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Some fixed point theorems for generalized contractive mappings in complete metric spaces

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Abstract
We introduce new concepts of generalized contractive and generalized α-Suzuki type contractive mappings. Then, we obtain sufficient conditions for the existence of a fixed point of these classes of mappings on complete metric spaces and b-complete b-metric spaces. Our results extend the theorems of Ćirić, Chatterjea, Kannan and Reich.

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1 Introduction and preliminaries

The theoretical framework of fixed point theory has been an active research field over the last three decades. Of course, the Banach contraction mapping principle [1] is the first important result on fixed points for contractive-type mappings. This well-known theorem, which is an essential tool in many branches of mathematical analysis, first appeared in an explicit form in Banach's thesis in 1922, where it was used to establish the existence of a solution for an integral equation. So far, according to its importance and simplicity, several authors have obtained many interesting extensions and generalizations of the Banach contraction principle (see [2–10] and the references therein). Some of such generalizations are obtained by contraction conditions described by rational expressions (see [11–13]).

Throughout this paper, X is assumed to be a nonempty set. Then, the concepts of T-contraction and C-contraction have been introduced, respectively, by Kannan [14] and Chatterjea [15] as follows.

Definition 1.1 Let (X,d) be a metric space. A mapping $f: X \to X$ is said to be:

(i) a C-contraction (see [15]) if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(fx,fy) \leq \alpha\bigl[d(x,fy) + d(y,fx)\bigr];$$

(ii) a K-contraction (see [14]) if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:

$$d(fx,fy) \leq \alpha\bigl[d(x,fx) + d(y,fy)\bigr];$$

(iii) a Reich contraction (see [9]) iff for all $x, y \in X$ there exist nonnegative numbers q, r, s such that $q + r + s < 1$ and

$$d(fx,fy) \leq qd(x,y) + rd(x,fx) + sd(y,fy);$$

(iv) a Ćirić contraction (see [3]) iff for all $x, y \in X$ there exist nonnegative numbers q, r, s and t such that $q + r + s + 2t < 1$ and

$$d(fx,fy) \leq qd(x,y) + rd(x,fx) + sd(y,fy) + t\bigl[d(x,fy) + d(y,fx)\bigr].$$

In 1968 Kannan (see [14]) established a fixed point theorem for a K-contraction. Also, in 1972 Chatterjea (see [15]) proved that if (X,d) is a complete metric space, then every C-contraction on X has a unique fixed point.

Let $\mathfrak{S}$ denote the class of all real functions $\beta : [0,\infty) \to [0,1)$ satisfying the condition

$$\beta(t_n) \to 1 \quad \text{implies that} \quad t_n \to 0 \quad \text{as } n \to \infty.$$

One of the interesting results which generalizes the Banach contraction principle was given by Samet *et al.* [16] by defining α-ψ-contractive mappings.

Definition 1.2 (see [16]) Let $f : X \to X$ be a mapping, and let $\alpha : X \times X \to [0,\infty)$ be a function. We say that f is an α-admissible mapping if

$$x,y \in X, \quad \alpha(x,y) \geq 1 \quad \Longrightarrow \quad \alpha(fx,fy) \geq 1.$$

Denote by Ψ' the family of all nondecreasing functions $\psi : [0,\infty) \to [0,\infty)$ such that $\sum_{n=1}^{\infty} \psi^n(t) < \infty$ for all $t > 0$, where ψ^n is the nth iterate of ψ.

Theorem 1.3 (see [16]) *Let (X,d) be a complete metric space, and let f be an α-admissible mapping. Assume that*

$$\alpha(x,y)d(fx,fy) \leq \psi\bigl(d(x,y)\bigr),$$

where $\psi \in \Psi'$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$*;

(ii) *either f is continuous, or, for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$ such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

Definition 1.4 (see [17]) Let $f : X \to X$ and $\alpha : X \times X \to [0,+\infty)$. We say that f is a triangular α-admissible mapping if

(T1) $\alpha(x,y) \geq 1$ implies $\alpha(fx,fy) \geq 1$, $x, y \in X$;

(T2) $\left\{\begin{smallmatrix} \alpha(x,z) \geq 1, \\ \alpha(z,y) \geq 1 \end{smallmatrix}\right.$ implies $\alpha(x,y) \geq 1$, $x, y, z \in X$.

Lemma 1.5 (see [17]) *Let f be a triangular α-admissible mapping. Assume that there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$. Then*

$$\alpha(x_m,x_n) \geq 1 \quad \textit{for all } m,n \in \mathbb{N} \textit{ with } m < n.$$

In this paper, we introduce new concepts of generalized contractive and generalized α-Suzuki type contractive mappings. Then, we obtain sufficient conditions for the existence of a fixed point of these classes of mappings on complete metric spaces and b-complete b-metric spaces. In particular, our results extend the theorems of Ćirić, Chatterjea, Kannan and Reich.

2 Generalization of Ćirić, Chatterjea, Kannan and Reich contractions

In [18], Jleli and Samet introduced a new type of contractive mappings and established a new fixed point theorem for such mappings in the setting of generalized metric spaces.

Consistent with [18], we denote by Ψ the set of all functions $\psi : [0,\infty) \to [1,\infty)$ satisfying the following conditions:

(ψ_1) ψ is nondecreasing and $\psi(t) = 1$ if and only if $t = 0$;
(ψ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \psi(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(ψ_3) there exist $r \in (0,1)$ and $\ell \in (0,\infty]$ such that $\lim_{t\to 0^+} \frac{\psi(t)-1}{t^r} = \ell$;
(ψ_4) $\psi(a+b) \le \psi(a)\psi(b)$ for all $a,b > 0$.

