A note on strong protomodularity, actions and quotients

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Abstract

We investigate some properties of the fibration of points. We obtain a characterization of protomodular categories among pointed regular ones, and, in the semi-abelian case, a characterization of strong protomodularity. Everything is also stated in terms of internal actions.

1 Introduction

The present work originates from the investigation of the categorical properties related to two well-known features of group actions.

Actions on quotients

Suppose we are given a pair (ξ, g):

$$A \times Y \overset{\xi}{\dashrightarrow} Y \xrightarrow{g} Z\,,$$

where ξ is a left-action of groups, and g is a surjective homomorphism. We pose the following problem: under what conditions does the action ξ pass to the quotient?

Indeed, it is not difficult to see that ξ is well-defined on the cosets of Y *mod* $X = \mathrm{Ker}(g)$, precisely when it is well-defined on the 0-coset X, i.e. when it restricts to X. We state this property as follows:

(KC) An action passes to the quotient if, and only if, it restricts to the kernel.

One may ask if this property holds in semi-abelian categories, other than that of groups. The answer is positive whenever the category is also strongly protomodular. Furthermore, it turns out that the property (KC) characterizes strongly semi-abelian categories among semi-abelian categories.

Action of quotients

Suppose now that we are given a group action ξ as before, and a surjective group homomorphism $q\colon A \to Q$. A natural question arises: when does the given A-action induce a Q-action? In this case, the restriction to the kernel K of q always exists, and the condition under which the action of the quotient is well defined, amounts to the fact that the kernel of q acts trivially. Our result states that the same property holds in all strongly semi-abelian categories.

Many algebraic varieties of the universal algebra are strongly semi-abelian, as the category of groups, Lie algebras, rings and, more generally, all distributive Ω_2-groups, i.e. distributive Ω-groups with only unary and binary operations (see [14]), as for instance the categories of interest in the sense of G. Orzech ([15]). Our work confirms that strongly protomodular categories are a convenient setting for working with internal actions, and related constructions. Indeed, in the (strongly semi-abelian) varietal case, not only internal actions can be described externally, i.e. with suitable set-theoretical maps, but also, they behave *nicely* with respect to quotients. This make it possible to apply to the intrinsic setting several varietal techniques, and *vice-versa*.

The paper is organized as follows.

In the next section we recall the basic notions and fix the notation.

The third and the fourth are the core sections of the paper. Here we set up some general result that will be the key in order to approach the problems mentioned in the introduction. In the third section, we prove that, in the pointed regular context, protomodularity is equivalent to the fact that kernel functors reflect short exact sequences. Furthermore, we give a characterization of strongly semi-abelian categories among semi-abelian categories (Theorem 5.3). In the fourth section we approach the problem of determining the conditions that make it possible to define a covariant change of base functor between the fiber of the fibration of points.

Actions on quotients and actions of quotients are treated in section 5, where the results achieved in the previous sections are translated in terms of internal object actions.

2 Preliminaries

Here we recall some basic notions from [5], and fix the notation.

2.1 Kernel functors and protomodularity

Let $\mathcal{C}$ be a category with finite limits. We denote by $\mathbf{Pt}(\mathcal{C})$ the category with objects the four-tuples (B, A, b, s_b) in $\mathcal{C}$, with $b\colon B \to A$ and $b \cdot s_b = 1_A$, and with morphisms $(f, g)\colon (D, C, d, s_d) \to (B, A, b, s_b)$:

$$\begin{tikzcd} D \arrow[r, "f"] \arrow[d, shift right, "d"'] & B \arrow[d, shift right, "b"'] \\ C \arrow[r, "g"'] \arrow[u, shift right, "s_d"'] & A \arrow[u, shift right, "s_b"'] \end{tikzcd} \tag{1}$$

such that both the upward and the downward directed squares commute. The codomain assignment $(B, A, b, s_b) \mapsto A$ gives rise to a fibration, the so called *fibration of points*:

$$\mathcal{F} \colon \mathbf{Pt}(\mathcal{C}) \to \mathcal{C}.$$

For an object A of $\mathcal{C}$, we denote by $\mathbf{Pt}_A(\mathcal{C})$ the fiber of $\mathcal{F}$ over A. Cartesian morphism are given by commutative diagrams (1) with the downward directed square a pullback. This way, any morphism $g \colon C \to A$ defines a "change of base" functor $g^* \colon \mathbf{Pt}_A(\mathcal{C}) \to \mathbf{Pt}_C(\mathcal{C})$.

If the category $\mathcal{C}$ is finitely complete, also the fibers $\mathbf{Pt}_A(\mathcal{C})$ are, and every change of base functor is left exact.

When moreover the category $\mathcal{C}$ admits an initial object 0, then for any object A of $\mathcal{C}$, one can consider the change of base along the initial arrow $!_A \colon 0 \to A$. This defines the *kernel functor*, denoted by $\mathcal{K}_A$.

The category $\mathcal{C}$ is called pointed when the unique arrow $0 \to 1$ is an isomorphism. If this is the case, the domain functor $\mathbf{Pt}_0(\mathcal{C}) \to \mathcal{C}$ is an isomorphism of categories, so that we can consider

$$\mathcal{K}_A \colon \mathbf{Pt}_A(\mathcal{C}) \to \mathcal{C}\,.$$

Recall that a pointed category is called *protomodular* when $\mathcal{K}_A$ reflect isomorphisms, for any object A (see [5]).

