AN ALMOST NILPOTENT VARIETY OF EXPONENT 2

BY

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ABSTRACT

We construct a non-associative algebra A over a field of characteristic
zero with the following properties: if $\mathcal{V}$ is the variety generated by A, then
$\mathcal{V}$ has exponential growth but any proper subvariety of $\mathcal{V}$ is nilpotent.
Moreover, by studying the asymptotics of the sequence of codimensions of
A we deduce that $exp(\mathcal{V}) = 2$.

1. Introduction

Let F be a field of characteristic zero and A a not necessarily associative F-algebra. A natural and well established way of measuring the polynomial identities satisfied by A is through the study of the asymptotic behavior of its sequence of codimensions $c_n(A)$, $n = 1, 2, \ldots$. In characteristic zero, without loss of generality, one can study the multilinear identities satisfied by A and the sequence

of codimensions actually gives a measure of such identities. More precisely, let $F\{X\}$ be the free non-associative algebra in a countable set X and P_n the space of multilinear polynomials in the first n variables. If $Id(A)$ denotes the T-ideal of polynomial identites satisfied by A, one considers the relatively free algebra $F\{X\}/Id(A)$ and the sequence of subspaces $P_n/(P_n \cap Id(A))$, $n \geq 1$. Then

$$c_n(A) = \dim P_n/(P_n \cap Id(A))$$

is the n-th codimension of A

In the language of varieties, if $\mathcal{V} = var(A)$ is the variety generated A, then the growth of $\mathcal{V}$ is the codimension growth of the algebra A. Also, we write $Id(\mathcal{V}) = Id(A)$ and $c_n(\mathcal{V}) = c_n(A)$.

The asymptotic behavior of the sequence of codimensions has been extensively studied in recent years leading to classification theorems (see, for instance, [7], [9], [14]). The key result in this area says that the sequence of codimensions of an associative PI-algebra (algebra satisfying a non-trivial polynomial identity) is exponentially bounded ([13]). Moreover, its exponential rate of growth, $exp(A) = \lim_{n\to\infty} \sqrt[n]{c_n(A)}$, is an integer called the PI-exponent of A ([4], [5], see also [7]).

For non-associative PI-algebras, the corresponding codimensions in general are not exponentially bounded. The same conclusion can be drawn for Lie and Jordan algebras. In case of Lie algebras an interesting scale of functions between exponential and factorial has been provided by Petrogradsky in [12]. Even when the codimensions are exponentially bounded, the exponential growth rate can be not an integer (see [11]). In fact in [6] the authors constructed for any real number $\alpha > 1$ an algebra whose exponential growth of the codimensions is equal to α.

Here we are interested in the largest T-ideals, or equivalently in the smallest varieties. Recall that an algebra A is nilpotent if, for some $k \geq 1$, any product of k elements of A (with all possible arrangement of the brackets) is zero. Clearly if A is a nilpotent algebra then $c_n(A) = 0$, for n large. Accordingly we say that a variety is nilpotent if it is generated by a nilpotent algebra. Hence nilpotent varieties are the ones of smallest growth. Clearly next smallest varieties must have all proper subvarieties nilpotent. A variety with such a property is called almost nilpotent. If we consider varieties of associative algebras, it is easily seen (see Section 3) that the only almost nilpotent variety is the variety $\mathcal{V}$ of commutative algebras (the sequence of codimensions is $c_n(\mathcal{V}) = 1, n \geq 1$). In

case of varieties of Lie algebras, in [9] it was shown that there is also only one almost nilpotent variety: this is the metabelian variety, denoted $\mathcal{A}^2$, and in this case $c_n(\mathcal{A}^2) = n - 1$.

In the non-associative case the situation is unclear. Anyway, by a standard application of Zorn's lemma, here we show that any variety contains a subvariety which is almost nilpotent.

Next we consider the problem of constructing an almost nilpotent variety of exponential growth and this is the main result of the paper. In fact we explicitly construct a non-associative algebra A such that $\mathcal{V} = var(A)$ is almost nilpotent. Moreover, we extensively study the structure of the multilinear identities of A as a module of the symmetric group. We are able to determine almost all multiplicities in the n-th cocharacter of $\mathcal{V}$ and as a reward we prove that the variety $\mathcal{V}$ is almost nilpotent and $exp(\mathcal{V}) = exp(A)$ exists and is equal to 2.