Theorem 2.1 (see [18], Corollary 2.1) *Let (X,d) be a complete metric space and $f : X \to X$ be a mapping. Suppose that there exist $\psi \in \Psi$ and $k \in (0,1)$ such that*

$$x,y \in X, \quad d(fx,fy) \ne 0 \quad \Longrightarrow \quad \psi\bigl(d(fx,fy)\bigr) \le \bigl[\psi\bigl(d(x,y)\bigr)\bigr]^k.$$

Then f has a unique fixed point.

Observe that the Banach contraction principle follows immediately from the above theorem.

By introducing the following new concept, first we extend the result of Jleli and Samet, then we obtain some new generalizations of the Banach contraction principle.

Definition 2.2 Let (X,d) be a metric space, and let $f : X \to X$ be a mapping.

f is said to be a JS-contraction whenever there are a function $\psi \in \Psi$ and positive real numbers k_1, k_2, k_3, k_4 with $0 \le k_1 + k_2 + k_3 + 2k_4 < 1$ such that

$$\begin{aligned}\psi\bigl(d(fx,fy)\bigr) \le{}& \bigl[\psi\bigl(d(x,y)\bigr)\bigr]^{k_1}\bigl[\psi\bigl(d(x,fx)\bigr)\bigr]^{k_2}\bigl[\psi\bigl(d(y,fy)\bigr)\bigr]^{k_3} \\ &\times \bigl[\psi\bigl(d(x,fy)+d(y,fx)\bigr)\bigr]^{k_4}\end{aligned} \tag{2.1}$$

for all $x,y \in X$.

Our first result is the following.

Theorem 2.3 *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous JS-contraction. Then f has a unique fixed point.*

Proof Let $x_0 \in X$ be arbitrary. For $x_0 \in X$, we define the sequence $\{x_n\}$ by $x_n = f^n x_0 = fx_{n-1}$. Also, if there exists $n_0 \in \mathbb{N}$ such that $x_{n_0} = x_{n_0+1}$, then x_{n_0} is a fixed point of f, and we have nothing to prove. Thus, we assume that $x_n \ne x_{n+1}$, *i.e.*, $d(fx_{n-1},fx_n) > 0$ for all $n \in \mathbb{N} \cup \{0\}$.

Now, we will prove that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0.$$

Since f is a JS-contraction, then, by using condition (2.1), we obtain that

$$\begin{aligned}
&\psi\big(d(x_{n+1},x_n)\big)\\
&\quad= \psi\big(d(fx_n,fx_{n-1})\big)\\
&\quad\le \big[\psi\big(d(x_n,x_{n-1})\big)\big]^{k_1}\big[\psi\big(d(x_n,fx_n)\big)\big]^{k_2}\big[\psi\big(d(x_{n-1},fx_{n-1})\big)\big]^{k_3}\\
&\qquad\times \big[\psi\big(d(x_n,fx_{n-1}) + d(x_{n-1},fx_n)\big)\big]^{k_4}\\
&\quad\le \big[\psi\big(d(x_n,x_{n-1})\big)\big]^{k_1}\big[\psi\big(d(x_n,x_{n+1})\big)\big]^{k_2}\big[\psi\big(d(x_{n-1},x_n)\big)\big]^{k_3}\big[\psi\big(d(x_{n-1},x_{n+1})\big)\big]^{k_4}\\
&\quad\le \big[\psi\big(d(x_n,x_{n-1})\big)\big]^{k_1+k_3}\big[\psi\big(d(x_n,x_{n+1})\big)\big]^{k_2}\big[\psi\big(d(x_{n-1},x_n)\big)\big]^{k_4}\big[\psi\big(d(x_n,x_{n+1})\big)\big]^{k_4}.
\end{aligned}$$

Therefore, we write

$$1 < \psi\big(d(x_{n+1},x_n)\big) \le \big[\psi\big(d(x_n,x_{n-1})\big)\big]^{\frac{k_1+k_3+k_4}{1-k_2-k_4}} \le \big[\psi\big(d(x_1,x_0)\big)\big]^{(\frac{k_1+k_3+k_4}{1-k_2-k_4})^n}.$$

This gives us that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

by our assumptions about the function ψ. From similar arguments as in the proof of Theorem 2.1 of [18] it follows that there exists $n_1 \in \mathbb{N}$ such that

$$d(x_n,x_{n+1}) \le \frac{1}{n^{\frac{1}{r}}}$$

for all $n \ge n_1$.

Now, for $m > n > n_1$, we have

$$d(x_n,x_m) \le \sum_{i=n}^{m-1} d(x_i,x_{i+1}) \le \sum_{i=n}^{m-1} \frac{1}{i^{\frac{1}{r}}}.$$

Since $0 < r < 1$, then $\sum_{i=n}^{\infty} \frac{1}{i^{\frac{1}{r}}}$ converges and hence $d(x_n,x_m) \to 0$ as $m,n \to \infty$. Thus, we proved that $\{x_n\}$ is a Cauchy sequence. Completeness of (X,d) ensures that there exists $x^* \in X$ such that $x_n \to x^*$ as $n \to \infty$. Next, since f is a continuous mapping, then $x_{n+1} = fx_n \to fx^*$ as $n \to \infty$, *i.e.*, $x^* = fx^*$. Thus, f has a fixed point.

Finally, suppose that there exists $z \ne x^*$ such that $z = fz$. Clearly, $d(z,x^*) = d(fz,fx^*) \ne 0$ and so we can apply condition (2.1) for the pair (z,x^*). Now, by (2.1) we get

$$1 < \psi\big(d(z,x^*)\big) = \psi\big(d(fz,fx^*)\big) \le \big[\psi\big(d(z,x^*)\big)\big]^{k_1+2k_4} < \psi\big(d(z,x^*)\big),$$

which leads to contradiction. Thus, we have a unique fixed point of f in X. □

For specific choices of function ψ, we obtain some significant results. First, by taking $\psi(t) = e^{\sqrt{t}}$ in (2.1), we state a generalization of Ćirić result in [3].

