Electronic Journal of Differential Equations, Vol. 2014 (2014), No. 200, pp. 1–7.
ISSN: 1072-6691. URL: http://ejde.math.txstate.edu or http://ejde.math.unt.edu
ftp ejde.math.txstate.edu

EXISTENCE AND MULTIPLICITY OF SOLUTIONS FOR DIRICHLET PROBLEMS INVOLVING NONLINEARITIES WITH ARBITRARY GROWTH

GIOVANNI ANELLO, FRANCESCO TULONE

Abstract. In this article we study the existence and multiplicity of solutions for the Dirichlet problem

$$
\begin{gathered}
-\Delta_p u = \lambda f(x,u) + \mu g(x,u) \quad \text{in } \Omega, \\
u = 0 \quad \text{on } \partial\Omega
\end{gathered}
$$

where Ω is a bounded domain in $\mathbb{R}^N$, $f, g : \Omega \times \mathbb{R} \to \mathbb{R}$ are Carathèodory functions, and λ, μ are nonnegative parameters. We impose no growth condition at ∞ on the nonlinearities f, g. A corollary to our main result improves an existence result recently obtained by Bonanno via a critical point theorem for C^1 functionals which do not satisfy the usual sequential weak lower semicontinuity property.

1. Introduction

In this article we study the Dirichlet problem

$$
\begin{gathered}
-\Delta_p u = \lambda f(x,u) \quad \text{in } \Omega \\
u = 0 \quad \text{on } \partial\Omega,
\end{gathered} \tag{1.1}
$$

where $p \in]1, +\infty[$, $\Delta_p(\cdot) := \operatorname{div}(|\nabla(\cdot)|^{p-2}\nabla(\cdot))$ is the p-laplacian operator, Ω is a bounded smooth domain in $\mathbb{R}^N$, λ is a positive parameter, and $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathèodory function. We will establish some existence and multiplicity results for problem (1.1) for small values of the parameter λ by imposing only local conditions on the nonlinearity f, allowing this latter to be of arbitrary growth at ∞. In particular, our existence result improves and extends a recent result by Bonanno [5, Theorem 8.1] obtained as application of a critical point theorem for C^1 functionals, which may fail to be sequentially weakly lower semicontinuous, established by the same author. Here, we will apply classical variational methods, regularity theory and truncation arguments. To establish the multiplicity of solutions, we will make use of a Mountain Pass Theorem by Pucci-Serrin [12] which applies in the case in which the energy functional possesses at least two (not necessarily strict) local minima. In our case, the energy functional associated to problem (1.1), with f

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2000 *Mathematics Subject Classification.* 35J20, 35J25.
Key words and phrases. Existence and multiplicity of solutions; Dirichlet problem; growth condition; critical point theorem.

Submitted May 20, 2014. Published September 26, 2014.

suitably truncated, admits a global minimum with negative energy, and a local minimum at 0. Our multiplicity result extends to more general nonlinearities [4, Theorem 1]. We refer the reader to [1, 2, 3, 7, 10] for other existence and multiplicity results for problem (1.1) involving nonlinearities with arbitrary growth.

2. Main results

Throughout this section, Ω is a bounded smooth domain in $\mathbb{R}^N$, and $f : \Omega\times\mathbb{R} \to \mathbb{R}$ is a Carathèodory function. The solutions of problem (1.1) will be understood in the weak sense. Therefore, a function $u \in W_0^{1,p}(\Omega)$ is a (weak) solution of problem (1.1) if and only if, for every $v \in W_0^{1,p}(\Omega)$:

(1) the function $x \in \Omega \to f(x, u(x))v(x)$ is summable in Ω;
(2) $\int_\Omega |\nabla u(x)|^{p-2}\nabla u(x)\nabla v(x)dx - \lambda \int_\Omega f(x, u(x))v(x)dx = 0$.

2.1. **Existence of solutions.** The next Lemma follows by applying the well known Moser's iterative scheme ([6, 11]) and standard regularity results ([9]).

