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Nonlinear contractions involving simulation functions in a metric space with a partial order

Hajer Argoubi[a], Bessem Samet[b,*], Calogero Vetro[c]

[a] *FST Campus Universitaire, 2092-El Manar Tunis, Tunisia.*
[b] *Department of Mathematics, College of Science, King Saud University, P.O. Box 2455, Riyadh 11451, Saudi Arabia.*
[c] *Dipartimento di Matematica e Informatica, Università degli Studi di Palermo, via Archirafi 34, 90123 Palermo, Italy.*

Abstract

Very recently, Khojasteh, Shukla and Radenović [F. Khojasteh, S. Shukla, S. Radenović, Filomat, 29 (2015), 1189–1194] introduced the notion of $\mathcal{Z}$-contraction, that is, a nonlinear contraction involving a new class of mappings namely simulation functions. This kind of contractions generalizes the Banach contraction and unifies several known types of nonlinear contractions. In this paper, we consider a pair of nonlinear operators satisfying a nonlinear contraction involving a simulation function in a metric space endowed with a partial order. For this pair of operators, we establish coincidence and common fixed point results. As applications, several related results in fixed point theory in a metric space with a partial order are deduced.

Keywords: Partial order, nonlinear contraction, coincidence point, common fixed point, simulation function.
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1. Introduction

Fixed point theory is a very useful tool for several areas of mathematical analysis and its applications. Loosely speaking, there are three principal categories in this theory: the metric, the topological and the order-theoretic approach, where fundamental examples of these are: Banach's, Brouwer's and Tarski's theorems respectively.

In recent years, many results appeared related to metric fixed point theory in partially ordered sets. The first work in this direction was the 2004 paper of Ran and Reurings [11], where they established a fixed

*Corresponding author
Email addresses: hajer1004@yahoo.fr (Hajer Argoubi), bsamet@ksu.edu.sa (Bessem Samet), calogero.vetro@unipa.it (Calogero Vetro)

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point result, which can be considered as a combination of two fixed point theorems: Banach contraction principle and Knaster-Tarski fixed point theorem. Further, several results appeared in this direction, we mention [1, 2, 3, 4, 5, 7, 8, 9, 10, 13] and the references therein.

Very recently, Khojasteh, Shukla and Radenović [6] introduced the notion of $\mathcal{Z}$-contraction, that is, a nonlinear contraction involving a new class of mappings namely simulation functions. They studied the existence and uniqueness of fixed points for $\mathcal{Z}$-contraction type operators. This class of $\mathcal{Z}$-contractions includes a large types of nonlinear contractions existing in the literature. Thus, it is possible to treat several fixed point problems from a unique, common point of view.

In [12], Roldán *et al.* studied the existence and uniqueness of coincidence points of a pair of nonlinear operators satisfying a certain contraction involving simulation functions.

In this paper, we consider a pair of nonlinear operators satisfying a nonlinear contraction involving a simulation function in a metric space endowed with a partial order. For this kind of contractions, we establish coincidence and common fixed point results. As applications, several related results in fixed point theory in a metric space with a partial order are deduced.

2. The class of simulation functions

The class of simulation functions was introduced by Khojasteh *et al.* in [6] as follows.

Definition 2.1. A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the following conditions:

(ζ_1) $\zeta(0,0) = 0$;

(ζ_2) $\zeta(t,s) < s - t$, for all $t, s > 0$;

(ζ_3) if $\{t_n\}$, $\{s_n\}$ are sequences in $(0,\infty)$ such that $\lim_{n\to\infty} t_n = \lim_{n\to\infty} s_n = \ell \in (0,\infty)$, then

$$\limsup_{n\to\infty} \zeta(t_n, s_n) < 0.$$

The main result in [6] is the following.

Theorem 2.2. *Let (X,d) be a complete metric space and $T : X \to X$ be a $\mathcal{Z}$-contraction with respect to a certain simulation function ζ, that is,*

$$\zeta(d(Tx,Ty), d(x,y)) \geq 0, \text{ for all } x, y \in X. \tag{2.1}$$

Then T has a unique fixed point. Moreover, for every $x_0 \in X$, the Picard sequence $\{T^n x_0\}$ converges to this fixed point.

