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COMMON FIXED POINTS OF GENERALIZED MIZOGUCHI-TAKAHASHI TYPE CONTRACTIONS IN PARTIAL METRIC SPACES

Angela Urso and Calogero Vetro*
Department of Mathematics and Computer Science, Università degli Studi di Palermo, Via Archirafi, 34, 90123 Palermo, Italy
E-mails: angiolettaurso1988@libero.it; calogero.vetro@unipa.it

**Corresponding author.*

Abstract We give some common fixed point results for multivalued mappings in the setting of complete partial metric spaces. Our theorems extend and complement analogous results in the existing literature on metric and partial metric spaces. Finally, we provide an example to illustrate the new theory.

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1. Introduction and preliminaries

Nadler [21] established a fundamental theorem that combines the ideas of multivalued mapping and contractive condition as follows.

Theorem 1.1. *Let (X, d) be a complete metric space and let $T : X \to CB(X)$ be a multivalued mapping satisfying the contractive condition*

$$H(Tx, Ty) \leq kd(x, y), \tag{1.1}$$

for all $x, y \in X$, where k is a constant such that $k \in (0, 1)$ and $CB(X)$ denotes the family of non-empty closed and bounded subsets of X. Then T has a fixed point, that is, there exists a point $x^* \in X$ such that $x^* \in Tx^*$.

This result was successfully extended and applied by many authors [6, 7, 11, 14, 15, 18] in various abstract spaces. Here, we consider the notion of partial metric space. Precisely, Matthews [19] introduced the concept of partial metric as a part of the study of denotational semantics of dataflow networks. Then, the partial metric space became an useful setting to get generalizations of fixed point theorems [1, 4, 8–10, 22, 24, 25, 28].

Recently, Aydi et al. [5] introduced the concept of partial Hausdorff metric and extended the Nadler's fixed point theorem to such spaces. Some interesting contributions

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on multivalued mappings in partial metric spaces can be found in [2, 12, 26]. Finally, we point out that some authors [13, 17, 27] showed that a lot of fixed point theorems in partial metric spaces can be directly reduced to their metric counterparts.
Let $\mathbb{R}^+$ be the set of all non-negative real numbers and $\mathbb{N}$ the set of all positive integers. Mizoguchi and Takahashi [20] introduced one of the most interesting contractive condition in the classical setting of metric spaces. Then, Mizoguchi and Takahashi proved a generalization of Nadler's fixed point theorem, by changing the constant $k \in (0,1)$ in (1.1) with a function $\varphi : \mathbb{R}^+ \to [0,1)$ such that

$$\limsup_{r\to t^+} \varphi(r) < 1$$

for all $t \in \mathbb{R}^+$. We refer to the following fixed point theorem.

Theorem 1.2. *Let (X,d) be a complete metric space and $T : X \to CB(X)$ be a multivalued mapping. Assume that there exists a function $\varphi : [0,+\infty) \to [0,1)$, with $\limsup_{r\to t^+} \varphi(r) < 1$ for all $t \in [0,+\infty)$, such that*

$$H(Tx,Ty) \le \varphi(d(x,y))d(x,y),$$

for all $x,y \in X$, with $x \neq y$. Then T has a fixed point.

Very recently, Javahernia et al. [16] used the following definition to generalize Mizoguchi-Takahashi's theorem to establishing the existence of a common fixed point of two multivalued mappings in the setting of metric spaces.

Definition 1.3. A function $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ is a generalized Mizoguchi-Takahashi type function (for short, GMT-function) if the following conditions hold:

(ϑ1) $0 < \vartheta(t,s) < 1$ for all $t,s > 0$;

(ϑ2) for every bounded sequence $\{t_n\} \subset (0,+\infty)$ and every non-increasing sequence $\{s_n\} \subset (0,+\infty)$, one has

$$\limsup_{n\to+\infty} \vartheta(t_n,s_n) < 1.$$

We denote by $\widehat{GMT}(\mathbb{R})$ the class of functions satisfying Definition 1.3.

Example 1.4 (See [16]). Let $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be the function defined by

$$\vartheta(t,s) = \begin{cases} \dfrac{t}{s^2+1} & \text{if } 1 < t < s, \\ \\ \dfrac{\ln(s+10)}{s+9} & \text{otherwise.} \end{cases}$$

Then, $\vartheta \in \widehat{GMT}(\mathbb{R})$.

Moreover, Javahernia et al. [16] proved the following theorem.

Theorem 1.5. *Let (X,d) be a complete metric space and let $T,S : X \to CB(X)$ and suppose there exists $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that*

$$H(Tx,Sy) \le \vartheta(H(Tx,Sy), M_{T,S}(x,y))M_{T,S}(x,y))$$

for all $x,y \in X$, where

$$M_{T,S}(x,y) := \max\left\{d(x,y), d(x,Tx), d(y,Sy), \frac{d(x,Sy)+d(y,Tx)}{2}\right\}.$$

Then T, S have a common fixed point, that is, there exists a point $x^* \in X$ such that $x^* \in Tx^*$ and $x^* \in Sx^*$.

In this paper, we investigate the possibility to extend this theorem to the setting of partial metric spaces. Precisely, we give some common fixed point results for multivalued mappings in the setting of complete partial metric spaces. Our results extend and complement analogous results in the existing literature on metric and partial metric spaces. Finally, we provide an example to illustrate the new theory.

2. Partial metric

We collect some definitions and results on partial metrics and partial metric spaces.

Definition 2.1 ([19])**.** A partial metric on a non-empty set X is a function $p : X \times X \to \mathbb{R}^+$ such that, for all $x, y, z \in X$, the following conditions are satisfied:

(p1) $x = y \Longleftrightarrow p(x, x) = p(x, y) = p(y, y)$;
(p2) $p(x, x) \leq p(x, y)$;
(p3) $p(x, y) = p(y, x)$;
(p4) $p(x, y) \leq p(x, z) + p(z, y) - p(z, z)$.

Then, a non-empty set X equipped with a partial metric p is called a partial metric space.

