Mathematical Patterns and Cognitive Architectures

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Abstract. Mathematical patterns are an important subclass of the class of patterns. The main task of this paper is examining a particular proposal concerning the nature of mathematical patterns and some elements of the cognitive structure an agent should have to recognize them.

1 Introduction

As is well known, the main aim of pattern recognition is to determine whether, and to what extent, what we call 'pattern recognition' can be accounted for in terms of automatic processes. From this it follows that two of its central problems are how to: (i) describe and explain the way humans, and other biological systems, produce/discover and characterize patterns; and how to (ii) develop automatic systems capable of performing pattern recognition behaviour.

Having stated these important facts, we need to point out that at the foundations of pattern recognition there are two more basic questions which we can formulate in the following way: (a) what is a pattern? (b) how do we come to know patterns? And it is clear that, if we intend to develop a science of pattern recognition able to provide a rigorous way of achieving its main aim, and of pursuing its central objects of study, it is very important to address questions (a) and (b).

What we intend to do in this paper is tackling questions (a) and (b) not in their full generality, but in the privileged context provided by mathematics, where there exists a consolidated tradition which regards it as a science of patterns,[4] connecting the results of our enquiries to the appropriate levels of the cognitive architecture we propose for a cognitive agent.

[4] See on this [Oliveri, 1997], [Shapiro, 2000], [Resnik, 2001], [Oliveri, 2007], [Oliveri, 2012], [Bombieri, 2013].

2 A case study

If we are presented with the two following objects **a** and **b**, it is very difficult to see what interesting mathematical feature they might have in common, if any, let alone that they exemplify the same mathematical pattern:

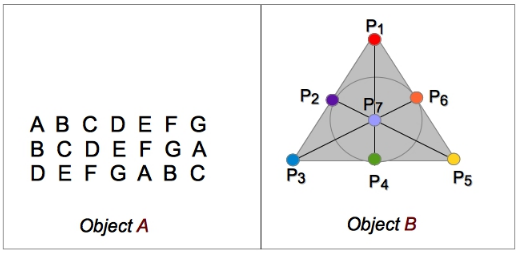

Fig. 1. Different Information Processing for two different Cognitive Agents

Indeed, whereas object **a** is a 3×7 matrix whose elements are the first seven letters of the Italian alphabet, object **b** is an equilateral triangle in which we have inscribed a circle, drawn three bisecting segments, and singled out the points of intersection of three curves.

However, the situation radically changes if we introduce the following formal system T with the appropriate interpretations.

Let T be a formal system such that the language of T contains a primitive binary relation 'x belongs to a set X' ($x \in X$), and its inverse 'X contains an element x' ($X \ni x$).

Furthermore, let us assume that D, the domain, is a set of countably many undefined elements $a_1, a_2, \ldots$; call 'm-set' a subset X of D; and consider the following as the axioms of T:

Axiom 1 If x and y are distinct elements of D there is at least one m-set containing x and y;
Axiom 2 If x and y are distinct elements of D there is not more than one m-set containing x and y;
Axiom 3 Any two m-sets have at least one element of D in common;
Axiom 4 There exists at least one m-set.
Axiom 5 Every m-set contains at least three elements of D;
Axiom 6 All the elements of D do not belong to the same m-set;
Axiom 7 No m-set contains more than three elements of D.[5]

[5] The case study discussed in this section has been taken from [Oliveri, 2012], §3, pp. 410-414. These axioms have been taken, with some minor alterations, from [Tuller, 1967], §2.10, p. 30.

At this point, if we put $I_1(a_1) = A, \ldots, I_1(a_7) = G$, we find that, under this interpretation, what corresponds to the m-sets are the columns of the matrix in fig. 1, and that object **a** is a model of T.

On the other hand, if we put $I_2(a_1) = P_1, \ldots, I_2(a_7) = P_7$, we find that, under this interpretation, what corresponds to the m-sets are the curves in fig. 1, and that object **b** is a model of T. But the surprises do not end up here, because we can now prove that the two models of T mentioned above are isomorphic to one another (see on this [Oliveri, 2012], §3, p. 413, footnote 12).

Several are the things that interest us in this example. First of all, the expression 'the pattern described by T' appears to refer to the mathematical structure which is realized/instantiated in objects **a** and **b**. What this seems to suggest is that, in the mathematical case, the concept of pattern coincides with that of mathematical structure.

Secondly, in the absence of our formal system T, we cannot see the pattern/structure instantiated by **a** and **b** because we are in no position for making the relevant observations concerning the salient features of the pattern/structure in question as is shown by the fact that, in particular, we are unable to make a number of fundamental distinctions such as that between part and whole, etc. etc.

Thirdly, the mathematical structure which becomes salient when we observe objects **a** and **b** *through* T depends not only on T, but also on **a** and **b**. In fact, given that we can prove in T that there exist exactly seven elements in D and seven m-sets if, for instance, the number of letters of the Italian alphabet we considered as elements of our matrix were different from seven, the matrix could not be a model of T (the same applies *mutatis mutandis* to the number of points of intersection of three curves in **b**).