Note that the condition (ζ_1) was not used for the proof of Theorem 2.2. However, taking $x = y$ in (2.1), we obtain $\zeta(0,0) \geq 0$. So if $\zeta(0,0) < 0$, then the set of operators $T : X \to X$ satisfying (2.1) will be empty.

Taking in consideration the above remark, we slightly modify the previous definition by removing the condition (ζ_1). So the following notion will be used throughout this paper.

Definition 2.3. A simulation function is a mapping $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfying the conditions (ζ_2) and (ζ_3).

Clearly, any simulation function in the original Khojasteh *et al.* sense (Definition 2.1) is also a simulation function in our sense (Definition 2.3), but the converse is not true, as we show in the following example.

Example 2.4. Let $\zeta_\lambda : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by

$$\zeta_\lambda(t,s) = \begin{cases} 1 & \text{if } (s,t) = (0,0), \\ \lambda s - t & \text{otherwise,} \end{cases}$$

where $\lambda \in (0,1)$. Then ζ_λ satisfies (ζ_2) and (ζ_3) with $\zeta_\lambda(0,0) > 0$.

3. Coincidence and common fixed points via simulation functions

Let (X,d) be a complete metric space. We suppose that the set X is endowed with a partial order $\preceq$. We recall the following definitions.

Definition 3.1 ([3])**.** Let $f,g : X \to X$ be two given mappings. We say that f is g-non-decreasing if

$$(x,y) \in X \times X,\ gx \preceq gy \Longrightarrow fx \preceq fy.$$

Definition 3.2. Let $f,g : X \to X$ be two given mappings. We say that $x \in X$

- is a coincidence point of f and g if $fx = gx$;
- is a common fixed point of f and g if $x = fx = gx$;
- is a fixed point of f if $fx = x$.

Now, we present our first main result in this paper.

Theorem 3.3. *Let $f,g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ *is closed;*

(iii) f *is g-non-decreasing;*

(iv) *there exists $x_0 \in X$ with $gx_0 \preceq fx_0$;*

(v) *if $\{gx_n\} \subset X$ is a non-decreasing sequence (w.r.t. $\preceq$) with $gx_n \to gz$ in $g(X)$, then $gz \preceq g(gz)$ and $gx_n \preceq gz$, for all $n \in \mathbb{N}$;*

(vi) *there exists a simulation function ζ such that for every $(x,y) \in X \times X$ with $gx \preceq gy$, we have*

$$\zeta\left(d(fx,fy), M(f,g,x,y)\right) \geq 0,$$

where

$$M(f,g,x,y) = \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\right\}.$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

In order to prove Theorem 3.3, some lemmas are needed.

Lemma 3.4. *Suppose that all the assumptions of Theorem 3.3 are satisfied. Let $\{x_n\}$ be a sequence in X such that*

$$gx_{n+1} = fx_n, \text{ for all } n \in \mathbb{N}. \tag{3.1}$$

Suppose that $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Then

$$\lim_{n\to\infty} d(gx_n, gx_{n+1}) = 0.$$

Proof. At first, observe that from (iii) and (iv), we have

$$gx_0 \preceq gx_1 \preceq \cdots \preceq gx_n \preceq gx_{n+1} \preceq \cdots \tag{3.2}$$

It follows from (3.2) and (vi) that for all $n \geq 1$, we have

$$\begin{aligned} 0 &\leq \zeta(d(fx_{n-1}, fx_n), M(f, g, x_{n-1}, x_n)) \\ &= \zeta(d(gx_n, gx_{n+1}), M(f, g, x_{n-1}, x_n)). \end{aligned}$$

Moreover, for all $n \geq 1$, we have

$$M(f, g, x_{n-1}, x_n) = \max\left\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1}), \frac{d(gx_{n-1}, gx_{n+1})}{2}\right\}.$$

The triangle inequality yields

$$\frac{d(gx_{n-1}, gx_{n+1})}{2} \leq \max\left\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\right\}.$$

Thus

$$M(f, g, x_{n-1}, x_n) = \max\left\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\right\}, \ n \geq 1.$$