Theorem 2.4 *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_1, k_2, k_3, k_4, with $0 \leq k_1 + k_2 + k_3 + 2k_4 < 1$, such that*

$$\sqrt{d(fx,fy)} \leq k_1\sqrt{d(x,y)} + k_2\sqrt{d(x,fx)} + k_3\sqrt{d(y,fy)} + k_4\sqrt{d(x,fy) + d(y,fx)} \tag{2.2}$$

for all $x,y \in X$. Then f has a unique fixed point.

Remark 2.5 Notice that condition (2.2) is equivalent to

$$\begin{aligned} d(fx,fy) \leq{}& k_1{}^2 d(x,y) + k_2{}^2 d(x,fx) + k_3{}^2 d(y,fy) + k_4{}^2\big[d(x,fy) + d(y,fx)\big] \\ &+ 2k_1k_2\sqrt{d(x,y)d(x,fx)} + 2k_1k_3\sqrt{d(x,y)d(y,fy)} \\ &+ 2k_1k_4\sqrt{d(x,y)\big[d(x,fy) + d(y,fx)\big]} + 2k_2k_3\sqrt{d(x,fx)d(y,fy)} \\ &+ 2k_2k_4\sqrt{d(x,fx)\big[d(x,fy) + d(y,fx)\big]} + 2k_3k_4\sqrt{d(y,fy)\big[d(x,fy) + d(y,fx)\big]}. \end{aligned}$$

Next, in view of Remark 2.5, by taking $k_1 = k_4 = 0$ in Theorem 2.4, we obtain the following extension of Kannan result.

Theorem 2.6 *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_2, k_3, with $0 \leq k_2 + k_3 < 1$, such that*

$$d(fx,fy) \leq k_2{}^2 d(x,fx) + k_3{}^2 d(y,fy) + 2k_2k_3\sqrt{d(x,fx)d(y,fy)} \tag{2.3}$$

for all $x,y \in X$. Then f has a unique fixed point.

On the other hand, by taking $k_1 = k_2 = k_3 = 0$ in Theorem 2.4, we obtain the following Chatterjea type result.

Theorem 2.7 *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous mapping. Suppose that there exists $k_4 \in [0, \frac{1}{2})$ such that*

$$d(fx,fy) \leq k_4{}^2\big[d(x,fy) + d(y,fx)\big]$$

for all $x,y \in X$. Then f has a unique fixed point.

From Theorem 2.4, by taking $k_4 = 0$, we obtain the extension of Reich contraction.

Theorem 2.8 *Let (X,d) be a complete metric space and $f : X \to X$ be a continuous mapping. Suppose that there exist positive real numbers k_1, k_2, k_3, with $0 \leq k_1 + k_2 + k_3 < 1$, such that*

$$\begin{aligned} d(fx,fy) \leq{}& k_1{}^2 d(x,y) + k_2{}^2 d(x,fx) + k_3{}^2 d(y,fy) \\ &+ 2k_1k_2\sqrt{d(x,y)d(x,fx)} + 2k_1k_3\sqrt{d(x,y)d(y,fy)} + 2k_2k_3\sqrt{d(x,fx)d(y,fy)} \end{aligned}$$

for all $x,y \in X$. Then f has a unique fixed point.

Finally, by taking $\psi(t) = e^{\sqrt[n]{t}}$ in (2.1), we have the following corollary.

Corollary 2.9 *Let* (X,d) *be a complete metric space and* $f : X \to X$ *be a continuous mapping. Suppose that there exist positive real numbers* k_1, k_2, k_3, k_4, *with* $0 \le k_1 + k_2 + k_3 + 2k_4 < 1$, *such that*

$$\sqrt[n]{d(fx,fy)} \le k_1 \sqrt[n]{d(x,y)} + k_2 \sqrt[n]{d(x,fx)} + k_3 \sqrt[n]{d(y,fy)} + k_4 \sqrt[n]{d(x,fy) + d(y,fx)}$$

for all $x, y \in X$. *Then* f *has a unique fixed point.*

3 Generalized α-Suzuki type contractions

Czerwik in [19] introduced the concept of b-metric space. Since then, several papers discussed fixed point results for single-valued and multi-valued operators in b-metric spaces (see, *e.g.*, [19, 20]).

Definition 3.1 Let X be a (nonempty) set and $s \ge 1$ be a given real number. A function $d : X \times X \to \mathbb{R}^+$ is a b-metric if, for all $x, y, z \in X$, the following conditions are satisfied:

(b_1) $d(x,y) = 0$ iff $x = y$,
(b_2) $d(x,y) = d(y,x)$,
(b_3) $d(x,z) \le s[d(x,y) + d(y,z)]$.

In this case, the pair (X,d) is called a b-metric space.

Definition 3.2 (see [21]) Let (X,d) be a b-metric space.

(i) A sequence $\{x_n\}$ in X is called b-convergent if and only if there exists $x \in X$ such that $d(x_n,x) \to 0$ as $n \to \infty$. In this case, we write $\lim_{n\to\infty} x_n = x$.
(ii) A sequence $\{x_n\}$ in X is said to be b-Cauchy if and only if $d(x_n,x_m) \to 0$ as $n, m \to \infty$.
(iii) The b-metric space (X,d) is b-complete if every b-Cauchy sequence in X is b-convergent.

Note that a b-metric need not be a continuous function. The following example (corrected from [22]) illustrates this fact.

Example 3.3 Let $X = \mathbb{N} \cup \{\infty\}$ and let $d : X \times X \to \mathbb{R}$ be defined by

$$d(m,n) = \begin{cases} 0 & \text{if } m = n, \\ |\frac{1}{m} - \frac{1}{n}| & \text{if one of } m,n \text{ is even and the other is even or } \infty, \\ 5 & \text{if one of } m,n \text{ is odd and the other is odd (and } m \ne n) \text{ or } \infty, \\ 2, & \text{otherwise.} \end{cases}$$

It can be checked that for all $m, n, p \in X$, we have

$$d(m,p) \le \frac{5}{2}\bigl[d(m,n) + d(n,p)\bigr].$$

Thus, (X,d) is a b-metric space (with $s = 5/2$). Let $x_n = 2n$ for each $n \in \mathbb{N}$. Then

$$d(2n,\infty) = \frac{1}{2n} \to 0 \quad \text{as } n \to \infty,$$

i.e., $x_n \to \infty$, but $d(x_n,1) = 2 \nrightarrow 5 = d(\infty,1)$ as $n \to \infty$.

