Step 10: Algebraically we work with Q, but we use the line to represent the numbers

Step 11: In the limit processes with function there is a continuous variation towards the point [dynamic]

Step 12: A continuous function is a function that makes correspond to small variation of x in a small interval, small variation in the functions' image interval (
/)

Step 13: Intervals of Q and R can be identified graphically, but it's full and I know it's R

Step 14: The segment is a set of points that correspond to rational and irrational numbers; the rational points are constructed dividing and multiplying, the irrationals we use at school by means of geometrical constructions

Step 15: /can be rational numbers but not all the limit processes lead to rational numbers, like 2

Step 16: Infinite rational steps don't imply something is rational

Step 17: The relation between continuous and discontinuous is complex: limits of functions and separated steps are not the same

Step 18: Intervals contains other intervals, smaller and smaller

Step 19: There is a parallelism between correspondence and postulates, R is in correspondence with the line

Step 20: The existence of infinite points is a geometrical axiom; the Archimedean axiom concerns the line and not only the abstract numerical set

Step 21: The domain we take for granted is R

2. Declared teachers' teaching sequences

a. Introduce real numbers as points of the line, taking it for granted
b. To construct a rigorous construction is a finesse; Euler didn't have the real numbers but he did so many wonderful things
c. We indeed use the approximations of irrational numbers
d. Introduce 2 for its property ie. the number whose square is 2

e. Use the properties :Q is dense in R, ordered a consistent way in respect of Q, complete, without expliciting them
f. Introduce continuous functions using limits in a dynamic form
g. Real number are the decimal numbers
h. To describe the properties of the line you go on reducing distances until the thickness of your pencil
i. You divide with your imagination zooming in more and more the intervals

3. Teachers' idiosyncratic idea

a. As human beings we only use Q, so we "play the play" of approximating R using Q.
b. R is important for the properties of irrational numbers, not for its decimal representation
c. The segment is a subset of R because I can draw it full
d. If I see a segment after I can't establish if this represent Q or R
e. You always leave with the double truth: the set of real numbers, that is an abstraction, but you operate with finite representations, draws and computations
f. Maybe numbers don't exist, they are only human inventions
g. R is the line and you have to make an act of faith that Q can be completed to R
h. The sequences must converge to something
i. The humans are limited, they can go imagining only until the thickness of a pencil; it would be sufficient Q to carry on this discourse
j. Metaphorically an irrational number is like the border of an abyss: we approach t something that doesn't exist; teachers don't understand very well too what happens; we get used at a certain point
k. The formalization was useful to clear Dedekind's conscience, but it's useful for anything
l. We don't have a symbol for Q intervals; we can't express a Q interval using irrational numbers as endpoints

4. Teachers' reports of students' difficulties with real numbers

a. The correspondence between numbers and points is innate and sufficient, it's a preconception and we can take it granted
b. What makes intelligible to the contemporaries the limits is to be potential
c. Persons thinks algebraically in terms of rational numbers but see them onto a line as real numbers
d. The students see the approach in a continuous variation towards a point when they talk about the limits
e. To understand the continuity is necessary a counterexample taking off a point
f. It's very hard for the students to understand the relation between Q and R
g. Only smart students are annoyed if you don't deepen the problems
h. Decimals are sequences, better series, but we don't say this to the students
i. The students have an intuition of the line as small ball, one close to the others
j. To deepen theoretical questions creates more doubts

5. Teachers' didactical goals
6. Teachers' orientations about teaching-learning sequences concerning real numbers

a. To be rigorous with real numbers is not necessary to do what we have to do in the high school
b. The existence of a correspondence between points and numbers is sufficient
c. To go beyond rational numbers we need Geometry; in Algebra it's not necessary; then he changes his idea thinking at radicals and functions.
d. In Mathematics we need not rational numbers since the 8th grade with radicals because the students see geometrical constructions connected to irrational numbers.
e. You can insert the irrational numbers between the rational being consistent with the order of Q without considering the algebraic structure of real numbers.
f. The Calculus is graphic-centered, the functions are identified with their graphic representation so R is the line.
g. To construct R formally has a few advantages and many prejudices.
h. The axiom of the order is something formal

i. What we need are the properties of numbers like 2 not their decimal representation
j. We can omit historical passages without losing something in terms of meaning; we just can choose what to do and what avoid to do in order to simplify Mathematics for the students
k. It's important to consider full intervals for the Calculus
l. You have to construct R equal to R for computations and order, that can contain Q nd irrationals
m. We teach a pseudo-Mathematics, you have to decide what to say to the poor students who trust you
n. Density, completeness, continuity are used without being explicit
o. You need real numbers to talk about sequences and series
p. The adults created a wonderful Mathematics, I can only show to the students some hints
q. We have to make Mathematics easier, to make it more understandable

Teacher 5: (The students have many difficulties with real numbers: maybe we simplify too much)

The teachers' interviews analysis consists in the following 8 steps:

1. Teachers' interviews' synthesis step by step

Step 1:
NZQR

Step 2: R is the union of rational and irrational numbers, the set of all the numbers existing in nature

Step 3: The diagonal of the unitary square is 2; we need a set that contains all the numbers

Step 4: Between 0 and 1 there infinite points

Step 5: The conventions used for the representations of intervals of R are worthful also for Q

Step 6: R is necessary for equations, inequalities, functions; quite for everything.

Step 7: The number line is important since we have to represent 2in the first year

Step 8: If we don't introduce R, the irrational number would remain punctual. We need R to complete the line as a continuum

Step 9: In R the consecutive of numbers doesn't exist

2. Declared teachers' teaching sequences

a. Introduce the set chain
b. Use as example of real number because they already know it
c. Introduce R as joint of rational and irrational using the metaphor of the
d. doll
e. Introduce the problem of the diagonal of the unitary square without proof and put into the line
f. The fraction are all the finite or periodic decimal numbers; some numbers that exist in the nature are not rational so we construct the set that contains all the numbers

3. Teachers' idiosyncratic idea

a. R is the set of all the numbers existing in the nature
b. We can't represent subsets of Q using
c. if the extremes are irrational numbers
d. R is necessary quite for all in Mathematics, in particular for real functions in the real domain

4. Teachers' reports of students' difficulties with real numbers

a. The students have many difficulties with real numbers
b. The students don't understand that between 0 and 1 there are infinite numbers
c. The students don't read the informations about the set of numbers in which they have to solve equations and inequalities
d. The students forget everything
e. The students' problems concern rational numbers
f. The students always ask how many points they must draw to trace the line

5. Teachers' didactical goals
6. Teachers' orientations about teaching-learning sequences concerning real numbers

a. It's very hard to introduce real numbers
b. Introduce real numbers in a very simple manner because she teaches in a Professional school; if she would have taught in a Scientific high school she would have used contiguous classes
c. It's impressive that in the Scientific high schools the problems are the same and also the teaching sequences are the same
d. Maybe our error is not to deepen enough, to simplify too much or to omit explanations
e. To use rational functions, or to us N, Z, Q with inequalities or functions is too complex, she never does it

Teacher 6: (The students forget everything: they remember only the last thing you teach, so they use real number and forget the rational ones)

1. Teachers' interviews' synthesis step by step

Step 1: Real numbers contains new elements: the roots and some other numbers like these

Step 2: A new set is created when an operation opens the path and create the exigence to extend the previous set

Step 3: The diagonal of the unitary square is an irrational numbers; after Ippaso the exigence of creating a new set emerge

Step 4: Between two numbers there are always infinite numbers

Step 5: The concept of infinitesimal can also be rational and sometimes it's 0, sometimes it's not

Step 6: A segment is full, it represents R

Step 7: R is necessary because we need algebraic numbers

Step 8: The concept of continuum and real number are known and they are both necessary in the Calculus

2. Declared teachers' teaching sequences

a. Introduce the numerical sets to create a set closed respect of the operations
b. Introduce the density of numbers and of the line at the same time
c. Represent the numbers using a line
d. Task: Take a segment in Geogebra and enlarge it more and more
e. Introduce R as the set that contains the roots
f. Introduce limits in a intuitive dynamic way
g. Rational problems are particular cases

3. Teachers' idiosyncratic idea

a. Mix operations and geometrical procedures, identifies numbers and segments

4. Teachers' reports of students' difficulties with real numbers

a. The students don't understand the infinitesimals and never understood that between 1 and 1,0001 there are infinite numbers
b. The students understand that in a segment there are infinite points
c. The students don't understand that they can divide a segment in parts and then divide it again and always find again infinite numbers, in particular when the points are very close
d. The students are more impressed by the graphical representation so I use the graphical representations for the numbers
e. The density presented through zoom in Geogebra are understood but they don't leave a trace
f. The students write x <
g. 5, in analogy with the equations
h. The students have an elementary knowledge because after some years they study only for the mark, not to learn really. They forgot, they are not able to use what they learn
i. The students have many difficulties when a teacher formalizes
j. A neighborhood have always infinite points: it's obvious, also his son understand
k. Potential infinity in form of recurrent procedures is easy to understand, while the infinity itself (infinite quantity) no
l. The problem for the students is working in Q, not in R, they are used to work in R and not used to work in Q
m. A student can operate with real numbers even if she doesn't know rational numbers
n. The students don't know that real numbers have no consecutive numbers
o. The students focus only the last thing on the last thing they study; they forget Q because you have introduced R as last.
p. The students have an intuition of the correspondence between numbers and points, for them the line is continuous

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. Real numbers are introduced in the first years of the high school
b. R is necessary to introduce functions and to solve equations
c. Infinitesimals are difficult to teach
d. R and the continuum are necessary in the Calculus, but the teachers don't need to recall them since they are known
e. The problem is the lack of time, not the fact that the students don't understand
f. A teacher must make choices, she can't do everything in a few hours
g. A teacher trusts the fact that the students knows real numbers, because they have studied them, even if the most of them have difficulties
h. At school R is used intuitively, using the continuum
i. Takes for granted that a neighborhood has infinite points
j. Takes for granted the students know R intervals
k. The textbook don't help the teachers: there are no exercises in Q or Z

Teacher 7: (We are responsible for the students difficulties with rational and real numbers)

1. Teachers' interviews' synthesis step by step

Step 1: There are infinite numbers like the root square of 2

Step 2: R is Q join to irrational numbers

2. Declared teachers' teaching sequences

a. She says there are infinite numbers like 3,
b. Task: she asks the students to invent irrational numbers in the decimal register

3. Teachers' idiosyncratic idea
4. Teachers' reports of students' difficulties with real numbers

a. My students would not have read x Q
b. The students have many difficulties with quadratic equations
c. The students solve the quadratic equation associated to quadratic inequalities and linear inequalities: this is the reason why they write x<5
d. The students have difficulties also to understand algebraic numbers
e. The choices depends on the kind of schools
f. For the students is very difficult to learn limits

5. Teachers' didactical goals
6. Teachers' orientations about teaching-learning sequences concerning real numbers

a. What do we do at school to make them doing such things?
b. Maybe we don't work enough in Q

Teacher 8: (The students get confused when we formalize: real numbers are intuitive)

1. Teachers' interviews' synthesis step by step

Step 1: Real numbers had already been constructed by Archimedes

Step 2:
NZQR

Step 3: Convergence of intervals on a rational point or on a "void space" using Q, you have to construct a number

Step 4: is not of the previous kind, you have to construct it in a different way

Step 5: You can find in the nature in so many beings

Step 6: 2is the diagonal of the unitary square

Step 7: R is necessary for exponentials to complete the graphic

Step 8: the topology of the line is necessary to introduce the limits

Step 9: R is completed also with transcendental numbers

Step 10: R is in correspondence with the points of a line

Step 11: On the line using R you lose the concept of order, you can't establish the consecutive number

Step 12: The separating elements of the line must exist

Step 13: R is necessary to solve quadratic equations

Step 14: e is a limit, while is different

Step 15: Hyperreal numbers: a number that has a strange circle around

Step 16: The model of continuity doesn't work for everything real

2. Declared teachers' teaching sequences

a. NZQ and then we need to go on enlarging because there is a number that is not rational
b. He introduces constructively using a geometrical procedure, saying you can fin in the nature a lot of times
c. He introduces2as the diagonal of the unitary square projected on the line
d. He has a phenomenological approach to the exponential, f.i. growth rhythm , that doesn't need real numbers, but you need real numbers in the middle
e. He says the line is complete adding to Q not only roots but also transcendental numbers
f. He introduces intuitively the topology of the line and its density in order to introduce limits
g. He introduces eas limit
h. The enlargement from Q to R is realized using the roots

3. Teachers' idiosyncratic idea

a. R is an Archimedean set
b. He got confused at the University when his Professor formalized real numbers
c. You can use numbers without formalizing
d. You can complete with continuity but there is the problem of
e. R is a set in correspondence with the line
f. Nature is analogic, is discrete
g. Irrational numbers aren't always between two rational numbers
h. R is a tool, we need to learn to use real numbers
i. Hyperreal numbers are strange, he's not sure they exist
j. The model of continuity doesn't work for everything real

4. Teachers' reports of students' difficulties with real numbers

a. He thinks that to formalize confuses the students
b. He tries to make them understand the concept of limit using / as rarely as possible
c. The students see the density of the line, they understand it; it's very intuitive
d. The separating element 2emerge in the practice of dividing segments: there must be a number whose square is 2
e. The students have a vague sensation of what R is
f. It's important that 2 exists, not what it is

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. He uses real numbers without formalizing
b. He uses R for everything; he introduces R before every other thing
c. Introduce 2as a diagonal of a square projected with a compass on the number line
d. The exponentials need R to complete the graphic
e. You need real numbers, a bit of topology of the line, to introduce limits
f. When you introduce the Calculus they already know real numbers that they studied in the first years
g. To say you can't establish the consecutive number is necessary for the limits
h. The new books simplify much more than the previous ones
i. We must take care of the goals we have to reach with these students
j. He doesn't talk about the different cardinalities even if it's in the book
k. The graphic representation for R is the best one, teachers should work more with it

Teacher 9: (The real numbers are important to do beautiful things in Geometry)

1. Teachers' interviews' synthesis step by step

Step 1: Irrational numbers

Step 2:

is different in respect of the irrationals found as limits

Step 3: The line is necessary for real numbers

Step 4: Real numbers are separating elements of contiguous classes

Step 5: Operations with real numbers (roots,)

Step 6: The Euclidean geometry's axioms contribute to create the properties of R as the set of points of the line

Step 7: Real numbers emerge in geometrical problems

Step 8: Irrational numbers are numbers we can't express using fractions; it's connected to Geometry with commensurable and incommensurable magnitudes

2. Declared teachers' teaching sequences

a. She introduces irrational numbers using the diagonal of the unitary square
b. She introduces real numbers using contiguous classes
c. She presents operations involving irrational numbers
d. She prove in the Euclidean geometry that between two point there is always a point and that every point has a precedent, without associating numbers to the points
e. She introduce real numbers in order to use roots in Geometry
f. Use historical approaches

3. Teachers' idiosyncratic idea

a. It's impressive the Ancient got so fast so good results
b. Real numbers are necessary in Geometry
c. The line has geometrical properties that in a second time

4. Teachers' reports of students' difficulties with real numbers

a. A students of her would answer a real number is a separating element for contiguous classes
b. The students "defoliate", go at the core the matter

5. Teachers' orientations about teaching-learning sequences concerning real numbers

a. Teachers only remember what they teach usually; the other things are vague and they have to recall them in their mind studying again
b. In the first years you teach irrational numbers, you don't go on; she doesn't know what her colleagues do in the further years
c. The Euclidean geometry's axioms contribute to create the properties of R as the set of points of the line
d. Textbooks simplify very much, both for R and for Q

e. You can do a lot of nice things in Geometry with real numbers, merely using Pythagoras' and Euclid's theorems in which roots are necessary

Teacher 10: (I understand my students: I studied the formal aspects of real numbers and I came back to the trace of a segment)

1. Teachers' interviews' synthesis step by step

Step 1: Real numbers as separating element of contiguous classes

Step 2: R is the line; the examples of irrational numbers are geometrical

Step 3: R is not necessary for the series; N is enough.

