From the fourteenth century to Cabrì: convoluted constructions of star polygons

Introduction

The first treatments of regular star polygons seem to date back to the fourteenth century, but a comprehensive theory on the subject was presented only in the nineteenth century by the mathematician Louis Poinsot.

After showing how star polygons are closely linked to the concept of prime numbers, I introduce here some constructions, easily reproducible with geometry software that allow us to investigate and see some nice and hidden property obtained by the scholars of the fourteenth century onwards.

Regular star polygons and prime numbers

Divide a circumference into n equal parts through n points; if we connect all the points in succession, through chords, we get what we recognize as a regular convex polygon. If we choose to connect the points, starting from any one of them in regular steps, two by two, or three by three or, generally, h by h, we get what is called a *regular star polygon*.

It is evident that we are able to create regular star polygons only for certain values of h.

Let us divide the circumference, for example, into 15 parts and let's start by connecting the points two by two. In order to close the figure, we return to the starting point after two full turns on the circumference. The polygon that is formed is like the one in Figure 1: a polygon of "order" 15 and "species" two. For h=7 we get again a regular star polygon of order 15 and species seven:

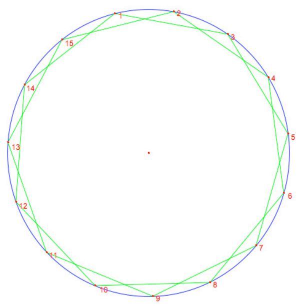


Figure 1: Regular star polygon with n=15 and h=2

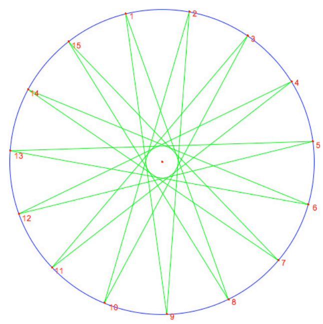


Figure 2: Regular star polygon with n=15 and h=7

If we try to connect the points three by three (h=3), the figure closes almost immediately and we finish before we touch all the points. By linking the points three by three, we return to the starting point at the fifth step and we draw only five segments. This happens because three is a divisor of 15. The same thing happens if we choose h=5 (also a divisor of 15): after three steps, we return to the starting point and we draw a triangle.

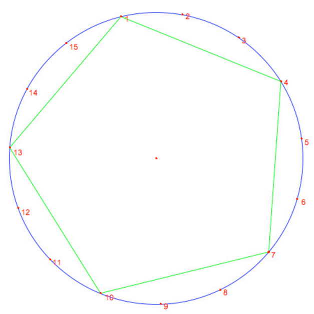


Figure 3: Regular polygon with n=15 and h=3

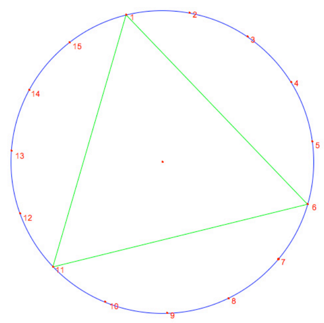


Figure 4: Regular polygon with n=15 and h=5

If we take a number that has no common divisors with 15 (for example, two), we return to the starting vertex and close the figure only if we touch all the points.

So it is possible to conclude that we are able to build regular star polygons only when n and h are coprime integers.

Through regular star polygons we can offer an alternative geometric instead of arithmetic definition of *coprime integer numbers* and also of *prime numbers*.

Mathematicians define polygons by *order*[1] and *species*: formally two polygons are of the same order if they have an equal number of sides while two polygons are of the same species if the sum of their angles is equal. The sum of angles varies from polygon to polygon within a given order depending on the value of h, which denotes the step chosen to connect the points.

A *prime number* (or a prime) is a natural number greater than *1* that has no positive divisors other than *1* and itself. A natural number greater than *1* that is not a prime number is called a composite number. Book VII of Euclid's Elements (circa 300 BC) contains the definition (number XI) and important theorems about primes, including the infinitude of primes and the fundamental theorem of arithmetic.