It is easy to prove the following lemma.

Lemma 3.4 *Let (X,d) be a b-metric space with $s \geq 1$. If a sequence $\{x_n\} \subseteq X$ is b-convergent, then it admits a unique limit.*

Now, we consider a new set of real functions, say Θ. Precisely, we modify the set Ψ by substituting the condition ψ_3 by another condition. Applying this condition we can have a wide range of functions. Thus, we denote by Θ the set of all functions $\theta : (0,\infty) \to (1,\infty)$ satisfying the following conditions:

(θ_1) θ is nondecreasing;
(θ_2) for each sequence $\{t_n\} \subseteq (0,\infty)$, $\lim_{n\to\infty} \theta(t_n) = 1$ if and only if $\lim_{n\to\infty} t_n = 0$;
(θ_3) θ is continuous.

Remark 3.5 It is clear that $f(t) = e^t$ does not belong to Ψ, but $f(t) = e^t \in \Theta$. Another examples are $f(t) = \cosh t$ and $f(t) = 1 + \ln(1 + t)$ for all $t > 0$.

In 1962, Edelstein (see [23]) proved an interesting version of the Banach contraction principle. In 2009, Suzuki (see [24]) proved certain remarkable results to improve the results of Banach and Edelstein (see also [22, 25–27]).

Now, we are ready to prove the following Suzuki-Edelstein type theorem. The values of $M(x,y)$ in the sequel appeared recently in [28]. Also, we assume that $\alpha : X \times X \to (0,\infty)$.

Theorem 3.6 *Let (X,d) be a b-complete b-metric space with $s > 1$, and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \quad \Longrightarrow \quad \alpha(x,y)\theta\bigl(s^2 d(fx,fy)\bigr) \leq \bigl[\theta\bigl(M(x,y)\bigr)\bigr]^k \tag{3.1}$$

for all $x,y \in X$ with $fx \neq fy$, where

$$M(x,y) = \max\biggl\{d(x,y), \frac{d(x,fx)d(x,fy) + d(y,fy)d(y,fx)}{1 + s[d(x,fx) + d(y,fy)]}, \frac{d(x,fx)d(x,fy) + d(y,fy)d(y,fx)}{1 + d(x,fy) + d(y,fx)}\biggr\}.$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$;*
(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

Proof Let $x_0 \in X$ be such that $\alpha(x_0,fx_0) \geq 1$. Define a sequence $\{x_n\}$ by $x_n = f^n x_0$ for all $n \in \mathbb{N}$. Since f is an α-admissible mapping and $\alpha(x_0,x_1) = \alpha(x_0,fx_0) \geq 1$, we deduce that $\alpha(x_1,x_2) = \alpha(fx_0,fx_1) \geq 1$. Continuing this process, we get that $\alpha(x_n,x_{n+1}) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$. Without loss of generality, we suppose that $x_n \neq x_{n+1}$ for all $\mathbb{N} \cup \{0\}$. We will do the proof in the following steps.

Step I: We will show that $\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$. Since $\alpha(x_n,x_{n+1}) \geq 1$ for each $n \in \mathbb{N}$, and $\frac{1}{2s}d(x_{n-1},fx_{n-1}) \leq d(x_{n-1},x_n)$ then by (3.1) we have

$$\begin{aligned}
\theta\big(d(x_n,x_{n+1})\big) &= \theta\big(d(fx_{n-1},fx_n)\big) \\
&\leq \alpha(x_{n-1},x_n)\theta\big(s^2 d(fx_{n-1},fx_n)\big) \\
&\leq \big[\theta\big(M(x_{n-1},x_n)\big)\big]^k \\
&= \big[\theta\big(d(x_{n-1},x_n)\big)\big]^k \\
&< \theta\big(d(x_{n-1},x_n)\big) \qquad (3.2)
\end{aligned}$$

because

$$\begin{aligned}
M(x_{n-1},x_n) &= \max\bigg\{d(x_{n-1},x_n), \frac{d(x_{n-1},fx_{n-1})d(x_{n-1},fx_n) + d(x_n,fx_n)d(x_n,fx_{n-1})}{1 + s[d(x_{n-1},fx_{n-1}) + d(fx_{n-1},fx_n)]}, \\
&\qquad \frac{d(x_{n-1},fx_{n-1})d(x_{n-1},fx_n) + d(x_n,fx_n)d(x_n,fx_{n-1})}{1 + d(x_{n-1},fx_n) + d(x_n,fx_{n-1})}\bigg\} \\
&= \max\bigg\{d(x_{n-1},x_n), \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})d(x_n,x_n)}{1 + s[d(x_{n-1},x_n) + d(x_n,x_{n+1})]}, \\
&\qquad \frac{d(x_{n-1},x_n)d(x_{n-1},x_{n+1}) + d(x_n,x_{n+1})d(x_n,x_n)}{1 + d(x_{n-1},x_{n+1}) + d(x_n,x_n)}\bigg\} \\
&= d(x_{n-1},x_n).
\end{aligned}$$

Therefore, we have

$$1 < \theta\big(d(x_{n+1},x_n)\big) \leq \big[\theta\big(d(x_n,x_{n-1})\big)\big]^k \leq \big[\theta\big(d(x_1,x_0)\big)\big]^{k^n}.$$

This gives us that

$$\lim_{n\to\infty} d(x_n,x_{n+1}) = 0$$

by our assumptions about function θ.

