Dual Boundary Element Method for Fatigue Crack Growth: Implementation of the Richard's Criterion

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Abstract. A new criterion for fatigue crack growth, whose accuracy was previously tested in the literature with the Finite Element Method, is here adopted with a Dual Boundary Element formulation. The fatigue crack growth of an elliptical inclined crack, embedded in a three dimensional cylindrical bar, is analyzed. In this way in addition to the propagation angle estimated by the Sih's criterion, it is possible to take into account a *twist* propagation angle. The two propagation criteria are compared in terms of shape of the propagated crack and in terms of SIFs along the crack front. The efficiency of the Dual Boundary Element Method in this study is highlighted.

Introduction

It is well established that computational modeling may provide a formidable tool for design and maintenance of engineering structures.
To date the boundary element method (BEM) has proved very effective for fracture mechanics problems, without the limitations of an extremely refined mesh in the crack tip, and the requirement of a continuous re-meshing for crack growth simulations typical of the finite element method (FEM) [1].
The Dual Boundary Element Method (DBEM), developed by Portela, Aliabadi and Rooke [2] for 2D problems and then by Mi and Aliabadi [3] for 3D problems, appears to be a more general and computationally efficient way to model crack problems with respect to the multi-region method developed by Blandford et al. [4]. The DBEM is a single region formulation, applying the displacement boundary equation on one crack surface and the traction boundary equation on the other. The advantage of the method is its robustness in modeling the crack growth, with need of little re-meshing of the original model.
It is well-known that a sequence of increasing and decreasing loads, could lead to an increase in the crack front at each step, even though the maximum stress intensity factor may be much less than the critical value; in the framework of the linear fracture mechanics it is possible to use the Paris law [5], that nowadays has gained general acceptance, to predict the crack incremental size.
On the other hand, there is no unique accepted criterion regarding crack growth direction, especially for 3D cases; the most common and used criterion, for its precision and versatility in numerical simulation but also for the possibility to use it both in two and three dimensions, is the minimum strain energy density criterion formulated by Sih [6]; even if it is the most used criterion, it has a some drawback: in three dimensional cases it is insensitive to Mode III stress intensity factors, and for this reason, the twist propagation angle is always equal to zero.
In the present study the numerical simulation of the crack growth of an embedded inclined elliptical crack subjected to Mixed Mode load conditions is presented, adopting a new criterion by Richard [7], that being sensitive to Mode III, estimates the crack's twisting. As results the propagated crack patterns for the two different criteria are shown. Furthermore, for the Richard's criterion, the trend of the crack growth angles and stress intensity factors along the crack front are presented, together with comparison graphics between Sih's and Richard's results.

Dual Boundary Element Method

Dual Boundary Integral Equation Formulation for Crack Problems. Let us consider a cracked body and let Γ^{+} and Γ^{-} be the upper and lower crack surfaces, and Γ^{e} the rest of the boundary. The DBEM formulation is obtained by collocating the displacement integral equation on the boundary and on one of the crack surfaces and the traction boundary integral equation on the other crack face.
The equation collocated on the boundary is the classical one and it is not recalled here, while the equations written for the crack surfaces are:

$$\frac{1}{2}\,u_i(\boldsymbol{x}^{+}) + \frac{1}{2}\,u_i(\boldsymbol{x}^{-}) + \int_{\Gamma} T_{ij}^{*}\,(\boldsymbol{x}^{+},\boldsymbol{x})u_i(\boldsymbol{x})d\Gamma(\mathbf{x}) = \int_{\Gamma} U_{ij}^{*}\,(\boldsymbol{x}^{+},\boldsymbol{x})t_j(\boldsymbol{x})d\Gamma(\mathbf{x}) \tag{1}$$

$$\frac{1}{2}\,t_j(\boldsymbol{x}^{-}) - \frac{1}{2}\,t_j(\boldsymbol{x}^{+}) + n_i(\boldsymbol{x}^{-})\int_{\Gamma} T_{ijk}^{*}\,(\boldsymbol{x}^{-},\boldsymbol{x})u_k(\boldsymbol{x})d\Gamma(\mathbf{x}) = n_i(\boldsymbol{x}^{-})\int_{\Gamma} U_{ijk}^{*}\,(\boldsymbol{x}^{-},\boldsymbol{x})t_k(\boldsymbol{x})d\Gamma(\mathbf{x}) \tag{2}$$