Two integers are said to be coprime if the only positive integer that evenly divides both of them is *1*.

The geometrical definitions for prime number and coprime numbers were given by the French mathematician Louis Poinsot (1777-1859) in *Mémoire sur les polygones et les polyedres*[2]: let n be the order of a regular star polygon and let h be its species. If we connect the n points A, B, C, … in regular steps h by h and we touch all the points before returning to the starting point, then number h is coprime to n.

But if we connect all the n points for any h but we can never return to the first one without going through all the others, then none of the h values divides n and n is a prime number.

In the latter case, obviously, we cannot draw any star polygon, but only convex polygons.

Several constructions from the fourteenth century

The ancient geometers studied only regular or irregular convex polygons and we must go back to Boezio's *De geometria*, to see what might be the first example of a regular star pentagon inscribed in a circle.[3]

At the beginning of the fourteenth century, Thomas Bradwardine (1290-1349)[4] created a theory of regular star polygons. He stated several propositions and reached some remarkable conclusions: the first regular star polygon of the second species is the five-sided polygon; the sum of the angles of the five-sided star polygon is equal to two right angles; heptagon is the first regular star polygon of the third species.

Bradwardine stated the general principle: the first regular star polygon of any species is obtained by extending the sides of the third constructible figure of the previous species.

For example, let species h=3; in order to build the first regular star polygon when h=3, we need to create the third regular star polygon of species h-1=2.

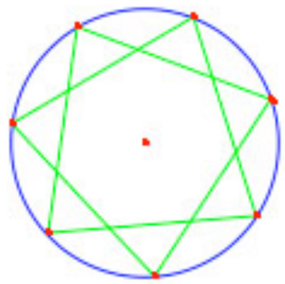
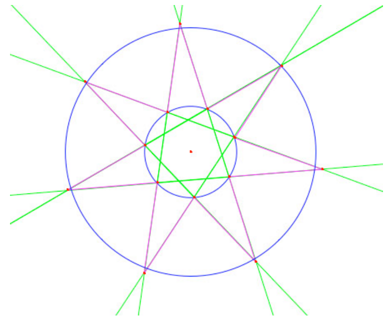
For h=2, the first regular star polygon is five-sided, the second one is the six-sided, and the third one is the seven-sided. By extending the sides of the seven-sided regular star polygon of species two, we are able to obtain the even-sided regular star polygon of species three:

[1] I.d. the number of its sides.

[2] [Poinsot 1809], pag.28.

[3] [Günther 1873] and [Boncompagni 1873].

[4] [Bradwardino 1496].

 Figure 5: The seven-sided regular star polygon when h=2	 Figure 6: The seven-sided regular star polygon when h=3

By induction, Bradwardine also proposed the following theorem: for any given species, the sum of the angles of the first regular star polygon constructible for this species always equals two right angles; the sum of the angles of all other regular star polygons constructible for this same species increases by two right angles as we move from one figure to the next.

Cardinal Daniele Barbaro (1514-1570) in his treatise on perspective[5] shows that regular polygons give rise to other polygons in two ways: the first way is to extend the sides until they meet; the meeting points are the vertices of a new polygon similar to the first one. The second way is to draw all the diagonals from each vertex to its non-adjacent vertices; their intersections form a second polygon similar to the original.

How to reproduce Barbaro's figures and constructions? One way is to exploit a plain geometry software like the Cabri®. To construct superior species heptagons, use the *regular polygon* instrument from *Lines* toolbar, make a regular heptagon (in blue on pic. 9); using the instrument *line* of *Lines* toolbar, it is possible to construct the extensions of each side of the heptagon, creating the lines passing for each couple of adjacent vertices that mark the polygon sides. Through *intersection point* from toolbox *Points* the intersection points are marked between the lines constructed. These will be the vertices of the new, superior order star polygons. To highlight them, using the instrument *polygon*, from toolbox *Lines*, make the polygon passing for all those ordered points, as in pictures 9 and 10 (violet).