If $p(x, y) = 0$, then (p1) and (p2) imply that $x = y$, but the converse does not hold true always. Each partial metric p on X generates a T_0 topology γ_p on X which has as a base the family of the open balls $\{B_p(x, \varepsilon) : x \in X, \varepsilon > 0\}$ where

$$B_p(x, \varepsilon) = \{y \in X : p(x, y) < p(x, x) + \varepsilon\}$$

for all $x \in X$ and $\varepsilon > 0$.

Definition 2.2 ([3, 19])**.** Let (X, p) be a partial metric space. Then a sequence $\{x_n\}$ is called:

(i) convergent, with respect to γ_p, if there exists some x in X such that $p(x, x) = \lim_{n\to+\infty} p(x, x_n)$;
(ii) Cauchy sequence if there exists (and is finite) $\lim_{n,m\to+\infty} p(x_n, x_m)$.

A partial metric space (X, p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges, with respect to γ_p, to a point $x \in X$ such that $p(x, x) = \lim_{n,m\to+\infty} p(x_n, x_m)$.

Let $CB^p(X)$ be the collection of all non-empty closed and bounded subsets of X with respect to the partial metric p. Consistent with Aydi et al. [5], closedness is taken from (X, γ_p). Moreover, boundedness is given as follows: A is a bounded subset in (X, p) if there exist $x_0 \in X$ and $M \geq 0$ such that for all $a \in A$, we have $a \in B_p(x_0, M)$, that is, $p(x_0, a) < p(x_0, x_0) + M$. Then, for $A, B \in CB^p(X)$, $x \in X$, the functions $\delta_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ and $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ are defined as follows

$$p(x, A) = \inf\{p(x, a) : a \in A\}, \quad p(A, B) = \inf\{p(x, y) : x \in A, y \in B\},$$

$$\delta_p(A, B) = \sup\{p(a, B) : a \in A\}, \quad \delta_p(B, A) = \sup\{p(b, A) : b \in B\}$$

and

$$H_p(A, B) = \max\{\delta_p(A, B), \delta_p(B, A)\}.$$

Proposition 2.3 ([5])**.** *Let (X, p) be a partial metric space. For all $A, B, C \in CB^p(X)$, we have the following:*

(i) $\delta_p(A, A) = \sup\{p(a, a) : a \in A\}$;
(ii) $\delta_p(A, A) \le \delta_p(A, B)$;
(iii) $\delta_p(A, B) = 0$ *implies that* $A \subseteq B$;
(iv) $\delta_p(A, B) \le \delta_p(A, C) + \delta_p(C, B) - \inf_{c \in C} p(c, c)$.

Proposition 2.4 ([5]). *Let* (X, p) *be a partial metric space. For all* $A, B, C \in CB^p(X)$*, we have the following:*

(h1) $H_p(A, A) \le H_p(A, B)$;
(h2) $H_p(A, B) = H_p(B, A)$;
(h3) $H_p(A, B) \le H_p(A, C) + H_p(C, B) - \inf_{c \in C} p(c, c)$;
(h4) $H_p(A, B) = 0 \Longrightarrow A = B$.

The mapping $H_p : CB^p(X) \times CB^p(X) \to \mathbb{R}^+$ is called the partial Hausdorff metric induced by p. Every Hausdorff metric is a partial Hausdorff metric but the converse is not true, see Example 2.6 in [5].

In the proofs of our theorems, we will use the following two lemmas.

Lemma 2.5 ([3]). *Let* (X, p) *be a partial metric space and* A *any non-empty set in* (X, p)*, then*

$$a \in \overline{A} \iff p(a, A) = p(a, a),$$

where $\overline{A}$ *denotes the closure of* A *with respect to the partial metric* p*. Notice that* A *is closed in* (X, p) *if and only if* $A = \overline{A}$.

Lemma 2.6 ([5]). *Let* (X, p) *be a partial metric space,* $A, B \in CB^p(X)$ *and* $h > 1$*, then for any* $a \in A$*, there exists* $b(a) \in B$ *such that* $p(a, b(a)) \le hH_p(A, B)$.

Thus, we have the following partial metric version of Nadler's fixed point theorem.

Theorem 2.7 ([5]). *Let* (X, p) *be a partial metric space. If* $T : X \to CB^p(X)$ *is a multivalued mapping such that for all* $x, y \in X$*, we have* $H_p(Tx, Ty) \le kp(x, y)$*, where* $k \in (0, 1)$*, then* T *has a fixed point, that is, there exists a point* $u \in X$ *such that* $u \in Tu$.

Finally, we recall the following lemma; see [23].

Lemma 2.8. *Let* (X, p) *be a partial metric space and* $T : X \to CB^p(X)$ *a multivalued mapping. If* $\{x_n\} \subset X$ *is a sequence,* $x_n \to u$ *and* $p(u, u) = 0$*, then*

$$\lim_{n \to +\infty} p(x_n, Tu) = p(u, Tu).$$

3. Main Results

According to [16] and [20], we introduce the contractive conditions considered in this paper.

Definition 3.1. Let (X, p) be a partial metric space. Two multivalued mappings $T, S : X \to CB^p(X)$ are a pair of generalized Mizoguchi-Takahashi type contractions (for short, CGMT-contraction) if there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that

$$H_p(Tx, Sy) \le \vartheta(H_p(Tx, Sy), p(x, y))p(x, y), \tag{3.1}$$

for all $x, y \in X$.

Definition 3.2. Let (X,p) be a partial metric space. Two multivalued mappings $T,S : X \to CB^p(X)$ are a pair of generalized weaker Mizoguchi-Takahashi type contractions (for short, CWGMT-contraction) if there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that

$$H_p(Tx,Sy) \leq \vartheta(H_p(Tx,Sy),M_{T,S}(x,y))M_{T,S}(x,y), \tag{3.2}$$

for all $x,y \in X$, where

$$M_{T,S}(x,y) := \max\left\{p(x,y),p(x,Tx),p(y,Sy),\frac{p(x,Sy)+p(y,Tx)}{2}\right\}.$$

Now, we state and prove our first result.