Therefore, from condition (ζ_2), we have

$$\begin{aligned} 0 &\leq \zeta\left(d(gx_n, gx_{n+1}), \max\left\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\right\}\right) \\ &< \max\left\{d(gx_{n-1}, gx_n), d(gx_n, gx_{n+1})\right\} - d(gx_n, gx_{n+1}), \end{aligned}$$

for all $n \geq 1$. The above inequality shows that

$$M(f, g, x_{n-1}, x_n) = d(gx_{n-1}, gx_n), \ n \geq 1,$$

which implies that $\{d(gx_{n-1}, gx_n)\}$ is a monotonically decreasing sequence of non-negative real numbers. So there is some $r \geq 0$ such that

$$\lim_{n\to\infty} d(gx_{n-1}, gx_n) = r.$$

Suppose that $r > 0$. It follows from the condition (ζ_3) that

$$0 \leq \limsup_{n\to\infty} \zeta\left(d(gx_n, gx_{n+1}), d(gx_{n-1}, gx_n)\right) < 0,$$

which is a contradiction. Then we conclude that $r = 0$, which ends the proof. □

Lemma 3.5. *Suppose that all the assumptions of Theorem 3.3 are satisfied. Let $\{x_n\}$ be a sequence in X such that* (3.1) *holds with $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Then $\{gx_n\}$ is bounded.*

Proof. Let us assume that $\{gx_n\}$ is not a bounded sequence. Then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $n_1 = 1$ and for each $k \in \mathbb{N}$, n_{k+1} is the minimum integer such that

$$d(gx_{n_{k+1}}, gx_{n_k}) > 1$$

and

$$d(gx_m, gx_{n_k}) \leq 1, \text{ for } n_k \leq m \leq n_{k+1} - 1.$$

By the triangle inequality, we obtain

$$\begin{aligned} 1 < d(gx_{n_{k+1}}, gx_{n_k}) &\leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + d(gx_{n_{k+1}-1}, gx_{n_k}) \\ &\leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + 1. \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using Lemma 3.4, we get

$$\lim_{k\to\infty} d(gx_{n_{k+1}}, gx_{n_k}) = 1. \tag{3.3}$$

Using the triangle inequality, we get

$$\begin{aligned} 1 < d(gx_{n_{k+1}}, gx_{n_k}) &\leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + d(gx_{n_{k+1}-1}, gx_{n_k}) \\ &\leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + d(gx_{n_{k+1}-1}, gx_{n_k-1}) + d(gx_{n_k}, gx_{n_k-1}) \\ &\leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + d(gx_{n_{k+1}-1}, gx_{n_k}) + 2d(gx_{n_k}, gx_{n_k-1}) \\ &\leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}) + 1 + 2d(gx_{n_k}, gx_{n_k-1}). \end{aligned}$$

Letting $k \to \infty$ in the above inequality and using Lemma 3.4, we get

$$\lim_{k\to\infty} d(gx_{n_{k+1}-1}, gx_{n_k-1}) = 1. \tag{3.4}$$

Again, the triangle inequality yields

$$|d(gx_{n_{k+1}-1}, gx_{n_k}) - d(gx_{n_k}, gx_{n_{k+1}})| \leq d(gx_{n_{k+1}-1}, gx_{n_{k+1}}).$$

Letting $k \to \infty$ in the above inequality, using Lemma 3.4 and (3.3), we obtain

$$\lim_{k\to\infty} d(gx_{n_{k+1}-1}, gx_{n_k}) = 1. \tag{3.5}$$

By a similar way, we have

$$|d(gx_{n_k-1}, gx_{n_{k+1}}) - d(gx_{n_k-1}, gx_{n_{k+1}-1})| \leq d(gx_{n_{k+1}}, gx_{n_{k+1}-1}).$$

Letting $k \to \infty$ in the above inequality, using Lemma 3.4 and (3.4), we obtain

$$\lim_{k\to\infty} d(gx_{n_k-1}, gx_{n_{k+1}}) = 1. \tag{3.6}$$

Now, using Lemma 3.4, (3.3), (3.4), (3.5) and (3.6), we obtain

$$\lim_{k\to\infty} M(f, g, x_{n_{k+1}-1}, x_{n_k-1}) = 1. \tag{3.7}$$

Using (vi), (3.2), (3.3), (3.7) and the condition (ζ_3), we obtain

$$0 \leq \limsup_{k\to\infty} \zeta\left(d(gx_{n_{k+1}}, gx_{n_k}), M(f, g, x_{n_{k+1}-1}, x_{n_k-1})\right) < 0,$$

which is a contradiction. This ends the proof. □

Lemma 3.6. *Suppose that all the assumptions of Theorem 3.3 are satisfied. Let $\{x_n\}$ be a sequence in X such that* (3.1) *holds with $gx_n \neq gx_{n+1}$ for all $n \in \mathbb{N}$. Then $\{gx_n\}$ is a Cauchy sequence.*