As to Barbaro's second method, always starting from the regular heptagon, all possible diagonals are drawn using the instrument *segment* from the toolbox *Lines*. Then, with *intersection point* all segment intersection points are marked. These are going to be the vertices of the new, superior order star polygons. To highlight them, using the instrument *polygon*, from toolbox *Lines*, make the polygon which passes for all those ordered points, as in pictures 11 and 12 (violet).

[5] [BARBARO 1569].

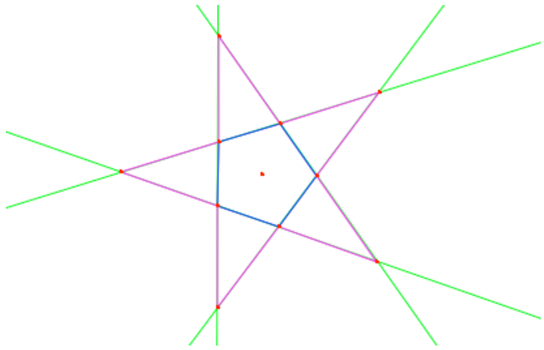


Figure 7: Regular star pentagon generated by extending sides of another polygon

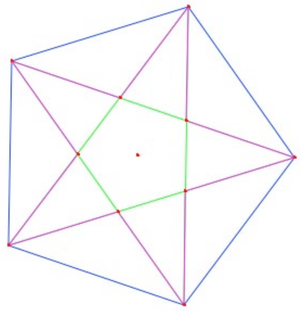


Figure 8: Regular star pentagon generated by drawing diagonals of another polygon

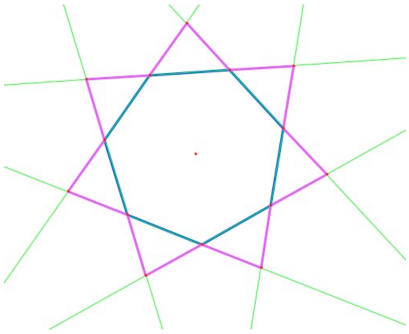


Figure 9: Regular star heptagon generated by extending sides of another polygon

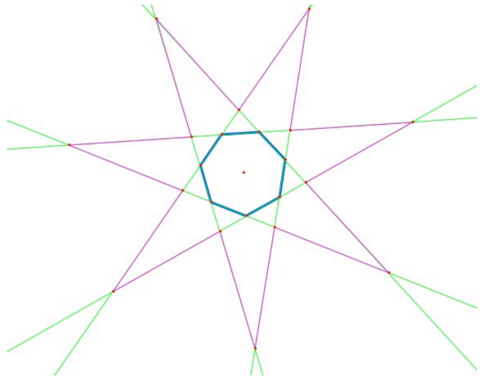


Figure 10: Regular star heptagon generated by extending sides of another polygon

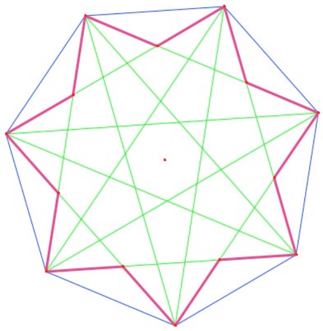


Figure 11: Regular star heptagon generated by drawing diagonals of another polygon

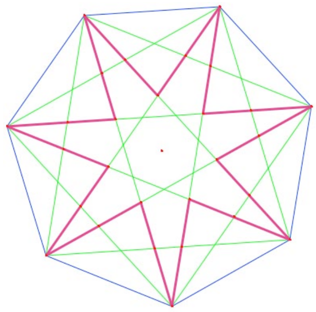


Figure 12: Regular star heptagon generated by drawing diagonals of another polygon