Lemma 2.1. *Let $\gamma > \max\{1, \frac{N}{p}\}$. For each $h \in L^\gamma(\Omega)$ (resp. $h \in L^\infty(\Omega)$) denote by $u_h \in W_0^{1,p}(\Omega)$ the (unique) weak solution of the problem*

$$\begin{gathered} -\Delta_p u = h(x) \quad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

Then $u_h \in C^1(\overline{\Omega})$ and

$$C_\gamma := \sup_{h\in L^\gamma(\Omega)\setminus\{0\}} \frac{\max_{\overline{\Omega}} |u_h|}{\|h\|_\gamma^{\frac{1}{p-1}}} \quad (\text{resp. } C_\infty := \sup_{h\in L^\infty(\Omega)\setminus\{0\}} \frac{\max_{\overline{\Omega}} |u_h|}{\|h\|_\infty^{\frac{1}{p-1}}})$$

is a positive finite constant.

Our existence result reads as follows:

Theorem 2.2. *Assume that the following conditions hold:*

(i) *there exist $C > 0$ and $\gamma \in]\max\{1, \frac{N}{p}\}, +\infty]$ such that $\sup_{|t|\le C} |f(\cdot, t)| \in L^\gamma(\Omega)$.*
(ii) *there exist a closed ball $B_r(x_0) \subset \Omega$ and $\eta \in \mathbb{R} \setminus \{0\}$, with $|\eta| \le C$, such that*

$$\begin{aligned} \Lambda_1(\eta) &:= p\big(\frac{r}{|\eta|}\big)^p \int_0^1 (1-t)^{N-1} \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x, \eta t)dt \\ &> \big(\frac{C_\gamma}{C}\big)^{p-1} \|\sup_{|t|\le C} |f(\cdot, t)|\|_\gamma =: \Lambda_2. \end{aligned}$$

Then, for each $\lambda \in]\Lambda_1(\eta)^{-1}, \Lambda_2^{-1}]$, problem (1.1) admits at least a weak solution $u_\lambda \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ such that

$$\frac{1}{p}\|u_\lambda\|^p < \lambda \int_\Omega \Big(\int_0^{u_\lambda(x)} f(x, t)dt\Big)dx. \tag{2.1}$$

Proof. Let $C > 0$ be as in the hypotheses and define

$$f_C(x, t) = \begin{cases} f(x, -C) & \text{if } (x, t) \in \Omega\times] - \infty, -C[, \\ f(x, t) & \text{if } (x, t) \in \Omega \times [-C, C], \\ f(x, C) & \text{if } (x, t) \in \Omega\times]C, +\infty[. \end{cases} \tag{2.2}$$

Moreover, for each $\lambda > 0$, put

$$\Psi_\lambda(u) = \frac{1}{p}\|u\|^p - \lambda \int_\Omega \Big(\int_0^{u(x)} f_C(x,t)dt\Big)dx \tag{2.3}$$

for every $u \in W_0^{1,p}(\Omega)$. From $i)$ and the definition of f_C, we have that Ψ_λ is of class C^1 in $W_0^{1,p}(\Omega)$, sequentially weakly lower semicontinuous and coercive. Hence, it admits a global minimum $u_\lambda \in W_0^{1,p}(\Omega)$ which is a weak solution of the problem

$$\begin{gathered} -\Delta_p u = \lambda f_C(x,u) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega. \end{gathered}$$

From assumption (i) and Lemma 2.1 we have $u_\lambda \in C^1(\overline{\Omega})$ and

$$\|u_\lambda\|_\infty \le C_\gamma \lambda^{\frac{1}{p-1}} \|\sup_{|t|\le C} |f(\cdot,t)|\|_\gamma^{\frac{1}{p-1}}.$$