Proof. Let

$$C_n = \sup\{d(gx_i, gx_j) : i, j \geq n\}, \ n \in \mathbb{N}.$$

From Lemma 3.5, we know that $C_n < \infty$ for every $n \in \mathbb{N}$. Since $\{C_n\}$ is a positive monotonically decreasing sequence, there is some $C \geq 0$ such that

$$\lim_{n\to\infty} C_n = C.$$

Let us suppose that $C > 0$. By the definition of C_n, for every $k \in \mathbb{N}$ ($k \geq 1$), there exists $n_k, m_k \in \mathbb{N}$ such that $m_k > n_k \geq k$ and

$$C_k - \frac{1}{k} < d(gx_{m_k}, gx_{n_k}) \leq C_k.$$

Letting $k \to \infty$ in the above inequality, we get

$$\lim_{k\to\infty} d(gx_{m_k}, gx_{n_k}) = C. \tag{3.8}$$

By the triangle inequality, we have

$$|d(gx_{m_k}, gx_{n_k}) - d(gx_{m_k-1}, gx_{n_k-1})| \leq d(gx_{m_k}, gx_{m_k-1}) + d(gx_{n_k}, gx_{n_k-1}).$$

Letting $k \to \infty$ in the above inequality, using (3.8) and Lemma 3.4, we get

$$\lim_{k\to\infty} d(gx_{m_k-1}, gx_{n_k-1}) = C. \tag{3.9}$$

Similarly, we have

$$\lim_{k\to\infty} d(gx_{m_k-1}, gx_{n_k}) = C \tag{3.10}$$

and

$$\lim_{k\to\infty} d(gx_{n_k-1}, gx_{m_k}) = C. \tag{3.11}$$

Using Lemma 3.4, (3.9), (3.10) and (3.11), we obtain

$$\lim_{k\to\infty} M(f, g, x_{m_k-1}, x_{n_k-1}) = C. \tag{3.12}$$

By (vi), (3.8), (3.12) and the condition (ζ_3), we get

$$0 \leq \limsup_{k\to\infty} \zeta(d(gx_{m_k}, gx_{n_k}), M(f, g, x_{m_k-1}, x_{n_k-1})) < 0,$$

which is a contradiction. Thus we have $C = 0$, that is,

$$\lim_{n\to\infty} C_n = 0.$$

This proves that $\{gx_n\}$ is a Cauchy sequence. □

Now, we are able to prove our main result given by Theorem 3.3.

Proof. At first, observe that if $gx_p = gx_{p+1}$ for some $p \in \mathbb{N}$, then $gx_p = fx_p$, that is, x_p is a coincidence point of f and g. In this case, the existence of a coincidence point is proved. So, we can suppose that $gx_n \neq gx_{n+1}$ for every $n \in \mathbb{N}$.

Since $g(X)$ is closed and (X, d) is complete, by Lemma 3.6, there exists some $z \in X$ such that

$$gx_n \to gz \text{ as } n \to \infty. \tag{3.13}$$

Now, we show that z is a coincidence point of f and g. Suppose that $d(gz, fz) > 0$. For all $n \in \mathbb{N}$, we have

$$M(f, g, x_n, z) = \max\left\{d(gx_n, gz), d(gx_n, gx_{n+1}), d(gz, fz), \frac{d(gx_n, fz) + d(gz, gx_{n+1})}{2}\right\}.$$

Letting $n \to \infty$ and using (3.13), we get

$$\lim_{n\to\infty} M(f, g, x_n, z) = d(gz, fz) > 0. \tag{3.14}$$

On the other hand, by (v), (vi), (3.13), (3.14) and the condition (ζ_3), we have

$$0 \le \limsup_{n\to\infty} \zeta(d(gx_{n+1}, fz), M(f, g, x_n, z)) < 0,$$

which is a contradiction. Thus we have $d(gz, fz) = 0$, and z is a coincidence point of f and g.