Where $i,j = 1,2,3$; $U_{ij}^{*}(\mathbf{x}^{+},\mathbf{x})$ and $T_{ij}^{*}(\mathbf{x}^{+},\mathbf{x})$ are respectively Kelvin displacement and traction fundamental solutions, $U_{ijk}^{*}(\mathbf{x}^{+},\mathbf{x})$ and $T_{ijk}^{*}(\mathbf{x}^{+},\mathbf{x})$ are obtained from the derivatives of the fundamental solutions; in eq.(1) the integral at the left-hand side is a Cauchy principal value integral as the integral at the right-hand side of eq.(2); while the first integral in eq.(2) stands for an Hadamard principal value integral.

Crack Modelling Strategy. The strategy used for DBEM modelling is given in [3] and it can be summarized in the following points

- The crack boundaries are modeled with discontinuous quadratic elements;
- The surfaces intersecting a crack surface are modeled with edge-discontinuous quadrilateral and triangular quadratic elements;
- Continuous quadratic elements are used along the remaining boundary;
- For collocation on the crack surface Γ^{+} the displacement equation in eq.(1) is applied;
- For collocation on the other crack surface Γ^{-} the traction equation in eq.(2) is applied;
- The usual boundary displacement equation is applied for collocation on all other surfaces.

For further details the interested reader is referred to [3].

Fatigue Crack Growth

When a cracked body is subjected to a generic loading system, the movements of the upper and lower surfaces of the crack with respect to each other can be described using three basic modes:

- Mode I or opening mode, where the two crack surfaces are pulled apart;
- Mode II or sliding mode, where the two crack surfaces slide over each other along the crack line;
- Mode III or tearing mode, where the crack surfaces slide over each other perpendicular to the crack line.

With the superposition of these three basic modes any crack deformation can be described, and under the hypothesis of linear elastic fracture mechanics (LEFM), the determination of the rate of crack growth in a loaded structure subjected to fatigue loading is correlated only to the knowledge of the stress intensity factors, in this way the behavior of the cracked body can be fully described.
The fatigue loading is a sequence of increasing and decreasing load (cyclic), that can lead to an increase in crack front even if the maximum stress intensity factor may be much less than the critical one.
The goal for the designer is to establish the necessary number of cycles for a crack to extend from some initial length to a pre-imposed one.
The typical Paris' sigmoidal curve relates the rate of crack growth per load cycle: da/dn, with the applied stress intensity factor range: $\Delta K = K_{max} - K_{min}$; in this log-log plot can be recognized three different zones [5]:

- The first region, where the crack growth goes asymptotically to zero as ΔK approaches a threshold value: ΔK_{th} ; this value represents the fatigue limit, for stress intensity factors below ΔK_{th} there is no crack growth;
- In the second region, the $\ln(da/dn)$ tends to vary linearly with respect to the $\ln \Delta K$;
- In the third region it can be seen a drastic acceleration as K_{max} approaches K_c the fracture toughness of the material.

To describe the behaviour in the linear region, Paris et al. [8] developed an empirical formula:

$$\frac{da}{dn} = f(\Delta K) \tag{3}$$

To take into account also other factors, among which: load frequency, environment and mean load; Paris and Erdogan [5] suggested another law depending on two empirical material constants: *C* and *m*.

$$\frac{da}{dn} = C\, \Delta K^m \tag{4}$$

Equation (4) is generally called Paris law, and has gained worldwide acceptance in engineering practice.
To obtain a generalized fatigue crack growth formula which takes into account the combined effect of Mode I and Mode II, Tanaka [9] proposed an expression for ΔK:

$$\Delta K_{eff} = (K_I^4 + 8\, K_{II}^4)^{1/4} \tag{5}$$

Criteria for Crack Growth Propagation

Spatial Mixed Mode problems are characterised by the superposition of the three fracture modes. Exist only few fracture criteria to describe 3D Mixed Mode problems, two of them will be described in the following with a brief comparison.