Theorem 3.3. *Let (X,p) be a complete partial metric space and $T,S : X \to CB^p(X)$ be two multivalued mappings. Assume that there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that the pair (T,S) is a CGMT-contraction. Then, the pair (T,S) has a common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point and $x_1 \in Sx_0$. Consequently, $p(x_1,x_1) = p(x_1,Sx_0)$, by Lemma 2.5.
Since $Tx_1 \neq \emptyset$, then we can choose $x_2 \in Tx_1$ (clearly, $p(x_2,Tx_1) = p(x_2,x_2)$).
Firstly, we assume $x_1 = x_2$. Then, by definition of partial metric, we have

$$p(x_1,Tx_1) = p(x_1,x_1) \leq p(x_1,Sx_1) = p(x_2,Sx_2).$$

If $p(x_2,Sx_2) = 0$, then $x_2 \in Sx_2$, that is x_2 is a common fixed point of the pair (T,S). Also, if $H_p(Tx_1,Sx_1) = 0$, then $Tx_1 = Sx_1$ and so x_1 is a common fixed point of the pair (T,S).
Thus, we suppose $H_p(Tx_1,Sx_2) \neq 0$ and $p(x_2,Sx_2) \neq 0$. Now, by using the contractive condition (3.1), we get

$$H_p(Tx_1,Sx_2) \leq \vartheta(H_p(Tx_1,Sx_2),p(x_1,x_2))p(x_1,x_2),$$

and hence, by property $(\vartheta 1)$, we have

$$p(x_1,x_2) = p(x_2,x_2) \leq p(x_2,Sx_2) \leq H_p(Tx_1,Sx_2) < p(x_1,x_2),$$

that is a contradiction.
Now, we assume $x_1 \neq x_2$, $H_p(Tx_1,Sx_2) > 0$ and $p(x_1,x_2) > 0$. Let

$$h_1 = \frac{1}{\sqrt{\vartheta(H_p(Tx_1,Sx_2),p(x_1,x_2))}} > 1$$

so that, by Lemma 2.6, there exists $x_3 \in Sx_2$ such that

$$p(x_2,x_3) \leq h_1 H_p(Tx_1,Sx_2).$$

By using the contractive condition (3.1), we get

$$\begin{aligned} p(x_2,x_3) \quad &\leq \frac{\vartheta(H_p(Tx_1,Sx_2),p(x_1,x_2))p(x_1,x_2)}{\sqrt{\vartheta(H_p(Tx_1,Sx_2),p(x_1,x_2))}} \\ &= \sqrt{\vartheta(H_p(Tx_1,Sx_2),p(x_1,x_2))}p(x_1,x_2). \end{aligned}$$

Choose $x_4 \in Tx_3$ with $p(x_3,x_4) > 0$ and $H_p(Tx_3,Sx_4) > 0$. Then, put

$$h_2 = \frac{1}{\sqrt{\vartheta(H_p(Tx_3,Sx_4),p(x_3,x_4))}} > 1.$$

Again, by Lemma 2.6, there exists $x_5 \in Sx_4$ such that

$$p(x_4, x_5) \leq h_2 H_p(Tx_3, Sx_4).$$

By using the contractive condition (3.1), we have

$$\begin{aligned} p(x_4, x_5) &\leq \frac{\vartheta(H_p(Tx_3, Sx_4), p(x_3, x_4))p(x_3, x_4)}{\sqrt{\vartheta(H_p(Tx_3, Sx_4), p(x_3, x_4))}} \\ &= \sqrt{\vartheta(H_p(Tx_3, Sx_4), p(x_3, x_4))}p(x_3, x_4). \end{aligned}$$

By induction, let $x_{2k-1}, x_{2k}, x_{2k+1} \in X$ such that $x_{2k} \in Tx_{2k-1}$, $x_{2k+1} \in Sx_{2k}$, $p(x_{2k-1}, x_{2k}) > 0$ and $H_p(Tx_{2k-1}, Sx_{2k}) > 0$ with

$$p(x_{2k}, x_{2k+1}) < \frac{H_p(Tx_{2k-1}, Sx_{2k})}{\sqrt{\vartheta(H_p(Tx_{2k-1}, Sx_{2k}), p(x_{2k-1}, x_{2k}))}}.$$

Then, letting $x_{2k+2} \in Tx_{2k+1}$ with $p(x_{2k+1}, x_{2k+2}) > 0$, $H_p(Tx_{2k+1}, Sx_{2k+2}) > 0$ and

$$h_{k+1} = \frac{1}{\sqrt{\vartheta(H_p(Tx_{2k+1}, Sx_{2k+2}), p(x_{2k+1}, x_{2k+2}))}} > 1,$$

there exists $x_{2k+3} \in Sx_{2k+2}$ such that

$$p(x_{2k+2}, x_{2k+3}) \leq h_{k+1} H_p(Tx_{2k+1}, Sx_{2k+2}).$$

By using the contractive condition (3.1) we get

$$p(x_{2k+2}, x_{2k+3}) \leq \sqrt{\vartheta(H_p(Tx_{2k+1}, Sx_{2k+2}), p(x_{2k+1}, x_{2k+2}))}p(x_{2k+1}, x_{2k+2}).$$

Thus, by induction, we construct a sequence $\{x_{2n+1}\}$ in X such that, for all $n \in \mathbb{N}$, $x_{2n} \in Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$, $H_p(Tx_{2n-1}, Sx_{2n}) > 0$ and $p(x_{2n-1}, x_{2n}) > 0$ with

$$p(x_{2n}, x_{2n+1}) < \frac{H_p(Tx_{2n-1}, Sx_{2n})}{\sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}}.$$

By using the contractive condition (3.1) we obtain

$$p(x_{2n}, x_{2n+1}) \leq \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}p(x_{2n-1}, x_{2n})$$

for all $n \in \mathbb{N}$. A similar reasoning, by interchanging the roles of S and T, shows that there exists $x_{2n+2} \in Tx_{2n+1}$ such that

$$p(x_{2n+1}, x_{2n+2}) \leq \sqrt{\vartheta(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n}, x_{2n+1}))}p(x_{2n}, x_{2n+1}).$$