Finally, kernel functors can be seen as the restrictions to the fibers of a more general obviously defined functor

$$\mathcal{K} \colon \mathbf{Pt}(\mathcal{C}) \to \mathcal{C}\,.$$

2.2 Kernel maps and normal monomorphisms

As usual, we call *kernel map* any $f \colon X \to Y$, pullback of an initial arrow:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & 0 \arrow[d, "!_Z"] \\ Y \arrow[r, "g"'] & Z \end{tikzcd}$$

In this case we write $f = \ker(g)$ or $f = k_g$, and $X = \mathrm{Ker}(g)$. We denote by $\mathcal{K}$ the class of kernel maps of a given category $\mathcal{C}$.

The dual construction defines cokernels maps, and the corresponding obvious notation is adopted.

In a pointed (regular) category $\mathcal{C}$, we call *short exact* ([6]) a sequence

$$X \xrightarrow{f} Y \xrightarrow{g} Z\ ,$$

such that g is a regular epimorphism, and $f = \ker(g)$. When $\mathcal{C}$ is also protomodular, every regular epimorphism is also a cokernel (of its kernel), so that (f, g) is short exact precisely when $f = \ker(g)$ and and $g = \mathrm{coker}(f)$ (see [9]).

In [8], Bourn introduces a more general notion of *normal monomorphism*.

In a category $\mathcal{C}$ with finite limits, a monomorphism $f\colon X \to Y$ is *normal* when it is an equivalence class of some equivalence relation R on Y. Indeed, when the category $\mathcal{C}$ is protomodular, normality becomes a property (if f is normal to a relation R, then R is unique), and if $\mathcal{C}$ is also pointed, we can reduce ourselves to considering just the zero class of the relation R: a morphism $f\colon X \to Y$ is normal if, and only if, there exists an equivalence relation (R, r_1, r_2) on Y, and a morphism $X \to R$, such that square

$$\begin{tikzcd}
X \arrow[r, "n"] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & R \arrow[d, "\langle r_1, r_2 \rangle"] \\
Y \arrow[r, "\langle 1,0 \rangle"'] & Y \times Y
\end{tikzcd}$$

is a pullback. We denote by $\mathcal{N}$ the class of normal monomorphisms.

Indeed, any kernel is normal to its associated kernel relation. On the other hand, not every normal monomorphism is a kernel, i.e. $\mathcal{K} \subseteq \mathcal{N}$, and the inclusion is strict, in general.

Recall that an equivalence relation is called *effective* when it is the kernel relation of some map. A category $\mathcal{C}$ is *regular*, if it is finitely complete, regular epimorphisms are stable for pullbacks, and effective equivalence relations admit coequalizers. $\mathcal{C}$ is *Barr exact* ([2]) when, moreover, all equivalence relations are effective. Finally we recall that pointed protomodular categories are called *homological* when they are also regular, they are called *semi-abelian* when they are Barr exact, with finite coproducts ([5]). An important feature of homological categories, is that in such contexts intrinsic versions of some classical lemmas of homological algebra holds. This is the case of the 3×3 lemma (see [9]).

Proposition 2.1. *Let $\mathcal{C}$ be a homological category with cokernels. The class of normal monomorphisms coincide with the class of kernels if, and only if, $\mathcal{C}$ is Barr exact.*

Proof. This is basically a reformulation of Corollary 3.1 in [13]. $\square$

2.3 Strong protomodularity

In [9] Bourn calls *normal*, a left exact functor that is conservative and reflects normal monomorphisms. A relevant application of this definition, is related to the fibration of points. Indeed, when all the change of base functors are normal, the category is called *strongly-protomodular* (see [7, 5]). In the pointed case, it suffices to consider the kernel functors $\mathcal{K}_A$, for every object A. A strongly protomodular semi-abelian category is termed *strongly semi-abelian*.

Bourn, in [7], gives a characterization of normal subobjects in $\mathbf{Pt}_A(\mathcal{C})$. When $\mathcal{C}$ is protomodular with zero object, $\varphi\colon (B, b, s_b) \to (C, c, s_c)$ is normal in $\mathbf{Pt}_A(\mathcal{C})$

if, and only if, $\varphi \cdot k_b$ is normal in $\mathcal{C}$, where $k_c = \ker(c)$:

$$
\begin{tikzcd}
X \arrow[r, "k_b"] \arrow[d, "f"'] & B \arrow[r, shift left, "b"] \arrow[d, "\varphi"] & A \arrow[l, shift left, "s_b"] \arrow[d, equal] \\
Y \arrow[r, "k_c"'] & C \arrow[r, shift left, "c"] & A \arrow[l, shift left, "s_c"]
\end{tikzcd}
$$

This, in turns, gives a criterion for strong protomodularity: it suffices to check, for every morphism of split short exact sequences as above, that if f is normal, then also $k_c \cdot f$ is.

According to Proposition 2.1, when $\mathcal{C}$ is Barr-exact, we can replace "normal morphisms" with "kernel maps" in the criterion above.

2.4 Internal actions

Semi-abelian categories are a convenient setting for working with *internal actions*. Here we briefly recall their definition from [4]. This will help in formulating property (KC) intrinsically.