Step 4: Exponential functions are defined in R
Step 5: Density of the line

Step 6: Dynamic conception of continuity; continuous segments can be traced with a pencil

Step 7: Formalization is not necessary

Step 8: Limit points as points we can approach without reaching them

Step 9: Irrational numbers are represented with other symbols (
,e,..)

Step 10: R is necessary to give an idea of C

2. Declared teachers' teaching sequences

a. She introduces the line, the diagonal at the unitary square and the Pythagoras' theorem or real numbers as separating elements of contiguous classes, depending on the students
b. To introduce the exponential functions you follow this path: 2N, 2Z, 2Qand the you go on with your fantasy. 22is a separating element between one point and another
c. She introduces the density of the line saying there are not consecutive numbers
d. To represent the correspondence between points and numbers she traces the segment and stop.
e. She uses the roots to enlarge Q, to say that other numbers exist, she doesn't go on
f. She uses the history of Mathematics to show that Mathematics is in neverending evolution
g.

3. Teachers' idiosyncratic idea

a. When she was attending the training courses, when the Professor talked about real numbers she did the same: she repeated the difficult lesson and then she forgot everything, going on drawing a continuous line
b. The Professor said these things are fundamental for a teacher, she got ashamed
c. Formalizations are due to a specific way to study at the University; you must tell the history of Mathematics but you don't need stuff like this at school

4. Teachers' reports of students' difficulties with real numbers

a. Only gifted students understand real numbers as separating elements of contiguous classes
b. Only two students understand the formalization; for the other f.i. if they write /3 + kthey don't know what k is

c. The students thinks at R as a continuous trace, also with the functions. They think at a small ball that moving leave a trace
d. The students don't pose the problem of connecting the point using a continuous line in the exponential; they don't care about the meaning of 22or of the existence of empty spaces
e. Only a few students understand, she reassures the others saying that everything works and it's OK
f. The students understand the limit points very intuitively
g. To the students who wants to know more or attend Mathematics courses at the University she answers the day after, after having restudied
h. The students' mistakes may be due to an excess of precision

5. Teachers' didactical goals
6. Teachers' orientations about teaching-learning sequences concerning real numbers

a. If you have gifted students you can introduce real numbers as separating elements of contiguous classes, otherwise you introduce the line, the diagonal at the unitary square and the Pythagoras' theorem
b. R is not necessary for the series; N is enough. for the other things R is necessary
c. Drawing a segment may represent the correspondence between numbers and points
d. The computer is different in respect of the pencil because it shows pixels
e. She knows that real numbers exist, like the golden ratio and so on, but when she teaches it's another thing
f. She never introduces function from N into N
g. To represent irrational numbers she avoids to use the decimal representation, uses symbols
h. If you go too in depth you risk that the students make mistakes
i. The graphic representation is the best because the crisis was geometrical and it's intuitive

Teacher 11: (Interdisciplinarity and real numbers)

1. Teachers' interviews' synthesis step by step

Step 1: Motion is described by means of functions more than continuous (
C2 at least), the dynamic representations requires continuous derivatives

Step 2: Real numbers may be necessary to complete functions continuously

Step 3: Qualitative variations of functions don't need real numbers and limits

Step 4: In Physics the teachers anticipate and use limit processes before the Calculus

Step 5: For the theorem of comparison between limits real numbers are necessary

Step 6: The Calculus is full of dynamical concepts with a static theory

Step 7: Sequences and functions defined in isolated points give a new idea of continuity

Step 8: The continuum is a limit of the discrete: here we miss something about real numbers

Step 9: The exponential functions is enlarged to conserve good properties also extending the domain, leaving aside the original definition and then it's completed continuously

Step 10: You need real numbers to extend continuously functions that are expressed analytically

2. Declared teachers' teaching sequences

a. Introduce R to complete exponential functions continuously

b. Many concepts we need in the Calculus are introduced before, also in other disciplines, in particular Physics
c. She ask the students to reason on variations, intuitive derivatives, before the Calculus
d. She introduces limit processes in Physics before introducing limits and the Calculus
e. She presents parametric functions, with a parameter that can make a function discontinuous in one point.
f. She says to the students that a function is obviously continuous in isolated points
g. She introduces the sequences. Would like to ask the students if a sequence is continuous or not
h. She introduces the asymptotic behavior
i. Task: She would now ask the students to draw f(n)=n2. Maybe they would trace a continuous graphic
j. Introduce the exponential function in the third class, before the limits and the Calculus: here she miss something about real numbers
k. She completes the functions continuously, from the discrete to the continuum, adding numbers in the functions' output corresponding to new numbers in the domain
l. To introduce the exponential function she recalls before the power's definition with natural numbers at the exponent (multiplying n times), proving their properties; then she says that to conserve the properties for other inputs the mathematicians had to renounce to the original definition; she misses the exigence to extend declaring only that her aim is to study the real exponential function. You define the exponential function to conserve its properties transforming the way you want the input value
m. She proved many times before that 2is irrational
n.

3. Teachers' idiosyncratic idea

a. The theorem of comparison for limits is the apotheosis of the necessity of real numbers
b. R is necessary to complete functions continuously
c. Sequences are natural; the students study discrete sets since the first year of primary school
d. She constructs the exponential function step by step, expanding the domain, and then completes it continuously
e. Functions associate input to outputs
f. The exponential function associate sums to products, this is its essential property, this becomes indeed its definition; thinking better this is true only until we use Q
g. We extend exponential function metaphorically continuously; from Q to R the previous scheme crashes
h. You have to invent irrational numbers to complete functions, it's good since we trace lines but we need to express them by means of symbols, we can't calculate them.
i. Asking to complete the functions continuously you take for granted they are continuous innately; what does it mean? It's a chaos, I must rethink to everything

4. Teachers' reports of students' difficulties with real numbers

a. The students are used to see trajectories more regular than we need for continuity
b. Classical Physics, in particular Mechanics, contributes to give to the students a too regular image of continuity
c. The students are ready to understand variations and derivatives even in the fourth year, before the Calculus and the limits; it's important to talk about qualitative variations and to compare variations at least as much as to talk about continuity
d. The too simplified models of function make the students wonder and ask why they have to consider limits in flat domains
e. A student asked: < If a function is continuous in a set, is this continuous also in its subsets?>. She always stress they have to complete continuously and maybe this is the reason why they wonder if the function was continuous before or in isolated points, that instead is obvious
f. The students' functions' outputs are never pathological

g. The students always ask what they have to do in the connection point between discontinuous functions

h. The students may react bad when they got aware there had been historical debates and that also the mathematicians may find something difficult to accept. They say: < If even they didn’t understand, how can I understand?>

i. The students in the primary school think naturally in discrete sets, but when we study the graphic of function they think that continuity is natural while restrictions of the domain are perceived as not natural

j. Maybe keeping on working on discretely the students would never fell it’s continuous

5. Teachers’ orientations about teaching-learning sequences concerning real numbers

a. Teachers are used to present too regular trajectories for continuity

b. We must take care of variations not only of punctual continuity

c. Before the Calculus we can prepare better the students; in particular we should find not regular functions and talk about them instead of regularize functions also when they are not regular, like we do modelling phenomena in Physics

d. It’s easy to present function with holes using parametric functions

e. If we present too regular functions, then we have continuously to motivated choices about the domains that are different from R

f. It’s important to work on sequences to deconstruct too simplified ideas of continuity

g. To introduce the complexity using the historical debates, like Berkeley’s critics to infinitesimals, is a double-edged sword

h. Introduce the asymptotic behavior very carefully; the teachers would need the help of an expert to plan this part.

i. Continuous functions, expressed analytically, must be completed continuously adding irrational numbers in the input

j. If we extend adding we never reach continuity

4. Conclusions and implications

Real numbers and real functions of real variables are included in all the national high school curricula of the world and also in all the STEM courses programmes at university. Also they are involved in all the STEM disciplines, given the use of continuous functions and intervals, in particular for modelization of the processes involving time and space. Also the dialectic discrete/continuous, typical of Calculus, is very important for Physics (Lesne, 2007). So didactical transposition of the set real numbers may impact significantly on students' learning in STEM disciplines.

Many papers addressed since the '70s the topic of students' difficulties in learning real numbers, with particular attention to irrational numbers and cardinality. Also the same researches showed that often pre-service teachers conserve doubts on real numbers and continuity after the Master and the pre-service courses.

Teachers answered a questionnaire designed to investigate formation (master degree, training courses attended) and knowledge (configurations of objects they associated to R set), and to explore the practices they associate to R (practices involving elements or subsets of R or objects traditionally used in the didactical transposition of real numbers like inequalities, Q etc) and the semiotic representations of subsets of rational and real numbers they consider best (semiotic representation of intervals). In Schoenfeld's model's (2011) words, the first questions about knowledge give information about resources, the following questions concerning knowledge are about goals, questions about practices and semiotic representations investigate orientations. 20 of these teachers were interviewed in focus groups (3 or 4 members) in which we guided a discussion on questionnaire answers in order to make them explicit their personal choices and the reasons of their choices. Also 11 teachers were interviewed in a further step during a training course concerning the teaching-learning of modern physics in which continuous models were explored.

From the data analysis emerges a variety of different resources of different teachers about real numbers. The lack of awareness of the necessity of a postulate of continuity is constant in the data. Also some of these teachers, not graduated in mathematics, showed a personal epistemological position on the nature - even on the existence - of real numbers and this may affect their goals and orientation. Also some of the declared orientations on real numbers and on the students' difficulties in learning processes, merely those about students' interpretation of the semiotic representations they use, have already been disconfirmed by previous research results. For example the graphic representation of continuity is considered the best one from the most the teachers; on the contrary visualization doesn't allow to perceive the properties both of dense and complete sets.

The teachers' choices observed in this research are often based on personal resources that are in general different from the institutional meaning of real numbers. Also they draw on beliefs that have already been disconfirmed by previous research results. Teachers' training should include these results as contents because teachers may base their decision on misleading beliefs.

References

Arrigo G., D'Amore B. (1999). 'Lo vedo ma non ci credo...'. Ostacoli epistemologici e didattici al processo di comprensione di un teorema di Georg Cantor che coinvolge l'infinito attuale. *L'insegnamento della matematica e delle scienze integrate.* 22B, 5,
465-494.

Arrigo G., D'Amore B. (2002). 'Lo vedo ma non ci credo...', seconda parte. Ancora su ostacoli epistemologici e didattici al processo di comprensione di alcuni teoremi di Georg Cantor. *La matematica e la sua didattica.* 1, 4-57.

Bagni G.T., (1999) Limite e visualizzazione: una ricerca sperimentale,
L'insegnamento della matematica e delle scienze integrate, 22B, 4 pp. 353-372

Bagni, G.T. (2000), Insiemi infiniti di numeri reali, *L'educazione matematica*, XXI,
VI, 2, 1, 22-46

Bozzi S., Mangione C. (1993). *Storia della Logica - Da Boole ai nostri giorni.* Milano: Garzanti

Bronner A. (2000). Analyse des contraintes et degrés de liberté de l'enseignant de mathématiques en formation initiale des professeurs. *Tréma* [En ligne], 18 | 2000, mis en ligne le 23 janvier 2013,

Cohen D., Fischbein E., Jemian (1995). The Concept of Irrational Numbers in High-School Students and Prospective Teachers. : Educational Studies in Mathematics, Vol. 29, No. 1 (Jul., 1995), pp. 29-44. Springer

D'Amore B. (2001). Concettualizzazione, registri di rappresentazioni semiotiche e noetica. *La matematica e la sua didattica.* 2, 150-173.

D'Amore B., Fandiño Pinilla M. I. (2012). Su alcune D in didattica della matematica: designazione, denotazione, denominazione, descrizione, definizione, dimostrazione. Riflessioni matematiche e didattiche che possono portare lontano. *Bollettino dei docenti di matematica*. [Bellinzona, Svizzera]. 64, 33-46. ISSN: 978-88-86486-85-9.

Fandino Pinilla M.I. (2007). Fractions: conceptual and didactic aspects. *Acta Didactica Universitatis Comenianae*. 7, 23-45.

Fiori C., Invernizzi S. (2009). *Numeri reali.* Bologna: Pitagora Editrice

Fischbein, E.; Jehiam, R. & Cohen, D. (1996), Il concetto di numero irrazionale
in studenti di scuola superiore ed in futuri insegnanti. *La matematica e la sua didattica, 3* (Educational studies in mathematics, 29, 1995, 29-44).

D'Amore, B., & Godino D.J. (2006). Punti di vista antropologico ed ontosemiotico in Didattica della Matematica. *La matematica e la sua didattica*. 1, 9-38.

D'Amore, B. (2001). Un contributo al dibattito su concetti e oggetti matematici: la posizione "ingenua" in una teoria "realista" vs il modello "antropologico" in una teoria "pragmatica". *La matematica e la sua didattica*. 1, 4-30.

Godino, J. D., & Batanero, C. (1998). Clarifying the meaning of mathematical objects as a priority area of research in Mathematics Education. In: A. Sierpinska & J. Kilpatrick (Eds.), *Mathematics education as a research domain: A search for identity*. 177-195. Dordrecht: Kluwer, A. P.