Suppose now that f and g commute at z. Set

$$w = gz = fz.$$

Then

$$fw = f(gz) = g(fz) = gw.$$

By (v), we have

$$gz \preceq g(gz) = gw.$$

On the other hand,

$$M(f, g, w, z) = d(w, gw).$$

Suppose that $d(w, gw) > 0$. Using (vi) and the condition (ζ_3), we obtain

$$0 \le \zeta(d(fw, fz), M(f, g, w, z)) = \zeta(d(w, gw), d(w, gw)) < 0,$$

which is a contradiction. Thus we have

$$w = gw = fw,$$

and w is a common fixed point of f and g. This ends the proof of Theorem 3.3. □

If g is the identity mapping, we obtain from Theorem 3.3 the following fixed point result.

Theorem 3.7. *Let $f : X \to X$ be a given mapping. Suppose that the following conditions hold:*

(i) $(x, y) \in X \times X,\ x \preceq y \Longrightarrow fx \preceq fy$;

(ii) *there exists $x_0 \in X$ with $x_0 \preceq fx_0$;*

(iii) *if $\{x_n\} \subset X$ is a non-decreasing sequence with $x_n \to z$, then $x_n \preceq z$, for all $n \in \mathbb{N}$;*

(iv) *there exists a simulation function ζ such that for every $(x, y) \in X \times X$ with $x \preceq y$, we have*

$$\zeta\left(d(fx, fy), M(f, x, y)\right) \ge 0,$$

where

$$M(f, x, y) = \max\left\{d(x, y), d(x, fx), d(y, fy), \frac{d(x, fy) + d(y, fx)}{2}\right\}.$$

Then $\{f^n x_0\}$ converges to a fixed point of f.

4. Coincidence and common fixed points via right-monotone simulation functions

Suppose now that $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ satisfies the following additional condition:
(ζ_4) for every $t \geq 0$, we have

$$s_1, s_2 \geq 0,\ s_1 \leq s_2 \Longrightarrow \zeta(t,s_1) \leq \zeta(t,s_2).$$

In this case, we say that ζ is a right-monotone simulation function.

Example 4.1. Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by

$$\zeta(t,s) = |\sin s| - t,\ \text{for all } t,s \geq 0.$$

Then ζ is a simulation function but it is not a right-monotone simulation function.

Example 4.2. Let $\zeta : [0,\infty) \times [0,\infty) \to \mathbb{R}$ be the function defined by

$$\zeta(t,s) = s - \frac{t+2}{t+1}t,\ \text{for all } t,s \geq 0.$$

Then ζ is a right-monotone simulation function.

From Theorem 3.3, we can deduce various coincidence and common fixed point results via right-monotone simulation functions.

Corollary 4.3. *Let $f,g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$*;*

(ii) $g(X)$ *is closed;*

(iii) f *is g-non-decreasing;*

(iv) *there exists $x_0 \in X$ with $gx_0 \preceq fx_0$;*

(v) *if $\{gx_n\} \subset X$ is a non-decreasing sequence (w.r.t. $\preceq$) with $gx_n \to gz$ in $g(X)$, then $gz \preceq g(gz)$ and $gx_n \preceq gz$, for all $n \in \mathbb{N}$;*

(vi) *there exists a right-monotone simulation function ζ such that for every $(x,y) \in X \times X$ with $gx \preceq gy$, we have*

$$\zeta\left(d(fx,fy), d(gx,gy)\right) \geq 0.$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Proof. Observe that

$$d(gx,gy) \leq M(f,g,x,y),$$

for all $x,y \in X$. Since ζ is a right-monotone simulation function, then

$$\zeta\left(d(fx,fy), d(gx,gy)\right) \geq 0 \Longrightarrow \zeta\left(d(fx,fy), M(f,g,x,y)\right) \geq 0.$$

Therefore the result follows from Theorem 3.3. $\square$

Similarly, we can deduce the following results.