Step II: Now, we prove that the sequence $\{x_n\}$ is a b-Cauchy sequence. Suppose the contrary, *i.e.*, $\{x_n\}$ is not a b-Cauchy sequence. Then there exists $\varepsilon > 0$ for which we can find two subsequences $\{x_{m_i}\}$ and $\{x_{n_i}\}$ of $\{x_n\}$ such that n_i is the smallest index for which

$$n_i > m_i > i \quad \text{and} \quad d(x_{m_i},x_{n_i}) \geq \varepsilon. \qquad (3.3)$$

This means that

$$d(x_{m_i},x_{n_i-1}) < \varepsilon.$$

From (3.3) and using (b_3), we get

$$\varepsilon \leq d(x_{m_i},x_{n_i}) \leq sd(x_{m_i},x_{m_i+1}) + sd(x_{m_i+1},x_{n_i}).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i}). \tag{3.4}$$

Remember that from (3.2) and (θ_1) we get

$$d(x_n, x_{n+1}) \leq d(x_{n-1}, x_n) \tag{3.5}$$

for all $n \in \mathbb{N}$. Suppose that there exists $i_0 \in \mathbb{N}$ such that

$$\frac{1}{2s} d(x_{m_{i_0}}, fx_{m_{i_0}}) > d(x_{m_{i_0}}, x_{n_{i_0}-1})$$

and

$$\frac{1}{2s} d(x_{m_{i_0}+1}, fx_{m_{i_0}+1}) > d(x_{m_{i_0}+1}, x_{n_{i_0}-1}).$$

Then, from (3.5), we have

$$\begin{aligned} d(x_{m_{i_0}}, x_{m_{i_0}+1}) &\leq s\bigl[d(x_{m_{i_0}}, x_{n_{i_0}-1}) + d(x_{m_{i_0}+1}, x_{n_{i_0}-1})\bigr] \\ &< s\left[\frac{1}{2s} d(x_{m_{i_0}}, fx_{m_{i_0}}) + \frac{1}{2s} d(x_{m_{i_0}+1}, fx_{m_{i_0}+1})\right] \\ &= \frac{1}{2}\bigl[d(x_{m_{i_0}}, x_{m_{i_0}+1}) + d(x_{m_{i_0}+1}, x_{m_{i_0}+2})\bigr] \\ &\leq \frac{1}{2}\bigl[d(x_{m_{i_0}}, x_{m_{i_0}+1}) + d(x_{m_{i_0}}, x_{m_{i_0}+1})\bigr] = d(x_{m_{i_0}}, x_{m_{i_0}+1}), \end{aligned}$$

which is a contradiction. Hence, either

$$\frac{1}{2s} d(x_{m_i}, fx_{m_i}) \leq d(x_{m_i}, x_{n_i-1})$$

or

$$\frac{1}{2s} d(x_{m_i+1}, fx_{m_i+1}) \leq d(x_{m_i+1}, x_{n_i-1})$$

holds for all $i \in \mathbb{N}$.

First suppose that

$$\frac{1}{2s} d(x_{m_i}, fx_{m_i}) \leq d(x_{m_i}, x_{n_i-1}) \tag{3.6}$$

holds for all $i \in J$, where J is an infinite set. As from Lemma 1.5, $\alpha(x_{m_i}, x_{n_i-1}) \geq 1$, according to (θ_1) we obtain that

$$\begin{aligned} \theta\left(s^2 \cdot \frac{\varepsilon}{s}\right) &\leq \theta\Bigl(s^2 \cdot \limsup_{i\to\infty, i\in J} d(x_{m_i+1}, x_{n_i})\Bigr) \\ &\leq \Bigl[\theta\Bigl(\limsup_{i\to\infty, i\in J} M(x_{m_i}, x_{n_i-1})\Bigr)\Bigr]^k \leq \bigl[\theta(\varepsilon)\bigr]^k \end{aligned}$$

because, from the definition of $M(x,y)$ and the above limits, we have

$$\begin{aligned}
&\limsup_{i\to\infty, i\in J} M(x_{m_i}, x_{n_i-1}) \\
&\quad = \limsup_{i\to\infty, i\in J} \max\left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, fx_{m_i})d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(fx_{m_i}, fx_{n_i-1})]}, \right. \\
&\quad \left. \frac{d(x_{m_i}, fx_{m_i})d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{n_i-1})d(x_{n_i-1}, fx_{m_i})}{1 + d(x_{m_i}, fx_{n_i-1}) + d(x_{n_i-1}, fx_{m_i})} \right\} \\
&\quad = \limsup_{i\to\infty, i\in J} \max\left\{ d(x_{m_i}, x_{n_i-1}), \frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+1})}{1 + s[d(x_{m_i}, x_{n_i-1}) + d(x_{m_i+1}, x_{n_i})]}, \right. \\
&\quad \left. \frac{d(x_{m_i}, x_{m_i+1})d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{n_i})d(x_{n_i-1}, x_{m_i+1})}{1 + d(x_{m_i}, x_{n_i}) + d(x_{n_i-1}, x_{m_i+1})} \right\} \\
&\quad \leq \varepsilon,
\end{aligned}$$

which implies that $\theta(s^2 \cdot \frac{\varepsilon}{s}) \leq [\theta(\varepsilon)]^k$, a contradiction.

