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Probabilistic characterization of nonlinear systems under α-stable white noise via complex fractional moments

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HIGHLIGHTS

- PDF of nonlinear systems under α-stable white noise is obtained in terms of CFMs.
- The Mellin Transform (MT) is applied to the Fractional Fokker–Planck equation.
- Relationship between MT for different values of parameter of MT is found.
- The PDF of the response is found for any power-law nonlinearities of the system.
- The accuracy of the PDF on the tails is guaranteed.

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ABSTRACT

The probability density function of the response of a nonlinear system under external α-stable Lévy white noise is ruled by the so called Fractional Fokker–Planck equation. In such equation the diffusive term is the Riesz fractional derivative of the probability density function of the response. The paper deals with the solution of such equation by using the complex fractional moments. The analysis is performed in terms of probability density for a linear and a non-linear half oscillator forced by Lévy white noise with different stability indexes α. Numerical results are reported for a wide range of non-linearity of the mechanical system and stability index of the Lévy white noise.

1. Introduction

The generalization of the Brownian motion $B(t)$ is the so-called α-stable Lévy motion $L_\alpha(t)$ $(0 < \alpha \leq 2)$ [1]. For $\alpha = 2$, $L_\alpha(t)$ reverts to the Brownian motion $B(t)$. $L_\alpha(t)$ and $B(t)$ experience some common features: (i) both have independent and orthogonal stationary increments with $L_\alpha(0) = 0$, $B(0) = 0$; (ii) both have continuous trajectories nowhere differentiable on time. The main differences are: (i) $B(t)$ is Gaussian while the Lévy motion $L_\alpha(t)$ is not; (ii) the mean square of $L_\alpha(t)$ no longer grows linearly in time as the Brownian motion. The smaller the α the greater the departure from Gaussianity is so generating a large variety of motions such as subdiffusive $(0 < \alpha < 1)$ or superdiffusive $(1 \leq \alpha \leq 2)$ ones (for more details see Ref. [2]).

The (formal) derivative of the Lévy motion gives the α-stable white noise. For Langevin differential equations enforced by normal white noise (formal derivative of the Brownian motion) the probability density function is ruled by the well known Fokker–Planck–Kolmogorov (FPK) equation [3]. If the input is the α-stable white noise, then the PDF of the response process is the so called Fractional Fokker–Planck (FFP) equation [4–6]. These equations differ from each other in the diffusive term.

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In the former case the diffusive term is the second derivative of the PDF of the response while in the latter the diffusive term is the Riesz fractional derivative of order α of the PDF. Since for $\alpha = 2$ the Riesz fractional derivative coalesces with the second derivative, then it may be concluded that the FFP equation is the generalization of the FPK equation. The fractional derivative reflects the nonlocal character in space of the diffusive term.

Various methods for finding approximate solution of the FPK equation have been proposed in the literature including path integral solution [7–11], Wiener Path integral method [12], stochastic averaging method [13,14], finite element method [15,16], and spectral stochastic finite element method [17]. Comparatively few papers have been devoted to the solution of the FFP equation see e.g. [4,7,18–25]. And in these papers the PDF of the response is given in exact form only for steady state solution and for particular classes of nonlinearities.

Recently the second author proposed an approximate method for finding the solution of the FPK equation in terms of Complex Fractional Moments (CFM) [26] that are moments of the type $E\left[|X|^{\gamma-1}\right]$, $\gamma \in \mathbb{C}$. These complex quantities are related to the Mellin transform of the PDF [27–29]. The appeal in using such quantities instead of integer or fractional moments with real exponent relies on the following points: (i) moments of the type $E\left[|X|^{\gamma-1}\right]$ never diverge provided the real parts of γ belong to the fundamental strip of the Mellin transform even for α-stable processes; (ii) both the PDF and Characteristic function are fully restored in the respective domains by inverse Mellin Transform theorem. Because of these remarkable properties in this paper the solution of the FFP equation in terms of complex fractional moments is proposed. The method is available for any kind of power law nonlinearity in the equation of the mechanical system as well as for non stationary input–output response.

2. Mellin transform and complex fractional moments

Let $p_X(x,t)$ be the probability density function of the random process $X(t)$. Let us now suppose that $p_X(x,t)$ is a symmetric distribution, that is $p_X(x,t) = p_X(-x,t)\,\forall t$.

The Mellin transform of $p_X(x,t)$ is:

$$\mathcal{M}\left\{p_X(x,t);\gamma\right\} = \int_0^\infty p_X(x,t)x^{\gamma-1}dx; \quad \gamma = \rho + I\eta. \tag{1}$$

In the following we denote the Mellin Transform of $p_X(x)$ as $M_p(\gamma-1,t)$. The inverse Mellin transform restitutes $p_X(x,t)$ in the form

$$p_X(x,t) = \frac{1}{2\pi i}\int_{\rho-i\infty}^{\rho+i\infty} M_p(\gamma-1)x^{-\gamma}d\gamma; \quad x>0. \tag{2}$$

It is to be emphasized that the integration is performed along the imaginary axis while ρ remains fixed. The condition for the existence of Eqs. (1) and (2) depends on the trend of $p_X(x,t)$ at $x=0$ and $x=\infty$. This condition is guaranteed if $-p<\rho<-q$ where p and q are the order of zero at $x=0$ and $x=\infty$ on the $p_X(x,t)$. Such an example if $p_X(x,t)$ is an α-stable distribution, then since the decay of the PDF is for x very large in the form $\tilde{x}^{-\alpha-1}(x>0)$, the order of zero at infinity is $-q=\alpha+1$, in zero the PDF is in general different from zero (order x^0) then for an α-stable distribution the existence condition of Eqs. (1) and (2) is $0<\rho<\alpha+1$. The admissible values for ρ, namely $-p<\rho<-q$ is the so called *Fundamental Strip* (FS) of the Mellin transform. Inspection of Eqs. (1) and (2) reveals that whatever the value of ρ chosen, provided it belongs to the FS, the value of $p_X(x,t)$ is fully restored in the whole range.

Eq. (2) can be discretized because $M_p(\gamma-1)\to 0$ when $\eta\to\pm\infty$ and then the summation can be truncated to a finite number of terms; so, the discretized version of Eq. (2), may be written as

$$p_X(x) = \frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} M_p(\gamma_k-1)x^{-\gamma_k}; \quad x>0;\ \gamma_k = \rho + ik\Delta\eta \tag{3}$$

where $\Delta\eta$ is the discretization of the η axis and $m\Delta\eta$ is a cut off value $\bar{\eta}=m\Delta\eta$ chosen in such a way that terms of higher order than $m\Delta\eta$ do not produce sensible variations on $p_X(x,t)$.

We now define

$$\int_{-\infty}^{\infty} p_X(x,t)|x|^{\gamma-1}dx = E\left[|X|^{\gamma-1}\right] \tag{4}$$

where $E[\cdot]$ means ensemble average. $E\left[|X|^{\gamma-1}\right]$ defined in Eq. (4) will be called CFM. On the other hand, in the case of symmetric distribution $E\left[|X|^{\gamma-1}\right]$ is related to the Mellin transform as

$$M_p(\gamma-1) = \frac{1}{2}E\left[|X|^{\gamma-1}\right]. \tag{5}$$

By inserting this relation into Eq. (3) we recognize the $p_X(x,t)$ is restored by the Complex Fractional Moments (CFM) of the PDF.