Taking stock of some of the main points made in this section in our study of the mathematical case, we need to say that: (i) we must distinguish between object and structure; (ii) there are strong reasons for identifying mathematical patterns with structures; (iii) necessary conditions for pattern recognition in mathematics are the existence of (1) an observer O; (2) a domain of objects D; and (3) a system of representation Σ, i.e. $(\mathrm{O}, \mathrm{D}, \Sigma)$.[6]

With regard to the problem of how we come to know mathematical patterns, given that mathematical patterns are neither sensible objects nor properties of sensible objects, e.g., what in our example we saw *as* a Euclidean equilateral triangle is not a perfect Euclidean equilateral triangle, because its sides do not have exactly the same length, do not contain an infinite number of points, are not breadthless, etc. (see on this [Oliveri, 2012], §§3 and 4, pp. 410-417), it follows that they are not given to us as a consequence of abstraction or induction/generalization carried out on pure observations. But, on the other hand, if mathematical patterns are (also) dependent on objects, as in the case of **a** and **b**, they cannot simply be in the eyes of the beholder either. They are given to

[6] Actually, the system of representation Σ is an ordered pair $\Sigma = (\mathrm{T}, I)$, where T is a set containing (as a subset) a recursive set of axioms $\mathcal{A}$ and all the logical consequences of $\mathcal{A}$, and I is an interpretation of T on to D.

us as a consequence of our activity of representing entities like **a** and **b** within a given system of representation Σ.

3 Patterns and conceptual spaces

Conceptual spaces (CS) were originally introduced by Gärdenfors as a bridge between symbolic and associationist models of information representation. This was part of an attempt to describe what he calls the 'geometry of thought'.

In [Gärdenfors, 2004] and [Gärdenfors, 2004a] we find a description of a cognitive architecture for modelling representations. The cognitive architecture is composed by three levels of representation: a *subconceptual level*, in which data coming from the environment (sensory input) are processed by means of a neural network based system; a *conceptual level*, where data are represented and conceptualized independently of language; and, finally, a *symbolic level* which makes it possible to manage the information produced at the conceptual level at a higher level through symbolic computations.

Gärdenfors' proposal of a way of representing information *via* his conceptual spaces exploits geometrical structures rather than symbols or connections between neurons. This geometrical representation is based on the existence/construction of a space endowed with a number of what Gärdenfors calls 'quality dimensions' whose main function is to represent different qualities of objects such as brightness, temperature, height, width, depth.

Moreover, for Gärdenfors, judgments of similarity play a crucial role in cognitive processes and, according to him, it is possible to associate the concept of distance to many kinds of quality dimensions. This idea naturally leads to the conjecture that the smaller is the distance between the representations of two given objects in a conceptual space the more similar to each other the objects represented are.

According to Gärdenfors, objects can be represented as points in a conceptual space, points which we are going to call 'knoxels',[7] and concepts as regions within a conceptual space. These regions may have various shapes, although to some concepts — those which refer to natural kinds or natural properties — correspond regions which are characterized by convexity.[8]

Of course, at this point a whole host of important questions come to the forefront, questions like how could a cognitive agent: (1) learn the appropriate conceptual spaces? (2) select between different spaces that could fill the data? (3) determine the possible dimensions for representing objects? etc. etc. And although all such questions are central to our attempt to use Gärdenfors conceptual spaces as part of the cognitive architecture of a conceptual agent — we have addressed some of them in [Augello et al., 2013a] and [Augello et al., 2013b] —

[7] The term 'knoxel' originates from [Gaglio, 1988] by the analogy with "pixel". A knoxel k is a point in Conceptual Space and it represents the epistemologically primitive element at the considered level of analysis.

[8] A set S is *convex* if and only if whenever $a, b \in S$ and c is between a and b then $c \in S$.

what we aim to do in this paper is: (α) showing the existence of at least three different pattern recognition procedures; and (β) individuating which of the 3 corresponding levels of the cognitive architecture of our cognitive agent is involved in the processing of mathematical patterns.

To do this consider the case study discussed in §2 (taken from [Oliveri, 2012], §3, pp. 410-414), and imagine we have before us a cognitive agent A endowed with level 1 information processing system. In this case A (its neural network) can be trained to recognize letters A, B, ..., G and distinguish them from one another; and do the same thing for the coloured round objects $P_1, P_2, \ldots, P_7$.

Furthermore, suppose that the letters and the coloured round objects are presented to A exactly as they are in fig. 1. Once more A, exploiting its level 2 information processing system, i.e. the conceptual spaces of letters and of colours, is able to give a correct representation of **a** and **b**, for example, by representing **a** and **b** in an appropriate finite-dimensional vector space using rigid motions and some operations which act on the spaces.

However, what A cannot do, if the formal system T (see §2) is absent from its symbolic level 3 information processing system, is recognizing that **a** and **b** exemplify/realize the same pattern. Therefore, if what we have argued so far is correct, it follows that in the dawning of a mathematical pattern all the three levels of information processing systems we mentioned above are involved

4 Conclusion

In this work we have revisited a three levels cognitive architecture as a foundational approach to pattern recognition for an agent. We have illustrated this possibility by exploiting a mathematical domain. We have also highlighted the relevance of a linguistic, symbolic, level in order to produce abstractions and see deeper mathematical patterns.

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