Now, if J is a finite set, then we can assume that

$$\frac{1}{2s} d(x_{m_i+1}, fx_{m_i+1}) \leq d(x_{m_i+1}, x_{n_i-1})$$

holds for all $i \in \mathbb{N}$. Further, from (3.3) and using (b_3), we get

$$\varepsilon \leq d(x_{m_i}, x_{n_i}) \leq sd(x_{m_i}, x_{m_i+2}) + sd(x_{m_i+2}, x_{n_i}).$$

Taking the upper limit as $i \to \infty$, we get

$$\frac{\varepsilon}{s} \leq \limsup_{i\to\infty} d(x_{m_i+2}, x_{n_i}).$$

Also, from (3.4) and using (b_3), we get

$$d(x_{m_i+1}, x_{n_i-1}) \leq sd(x_{m_i+1}, x_{n_i}) + sd(x_{n_i}, x_{n_i-1}).$$

Taking the upper limit as $i \to \infty$, we get

$$\limsup_{i\to\infty} d(x_{m_i+1}, x_{n_i-1}) \leq s\varepsilon.$$

From Lemma 1.5, $\alpha(x_{m_i+1}, x_{n_i-1}) \geq 1$, and so we have

$$\begin{aligned}
\theta\left(s^2 \cdot \frac{\varepsilon}{s}\right) &\leq \theta\left(s^2 \cdot \limsup_{i\to\infty} d(x_{m_i+2}, x_{n_i})\right) \\
&\leq \left[\theta\left(\limsup_{i\to\infty} M(x_{m_i+1}, x_{n_i-1})\right)\right]^k \leq \left[\theta(s\varepsilon)\right]^k
\end{aligned}$$

because

$$\begin{aligned}
&\limsup_{i\to\infty} M(x_{m_i+1}, x_{n_i-1}) \\
&\quad = \limsup_{i\to\infty} \max\left\{ d(x_{m_i+1}, x_{n_i-1}), \right.
\end{aligned}$$

$$\begin{aligned}
&\frac{d(x_{m_i+1},fx_{m_i+1})d(x_{m_i+1},fx_{n_i-1})+d(x_{n_i-1},fx_{n_i-1})d(x_{n_i-1},fx_{m_i+1})}{1+s[d(x_{m_i+1},x_{n_i-1})+d(fx_{m_i+1},fx_{n_i-1})]},\\
&\left.\frac{d(x_{m_i+1},fx_{m_i+1})d(x_{m_i+1},fx_{n_i-1})+d(x_{n_i-1},fx_{n_i-1})d(x_{n_i-1},fx_{m_i+1})}{1+d(x_{m_i+1},fx_{n_i-1})+d(x_{n_i-1},fx_{m_i+1})}\right\}\\
&=\limsup_{i\to\infty}\max\left\{d(x_{m_i+1},x_{n_i-1}),\right.\\
&\frac{d(x_{m_i+1},x_{m_i+2})d(x_{m_i+1},x_{n_i})+d(x_{n_i-1},x_{n_i})d(x_{n_i-1},x_{m_i+2})}{1+s[d(x_{m_i+1},x_{n_i-1})+d(x_{m_i+2},x_{n_i})]},\\
&\left.\frac{d(x_{m_i+1},x_{m_i+2})d(x_{m_i+1},x_{n_i})+d(x_{n_i-1},x_{n_i})d(x_{n_i-1},x_{m_i+2})}{1+d(x_{m_i+1},x_{n_i})+d(x_{n_i-1},x_{m_i+2})}\right\}\\
&\le s\varepsilon,
\end{aligned}$$

a contradiction. Therefore, in all cases $\{x_n\}$ is a b-Cauchy sequence, and hence b-completeness of X yields that $\{x_n\}$ b-converges to a point $x^* \in X$.

Remember that from (3.2) we get

$$d(x_n,x_{n+1}) \le d(x_{n-1},x_n) \tag{3.7}$$

for all $n \in \mathbb{N}$. Suppose that there exists $n_0 \in \mathbb{N}$ such that

$$\frac{1}{2s}d(x_{n_0},fx_{n_0}) > d(x_{n_0},x^*)$$

and

$$\frac{1}{2s}d(x_{n_0+1},fx_{n_0+1}) > d(x_{n_0+1},x^*).$$

Then from (3.7) we have

$$\begin{aligned}
d(x_{n_0},x_{n_0+1}) &\le s\left[d(x_{n_0},x^*)+d(x_{n_0+1},x^*)\right]\\
&< s\left[\frac{1}{2s}d(x_{n_0},fx_{n_0})+\frac{1}{2s}d(x_{n_0+1},fx_{n_0+1})\right]\\
&= \frac{1}{2}\left[d(x_{n_0},x_{n_0+1})+d(x_{n_0+1},x_{n_0+2})\right]\\
&\le \frac{1}{2}\left[d(x_{n_0},x_{n_0+1})+d(x_{n_0},x_{n_0+1})\right]=d(x_{n_0},x_{n_0+1}),
\end{aligned}$$

which is a contradiction. Hence, either

$$\frac{1}{2s}d(x_n,fx_n) \le d(x_n,x^*)$$

or

$$\frac{1}{2s}d(x_{n+1},fx_{n+1}) \le d(x_{n+1},x^*)$$

holds for all $n \in \mathbb{N}$. First, suppose that

$$\frac{1}{2s}d(x_n,fx_n) \le d(x_n,x^*)$$

holds for infinitely many values of n, say $n \in J$. Then, from (3.1), we have

$$\theta\bigl(d\bigl(fx^*,fx_n\bigr)\bigr) \leq \bigl[\theta\bigl(M\bigl(x^*,x_n\bigr)\bigr)\bigr]^k$$

for all $n \in J$ because

$$\begin{aligned} M\bigl(x^*,x_n\bigr) = \max\biggl\{ & d\bigl(x^*,x_n\bigr), \frac{d(x^*,fx^*)d(x^*,fx_n) + d(x_n,fx_n)d(x_n,fx^*)}{1 + s[d(x^*,x_n) + d(fx^*,fx_n)]}, \\ & \frac{d(x^*,fx^*)d(x^*,fx_n) + d(x_n,fx_n)d(x_n,fx^*)}{1 + d(x^*,fx_n) + d(x_n,fx^*)}\biggr\} \end{aligned}$$

for all $n \in \mathbb{N}$. Taking the limit as $n \to \infty$, with $n \in J$, in the above inequality we get that

$$\lim_{n\to\infty,n\in J} \theta\bigl(d\bigl(fx^*,fx_n\bigr)\bigr) = 1.$$

This implies that $\lim_{n\to\infty,n\in J} d(fx^*,fx_n) = 0$. Applying Lemma 3.4, we deduce that

$$fx^* = x^*.$$

By a similar method we can obtain $fx^* = x^*$ when

$$\frac{1}{2}d(x_{n+1},fx_{n+1}) \leq d\bigl(x_{n+1},x^*\bigr)$$

holds for infinitely many values of n. Hence, we proved that x^* is a fixed point of f. □

Analogously, we can prove the following theorems.