Godino, J. D., Batanero, C., & Font V. (2007). The onto-semiotic approach to research in mathematics education. *ZDM. The International Journal on Mathematics Education*. 39, 1-2, 127-135.

Godino, J. D., Ortiz, J. J., Roa, R., Wilhelmi M. R. (2011). Models for Statistical Pedagogical Knowledge. In: Batanero, C., *Teaching Statistics in School Mathematics-Challenges for Teaching and Teacher Education. New ICMI Study Series.* 14, 271-282

Margolinas C.(1988). Une étude sur les difficultés d'enseignement des nombres réels, Petit x n◦16, pp 51-68. IREM de Grenoble

Núñez, R., Edwards, L., Matos, J-F. (1999). Embodied Cognition as Grounding for Situatedness and Context in Mathematics Education. *Educational Studies in Mathematics*, 39 (1-3), 45-66. .

Lakoff, G. & Núñez, R. (2000). *Where mathematics comes from: How the embodied mind brings mathematics into being.* New York: Basic Books.

Santi, G. (2010). *Changes in meaning of mathematical objects due to semiotic transformations: a comparison between semiotic perspectives*. PhD dissertation. University of Palermo.

Sbaragli S. (2007). Le convinzioni di allievi e insegnanti sullinfinito matematico. In: Marazzani I. (2007). *La matematica e la sua didattica*. 87-91.

Sbaragli S., Santi G. (2011). Teacher's choices as the cause of misconceptions in the learning of the concept of angle. *International Journal for Studies in Mathematics Education*. Rivista online: http://periodicos.uniban.br/index.php/JIEEM/article/view/194/196**,** San Paolo, Brasile: Uniban. 4 2, 117-157.

Schwarzenberger R.L.E. and Tall D.O. 1978: 'Conflicts in the learning of real numbers andlimits', Mathematics Teaching, 82, 44-49

Sirotic, N. & Zazkis, R. (2007b). Irrational numbers: The gap between formal and intuitive knowledge. *Educational Studies in Mathematics, 65*(1), 49-76.

Sirotic N. , Zazkis R. (2007b). Irrational numbers on the number line – where are they?. *International Journal of Mathematical Education in Science and Technology, Vol. 38, No. 4, 15 June 2007, 477–488*

Tall D., Vinner S. (1981). Concept Image and Concept Definition in Mathematics with Particular Reference to Limits and Continuity. *Educational Studies in Mathematics*, Vol. 12, No. 2, pp. 151-169. Springer

Villani V. (2003). *Cominciamo da Zero. Domande, risposte e commenti per saperne di più sui perché della Matematica (Aritmetica e Algebra).* Bologna: Pitagora Editrice

Voskoglou, M. Gr. & Kosyvas, G. (2012), *Analyzing students difficulties in understanding real numbers*, Journal of Research on Mathematics Education, 1(3), 301-336.

Zazkis, R. (2005). Representing numbers: Prime and irrational. *International Journal of Mathematical Education in Science and Technology, 36* (2-3), 207-218.

Cinti F.(2013). Storia e sviluppo del concetto di limite : fra matematica, filosofia e didattica. Master degree dissertation. AMS Alma mater studiorum, Bologna [http://amslaurea.unibo.it/6325/1/cinti_fabiana_tesi.pdf]

Giusti, E. (1988). Il calcolo infinitesimale tra Leibniz e Newton, Rendiconti del Seminario matematico del Politecnico di Torino, 46, pp. 1-29.

E Borel, *Les nombre inaccessible* (Gauthier-Villiars, Paris, 1952).

http://www.mathedpage.org/infinity/galileo.html

http://progettomatematica.dm.unibo.it/insieminumerici/reali.htm

http://www.dm.uniba.it/~pisani/scienzenaturali/numerireali.pdf

http://www.batmath.it/corsi_uni/mat_uno/teoria_2.pdf

http://www.dm.unipi.it/~acquistp/ana1.pdf

http://patto.unite.it/pluginfile.php/1783/mod_resource/content/1/02_numeri_reali_funzioni.pdf

http://www.carlofelicemanara.it/public/file/File/Divulgazione/Collaborazioni%20a%20riviste/Periodico%20IMSI/La%20continuita%20in%20geometria.pdf

http://calvino.polito.it/Mat+5/incontri/note1.pdf

http://www.mat.uniroma3.it/scuola_orientamento/alumni/laureati/masi/masi.pdf

http://www.dmi.units.it/~inverniz/download/M/Rfield.pdf

http://www-groups.dcs.st-and.ac.uk/~history/HistTopics/

http://arxiv.org/pdf/math/0411418.pdf

http://fromdeathtolife.org/cphil/continuity.html

Appendices

Appendix A: Full reports of the analyzed interviews

Interview with Teacher 1

I: “Do you introduce real numbers?”

T: “What a hard question!”

I:”In which contexts do you introduce real numbers? How do you explain the students why you introduce real numbers and how do you represent them?”

T:”I would try … It’s hard … anyway quite soon. Not from the first years but very soon. I Think I would introduce them , very easily, showing that not all the numbers are rational. I would start from 2, they has been presented also in the grade 8. The existence of numbers that are not ratios of whole numbers…. thus a quite historical approach.”

I:”Which properties would you like them to know? Why do you introduce real numbers? Where are they necessary? Many teachers say that they didn’t succeed in introducing real numbers since they are too difficult..”

T:”What? Didn’t they succeeded? No, no, no...

I:”Maybe the Cauchy-convergence... ”

T:”For sure real numbers exists as numbers in the reality. I would start quite quickly since it’s sufficient to have used Pytagoras [theorem, nba] to be obliged to speak of real numbers. Also in Algebra [I would introduce real numbers] quite soon since the solutions of inequalities with x Q bother me...”

I:”Why?”

T:”Mmm .. they don’t… it’s very interesting, once introduced the real numbers, to solve inequalities in Q or in N, showing the difference, but in that case we ask to solve them in Q or in N because formally we didn’t introduce the real numbers, but there is a representation like this [trace a segment] of the solutions. So if you represent the solutions this way you can work with R for sure from the graphical point of view because the property that is congenial at an operational level is the correspondence between points of the line and numbers; for them [the students] is this one. They have naturally in their mind R. A student, when thinks at numbers, has in his/her mind a real number, i.e. he/she perceives the numbers in a continuous way, not in a discontinuous manner. He/she doesn’t perceive that there is a hole between a number and the consequent because you think at everything you see in a continuous manner. Without holes, without spaces between a thing and the other, it’s much more natural for them. They use rational numbers because when they were younger, in the very beginning, they had not confidence with numbers that were not rational because in the reality every real numbers is approximated with a rational one. But in my mind in their mind there is already a real number considered as the sequence of the numbers… once they overcome the question of the naturals, passing from N to Q, for them everything is become a bit .. joined … so quite soon it’s possible to highlight the difference between R and Q”.

I:”Why this [a segment] can’t represent Q?”

T:”Because in the middle there are other numbers… this way …2.. it’s all that.. if you draw them this way you take everything. There, in the middle, there are things that I don’t want. It’s obvious that I can’t draw it!”

I:”Are you saying that you can’t represent it in a graphic way?”

T:”Yes, and I would say this to them quite soon. Another thing that I don’t like very much is that it’s impossible to put in correspondence it [the segment] with Q. Maybe in the beginning I would say it in an informal way, but quite quickly… because I don’t know if the fact that it’s not in correspondence with Q has everything to do with the the fact that it’s continuous in this sense because they are two different things.”

I:"Are you saying that the correspondence is postulated but is not provable? That once constructed R we are not sure it's in correspondence with the line? The line has not indeed a definition. There is a different nature between the motion, the numbers and the algebraic structure. We have to postulate that a correspondence exists, you can't say that R complete the line "

T:"You could put other things in… But I would do this way: I would show that N and Q are in correspondence, that is not hard, while Q and R no.. to show that there is an enormous jump."

I:"Would you use in the last year [grade 13, nba] some construction of R to introduce the Calculus? Is there something that needs the properties of Q and R?"

T:"The mathematization of a primitive intuition. In my classroom this year, in the Calculus..."

I:"Didn't you recall R? Did you find difficulties?"

T:"No, because it's quite natural. Probably I did many examples… for instance.. that emerges a bit.. when we talked about limit points, the book didn't propose a definition …. instead I proposed a definition .. we talked for some minutes and since they had studied the sequences I tried to make them reflect about the fact that, while when you calculate the limit of a sequence only to the infinite, when you study a function on the real numbers you calculate limits also in finite points, because the only limit point for N is the infinite, while for R… But, in the reality, it's a thing that only a few students gather .. the straight line topology is so banal… to be honest no… but it's transformed by the books in a such banal thing that all [the teachers? the books?] always present all the theorems on the neighbourhoods, on the intervals.. because the interval at any rate works… so you don't gather intuitively… The Calculus is so complex for them that, being asked to learn [to use?] new tools, that a discourse these things may be very good for a few students … In fact I presented a lesson about the topology of the straight line, but I had not time enough. But I was very surprised by the fact that in the book there wasn't the definition of limit point.. there was nothing.. it asks to compute the limit, but didn't explain why you have to calculate it, everything is taken for granted. They had talked about contiguous classes, they see the continuum [represents simultaneously a limit point on the line]. They see all but that .. in the sense.. the problem of discontinuity. For them there is all. This is the reason why I say that I'm sure that for them these are real numbers [traces a segment] and that, however I take out stuff there other stuff remains here, close, this is sure. They didn't understand the sense of talking about limit points. I think that for them this is so banal that if you say < I take a limit point and however I choose some points around I find other points >. If they have a strong image of the continuum and not the same for the discontinuum, everything is a limit point, where is the problem?. In my mind for the 'mean student' while the image of the continuum is very strong, it's stranger the image of the discontinuum. For them the Dirichlet's function is something horrible obviously. Rightly they asked me why in this continuous function there are infinite points of not-derivability and then one asked <What are these things for in real life?> Very sincerely I answered .. [laughs]... then another student said: < Let's bet that there are no more functions like this>. It will be useful to introduce them very soon because once you have introduced them you can show the strong difference between this [draws a segment and writes R] and this [traces a segment with some points and writes Q]. This [R] is much more natural than this [Q], very natural"

I:"One is from before Christ and the other is from the end of '900"

T:"From ad geometrical point of view this is before Christ [Q] but there is a great difference. The Greeks thought that this [Q] was this [R]. The problem of 2, this is crucial.. Many students write ln2 = and wrote down an approximation. Why? Because this is what he read in the calculator and then he wrote a further approximation. There is a strong identification and it's not only a matter of symbols because when I correct it they look at me as a fussy person. It's identified, it's not only a symbol but it stems from.. it's like to say that 2 .. you do this procedure and find an irrational. How can you that all are like 2? The problem is that they don't have the concept of approximation."

I: "So they work at the numeric level with Q."

T:"Inevitably! With a Q with 2 or 3 digits..."

I:"Are you saying that there is a difference between the numerical and the geometrical procedures?"

T:"They study Physics from the first year [grade 9, nba], they only work with the calculator, because in Physics they use sinus, cosinus, logarithms ignoring their meaning, then they take the significant digits, rightly..."

I:”But without expliciting what they are doing..”

T:”Exactly! There should be upstream someone who explain quite soon which is the essential difference between the real numbers and their practical uses. My students in the end of the 5 years [9-13 grades, nba] don’t know it. I would expect them to know this at least: that they have understood the difference between a real number and one of its approximations. A lot of them.. it’s something subtle for them, but it shouldn’t be a subtle thing! This is what should emerge as the strongest thing, i.e. the difference between a real number and its approximations… don’t forget that there is an infinity of approximations..”

I:”To distinguish the continuum and the physical measures?”

T:”They don’t know the great difference between continuum and discrete, interpreted precisely as the number and the approximation.”

I:”Should we introduce it before?”

T:”Yes! In the same way we should do … the discrete set N, Z .. otherwise they don’t see the difference.”

I:”So you are suggesting to start from all the numbers and then to select different kinds of numbers? ”
T:”Yes.”

Interview with Teacher 2

Commenting the questionnaire:

T:”Properties? What can they be useful for? I need them for the measure (I use the properties) [Silence] In this point I have a doubt: it’s necessary to use the real numbers for the limit point? For me it’s necessary choose R, but at the same time between two rational numbers there is a rational number. R is necessary.”

Looking at the first video:

T:”This was nice. Introduction… Considering that there are many problems concerning reality that makes sense.,”

Looking at the second video:

T:”The real numbers doesn’t appear! No!”

Looking at the third video:

T:”Between 2 and 4… between -2 and 6… The only correct one.. but.. this not wrong… this one NEVER … I don’t like the hatching in the second. are they all acceptable? No, in this one the endpoints are wrong. We could say < It helps for reasoning>. This way it seems that all the students have to choose that one, but I like very much also this one : [2.4]. No one is acceptable in the last one. In all of these there are holes in the middle, aren’t there? Since there is Q.. ”

I:”What problems do you choose to introduce real numbers? What do you introduce exactly and when? Do you think that a formalization is possible to propose, at least in the last year [grade 13, nba]? Also please tell me every time which representation do you use, for instance, the line.”

T:”My first approach to real numbers is in the first lesson about numbers; I always see it as enlargement. I say <There is N, then there is Q, then there is R.> and I say < Which is the necessity?> and then I introduce also C at the same time. At least I give an idea and I always say … In the first year we work essentially with Q, we introduce a part of the real numbers in the second year. In the meanwhile I say why we passed from N to Z and I introduce them as necessary to close the operations in the sets, in this case in respect of subtraction, then in respect of division, square and I say that there are numbers that are not included anymore in the sets. I prove this in the first class, it depends on the classrooms. Usually I do this proof, first of all to introduce the absurde proof since in this case it’s easy to understand and also I need it to say to them: < You can’t take everything for granted but it’s necessary to prove in a Liceo scientifico.>. It depends on the classroom, if the students follow me, if I don’t waste too many time, sometimes I proposed it only to a student: < Look in the internet for this proof and then explain it to the classroom>. In the first year I usually do it also when I talk about Euclidean geometry, thus not exactly in the beginning when I talk about segments. I say: < It’s incredible how a finite thing has in itself something infinite> and I enjoy it very much because I say that in my mind the segment is the paradigm of life, in the sense that we can see the man’s life in the same way, as much as our desire … our desires are infinite. They can say other things. I show why I need segments and I say that there is a correspondence. In fact then I say : <It’s true here but also it’s true between 0 and 1. But I could also do it in Q.> I say: < If you take 0 and 1 there is always a point between 0 and 1. If I talked about an half part I stop at Q, but when they understand that there are other things in the middle, that there is always a point in the middle, that is always the half part and they can go on infinitely, this gives to me the idea of limit point in a certain sense. This is the reason why I had that doubt before, because this is an open discourse. I could also consider in Q a limit point, that is 0, because there is always something that stays… but at the same time there are also points that are not… there is also the square of ½, for instance. If I think at the square of ½ there is something..”