2. Preliminaries

Throughout, F will be a field of characteristic zero, $X = \{x_1, x_2, \dots\}$ a countable set and $F\{X\}$ the free non-associative algebra on X over F. For every $n \geq 1$, we consider P_n, the space of multilinear polynomials of $F\{X\}$ in the first n variables $x_1, x_2, \dots, x_n$. Notice that since the number of distinct arrangements of parentheses on a monomial of length n is the Catalan number $\frac{1}{n}\binom{2n-2}{n-1}$, it readily follows that $\dim_F P_n = \binom{2n-2}{n-1}(n-1)!$.

Given an algebra A, let $Id(A) = \{f \in F\{X\} | f \equiv 0 \text{ on } A\}$ be the T-ideal of $F\{X\}$ of polynomial identities of A. Since char$F = 0$, it is well known that the sequence of spaces $P_n \cap Id(A)$, $n = 1, 2, \dots$, carries all information about $Id(A)$. The symmetric group S_n acts on P_n by permuting variables: if $\sigma \in S_n$, $f(x_1, \dots, x_n) \in P_n$,

$$\sigma f(x_1, \dots, x_n) = f(x_{\sigma(1)}, \dots, x_{\sigma(n)}).$$

The space $P_n \cap Id(A)$ is invariant under this action and one studies the structure of $P_n(A) = P_n/(P_n \cap Id(A))$ as an S_n-module. The S_n-character of $P_n(A)$, denoted $\chi_n(A)$, is called the nth cocharacter of A. Its degree $c_n(A) = \chi_n(A)(1)$ is the nth codimension of A. By complete reducibility one writes $\chi_n(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda$, where χ_λ is the irreducible S_n-character corresponding to the partition λ of n and $m_\lambda \geq 0$ is the multiplicity of χ_λ (see, for example, [8] for the representation theory of the symmetric group).

If $\mathcal{V} = var(A)$ is the variety generated by the algebra A, we define $P_n(\mathcal{V}) = P_n(A)$ and $\chi_n(\mathcal{V}) = \chi_n(A)$.

In case $\mathcal{V}$ is a proper variety of associative algebras, by [1], the multiplicities m_λ are polynomially bounded. If $A = F\langle X \rangle$ is the free associative algebra we have that in $\chi_n(A)$, $m_\lambda = d_\lambda$ where $d_\lambda = \deg \chi_\lambda$ is the degree of the character χ_λ. In the non-associative case these relations do not hold any more (see, for instance, [10], section 2). For instance, for the free non-associative algebra $A = F\{X\}$ we have that in $\chi_n(A)$, $m_\lambda = C_n d_\lambda$ where C_n is the nth Catalan number.

We next recall some basic properties of the representation theory of the symmetric group that we shall use in the sequel. Let $\lambda \vdash n$ be a partition and let T_λ be a Young tableau of shape $\lambda \vdash n$. We denote by e_{T_λ} the corresponding essential idempotent of the group algebra FS_n. Recall that $e_{T_\lambda} = \bar{R}_{T_\lambda} \bar{C}_{T_\lambda}$ where

$$\bar{R}_{T_\lambda} = \sum_{\sigma \in R_{T_\lambda}} \sigma, \quad \bar{C}_{T_\lambda} = \sum_{\tau \in C_{T_\lambda}} (\mathrm{sgn}\,\tau)\tau$$

and R_{T_λ}, C_{T_λ} are the row and column stabilizers of T_λ, respectively. Recall that if M_λ is an irreducible S_n-submodule of $P_n(\mathcal{V})$ corresponding to λ, then there exists a polynomial $f(x_1, \ldots, x_n) \in P_n$ and a tableau T_λ such that $e_{T_\lambda} f(x_1, \ldots, x_n) \notin Id(\mathcal{V})$.

3. Almost nilpotent subvarieties

Let $\mathcal{V}$ be a variety of algebras. We say that $\mathcal{V}$ is a nilpotent variety if there exists $k \geq 1$ such that the product of k variables (with all possible bracketing) is an identity of $\mathcal{V}$. Also, we say that a variety $\mathcal{V}$ is almost nilpotent if $\mathcal{V}$ is not a nilpotent variety but every proper subvariety $\mathcal{U} \subset \mathcal{V}$ is nilpotent.

We start this section by proving the following

Remark 1: Let $\mathcal{V}$ be the variety of associative commutative algebras. Then $\mathcal{V}$ is the only almost nilpotent variety of associative algebras.