In particular, if $\lambda \le \Lambda_2^{-1}$ we obtain $\|u_\lambda\|_\infty \le C$. Consequently, u_λ is a weak solution of problem (1.1). Now, let η and $B_r(x_0)$ be as in the hypotheses. Let us to show that, if $\lambda > \Lambda_1^{-1}$, then inequality (2.1) holds. To this end, it is sufficient to show that $\Psi_\lambda(\varphi) < 0$ for some $\varphi \in W_0^{1,p}(\Omega)$. Define

$$\varphi(x) = \begin{cases} \frac{\eta}{r}(r - |x-x_0|) & \text{if } x \in B_r(x_0), \\ 0 & \text{if } x \in \Omega \setminus B_r(x_0). \end{cases}$$

Observe that $\varphi(x) \in [0, C]$ for all $x \in \Omega$. Thus, if we denote by ω_N the volume of the unit ball in $\mathbb{R}^N$ and use the polar coordinates and the integration by parts formula, we can compute $\Psi_\lambda(\varphi)$ as follows

$$\begin{aligned} &\Psi_\lambda(\varphi) \\ &= \frac{1}{p}\omega_N r^{N-p}|\eta|^p - \lambda \int_{B_r(x_0)} \Big(\int_0^{\varphi(x)} f(x,t)dt\Big)dx \\ &\le \frac{1}{p}\omega_N r^{N-p}|\eta|^p - \lambda N \omega_N \int_0^r \Big(\int_0^{\eta(1-\frac{\rho}{r})} \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,t)dt\Big)\rho^{N-1}d\rho \\ &= \frac{1}{p}\omega_N r^{N-p}|\eta|^p - \lambda N \omega_N r^N \int_0^1 \Big(\int_0^{\eta\rho} \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,t)dt\Big)(1-\rho)^{N-1}d\rho \\ &= \frac{1}{p}\omega_N r^{N-p}|\eta|^p - \lambda \omega_N r^N \int_0^1 (1-t)^N \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,\eta t)dt \end{aligned}$$

From $\lambda > \Lambda_1^{-1}$, we promptly obtain $\Psi_\lambda(\varphi) < 0$. □

Remark 2.3. Observe that the key inequality $\Lambda_1 > \Lambda_2$ in Theorem 2.2 is automatically satisfied if $\limsup_{\eta\to 0} \Lambda_1(\eta) = +\infty$. This is true, for instance, if

$$\lim_{\xi\to 0+} \frac{\int_0^\xi \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,t)dt}{|\xi|^p} = +\infty. \tag{2.4}$$

Indeed, putting $F(\xi) = \int_0^\xi \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,t)dt$ for short, we have

$$\frac{\int_0^1 (1-t)^N \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,\eta t)dt}{|\eta|^p} = N\frac{\int_0^\eta (\eta-\xi)^{N-1}F(\xi)d\xi}{|\eta|^{N+p}}. \tag{2.5}$$

Moreover, one has

$$\frac{d^i}{d\eta^i}\int_0^\eta (\eta-\xi)^{N-1}F(\xi)d\xi = (N-1)\cdots(N-i)\int_0^\eta (\eta-\xi)^{N-i-1}F(\xi)d\xi$$

for all $i = 1,\dots,N-1$, and

$$\frac{d^N}{d\eta^N}\int_0^\eta (\eta-\xi)^{N-1}F(\xi)d\xi = (N-1)!F(\eta).$$

Therefore, using (2.4), (2.5) and the de L'Hopital rule, we easily obtain

$$\lim_{\eta\to 0}\frac{\int_0^1 (1-t)^N \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,\eta t)dt}{|\eta|^p} = +\infty,$$

that is to say $\lim_{\eta\to 0}\Lambda_1(\eta) = +\infty$.