Bradwardine's theory of polygons is treated by the Polish Jo Brosius (1585-1652) in his work *Apologia pro Aristotele et Euclide contra Petrum Ramum*.[6] He also conceived a special procedure for the construction of star polygons. For example, take a regular convex heptagon and divide all sides through the midpoint. Brosius joins two consecutive midpoints with segments and overturns the small triangles generated on the heptagon along the segments, creating a fourteen-sided polygon. He consecutively connects the inner vertices and overturns the small triangles generated along the segments, creating a new fourteen-sided polygon. The two figures built are two regular star polygons, heptagons of the second and third species. Brosius' construction of polygons with different areas is an interesting example as every polygon is included in the previous one, but has the same perimeter.

[6] [BROSIUS 1652].

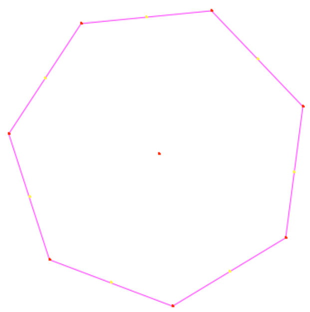


Figure 13: Convex heptagon of species I

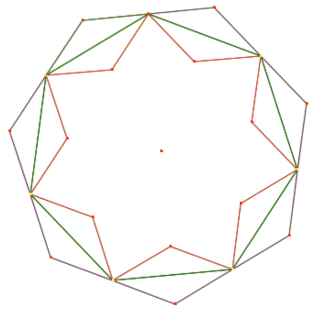


Figure 14: Heptagon of species II

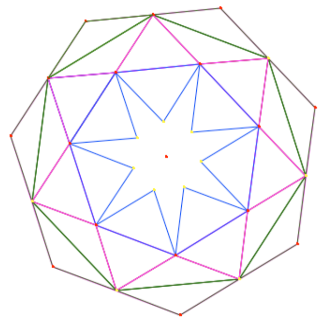


Figure 15: Heptagon of species III

Also in order to build isoperimetrical polygons the Cabri® can be used as well as to verify their properties. Using the instrument *regular polygon*, make a regular polygon. With the instrument *distance* or *length* from toolbox *Measurement*, measure its perimeter. Then, build the midpoints of each side with the instrument *Midpoint*, from toolbox *Construction*. After that, with the instrument *triangle* from toolbox *Lines*, for each vertex of the heptagon, build a triangle with vertex and the two midpoints being adjacent to it.
With the function *reflection*, from toobox *Transformations*, reverse each isosceles triangle from its basis. Use the function *polygon* to join in sequence all the triangles obtained. Thus, a II species heptagon is built (in red on picture 14). Then measure the polygon perimeter.
It is possible to continue by joining in ordered couplet the "inner" vertices of the polygon obtained. Like before, all newly formed small triangles are reversed from their basis. Use the function *polygon* to join the vertices of all formed triangles. A new star heptagon is then created (in blue on pic. 15). Measure the polygon perimeter to ensure it equals that of previous heptagons. With the instrument *area* from toolbox *Measurement* we can assess the areas of the three heptagons obtained to ascertain as each is smaller than the its previous correspondent.

CONCLUSION

In primary and secondary level schools, regular stars polygons are only briefly explained and, when they are, it is not for their geometric peculiarities, but for their relevance to prime numbers. Even now, the argument is not extensively treated. Notwithstanding, one can see from the constructions presented above that regular star polygons allow digressions and insights into various branches of mathematics, ranging from arithmetic to geometry. Finally, by using interactive geometry software (here we used Cabrì®), the buildings are fun and simple and help to highlight special properties of polygons.

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[POINSOT 1809] L. POINSOT, *Mémoire sur les polygones et les polyedres*, «Journal de l'École Polytechnique», Paris, Imprimerie Imperiale, 1809, Cahier IX; pag. 16-48.