Some very useful properties of the Mellin transform may be easily demonstrated: (i) $M_p(\gamma - 1)$ is holomorph into the fundamental strip; (ii) $E\left[|X|^{\rho+i\eta-1}\right] = E\left[|X|^{\rho-i\eta-1}\right]^*$ where the star means complex conjugate; (iii) every distribution possesses a finite fundamental strip since $p_X(x)$ is positive and its total area is finite (unitary area); (iv) because of the property (i) as $\eta \to \infty$, $E\left[|X|^{\rho+i\eta-1}\right] \to 0$; (v) the result of Eq. (3) is independent of the value of ρ selected, provided it belongs to the FS.

3. Fractional Fokker–Planck equation

Let the nonlinear Langevin equation enforced by the α-stable white noise $W_\alpha(t)$ be given in the form

$$\begin{cases} \dot{X} = f(X,t) + W_\alpha(t) & \text{(a)} \\ X(0) = X_0 & \text{(b)} \end{cases} \tag{6}$$

where $f(X,t)$ is any nonlinear function of the response process $X(t)$ and X_0 is a random variable with assigned probability density function. For simplicity sake we assume that $f(X,t) = -f(-X,t)$, $p_X(x,0) = p_{X_0}(x)$ is a symmetric PDF and W_α, formal derivative of the Lévy α-stable process $L_\alpha(t)$, is a symmetric α-stable ($S_\alpha S$) distribution. With these assumptions $p_X(x,t) = p_X(-x,t)\forall t$. The more general case of non symmetric distribution may be treated in a similar way [29].

The Itô equation associated to Eq. (6)(a) may be written in the form

$$dX(t) = f(X,t)dt + dL_\alpha(t) \tag{7}$$

where the characteristic function (CF) of $dL_\alpha(t)$ (for $S_\alpha S$ process) is in the form

$$\phi_{dL_\alpha}(\theta) = \exp(-dt\sigma|\theta|^\alpha) \tag{8}$$

where σ is the *scale factor* and α is the *stability index*. The equation ruling the evolution of the PDF of the response process $X(t)$, namely the FFP equation, is given in the form (see [5,6])

$$\frac{\partial p_X(x,t)}{\partial t} = -\frac{\partial}{\partial x}\left(f(x,t)p_X(x,t)\right) + \sigma^\alpha D_x^\alpha\left(p_X(x,t)\right). \tag{9}$$

The symbol $D_x^\alpha(\cdot)$ denotes the Riesz fractional derivative defined as

$$D_x^\alpha\left(u(x,t)\right) = \begin{cases} -\dfrac{1}{2\cos\left(\frac{\pi\alpha}{2}\right)}\left[D_{x+}^\alpha\left(u(x,t)\right) + D_{x-}^\alpha\left(u(x,t)\right)\right]; \quad \alpha \neq 1 & \text{(a)} \\ -\dfrac{d}{dx}\mathcal{H}\left[u(x,t)\right]; \quad \alpha = 1. & \text{(b)} \end{cases} \tag{10}$$

D_{x+}^α and D_{x-}^α are the left and the right hand side Liouville–Weyl derivatives given as

$$D_{x+}^\alpha\left(u(x,t)\right) = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dx^n}\int_{-\infty}^{x}\frac{u(\xi,t)}{(x-\xi)^{\alpha-n+1}}d\xi \qquad n = [\alpha]+1 \tag{11a}$$

$$D_{x-}^\alpha\left(u(x,t)\right) = \frac{1}{\Gamma(n-\alpha)}\left(-\frac{d}{dx}\right)^n\int_{x}^{\infty}\frac{u(\xi,t)}{(\xi-x)^{\alpha-n+1}}d\xi \tag{11b}$$

where $\Gamma(\cdot)$ is the Euler Gamma function and the symbol $[\alpha]$ denotes the integer part of the real number α. It follows that for subdiffusive input ($0 < \alpha < 1$) then $n = 1$, while for superdiffusive ones ($1 \le \alpha < 2$)$n = 2$. In Eq. (10)(b) the symbol $\mathcal{H}[\cdot]$ is the Hilbert transform operator

$$\mathcal{H}\left[u(x,t)\right] = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty}\frac{u(\xi,t)}{|x-\xi|}d\xi \tag{12}$$

and $\mathcal{P}$ stands for the principal value.

It may be demonstrated that the remarkable property

$$\mathcal{F}\left\{D_x^\alpha\left(u(x,t)\right),\theta\right\} = -|\theta|^\alpha\,\mathcal{F}\left\{u(x,t),\theta\right\} \tag{13}$$

holds true, where $\mathcal{F}\{u(x,t),\theta\}$ is the Fourier transform operator

$$\mathcal{F}\left\{u(x,t),\theta\right\} = \int_{-\infty}^{\infty}e^{i\theta x}u(x,t)dx. \tag{14}$$

Because of the property expressed in Eq. (13), the spectral counterpart of Eq. (9), namely the equation ruling the evolution of the characteristic function $\phi_X(\theta,t)$, is given as

$$\begin{cases} \dfrac{\partial \phi_X(\theta,t)}{\partial t} = i\theta E\left[f(X,t)\exp(i\theta X)\right] - \sigma^\alpha \left|\theta\right|^\alpha \phi_X(\theta,t) & \text{(a)} \\ \phi_X(\theta,0) = \phi_{X_0}(\theta) & \text{(b)} \end{cases} \tag{15}$$

where $\phi_{X_0}(\theta)$ is the Cf at $t=0$.

If the nonlinear function $f(x,t)$ belongs to the general class

$$f(x,t) = -\sum_{j=0}^{r} c_j \left|x\right|^{\beta_j} \operatorname{sgn}(x), \tag{16}$$

then Eq. (15)(a) is given in the form

$$\frac{\partial \phi_X(\theta,t)}{\partial t} = \theta \sum_{j=0}^{r} c_j \tilde{D}_\theta^{\beta_j} \phi_X(\theta,t) - \sigma^\alpha \left|\theta\right|^\alpha \phi_X(\theta,t) \tag{17}$$

where $\tilde{D}_\theta^{\beta_j}(\cdot)$ is the complementary Riesz fractional derivative defined as

$$\tilde{D}_\theta^{\beta_j}\phi_X(\theta,t) = \begin{cases} -\dfrac{1}{2\sin(\frac{\pi\beta_j}{2})}\left[D_{\theta+}^{\beta_j}(\phi_X(\theta,t)) - D_{\theta-}^{\beta_j}(\phi_X(\theta,t))\right]; \quad \beta_j \neq 1,3,\ldots & \text{(a)} \\ i^{\beta_j}\dfrac{\partial^{\beta_j}}{\partial\theta^{\beta_j}}\phi_X(\theta,t); \quad \beta_j = 1,3,\ldots. & \text{(b)} \end{cases} \tag{18}$$

The inverse Fourier transform of the complementary Riesz fractional derivative is

$$\begin{cases} \mathcal{F}^{-1}\left\{D_\theta^{\beta_j}\phi_X(\theta,t);x\right\} = -i\operatorname{sgn}(x)\left|x\right|^{\beta_j} p_X(x,t); \quad \beta_j \neq 1,3,\ldots & \text{(a)} \\ \mathcal{F}^{-1}\left\{i^{\beta_j}\dfrac{\partial^{\beta_j}}{\partial\theta^{\beta_j}}\phi_X(\theta,t);x\right\} = -x^{\beta_j} p_X(x,t); \quad \beta_j = 1,3,\ldots. & \text{(b)} \end{cases} \tag{19}$$

4. Solution of fractional Fokker–Planck equation in terms of CFM

In this section solution in terms of CFM for the FFP equation is pursued.