It follows that, for all $n \in \mathbb{N}$,

$$p(x_{2n}, x_{2n+1}) \leq \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}p(x_{2n-1}, x_{2n})$$

and

$$p(x_{2n+1}, x_{2n+2}) \leq \sqrt{\vartheta(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n}, x_{2n+1}))}p(x_{2n}, x_{2n+1}).$$

Thus, $\{p(x_n, x_{n-1})\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence. Therefore, we get

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) := \lambda \geq 0.$$

On the other hand, it is easy to show that $\{H_p(Tx_{2n-1}, Sx_{2n})\}_{n\in\mathbb{N}}$ is a bounded sequence. In fact, by hypothesis, we have

$$H_p(Tx_{2n-1}, Sx_{2n}) \leq \vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))p(x_{2n-1}, x_{2n}).$$

Now, since
$$\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}) < 1$$
then
$$H_p(Tx_{2n-1}, Sx_{2n}) \leq p(x_{2n-1}, x_{2n}),$$
and hence the sequence $\{H_p(Tx_{2n-1}, Sx_{2n})\}_{n\in\mathbb{N}}$ is bounded. Similarly, the sequence $\{H_p(Tx_{2n+1}, Sx_{2n})\}_{n\in\mathbb{N}}$ is bounded. By the property (ϑ_2),
$$\limsup_{n\to+\infty} \vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n})) < 1.$$
Next, we show that $\lambda = 0$.
From
$$p(x_{2n}, x_{2n+1}) \leq \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}\, p(x_{2n-1}, x_{2n}),$$
taking the upper limit, we get the contradiction
$$\lambda \leq \sqrt{\limsup \vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}\, \lambda < \lambda.$$
We show that the sequence $\{x_n\}$ is Cauchy in X.
Put
$$\mu_{2n-1} := \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}$$
and
$$\mu_{2n} := \sqrt{\vartheta(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n+1}, x_{2n}))}.$$
Clearly, $\mu_n \in (0, 1)$ for all $n \in \mathbb{N}$. Since $\limsup_{n\to\infty} \mu_n < 1$, we deduce that there exist $t \in [0, 1)$ and $n_0 \in \mathbb{N}$ such that $\mu_n \leq t$ for all $n \in \mathbb{N}$, $n \geq n_0$. Consequently, we write
$$p(x_{2n}, x_{2n+1}) \leq \mu_{2n-1} p(x_{2n-1}, x_{2n})$$
and
$$p(x_{2n+2}, x_{2n+1}) \leq \mu_{2n} p(x_{2n+1}, x_{2n}).$$
Therefore, we have
$$p(x_{2n}, x_{2n+1}) \leq t^{2n-n_0} p(x_1, x_2)$$
and
$$p(x_{2n+2}, x_{2n+1}) \leq t^{2n-n_0+1} p(x_1, x_2).$$
Also, for all $m > n > n_0$, we have
$$\begin{aligned} p(x_{2n}, x_{2m+1}) &\leq \sum_{i=n}^{m} p(x_{2i}, x_{2i+1}) + \sum_{i=n+1}^{m} p(x_{2i}, x_{2i-1}) \\ &\leq \sum_{i=n}^{m} t^{2i-n_0} p(x_1, x_2) + \sum_{i=n+1}^{m} t^{2i-n_0+1} p(x_1, x_2). \end{aligned}$$
It follows that
$$\limsup_{n\to+\infty} p(x_{2n}, x_{2m+1}) = 0$$
and hence the sequence $\{x_n\}$ is Cauchy. Since (X, p) is complete, then the sequence $\{x_n\}$ converges to a point $x^* \in X$ such that
$$\lim_{n\to+\infty} p(x^*, x_n) = p(x^*, x^*) = 0.$$

Finally, we have to show that $x^* \in Tx^*$, that is $p(x^*, Tx^*) = 0$. From

$$\begin{aligned} p(x^*, Tx^*) &\leq p(x^*, x_{2n+1}) + p(x_{2n+1}, Tx^*) \\ &\leq p(x^*, x_{2n+1}) + H_p(Tx^*, Sx_{2n}) \\ &\leq p(x^*, x_{2n+1}) + \vartheta(H_p(Tx^*, Sx_{2n}), p(x^*, x_{2n}))p(x^*, x_{2n}) \\ &\leq p(x^*, x_{2n+1}) + p(x^*, x_{2n}), \end{aligned}$$

for n to infinity, we get

$$p(x^*, Tx^*) \leq 2p(x^*, x^*) = 0.$$

Clearly $p(x^*, Tx^*) \geq 0$ and so

$$p(x^*, Tx^*) = p(x^*, x^*) = 0.$$

By Lemma 2.5 we deduce that $x^* \in Tx^*$.
By repeating the same reasoning for the multivalued mapping S, we get $x^* \in Sx^*$. We conclude that x^* is a common fixed point of the pair (T, S).

∎

Now, we state and prove a common fixed point theorem for a CWGMT-contraction.

Theorem 3.4. *Let (X, p) be a partial metric space and $T, S : X \to CB^p(X)$ be two multivalued mappings. Assume that there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that the pair (T, S) is a CWGMT-contraction. Then, the pair (T, S) has a common fixed point.*

Proof. Let $x_0 \in X$ be an arbitrary point and $x_1 \in Sx_0$. Consequently, $p(x_1, x_1) = p(x_1, Sx_0)$, by Lemma 2.5.
Since $Tx_1 \neq \emptyset$, then we can choose $x_2 \in Tx_1$ (clearly, $p(x_2, Tx_1) = p(x_2, x_2)$). Firstly, we assume $x_1 = x_2$. Then

$$\begin{aligned} &M_{T,S}(x_1, x_2) \\ &= \max\left\{p(x_1, x_2), p(x_1, Tx_1), p(x_2, Sx_2), \frac{p(x_1, Sx_2) + p(x_2, Tx_1)}{2}\right\} \\ &= \max\left\{p(x_1, x_1), p(x_1, Tx_1), p(x_1, Sx_1), \frac{p(x_1, Sx_1) + p(x_1, Tx_1)}{2}\right\} \\ &\leq \max\left\{p(x_1, Tx_1), p(x_2, Sx_2)\right\}. \end{aligned}$$