Let $\mathcal{C}$ be a finitely complete, pointed category with pushouts of split monomorphisms. Then, for every object A of $\mathcal{C}$, the functor $\mathcal{K}_A$ has a left adjoint Σ_A: for an object X of $\mathcal{C}$, $\Sigma_A(X)$ is the pointed object $\begin{tikzcd} A+X \arrow[r, shift left, "{[1,0]}"] & A \arrow[l, shift left, "i_A"] \end{tikzcd}$. The monad corresponding to this adjunction is denoted by $A\flat(-)$, and for any object X of $\mathcal{C}$ one gets a kernel diagram:

$$A\flat X \xrightarrow{\kappa_{A,X}} A + X \xrightarrow{[1,0]} A \ .$$

The $A\flat(-)$-algebras are called internal A-actions (see [4, 11]). The category $\mathbf{Alg}(A\flat(-))$ of such algebras will be more conveniently denoted by $\mathbf{Act}(A, -)$.

When the kernel functor $\mathcal{K}_A$ is monadic, then $\mathcal{C}$ is said to be a category with semi-direct products, and the canonical comparison

$$\Xi\colon \mathbf{Pt}_A(\mathcal{C}) \to \mathbf{Act}(A, -)\,, \tag{2}$$

establishes an equivalence of categories. All semi-abelian categories satisfy this condition.

For an action $\xi\colon A\flat X \to X$, the semi-direct product of X with A, with action ξ is the split epimorphism corresponding to ξ via Ξ. It can be computed explicitly (see [14]) by means of the coequalizer diagram:

$$
\begin{tikzcd}
A\flat X \arrow[r, shift left, "\kappa_{A,X}"] \arrow[r, shift right, "i_X \cdot \xi"'] & A + X \arrow[r, "q_\xi"] & A \rtimes_\xi X \,.
\end{tikzcd}
$$

Example 2.2. For objects A and X, the *trivial action* of A on X the composition

$$\rho_{A,X} = \rho_X \colon\ A\flat X \xrightarrow{\kappa_{A,X}} A + X \xrightarrow{[0,1]} X \ .$$

The map $\rho_{A,X}$ is natural in the two variables A and X. The corresponding split epimorphism is given by the cartesian product with the canonical section:

$$
\begin{tikzcd}
X \times A \arrow[r, shift left, "\pi_2"] & A \arrow[l, shift left, "\langle 0,1 \rangle"]
\end{tikzcd} .
$$

Example 2.3. Every object X acts on itself by conjugation. This is given by the composition

$$
\chi_X \colon\ X \flat X \xrightarrow{\kappa_{X,X}} X + X \xrightarrow{[1,1]} X .
$$

The map χ_X is natural in the variable X. The corresponding split epimorphism isomorphic to the cartesian product with the diagonal section:

$$
\begin{tikzcd}
X \times X \arrow[r, shift left, "\pi_2"] & X \arrow[l, shift left, "\langle 1,1 \rangle"]
\end{tikzcd} .
$$

3 Exactness properties of kernel functors

In this section, we analyze some issues related to the behavior of kernel functors with respect to kernels, cokernels and short exact sequences, in the pointed regular setting.

We have just recalled Bourn's characterization of normal subobject in $\mathbf{Pt}_A(\mathcal{C})$. In the homological setting, one can recover a similar characterization for kernels.

Lemma 3.1. *Let $\mathcal{C}$ be homological. Then $\varphi\colon (B, b, s_b) \to (C, c, s_c)$ is a kernel in $\mathbf{Pt}_A(\mathcal{C})$ if, and only if, $\varphi \cdot k_b$ is a kernel in $\mathcal{C}$, where $k_c = \ker(c)$.*

$$
\begin{tikzcd}
X \arrow[r, "k_b"] \arrow[d, "f"'] & B \arrow[r, shift left, "b"] \arrow[d, "\varphi"] & A \arrow[l, shift left, "s_b"] \arrow[d, equal] \\
Y \arrow[r, "k_c"'] & C \arrow[r, shift left, "c"] & A \arrow[l, shift left, "s_c"]
\end{tikzcd}
\tag{3}
$$

Proof. The fact that φ is a kernel, amounts precisely to the existence of a morphism of points $\gamma\colon (C, c, s_c) \to (D, d, s_d)$, such that the commutative square $\gamma \cdot \varphi = s_d \cdot b$ is a pullback in $\mathcal{C}$. Then, pasting it with the kernel diagram of (k_b, b), one easily sees that $k_c \cdot f = \varphi \cdot k_b = \ker(\gamma)$ in $\mathcal{C}$.

Conversely, let us assume that $k_c \cdot f = \varphi \cdot k_b$ is a kernel. Immediately one sees that also f is a kernel. We consider $(D, \gamma) = \operatorname{coker}(\varphi \cdot k_b)$ and $(Z, g) = \operatorname{coker}(f)$, and the complex (α, β), which is obtained by the universal property of such cokernels:

$$
\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "\varphi \cdot k_b"] & 0 \arrow[d] \\
Y \arrow[r, "k_c"'] \arrow[d, "g"'] & C \arrow[r, shift left, "c"] \arrow[d, "\gamma"'] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\
Z \arrow[r, "\alpha"'] & D \arrow[r, "\beta"'] & A
\end{tikzcd}
\tag{4}
$$