Proof. Let $\mathcal{V}$ be the variety of associative commutative algebras. Then $\mathcal{V}$ can be generated by the algebra $A = Fe$ where $e^2 = e$. Let $\mathcal{U}$ be a variety of associative algebras and suppose that $\mathcal{V} \not\subseteq \mathcal{U}$. Then $Id(\mathcal{U}) \not\subseteq Id(\mathcal{V})$ says that there exists a (multilinear) polynomial $f(x_1, \ldots, x_k) \in Id(\mathcal{U}) \setminus Id(\mathcal{V})$.

Since $f \notin Id(\mathcal{V}) = Id(A)$, f does not vanish on the basis $\{e\}$ of A. Thus if $f = \sum_{\sigma \in S_k} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(k)}$, we get $0 \neq f(e, \ldots, e) = \sum_{\sigma \in S_k} \alpha_\sigma e^k = \sum_{\sigma \in S_k} \alpha_\sigma e$. It follows that $\alpha = \sum_{\sigma \in S_k} \alpha_\sigma \neq 0$. Now let $\mathcal{U} = \mathrm{var}(B)$ and take any $b \in B$. Then $0 = f(b, \ldots, b) = \alpha b^k$, which implies $b^k = 0$. We have proved that if $\mathcal{U}$ is a variety of associative algebras and $\mathcal{V} \not\subseteq \mathcal{U}$, then $\mathcal{U}$ satisfies the identity $x^k \equiv 0$, for some $k \geq 1$. But then by the Nagata–Higman theorem (see [3]) $\mathcal{U}$ satisfies $x_1 \cdots x_t \equiv 0$ for some $t \geq 1$, i.e., $\mathcal{U}$ is a nilpotent variety. This proves that any proper subvariety of $\mathcal{V}$ is nilpotent and $\mathcal{V}$ is the only almost nilpotent variety of associative algebras. ∎

THEOREM 1: *Let $\mathcal{V}$ be a non-nilpotent variety of algebras. Then there exists a subvariety $\mathcal{W}$ of $\mathcal{V}$ such that $\mathcal{W}$ is almost nilpotent.*

Proof. Let S be the family of T-ideals Q of $F\{X\}$ such that $Q \supseteq Id(\mathcal{V})$ and $\frac{F\{X\}}{Q}$ is not nilpotent; S is non-empty since it contains the T-ideal $Id(\mathcal{V})$. We consider S partially ordered by inclusion, and we let $\{Q_i\}_{i \in I}$ be a chain in S. If $Q = \bigcup_{i \in I} Q_i$, we claim that $Q \in S$, i.e., $Q \supseteq Id(\mathcal{V})$ and $\frac{F\{X\}}{Q}$ is not nilpotent.

In fact if not, there exists $k \geq 1$ such that any multilinear monomial in $x_1, \ldots, x_k$ (with any arrangement of the brackets) belongs to Q. Let C_k be the k-th Catalan number and let us denote by $\{w_1, \ldots, w_{C_k}\}$ the set of monomials obtained from $x_1 \cdots x_k$ by placing the brackets in all possible ways. Then for every i, there exists $Q_{s_i}, s_i \in I$, such that $w_i \in Q_{s_i}$. If $s = \max_{i=1,\ldots,C_k} s_i$, then $w_1, \ldots, w_{C_k} \in Q_s$ and so $\frac{F\{X\}}{Q_s}$ is nilpotent, a contradiction.

By applying Zorn's lemma to S it follows that there exists a maximal element $M \in S$. If $\mathcal{W}$ is the variety such that $Id(\mathcal{W}) = M$, then $\mathcal{W}$ is not nilpotent but any proper subvariety is nilpotent. ∎

4. The algebra A

In this section we define a non-associative algebra A that will be the main object of our investigation.

Let us denote by R_a the operator of right multiplication by a.

Definition 1: A is the algebra over F generated by the elements $\{a, b, z\}$ satisfying the following relations:

(1) $a^2 = b^2 = ab = ba = az = bz = 0$;

(2) $(zw(R_a, R_b))(zw'(R_a, R_b)) = 0$, for any fixed eventually empty words w, w' in R_a and R_b;

(3) for all $k \geq 0$,

$$z(R_a R_b)^k R_a R_b + z(R_a R_b)^k R_b R_a = 0,$$

$$z(R_a R_b)^k R_a^2 = z(R_a R_b)^k R_b^2 = 0.$$

The following lemma easily follows from the above relations.