By the definition of partial metric, we have

$$p(x_1, Tx_1) = p(x_1, x_1) \leq p(x_1, Sx_1) = p(x_2, Sx_2)$$

and hence

$$\max\{p(x_1, Tx_1), p(x_2, Sx_2)\} = p(x_2, Sx_2).$$

If $p(x_2, Sx_2) = 0$, then $x_2 \in Sx_2$, that is x_2 is a common fixed point of the pair (T, S).
Also, if $H_p(Tx_1, Sx_2) = 0$, then $Tx_1 = Sx_2$. In fact, we have

$$0 = H_p(Tx_1, Sx_2) = \max\{\delta_p(Tx_1, Sx_2), \delta_p(Sx_2, Tx_1)\}$$

implies $\delta_p(Tx_1, Sx_2) = 0$ and $\delta_p(Sx_2, Tx_1) = 0$, that is $Tx_1 \subseteq Sx_2$ and $Sx_2 \subseteq Tx_1$.
Thus, we deduce $x_2 \in Tx_1 = Tx_2 = Sx_2$, that is x_2 is a common fixed point of the pair (T, S). Until now, we have not used the properties of the function ϑ.

We can assume $p(x_2, Sx_2) \neq 0$ and $H_p(Tx_1, Sx_2) \neq 0$. By using the contractive condition (3.2), we have

$$\begin{aligned} p(x_1, x_2) &= p(x_2, x_2) \\ &\leq p(x_2, Sx_2) \\ &\leq H_p(Tx_1, Sx_2) \\ &\leq \vartheta(H_p(Tx_1, Sx_2), p(x_2, Sx_2))p(x_2, Sx_2). \end{aligned}$$

By the property ($\vartheta 1$), that is $\vartheta(H_p(Tx_1, Sx_2), p(x_2, Sx_2)) < 1$, we get the contradiction $p(x_2, Sx_2) < p(x_2, Sx_2)$.
Assume that $x_1 \neq x_2$ with $M_{T,S}(x_1, x_2) > 0$ and $H_p(Tx_1, Sx_2) > 0$.
Put

$$h_1 = \frac{1}{\sqrt{\vartheta(H_p(Tx_1, Sx_2), M_{T,S}(x_1, x_2))}} > 1.$$

By using Lemma 2.6, there exists $x_3 \in Sx_2$ such that

$$p(x_2, x_3) \leq h_1 H_p(Tx_1, Sx_2).$$

By using the contractive condition (3.2), we get

$$\begin{aligned} p(x_2, x_3) &\leq \frac{\vartheta(H_p(Tx_1, Sx_2), M_{T,S}(x_1, x_2))M_{T,S}(x_1, x_2)}{\sqrt{\vartheta(H_p(Tx_1, Sx_2), M_{T,S}(x_1, x_2))}} \\ &= \sqrt{\vartheta(H_p(Tx_1, Sx_2), M_{T,S}(x_1, x_2))}M_{T,S}(x_1, x_2). \end{aligned}$$

By a similar reasoning, let $x_4 \in Tx_3$ such that $M_{T,S}(x_3, x_4) > 0$ and $H_p(Tx_3, Sx_4) > 0$. Also, put

$$h_2 = \frac{1}{\sqrt{\vartheta(H_p(Tx_3, Sx_4), M_{T,S}(x_3, x_4))}} > 1.$$

By Lemma 2.6, there exists $x_5 \in Sx_4$ such that

$$p(x_4, x_5) \leq h_2 H_p(Tx_3, Sx_4).$$

By using the contractive condition (3.2), we have

$$H_p(Tx_3, Sx_4) \leq \vartheta(H_p(Tx_3, Sx_4), M_{T,S}(x_3, x_4))M_{T,S}(x_3, x_4),$$

and hence

$$\begin{aligned} p(x_4, x_5) &\leq \frac{\vartheta(H_p(Tx_3, Sx_4), M_{T,S}(x_3, x_4))M_{T,S}(x_3, x_4)}{\sqrt{\vartheta(H_p(Tx_3, Sx_4), M_{T,S}(x_3, x_4))}} \\ &= \sqrt{\vartheta(H_p(Tx_3, Sx_4), M_{T,S}(x_3, x_4))}M_{T,S}(x_3, x_4). \end{aligned}$$

By induction, let $x_{2k-1}, x_{2k}, x_{2k+1} \in X$ such that $x_{2k} \in Tx_{2k-1}$, $x_{2k+1} \in Sx_{2k}$, $M_{T,S}(x_{2k-1}, x_{2k}) > 0$ and $H_p(Tx_{2k-1}, Sx_{2k}) > 0$ with

$$p(x_{2k}, x_{2k+1}) < \frac{H_p(Tx_{2k-1}, Sx_{2k})}{\sqrt{\vartheta(H_p(Tx_{2k-1}, Sx_{2k}), M_{T,S}(x_{2k-1}, x_{2k}))}}.$$

Then, letting $x_{2k+2} \in Tx_{2k+1}$, with $M_{T,S}(x_{2k+1}, x_{2k+2}) > 0$, $H_p(Tx_{2k+1}, Sx_{2k+2}) > 0$ and

$$h_{k+1} = \frac{1}{\sqrt{\vartheta(H_p(Tx_{2k+1}, Sx_{2k+2}), M_{T,S}(x_{2k+1}, x_{2k+2}))}} > 1,$$

there exists $x_{2k+3} \in Sx_{2k+2}$ such that

$$p(x_{2k+2}, x_{2k+3}) \leq h_{k+1} H_p(Tx_{2k+1}, Sx_{2k+2}).$$

By using the contractive condition (3.2) we get

$$p(x_{2k+2}, x_{2k+3}) \leq \sqrt{\vartheta(H_p(Tx_{2k+1}, Sx_{2k+2}), M_{T,S}(x_{2k+1}, x_{2k+2}))} M_{T,S}(x_{2k+1}, x_{2k+2}).$$