Then the three columns and the first and the second rows are short exact; by the 3×3 lemma (see [9]) one concludes that also the last row is short exact. Moreover β is split by $\gamma \cdot s_c$, and this makes γ is a morphism in $\mathbf{Pt}_A(\mathcal{C})$. Now, it is routine to show that φ is the kernel of γ in $\mathbf{Pt}_A(\mathcal{C})$, i.e. that the square square $\gamma \cdot \varphi = s_d \cdot b$ is a pullback in $\mathcal{C}$: it suffices to repeat the argument of the if-part of the proof, backwards. First of all, the square commutes, as one can prove by pre-composition with the jointly epic pair (k_b, s_b). Then, one should use the pullback cancelation property of protomodular categories (see [6]), thus obtaining the result. $\square$

The preservation property described in the next proposition is little more than a reformulation of several properties of short exact sequences analyzed in [9]. Nonetheless, we include it in the present work as, at the best of our knowledge, it is not stated explicitly elsewhere in the literature.

Proposition 3.2. *Let $\mathcal{C}$ be homological. Then $\mathcal{K}_A$ preserves short exact sequences, for every A in $\mathcal{C}$.*

Proof. Let us consider a short exact sequence (φ, γ) in $\mathbf{Pt}_A(\mathcal{C})$, with its restriction to kernels, namely the short exact sequence (f, g) in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "k_b"] \arrow[d, "f"'] & B \arrow[r, shift left, "b"] \arrow[d, "\varphi"'] & A \arrow[l, shift left, "s_b"] \arrow[d, equal] \\ Y \arrow[r, "k_c"'] \arrow[d, "g"'] & C \arrow[r, shift left, "c"] \arrow[d, "\gamma"'] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\ Z \arrow[r, "k_d"'] & D \arrow[r, shift left, "d"] & A \arrow[l, shift left, "s_d"] \end{tikzcd} \tag{5}$$

This means precisely that the square

$$\begin{tikzcd} B \arrow[r, "b"] \arrow[d, "\varphi"'] & A \arrow[d, "s_d"] \\ C \arrow[r, "\gamma"'] & D \end{tikzcd} \tag{6}$$

is a pushout and a pullback in $\mathbf{Pt}_A(\mathcal{C})$. Now, $\mathcal{K}_A$ clearly preserves the zero object, and since it has a left adjoint, it preserves limits, so that the square

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"'] & 0 \arrow[d] \\ Y \arrow[r, "g"'] & Z \end{tikzcd} \tag{7}$$

is a pullback in $\mathcal{C}$. Since $\mathcal{K}_A$ reflects regular epimorphisms (by Proposition 8 in [9]), g is a regular epimorphism, and the sequence (f, g) is short exact. $\square$

The following lemma is useful in applications. It can be considered as a sort of dual of Lemma 3.1.

Lemma 3.3. *Let $\mathcal{C}$ be homological with cokernels and let us consider a situation as described by diagram (5) above, with φ a kernel map in $\mathbf{Pt}_A(\mathcal{C})$. Then γ is the cokernel of φ in $\mathbf{Pt}_A(\mathcal{C})$ if, and only if, γ is the cokernel of $\varphi \cdot k_b$ in $\mathcal{C}$.*

Proof. In order to prove that the condition is necessary, we start with a pair (φ, γ) short exact in $\mathbf{Pt}_A(\mathcal{C})$. By carrying on the same arguments as in the proof of Proposition 3.2 above, we can paste the pushout diagram (6) with the pushout diagram below, and get the result :

$$\begin{tikzcd} X \arrow[r] \arrow[d, "k_b"'] & 0 \arrow[d] \\ B \arrow[r, "b"'] & A \end{tikzcd} \tag{8}$$

Conversely, let us consider a morphism of pointed objects $\omega \colon C \to E$, with $E = (E, e, s_e)$, such that $\omega \cdot \varphi = 0_A$ in $\mathbf{Pt}_A(\mathcal{C})$. Equivalently, $\omega \cdot \varphi = s_e \cdot b$ in $\mathcal{C}$. Pre-composing we get $\omega \cdot \varphi \cdot k_b = s_e \cdot b \cdot k_b = 0$, so that there exists a unique $\delta \colon D \to E$ such that $\delta \cdot \gamma = \omega$. Now δ underlies a morphism of pointed objects. Indeed one can prove $d = e \cdot \delta$ by means of the universal property of the cokernel γ. Moreover, $\delta \cdot s_d = \delta \cdot \gamma \cdot \varphi \cdot s_b = \omega \cdot \varphi \cdot s_b = s_e \cdot b \cdot s_b = s_e$. □

As a consequence, in the homological setting, kernel functors also reflect short exact sequences. Moreover, this property characterizes homological categories among pointed regular ones.