LEMMA 1: *A basis of A is given by the elements*

$$a, b, z(R_a R_b)^k, z(R_a R_b)^k R_a, z(R_a R_b)^k R_b,$$

for all $k \geq 0$.

Proof. By the defining relations it is clear that the above elements span A. Next we prove that they are linearly independent over F. Let

$$(1) \quad \alpha_1 a + \alpha_2 b + \sum_i \beta_i z(R_a R_b)^i + \sum_j \gamma_j z(R_a R_b)^j R_a + \sum_k \delta_k z(R_a R_b)^k R_b = 0.$$

Applying the above on the right to z, we obtain that $\alpha_1 za + \alpha_2 zb = 0$. Hence $\alpha_1 za R_a + \alpha_2 zb R_a = \alpha_2 zb R_a = 0$, and so $\alpha_2 = \alpha_1 = 0$. If we now multiply (1) by R_a on the right it follows that

$$\sum_i \beta_i z(R_a R_b)^i R_a - \sum_k \delta_k z(R_a R_b)^{k+1} = 0.$$

By multiplying again by R_a on the right we obtain $\sum_k \delta_k z(R_a R_b)^{k+1} R_a = 0$ and so $\delta_k = 0, \forall k$. Similar calculations complete the proof. ∎

The following remark will be of use in what follows.

Remark 2: Let $f(R_a, R_b)$ be a not necessarily homogeneous polynomial in R_a, R_b. If $zf(R_a, R_b) \neq 0$ then $z(R_a R_b)f(R_a, R_b) \neq 0$.

We next prove

LEMMA 2: $x_1(x_2 x_3)$ *is an identity of A.*

Proof. Notice that if I is the ideal generated by z, then $\dim A/I = 2$.

Let $\text{ann}_r(A) = \{x \in A | Ax = 0\}$ be the right annihilator of A. Since $I \subseteq \text{ann}_r(A)$ and, by definition of A, $A^2 \subseteq I$, we obtain that $0 = AI \supseteq AA^2$ and so $x_1(x_2 x_3) \equiv 0$ holds on A. ∎

From Lemma 2 it is clear that only left normed polynomials can have a non-zero evaluation on A. Throughout we shall omit the parenthesis in left normed monomials.

LEMMA 3: *The left normed monomial x_0xxx is an identity of A.*

Proof. Let $\varphi : F\{X\} \to A$ be an evaluation on A. Since x_0xxx is linear in x_0 we may evaluate x_0 into a basis element of A.

If $\varphi(x_0) = a$ or b then $\varphi(f) = 0$.

Hence we assume that $\varphi(x_0) = zw(R_a, R_b)$, for some word w in R_a and R_b. Let $\varphi(x) = \alpha a + \beta b + c$, for some $\alpha, \beta \in F$ and $c \in I$, the ideal generated by z. Then, since $I \subseteq \mathrm{ann}_r(A)$, we may assume that $\varphi(x) = \alpha a + \beta b$. Now by direct inspection it is easily shown that $\varphi(x_0xxx) = 0$. For instance, if $\varphi(x_0) = z(R_aR_b)^kR_a$,

$$\begin{aligned}
\varphi(x_0xxx) =& z(R_aR_b)^kR_a(\alpha^2R_a^2 + \beta^2R_b^2 + \alpha\beta(R_aR_b + R_bR_a))(\alpha R_a + \beta R_b)\\
=& (\beta^2z(R_aR_b)^{k+1}R_b + \alpha\beta z(R_aR_b)^{k+1}R_a)(\alpha R_a + \beta R_b)\\
=& \alpha\beta^2z(R_aR_b)^{k+1}R_bR_a + \alpha\beta^2z(R_aR_b)^{k+1}R_aR_b = 0. \quad \blacksquare
\end{aligned}$$

By applying a partial linearization to the identity of Lemma 3 we have the following

Remark 3: $x_0x_1x_1x_2 \equiv -x_0x_1x_2x_1 - x_0x_2x_1x_1 \pmod{Id(A)}$.

LEMMA 4: $x_0xxy_1 \cdots y_{2s+1}yy$ *is an identity of A.*

Proof. Let φ be an evaluation on A. By linearity on $x_0, y_1 \cdots y_{2s+1}$ and by what we remarked in the previous lemma, we may assume that $\varphi(x_0) = zw(R_a, R_b)$, for some w, $\varphi(x) = \alpha a + \beta b$, $\varphi(y) = \gamma a + \delta b$ and $\varphi(y_i) = \overline{y}_i = a$ or b, where $\alpha, \beta, \gamma, \delta \in F$.