Corollary 4.4. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ *is closed;*

(iii) f *is* g*-non-decreasing;*

(iv) *there exists* $x_0 \in X$ *with* $gx_0 \preceq fx_0$;

(v) *if* $\{gx_n\} \subset X$ *is a non-decreasing sequence (w.r.t.* $\preceq$*) with* $gx_n \to gz$ *in* $g(X)$*, then* $gz \preceq g(gz)$ *and* $gx_n \preceq gz$*, for all* $n \in \mathbb{N}$;

(vi) *there exists a right-monotone simulation function* ζ *such that for every* $(x, y) \in X \times X$ *with* $gx \preceq gy$*, we have*
$$\zeta\left(d(fx, fy), \max\{d(gx, fx), d(gy, fy)\}\right) \geq 0.$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Corollary 4.5. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$;

(ii) $g(X)$ *is closed;*

(iii) f *is* g*-non-decreasing;*

(iv) *there exists* $x_0 \in X$ *with* $gx_0 \preceq fx_0$;

(v) *if* $\{gx_n\} \subset X$ *is a non-decreasing sequence (w.r.t.* $\preceq$*) with* $gx_n \to gz$ *in* $g(X)$*, then* $gz \preceq g(gz)$ *and* $gx_n \preceq gz$*, for all* $n \in \mathbb{N}$;

(vi) *there exists a right-monotone simulation function* ζ *such that for every* $(x, y) \in X \times X$ *with* $gx \preceq gy$*, we have*
$$\zeta\left(d(fx, fy), \max\{d(gx, gy), d(gx, fx), d(gy, fy)\}\right) \geq 0.$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Note that the above results (Corollaries 4.3, 4.4 and 4.5) can be established independently for any simulation function that is not necessarily right-monotone.

Example 4.6. Let $X = [0, \infty)$ be endowed with the metric $d : X \times X \to \mathbb{R}$ given by

$$d(x, y) = \begin{cases} 0 & \text{if } x = y, \\ \max\{x, y\} & \text{if } x \neq y. \end{cases}$$

Now, consider the usual order of real numbers and define the mappings $f, g : X \to X$ by $fx = x$ and $gx = 2x$, for all $x \in X$. Clearly, by above definitions, conditions (i)-(v) of Corollary 4.3 hold true, with $x_0 = 0$. Next, let $\zeta : X \times X \to \mathbb{R}$ be given by

$$\zeta(t, s) = s - \frac{t+2}{t+1}t.$$

Then, we have

$$\zeta(d(fx, fy), d(gx, gy)) = 2y - \frac{y+2}{y+1}y = \frac{2y(y+1) - y(y+2)}{y+1} = \frac{y^2}{y+1} \geq 0$$

for every $(x, y) \in X \times X$, with $x \leq y$. Thus, by an application of Corollary 4.3, we get that f and g have a coincidence point, say $z = 0$. Also, since f and g commute at z, then f and g have a common fixed point.

5. Applications

In this section, as applications, we obtain some results in fixed point theory in partially ordered metric spaces via specific choices of simulation functions.

Let (X, d) be a complete metric space. We suppose that the set X is endowed with a partial order $\preceq$.

Corollary 5.1. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is closed;*

(iii) *f is g-non-decreasing;*

(iv) *there exists $x_0 \in X$ with $gx_0 \preceq fx_0$;*

(v) *if $\{gx_n\} \subset X$ is a non-decreasing sequence (w.r.t. $\preceq$) with $gx_n \to gz$ in $g(X)$, then $gz \preceq g(gz)$ and $gx_n \preceq gz$, for all $n \in \mathbb{N}$;*

(vi) *there exists some $k \in (0, 1)$ such that for every $(x, y) \in X \times X$ with $gx \preceq gy$, we have*

$$d(fx, fy) \leq k \max\left\{d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2}\right\}.$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Proof. The result follows from Theorem 3.3 by taking as simulation function

$$\zeta(t, s) = ks - t,$$

for all $t, s \geq 0$. □

Corollary 5.2. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is closed;*

(iii) *f is g-non-decreasing;*

(iv) *there exists $x_0 \in X$ with $gx_0 \preceq fx_0$;*

(v) *if $\{gx_n\} \subset X$ is a non-decreasing sequence (w.r.t. $\preceq$) with $gx_n \to gz$ in $g(X)$, then $gz \preceq g(gz)$ and $gx_n \preceq gz$, for all $n \in \mathbb{N}$;*