Proposition 3.4. *Let $\mathcal{C}$ be a pointed regular category. The following statements are equivalent:*

(1) *$\mathcal{C}$ is protomodular,*

(2) *$\mathcal{K}_A$ reflects short exact sequences, for every A in $\mathcal{C}$.*

Proof. For a protomodular category $\mathcal{C}$, let us consider a pair of morphisms (φ, γ) in $\mathbf{Pt}_A(\mathcal{C})$, such that the restriction to kernels (f, g) is a short exact sequence in $\mathcal{C}$ (see diagram (5)). Since $g \cdot f = 0$, $\gamma \cdot \varphi$ factors through A. More precisely, $\gamma \cdot \varphi = s_d \cdot b$, as one can prove by pre-composing this equality with the jointly epic pair (k_b, s_b). Then we consider diagram below:

$$\begin{tikzcd} X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "\varphi \cdot k_b"] & 0 \arrow[d] \\ Y \arrow[r, "k_c"'] \arrow[d, "g"'] & C \arrow[r, shift left, "c"] \arrow[d, "\gamma"'] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\ Z \arrow[r, "k_d"'] & D \arrow[r, shift left, "d"] & A \arrow[l, shift left, "s_d"] \end{tikzcd} \tag{9}$$

We can apply the 3×3 lemma: the three rows are exact, and so are the leftmost and the rightmost columns. The middle column is zero, so that we can conclude that it is short exact. The fact that $k_c \cdot f = \varphi \cdot k_b$ is normal in $\mathcal{C}$, is equivalent to φ being normal in $\mathbf{Pt}_A(\mathcal{C})$. Indeed it is a kernel: diagram (6) is a pullback in $\mathcal{C}$ by the pullback cancelation property of protomodular categories (see [6]). In fact, diagram (8) is a pullback by hypothesis, and (8) + (6) is a pullback because $\ker(\gamma) = \varphi \cdot k_b$. Applying Lemma 3.3, we conclude that (φ, γ) is a short exact sequence in $\mathbf{Pt}_A(\mathcal{C})$.

Conversely, we have to prove that, for any object A, the kernel functor $\mathcal{K}_A$ reflects isomorphisms. To this end, suppose we are given a map φ in $\mathbf{Pt}_A(\mathcal{C})$ such that its restriction to kernels is an isomorphism f. Then, the cokernel of f is trivial, and one can consider the following diagram:

$$\begin{tikzcd}
X \arrow[r, "k_b"] \arrow[d, "f"', "\simeq"] & B \arrow[r, shift left, "b"] \arrow[d, "\varphi"'] & A \arrow[l, shift left, "s_b"] \arrow[d, equal] \\
Y \arrow[r, "k_c"'] \arrow[d] & C \arrow[r, shift left, "c"] \arrow[d, "c"'] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\
0 \arrow[r] & A \arrow[r, shift left, "1_A"] & A \arrow[l, shift left, "1_A"]
\end{tikzcd}$$

Now, applying the hypothesis, we obtain that the sequence (φ, c) is short exact in $\mathbf{Pt}_A(\mathcal{C})$, so that φ is the pullback of 1_A along c, hence an isomorphism.

□

Proposition 3.2 and Proposition 3.4 together, imply immediately the following corollary.

Corollary 3.5. *Let $\mathcal{C}$ be homological. Then for any map $e \colon E \to A$, the change of base $e^* \colon \mathbf{Pt}_A(\mathcal{C}) \to \mathbf{Pt}_E(\mathcal{C})$ preserves and reflects short exact sequences.*

In the last part of this section we would like to examine the behavior of the kernel functors with respect to kernels and (some specific class of) cokernels. We start by giving a definition.

Definition 3.6. *We call a map φ in $\mathbf{Pt}_A(\mathcal{C})$ a* $\mathcal{K}_A$-kernel*, if its restriction to the kernel functor $\mathcal{K}_A$ is a kernel map in $\mathcal{C}$.*

Of course, any kernel φ in $\mathbf{Pt}_A(\mathcal{C})$ is a $\mathcal{K}_A$-kernel. It is interesting to investigate when the other implication holds. This is done in the next proposition.

Proposition 3.7. *Let $\mathcal{C}$ be homological, and let $\mathcal{K}_A \colon \mathbf{Pt}_A(\mathcal{C}) \to \mathcal{C}$ be the kernel functor, relative to an object A in $\mathcal{C}$. Then the following statements are equivalent:*

(1) *every $\mathcal{K}_A$-kernel is a kernel map in $\mathbf{Pt}_A(\mathcal{C})$,*

(2) *the kernel functor $\mathcal{K}_A$ reflects kernel maps,*

(3) *the kernel functor* $\mathcal{K}_A$ lifts *the cokernels of* $\mathcal{K}_A$*-kernels, i.e.*

(C$*$) *for every* φ *in* $\mathbf{Pt}_A(\mathcal{C})$ *such that* $\mathcal{K}_A(\varphi) = f$ *is a kernel map in* $\mathcal{C}$*, there exists a unique* γ *in* $\mathbf{Pt}_A(\mathcal{C})$ *such that* $\gamma = \mathrm{coker}(\varphi)$*, and* $\mathcal{K}_A(\gamma) = \mathrm{coker}(f)$*.*

For the notion of (co)limit lifting functor, the reader can refer to Definition 13.17 of [1].

Proof. The equivalence between (1) and (2) is tautological, and it motivates the choice of the notation adopted.

(2) $\Rightarrow$ (3). Let us assume that $\mathcal{K}_A$ reflects kernel maps, and (f, g) is short exact in $\mathcal{C}$. Then φ is a kernel map in $\mathbf{Pt}_A(\mathcal{C})$, and, by Lemma 3.1, also $k_c \cdot f$ is. Let (E, e) be the cokernel of $k_c \cdot f$, so that the pair $(k_c \cdot f, e)$ is a short exact sequence. We can consider the following commutative diagram

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "k_c \cdot f"] & 0 \arrow[d] \\
Y \arrow[r, "k_c"] \arrow[d, "g"'] & C \arrow[r, shift left, "c"] \arrow[d, "e"] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\
Z \arrow[r, "\alpha"'] & E \arrow[r, "\beta"'] & A
\end{tikzcd}$$

where α and β are obtained by the universal properties of the cokernels (E, e) and (Z, g). Since $\mathcal{C}$ is homological, we can apply the 3×3 lemma, and conclude that the sequence (α, β) is short exact. Moreover, β is split by $e \cdot s_c$, and this concludes this part of the proof.