If f is nonnegative, i.e., if F is nondecreasing (and so nonnegative in $[0,+\infty[$ and non-positive in $]-\infty,0]$), then to guarantee the limit $\limsup_{\eta\to 0}\Lambda_1(\eta) = +\infty$ it is sufficient requiring that

$$\limsup_{\xi\to 0}\frac{F(\xi)}{|\xi|^p} = +\infty. \tag{2.6}$$

Indeed, let $\{\xi_n\}\subset\mathbb{R}\setminus\{0\}$ be a sequence such that $\xi_n\to 0$ and

$$\frac{F(\xi_n)}{|\xi_n|^p}\to +\infty. \tag{2.7}$$

Without loss of generality, we can suppose $\xi_n > 0$, for all $n\in\mathbb{N}$. Then, we have

$$\begin{aligned}
\frac{\int_0^{2\xi_n}(2\xi_n-\xi)^{N-1}F(\xi)d\xi}{(2\xi_n)^{N+p}} &\ge \frac{\int_{\xi_n}^{2\xi_n}(2\xi_n-\xi)^{N-1}F(\xi)d\xi}{(2\xi_n)^{N+p}}\\
&\ge \frac{F(\xi_n)}{(\xi_n)^p}\cdot\frac{\int_{\xi_n}^{2\xi_n}(2\xi_n-\xi)^{N-1}d\xi}{2^{N+p}\xi_n^N}\\
&= \frac{1}{N2^{N+p}}\frac{F(\xi_n)}{(\xi_n)^p}
\end{aligned}$$

for all $n\in\mathbb{N}$. Hence, in view of (2.5) and (2.7), we have

$$\lim_{n\to+\infty}\frac{\int_0^1 (1-t)^N \operatorname{ess\,inf}_{x\in B_r(x_0)} f(x,2\xi_n t)dt}{|2\xi_n|^p} = +\infty,$$

that is to say $\limsup_{\eta\to 0}\Lambda_1(\eta) = +\infty$.

Remark 2.4. For applications of Theorem 2.2, it is useful to have upper estimates of the constant C_γ ($\gamma\in]\max\{1,\frac{N}{p}\},+\infty]$). For the constant C_∞ an upper estimate is easy to find. Indeed, let $\bar{x}\in\mathbb{R}^n$ and $R>0$ such that $B_R(\bar{x})\supseteq\Omega$ and define

$$u_R(x) = R^{\frac{p}{p-1}} - |x-\bar{x}|^{\frac{p}{p-1}},\quad \text{for all } x\in B_R(\bar{x}).$$

Then, $u_R\in C_0^1(\overline{B_R(\bar{x})})$ and a simple computation shows that

$$-\Delta_p u_R(x) = N\big(\frac{p}{p-1}\big)^{p-1}\quad \text{for all } x\in B_R(\bar{x}).$$

Now, let $h\in L^\infty(\Omega)$ and put $M = \operatorname{ess\,sup}_\Omega |h| = \|h\|_\infty$. Also, let u_h be the unique solution of the problem

$$-\Delta_p u = h(x)\quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \partial\Omega\,.$$

Then, we have

$$-\Delta_p\Big(\frac{u_h(x)}{M^{\frac{1}{p-1}}}\Big) = \frac{h(x)}{M} \le 1 = \frac{1}{N}\big(\frac{p-1}{p}\big)^{p-1}(-\Delta_p u_R(x)) = -\Delta_p\Big(\frac{p-1}{pN^{\frac{1}{p-1}}}u_R(x)\Big),$$

for all $x \in \Omega$. Since

$$\frac{u_h(x)}{M^{\frac{1}{p-1}}} \le \frac{p-1}{pN^{\frac{1}{p-1}}}u_R(x), \quad \text{for all } x \in \partial\Omega,$$

by the comparison principle for the p-Laplacian, one has

$$u_h(x) \le \frac{p-1}{pN^{\frac{1}{p-1}}}M^{\frac{1}{p-1}}u_R(x) \le \frac{p-1}{pN^{\frac{1}{p-1}}}R^{\frac{p}{p-1}}\|h\|_\infty^{\frac{1}{p-1}}, \quad \text{for all } x \in \Omega.$$