If $\deg w = 2k$, then $w = (R_aR_b)^k$ and by the defining relations we get

$$\varphi(x_0xxy_1 \cdots y_{2s+1}yy) = z(R_aR_b)^k(\alpha^2R_a^2 + \beta^2R_b^2 + \alpha\beta(R_aR_b + R_bR_a))\cdots = 0.$$

If $\deg w = 2k + 1$, then either $w = (R_aR_b)^kR_a$ or $w = (R_aR_b)^kR_b$. In any case, by the defining relations,

$$\varphi(x_0xxy_1 \cdots y_{2s+1}yy) = \mu z(R_aR_b)^{k+s+1}(\gamma^2R_a^2 + \delta^2R_b^2 + \gamma\delta(R_aR_b + R_bR_a)) = 0,$$

for some $\mu \in F$, and the lemma is proved. $\quad \blacksquare$

From now on we shall write $X_i = R_{x_i}$ the right multiplication by the variable x_i.

LEMMA 5: *Let* $w = w(X_1, X_2)$. *If* $\deg_{x_1} w - \deg_{x_2} w \geq 3$, *then* $x_0 w(X_1, X_2) \equiv 0$ *on* A.

Proof. The proof is by induction on $k = \deg_{x_2} w$.

If $k = 0$, we are done by Lemma 3. Suppose that the conclusion of the lemma holds for $\deg_{x_2} w = k > 0$ and let $\deg_{x_2} w = k + 1$.

If $x_0 w(X_1, X_2)$ is equal to $x_0 X_2 w'$ or to $x_0 X_1 X_2 w'$, for some w', then since $\deg_{x_2} w' < k + 1$, we are done by induction.

Therefore we may assume that $x_0 w(X_1, X_2) = x_0 X_1 X_1 w''$, for some w''. Also, by Lemma 2, we may assume that $x_0 w(X_1, X_2) = x_0 X_1 X_1 X_2 w'''$, for some w'''. Now, by Remark 3, we can write

$$x_0 X_1 X_1 X_2 \equiv -x_0 X_1 X_2 X_1 - x_0 X_2 X_1 X_1 \pmod{Id(A)},$$

and the proof follows from the previous cases. ∎

5. Some numerical invariants

We start this section by proving some combinatorial propositions that will allow us to compute most multiplicities in the cocharacter of $\mathcal{V} = var(A)$ for most partitions.

Recall that if $\lambda \vdash n$ is a partition of n and T_λ is a Young tableau of shape λ, we denote by $e_{T_\lambda} \in FS_n$ the corresponding essential idempotent of FS_n. In what follows we shall identify e_{T_λ} with the polynomial $e_{T_\lambda}(y_0 y_1, \ldots, y_n) = y_0 e_{T_\lambda} y_1 \cdots y_n$ obtained by acting with e_{T_λ} on the left normed monomial $y_1 \cdots y_n$. We shall then identify all variables corresponding to each row of the tableau.

Let $Q_n = \text{span}\{x_0 x_{\sigma(1)} \cdots x_{\sigma(n)} | \sigma \in S_n\}$ be the space of multilinear left normed monomials in $x_0, \ldots, x_n$ with x_0 as the first variable. The symmetric group S_n acts on Q_n by permuting the variables $x_1, \ldots, x_n$ and $Q_n \cap Id(A)$ is invariant under this action. Hence we have that

$$Q_n(A) = \frac{Q_n}{Q_n \cap Id(A)}$$

is an S_n-module and we denote by

$$\chi_n^Q(A) = \chi^{S_n}\left(\frac{Q_n}{Q_n \cap Id(A)}\right)$$

its character. We decompose $\chi_n^Q(A)$ into irreducible S_n-characters and we write

$$\chi_n^Q(A) = \sum_{\lambda \vdash n} m_\lambda^Q \chi_\lambda,$$

where m_λ^Q is the multiplicity of χ_λ in $\chi_n^Q(A)$.