I:”So more in the field of the theory of numbers rather than in Geometry”

T:”Yes because I .. they.. they understand that the segment that doesn’t.. because I say: < We don’t have a so subtle pencil.. This thing already shocks them. When I ask? Finite or infinite? they all think at a finite thing. The line no, but the segment.. because they see it as limited… and so this is a concept that open their mind very much. They propose many questions on this topic.. and I like it very much because it’s a first approach to the infinite.”

I:”Some teachers say that it’s not worthwhile to talk about real numbers in the secondary school..”

T:”It’s difficult to think to work without.. this things are anyway said a bit imprecisely, a part from the proof, but in the second year.. If you want you can do something also in the fifth, since with the limits… but a few times. In the second year I recall it < Do you remember?>, because if I have x2 =4there are no big problems, but if we arrive at x2 =3, What is there? Furthermore I recall this discourse in the first year with the Pythagoras’ theorem”.

I:”Once again with a geometrical approach?”

T:”Yes”

I:”So you carry out two parallel discourses.”

T:”Yes, always. Also with the Pythagoras’ theorem, I work very much also with an historical approach. and here I say precisely that it’s necessary, formally, even if Pythagoras didn’t name it square of 3 or square of 2. How we call it is not so interesting for us… But here it’s really necessary. ”

I:”And the transcendental numbers?”

T:”Okay… In the first and second here no, also , if I have a second year class in which I didn’t work enough on the circle I’ve not even introduced . In other classes I turned out to introduce .. we did a very nice work about the approximation of like is presented in the Museo del Calcolo in Pennabilli. I like so much the question of the introduction of because it gives precisely the idea of a little thing that is so close to x0and allows us to see what the function does in that point. To quantify this , >0, that is real, gives exactly the idea of the fact that we get closer and closer because this tends to 0. As representation…..”

I:”With the sequences.. how can we represent Q and construct R? It may seem forced but when I draw the segment if there are only point s of Q or R I can’t see the difference..”

T:”They’re so small that I don’t see them. So I could use this representation [traces a segment] and say that they are of Q, like you did here but with x Q. We don’t have a way to represent Q”

I:” Here, if one would suggest the extension from Q to R starting from this point, how could he/she do?”

T:”I see it well, graphically it’s understandable that there are many real numbers here inside the segment so this is not exactly Q this one, there’s something more. But you can’t take it out doing some white small empty spaces so sincerely I should reflect. I don’t know how to see another different representation of Q, do you understand?”

I:”Every time we try to represent Q we fall down into the representation of real numbers… but if we want to enlarge Q to R, how can we do?”

I:”In fact.. at this level.. going further the square… this opens the door! The question of the correspondence is very important, it has the same kind of infinite.. it’s a bit difficult,, it depends on the classrooms”.

Interview with Teacher 3

T:" So ... How was it? Which mark did I get? [he laughs]"

I:"10! You have been one of the most metacognitive! You tried to look the questions from an high perspective and to think."

T:"The seventh question.. Which was it?"

I:" it was the one in which you were asked to say when the properties of real numbers are necessary"

T:"Yes, Yes! That one. I already wrote in the answers.. It's clear that they're important but in the different things I did about the topic Mathematics and music, when you work with the notes sa a discrete set and you work with rational Numbers, fractions.. You may show to the students... Wait.. I show you a slide, I showed them a keyboard, the order of the notes is clearly that of the keys, they have an arithmetic progressione while the frequencies grow by multiplications. If you say that every octave the frequency is doubled, you may show that at every octave, you have a power in base 2 while the logarithm is the exponent you raise it to. So you don't need.. I'm speaking only of an introduction, as a Concept, it would not be possible ... The logarithmic function is another thing, but as an introduction, you were talking about introduction, weren't you? Did I write these things? "

I:"Yes, you did. Also you added the exponential function, while you said that to introduce the Integral and differential Calculus, as long as to introduce limits and intervals, you need R. I agree with you, this was a sort of provoking question since it's not necessary to introduce the properties of R to introduce exponential and logarithmic functions. Of course this is my opinion. Every teacher has her own logic and it's important that every teacher follows her fil-rouge but I only observe that to talk about the exponential function in the real domain is such a problem... A deep knowledge of real Numbers is needed in this second case. When you raise to the square root of 2 you don't have concrete examples, but however you choose a method, contiguous classes, cuts, limits, ... You say <I know I can go on with the exponents approximating it more and more accurately, the exponent tends to the square root of 2 and so... While the concept of exponential doesn't need the properties, the exponential function on R is not exactly trivial. On the contrary sometimes you can find people Who believe that R has to come before every other content."

T:"no no no! On the contrary in my mind precisely exponentials and logarithms are useful for ... You do it usually in the middle of the fourth year [grade 12, nba] while I introduce them in the end of the year because immediately after I start introducing the limits and these are prototypical real functions with real domain. Furthermore what you are saying is also more true because I highlight the difference between the power function and the exponential function, where the X is at he exponent and then I present X raised at X... With this we really go in the hyperuranium! X raised at X drive them crazy!

I:"I can imagine...."

T:"I ask them which is the domain ... These are for sure some challenges"

I:"very good.. Fundamentally the aim of this conversation is to fix some things that seemed to me to emerge in your questionnaire and to ask you something about the way you introduce the real Numbers, what strategies do you use, in what moments you decide to introduce them, what interdisciplinary examples you propose, since you were talking about history and philosophical issues. Let's start from the fundamental properties of real Numbers, you wrote: operations, order and completeness axioms"

T:"Yes"

I:"of course there was no room for deeper questions, but I ask you now If, over all about the completeness, in the moment you construct real Numbers you ... Since you said you would also take in account the historical happenings, how you would present this completeness? There could be many approaches, you for instance talked about axioms that leads more towards the axiomatic approach than to the construction of real

Numbers. How do you try to explain to the students what is this completeness, thinking at what they need."
T:"Yes , it's clear that to formalize in a very hard way .. For the students, also of the fifth year [grade 13, nba] Who won't attend a scientific course at the University I don't think that a too formal didactical approach is not 100% effective. There is the risk that I cause them some brains imbalances! I begin in the third year [grade 11] when they study the Zeno's paradox I start with a first idea. This becomes in the fourth year, when you approach the definition of limit, when you talk about the topology of the line you to talk about neighbourhoods and the basic concept that is interesting to me is that this representation of the real Numbers by means of a line is not a trivial thing. Once I asked a student <Is there a bijection or are they the same thing? How can you absurde that there is a bijection?>. To say that infinite point generate a set with dimension 1, that something with dimension 0 could generate something with dimension 1 do shock them. We don't enter the labyrinths of the topology or metrical spaces. When I was at the University I was shocked by the Geometry's Assistant professor who said: < On this line there are no holes, even If I'm not so convinced.. > Ahah it was wonderful! Maybe I won't say this... But I'll try to make them understand that the fact to find always, every time I take a point on the line, a real number this seems to me already a conquest. Also I like very much to talk about neighbourhoods with holes, without their centers, that is very frequent in my tests about the limits, but also it would be enough to talk about open limited intervals. To make them understand that you can approach one or the other endpoints but without reaching it it’s an important thing. the line without a point, for instance.. it’s important that they understand that is something that we can’t visualize, we have to reconstruct it as a mental idea. Graphically, using Geogebra for instance, the line without a point is not different from the line with that point.”
I:”Yes, if you take out a point, graphically, it’s like to take out an infinity of points..”
T:”Exactly. It’s a conceptual conquest, the history of mathematics… in the history this conquest took 2000, 2500 years.”
I:”Well.. and.. working without a ‘mean’ class what do you do in practice usually, how do you… which strategy do you use? You’re one of the small part of the teachers I interviewed that is aware of the limitations and the problems due to the use of the graphic representation. Also the books usually the line without a point is represented with a line with an empty circle.”
T:”well, it’s clear that the graphic representation is a synthetic representation that is important, as I wrote in the questionnaire, but it can’t be the only representation! You need the analytical one, the inequalities, the limits, that also are linked to inequalities.. An aspect that could be associated to Zeno are the contiguous classes, for instance, but I propose it only a little and reluctantly. I propose it only.. because I’m obliged! Only to take in account the history of mathematics… I try to talk about it once and stop! I also avoid to work with the sequences and I immediately shift to the limits of functions because to talk about function from N to R, as sequences are, try to create short circuits between the continuous and the discrete, and coming back and so on, I noticed that often provokes some misconceptions… I like to say to them an irrational number, or a transcendental one, has infinite digits without repetitions and this helps me when… we organize for instance the pi-day and I always try to recall its history .. also in this occasion to say that the number has infinite digits, or that the we can’t rectify the circle, or that we can’t square the circle.. these are two approaches to the same problem ...”
I:”And.. about the misconceptions you were talking about before.. what are the most interesting? It seemed to me that you were saying - and I found it very interesting - if I try to propose a formal approach, I have to present it but in the end it seems that the students understand less than before. What happened in your classrooms when you tried to do something like that?”
T:”Once I tried… Wait… before I tell you this.. when I was in high school I remember that my teacher, trying to present to me
2, presented to me the sequence with the contiguous classes, 1 ; 1,4; 1,41 and so on, and the fact that I can reduce as much as I want this difference…. I understood it, but I needed two years of University to

understand the deepest meaning! I like, maybe because I graduated with a thesis in Numerical Analysis, the numerical-geometrical approach for instance starting from the exhaustion, that is the Archimedes' method with approximations, for instance with pi, using inscribed and circumscribed polygons. They know what a perimeter and an area are, that is a number included by… that this are contiguous classes if we want, they are classes of rational numbers and you can express a perimeter.. then strange polygons, in which also irrational numbers appears, I don't talk about contiguous classes but they understand that these are two sequences that approximate the number, one in excess and one in deficiency, and that the difference can be reduced as much as they want. Also in this case this approach is something that yearns more for a basic concepts, an impression, for a platonic idea, their idea of number with infinite digits… I don't know.. the Cauchy's sequences, you can take the neighbourhoods of a point, as small as you want, the points are as close as you want... you don't call them Cauchy's sequences but … they see it, they see it by mean of inequalities that we solve numerically or that we solve algebraically.."

I."You are saying that there are properties of real numbers that could cause difficulties if we introduce them formally but all these properties are not so necessary in the high school, so we introduce only some things that are functional to the some goals, aren't you?"

T:"Exactly"

I: "And you said that one of the most important thing is that its graphical version can contain all the numbers, so discrete and continuous functions can be contained in the same representation. You said: why to study functions with discrete domains? The continuum is a kind of support for the discrete, it's a wonderful world in which you can do what you want. Let's think about the discrete problem presented this year in the final exam."

T:"This was a double-edge sword, because the students are not used … maybe I decide not to present the sequences, I work with it in the project Mathematics and reality, and here I can spend more time on this topic, but the final exam's problem was misleading. I draw on another question you posed in the questionnaire. Another approach I use is the field, or better the closure in respect of the operations. This helps me first of all as an historical approach, a construction of the numbers from the naturals, rationals and then I start talking about Pythagoras and I never stop "

I:"Because it concerns the music!"

T:"What I presented in my papers and conferences about the music is precisely starting from the musical scale from the point of view of the enlargement of numerical sets. You begin talking about naturals, then Pythagoras switch to rationals, and 2 that comes out … and then 122that you can't represent geometrically, and this is the Bach's semitone, the waves' equations, the logarithm I showed before, until the Fourier's series and the complex numbers. So.."

I:"Do you present this topic to teachers or to students or the both of them?"

T:"There is also a video… here you can find a way to reach R. I also always introduce, even if it's not in the national curricula, the Algebra's fundamental theorem. It's the minimum to do it! Joint to the Arithmetics' fundamental theorem.. is one of the conquests of humanity! They can't finish the high school without these results is unconceivable! To know that you can decompose every polynomial in C … "

I:"Yes. In particular if you say that you want to create a field in which you can define the square root of every number..."

T:"Of course! One of the most important problem in the history was to solve polynomial equations, Tartaglia, Gauss… If they studying polynomial functions don't know that they can find 1, 3, 5 zeros and that if the grade is odd one is surely real … also this means to have understood the real numbers."

I:"So you are saying that there are also all the numerical and algebraic aspects of the real numbers, not only the graphic one."

T:"Yes, exactly! the axioms.. ok.. you say the field is ordered, archimedean and so on… so what? Of course! You are happy.. an example of field, this [R], an example of vector space, this [R], or Rn, but it's the rest of mathematics that then was adapted to the properties of R and not vice versa. Since it has some properties more and more of its properties has then been found."

I:"A question, that you suggested. In my thesis I got aware that many conceptions of continuum exist and that real numbers are only one of the possible aspects of continuity, a sort of attempt to make static on one hand the processes' continuity, also in the physical sense (to fill all the gaps, to pass through a segment, ...) but also the other process that is that of divisibility. Divide here, divide there, you always find something. Another aspect is that of equations. There is thus a plurality of meanings, as you were saying, not only the differential, or the topological aspect, but also to understand that this set also allow to solve numerical problems. So I decided to study more the practices linked to continuity in the history and what emerged is that to be honest a rift last in time between the numerical and the geometrical approach, that the Greeks had showed since the very beginning and that had been a bit obscured by Descartes and Viete when put together numbers as solutions of equations and the geometrical construction, making a sort of outrage to the Greek's deny to talk about numbers and magnitudes as they were the same things, put a symbol in the place of a number, that can represent a magnitude or a number - as nowadays we do continuously, as it was normal - but to do this requires to put together two worlds that are a bit different. At school this is done in a quite tacit manner and it's not usual to reflect about the meaning of putting numbers on a line and to say that points and numbers are the same thing. What I want to ask you. You turn out in showing the most of this different aspects. Do you find some of this aspects more difficult for the students to learn? which strategies could you suggest to other teachers?"

T:"First of all when I indicate a letter, or a number on the axes I don't say: < This is the number three, this the number four, ...> and so on but I say this is always a representation of the number. Also 'Pythagorically' speaking it's an abstract idea, something that comes from elsewhere, I realize a representation first of all because this is one… also I use Geogebra, think at the name,
from a Cartesian point of view, as you were saying, it's true they are joint but there also two separated representations, there are two windows, the algebraic one and the graphic one. I try to use always the both of them."