Theorem 3.7 *Let (X,d) be a b-complete b-metric space with $s > 1$, and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \quad\Longrightarrow\quad \alpha(x,y)\theta\bigl(s^2d(fx,fy)\bigr) \leq \bigl[\theta\bigl(M(x,y)\bigr)\bigr]^k$$

for all $x,y \in X$ with $fx \neq fy$, where

$$M(x,y) = \max\biggl\{d(x,y), \frac{d(x,fx)d(y,fy)}{1 + d(x,y)}, \frac{d(x,fx)d(y,fy)}{1 + d(fx,fy)}\biggr\}.$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$;*
(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

Theorem 3.8 *Let (X,d) be a b-complete b-metric space with $s > 1$, and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$ and $k \in (0,1)$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \quad\Longrightarrow\quad \alpha(x,y)\theta\bigl(s^2d(fx,fy)\bigr) \leq \bigl[\theta\bigl(M(x,y)\bigr)\bigr]^k$$

for all $x, y \in X$ with $fx \neq fy$, where

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+s[d(x,y)+d(x,fy)+d(y,fx)]}, \frac{d(x,fy)d(x,y)}{1+sd(x,fx)+s^3[d(y,fx)+d(y,fy)]}\right\}.$$

Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*

(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

The following corollaries are immediate consequences of the obtained theorems.

Corollary 3.9 *Let (X, d) be a b-complete b-metric space with $s > 1$, and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$, $k \in (0,1)$ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \implies \alpha(x,y)\theta\bigl(s^2 d(fx,fy)\bigr) \leq \left[\theta\left(\alpha d(x,y) + \beta\frac{d(x,fx)d(y,fy)}{1+d(x,y)} + \gamma\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right)\right]^k$$

for all $x, y \in X$ with $fx \neq fy$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*

(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

Corollary 3.10 *Let (X, d) be a b-complete b-metric space with $s > 1$, and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$, $k \in (0,1)$ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \implies \alpha(x,y)\theta\bigl(s^2 d(fx,fy)\bigr) \leq \left[\theta\left(\alpha d(x,y) + \beta\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,fx)+d(y,fy)]} + \gamma\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\right)\right]^k$$

for all $x, y \in X$ with $fx \neq fy$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0, fx_0) \geq 1$;*

(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n, x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n, x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

Corollary 3.11 *Let (X, d) be a b-complete b-metric space with $s > 1$, and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$, $k \in (0,1)$ and $\alpha, \beta, \gamma \in [0,1)$*

with $\alpha + \beta + \gamma < 1$ *such that*

$$\begin{aligned}&\frac{1}{2s}d(x,fx) \leq d(x,y)\\&\quad\Longrightarrow\quad \alpha(x,y)\theta\bigl(s^2 d(fx,fy)\bigr) \leq \biggl[\theta\biggl(\alpha d(x,y) + \beta\frac{d(x,fx)d(y,fy)}{1+s[d(x,y)+d(x,fy)+d(y,fx)]}\\&\qquad\qquad + \gamma\frac{d(x,fy)d(x,y)}{1+sd(x,fx)+s^3[d(y,fx)+d(y,fy)]}\biggr)\biggr]^k\end{aligned}$$

for all $x,y \in X$ *with* $fx \neq fy$. *Also, suppose that the following assertions hold*:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0,fx_0) \geq 1$;

(ii) *for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n,x_{n+1}) \geq 1$, *for all* $n \in \mathbb{N} \cup \{0\}$, *such that* $x_n \to x$ *as* $n \to \infty$, *we have* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then f *has a fixed point.*

Taking $\theta(t) = e^t$ for all $t > 0$, in the above corollaries we obtain the following new results.

Corollary 3.12 *Let* (X,d) *be a b-complete b-metric space* (*with parameter* $s > 1$), *and let* f *be a triangular* α*-admissible mapping. Suppose that there exist* $\theta \in \Theta$, $k \in (0,1)$ *and* $\alpha,\beta,\gamma \in [0,1)$ *with* $\alpha + \beta + \gamma < 1$ *such that*

$$\begin{aligned}&\frac{1}{2s}d(x,fx) \leq d(x,y)\\&\quad\Longrightarrow\quad \ln\alpha(x,y) + s^2 d(fx,fy) \leq k\biggl[\alpha d(x,y) + \beta\frac{d(x,fx)d(y,fy)}{1+d(x,y)} + \gamma\frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\biggr]\end{aligned}$$

for all $x,y \in X$ *with* $fx \neq fy$. *Also, suppose that the following assertions hold*:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0,fx_0) \geq 1$;

(ii) *for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n,x_{n+1}) \geq 1$, *for all* $n \in \mathbb{N} \cup \{0\}$, *such that* $x_n \to x$ *as* $n \to \infty$, *we have* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then f *has a fixed point.*