It follows that

$$C_\infty \le \frac{p-1}{pN^{\frac{1}{p-1}}}R^{\frac{p}{p-1}}\,.$$

Remark 2.5. Note that, if $f(x,t) = 0$ for all $(x,t) \in \Omega\times]-\infty, 0]$ and $f(x,t) \ge 0$ for all $(x,t) \in \Omega\times]0, +\infty]$, the nonzero solutions of problem (1.1) are positive in Ω by the Strong Maximum Principle. Thus, if f satisfies the above condition, we can compare Theorem 2.2 with [5, Theorem 8.1]. In our case, differently to [5], where a polynomial growth up to the critical exponent on f was imposed (being the same function independent of $x \in \Omega$), to guarantee the existence of a positive solution for small $\lambda's$, besides (2.6) and the summability condition i), no other condition is required on f.

2.2. **Multiplicity of solutions.** We now state and proof our multiplicity result.

Theorem 2.6. *Assume that f satisfies* (i) *and* (ii) *of Theorem 2.2. Moreover, suppose that there exists $\delta > 0$ such that*

$$\operatorname{ess\,sup}_{x\in\Omega}\int_0^\xi f(x,t)dt \le 0, \quad \textit{for all } \xi \in [-\delta, \delta]. \tag{2.8}$$

Then, for each $\lambda \in]\Lambda_1(\eta)^{-1}, \Lambda_2^{-1}]$, problem (1.1) *admits at least two weak solutions $u_\lambda, v_\lambda \in W_0^{1,p}(\Omega) \cap C^1(\overline{\Omega})$ such that*

$$\frac{1}{p}\|u_\lambda\|^p < \lambda\int_\Omega\Big(\int_0^{u_\lambda(x)} f(x,t)dt\Big)dx, \quad \frac{1}{p}\|v_\lambda\|^p > \lambda\int_\Omega\Big(\int_0^{v_\lambda(x)} f(x,t)dt\Big)dx.$$

Proof. Let f_C be as in (2.2) and, for $\lambda \in]\Lambda_1(\eta)^{-1}, \Lambda_2^{-1}]$, let Ψ_λ be as in (2.3). From the proof of Theorem 2.2, we know that Ψ_λ is a C^1-functional that admits a global minimum $u_\lambda \in W_0^{1,p}(\Omega)$ such that $\Psi_\lambda(u_\lambda) < 0$. Moreover, again from the proof of Theorem 2.2, we have that every critical point of Ψ_λ is a weak solution of problem (1.1). Thus, if we show that $u = 0$ is a local minimum for Ψ_λ, conclusion follows by the mountain pass theorem of Pucci-Serrin [12]. To this end, it is sufficient to show that $u = 0$ is a local minimum for Ψ_λ in the $C_0^1(\overline{\Omega})$ topology (see [8, Theorem 3.1]). Indeed, for each sequence $\{u_n\}_{n\in\mathbb{N}}$ in $C_0^1(\overline{\Omega})$ such that $\lim_{n\to+\infty}\|u_n\|_{C^1(\overline{\Omega})} = 0$, we have, thanks to (2.8), $\Psi_\lambda(u_n) \ge 0$ for $n \in \mathbb{N}$ large enough. Hence, 0 is a local minimum for Ψ_λ. $\square$

Here is a consequence of Theorem 2.6.

Corollary 2.7. *Let $R > 0$ be the radius of the smallest ball containing Ω and let $h, g : [0, +\infty[\to \mathbb{R}$ be two continuous functions such that $h(0) = g(0) = 0$ and*

$$\lim_{\xi\to 0^+} \frac{\int_0^\xi h(t)dt}{\xi^p} = +\infty, \tag{2.9}$$

$$\lim_{\xi\to 0^+} \frac{\int_0^\xi g(t)dt}{\xi^s} = +\infty, \quad \text{for some } s \in]0, p[. \tag{2.10}$$

Finally, let

$$M = \sup_{C>0} \big\{ \frac{N}{R^p} \big(\frac{Cp}{p-1} \big)^{p-1} \big(\sup_{0\le t\le C} |h(t)| \big)^{-1} \big\}.$$

Then, for each $\lambda \in]0, M[$, there exists $\mu_\lambda > 0$ such that, for each $\mu \in]0, \mu_\lambda[$, the problem

$$\begin{gathered} -\Delta_p u = \lambda(h(u) - \mu g(u)) \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

admits at least two nonzero and nonnegative solutions.