Let $f \in Q_n$ and suppose that f has a non-zero evaluation on A. Being multi-linear f has a non-zero evaluation on a basis of A. Hence we must evaluate x_0 into $z(R_a R_b)^k$ or $z(R_a R_b)^k R_a$ or $z(R_a R_b)^k R_b$. It follows that any other variable of f must be evaluated into a or b. As a consequence we get that any multilinear polynomial of Q_n alternating on 3 variables is an identity of A. By standard arguments, this says that if the diagram of $\lambda \vdash n$ has more than two rows, then the corresponding S_n-character χ_λ appears with zero multiplicity in $\chi_n^Q(A)$. Also, let $(\lambda_1, \lambda_2) \vdash n$ be such that $\lambda_1 - \lambda_2 \geq 3$. Then for any tableau T_λ, if f is the polynomial obtained from $e_{T_\lambda}(x_0 x_1, \ldots, x_n)$ by identifying the variables corresponding to each row, we get that $\deg_{x_1} f - \deg_{x_2} f \geq 3$ and, by Lemma 5, we obtain that $m_\lambda^Q = 0$. Thus we obtain the decomposition

$$\chi_{2k}^Q(A) = m_{(k,k)}^Q \chi_{(k,k)} + m_{(k+1,k-1)}^Q \chi_{(k+1,k-1)}$$

and

$$\chi_{2k+1}^Q(A) = m_{(k+1,k)}^Q \chi_{(k+1,k)}.$$

In order to simplify the notation we shall also use the following convention: a monomial M in which some variables are overlined by the same sign (say tilda, hat, etc.) must be read as the polynomial in which those variables are alternated.

We start by analyzing the partitions of the type $\lambda = (k+1, k-1), k \geq 1$. We have the following

PROPOSITION 1: *If* $\lambda = (k+1, k-1), k \geq 1$, *then* $m_{(k+1,k-1)}^Q = 1$.

Proof. Let w_1, w_2 be polynomials, obtained from the essential idempotents corresponding to two different tableaux of shape λ by identifying the elements in each row of λ. Then $\deg_{x_1} w_1 - \deg_{x_2} w_1 = \deg_{x_1} w_2 - \deg_{x_2} w_2 = 2$.

First we shall prove that $m_{(k+1,k-1)}^Q \leq 1$.

To this end we assume that $x_0 w_1$ and $x_0 w_2$ are not polynomial identities of A and we need to prove that $x_0 w_1$ and $x_0 w_2$ are linearly dependent (mod. $Id(A)$). We prove our claim by induction on $t = \deg_{x_2} w_1 = \deg_{x_2} w_2$.

If $t = 0$, then $w_1 = w_2 = x_1 x_1$ and we are done.

Let $\deg_{x_2} w = k > 0$, where w is a monomial, and suppose first that $x_0 w = x_0 X_2 w'$ for some w'. Since $\deg_{x_1} w' - \deg_{x_2} w' = 3$ then, by Lemma 5, $x_0 w \equiv 0 \pmod{Id(A)}$. Therefore we must have that either $x_0 w = x_0 X_1 X_2 w'$ or $x_0 w = x_0 X_1 X_1 w''$, for some w', w''. In the second case, from Lemma 3, it follows that $x_0 w = x_0 X_1 X_1 X_2 w'''$. Moreover, by Remark 3, we have that $x_0 w = x_0 X_1 X_1 X_2 w''' \equiv -x_0 X_1 X_2 X_1 w''' \pmod{Id(A)}$.

Hence we may assume that $x_0 w_1 = x_0 X_1 X_2 w_1'$. Similar considerations concerning w_2 lead to $x_0 w_2 = x_0 X_1 X_2 w_2'$. But then we are done by induction. We have proved that $m_{(k+1,k-1)}^Q \leq 1$.

Next we prove that $m_{(k+1,k-1)}^Q = 1$.

Let

$$g_k = x_0 x_1 x_1 \underbrace{\bar{x_1} \bar{x_2} \cdots \hat{x_1} \hat{x_2}}_{k-1 \text{ pairs}}$$

be the left-normed polynomial associated to $\lambda = (k+1, k-1)$, where $\bar{}, \hat{}$ mean alternation on the corresponding elements. Notice that the polynomial g_k is obtained from the essential idempotent corresponding to λ by identifying all the elements in each row of λ.

We claim that if φ is an evaluation on A such that $\varphi(x_0) = za, \varphi(x_1) = a + b, \varphi(x_2) = b$, then $\varphi(g_k) = (-1)^{k-1} z(ab)^k(a+b) \neq 0$.