Let Eq. (9) be particularized for the nonlinearity expressed in Eq. (16), that is

$$\frac{\partial p_x(x,t)}{\partial t} = \sum_{j=1}^{r} c_j \frac{\partial}{\partial x}\left(\left|x\right|^{\beta_j}\operatorname{sgn}(x) p_X(x,t)\right) + \sigma^\alpha D_x^\alpha(p_X(x,t)). \tag{20}$$

The Mellin transform of this equation is

$$\begin{aligned} \frac{\partial M_p(\gamma-1,t)}{\partial t} &= \sum_{j=1}^{r} c_j \left[x^{\gamma-1+\beta_j} p_X(x,t)\right]_0^\infty - (\gamma-1)\sum_{j=1}^{r} c_j M_p(\gamma-2+\beta_j,t) \\ &\quad - \sum_{k=0}^{n-1}\frac{\Gamma(\gamma-1+k)}{\Gamma(\gamma-1)}\left[\frac{d^{n-k-1}}{dx^{n-k-1}}\left(\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(x-\xi)^{\alpha-n+1}}d\xi\right.\right. \\ &\quad \left.\left. + (-1)^n\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(\xi-x)^{\alpha-n+1}}d\xi\right)x^{\gamma-k-1}\right]_0^\infty - \sigma^\alpha\frac{\nu_c(\gamma)}{\nu_c(\gamma-\alpha)}M_p(\gamma-1-\alpha,t) \end{aligned} \tag{21}$$

where $\nu_c(\gamma) = \Gamma(\gamma)\cos(\pi/2\gamma)$ and the terms in the second square brackets of r.h.s of Eq. (21) are the integral of integer order of the Riesz fractional derivative. By properly selecting the value of ρ into the FS the first and the third term of Eq. (21) vanish (see Appendix) and Eq. (21) simplifies to

$$\frac{\partial M_p(\gamma-1,t)}{\partial t} = -(\gamma-1)\sum_{j=1}^{r} c_j M_p(\gamma-2+\beta_j,t) - \sigma^\alpha\frac{\nu_c(\gamma)}{\nu_c(\gamma-\alpha)}M_p(\gamma-1-\alpha). \tag{22}$$

This equation may be particularized for different values of γ, say $\gamma_k = \rho + ik\Delta\eta$ $(k=-m,\ldots,0,\ldots,m)$ obtaining a set of $2m+1$ ordinary linear differential equations. At this point it is not possible to solve such a set of differential equations because the fractional moments are evaluated for different values of ρ. This problem is analogous to that of the well known infinite hierarchy when we are dealing with moment equations. To overcome this problem we observe that Eq. (3) can be written for different values of ρ, provided they belong to the FS; so we can equate Eq. (3) particularized for two different values of ρ, $\rho_1 = \rho$ and $\rho_2 = \rho + \Delta\rho$ and we indicate with $M_p(\gamma_k^{(1)} - 1, t), M_p(\gamma_k^{(2)} - 1, t)$ the CFM evaluated in $\gamma_k^{(j)} = \rho_j + ik\Delta\eta$, with $j = 1, 2$:

$$x^{-\rho_1}\frac{\Delta\eta}{2\pi}\sum_{s=-m}^{m} M_p(\gamma_s^{(1)} - 1, t)x^{-is\frac{\pi}{b}} = x^{-\rho_2}\frac{\Delta\eta}{2\pi}\sum_{k=-m}^{m} M_p(\gamma_k^{(2)} - 1, t)x^{-ik\frac{\pi}{b}} \tag{23}$$

where $b = \pi/\Delta\eta$. By multiplying both sides of this equation for $x^{-1/2}$ we can write:

$$x^{-\rho}\sum_{s=-m}^{m} M_p(\gamma_s^{(1)} - 1, t)e^{-is\frac{\pi}{b}\ln x} = x^{-(\Delta\rho+1/2)}\sum_{k=-m}^{m} M_p(\gamma_k^{(2)} - 1, t)e^{-ik\frac{\pi}{b}\ln x} \quad x > 0. \tag{24}$$

It is to be remarked that Eq. (24) strictly holds for $x > 0$ because in zero singularities appear. Now we suppose $M_p(\gamma_k^{(2)} - 1, t)$ is known and we want evaluate $M_p(\gamma_k^{(1)} - 1, t)$. Because Eq. (3) is an approximation, we require that Eq. (24) holds true in a weak sense in the interval $x_1 > 0, x_2 \gg 0$:

$$\int_{x_1}^{x_2}\frac{1}{x}\left\{\left[\sum_{s=-m}^{m} M_p(\gamma_s^{(1)} - 1, t)e^{-is\frac{\pi}{b}\ln x} - x^{-(\Delta\rho+1/2)}\sum_{k=-m}^{m} M_p(\gamma_k^{(2)} - 1, t)e^{-ik\frac{\pi}{b}\ln x}\right][C.C.]\right\}dx$$
$$= \min(M_p(\gamma_k^{(1)} - 1, t)) \tag{25}$$

where $[C.C.]$ stands for complex conjugate. Now we make a change of variables by putting

$$\xi = \ln x, \quad d\xi = \frac{dx}{x}; \qquad \xi_j = \ln x_j, \quad j = 1, 2. \tag{26}$$

In order to find $M_p(\gamma_s^{(1)}, t)$ as a linear combination of $M_p(\gamma_k^{(2)}, t)$ we perform variations and we put $x_1 = e^{-b}, x_2 = e^{b}$ instead of $x_1 = 0, x_2 = \infty$. The latter conditions may not be enforced since in zero singularities appear. With these positions two goals are achieved: (i) since $b = \pi/\Delta\eta$, and $\Delta\eta$ is of order $0.3 \div 0.5$, the interval $e^{-b} \div e^{b}$ is very large ($\Delta\eta = 0.3$, $e^{-b} \cong e^{-10}$, $e^{b} \cong e^{10}$), that is we require that $p_X(x, t)$ will match the effective PDF in a very wide range but the singularity in zero is excluded; (ii) in the ξ domain the integral in Eq. (25) is performed in the range $-b \div b$

$$\int_{-b}^{b}\left\{\left[\sum_{s=-m}^{m} M_p(\gamma_s^{(1)} - 1, t)e^{-is\frac{\pi}{b}\ln x} - e^{-\Delta\rho\xi}\sum_{k=-m}^{m} M_p(\gamma_k^{(2)} - 1, t)e^{-ik\frac{\pi}{b}\ln x}\right][C.C.]\right\}dx$$
$$= \min(M_p(\gamma_s^{(1)} - 1, t)). \tag{27}$$

Now we can take advantage of the orthogonality condition of $e^{ik\frac{\pi}{b}\xi}$ in $-b \div b$ so that after minimization we obtain

$$M_p(\gamma_s^{(1)} - 1, t) = \sum_{k=-m}^{m} M_p(\gamma_k^{(2)} - 1, t)a_{ks}(\Delta\rho) \tag{28}$$

where

$$a_{ks}(\Delta\rho) = \frac{1}{2b}\int_{-b}^{b} e^{-\Delta\rho\xi}e^{-i(s-k)\frac{\pi}{b}\xi}d\xi = \frac{\sin[\pi(k-s) - ib\Delta\rho]}{\pi(k-s) - ib\Delta\rho}. \tag{29}$$

With the aid of Eq. (28) we can obtain $M_p(\gamma_s^{(1)} - 1, t)$ as a linear combination of $M_p(\gamma_k^{(2)} - 1, t)$ and solve differential equations by using Mellin transform. With these results we may solve also the FFP equation. Eq. (22) requires $M_p(\gamma_k - 1, t)$, $M_p(\gamma_k - 2 + \beta_j, t)$ and $M_p(\gamma_k - 1 - \alpha, t)$; we select a proper initial value of ρ in order to fulfill the limitations $\rho - 1 + \beta_j > 0$, $\rho - 1 + \beta_j - u < 0$ and the general limitation $\rho < 1 + \alpha$ and we evaluate

$$M_p(\gamma_s + \beta_j - 2, t) = \sum_{k=-m}^{m} M_p(\gamma_k - 1, t)a_{ks}(1 - \beta) \tag{30a}$$

$$M_p(\gamma_s - 1 - \alpha, t) = \sum_{k=-m}^{m} M_p(\gamma_k - 1, t)a_{ks}(\alpha). \tag{30b}$$