(3) $\Rightarrow$ (2). In order to prove this implication, we start with a morphism φ as above, together with its restriction to kernels f. Let us assume that f is a kernel, and we write (Z, g) for the cokernel of f. Now, using the condition (C*), we can consider the commutative diagram below

$$\begin{tikzcd}
X \arrow[r, equal] \arrow[d, "f"'] & X \arrow[r] \arrow[d, "k_c \cdot f"] & 0 \arrow[d] \\
Y \arrow[r, "k_c"'] \arrow[d, "g"'] & C \arrow[r, shift left, "c"] \arrow[d, "\gamma"'] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\
Z \arrow[r, "k_d"'] & D \arrow[r, shift left, "d"] & A \arrow[l, shift left, "s_d"]
\end{tikzcd}$$

Since $\gamma \cdot k_c \cdot f = k_d \cdot g \cdot f = 0$, again we can apply 3×3 lemma, thus obtaining that $(k_c \cdot f, \gamma)$ is short exact, so that $k_c \cdot f$ is a kernel in $\mathcal{C}$, i.e. φ is a kernel in $\mathbf{Pt}_A(\mathcal{C})$. $\square$

When $\mathcal{C}$ is Barr exact, condition (2) of Proposition 3.7 above, expresses precisely the strong protomodularity axiom. This is stated in the following proposition.

Proposition 3.8. *Let $\mathcal{C}$ be a semi-abelian category. The following statements are equivalent*

(1) *$\mathcal{C}$ is strongly semi-abelian,*

(2) *for every object A of $\mathcal{C}$, the kernel functor $\mathcal{K}_A$ lifts the cokernels of $\mathcal{K}_A$-kernels, i.e. condition $(C*)$ of Proposition 3.7 is satisfied.*

4 A push forward property for the change of base functor

As we recalled in Section 2, for any map $f\colon E \to A$, the change of base functor $f^*\colon \mathbf{Pt}_A(\mathcal{C}) \to \mathbf{Pt}_E(\mathcal{C})$, is defined by pulling back along f. In other term, f defines a functor between the fibers that moves *backward*, with respect to the direction of f. A quite natural question to ask is whether there are conditions allowing to *push forward* along a map. More precisely, given a map $q\colon A \to Q$, we aim to define a functor $q_*\colon \mathbf{Pt}_A(\mathcal{C}) \to \mathbf{Pt}_Q(\mathcal{C})$.

In the present work, we restrict our attention to the case when q is a regular epimorphism. The following result holds:

Proposition 4.1. *In a strongly semi-abelian category $\mathcal{C}$, we consider a pointed object (C, c, s_c, A), and a regular epimorphism $q\colon A \to Q$. Then, if we denote by (K, k) and (Y, k_c) the kernels of q and of c respectively, the following statements are equivalent:*

(1) *the pullback along k of (C, c, s_c, A) is the pointed object*

$$(Y \times K, \pi_2, \langle 0, 1\rangle, K)\,,$$

(2) *there exist a pointed object (D, d, s_d, Q) and a cartesian morphism*

$$(\gamma, q)\colon (C, c, s_c, A) \to (D, d, s_d, Q)\,.$$

The situation is described by the following diagram, that includes the kernels involved.

$$\begin{tikzcd}
Y \arrow[r, "k_{\pi_2}"] \arrow[d, equal] & Y \times K \arrow[r, shift left, "\pi_2"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "(\lozenge)"] & K \arrow[l, shift left, "\langle 0,1\rangle"] \arrow[d, "k"] \\
Y \arrow[r, "k_c"'] \arrow[d, equal] & C \arrow[r, shift left, "c"] \arrow[d, dashed, "\gamma"'] \arrow[dr, phantom, "(\lozenge\lozenge)"] & A \arrow[l, shift left, "s_c"] \arrow[d, "q"] \\
Y \arrow[r, dashed, "k_d"'] & D \arrow[r, dashed, shift left, "d"] & Q \arrow[l, dashed, shift left, "s_d"]
\end{tikzcd} \tag{10}$$

Proof. (1) $\Rightarrow$ (2). By the assumption in (1), φ is a kernel, since it is the pullback of a kernel. Now, we can consider the diagram below, that extends the square

($\Diamond$) by taking the kernels of the horizontal split epimorphisms and the cokernels of the vertical monomorphisms. Since the base category is homological, not only we can say that such kernels are isomorphic, but, by 3×3 lemma, also the cokernels of k and φ are:

$$\begin{tikzcd}
Y \arrow[r, "{\langle 1,0\rangle}"] \arrow[d, equal] & Y \times K \arrow[r, shift left, "\pi_2"] \arrow[d, "\varphi"'] & K \arrow[l, shift left, "{\langle 0,1\rangle}"] \arrow[d, "k"] \\
Y \arrow[r, "k_c"] \arrow[d] & C \arrow[r, shift left, "c"] \arrow[d, "q\cdot c"'] & A \arrow[l, shift left, "s_c"] \arrow[d, "q"] \\
0 \arrow[r] & Q \arrow[r, equal] & Q
\end{tikzcd}$$