I:"If you use the both of them… it's not so frequent.."

T:"for instance with my students in the third year, try to enlarge the point. What is this a point or a circle? What is this? Euclid says that the point has no dimension… what are we doing? but we are not representing the point, we're representing one of its representations… the point itself, that is a geometrical entity… you can't draw it, you only can represent it! It's like the graph of the line, or the graph of the curve .. to visualize better it on Geogebra you can increase the thickness of the line.. but the line have dimension 1, you can't make it larger!"

I:"So you are saying that your students feel the need of understanding better, if you create the occasion, if you stimulate them and then you leave them free to ask.."

T."Yes.. but I want that a thing is clear. I don't think this is a trivial question."

I:"Sometimes it seems that this is lost time"

T:"No, no.. furthermore in my school I'm one of the few teachers that use it often. Geogebra to do, for instance.. I'm working on circumferences in the third class. I proposed this very simple example. The systems with three unknowns.. the linear systems you studied are also useful for this. Now open geogebra, use the tool! Draw a point. what happens? Nothing. Draw two points. What happens? Geogebra draws a line. Draw another point. What happens? Geogebra draws a circle. This takes 10 seconds, but it's very significant..Also I prepare some Geogebra files to do many other things. For instance I show to them the Euler's line, one of the most important post-euclidean geometry results. I think we should do it. It's wonderful… it takes a few time. Both of the approaches are important… What is important is to give the right weight to the kind of approach you are you are using. You have always to stress that you are always representing! Also the triangle was not a triangle, was a representation.. the triangle isn't a draw, it's abstract.

I link for instance Donald duck - Mathemagic land. It's wonderful!"
I:"I agree with you. I propose you last question concerning the questionnaire. It concerns the core question of the last part. It concerns Q, its representations and the fact to enlarge Q to R. The idea .. it seems that the most frequent difficulties concern the irrational numbers, infinite digits, radicals.. but I conjecture that the students have more troubles working with Q in some representations rather then working with R. A colleague for instance said that, if you reason from a topological point of view, R is a very complex structure but if you think at high school mathematics R is intuitive and Q is not intuitive. What do you think about it? This remind me what you were saying concerning the discrete sets. Q is not discrete but it's more complicate to think from some points of view. The problem that it seems to me that is emerging, if I want to construct R starting from Q, following a construction, intuitive or one of the formal classical one, I need before to know what is Q, what a Q interval is. Do you turn out to work with the students on Q and to construct R starting from q or is the process much softer than a 'Hankel enlargement'?"
T:"I do this: I have one or two lesson on the numerical chain from an historical point view. Indeed I'm reflecting on what you are saying… It's not trivial to represent Q but you could represent Q in the extensive form, as a set, you wrote is as a fraction between whole numbers, or a whole and a natural number"
I:"In the numerical register, in the world you were talking about before."
T:"I do this, also in my conferences about music and mathematics. When Pythagoras start to use fractions, rational numbers.. for instance Odifreddi talks about ratio, reason, connected with the word logos, that is one of the ancient principles, while I prefer to stress the analogy between ratio and relation. When you put two numbers one onto the other, you are creating a ratio but also a relation. They understand that there is a reason in the harmony of the spheres, like Pythagoras said, and also there is an attempt before Tolomeus and then reconsidered in the '500 to simplify some fractions that came out from the model, like 15/8. This implied the simplification of the musical scale. So the cognitive process that is the reduction of fractions to minimal terms becomes not only a technical procedure but also an... estetical necessity.. it's not a trivial thing. I think I have problems with them when they say… is it possible that these numbers came out in the end of the exercise? But this don't shock them as much as the fact that we can't represent the cubic root of 2. about the representation of Q… I don't know… more with Cantor, you can come up to the discourse of density.. the representation of Q is a fact very connected to the geometric representation… two edges that are commensurable"
I:"Maybe you are saying that the representation of Q is linked to the geometrical constructions but it's anyway something abstract, aren't you? The problems may emerge when … also with the teachers.. you say Q is dense in R. Between two rational numbers there is always a real number.. but the way we say this can be also misleading. For instance a teacher expressed the idea in a manner that suggested that there is a rational number, then there is an irrational numbers, then another rational and so on. In the moment in which you represent a number with a point, that indeed is a segment's endpoint, you don't represent the two magnitudes you are comparing but you somehow obscure the unity and what you really do is to identify a number with a point. to visualize the density of this set of point is not exactly trivial..."
T: "In my mind the error is to try to represent Q on the line, in the sense that the more effective geometrical representation of Q is the one I told you before. I don't want to represent a fraction as a number, if I'm not looking for solutions of equations of course, 22/7 on the line but I represent it, if I need...or the approach, for instance, but it's more used in Physics... you can transform in the decimal representation and this way clearly the form is that of a real number, etc etc "
I:"Maybe the decimal representation… but when we have to enlarge Q or we stay in the numerical register, but you often usually return to the graphical example. You say that the best way to represent Q is numerical or with an abstract idea of ratio"
T:"The commensurability, Euclid, Pythagoras. I always talk about the problem of the cube's duplication… wait. all of them know the history of the square root of 2, Ippaso… but the cubic square of 2 is not π, is not ,

it’s there, in the difficulty chain, before you present the square root of 2, 3, 4, 5, you present it as a sequence, you always add a unit more.. for the square roots of whole numbers… how can I represent it? I take a compass, I draw the diagonal of a square, then I report the segment on the line, while in the cube’s duplication, the cubic root of 2 is not representable using the ruler and the compass.. the fact to say that there are numbers that they can compute using the calculator. I ask which is the value of the cubic root of 2? They immediately answer… this is 1, … and I say < No! This is ABOUT 1,...>. It’s important that they know that there are many numbers that exists and that we can’t represent geometrically, because Q, all the rational numbers can be represented geometrically, as ratios between edges, two whole numbers, you put them in relation..this is something that you can see somehow. So I won’t… it’s true that at the University you are asked to know what density is … to explain, to prove, but I prefer..”

Interview with Teacher 4

Silvano:

I:"In your opinion in which contexts is necessary to use real numbers, when it's necessary to define them at school?"

T:"It's not important to define them in a rigorous way. To do what we have to do we can use the preconception of the correspondence between real numbers and line. We can take it for granted and act consequently. Beyond the rational numbers, excluding π, a teacher can't be subtle and realize a rigorous construction. One is content with the use of an approximation. Euler didn't have a rigorous definition but he made so many good things. I would rely very much on geometry to go beyond the rational numbers. In Algebra it's not necessary but with the radicals, the operations.. but there also functions.. In mathematics we need to use numbers that aren't rational. In the first year [grade 8, nba] .. radicals. If they are square we can represent them very well geometrically. If the radicals are not squares you can insert them in a consistent order with that of rationals and you don't need the structure of real numbers. Real function with real domain.. the graphic.. the Calculus is graphic-centered, indeed the way real numbers are imagined is the real line, with an abuse of language, that in my mind makes a few damages at this level but may have many advantages. We need a representation of the total order as linear order, it's not interiorized as the axiom of the order. In R we need the order and then the operations with R are compatible with the operation in Q. We, as human being, can solve operations only with rational numbers, rather in Z so we play this play to approximate R with Q continuously."

I:"Why don't we use only Q?"

T:"Because $\sqrt{2}$ is not in Q! "

I:"Why don't use its approximation in Q?"

T:"Because.. in some cases... for instance we don't know $\sqrt{2}$, but we know one of its properties. When we use this number or when we make it appear in some expressions or something like that we use the fact that it has a property, that is that raised to the power 2 it's w, but this is important because the most of the real numbers doesn't have a property like this one and we don't use them at school."

I: "And in the Calculus? Do we use them?"

T: "Yes, in the Calculus yes!"

I: "When?"

T: "If one wants to work seriously with the Calculus he has to introduce real numbers in a rigorous way."

I: "For instance, in the last year, when you have to introduce real functions in the real domain? What about the graphic of the function? And what about limit points, limits, integrals? Do we need R? Which properties?"

T: "I introduce the limit of a real function in the real domain in every point x0 of R. I'll yìthink now to how I would go on. The properties... What is useful to know to work with Calculus? 1. Q is dense in R. 2 The order

in Q and R are compatible". 3. Complete.
I: "Complete?"
T: "Yes.. many things are used without expliciting. Limits exists and are unique, the upper extreme exists. Is there all the construction of real numbers - whatever it is - in the national curriculum? Before or after the Calculus? [reads again the question concerning the national curriculum] How can we talk about trascendental numbers without... Mmm.. a preliminary understanding of the problem of formalization. But.. who wrote this? Elements of the approximated computation. The mathematical infinity? The potential infinite, the actual infinite: a number as big as we want or an infinite number.... The definition of limit avoids the actual infinite, the reason why it's acceptable for a modern eye is exactly this: it doesn't imply that there is the infinite but that we can go beyond a limit that a person establish (potential infinite)"
I: "What does R is necessary for in this discourse?"
T: "To be honest a person thinks algebrically in terms of rational numbers but he sees them in terms of real numbers i.e. of the continuity of the line, images them onto the line. I have always to add that a limit point ... one sees this approaching to the limit point without reaching it but in a continuous variation. To know what does this continuity mean or its rigorous definition is left..."
I: "What do you mean with continuity? Passing through, approaching without leaving back holes? Graphically how would you do it? Think about the limit. What is there of R in this representation?"
T: "Mmm.. a function that tends to x0, a limit l that is not a value of the function. You choose an interval that goes from l + epsilon to l - epsilon, [he draws the interval and the axes] then the idea is to have for every choice of this epsilon a delta here in the bottom, x0 - delta, x0 + delta in an interval centered in x0 [he draws a segment] and all the image of the f lies inside the chosen interval. What am I using of R and why this representation? I'm sure that it's real with R domain ... but also tracing Q lead to fill it..."
I:"Let's imagine we are working with R"
T: "Yes because in the definition of limit it's useful to have a counterexample, the function jumps here, so here you understand that for every interval of this value, doesn't matter how small it is around x0, there is a part that goes beyond. Every time I speak about intervals I mean 'without holes' i.e. including rational numbers, i.e. the points that correspond to rational and irrational numbers. It would be important to consider a R interval."
I:"How can we represent the difference?"
T: "There are points that we can't refer to rational numbers, i.e. we say that to make it correspond to Q, in the correspondence between R and the line, you could begin to see which points correspond to Z, choosing 0 and 1, using the compass and the ruler going on putting fractions, divisions of segments. Q is consistent with infinite points onto the line ordered like the rational numbers. But then there is Ippaso, who says that not all the real numbers, i.e. all the lenghts, correspond to numbers that we can construct this way. So, for instance, the lenght of the diagonal with ruler and compass is not only one of these infinite points far a finite number of step from the unitary segment. It's possible to construct, not rational but from the rationals. All the numbers like this... "

Focus group with teachers 5-6-7

I:"I'm working in my PhD thesis on didactical transposition of real numbers in the high school. I'm interested in this topic since the students don't understand usually real numbers even in the end of the University sometimes."

T2:"Absolutely. "

I:"Does it happen to you too?"

T2: "It's a big problem. Yes."

I:"Also students that attend Mathematics' courses sometimes, in front of some questions, have many doubts. What are you greater difficulties in your classrooms?"

T2:"I work in a Professional Institute, it's a reality complex itself. I start introducing all the numeric sets, in the end I don't present R with the contiguous classes, as one could do in the University or in a Scientific high school, I work in a very simple manner. I define all the numbers, natural numbers and so on, then I say, guys, there are also the irrational ones, I show that not all the numbers are possible to express in form of a fraction, then I introduce the irrational numbers. "

I:"With proof that root square of 2….?"

T2:"Yes, yes. The proof is very difficult for them, I talk with them of the famous Pi, I show that not all the numbers are rational using the rules, and then I say: < Guys, there are the numbers.. the Matrioska. There is this big Matrioska, that are the irrational numbers, then there are the rational numbers; joining this two we obtain R, that are all the numbers that exist in the Nature> , in a very simple manner, depending on the the audience. I don't use the contiguous classes and so on. "

T3: "I work in a Scientific high school."

I: "So the situation is a bit different, isn't it? Do you turn out to introduce real numbers?"

T3: "Yes. I work in the first two years and they are already included, because there are already the roots, things like that. In general I feel the need to reconsider them, when numbers like e, Pi, come out. So usually what I do is to discuss about the sets. I start with the natural numbers set, then in the moment in which we have operations that we can't execute in this set the exigence to introduce a new set emerge. "

T2:"So the Matrioska dolls, the dolls.. "

T3: "To introduce the concept of real numbers I use the classical example of the diagonal of the square. I tell us a bit of history, Pythagoras, Pythagoras' history "

T2: "Exactly."

T3: "Then the name of the person who… I don't remember the name … who had revealed the history of the segment. In the end a discourse emerge concerning operations that we can't execute in this set. So the exigence to construct a new set emerges. Then I talk a bit about complex numbers because it arises that this root of -5.. at a certain point you say that there are complex numbers."

T2: "So you do it like I do! You don't present the contiguous classes, all that discourse… so it's not only for the teachers who teach in a Professional school. It's the same for all."

T3: "Yes."

T2:"I present them as rational numbers joint to irrational numbers and I don't prove it with root square of 2, I use Pi that is a number that they know."

T3: "No, from the exigence of create new sets."

T2: "I say: < And now what do we do? Let's enlarge> Yes, yes, I do the same. I'm comforted…"

T3: "I need go further, and to introduce the complex numbers."

T2: "Yes, the last exigence is to introduce the square root of a negative number.. and so there is this particle , that raised at 2 gives -1. Ok, Ok."

I: "What are the students' greatest difficulties with real numbers? Why don't you introduce them as contiguous classes, as complete set, and so on?"

T3: "Many times the concept… the concept.. the greatest difficulties they have is with the concept of infinitesimal quantity. To understand that if I take 1 and 1,0001 I find between these two numbers an infinity of numbers."

T2: "Between 0 and 1 there are 0,1, 0,2, … ten numbers according to my students!"

T3: "So when … to understand that … if I take two numbers, between those numbers I can find an infinity of numbers .."

T2: "No they don't understand this!"

T3: "The concept that a part of an infinite can be again infinite. For instance the concept that has already been introduced in the first two years that a segment contains an infinity of points. When I say that a segment contains an infinity of points it's OK, but when I take 2 points and I draw on a blackboard two very close points they don't understand that.. they have difficulties in understanding that between two points that are so close there are infinite elements."

I: "You work with a graphical representation, is it right?"

T3: "Also, you give also a theoretical justifications, but what impress them more is the graphic representation."