By inserting these equations in Eq. (22) particularized for $\gamma = \gamma_s$, we get a set of $2m + 1$ complex ordinary differential equations in the unknown $M_p(\gamma_s - 1, t)$. To find the correct solution of such a system it is necessary to impose that the area under the PDF in the interval $e^{-b} \div e^{b}$ is 1/2. This condition may be enforced very easily, taking into account Eq. (3), as follows:

$$\frac{1}{2b}\sum_{s=-m}^{m} M_p(\gamma_s-1,t)\int_{e^{-b}}^{e^{b}} x^{-\gamma_s}dx=\frac{1}{2}. \tag{31}$$

From this equation we get

$$M_p(\gamma_0-1,t)=b-\sum_{s=-m}^{m} M_p(\gamma_s-1,t);\quad s\neq 0. \tag{32}$$

By inserting this condition into Eq. (22) and inserting Eqs. (30) particularized for $s=-m,\dots,-1,1,\dots,m$ we get a set of $2m+1$ differential equations ruling the evolution of the CFM; the sth equation is given in the form

$$\frac{\partial M_p(\gamma_s-1,t)}{\partial t}=-(\gamma_s-1)\sum_{j=1}^{r} c_j\sum_{k=-m}^{m} M_p(\gamma_k-1,t)a_{ks}(1-\beta_j)-\sigma^{\alpha}\frac{\nu_c(\gamma_s)}{\nu_c(\gamma_s-\alpha)}\sum_{k=-m}^{m} M_p(\gamma_k-1,t)a_{ks}(\alpha). \tag{33}$$

After the solution of such a system the PDF at any instant t can be reconstructed by using Eq. (3). In the next section some applications are presented for various values of α and β.

5. Numerical application

In this section the solution of the FFP equation for linear ($\beta=1$) and non-linear ($\beta=0, 0.3, 0.5, 1.5, 3$) half oscillator forced by α-stable white noise with various values of stability index ($\alpha=2, 1.5, 1, 0.8, 0.5$) is presented. The case $\alpha=2$ corresponds to the Fokker–Planck equation yet solved with the same method in Ref. [26] and is reported for completeness. In all the examples reported the oscillator is quiescent for $t\leq 0$, so the PDF is a Dirac Delta and the CFMs are all zero; at $t=0$ we apply an α-stable white noise with scale factor $\sigma=1$, skewness $\beta=0$, shift $\mu=0$. The solution obtained with the proposed method is contrasted with the PDF constructed by digital simulations with 10^6 samples and/or the exact when available.

5.1. $\alpha=2$

This case corresponds to the Fokker–Planck equation. It is the simplest case because the fundamental strip of a Gaussian distribution is $0\div\infty$, because of the absence of long tails in this distribution and the CFMs never diverge, so we can use any value of ρ provided that it satisfies the condition $\rho-1+\beta_j>0$ (see Appendix). For this value of the stability index α we know some exact solution:

- for $\beta\neq 1$ we know the stationary solution given as

$$p_X(x,\infty)=\nu\exp\left(-\frac{2c|x|^{\beta+1}}{2(\beta+1)\sigma^2}\right) \tag{34}$$

where ν is a normalization constant given as

$$\nu=\frac{1}{2\int_0^{\infty}\exp\left(-\frac{2c|x|^{\beta+1}}{2(\beta+1)\sigma^2}\right)dx}=\frac{1}{2^{\frac{\beta}{\beta+1}}\left[\frac{c}{(\beta+1)2\sigma^2}\right]^{-\frac{1}{\beta+1}}\Gamma\left(1+\frac{1}{\beta+1}\right)} \tag{35}$$

- for $\beta=1$ it is well known the evolutionary solution given as

$$p_X(x,t)=\frac{1}{\sqrt{\pi}2\sigma(t)}\exp\left(-\frac{x^2}{4\sigma^2(t)}\right) \tag{36}$$

where

$$\sigma(t)=\sqrt{\frac{\sigma^2}{2c}(1-\exp(-2ct))} \tag{37}$$

is the scale at the time t. It is to be stressed that σ is not the standard deviation, but the scale of the Lévy (normal) process.

Solution at various instants is shown for different values of β (see Fig. 1).

5.2. $\alpha=1.5$

This value of α has been investigated as a general case in the range $1\div 2$. When stability index is lesser than 2 the fundamental strip depends on the values of α and β, because of the decay of the PDF for $x\to\infty$. In particular in Ref. [4] it has been demonstrated that for α-stable input, the tails of the PDF of the output decay as a power law x^{-u}, being $u=\alpha+1$ for the linear half-oscillator ($\beta=1$) and $u=\alpha+3$ for the quartic oscillator ($\beta=3$), so in both cases $u=\alpha+\beta$. This allows us to do some considerations on the FS that in unknown. These considerations, reported in the Appendix, result in

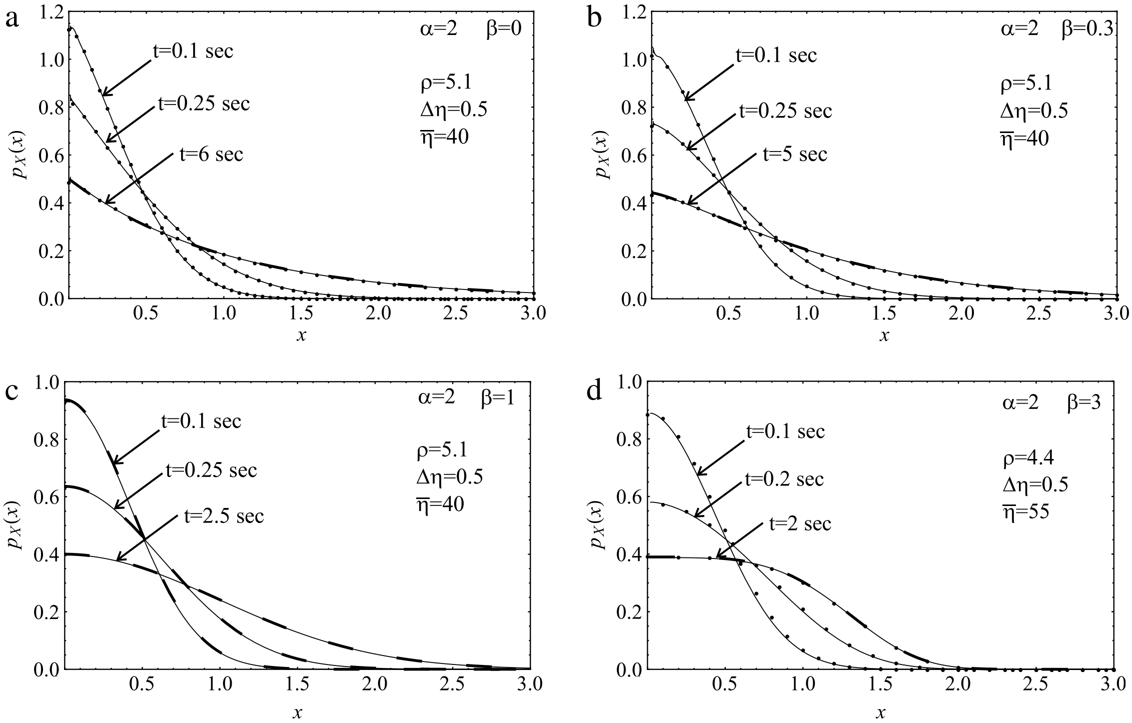

Fig. 1. CFM solution (continuous line) versus PDF by digital simulation (dotted curves) for $\alpha = 2$ at various instants and for different β.

the choice of ρ in the range $0 \div 1 + \alpha$. From this descends that, since in the Mellin transform of FFP equation there are CFMs evaluated for different values of ρ, we cannot solve the system with some values of $\beta > 1.7$ because CFMs from the drift term and from diffusive term are evaluated in value of ρ outside the FS. In the following results for $\alpha = 1.5$ are reported in Fig. 2.