Let us consider the following morphism of short exact sequences:

$$\begin{tikzcd}
K \arrow[r, "k"] \arrow[d, "{\langle 0,1\rangle}"'] & A \arrow[r, "q"] \arrow[d, "s_c"] & Q \arrow[d, equal] \\
Y \times K \arrow[r, "\varphi"'] & C \arrow[r, "q\cdot c"'] & Q
\end{tikzcd}$$

Since $\langle 0, 1\rangle$ is a kernel, applying Axiom M1.2 of [16] (which holds in every strongly semi-abelian category) we deduce that also $s_c \cdot k = \varphi \cdot \langle 0, 1\rangle$ is a kernel. Let us compute the cokernel $q' = \mathrm{coker}(s_c \cdot k)$, and arrange our data in the diagram below:

$$\begin{tikzcd}
K \arrow[r, "{\langle 0,1\rangle}"] \arrow[d, equal] & Y \times K \arrow[r, "\pi_1"] \arrow[d, "\varphi"'] & Y \arrow[d, "\alpha"] \\
K \arrow[r, "s_c cdotk"'] \arrow[d] & C \arrow[r, "\gamma"'] \arrow[d, "q\cdot c"] & D \arrow[d, "\beta"] \\
0 \arrow[r] & Q \arrow[r, equal] & Q
\end{tikzcd}$$

where α and β are obviously obtained by the universal property of the cokernels involved. By the 3×3 lemma, we deduce that the sequence (α, β) is short exact. Finally, the square $\beta \cdot \gamma = q \cdot c$ is a pullback, since $\mathrm{Ker}(\beta) = \mathrm{Ker}(c)$:

$$\begin{tikzcd}
 & K \arrow[r, equal] \arrow[d, "s_c\cdot k"'] & K \arrow[d, "k"] \\
Y \arrow[r, "k_c"] \arrow[d, equal] & C \arrow[r, shift left, "c"] \arrow[d, "\gamma"] & A \arrow[l, shift left, "s_c"] \arrow[d, "q"] \\
Y \arrow[r, "\alpha"'] & D \arrow[r, "\beta"'] & Q
\end{tikzcd}$$

Then also $(\mathrm{Ker}\gamma) = \mathrm{Ker}(q)$. Moreover β is a split epimorphism. In order to prove this assertion, we notice that $(\gamma \cdot s_c) \cdot k = \gamma \cdot (s_c \cdot k) = 0$, and by the universal property of the cokernel q, there exists a (unique) map $\sigma\colon Q \to D$ such that $\sigma \cdot q = \gamma \cdot s_c$. Hence $\beta \cdot \sigma \cdot q = \beta \cdot \gamma \cdot s_c = q \cdot c \cdot s_c = q$, and , since q is epic, we get $\beta \cdot \sigma = 1_Q$.

(2) $\Rightarrow$ (1). Assume we are in the situation as described by the diagram below

$$\begin{tikzcd}
B \arrow[r, shift left, "b"] \arrow[d, "\varphi"'] \arrow[dr, phantom, "(i)"] & K \arrow[l, shift left, "s_b"] \arrow[d, "k"] \\
C \arrow[r, shift left, "c"] \arrow[d, "\gamma"'] \arrow[dr, phantom, "(ii)"] & A \arrow[l, shift left, "s_c"] \arrow[d, "q"] \\
D \arrow[r, shift left, "d"] & Q \arrow[l, shift left, "s_d"]
\end{tikzcd}$$

with (i) and (ii) pullbacks, and (k, q) short exact. Let us denote by Y the kernel of c. Then $(i) + (ii)$ is a pullback, and since $q \cdot k$ factors through 0, the pointed object (B, b, s_b, K) is isomorphic to the product projection $(Y \times K, \pi_2, \langle 0, 1\rangle, K)$. $\square$

For a map k with codomain A, we denote by $\mathbf{Pt}_A(\mathcal{C})|_k$ the full subcategory of $\mathbf{Pt}_A(\mathcal{C})$, with objects those split epimorphisms such that the change of base along k gives a product projection. Then it is easy to prove that Proposition 4.1 above can be extended to the following

Corollary 4.2. *Let $\mathcal{C}$ be strongly semi-abelian, and let $q\colon A \to Q$ be a regular epimorphism of $\mathcal{C}$ with kernel (K, k). Then q induces a functor:*

$$q_*\colon \mathbf{Pt}_A(\mathcal{C})|_k \to \mathbf{Pt}_Q(\mathcal{C})\,.$$

5 Back to action(s)

In this section we return to the problems described in the introduction, now set in the semi-abelian context.

5.1 Property (KC) for split epimorphisms

From now on, we consider $\mathcal{C}$ semi-abelian. In this setting, we will first formulate our property (KC) in terms of split epimorphisms, according to the equivalence (2) between actions and points. Then, in the next section, we will go back to the original formulation of the problem.