Proof. Let $\lambda \in]0, M[$ and let $C > 0$ be such that

$$\lambda < \frac{N}{R^p} \Big(\frac{Cp}{p-1} \Big)^{p-1} \Big(\sup_{0\le t\le C} |h(t)| \Big)^{-1}.$$

Put $f(x, t) = h(t)$ for each $(x, t) \in \Omega \times [0, +\infty[$ and $f(x, t) = 0$ for each $(x, t) \in \Omega \times [-\infty, 0[$. Let $B_r(x_0)$ be a closed ball contained in Ω. Thanks to (2.9) and Remark 2.3, we have

$$\lim_{\eta\to 0^+} \Lambda_1(\eta) = \lim_{\eta\to 0^+} p\big(\frac{r}{|\eta|}\big)^p \int_0^1 (1-t)^{N-1} h(\eta t)dt = +\infty.$$

Therefore, we can find $\eta_0 \in]0, C[$ and $\mu_\lambda > 0$ such that

$$\begin{aligned} &\Big[p\big(\frac{r}{\eta_0}\big)^p \int_0^1 (1-t)^{N-1}(h(\eta_0 t) - \mu g(\eta_0 t))dt \Big]^{-1} \\ &< \lambda < \frac{N}{R^p} \big(\frac{Cp}{p-1} \big)^{p-1} \Big(\sup_{0\le t\le C} |h(t) - \mu g(t)| \Big)^{-1}. \end{aligned}$$

for all $\mu \in]0, \mu_\lambda[$. From Remark 2.4, it turns out that

$$\frac{N}{R^p} \big(\frac{Cp}{p-1} \big)^{p-1} \Big(\sup_{0\le t\le C} |h(t) - \mu g(t)| \Big)^{-1} < \Big[\big(\frac{C_\infty}{C} \big)^{p-1} \sup_{0\le t\le C} |h(t) - \mu g(t)| \Big]^{-1}.$$

Moreover, from (2.9) and (2.10), for each $\mu \in]0, \mu_\lambda[$, there exists $\delta_\mu > 0$ such that

$$\int_0^\xi (h(t) - \mu g(t))dt \le 0,$$

for each $\xi \in [0, \delta_\mu]$. Conclusion now follows from Theorem 2.6 applied to the function $h(t) - \mu g(t)$, extended by continuity to the whole real axis by putting $h(t) - \mu g(t) = 0$ for all $t \in] - \infty, 0[$, and from the maximum principle. □

Example 2.8. Let $R > 0$ be as in Corollary 2.7. Moreover, let $s \in]1, p[$ and $r \in]1, s[$. Then, Corollary 2.7 can be applied to the functions $h(t) = t^{s-1}e^t$ and

$g(t) = t^{r-1}e^t$. In this case, the constant M can be explicitly computed and one has:

$$M = \sup_{C>0} \Big\{ \frac{N}{R^p} \big(\frac{Cp}{p-1}\big)^{p-1} \Big(\sup_{0\le t\le C} |h(t)| \Big)^{-1} \Big\} = \frac{N}{R^p} \big(\frac{p}{p-1}\big)^{p-1} \big(\frac{p-s}{e}\big)^{p-s}.$$

We conclude that, for each $\lambda \in]0, M[$, there exists $\mu_\lambda > 0$ such that for each $\mu \in]0, \mu_\lambda[$, the problem

$$\begin{gathered} -\Delta_p u = \lambda(u^{s-1} - \mu u^{r-1})e^u \quad \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega \end{gathered}$$

admits at least two nonzero and nonnegative solutions.

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