I: "And in the graphic one they have more difficulties."

T3: "Yes. So many times to understand such a thing there are the computers, if I draw two point on Geogebra, if I zoom the page they see that these two points are close, if I reduce or enlarge, between them there is already room. You take the points that are in the middle and I reduce more and more there is always a room so in theory between these two points there is always something. Then Achilles and the turtle…"

I: "Is this accepted by the students?"

T3: "ehmm… yes.. but in the moment I turn out to do it graphically.."

I: "When they study the Calculus, do they recall these concepts?"

T3: "Yes, indeed I talk merely for the Calculus, because the concept of limit, the concept of infinitesimal, all these things.. for instance I say that in the limit 0 is not 0 really."

T4: "We talked about it already in the previous focus group. I was in Bologna. Here we talked merely about the intervals, the inequalities. It was about solving inequalities in R and in Q."

I: "Yes, let's talk about the students' answers I presented in the questionnaire. We asked to represent the solution of this inequality in Q. Let's discuss about it. We presented it because usually this is the way we represent the sets we need to introduce R as Dedekind' rational cuts. We take a subset of Q that approximated. The problem is that the students interpreted it as an hybrid sets including the root square of 5."

T3: "2 and 5 could be considered correct specifying that x belongs to Q; 1 in my mind is not correct because a segment, with the all points represented is not Q, not all the points are in Q. The other could all be correct, acceptable, even that with extremes included if I explicit that the solutions belong to rational numbers."

T2: "Wait! With the extremes included?"

T3: "Yes, if you say x belongs to Q"

T2: "Ok, ok… you specify it, all right. But I don't think they have written x belongs to Q. "

T3: "In the answers it's not explicated. From the second to the last all could be correct, but only if you explicit that the solution must belong to Q."

T2:"Yes, yes."

I: "and what about the graphic representation? What should have they done? How should they represent it graphically?"

T3: "I don't know how to say it… only I could say that also in this case they have to write that the points are of Q."

I: "We should also include you in the discourse [referring to T5]."

T5: "Eh, yes."

I: "This is a crucial point in the teaching experiment. The students in some occasion don't recognize the necessity of constructing a set of real numbers while they are calling real numbers another set they have been using every day for several years. We had the problem to enlarge a set they have already enlarged. How can we explain that Q is different from R."

T5: "My students would have chosen the first, because they don't read x belong to Q."

T2: "This is exactly what I think of my students. For sure they would not have mentioned it at all. They solve x equal to + or – square root of 5. They wouldn't have read x in Q, or even if they read it, they don't take care of it."

I: “Don’t they take it in account?”

T2: “Absolutely.”

I: “In your mind why?”

T5: “Because at school what do we usually do? ”

T3: “I get very angry because they solve it writing x < + or – square root of 5.”

T2: “No!! But what can we do to make them understand? I say to them the famous concordance or discordance of signs.”

T3: “The associated equation.”

T2: “The associated equation! I have to repeat it every time! How can we solve this one?”

T5: “Because at school they pass immediately… the quadratic inequalities have many difficulties… for instance $x^2 > 5$, they solve the equations before.. $x^2 > 4$, $x > +-2$. And this a frequent error. Doing the inequalities after the equations, they solve them in the same way combining quadratic equations and linear inequalities. ”

T3: “I ask them: < Is this lower than + 5 or lower than -5. What is this number? ”

T2: “It’s impressive that we have the same difficulties. At different level we have the same difficulties. ”

T3: “The problem with this kind of tasks is that they write x < +- square root of 5.”

T5: “This is what I was saying. If you ask $x^2 > 4$, they write $x > +-2$. ”

T2: “Yes. They use the < as it is a =.”

T5: “Maybe because of the linear inequalities, because you solve them..”

T2: “I’ve so many difficulties.”

T5: “After some months some exercises appears …”

T3: “Another important thing in my mind is the same problem that some adults are showing, i.e. they have an elementary knowledge. After the primary school the students work with airtight containers, they study for the mark, not to learn. Many times… for instance.. in grade 7, they work 4 months with the proportions, they become good in using proportions, and the next year they don’t remember, they don’t care about them anymore!”

T2: “It’s true.”

T3: “In the first year of high school if you talk about proportions they ask you: <What is a proportion?>. They have been working for four months and don’t remember anything.

T2: “It’s true, they don’t remember anything!”

T3: “So.. since at the Scientific high school the quadratic inequalities are presented in the second year , when they arrive at the fourth, fifth year and they have to analyze functions they have to apply these things… they easily do things like $x <+-2$ and so on. Probably when they studied inequalities they didn’t do such a thing! The core problem is that they actually study not to learn but for the mark.”

I: “So actually they are not able to use what they studied before.”

T3: “How many times? Of course I’m referring to the mass, to the most of them , there are always exceptions, but an amount of students do this.”

I: “Another question. Since it’s so hard to introduce real numbers, what about not introducing real numbers? What are the activities in which they’re really necessary?”

T3, T2: “Absolutely no.”

T3: “In what sense? To pretend they don’t exist?”

I: “To work in other sets, not in R.”

T2: “No! And what about the real functions in the real domain?”

T3: “And the algebraic numbers?”

T5: “And how can you solve quadratic equations?”

I: “You could introduce algebraic numbers and avoid to introduce real numbers..”

T5: “There are difficulties also with algebraic numbers..”

[Bustle]

I: “To work with algebraic numbers and to introduce completeness is very different. The algebraic numbers are nor complete nor continuous. Sometimes we say we work with real numbers but numerically we introduce only algebraic numbers or rational numbers. To introduce real numbers would mean to construct a field with all the properties: complete, totally ordered, Archimedean..”

T3: “When you use Geometry, many times you introduce the postulates, the contiguous elements and so on, but don’t think that many students understand these things. So what should we do? Delete all Geometry? ”

I: “It was just a proposal. Since it seems that no one understand what real number are.”

T3: “Not no one… but when …”

[Bustle]

T5: “Sometimes the students don’t understand what rational, irrational means.”

T2: “Irrational. The distinction between rational and irrational.”

T5: “They don’t understand, they don’t understand… Maybe we should just say it, without deepen.”

T2: “Maybe the error is that we don’t deepen… ”

T5: “Eh.. this could also be.”

T3: “I’ve also lost a bit the motivation. I remember that some years ago I taught also in the first two years I remember that there were these difficulties”.

I: “Some of your colleagues reported their experiences and the most of them told me that when they try to introduce R using contiguous classes, Dedekind’s cuts..”

T3: “The concept of continuum more or less is known. In the last year, teaching the Calculus, I would like to see how do you do without… ”

I: “For instance, standing on the national curricula, in the last year the teachers are asked to formalize R. But many teachers say to me that in the practice it’s quite impossible to do it since the students don’t understand. ”

T3: “They are evolved primates, maybe we should have more [don’t understand, nba].. I have many difficulties… ”

T5: “It also depend on the kind of school.”

T3: “The problem is not that they don’t understand, the problem is the time. We don’t want to fuel an argument, we are teachers that also attend training courses and aim at training other teachers but when the national school reform instead of potentiating the mathematics curricula reduced the number of hours, I don’t know what can I do.”

T5: “In the Liceo Classico with 2 hours a week… “

T3: “Also the concept of irrationality, also the mathematics seen from the foundational point of view. In the Liceo Classico, with 2 hours a week; you have to make choices, priority choices, you can’t lose one week…”

I: “So to work with Calculus it’s sufficient that they know some irrational numbers.”

T3: “You trust in the fact they have studied them. Not only some real numbers.. Pi is a transcendental number, is irrational… I can’t show the Taylor’s series! I’ve also proposed it in the previous years in particular courses. You say.. you go on with a sequence, you arrive at the fiftieth digits, we are human.. ”

T5: “It’s very hard. It depends on the point of view.”

I: “Is this sufficient.. some irrational numbers are sufficient for the Calculus?”

T3: “We always work with real number but with a quite intuitive approach, because if you approach it from another point of view…”

I: ”Not formalizing.. Ok. Maybe some problems may emerge you have to take for granted that in a neighborhood of a point there are infinite points, finite points.. ”

T3: “It’s possible to do it, with a recurrent procedure… it’s quite understandable. Even my son understood it! You can also use the intervals.. ”

I: ”How? The interval may also be critical, since we should introduce it only in R. But if we have to enlarge Q we try to represent Q-intervals and to construct R starting from Q, that is different from adding some irrational elements.”

T3: “We are talking about students of the fourth or fifth year, in which the concept of real numbers is already known.”

I: “But we have said that the students don’t have this concept.”

T3: “They have difficulties, but the concept has already been introduced. When there are roots, we have to talk about the real numbers.. so when we talk about intervals, an interval is a set of real numbers, the concept has already been introduced in a certain sense. We, in the last three years, take for granted that it has already been introduced.”

I: “How much do they know, what do they know about real numbers? An example.. this set and its complementary are or are not a Dedekind’s cut. The students answered <No> because A intersected with B is not empty but instead it’s equal to root square of 5. The most of the students didn’t same to be able to reason in Q, but rather they use a set that is bigger than Q and is not R. Which is the true set they use?”

T3: “Maybe the problem is not in the set of real numbers, but in set of rational numbers.”

T5: “Yes.”

T2: “Absolutely.”

T3: “Because many times the students, many times the set of rational numbers is not introduced, we take it for granted. We talk a lot about natural numbers, we talk a lot about real numbers, we don’t talk about rational numbers. Maybe the problem is Q, but since usually we introduce fractions, in the middle school, sometimes in the primary school, Q is very neglected. Also because the concept of set in that year is hard to introduce.”

I: “Intuitively how do you introduce the real numbers in the first years?”

T2: “In a second class I’m now introducing the geometrical construction of square root of 2, I use the line.”

I: “So you say a not rational number exists, and then..”

T5: “Infinite numbers like this one, root square of 3, Pi…”

T2: “Starting from the fractions you say: < Guys, there are limited rational numbers that I transform this way, there are the periodic numbers, and there is a set numbers with whom I can’t do this. They are unlimited and not periodic. Is there this number? Is there Pi? Not all the numbers can stay here, so let’s construct the set of all the numbers that exist in the Nature and we call them real numbers. ”

T5: “An activity that I propose is: < Let’s invent some irrational numbers.> Otherwise…”

T2: “Otherwise they are only punctual.”

T5: “<Let’s invent number with an infinity of digits, 0,123456.. a sequence.. numbers that has an infinity of not periodic digits, it’s a way to lead them to touch a bit.. ”

I: “And how would you connect it with the square, the diagonal? How do you pass from this discourse to the other?”

T5: “What she [T2] were saying.”

T2: “This is just a first approach, to present the .. between 1 and 2 where this root square of 2 is placed, with the ruler you can measure 1, and root square of 2? How can you do it? It has an infinity of digits. You can realize the geometrical construction. Just now I'm going to divide them in small groups and to ask them to construct other real number using Geogebra."

I: "So in your mind it's necessary to work with real numbers."

T3: "Essentially we work with real numbers, the problem is not to work with rational numbers."

T5: "And how could we do?"

T2: "To use rational functions in the rational domain? No! No! Much more complicated!"

T5: "The equations and the inequalities are not usually solved in a numerical set. In my mind this is important."

I: "Could this be important? Don't you say the inequalities are solved in Q, R, N?"

T2: "No, only in particular cases."

T3: "Only with particular problems, if we need a rational number."

T2: "N, Z.. I never do this. I don't say: < The solution is in Q, the solution is in N>.. it's normal that it's R. sometimes I do it to do something different, but generally no. "

T3: "Only once I found in a book exercises to solve not in R."

T6: "sometimes there are problems in N. Find that natural number…"

T3: "No I say precisely the equations. Not in all the books."

T5: "With the equations it could happen."

T2: "Yes, with but the inequalities no.."

T6: "In my mind when the students are used to work in R, that is a bigger set, that you finally reach… the problem is the contrary, the concrete problem… for instance a discrete problem, a merchant sells some bottle of wine. To see that the function isn't continuous, but it's discrete, here it becomes very complicated because they're used to draw lines instead of only drawing points. In my mind this is a risk we run while the process should be the contrary, if one starts from R, one could say : < Stop! I'm arrived. Why should I come back?".

T5: "It's true. The problem may be that one pass from natural numbers to rational numbers, some properties are true in the set of natural numbers, that the students supposed to be valid also in the set of rational numbers. The consecutive for instance."

I: "This property is often easily transported also into R."

T2: "This problem of the consequent indeed is.. "

T3: "The consequent of 0,2 is 0,3. The consequent of ¾ is 4/4."

I: "So you are confirming that they have problems in Q. "

T3: "They are used to concentrate on the last thing they are doing. The last thing we do are real numbers, what you did before, the rational numbers, is overcome. What they know relatively well all the real numbers, because in the moment in which … 1 ,2 , 3 the primary's school teachers taught them.."

T5: "Be careful, sometimes the primary's school teachers don't know these things."

T3: "The concept of relative numbers already is.. the rational numbers… they know this is a subset of real numbers. The problem is not R."

I: "So they work without knowing in which set they work."

T3: "No. They know they're working in the set of real numbers. If you say you are going to work with rational numbers they are in crisis. < Does a set called rational numbers exist?>. They know they it's a continuum, that there is an association between points and numbers."

I: "Well, this is not so trivial from a mathematical point of view."

T3: "They have intuitions. Every point is associated to a number, that I can represent number on a line, that there is this continuum. The rational numbers are forgotten.”

T6: “Do they ask you how many points have they to use? 2,3 4 infinite. Even if say it thousands times they always ask it to you. They are scared, this is a recurrent question.”

T2: “It’s true. Can I use the number of points I want?”

T6: “In the moment itself they understand, as he was saying [T3], after a week they ask you again.”

I: “What do they ask?”

T2: “They draw two points and then the line ends.”

T6: “No, two no. I don’t want.”

T2: “Why not two?”

Focus group with teachers 8-9-10

T1: “I would not have done it that way. This is not the graphic solution. Graphic is when you have the intersection in a point. Y = x, y = 22/17, solution, intersection. If we want to make it graphic. This is not graphic.”

I: “But it’s widespread.”

T1: “It’s awful. You have to put f(x)=g(x). Then <, >, … They have to be precise with the graphics. These are intervals, what does it mean? [..] Was the discussion at the Master interesting?”

I: “Yes, we couldn’t stop it even if it was Saturday. The debate was very heated. A physician asserted that real numbers don’t exist and so it’s senseless to talk about them. So he animated the discussion in a very interesting way. It was wonderful.”

T1: “This was a bit drastic. Yes, if he thinks at the measure in that sense under certain aspects he’s right. I’ve never an infinity of digits, I can have 1,4 1,42 maybe he thinks at this … boh.. He was drastic.”