In fact, by induction on k, if $k = 1$ then $\varphi(g_1) = \varphi(x_0 x_1 x_1) = zab(a+b)$. Now, in general,

$$\varphi(g_{k+1}) = \varphi(g_k)\widehat{(a+b)}\widehat{b} = (-1)^{k-1} z(ab)^k(a+b)\widehat{(a+b)}\widehat{b}.$$

Since by (3) of Definition 1

$$z(ab)^k(a+b)(a+b)b \equiv 0 \pmod{Id(A)},$$

it follows that $\varphi(g_{k+1}) = (-1)^k z(ab)^{k+1}(a+b) \neq 0$, and we are done. ∎

PROPOSITION 2: *If $\lambda = (k+1, k)$ then $m_{(k+1,k)}^Q = 2$.*

Proof. By induction on k. If $k = 1$, since $d_{(2,1)} = \deg \chi_{(2,1)} = 2$, we are done.

Let $k \geq 2$, and let $w = w(x_1, x_2)$ be a homogeneous polynomial such that $\deg_{x_1} w - \deg_{x_2} w = 1$. We claim that $x_0 w$ can be written as a linear combination of polynomials of the type $x_0 X_2 w'$ and $x_0 w' X_2$ for some w'.

In fact, if we apply partial linearization to the identity of Lemma 3 we obtain

$$x_0 X_1 X_1 X_2 \equiv -x_0 X_1 X_2 X_1 - x_0 X_2 X_1 X_1 \pmod{Id(A)}$$

and

$$x_0 u X_2 X_1 X_1 \equiv -x_0 u X_1 X_2 X_1 - x_0 u X_1 X_1 X_2 \ (\text{mod. } Id(A)),$$

for any monomial u. It follows that $x_0 w$ is a linear combination of polynomials of the type $x_0 X_2 w'$, $x_0 w' X_2$ and $x_0 X_1 X_2 v X_2 X_1$ for some w', v with $\deg v = 2k-3$.

But linearizing the identity of Lemma 4, for any v with $\deg v = 2k-3$ we get

$$x_0 X_1 X_2 v X_2 X_1 \equiv -x_0 X_2 X_1 v X_1 X_2 - x_0 X_1 X_2 v X_1 X_2 - x_0 X_2 X_1 v X_2 X_1$$

and the claim is proved.

Let $H_{k+1,k-1}$ be the space of homogeneous polynomials in x_1, and x_2 of degree $k+1$ in x_1 and $k-1$ in x_2. Then Proposition 1 says that $\dim x_0 H_{k+1,k-1} \leq 1$ and $\dim x_0 H_{k+1,k-1} X_2 \leq 1 \ (\text{mod. } Id(A))$. Hence from the previous claim we get that $m^Q_{(k+1,k)} \leq 2$.

Define the following two polynomials:

$$g_1 = x_0 x_1 \bar{x_1} \bar{x_2} \cdots \widetilde{x_1} \widetilde{x_2} = x_0 g_1'$$

and

$$g_2 = x_0 \bar{x_1} \bar{x_2} \cdots \widetilde{x_1} \widetilde{x_2} x_1 = x_0 g_2'.$$

The polynomials g_1', g_2' are obtained from the essential idempotents corresponding to the tableaux

$$T^1_\lambda = \begin{array}{|c|c|c|c|c|} \hline 2 & 4 & \cdots & n-1 & 1 \\ \hline 3 & 5 & \cdots & n \\ \hline \end{array} \quad \text{and} \quad T^2_\lambda = \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & \cdots & n-2 & n \\ \hline 2 & 4 & \cdots & n-1 \\ \hline \end{array}$$

by identifying the integers in each row of λ.

In order to complete the proof of the proposition it is enough to prove that g_1, g_2 are linearly independent $(\text{mod. } Id(A))$. Write $g = \alpha g_1 + \beta g_2 \equiv 0$ $(\text{mod. } Id(A))$, for some $\alpha, \beta \in F$.

Let φ be an evaluation on A such that $\varphi(x_1) = a, \varphi(x_2) = b, \varphi(x_0) = z$.