5.3. $\alpha = 1$

This is the case when the input is a Cauchy process. Consideration on the fundamental strip in this case is the same as of Section 5.2. In this case the steady state solution for $\beta = 1$ is known as

$$p_X(x, \infty) = \frac{\sigma c}{\pi(\sigma^2 + c^2x^2)}. \tag{38}$$

In the following results for various values for β at different instants are reported (see Fig. 3).

5.4. $\alpha = 0.8$

This case is taken as a general case in the range $0 \le \alpha \le 1$. In the following results for various values for β at different instants are reported (see Fig. 4).

5.5. $\alpha = 0.5$

In this case the input is a symmetric Lévy process. For this value of α we are actually able to solve only the linear case for which the steady state solution may be obtained in the following form

$$p_X(x, \infty) = \sqrt{\frac{\bar{\sigma}}{2\pi|x|^3}} \left(\cos\left(\frac{\bar{\sigma}}{4x}\right) \left(\frac{1}{2} - F_c\left(\sqrt{\frac{\bar{\sigma}}{2\pi|x|}}\right)\right) + \sin\left(\frac{\bar{\sigma}}{4x}\right) \left(\frac{1}{2} - F_s\left(\sqrt{\frac{\bar{\sigma}}{2\pi|x|}}\right)\right)\right) \tag{39}$$

where $F_c(\cdot)$ and $F_s(\cdot)$ are the Fresnel integrals defined as follows:

$$F_c(x) = \int_0^x \cos\left(\frac{\pi t^2}{2}\right) dt \tag{40a}$$

$$F_s(x) = \int_0^x \sin\left(\frac{\pi t^2}{2}\right) dt \tag{40b}$$

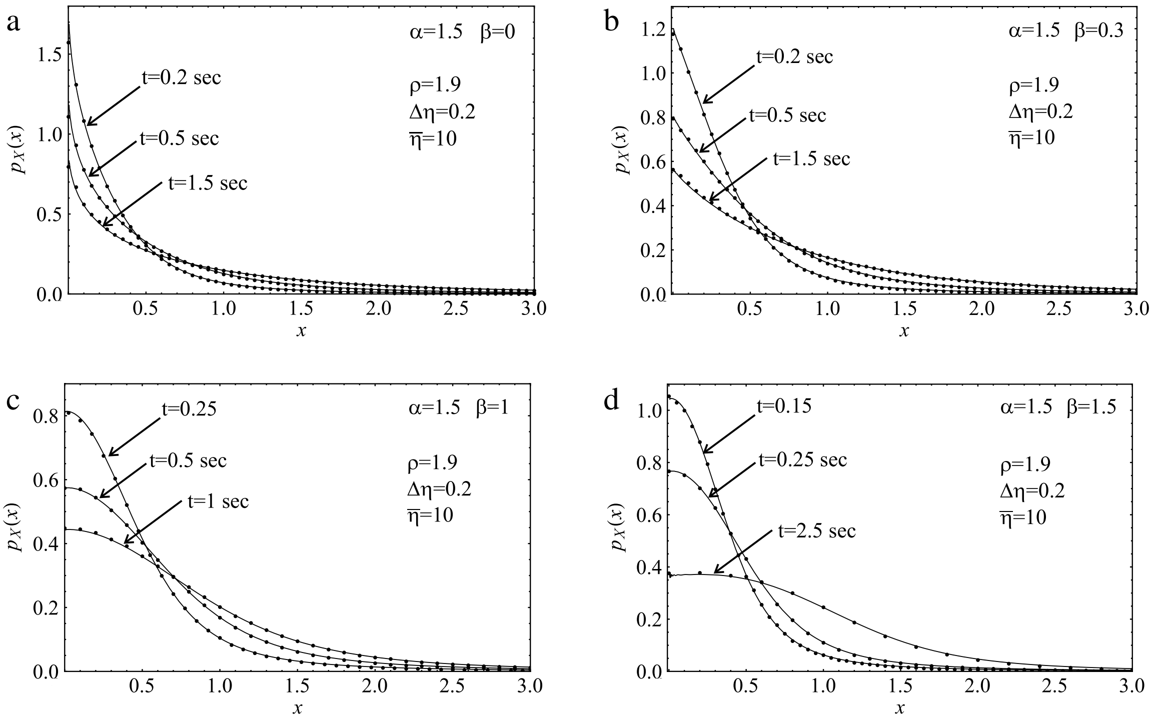

Fig. 2. CFM solution (continuous line) versus PDF by digital simulation (dotted curves) for $\alpha = 1.5$ at various instants and for different β.

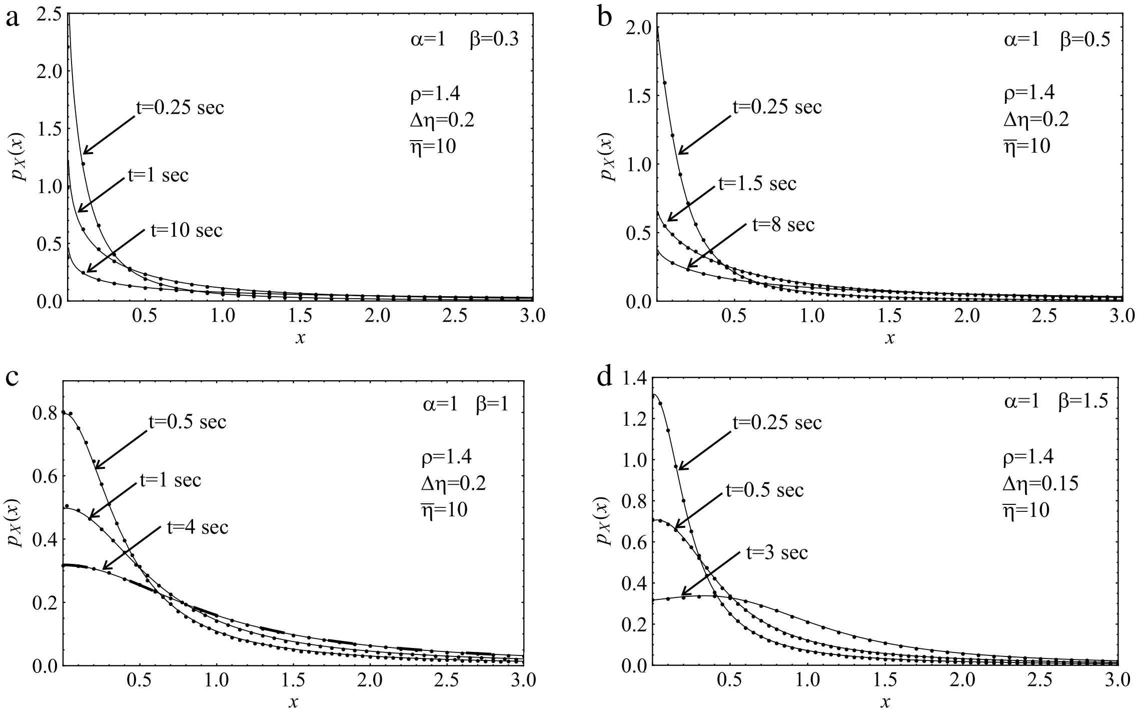

Fig. 3. CFM solution (continuous line) versus PDF by digital simulation (dotted curves) for $\alpha = 1$ at various instants and for different β.