Thus, by induction, we construct a sequence $\{x_{2n+1}\}$ in X such that, for all $n \in \mathbb{N}$, $x_{2n} \in Tx_{2n-1}$, $x_{2n+1} \in Sx_{2n}$, $H_p(Tx_{2n-1}, Sx_{2n}) > 0$ and $M_{T,S}(x_{2n-1}, x_{2n}) > 0$ with

$$p(x_{2n}, x_{2n+1}) \leq \frac{H_p(Tx_{2n-1}, Sx_{2n})}{\sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n}))}}.$$

By using the contractive condition (3.2), we obtain

$$p(x_{2n}, x_{2n+1}) < \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n}))} M_{T,S}(x_{2n-1}, x_{2n})$$

for all $n \in \mathbb{N}$. A similar reasoning, by interchanging the roles of S and T, shows that there exists $x_{2n+2} \in Tx_{2n+1}$ such that

$$p(x_{2n+1}, x_{2n+2}) < \sqrt{\vartheta(H_p(Tx_{2n+1}, Sx_{2n}), M_{T,S}(x_{2n+1}, x_{2n}))} M_{T,S}(x_{2n+1}, x_{2n}).$$

Notice that

$$\begin{aligned}
& p(x_{2n-1}, x_{2n}) \\
& \leq M_{T,S}(x_{2n-1}, x_{2n}) \\
& = \max \Big\{ p(x_{2n-1}, x_{2n}), p(x_{2n-1}, Tx_{2n-1}), p(x_{2n}, Sx_{2n}), \\
& \qquad \frac{p(x_{2n-1}, Sx_{2n}) + p(x_{2n}, Tx_{2n-1})}{2} \Big\} \\
& \leq \max \Big\{ p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \frac{p(x_{2n-1}, x_{2n+1}) + p(x_{2n}, x_{2n})}{2} \Big\} \\
& \leq \max \Big\{ p(x_{2n-1}, x_{2n}), p(x_{2n}, x_{2n+1}), \\
& \qquad \frac{p(x_{2n-1}, x_{2n}) + p(x_{2n}, x_{2n+1}) - p(x_{2n}, x_{2n}) + p(x_{2n}, x_{2n})}{2} \Big\} \\
& = p(x_{2n-1}, x_{2n}).
\end{aligned}$$

It follows that

$$M_{T,S}(x_{2n-1}, x_{2n}) = p(x_{2n-1}, x_{2n}).$$

Also, we have

$$\begin{aligned}
&p(x_{2n+1}, x_{2n}) \\
&\le M_{T,S}(x_{2n+1}, x_{2n}) \\
&= \max\Big\{p(x_{2n+1}, x_{2n}), p(x_{2n+1}, Tx_{2n+1}), p(x_{2n}, Sx_{2n}), \\
&\qquad \frac{p(x_{2n+1}, Sx_{2n}) + p(x_{2n}, Tx_{2n+1})}{2}\Big\} \\
&\le \max\Big\{p(x_{2n+1}, x_{2n}), p(x_{2n+1}, x_{2n+2}), \frac{p(x_{2n+1}, x_{2n+1}) + p(x_{2n}, x_{2n+2})}{2}\Big\} \\
&\le \max\Big\{p(x_{2n+1}, x_{2n}), p(x_{2n+1}, x_{2n+2}), \\
&\qquad \frac{p(x_{2n}, x_{2n+1}) + p(x_{2n+1}, x_{2n+2}) - p(x_{2n+1}, x_{2n+1}) + p(x_{2n+1}, x_{2n+1})}{2}\Big\} \\
&= p(x_{2n+1}, x_{2n}).
\end{aligned}$$

Then,

$$M_{T,S}(x_{2n+1}, x_{2n}) = p(x_{2n+1}, x_{2n}).$$

It follows that, for all $n \in \mathbb{N}$,

$$p(x_{2n}, x_{2n+1}) < \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}p(x_{2n-1}, x_{2n})$$

and

$$p(x_{2n+1}, x_{2n+2}) < \sqrt{\vartheta(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n}, x_{2n+1}))}p(x_{2n}, x_{2n+1}).$$

Thus, the sequence $\{p(x_n, x_{n-1})\}_{n\in\mathbb{N}}$ is strictly decreasing.

On the other hand, it is easy to show that the sequences $\{H_p(Tx_{2n-1}, Sx_{2n})\}_{n\in\mathbb{N}}$ and $\{H_p(Tx_{2n+1}, Sx_{2n})\}_{n\in\mathbb{N}}$ are bounded. In fact, by hypothesis, we have

$$H_p(Tx_{2n-1}, Sx_{2n}) \le \vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n}))M_{T,S}(x_{2n-1}, x_{2n}).$$

Since

$$\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n}) < 1$$

and

$$M_{T,S}(x_{2n-1}, x_{2n}) = p(x_{2n-1}, x_{2n}),$$

it follows that

$$H_p(Tx_{2n-1}, Sx_{2n}) \le p(x_{2n-1}, x_{2n}),$$

and hence $\{H_p(Tx_{2n-1}, Sx_{2n})\}_{n\in\mathbb{N}}$ is a bounded sequence. Similarly, the sequence $\{H_p(Tx_{2n+1}, Sx_{2n})\}_{n\in\mathbb{N}}$ is bounded.