Let a short exact sequence $X \xrightarrow{f} Y \xrightarrow{g} y$ be given, and consider a split epimorphism (C, c, s_c), with kernel Y and codomain A. Let ξ be the corresponding action.

$$
\begin{tikzcd}
X \arrow[r, dashed, "k_b"] \arrow[d, "f"'] & B \arrow[r, dashed, shift left, "b"] \arrow[d, dashed, "\varphi"'] \arrow[dr, phantom, "(\dagger)"] & A \arrow[l, dashed, shift left, "s_b"] \arrow[d, equal] \\
Y \arrow[r, "k_c"'] \arrow[d, "g"'] & C \arrow[r, shift left, "c"] \arrow[d, dashed, "\gamma"'] \arrow[dr, phantom, "(\ddagger)"] & A \arrow[l, shift left, "s_c"] \arrow[d, equal] \\
Z \arrow[r, dashed, "k_d"'] & D \arrow[r, dashed, shift left, "d"] & A \arrow[l, dashed, shift left, "s_d"]
\end{tikzcd}
\tag{11}
$$

Then, with reference to the diagram above, the fact that the action ξ restricts to the kernel X, amounts to the fact that there exists a morphism φ of split epimorphisms ($\dagger$), that restricts to f, while the fact that the action ξ passes to the quotient $Z = Y/X$ amounts to the fact that there exists a morphism γ of split epimorphisms ($\ddagger$), that restricts to g. In this fashion, with a little abuse of language, we can translate property (KC) in the double implication ($\dagger$) $\Leftrightarrow$ ($\ddagger$).

Remark 5.1. Indeed, the implication ($\ddagger$) $\Rightarrow$ ($\dagger$) holds in any pointed category with finite limits, with no assumption of g. In other words, this is the trivial part of our starting problem, and it has nothing to do with actions, etc.

Remark 5.2. On the other hand, the implication ($\dagger$) $\Rightarrow$ ($\ddagger$) is precisely the condition ($C*$) stated in Proposition 3.7.

5.2 Actions on quotients

We are ready to return to our initial problem, and to formulate it in terms of internal actions.

In a pointed regular category with semi-direct products, we consider an A-action ξ on Y and a short exact sequence, (f, g).

$$
\begin{tikzcd}
A\flat X \arrow[r, "1\flat f"] \arrow[d, dashed, "\xi_|"'] \arrow[dr, phantom, "(\dagger)"] & A\flat Y \arrow[r, "1\flat g"] \arrow[d, "\xi"'] \arrow[dr, phantom, "(\ddagger)"] & A\flat Z \arrow[d, dashed, "\overline{\xi}"'] \\
X \arrow[r, "f"'] & Y \arrow[r, "g"'] & Z
\end{tikzcd}
\tag{12}
$$

We can state the implications above using internal actions, as follows:

($\ddagger$) $\Rightarrow$ ($\dagger$) If ξ passes to the quotient Z, then it restricts to the kernel X. In other words, if there exists an action $\overline{\xi}$ such that the square on the right commutes, then there exists an action $\xi_|$ such that the square on the left commutes.

($\dagger$) $\Rightarrow$ ($\ddagger$) If ξ restricts to the kernel X, then it passes to the quotient Z. In other words, if there exists an action $\xi_|$ such that the square on the left commutes, then there exists an action $\overline{\xi}$ such that the square on the right commutes.

Of course, the implication (‡) ⇒ (†) does hold in any pointed category with semi-direct products. For what concerns property (†) ⇒ (‡), we can translate Proposition 3.8 accordingly, in terms of internal actions. This is summarized in the following Theorem, that can be derived directly from Proposition 3.8.

Theorem 5.3. *Let $\mathcal{C}$ be a semi-abelian category. The following statements are equivalent:*

(1) *$\mathcal{C}$ is strongly protomodular,*

(2) *(†) ⇒ (‡), i.e. for any A-action ξ on an object Y, and for any normal subobject X of Y such that ξ restricts X, ξ passes to the quotient Y/X.*

5.3 Action of quotients

So far we discussed the conditions under which an action on a given object extends to a quotient of that object. Now we change our point of view: we fix the *acted* object, and we consider when an action *of* a given object, induces an action of a quotient of that object.

More precisely, let a short exact sequence

$$K \xrightarrow{k} A \xrightarrow{q} Q$$

be given, and let us consider an action $\xi\colon A\flat Y \to Y$. We pose the following question: when does the action ξ induces an action $Q\flat Y \to Y$?

The answer, in the strongly semi-abelian context, involves the restriction to the kernel K: likewise in the case of groups, ξ induces an action $q_*(\xi)$ of the quotient Q, precisely when $\xi \cdot (k\flat 1)$ is trivial.

Proposition 5.4. *Let $\mathcal{C}$ be strongly semi-abelian, (k, q) a short exact sequence and ξ an action, as above. Then the following conditions are equivalent:*

(1) *$\xi \cdot (k\flat 1) = \rho_K$, i.e. the trivial action on K,*

(2) *there exists an action $q_*(\xi)\colon Q\flat Y \to Y$ such that $\xi = q_*(\xi) \cdot (q\flat 1)$.*

Proof. This is nothing but the formulation of Proposition 4.1 in terms of internal actions. □

Further developments

Theorem 5.3, together with Proposition 5.4, seems to suggest that strongly semi-abelian categories are a convenient setting for developing homological algebra of internal (pre)crossed modules. Moreover, for all distributive Ω_2-groups, it is possible to easily translate conditions involving internal actions, in conditions involving *external* (i.e. usual) actions, thus making the theory manageable in many algebraic situation of interest. This arguments encourage the further investigations classical homological constructions, such as short exact sequences of (pre)crossed modules, central extensions, etc.

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