and $\bar{\sigma}$ is the scale factor of the output defined as

$$\bar{\sigma} = \sigma \left(\frac{c}{2}\right)^{-2}. \tag{41}$$

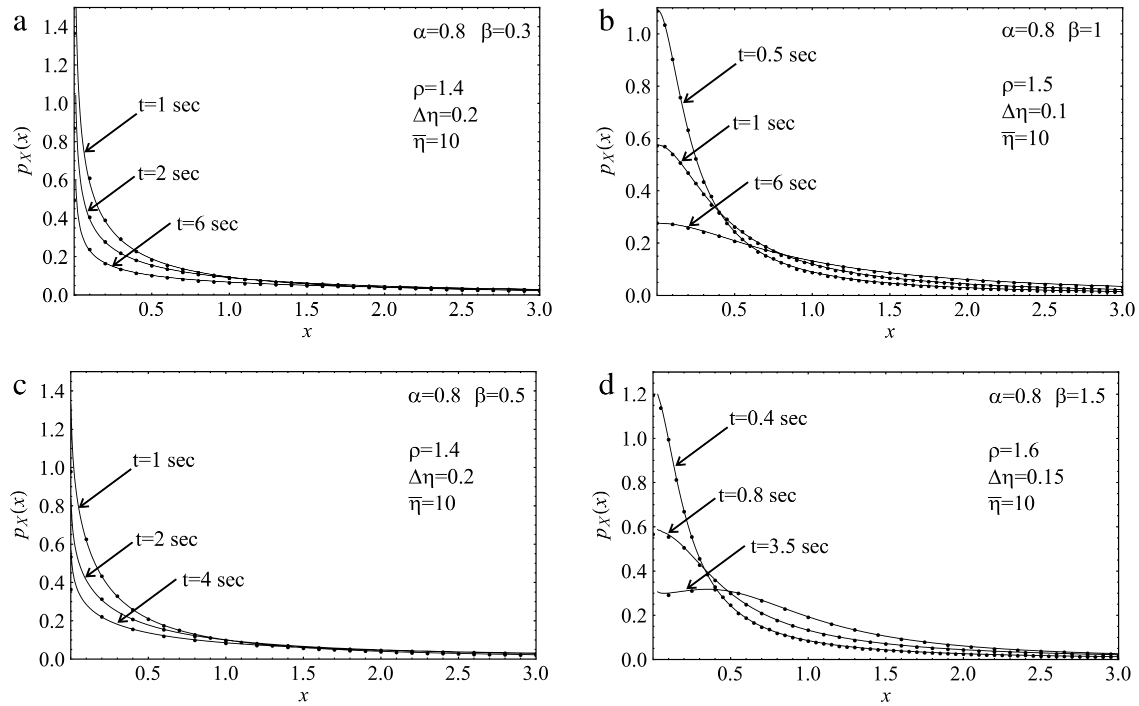

Fig. 4. CFM solution (continuous line) versus PDF by digital simulation (dotted curves) for $\alpha = 0.8$ at various instants and for different β.

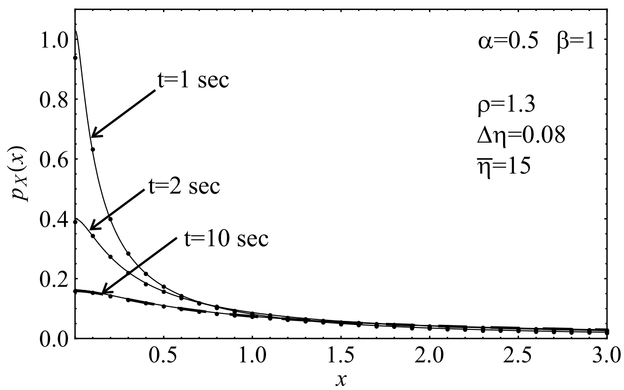

Fig. 5. CFM solution (continuous line) versus PDF by digital simulation (dotted curves) at various instants for $\alpha = 0.5$, $\beta = 1$.

In Fig. 5 results for $\beta = 1$ are reported.

5.6. Trend of the PDF at ∞

Fig. 6 shows in logarithmic plot the stationary solution of the FFP equation for the linear case ($\beta = 1$) and for two different values of α (1 and 0.5) for which the stationary solution is known in analytical form. From these figures it is possible to observe that the solution provided by the proposed method coalesces with the exact one also for large values of *x*. This fact is very important because other methods of solution fail in the description of the long tails of the PDF.

5.7. Choice of the parameter *m*

In order to properly define the parameter m, some considerations are necessary: (i) the choice of m strictly depends on $\Delta\eta$ since $m\Delta\eta = \bar{\eta}$ is the truncation of $M_p(\gamma - 1)$ that in turns depends on the value of ρ selected; (ii) higher value of ρ, at a parity of the PDF at hand produces oscillations in $M_p(\gamma - 1)$ as shown in Fig. 7 in which CFM is reported for different values of ρ ($\rho = 0.5$; $\rho = 3.5$). It follows that in order to properly discretize the inverse Mellin transform it is necessary to have a smaller value of $\Delta\eta$ as ρ increases. In the case of α-stable Lévy white noise the selection of ρ is obligated by the limitations

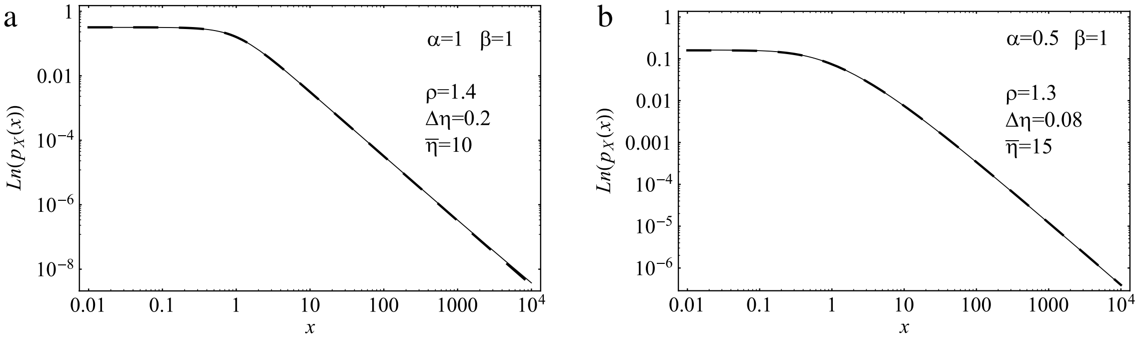

Fig. 6. Log–Log plot of the stationary solution for $\beta = 1$ and $\alpha = 1, 0.5$ contrasted with exact steady state solution.