Corollary 3.13 *Let* (X,d) *be a b-complete b-metric space* (*with parameter* $s > 1$), *and let* f *be a triangular* α*-admissible mapping. Suppose that there exist* $\theta \in \Theta$, $k \in (0,1)$ *and* $\alpha,\beta,\gamma \in [0,1)$ *with* $\alpha + \beta + \gamma < 1$ *such that*

$$\begin{aligned}&\frac{1}{2s}d(x,fx) \leq d(x,y)\\&\quad\Longrightarrow\quad \ln\alpha(x,y) + s^2 d(fx,fy) \leq k\biggl[\alpha d(x,y) + \beta\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+s[d(x,fx)+d(y,fy)]}\\&\qquad\qquad + \gamma\frac{d(x,fx)d(x,fy)+d(y,fy)d(y,fx)}{1+d(x,fy)+d(y,fx)}\biggr]\end{aligned}$$

for all $x,y \in X$ *with* $fx \neq fy$. *Also, suppose that the following assertions hold*:

(i) *there exists* $x_0 \in X$ *such that* $\alpha(x_0,fx_0) \geq 1$;

(ii) *for any sequence* $\{x_n\}$ *in* X *with* $\alpha(x_n,x_{n+1}) \geq 1$, *for all* $n \in \mathbb{N} \cup \{0\}$, *such that* $x_n \to x$ *as* $n \to \infty$, *we have* $\alpha(x_n,x) \geq 1$ *for all* $n \in \mathbb{N} \cup \{0\}$.

Then f *has a fixed point.*

Corollary 3.14 *Let (X,d) be a b-complete b-metric space (with parameter $s > 1$), and let f be a triangular α-admissible mapping. Suppose that there exist $\theta \in \Theta$, $k \in (0,1)$ and $\alpha, \beta, \gamma \in [0,1)$ with $\alpha + \beta + \gamma < 1$ such that*

$$\begin{aligned}
&\frac{1}{2s} d(x,fx) \leq d(x,y) \\
&\quad \Longrightarrow \quad \ln \alpha(x,y) + s^2 d(fx,fy) \leq k\left[\alpha d(x,y) + \beta \frac{d(x,fx)d(y,fy)}{1 + s[d(x,y) + d(x,fy) + d(y,fx)]}\right. \\
&\qquad\qquad \left. + \gamma \frac{d(x,fy)d(x,y)}{1 + sd(x,fx) + s^3[d(y,fx) + d(y,fy)]}\right]
\end{aligned}$$

for all $x,y \in X$ with $fx \neq fy$. Also, suppose that the following assertions hold:

(i) *there exists $x_0 \in X$ such that $\alpha(x_0,fx_0) \geq 1$;*

(ii) *for any sequence $\{x_n\}$ in X with $\alpha(x_n,x_{n+1}) \geq 1$, for all $n \in \mathbb{N} \cup \{0\}$, such that $x_n \to x$ as $n \to \infty$, we have $\alpha(x_n,x) \geq 1$ for all $n \in \mathbb{N} \cup \{0\}$.*

Then f has a fixed point.

The following example supports our results.

Example 3.15 Let $X = \mathbb{R}$. Define a metric d on X by $d(x,y) = (x - y)^2$. Clearly, (X,d) is a complete b-metric space, with $s = 2$. Also, let $k = \frac{1}{2}$ and define $f : X \to X$, $\theta : (0,\infty) \to (1,\infty)$ and $\alpha : X \times X \to [0,\infty)$ by

$$f(x) = \begin{cases} \frac{1}{4}x & \text{if } x \in \{0,4,5\}, \\ 2x, & \text{otherwise}, \end{cases} \qquad \theta(t) = \cosh t,$$

$$\alpha(x,y) = \begin{cases} 1 & \text{if } x,y \in \{0,4,5\}, \\ 0, & \text{otherwise}. \end{cases}$$

First, we assume that $\frac{1}{4}d(x,fx) \leq d(x,y)$ and $\alpha(x,y) \geq 1$ with $fx \neq fy$. Then,

$$(x,y) \in \bigl\{(0,4),(0,5),(4,0),(4,5),(5,0),(5,4)\bigr\}.$$

Now, we consider the following cases:

- Let $(x,y) = (0,4)$, then

$$\begin{aligned}
\alpha(x,y)\theta\bigl(s^2 d(fx,fy)\bigr) &= \cosh\bigl(4 \cdot d(0,1)\bigr) = 27.3082328 \\
&\leq \sqrt{\cosh(16)} = \sqrt{4443055.26} \\
&= 2107.85561 \leq \bigl[\theta\bigl(M(0,4)\bigr)\bigr]^k.
\end{aligned}$$

- Let $(x,y) = (0,5)$, then

$$\begin{aligned}
\alpha(x,y)\theta\bigl(s^2 d(fx,fy)\bigr) &= \cosh\left(4 \cdot d\left(0, \frac{5}{4}\right)\right) = 259.007378 \\
&\leq \sqrt{\cosh(25)} = \sqrt{3.60024497 \times 10^{10}} \\
&= 189743.115 \leq \bigl[\theta\bigl(M(0,5)\bigr)\bigr]^k.
\end{aligned}$$

- Let $(x,y) = (4,5)$, then

$$\begin{aligned}\alpha(x,y)\theta\left(s^2 d(fx,fy)\right) &= \cosh\left(4\cdot d\left(1,\frac{5}{4}\right)\right) = 1.0314131 \\ &\leq \sqrt{\cosh(1)} = \sqrt{1.54308063} \\ &= 1.24220797 \leq \left[\theta\left(M(4,5)\right)\right]^k.\end{aligned}$$

We deduce that

$$\frac{1}{2s}d(x,fx) \leq d(x,y) \quad \Longrightarrow \quad \alpha(x,y)\theta\left(s^2 d(fx,fy)\right) \leq \left[\theta\left(M(x,y)\right)\right]^k$$

for all $x, y \in X$ with $fx \neq fy$, where (we recall)

$$M(x,y) = \max\left\{d(x,y), \frac{d(x,fx)d(y,fy)}{1+d(x,y)}, \frac{d(x,fx)d(y,fy)}{1+d(fx,fy)}\right\}.$$

Therefore, all conditions of Theorem 3.7 hold true and f has a fixed point. Here, $x = 0$ is a fixed point of f.

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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