I: “A statement like this animates a discussion.”

T1: “Maybe he was wishing to provoke.”

I: “Another said to me: < I challenge you to construct real numbers!>”

T1: “Beh, it from the epoch of … Archimedes was already able to do it! Archimedes… it lasts still nowadays. The Archimedean axiom. If you place them in the correct way, given two numbers we always find another number that multiplied by another becomes bigger. He was able, think if we are not!”

I: “It’s a debated topic.”

T1: “Yes, but how did he study these real numbers? I understand that one can be confused as we were studying these numbers with Pini [his Professor of Calculus at the University]. You had to start from N, then you had to construct Z, Z contained something that was equipotential to N. Then from Z you had to construct Q; then there was something more like root square of 2, so you … I remember there was a proof… the convergence of that intervals that seemed to arrive there but the convergence was on the void, there was nothing. So you have to construct something. Then there was another way to construct Pi. Pi… how was it? Where does it come from? It’s transcendent. I’m old… I studied Mathematics at the University.”

I: “At school what do you turn out to do about real numbers.”

T1: “Uh.. only a few. You told them some stories, the classical approach in the third year with root square of 2, that is the classical absurd proof, with the bisection method, that is given two rational numbers there is always a number in the middle but in some cases there isn’t even if the intervals converge, so what I remember from Pini is that intuitively you can see it, while I show that you can’t obtain Pi this way, it’s another kind of number that is there, inside, and that you need because otherwise the ratio between circle and ray is not there. It comes out from this kind of stuff. Sometimes I look my old notes but … mah… I think they are so complicated.”

I: “And does the students understand?”

T1: “The root square of 2, yes. It’s not a problem. Pi .. they remain a bit more confused. I’ve a French book in which you see that Pi comes out in so many ways… bodies, organisms, leaves… an infinity of stuff. It’s really an extraordinary number.. it’s call

…”

T3: “Let’s go on with the interview.”

I: “Let’s discuss a bit about the questionnaire.”

T3: “Well, so we are going to relax”

I: “Yes.”

T2: “I work in the first two year, I don’t remember irrational inequalities..”

T1: “What?”

T2: “I do the irrational equations, not the irrational inequalities. You present in the second year the irrational numbers, but you don’t spend too time. You explain well the ..”

I: “We are going to discuss about the real numbers since no one knows how to introduce them in the high school, there are many opinions and difficulties, the national curricula says you should formalize real numbers in the last year, but it’s not trivial.”

T1: “No, no, no. You use them but you don’t formalize.”

I: “I re-pose a question. How do you introduce real numbers? In the questionnaire did you answer thinking at mathematics or thinking at your students?”

T2: “I thought at the students. I work in the first two years, maybe my colleague who work in the last three years may say more. In the beginning of the second year you, depending on the classes...”

T3:”I remember a class with a students, who was a genius, in that class we introduced the separating element, the contiguous classes and things like that. We had the Mereu [textbook]. In other classes in which you have difficulties, you show the line, you show that if you consider a square, you take the diagonal, you use the Pythagoras’ theorem that is known since the previous years, then you stop.”

T1: “We also do the same.”

T3: ”If you have another class, with better students, you can do something more.”

I: “In the question concerning the usefulness of real numbers, exponential, logarithmic. What did you choose?”

T1, T2, T3: “Quite all!”

T3: “Maybe the series can also be introduced using N”

T1: “No, the series…”

T3: “We chose everything.”

T1: “The exponential… for sure! And how could you do?”

I: “Well. And how do introduce it exactly?”

T1: “My approach is quite phenomenological. I present the growth rhythms. I show or the vaccines, how do leaves grow in a tree, and so on. So you have or the history of the merchant who met a man who said to me: < I give you a million, you give me 1 euro, 2 euro and so on for a month. Doing this the real numbers aren’t involved since this is a sequence of whole numbers, but when you go and see what happens in the middle then you need the real numbers.”

I: “So how do you manage expressions like 2 raised at the square root of 2?”

T3: “I present 2 raised at N, 2 raised at Z, 2 raised at 2, and then you go on with your fantasy! ”

T1: “Yes, you do this way.”

I: “The concept of power is not so easily associated to repeat n times, …”

T3: “You present 2 raised at N, 2 raised at Z, 2 raised at 2 and then you give 2 raised at the root square of 2 and you explain that this becomes a matter of separation between one and the other, the way you can define it. ”

T1: “Yes , you complete. There’s always the problem of Pi, that you can’t construct this way. The transcendent… is not possible to construct it this way… it’s another thing.”

T2: “It’s another thing.”

T1: “But it arise in a wonderful way when you present the complex numbers with Euler… it’s wonderful. I always do this because I say…”

T2: “When do you do this? In the third, the fourth year?’”

T1: “In the fourth.”

I: “And the limits? You need the real numbers also to introduce the limits?”

T2: “Yes, I chose it.”

T1: “Of course! Yes! You need a minimum of topology of the line.”

I: “And how do you do it?”

T1: “I try to avoid as much as I can the epsilon/delta, because they get confused. The classical definition that a student see usually in the first Calculus course at the University, that is epsilon/delta.. at a certain moment I introduce it, but before I try to make them understand the concept.. quite … the neighborhood, an open interval, what it means. ”

I: “do you introduce the concept of limit point before or after the introduction of R?”

T1: “Yes, they have already R”.

I: “And do you present R?”

T1: “R had already been constructed in the second year! You have to complete Q, then you have already put inside the transcendent numbers. But you don’t formalize, you give a set in a correspondence with the line.”

I: “One of your students how would answer the question < What is a real number?>”

T2: “In the first two years he would answer it’s a separating element of contiguous classes, then I don’t know.”

T1: “Nothing more.”

T3: “We take for granted that they know what a real number is… and that the teacher also know it!”

T2: “Because after, always in the first two years, you execute the operations with the real numbers.”

T1: “The only thing that I stress very much is that you lose the conception of order, i.e. given a real numbers you don’t know what is its consecutive number, you can’t establish it.”

T3: “That there is not a precedent nor a consecutive number”

T1: “That you have difficulties with this.. the only thing I say is that given a real number we don’t know which is its consecutive number. What comes after is vague, it’s the only thing I give because I need it for the limit points, the extremes, but these are always chats very intuitive.”

I: “Do they work on the line? Do they understand?”

T1: “In an intuitive way they see, yes. They understand… they see.”

I: “And what about the infinity of the digits, the decimal representation? Do they understand in this way? Because changing representation something could change. Is it possible to make them understand in this register what is the continuity, the postulate of continuity, …”

T1: “Well, no. I do this at the beginning of the fifth year.”

I: “Think for instance at the video with the slider.”

T2: “I didn’t understand it.”

T1: “He was playing cheating, because you have a slider that is moving on the number line, the other on a small line but indeed it’s the same thing. It’s cheating.”

I: “How would you explain that the line is complete, that the holes are all filled?”

T3: “That way. You take a marker, you draw a line on the blackboard and stop.”

I: “You do something like that in your lessons, don’t you?”

T3: “Yes, while it’s different using the computer because the computer shows pixels while …”

T1: “Yes. Now I understand why that physician was arguing that real numbers don’t exist! Because the nature is not analogic, is digital, discrete!”

T2: “You have to take in account that in the first two years they study Euclidean geometry. When I introduce the line, you know that there are the axioms. When they are in the first year. It’s the first proof they see. You show that every point has a precedent, so.. that concept that between two points you have always a point … inside.. now I’m saying it bad, and you show this in the first year.”

T1: “Wait.. but if you show it this way, this should imply that root square is always between two rational numbers.”

T2: “No, but there you do this without associating numbers to points, but when you show …”

T1: “But…”

T2: “The Euclidean geometry.. they do this! So you show exactly this..”

T1: “But if you do the segment root square of 2 using the roots it seems that here inside a number of that kind there should be. The separating elements emerge.”

T2: “Yes, yes.”

T1: “Yes…”

I: “All the aspects like separating elements, postulates of continuity, cuts, ..”

T1: “Postulate of continuity is given. A few… very few. ”

T3: “Let’s talk about the books. My book of the first two years don’t have the separating elements.”

T1: “There was in other books, also the book of the last three years, Mereu, was much better than this one.”

T3: “Why did we change it?”

I: “So the book is not useful to talk about real numbers?”

T3: “Wait. What do you mean with talking about real numbers? When you are at school. I understand because I studied too that things, the real number exist, the golden ratio and so on but when you work ”

I: “I’m also interested in this.”

T3: “This things we are saying are absorbed by two students. The other students, for instance you write pi/3 + k*pi , that k, I have ten students that don’t know what it is, do you understand?”

T1: “Yes”

I: “so when they say real function in the real domain what do the really think at?”

T3: “They think at a continuous trace and at a small ball, because indeed we never show functions from N into N. They don’t pose the problem to connect the exponential. When you pose the problem : < What is e raised at the square root of 2 they don’t pose the problem. They immediately say: < What are you asking us?> and put all the points, they trace all the line and draw it complete, they don’t pose the problem that there could be some empty spaces. You pose the problem, some of them understand that this could be a problem and we should understand what we are talking about, you reassure them, my colleagues know what I am talking about, and they go on more serene, they know.. indeed this is what I also do because when Coen defined the real numbers in the training course, with all that classes and stuff like this, I studied them , I repeated and stop, you can forget it my dear, come back to draw the continuous line… in the sense that it’s the same to me too, I confess.”

I: “And when you work with the limit, the idea to approach without reaching a point. Is this accepted?”

T1: “No, no, no, no…“

T3: “Mmm.. it’s accepted from a point of view.. ”

T1: “Intuitive, but…”

T3: “Yes, it’s exactly like this.”

I: “You’re saying that in the end R is always used by no one know what R is, aren’t you?”

T3: “Yes.”

I: “R is always used but a sense of vague remains.”

T1: “Yes, in a precise manner, so defined no.”

I: “What about the different cardinality?”

T1, T3: “No!”

I: “Sometimes in the books I read: the power of the continuum, different cardinalities and I think: <Does anyone really talk about this topic?>”

T1: “You always have to take in account where we should take this students. The problem is that they have to..”

T3: “In the previous years this was in the curricula. Now only the students who wants Mathematics courses at the University, and you say: <Wait I go home and I explain it to you tomorrow.>”

T1: “We have to do so many things, they need to know how to use that tool. That numerical set…”

T2: “However the textbooks are much more simplified, the rigorous aspect you had in textbooks now is disappeared. Also in the Scientific high school. Also in the textbook the rational numbers are a bit underestimated.”

T1: “That book is worse than the other.”

I: “They need to know how to use them…. I’m not so sure we need to introduce the real numbers. What do we really need real numbers.”

T2: “You do a lot of nice things.”

T1: “x^2=2 has a solution.”

T2: “We do a lot of Geometry in the first two years, the problems where you use Pythagoras, Euclid, so many roots emerge.”

T1: “You need to know that there are numbers that give you the solution. Somehow you write them with a finite number of digits or not. What is important is that it exists.”

T3: “A problem is that they see them represented with e, pi… these irrational are again different.”

T1: “These are different.”

T3: “You don’t find them in equations.”

T1: “Exactly; e emerges as limit, but pi, pi is not like this..”

T3: “Here these is also..”

I: “What about the idea of enlargement, from Q to R? How do you present it?”

T1, T3: “Using the roots.”

I: “After the roots. From N to Z you have the need to enlarge because of the subtraction, from Z to Q because of the division. But from Q to R? You show the roots but..”

T3: “You stop there. The goal you have is that other numbers exist but you don’t go on.”

T2: “There are numbers that are not rational, there isn’t a fraction that express them. In Geometry you connect the discourses because you present commensurable and incommensurable segments."

T1: "Yes."

T3: "If you don't introduce R you can't give an idea of C."

I: "Talking about representation is there a register in which it's better, it's easier to talk about real numbers?"

T1: "The graphical one! The graphical one is surely the easiest."

I: "Do you turn out…?"

T1: "Yes, yes. With the graphic is surely easier. The graphic to represent R but also C is surely the easiest. On an axes the imaginary, the real on the other. Absolutely, absolutely, this is the best register. Indeed I think we should work also more in this register, because they see it a lot of years before, in the first two years."

I: "A student in the graphic register showed a particular conception of the line. I asked: <Does the consecutive number of 0 in Q?>. He answered: < No, because Q is dense>. <And what about 0 in R?> >The consecutive of R exists and is the minimum of the number bigger than 0.>"

T1: "Of which interval?"

T3: "What's the age of the student?"

I: "He's a student of the fourth year."

T3: "He expressed well his thought…"

I: "They had studied it."

T3: "ahh.."

T1: "Superior extreme, inferior extreme… "

I: "In your mind why he said this? If the consecutive number doesn't exist in Q, how can it exist in R? In my mind he reasoned using fractions in the first case and using the line in the second case. He imagines that the line is full of points one after the other, so it's impossible for him to put another point between 0 and the minimum."

T1: "There is another crazy person who does things using the hyperreals, who encircle the points with a strange cloud of other strange points, like a monad, I read something but I never understood what it was. There are those numbers, they exist."

T3: "Yes, they exist. "

T1: "Mm.. they exist.. I don't know."

T3: "I appreciate the historical version of these proposals.

T2: "Yes, in my mind it's useful."

T3."In my mind you have to present it also in order to .. from the view of the culture that is enlarged, of the mathematics that is in evolution, things that are hard to understand because they need definitions that make the students struggle, that also make us struggle, in this sense. Then to go into the deep sense precisely we risk this kind of things to happen … we risk these things to happen if you go too much in depths. They don't have the tools yet."

T1: "You create confusion."

T3: "Maybe we ourselves … when you go the university you are prone to learn in a certain way, they try always…"

T2: "They defoliate, they look for the heart of the matter."

T1: "The better register is the graphic."

T3: "The graphic, because it's intuitive and as she were saying you have to start from the Geometry, that was the crisis, the lead to.."

T2: "Yes, also because historically this was the evolution. It's also incredible that they got it so fast. This discovery..”

T1: “When you tell that there was a secret…”

I: “Me too. I fell in love with this topic during the lessons of the professor Coen, who explained that in Dedekind there is indeed only a little more than what we can read in Euclid.”

T2: “It’s incredible.”

T3: “Coen in his lesson was used to say: < It’s incredible that the teachers don’t know… > and I thought < Oh my God, what a shame…>. I didn’t remember anything of that classes… You say the same things he said, and then you forget them again.”

T1: “Yes, yes.”