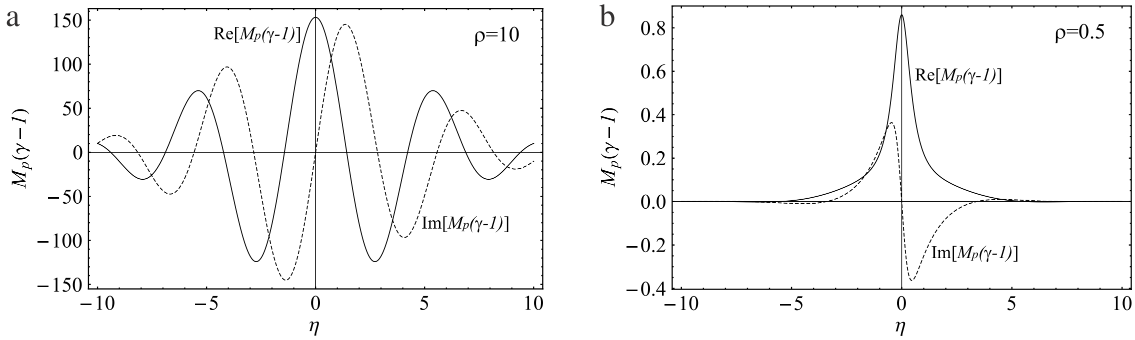

Fig. 7. CFMs of a Gaussian distribution with $\sigma = 1$ and different values of ρ.

of the FS of the Mellin transform and on the non-linearity. So because m and consequently $\Delta\eta$ depend on many parameters we can proceed with trial and error (two or three attempts are enough) or if we have a crude estimation on the PDF of the response at steady state by using approximate techniques (like stochastic linearization) then a preliminary choice of m may be readily performed. In quiescent systems, since in $t = 0$ all CFMs are zero and the tails of the PDF increase then the worst situation will remain the PDF at the steady state or when the scale attains the maximum value. It follows that as the PDF is well reproduced for the steady state or in correspondence of the maximum scale (or of the variance if it exists) then all the parameters (m and $\bar{\eta}$) may be used also in the transient zone.

6. Conclusion

In this paper the response in terms of PDF of a non-linear system forced by an external α-stable white noise is studied. The governing equations of such a system in terms of PDF is the so called Fractional Fokker–Planck equation obtained with the aid of Itó calculus. The solution of these equation is found by the Mellin transform operator that allows us to transform the terms of the FFP equation in complex fractional moments of the PDF; these moments are finite only if the Mellin transform is performed with a value of ρ belonging to the fundamental strip of the PDF. The key of the method is to write the CFMs calculated for different values of ρ as a linear combination of the CFMs of an arbitrary value of ρ; once this aim is obtained the FFP equation is transformed into a set of complex ordinary differential equations that can be easily solved with a symbolic or numeric package. The numerical applications show the accuracy of the prediction of the PDF in any instant of the temporal range at hand; this is the main advantage of this method, in fact other solutions have been found for Fractional Fokker–Planck equation, but only in the stationary state and for particular form of the drift term. Lastly, it is important to underline that this method can be easily extended to systems forced by asymmetric α-stable white noise and to more complex systems also involving fractional terms. As a concluding remark the effectiveness of the proposed method relies on the following points: (i) the method is robust in the sense that it works for any kind of nonlinearity in the stochastic differential equation; (ii) the method works for Gaussian or non Gaussian white noise input; (iii) the method ensures us the trend of the response at ∞ and the latter aspect is very important for the reliability analysis.

Appendix

In this appendix two issues about the choice o ρ are discussed. The first is about the FS of the Mellin transform of an α-stable distribution, because if we select a value of ρ not belonging to the FS, CFMs do not exist. The second is about the conditions for which the terms in square brackets of Eq. (21) vanish; this is also an important issue since if these terms have values different from zero, the FFP equation in terms of CFMs cannot be solved.

Fundamental strip

When we solve the Fractional Fokker–Planck equation we only know the FS of the distribution of the input in the form $-p < \rho < -q$; these limits can be found with considerations on the behavior of the distribution for $x \to 0$ (for p) and for $x \to \infty$ (for q): since in zero a symmetric distribution has a finite value, the order of $p_X(x)$ for $x \to 0$ is $\mathcal{O}(x^0)$, then $p = 0$; thanks to previous works (see e.g. [4]) we know that $p_X(x)$ for $x \to \infty$ behaves like $x^{-(\alpha+1)}$, then $q = 1 + \alpha$. From these considerations we may write

$$0 < \rho < 1 + \alpha. \tag{A.1}$$

It has to be noted that if $\alpha = 2$ the distribution is Gaussian; this distribution for $x \to \infty$ behaves like $x^{-\infty}$, then the fundamental strip is

$$0 < \rho < \infty. \tag{A.2}$$

However it is necessary to know the FS of the output to select the proper value of ρ. We have three different possibilities

- If the mechanical system is linear ($\beta = 1$), it is well known the output has an α-stable distribution with the same stability index α of the input; so the FS is the same as the input

$$0 < \rho < 1 + \alpha. \tag{A.3}$$

- If the mechanical system is nonlinear with $\beta > 1$, the output has a distribution that decays more quickly than the case $\beta = 1$, clearly the upper bound of the FS is for sure greater than $1 + \alpha$. In Ref. [4] it has been demonstrated that for α-stable input, the tails of the PDF decay as a power law x^{-u}, being $u = \alpha + 1$ for the linear half-oscillator ($\beta = 1$) and $u = \alpha + 3$ for the quartic oscillator ($\beta = 3$). If we extend these concepts to real positive values of β we can assert that $u = \alpha + \beta$ and then we can assume

$$0 < \rho < \beta + \alpha. \tag{A.4}$$

- If the mechanical system is nonlinear with $\beta < 1$, the output has a distribution that decays slowly than the case $\beta = 1$, clearly the upper bound of the FS is for sure smaller than $1 + \alpha$ and then we must assume an FS with an upper bound smaller than the input one; based on considerations made in the previous point, also in this case we can assume

$$0 < \rho < \beta + \alpha. \tag{A.5}$$

Other limitations on the choice of ρ

As already written at the beginning of this appendix, other limitations on the choice of ρ come out from the terms in square brackets of Eq. (21). One of these terms, the first, is due to integration by parts when we perform the Mellin transform of the drift term, while at least other two terms can appear form integration by part when we perform Mellin transform of the diffusive term; in particular if $0 < \alpha \le 1$ the Mellin transform of diffusive term generates only one term, while if $1 < \alpha \le 2$ the Mellin transform generates two terms.

Let us start with the term due to the drift term in the FFP equation

$$\left[x^{\gamma-1+\beta_j} p_X(x,t)\right]_0^{\infty}. \tag{A.6}$$

Since in zero the PDF of the output has a finite value, this term vanishes in zero if $\rho - 1 + \beta_j > 0$, then the limit is $\rho > 1 - \beta_j$; at ∞, if $\lim_{x\to\infty} p_x(x,t) = \mathcal{O}(x^{-u})$, the term vanishes if $\rho - 1 + \beta_j - u < 0$. Since we assume $u = \alpha + \beta_j$, the limit is $\rho < 1 + \alpha$.

The terms coming from integration by part of the diffusive term are expressed as follows:

$$\sum_{k=0}^{n-1}\left[\frac{d^{n-k-1}}{dx^{n-k-1}}\left(\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(x-\xi)^{\alpha-n-1}}d\xi + (-1)^n\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(\xi-x)^{\alpha-n-1}}d\xi\right)x^{\gamma-k-1}\right]_0^{\infty}. \tag{A.7}$$

If $0 < \alpha \le 1$ this summation represents only one term ($n = 1$); if $1 < \alpha \le 2$, then $n = 2$ and the summation has two terms. In the following we discuss separately the case $n = 1$ and $n = 2$.

For $n = 1$ the summation has only one term that is the integral of order 1 of Riesz fractional derivative of the unknown PDF, then it can be written as

$$\left[\left(\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(x-\xi)^{\alpha}}d\xi - \int_{-\infty}^{x}\frac{p_X(\xi,t)}{(\xi-x)^{\alpha}}d\xi\right)x^{\gamma-1}\right]_0^{\infty}. \tag{A.8}$$

This term vanishes in $x = 0$ if $\rho > 1$; for $x \to \infty$ we can make some consideration: since the PDF of the output behaves like $x^{-(\beta+\alpha)}$ we may assume that the Riesz fractional derivative of orderα of the PDF of the output behaves like $x^{-(\beta+2\alpha)}$. The integral of order 1 of this power law is $x^{-(\beta+2\alpha)+1}$, so the term vanishes at ∞ if $\rho - 1 < \beta + 2\alpha - 1$, then the condition is $\rho < \beta + 2\alpha$.