Minimum Strain Energy Density Criterion (S-Criterion). Formulated by Sih [6] to date is the most popular and used criterion for 3D problems, because it seems to be able to handle very well three dimensional crack growth, taking into consideration the three stress intensity factors.
The explicit expression for the strain energy density around the crack front can be written as a function of the *strain energy density factor*: *S* [7]

$$\frac{dW}{dV} = \frac{S(\varphi, \psi)}{r} + O(1) \tag{6}$$

where $S(\varphi, \psi)$ is given by

$$S(\varphi, \psi) = \frac{1}{\cos \psi}\ (a_{11}K_I^2 + 2a_{12}K_I K_{II} + a_{22}K_{II}^2 + a_{33}K_{III}^2) \tag{7}$$

Where $a_{11}, a_{12}, a_{22}, a_{33}$ depend only by φ, υ and μ that is the shear modulus of elasticity and υ is the Poisson's ratio. The crack angles φ_0 and ψ_0 (elevation and twisting) shown in Fig.1 are derived by minimizing S of eq.(7):

$$\left.\frac{\partial S}{\partial \varphi}\right|_{\varphi=\varphi_0} = 0 \qquad \text{and} \qquad \left.\frac{\partial S}{\partial \psi}\right|_{\psi=\psi_0} = 0 \tag{8}$$

It is apparent from eq.(7), that the minimum of $S(\varphi)$ always occurs in the normal plane of the crack front curve, namely when $\cos \psi = 1 \Rightarrow \psi = 0$, independently of the Mixed Mode combination; also the partial derivative of S by φ is independent of ψ_0 as well as K_{III}.

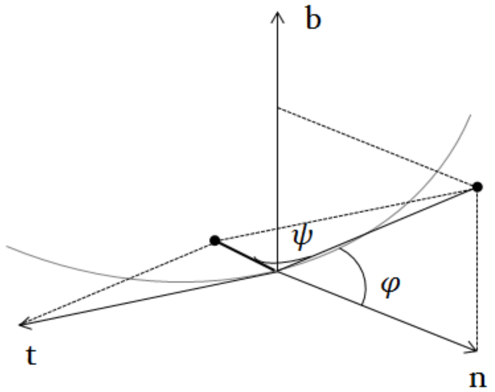


Figure 1: The two different rotations: $\boldsymbol{\varphi}$, $\boldsymbol{\psi}$.

Therefore this criterion is **insensitive** to Mode III [7].

Richard's Criterion. In order to take into account the twist rotation (ψ angle), considering in this way also the Mode III, and to simplify the prediction of crack growth, Richard developing approximation functions, proposed a criterion whose efficiency is tested experimentally.
The angles can be written as:

$$\varphi_0 = \mp\left[A\frac{|K_I|}{K_I + |K_{II}| + |K_{III}|} + B\left(\frac{|K_{II}|}{K_I + |K_{II}| + |K_{III}|}\right)^2\right] \tag{9}$$

where $\varphi_0 < 0°$ for $K_{II} > 0$ and $\varphi_0 > 0°$ for $K_{II} \leq 0$.

$$\psi_0 = \mp\left[C\frac{|K_{III}|}{K_I + |K_{II}| + |K_{III}|} + D\left(\frac{|K_{III}|}{K_I + |K_{II}| + |K_{III}|}\right)^2\right] \tag{10}$$

where $\psi_0 < 0°$ for $K_{III} > 0$ and $\psi_0 > 0°$ for $K_{III} \leq 0$.
With $A = 140°, B = -70°, C = 78°, D = -33°$ eq.(9,10) are in good agreement with the crack deflection angles predicted by another criterion: the Schöllmann criterion [7].
Richard proposes also an expression for an equivalent stress intensity factor: K_v to evaluate if an unstable crack growth will occur:

$$K_v = \frac{K_I}{2} + \frac{1}{2}\sqrt{K_I^2 + 4\left(\frac{K_{IC}}{K_{IIC}}K_{II}\right)^2 + 4\left(\frac{K_{IC}}{K_{IIIC}}K_{III}\right)^2} \geq K_{IC} \tag{11}$$

where K_{IC}, K_{IIC} and K_{IIIC} are the fracture toughness; also for this expression, if $K_{IC}/K_{IIC} = 1.155$ and $K_{IC}/K_{IIIC} = 1.0$; eq.(11) is in a good agreement with the K_v predicted by the Schöllmann criterion.
In the same way the fatigue crack growth is possible, only if, defining the cyclic stress intensity factor as in eq.(12), it exceeds the threshold value $\Delta K_{I,th}$ and is smaller than $\Delta K_{I,c}$.

$$\Delta K_v = \frac{\Delta K_I}{2} + \frac{1}{2}\sqrt{\Delta K_I^2 + 4\left(\frac{\Delta K_{IC}}{\Delta K_{IIC}}\Delta K_{II}\right)^2 + 4\left(\frac{\Delta K_{IC}}{\Delta K_{IIIC}}\Delta K_{III}\right)^2} \tag{12}$$

To test the efficiency of this criterion, simulations were performed using the Finite Element Method and the program system ADAPCRACK3D [10].