Consequently, there exists $\lambda \ge 0$ such that

$$\lim_{n\to+\infty} p(x_n, x_{n+1}) = \inf_{n\in\mathbb{N}} p(x_n, x_{n+1}) := \lambda.$$

By using the property ($\vartheta 2$), we obtain

$$\limsup_{n\to+\infty} \vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n})) < 1$$

and

$$\limsup_{n\to+\infty} \vartheta(H_p(Tx_{2n+1}, Sx_{2n}), M_{T,S}(x_{2n+1}, x_{2n})) < 1.$$

Now, we show that $\lambda = 0$.
From

$$p(x_{2n}, x_{2n+1}) \leq \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n}))}\, M_{T,S}(x_{2n-1}, x_{2n}),$$

taking the upper limit, we get the contradiction

$$\lambda \leq \sqrt{\limsup \vartheta(H_p(Tx_{2n-1}, Sx_{2n}), M_{T,S}(x_{2n-1}, x_{2n}))}\, \lambda < \lambda.$$

Next step is to show that the sequence $\{x_n\}$ is Cauchy.
Put

$$\mu_{2n-1} := \sqrt{\vartheta(H_p(Tx_{2n-1}, Sx_{2n}), p(x_{2n-1}, x_{2n}))}$$

and

$$\mu_{2n} := \sqrt{\vartheta(H_p(Tx_{2n+1}, Sx_{2n}), p(x_{2n}, x_{2n+1}))}.$$

Clearly, $\mu_n \in (0,1)$ for all $n \in \mathbb{N}$. Since $\limsup_{n\to+\infty} \mu_n < 1$, we deduce that there exist $t \in [0,1)$ and $n_0 \in \mathbb{N}$ such that $\mu_n \leq t$ for all $n \in \mathbb{N}$, $n \geq n_0$. Consequently, we write

$$p(x_{2n}, x_{2n+1}) \leq \mu_{2n-1} p(x_{2n-1}, x_{2n})$$

and

$$p(x_{2n+2}, x_{2n+1}) \leq \mu_{2n} p(x_{2n+1}, x_{2n}).$$

Therefore, we have

$$p(x_{2n}, x_{2n+1}) \leq t^{2n-n_0} p(x_1, x_2)$$

and

$$p(x_{2n+2}, x_{2n+1}) \leq t^{2n-n_0+1} p(x_1, x_2).$$

Also, for all $m > n > n_0$, we have

$$\begin{aligned} p(x_{2n}, x_{2m+1}) \quad & \leq \sum_{i=n}^{m} p(x_{2i}, x_{2i+1}) + \sum_{i=n+1}^{m} p(x_{2i}, x_{2i-1}) \\ & \leq \sum_{i=n}^{m} t^{2i-n_0} p(x_1, x_2) + \sum_{i=n+1}^{m} t^{2i-n_0+1} p(x_1, x_2). \end{aligned}$$

It follows that

$$\limsup_{n\to+\infty} p(x_{2n}, x_{2m+1}) = 0$$

and hence the sequence $\{x_n\}$ is Cauchy. Since (X, p) is complete, then the sequence $\{x_n\}$ converges to a point $x^* \in X$ such that

$$\lim_{n\to+\infty} p(x^*, x_n) = p(x^*, x^*) = 0.$$

We show that $x^* \in Tx^*$ and hence $p(x^*, Tx^*) = 0$. Reasoning by absurd, suppose $p(x^*, Tx^*) \neq 0$. By using the properties of ϑ, of p and the contractive condition (3.2), we write

$$
\begin{aligned}
& p(x^*, Tx^*) \\
& \leq p(x^*, x_{2n+1}) + p(x_{2n+1}, Tx^*) \\
& \leq p(x^*, x_{2n+1}) + H_p(Tx^*, Sx_{2n}) \\
& \leq p(x^*, x_{2n+1}) + \vartheta(H_p(Tx^*, Sx_{2n}), M_{T,S}(x^*, x_{2n}))M_{T,S}(x^*, x_{2n}) \\
& \leq p(x^*, x_{2n+1}) + \vartheta(H_p(Tx^*, Sx_{2n}), M_{T,S}(x^*, x_{2n})) \max\Big\{p(x^*, x_{2n}), \\
& \qquad p(x^*, Tx^*), p(x_{2n}, Sx_{2n}), \frac{p(x^*, Sx_{2n}) + p(x_{2n}, Tx^*)}{2}\Big\}.
\end{aligned}
$$

Taking the limit as n to infinity, we get the contradiction

$$p(x^*, Tx^*) < p(x^*, Tx^*)$$

and hence $p(x^*, Tx^*) = 0$. Then, we have

$$p(x^*, Tx^*) = p(x^*, x^*).$$

By using Lemma 2.5, we deduce that $x^* \in Tx^*$. The analogous reasoning for the multivalued mapping S shows that $x^* \in Sx^*$. We conclude that x^* is a common fixed point of the pair (T, S). ∎

4. Consequences and example

As consequences of Theorem 3.4 we give two corollaries, which are generalizations of Nadler's theorem and Mizoguchi-Takahashi's theorem, respectively.

Corollary 4.1. *Let (X, p) be a complete partial metric space and $T, S : X \to CB^p(X)$ be two multivalued mappings such that, for all $x, y \in X$, we have*

$$H_p(Tx, Sy) \leq kM_{T,S}(x, y)$$

with $k \in (0, 1)$. Then, the pair (T, S) has a common fixed point.

Proof. Consider the function $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given by

$$\vartheta(t, s) = k$$

for all $t, s \in \mathbb{R}$. Clearly, $\vartheta \in \widehat{GMT}(\mathbb{R})$ and so the proof follows by an application of Theorem 3.4. ∎

Corollary 4.2. *Let (X, p) be a complete partial metric space and $T, S : X \to CB^p(X)$ be two multivalued mappings such that, for all $x, y \in X$, we have*

$$H_p(Tx, Sy) \leq \varphi(M_{T,S}(x, y))M_{T,S}(x, y),$$

where $\varphi : \mathbb{R}^+ \to [0, 1)$ is a function such that $\limsup_{r\to t^+} \varphi(r) < 1$ for all $t \in \mathbb{R}^+$. Then, the pair (T, S) has a common fixed point.

Proof. Consider the function $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ given bye

$$\vartheta(t, s) = \varphi(s)$$

for all $t, s \in \mathbb{R}$. Clearly, $\vartheta \in \widehat{GMT}(\mathbb{R})$ and so the proof follows by an application of Theorem 3.4. ∎

Also, if we assume $T = S$ in Definitions 3.1 and 3.2, then we retrieve the following definitions.

Definition 4.3. Let (X, p) be a partial metric space. A multivalued mapping $T : X \to CB^p(X)$ is a GMT-contraction if there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that

$$H_p(Tx, Ty) \leq \vartheta(H_p(Tx, Ty), p(x, y))p(x, y),$$

for all $x, y \in X$.