For $n = 2$ the summation has two terms, the term for $k = 0$ is the integral of order 1 of the Riesz fractional derivative, while the term for $k = 1$ is the integral of order 2 of Riesz fractional derivative

$$\sum_{k=0}^{1}\left[\frac{d^{1-k}}{dx^{1-k}}\left(\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(x-\xi)^{\alpha}}d\xi+\int_{-\infty}^{x}\frac{p_X(\xi,t)}{(\xi-x)^{\alpha}}d\xi\right)x^{\gamma-k-1}\right]_0^{\infty}. \tag{A.9}$$

The term for $k = 0$ vanishes in 0 if $\rho - 1 > 0$; for $x \to \infty$ the Riesz fractional derivative of the power law $x^{-(\alpha+\beta)}$ is $x^{-(2\alpha+\beta)}$, its integral is $x^{-(2\alpha+\beta)+1}$, then the term vanishes if $\rho < 2\alpha + \beta$.

With analogous considerations we find that the term for $k = 1$ vanishes in 0 if $\rho > 2$ and for $x \to \infty$ if $\rho < 2\alpha + \beta$.

Summing up, from the drift term descends the following conditions

$$x = 0 \quad \Rightarrow \quad \rho > 1 - \beta \tag{A.10a}$$
$$x \to \infty \quad \Rightarrow \quad \rho < 1 + \alpha. \tag{A.10b}$$

If $n = 1$, from the diffusive term descends the following conditions

$$x = 0 \quad \Rightarrow \quad \rho > 1 \tag{A.11a}$$
$$x \to \infty \quad \Rightarrow \quad \rho < 2\alpha + \beta. \tag{A.11b}$$

If $n = 2$, from the diffusive term descends the following conditions

$$x = 0 \quad \Rightarrow \quad \rho > 2 \tag{A.12a}$$
$$x \to \infty \quad \Rightarrow \quad \rho < 2\alpha + \beta. \tag{A.12b}$$

These restrictions cannot actually be eliminated and make impossible to solve some cases in which some values of ρ in the CFMs of the FFP equation are outside the FS.

It is to be remarked that the restrictions coming from the behavior of the distribution in $x = 0$ are not respected in our numerical examples, since by considering the range $e^{-b} \div e^{b}$ we exclude the value in zero.

References

[1] G. Samorodnitsky, S.M. Taqqu, Stable Non-Gaussian Random Processes. Stochastic Models with Infinite Variance, Chapman and Hall/CRC, USA, 1994.
[2] R. Metzler, J. Klafter, The random walk's guide to anomalous diffusion: A fractional dynamics approach, Phys. Rep. 399 (2000) 1–77.
[3] H. Risken, The Fokker Planck Equation, Springer, Berlin, 1996.
[4] A. Chechkin, V. Gonchar, J. Klafter, R. Metzler, L. Tanatarov, Stationary state of non-linear oscillator driven by Lévy noise, Chem. Phys. 284 (2002) 233–251.
[5] H. Fogedby, Lévy flights in random environments, Phys. Rev. Lett. 73 (19) (1994) 2317–2520.
[6] R. Metzler, E. Barkai, J. Klafter, Deriving fractional Fokker–Planck equations from a generalised master equation, Europhys. Lett. 46 (4) (1999) 431–436.
[7] F. Mainardi, Y. Luchko, G. Pagnini, The fundamental solution of the space–time fractional diffusion equation, Fract. Calc. Appl. Anal. 4 (2001) 153–192.
[8] R. Iwankiewicz, S.R.K. Nielsen, Solution techniques for pulse problems in non-linear stochastic dynamics, Probab. Eng. Mech. 15 (2000) 25–36. original research article.
[9] A. Naess, V. Moe, Efficient path integration methods for nonlinear dynamic systems, Probab. Eng. Mech. 15 (2000) 221–231.
[10] E. Mamontov, A. Naess, An analytical–numerical method for fast evaluation of probability densities for transient solutions of nonlinear itô's stochastic differential equations, Int. J. Eng. Dyn. 47 (2009) 116–130.
[11] A. Pirrotta, R. Santoro, Probabilistic response of nonlinear systems under combined normal and Poisson white noise via path integral method, Probab. Eng. Mech. 25 (2011) 25–32.
[12] I.A. Kougioumtzoglou, P.D. Spanos, An analytical Wiener path integral technique for non-stationary response determination of nonlinear oscillators, Probab. Eng. Mech. 28 (2012) 125–131.
[13] W.Q. Zhu, Stochastic averaging method in random vibration, Appl. Mech. Rev. 41 (1988) 189–199.
[14] J.B. Roberts, P.D. Spanos, Stochastic averaging: An approximate method of solving random vibration problems, Int. J. Non-Linear Mech. 21 (1986) 111–134. review article.
[15] L.A. Bergman, B.F. Spencer, On the numerical solution of the Fokker–Planck equation for nonlinear stochastic systems, Nonlinear Dynam. 4 (1993) 357–372.
[16] G. Stefanou, The stochastic finite element method: Past, present and future review article, Comp. Methods Appl. Mech. Engrg. 198 (2009) 1031–1051.
[17] R. Ghanem, P.D. Spanos, Spectral stochastic finite element formulation for reliability analysis, J. Eng. Mech. 117 (10) (1991) 2351–2372.
[18] M. Di Paola, G. Failla, Stochastic response analysis of linear and nonlinear systems to α-stable Lévy white noise, Probab. Eng. Mech. 20 (2005) 128–135.
[19] M. Di Paola, A. Pirrotta, M. Zingales, Itô calculus extended to systems driven by α-stable Lévy white noises (a novel clip on the tails of Lévy motion), Int. J. Non-Linear Mech. 42 (2007) 1046–1054.
[20] B. Dybiec, I. Sokolov, A.V. Chechkin, Stationary states in single-well potentials under symmetric Lévy noise, J. Stat. Mech.: Theory Exp. 7 (2010) P07008.
[21] V.Y. Gonchar, L.V. Tanatarov, A.V. Chechkin, Stationary solutions of the fractional kinetic equation with a symmetric power-law potential, Theoret. Math. Phys. 131 (1) (2002) 582–594.
[22] M. Grigoriu, Equivalent linearization for systems driven by Lévy white noise, Probab. Eng. Mech. 15 (2000) 185–190.
[23] M. Grigoriu, Characteristic function equation for the state of dynamic system with Gaussian, Poisson and Lévy white noise, Probab. Eng. Mech. 19 (2004) 449–461.
[24] G. Samorodnitsky, M. Grigoriu, Characteristic function for the stationary state of a one dimensional dynamical system with Lévy noise, Theoret. Math. Phys. 150 (2007) 332–346.
[25] G. Cottone, Statistics of nonlinear dynamical systems under Lévy noises by a convolution quadrature approach, J. Phys. A 44 (18) (2011) 185001.
[26] M. Di Paola, Fokker Planck equation solved in terms of complex fractional moments, Probab. Eng. Mech.
[27] M. Di Paola, G. Cottone, On the use of fractional calculus for the probabilistic characterization of random variable, Probab. Eng. Mech. 24 (2009) 321–330.
[28] M. Di Paola, G. Cottone, R. Metzler, Fractional calculus approach to the statistical characterization of random variables and vectors, Phys. A 389 (2010) 909–920.
[29] M. Di Paola, F.P. Pinnola, Riesz fractional integrals and complex fractional moments for the probabilistic characterization of random variables, Probab. Eng. Mech. 29 (2012) 149–156.