T3: “You forget them again because if you don’t teach them every day … he talked about them as something essential.. ”

T2: “Or you do like I do. I’m stopped at the first two years, apart from some experiences, the next year I will try the fifth year.. You have to recall everything because.. there is nothing to do.. you forget them. I’m graduated in Statistics. If you ask me the Gaussian, maybe I remember it, but all the rest..”

T3: “Sorry I have to go, thank you.”

T1: “I understood why the physician said that. Because the universe is digital, is not analogic. He says the real numbers don’t exist, the model of continuity doesn’t work at all.”

I: “Maybe but the fact that you can reduce the interval as long as you want in the model because you know you can do it.”

T1: “But, for instance, they are only approximations of what really happen. I see my son, when have any computation, he develop in Fourier’s series, Laplace and so on because he’s only interested in aparticular interval of precision, so.. ”

I: “Mathematics has a different ontology, you can’t touch anything.”

T1: “In physics too. What do you touch?”

I: “It’s exactly what I’m saying. Real numbers don’t exist in the same way a wave doesn’t exist, a tensor doesn’t exist.”

T1: “He was provoking.”

Interview with Teacher 11

I: “In your mind is there a possible interdisciplinarity between functions in physics and in mathematics?”

T1. “My teacher in the training course always said that the concept of continuous function in Physics is always also derivable since they continuous phenomena. The continuity, a continuous dependence in Physics must also be derivable.”

I: “Yes, of course. In the motion we also need speed and acceleration.”

T: “Exactly.”

I: “These things are interesting. You are used to see trajectories that are always more regular than the continuous ones.”

T: “The classical trajectory is a continuous trajectory.”

I: “Not only continuous, more regular, also derivable at least times continuously. So in physics you see the functions before and they are much more regular than the continuous. Obviously we don't say it. ”

T: “They have this kind of concept of function. In my mind this is a misconception that when you teach Calculus you find in all the students, in all the students! They don't understand which is the sense of take a function defined in a set different from R, in R without a point.”

I: “In my mind this could be a connection. In Physics if you don't separate model and reality they see this, the function is identified with the phenomenon. So you associate to a point a function.”

T: “At least in the classical physics.”

I: “So the function coincides with the reality. Maybe if you stress that the trajectory and its mathematical description are something different, that mathematics give a model of that motion, these could unchain mathematics from its constraints to be necessary linked to reality. The trace is not the motion but a way to mathematize it.”

T: “We should stress this is a useful semiotic register”

I: “Very useful.”

T: “A way to see some properties.”

I: “If you turn out to stress this is a model.”

T: “If you work well since the first year, they should be ready to learn the concept of derivatives already in the fourth year.”

I: “Maybe. If you don't drive it to the limit or don't stress the computation.”

T: “I also with my students of the fifth year in Physics, even if they had not studied limits and integral I proved the Gauss theorem for the electric field's flow.”

I: “You already summed infinite infinitesimals!”

T: “The concept of differential ”

I: “In my mind too perfect functions are showed even where there are not all the regularities we listed before. For instance in the Clapeyron's representation of thermodynamics transformations what we know is a list of couple of variables measured in equilibrium states, while the transformations are very often represented by means of lines, in which every point is supposed to be known and always included in a small neighborhood of the previous point. I said this immediately to my students.”

T: “This could be a good example, the students have a too simplified model of function, that generates misconceptions. I love the Calculus itself and every particular function I interesting to me but they have too simplified models and always one asks me: < Why do we study functions with a different domain, with a hole in the domain?>”

I: “You approximate the phenomena with continuous functions but you never say it.”

T: “Taking it for granted”

I: “There is a list of assumptions the textbook obscures. Clapeyron of course was aware of it!”

T: “The really important thing is the awareness. This is a good example. The students are less and less available to play this game. They don't go beyond with the play of thinking.”

I: “We should find not regular functions.”

T: “We should find significant examples that model situations... My students for instance often ask me why we have to consider a domain without a point... Also I ask the students to find some values but also to evaluate their trend. For instance, what does the sinus do from this point to this one? It grows. I demand them to see that at every number from 0 to Pi/2, at every number, that can be thought as the measure in radiant of an angle, to which correspond a number from 0 to 1 and grows. ”

I: “If you want to pass from the points to the line, if need some tools. At least the trend permit to know you don't have to take in account relative maximum and minimum points.”

T: “Then we don't draw it? Then.. how does it grow? Do you see it emerges the concept of derivative? If I take 10°, here, or if I increase of 10° after, they immediately see that here the function grows more here. Many students say to me: < Here it's quite horizontal!>. They knows it.. I work a lot on the concept of inclination. They know that this line … all the points have the same variation speed, so this is not a good model, but it's very hard to lead them to reason this way.”

I: “Maybe posing the problem as an open question...”

T: “Exactly. Then.. I don't know.. there is also Geogebra.”

I: “To make understand that there are the holes you can start from N.. what do we have?”

T: “But this way you have only isolated points in the domain. ... the study of a domain without a point is different.”

I: “It's to make them imagine domains that are different from R.”

T: “The line with a hole always comes out... when you study the hyperbolic curve with parameters <For which values of k..? The classical omographic function in which you are asked to say what you obtain varying the parameter. I'm now thinking for the first time at this example. Which is the usual situation. I need this x at the denominator. If this is 0 it's a line, otherwise the line has a hole because you have existence conditions. In this case, I never thought to this, the unknown is the parameter itself.”

I: “Yes.”

T: “Here there is a hole for k=1/3.”

I: “to make them understand why we don't consider whole intervals we could also provide example of condition in applied problems in which there are constraints. You can wish to consider the function in a restricted domain. Otherwise why do you exclude a point? Think at your student's question this morning. <Why should we take off a point>?”

T: “Yes, in that case I was exemplifying a function .. motivate choices about the domain. ”

I: “What could be other problems? You were saying, to present function that has a domain without some points.”

T: “They today asked me another thing, I took it for granted. If a function is continuous in a set, is this continuous also in every subset? One of my students... I had highlighted so much that if you add points to the domain this can become continuous.”

I: “And if I reduce it?”

T: “Sometimes I think that to spend to much time in analyzing these concepts..”

I: “The question was well posed.. Maybe you think they are thinking at the limits, while probably there are using more intuitive concepts. As if they thinks that if it's a strange case we use limits, if they see it...”

T: “The continuity is this discourse..”

I: “It's like they have different theories, different methods for different problems. You sometimes work formally, sometimes intuitively. They could think these are different problems.”

T: “Not only. You create a visual image showing practical examples, then, if you think at it, you demand them to do the contrary, from the equation recognize if the graph they have to construct is continuous or not. So you work in a register that is what you ask them to produce when they apply the concept.”

I: “Yes.”

T: “Their output...”

I: “So in your mind it's also a matter of registers.”

T: “I'm thinking now... their output is never pathological, because they are asked to work with usual elementary functions. In the connection points.. they are not prone, when the input you give has different forms.”

I: “It could be.”

T: “This question are recurrent, they ask often.”

I: “Are there other questions heard today that can give us a feedback from the students? I remember a student who said: < If it's a real function in the real domain it's continuous.> and the said: <I can't imagine another way. < This never comes to my mind>. He said exactly this sentence. Maybe we should not only present function in the real domain in the beginning.”

T: “I introduced the sequences just this year with them.”

I: “Maybe they are used from a lot of years to imagine continuous graphs.”

T: “I should try to give them a sequence and ask if it's continuous or not.”

I: “This could be a critical example. Let's write it down as example.”

T: “f(n)=n^2?”

I: “We should find a case in which without the theorems you can't answer, in which the theoretical approach is necessary.”

T: “Limits.. the theorem of comparison is the apotheosis.”

I: “An example in which if they only use the visual approach make mistakes. I have a thesis in which the discussion about infinitesimals is reported.. Leibniz, Berkeley, Newton. If the infinitesimal have or not an extension.”

T: “Analysis standard and non-standard. In Physics! For the precision you need that the infinitesimal is measurable, but for the precision of your final result it's more precise when it tends to 0.”

I: “This is the point, but there it was approached from the philosophical point of view.”

T: “It would be interesting because you are selling dynamical concepts with a static theory.”

I: “This is crucial, the discussions are indeed about dynamic/static, concrete/abstract (atomic approach), exist/not exist (if it's null, it doesn't exist, but if it doesn't exist it's not bigger than 0). The ontological may interest students.”

T: “The history maybe a support for their trust. I said to Martina: < We introduced in these two months concepts that lead to discussion … sometimes I say this to encourage but sometimes I have the opposite reaction, like : < If they didn't understand, how can I understand?>. It's a double-edged sword.”

I: “Maybe we should recreate the historical critic points. Newton approached a physical problem, Leibniz a mathematics problem. Both of them reach the same conclusions but from different point of view. Euler then intervene considering 0 the infinitesimal, Berkeley criticized it vehemently. This could help the students who don't accept what you're saying, because they have any doubt.”

T: “Also the concept of asymptotic behavior. These are conceptual nodes that should be approached with the accuracy of a teacher trainer. I'm now thinking to the sequence, this is more spontaneous...”

I: “Maybe they would not represent it graphically.”

T: “Graphically maybe they would trace the parabolic curve continuously.”

I: “It would be interesting to ask them.”

T: “If you think at it they have studied this before, think at the primary school, and this is natural concept. Is natural the extension or the restriction. Now they think at the restriction as not natural, while the construction of the exponential function....”

I: “This could be a critical point, the dialectic relation between extension and restriction.”

T: “Is there a natural direction? The path of the exponential function is an expansion.”

I: “How do you present it?”

T: “I work on the extension by continuity.”

I: “Should we go from discrete to continuous or from the continuous to the discrete?”

T: “I go from the discrete to the continuous.”

I: “But here you go in the backward.”

T: “Here I miss something about the real numbers... I tell you how I did it.. I'm gonna do it now in my third class. I reconsider the scholastic history of powers. First of all we consider the variable at the exponent, then we recall the first misconception, i.e. power is multiplying a number by itself n times. Then I ask them the properties and to prove them. I focus on 2 raised at 0. What does it mean? A student once asked me: < My teacher explained to me that I multiply it 0 times, then I mustn't write it down, but since we're working with multiplications when we have nothing, it's 1, the neutral element...”

I: “Sometimes a teacher can explain something to herself and then explain it to the students in the same way. Maybe sometimes I do it teaching Physics [I laughs]”

T: “Me too. [T laughs]. I say here two things concerning powers. We have multiplications, the properties and so on. We reach a point in which we have to renounce at this. I miss here which is the exigence to do this, because you finalize it to study the exponential function in the real domain!”

I: “I think that if you divide a number by itself to conserve the formal properties you subtract the exponents and obtain the conventional expression 2 raised at 0.”

T: “I explain to them that to save the properties the mathematicians were forced to abandon the original conception. Paradoxically you define the exponential functions by means of the properties you want it to have. You transform the input value. ”

I: “You build what you want it to be and then you use the properties as a definition.”

T: “Paradoxically this become its definition. The exponential function is.. to the input sums.. that machine that giving sums returns products. Once understood this they don't forget the properties of logarithms.”

I: “I never saw the exponential this way.”

T: “This becomes the new definition because you define this but what is the sense? We do this to conserve that properties. The properties says to us that we can interpret 2 raised at -n as ½ raised at n.”

I: “Are you saying we need a structure?”

T: “Exactly. We renounce to the nursery rhyme: < A power 2 raised at n is the multiplication...> but you have the features we want. Here returns the concept of extending by continuity, even if it's another kind of continuity.”

I: “Maybe the extension by continuity is strange for them because they never extended by continuity what was obtained by means of this process. I take 2 raised at 0, 3, ½, and so on but these are discrete numbers. To trace the line you have to..”

T: “You need the real numbers”

I: “You have to add values that makes it continuous.”

T: “I say <Let's call..>, wait, I'm thinking now. This lead you to 2 raised at x with x rational, but the passage to rational to reals, this scheme cracks.. the powers' properties help you until x rational.”

I: “It's analogous to what Dedekind said about the fact we have to invent numbers. I put here a real number, I have 2 raised at x, but if x is unknown, you invent something that makes continuous the functions.”

T: “I like so much this idea of inventing since its trace remains in the moment in which we have to write that numbers, because we use symbols. To express the irrational numbers you have to use symbols I can't calculate them in their entirety. For instance root square of 2, Pi, 2 raised at Pi.”

I: “In this moment a teacher could read Dedekind, when he wrote that the construction is a free act of the thought. Also a mathematician talking about irrational ...”

T: “What is not obvious is making them aware that here is a hole! The density of Q makes it not obvious. Overall when you want to represent an infinitesimal visually … it's not easy. I travel again through the powers' properties, they drive me until here, but I say. I know that also 2 raised at root square of 2 exists, so how can we give sense to this?”

I: “The know that root square of 2 is not rational?”

T: “Yes, I prove it many times.”

I: “You say: < Make it full>. You put there 2, and then how do you extend it by continuity? If you ask them to it associating values. You could ask if in your mind this is complete. If they answer yes, we can conjecture that the choice of representing number on the line by means of collection of points is a big mistake. They should take something for every point. Or they have a misconception concerning Q that fill the line R you have gratis the problem to solve constructing real numbers.”

T: “The root square of 2 is known, they are able to construct.”

I:. “You could ask them to complete a function in an interval rather than in a point. They could understand the need of adding points to complete by continuity but also associate a value to a limit point. If you are on the rational... you create a critical situation in which to have the definition of continuity as limit is necessary.”

T: “Also if you complete it by continuity you're taking for granted it's continuous innately. What does it mean to complete by continuity? It's a chaos. We have to rethink the previous paths because you arrive..”

I: “You have a function presented like this and ask them to complete it by continuity.”

T: “It would also be nice.. Wait, now I tell you which my exigence is. First of all to investigate their image of continuity, so this could be an exercise: I give three images and you say if it' continuous and why, because the mathematical continuity is also this one [two pieces defined o intervals]. Do you bet they think that this is continuous? [one pieces]. In the assessment of the next week in the fifth class next week I'll put one question of this kind. Here the function is defined. F is continuous on R. What of these graphics are acceptable? In my mind this [cuspid] is not associated to continuity because they associate it to be smooth, with a unique equation... In fact they take for granted that.. they have a concept of similarity added to not to interrupt the trace.”

I: “Maybe in their mind the function stops here and restart there ...”

T: “The word continuity is used very soon, also in the previous years. We have to investigate where you name continuity.”

I: “Maybe the line is the first critical point. You draw I thanks to an axiom. Then the hyperbolic curve is the first critical function. In physics they only see a part with the inverse proportionality. When do we transform first the line in a function at school?”

T: “With the conic.”

Appendix B: Real numbers historical development