Definition 4.4. Let (X, p) be a partial metric space. A multivalued mapping $T : X \to CB^p(X)$ is a weaker generalized Mizoguchi-Takahashi type contraction (for short, WGMT-contraction) if there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that

$$H_p(Tx, Ty) \leq \vartheta(H_p(Tx, Ty), M_T(x, y))M_T(x, y),$$

for all $x, y \in X$, where

$$M_T(x, y) := \max\left\{p(x, y), p(x, Tx), p(y, Ty), \frac{p(x, Ty) + p(y, Tx)}{2}\right\}.$$

Clearly, by using the Definitions 4.3 and 4.4, we can state (without proofs) the following particular case of Theorems 3.3 and 3.4, respectively.

Corollary 4.5. *Let (X, p) be a complete partial metric space and $T : X \to CB^p(X)$ be a multivalued mapping. Assume that there exists a function $\vartheta \in \widehat{GMT}(\mathbb{R})$ such that T is a GMT-contraction (resp., WGMT-contraction). Then, T has a fixed point.*

Finally, we give an illustrative example of Theorem 3.4.

Example 4.6. Let $X = [0, 1] \cup \{4\}$. Let $T, S : X \to CB^p(X)$ be two multivalued mappings defined by

$$Tx = \left[0, \frac{x}{4}\right], \quad \text{for all } x \in X$$

and

$$Sy = \left\{\frac{y}{4}\right\}, \quad \text{for all } y \in X.$$

Clearly, (X, p) is a complete partial metric space, where the partial metric $p : X \times X \to \mathbb{R}^+$ is given by

$$p(x, y) = \frac{1}{4}|x - y| + \frac{1}{2}\max\{x, y\}, \quad \text{for all } x, y \in X.$$

Let $\vartheta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be defined by

$$\vartheta(t, s) = \begin{cases} \frac{2}{3} & \text{if } s \in [0, 1], \\ \frac{1}{3} & \text{otherwise,} \end{cases}$$

for all $t \in \mathbb{R}^+$. Clearly $\vartheta \in \widehat{GMT}(\mathbb{R})$. Now, we show that the pair (T, S) is a CWGMT-contraction. Then, we distinguish the following four cases:

Case 1. If $x = 4$ and $y \in [0, 1]$, then $T4 = [0, 1]$ and $Sy = \left\{\frac{y}{4}\right\}$. Then

$$M_{T,S}(4, y) \geq p(4, y) = 3 - \frac{1}{4}y$$

and hence

$$\begin{aligned} H_p\left([0,1],\left\{\frac{y}{4}\right\}\right) &= \frac{3}{4}-\frac{1}{16}y \\ &\leq \frac{1}{3}\cdot\left(3-\frac{1}{4}y\right) \\ &\leq \vartheta\left(H_p\left([0,1],\left\{\frac{y}{4}\right\}\right), M_{T,S}(4,y)\right) M_{T,S}(4,y). \end{aligned}$$

It follows that the contractive condition (3.2) holds true.

Case 2. If $x \in [0,1]$ and $y=4$, then $Tx=\left[0,\frac{x}{4}\right]$ and $S4=\{1\}$. Then

$$M_{T,S}(x,4) \geq p(x,4) = 3-\frac{1}{4}x$$

and hence the contractive condition (3.2) holds true. In fact

$$\begin{aligned} H_p\left(\left[0,\frac{x}{4}\right],\{1\}\right) &= \frac{3}{4} \\ &\leq \frac{1}{3}\cdot\left(3-\frac{1}{4}x\right) \\ &\leq \vartheta\left(H_p\left(\left[0,\frac{x}{4}\right],\{1\}\right), M_{T,S}(x,4)\right) M_{T,S}(x,4). \end{aligned}$$

Case 3. If $x,y \in [0,1]$, then $Tx=\left[0,\frac{x}{4}\right]$ and $Sy=\left\{\frac{y}{4}\right\}$. Then

$$M_{T,S}(x,y) \geq p(x,y) = \begin{cases} \frac{3}{4}x-\frac{1}{4}y & \text{if } x \geq y, \\ \frac{3}{4}y-\frac{1}{4}x & \text{if } x < y, \end{cases}$$

and hence, if $x \geq y$, we have

$$\begin{aligned} H_p\left(\left[0,\frac{x}{4}\right],\left\{\frac{y}{4}\right\}\right) &= \max\left\{\frac{3}{16}y, \frac{3}{16}x-\frac{1}{16}y\right\} \\ &\leq \frac{2}{3}\cdot\left(\frac{3}{4}x-\frac{1}{4}y\right) \\ &\leq \vartheta\left(H_p\left(\left[0,\frac{x}{4}\right],\left\{\frac{y}{4}\right\}\right), M_{T,S}(x,y)\right) M_{T,S}(x,y). \end{aligned}$$

Otherwise, if $x<y$, we write

$$\begin{aligned} H_p\left(\left[0,\frac{x}{4}\right],\left\{\frac{y}{4}\right\}\right) &= \frac{3}{16}y \\ &\leq \frac{2}{3}\cdot\left(\frac{3}{4}y-\frac{1}{4}x\right) \\ &\leq \vartheta\left(H_p\left(\left[0,\frac{x}{4}\right],\left\{\frac{y}{4}\right\}\right), M_{T,S}(x,y)\right) M_{T,S}(x,y). \end{aligned}$$

It follows that the contractive condition (3.2) holds true.

Case 4. If $x=y=4$, then $Tx=[0,1]$ and $Sy=\{1\}$. Then

$$M_{T,S}(4,4) \geq p(4,[0,1]) = \frac{11}{4}$$

and hence the contractive condition (3.2) holds true. In fact

$$\begin{aligned} H_p([0,1],\{1\}) &= \frac{3}{4} \\ &< \frac{1}{3}\cdot\frac{11}{4} \\ &\leq \vartheta\left(H_p([0,1],\{1\}), M_{T,S}(4,4)\right) M_{T,S}(4,4). \end{aligned}$$

We conclude that all the hypotheses of Theorem 3.4 are satisfied and hence the pair (T, S) has a common fixed point.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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