Numerical Simulation of Crack Growth

Experiments are surely a fundamental part to understand the crack behaviour, but they are in most cases very expensive; for this reason, becomes necessary the developing of numerical techniques which allow the engineer to predict the crack evolution.
The Finite Element Method has been used in a lot of numerical simulations, by many researchers [11]; but the common drawback of this method is the need of a continuous crack re-meshing to follow the crack extension, especially when a 3D problem is described.
The main advantage of the Dual Boundary Element Method, is that the need of a re-meshing procedure is practically negligible, by virtue of the boundary formulation, and of the intrinsic characteristic of the single region analysis; in this way both embedded and edge crack can be studied with only a localized re-meshing on the free surfaces on the breaking cracks as the growth takes place [3,12].
To illustrate this procedure a mixed mode 3D crack growth problem is considered here: the fatigue cracking process is generated by a constant amplitude cyclic tensile loading applied on the upper and lower surfaces of a cylindrical bar of radius R and height h with an embedded inclined elliptical crack [3].
Initially crack surfaces are defined and the DBEM, is used to analyze the stress system; the three stress intensity factors are evaluated in three nodes for each element that forms the crack front; for a total of 16 elements and 48 nodes. The incremental direction was calculated using both the Sih and Richard's criterion, to show as the second is sensitive to the Mode III and thus to the twisting rotation. Four incremental steps were performed, fixing at each step the maximum incremental crack length at 0.2 times the crack semi-major axis. The incremental part of the crack is constructed using the incremental direction and size in the form of piecewise surfaces which vary linearly along the crack growth direction.
After the necessary modification of the boundary mesh, the analysis carries on taking into account the new configuration.
This method can be utilized until the predefined crack length is reached or the effective stress intensity factor has exceeded the fracture toughness of the material.

Determination of Incremental Direction and Size. To perform the incremental analysis is necessary to know two different parameters: the direction and the size of the crack incremental extension.
Using The **Sih criterion**, φ_0 is evaluated in the local coordinate plane perpendicular to the crack front Fig.1, minimizing numerically S with respect to φ. The resultant propagation directions can then be referred to the global system of coordinates and expressed as propagation unit vectors.
The size of the increment is evaluated using the Paris law eq.(4), and since linear elasticity is considered in this problem the maximum amount Δa_{max} of the increment corresponds to the crack front point where the maximum value of ΔK is reached.
Therefore, the incremental size at each step, and at each node can be evaluated by :

$$\Delta a = \Delta a_{max} \left(\frac{\Delta K}{\Delta K_{max}} \right) \tag{13}$$

where for ΔK was used the expression proposed by Gerstle, that takes into account all the three stress intensity factors:

$$\Delta K_{eff}^2 = (K_I + B\,|K_{III}|)^2 + 2K_{II}^2 \tag{14}$$

The same procedure is adopted for the **Richard's criterion**: for each node the three stress intensity factors are evaluated, and so, using eq.(9,10) φ_0 and ψ_0 are known; thus the resultant propagation vector will take into account not only the propagation angle evaluated by Sih but also the twisting rotation as shown in Fig.1.
In this way considering also the Mode III, the pattern of the deformed crack, will be surely more realistic and precise. A comparison between the Sih's and the Richard's deformed crack pattern, is shown in Fig.2.