(vi) *there exists a lower semi-continuous function $\varphi : [0, \infty) \to [0, \infty)$ with $\varphi^{-1}(0) = \{0\}$ such that for every $(x, y) \in X \times X$ with $gx \preceq gy$, we have*

$$\begin{aligned} d(fx, fy) \leq \max &\left\{ d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2} \right\} \\ &- \varphi\left(\max\left\{ d(gx, gy), d(gx, fx), d(gy, fy), \frac{d(gx, fy) + d(gy, fx)}{2} \right\} \right). \end{aligned}$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Proof. The result follows from Theorem 3.3 by taking as simulation function

$$\zeta(t, s) = s - \varphi(s) - t,$$

for all $t, s \geq 0$. □

Corollary 5.3. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is closed;*

(iii) *f is g-non-decreasing;*

(iv) *there exists $x_0 \in X$ with $gx_0 \preceq fx_0$;*

(v) *if $\{gx_n\} \subset X$ is a non-decreasing sequence (w.r.t. $\preceq$) with $gx_n \to gz$ in $g(X)$, then $gz \preceq g(gz)$ and $gx_n \preceq gz$, for all $n \in \mathbb{N}$;*

(vi) *there exists a function $\varphi : [0, \infty) \to [0, 1)$ with $\lim_{t \to r^+} \varphi(t) < 1$ for all $r > 0$ such that for every $(x, y) \in X \times X$ with $gx \preceq gy$, we have*

$$d(fx,fy) \leq \varphi\left(\max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\right\}\right)$$
$$\max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\right\}.$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Proof. The result follows from Theorem 3.3 by taking as simulation function

$$\zeta(t,s) = s\varphi(s) - t,$$

for all $t, s \geq 0$. □

Corollary 5.4. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is closed;*

(iii) *f is g-non-decreasing;*

(iv) *there exists $x_0 \in X$ with $gx_0 \preceq fx_0$;*

(v) *if $\{gx_n\} \subset X$ is a non-decreasing sequence (w.r.t. $\preceq$) with $gx_n \to gz$ in $g(X)$, then $gz \preceq g(gz)$ and $gx_n \preceq gz$, for all $n \in \mathbb{N}$;*

(vi) *there exists an upper semi-continuous function $\eta : [0,\infty) \to [0,\infty)$ with $\eta(t) < t$ for all $t > 0$ and $\eta(0) = 0$ such that for every $(x,y) \in X \times X$ with $gx \preceq gy$, we have*

$$d(fx,fy) \leq \eta\left(\max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\right\}\right).$$

Then f and g have a coincidence point. Further, if f and g commute at their coincidence points, then f and g have a common fixed point.

Proof. The result follows from Theorem 3.3 by taking as simulation function

$$\zeta(t,s) = \eta(s) - t,$$

for all $t, s \geq 0$. □

Corollary 5.5. *Let $f, g : X \to X$ be two given mappings. Suppose that the following conditions hold:*

(i) *$f(X) \subseteq g(X)$;*

(ii) *$g(X)$ is closed;*

(iii) *f is g-non-decreasing;*

(iv) *there exists* $x_0 \in X$ *with* $gx_0 \preceq fx_0$*;*

(v) *if* $\{gx_n\} \subset X$ *is a non-decreasing sequence (w.r.t.* $\preceq$*) with* $gx_n \to gz$ *in* $g(X)$*, then* $gz \preceq g(gz)$ *and* $gx_n \preceq gz$*, for all* $n \in \mathbb{N}$*;*

(vi) *there exists a function* $\phi : [0,\infty) \to [0,\infty)$ *with* $\phi \in L^1_{loc}[0,\infty)$ *and*

$$\int_0^{\varepsilon} \phi(u)\, du > \varepsilon,$$

for every $\varepsilon > 0$*, such that for every* $(x,y) \in X \times X$ *with* $gx \preceq gy$*, we have*

$$\int_0^{d(fx,fy)} \phi(u)\, du \le \max\left\{d(gx,gy), d(gx,fx), d(gy,fy), \frac{d(gx,fy)+d(gy,fx)}{2}\right\}.$$

Then f *and* g *have a coincidence point. Further, if* f *and* g *commute at their coincidence points, then* f *and* g *have a common fixed point.*

Proof. The result follows from Theorem 3.3 by taking as simulation function

$$\zeta(t,s) = s - \int_0^t \phi(u)\, du,$$

for all $t, s \ge 0$. □

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