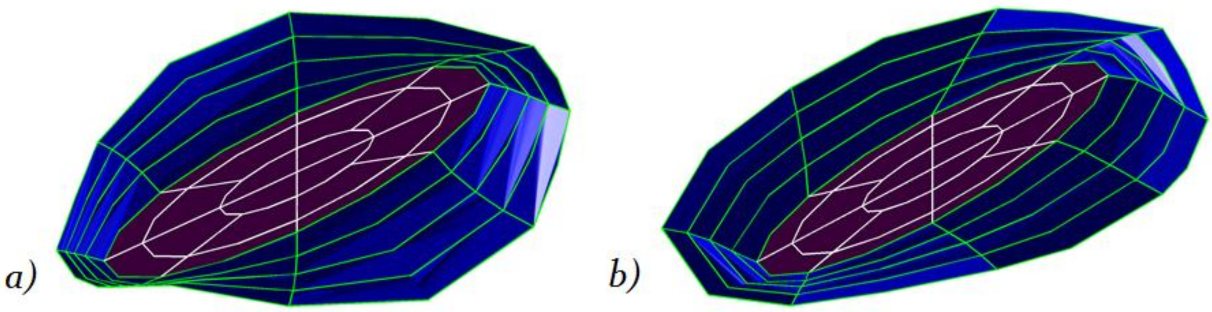
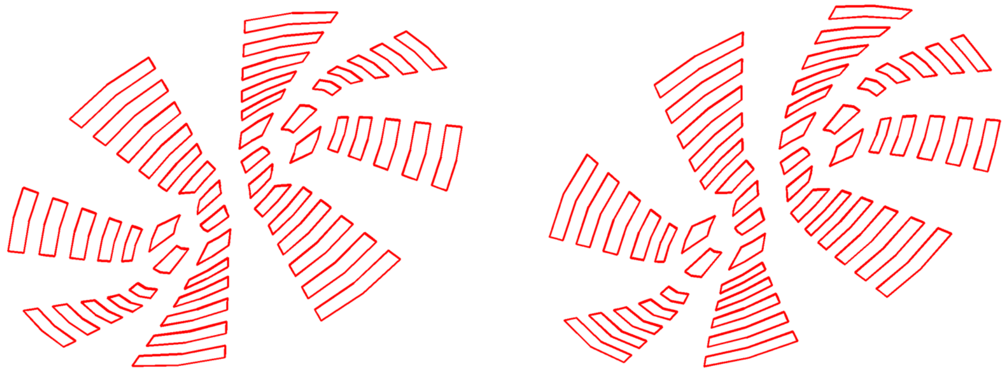


Figure 2: Comparison between the crack growth path with the Sih's criterion a) and the Richard's Criterion b)

The deformed crack surfaces after the last analysis are shown in Fig.3; in Fig.4 are also presented the crack growth angles: φ and ψ, along the crack front computed with the Richard's criterion, and in Fig.5 the three stress intensity factors: K_I, K_{II} and K_{III}, corresponding to each analysis normalized by the value of the stress applied.
It is worth to be notice that they are in good agreement with those found by Mi for the same case, using the Sih's criterion [3].



a) *b)*

Figure 3: Comparison between the deformed crack surfaces with the Sih's criterion a) and the Richard's Criterion b)

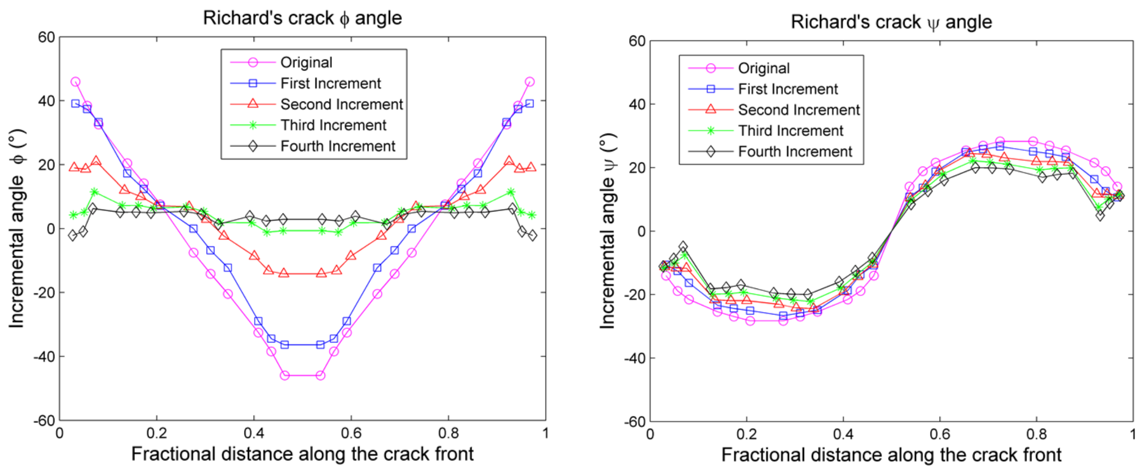


Figure 4: Crack growth angles along the front estimated with the Richard's criterion

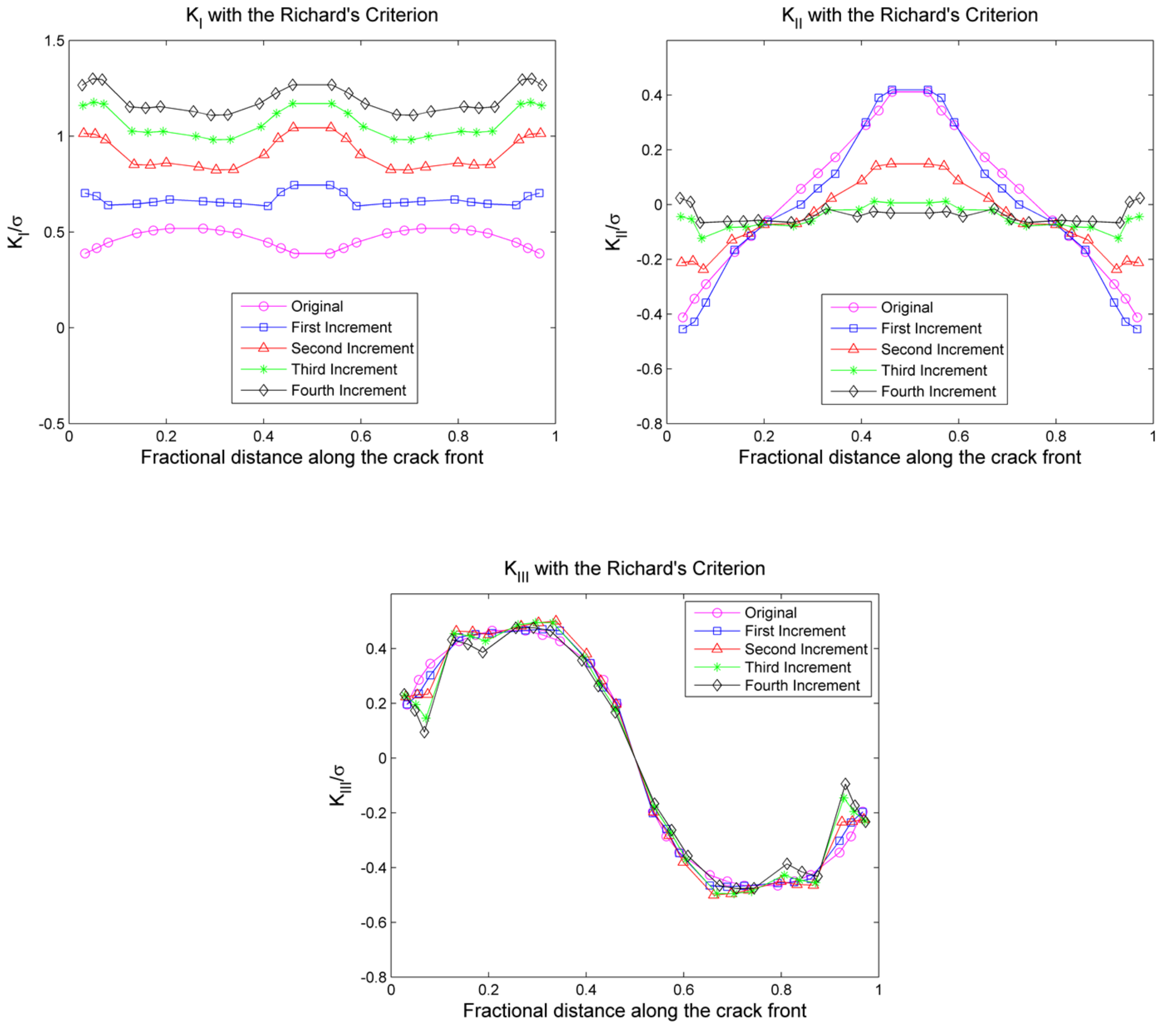


Figure 5: Trend of the normalized Stress Intensity Factors: K_I, K_{II} and K_{III}

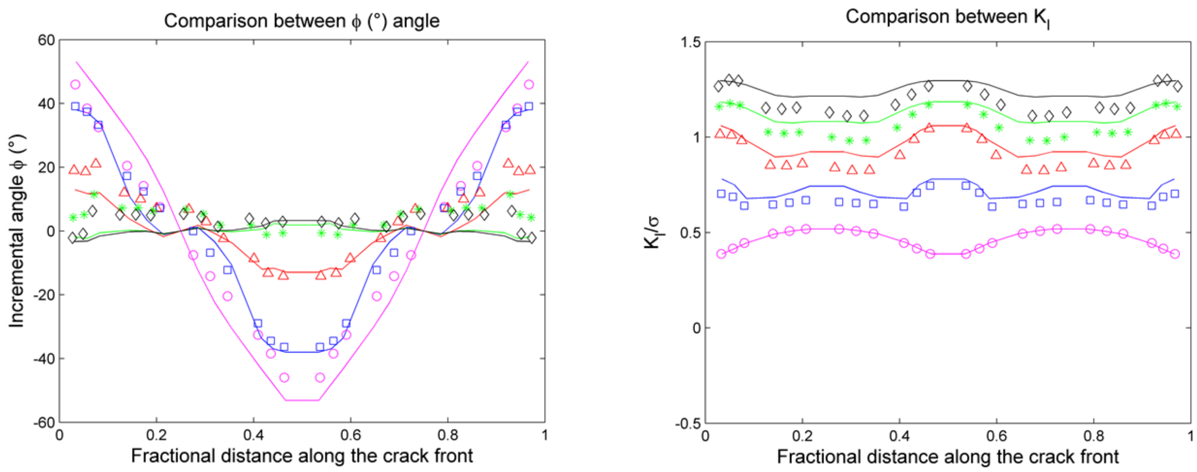


Figure 6: Comparison between the results obtained from Sih (solid lines) and Richard (markers)

Fig.(6-7) show respectively the comparison between the φ angles and the different stress intensity factors, found for each analysis with the Sih's criterion, represented by the solid lines, and those estimated with the Richard's criterion, represented by the markers.

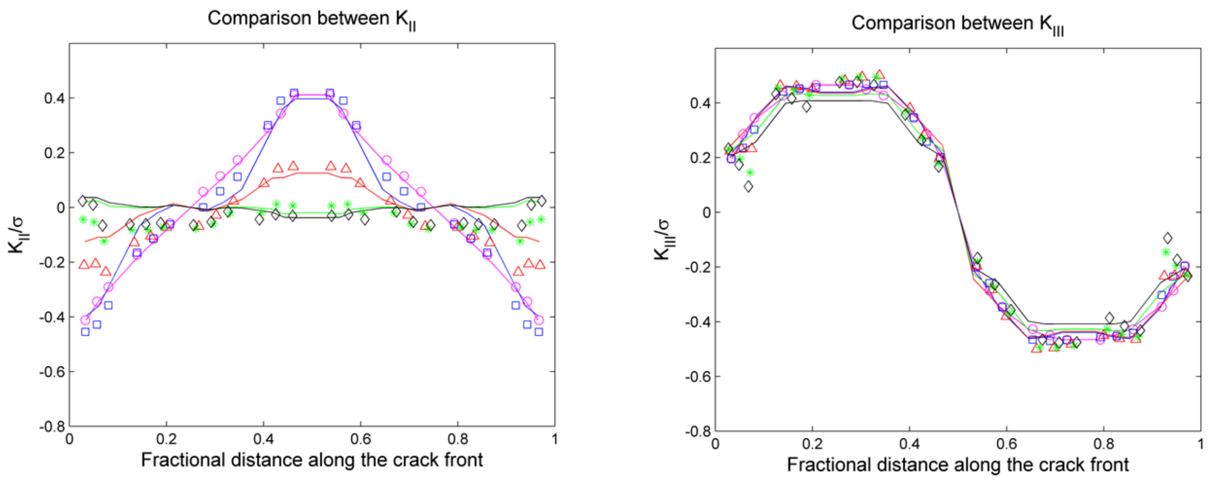


Figure 7: Comparison between the results obtained from Sih (solid lines) and Richard (markers)

Summary

A three-dimensional dual boundary element formulation for the analysis of the fatigue crack growth of an embedded inclined elliptical crack subjected to a fracture Mixed Mode has been adopted, to carry out a numerical simulation and thus, to estimate the new propagated crack pattern. Two different criteria have been used to calculate the crack growth direction, from them comparison a more realistic crack behavior is shown using the Richard's one. In this article are also shown the trends and the comparisons of the crack angles and the three stress intensity factors along the crack front, for each propagation.

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