Since by the defining relation on A,

$$z(R_a R_b)^k R_a R_b \equiv -z(R_a R_b)^k R_b R_a \ (\text{mod. } Id(A)),$$

and by Remark 3,

$$zaba \equiv -zaab - zbaa \equiv -zbaa \ (\text{mod. } Id(A)),$$

we obtain

$$\varphi(g) = (-1)^k \alpha z abab \cdots aba + 2^k \beta z abab \cdots aba \equiv 0 \ (\text{mod. } Id(A))$$

and so
$$(-1)^k \alpha + 2^k \beta = 0.$$
In the same way, by putting $\varphi(x_1) = a, \varphi(x_2) = b, \varphi(x_0) = zb$ we have
$$(2^k \alpha + (-1)^k \beta)zbabab \cdots ab \equiv 0 \pmod{Id(A)}.$$
It follows that
$$2^k \alpha + (-1)^k \beta = 0.$$
Hence from the above we have that $\alpha = \beta = 0$, and this implies that $m^Q_{(k+1,k)} = 2$.   ∎

PROPOSITION 3: *If $\lambda = (k, k)$ then $0 < m^Q_{(k,k)} \leq 2$.*

Proof. First we prove that $m^Q_{(k,k)} > 0$. Let $n = 2k$ and define the following polynomial:
$$g = x_0 \bar{x}_1 \bar{x}_2 \cdots \widetilde{x_1} \widetilde{x_2} = x_0 g',$$
where we have k pairs $x_1 x_2$.

The polynomial g' is obtained from the essential idempotent corresponding to the tableaux

$$T_\lambda = \begin{array}{|c|c|c|c|}
\hline
1 & 3 & \cdots & n-1 \\
\hline
2 & 4 & \cdots & n \\
\hline
\end{array}$$

by identifying the integers in each row. If $m_{(k,k)} = 0$ then $g \equiv 0 \pmod{Id(A)}$. From this it follows that $g_1 \equiv 0$ and $g_2 \equiv 0 \pmod{Id(A)}$, where g_1 and g_2 are the polynomials defined in the proof of Proposition 2, a contradiction. Hence $m^Q_{(k,k)} > 0$.

Let w be a polynomial in x_1 and x_2, with $\deg_{x_1} w = \deg_{x_2} w = k$. By using a partial linearization of the identity in Lemma 3, it easily follows that mod. $Id(A)$, w can be written as a linear combination of monomials of the type
$$x_1 x_2 x_2 w', x_2 x_1 x_1 w'', x_1 x_1 x_2 w''', x_2 x_2 x_1 v,$$
for some w', w'', w''', v.

By a repeated application of this argument we get the following.

If $k = 2m$, then w can be written as a linear combination of the following 4 monomials,
$$w_1 = x_1 x_1 x_2 x_2 \cdots x_1 x_1 x_2 x_2,$$
$$w_1' = x_2 x_2 x_1 x_1 \cdots x_2 x_2 x_1 x_1,$$
$$w_2 = x_1 x_2 x_2 x_1 \cdots x_1 x_2 x_2 x_1,$$
$$w_2' = x_2 x_1 x_2 x_1 x_2 \cdots x_2 x_1 x_1 x_2.$$

If $k = 2m + 1$, then w can be written as a linear combination of the following monomials:

$$\bar{w}_1 = x_1 x_1 x_2 x_2 \cdots x_1 x_1 x_2 x_2 x_1 x_2,$$

$$\bar{w}_1' = x_2 x_2 x_1 x_1 \cdots x_2 x_2 x_1 x_1 x_2 x_1,$$

$$\bar{w}_2 = x_1 x_2 x_2 x_1 \cdots x_1 x_2 x_2 x_1 x_1 x_2,$$

$$\bar{w}_2' = x_2 x_1 x_1 x_1 x_2 \cdots x_2 x_1 x_1 x_2 x_2 x_1.$$

Let g be a highest weight vector corresponding to the partition $\lambda = (k, k)$. If $k = 2m$, then we can write $g \equiv \alpha w_1 + \beta w_1' + \gamma w_2 + \delta w_2'$, for some $\alpha, \beta, \gamma, \delta \in F$. Since g is alternating on k pairs $\{x_1, x_2\}$, we get that

$$g(x_1, x_2) \equiv (-1)^{2m} g(x_2, x_1) = \alpha w_1' + \beta w_1 + \gamma w_2' + \delta w_2.$$

Hence

$$g \equiv \frac{\alpha + \beta}{2}(w_1 + w_1') + \frac{\gamma + \delta}{2}(w_2 + w_2') \ (\text{mod.} \ Id(A)).$$

It follows that any highest weight vector corresponding to the partition $\lambda = (k, k)$ is a linear combination of $(w_1 + w_2)$ and $(w_1' + w_2')$ and so $m_{(k,k)}^Q \leq 2$.

Similarly, if $k = 2m + 1$, write

$$g(x_1, x_2) \equiv \alpha \bar{w}_1 + \beta \bar{w}_1' + \gamma \bar{w}_2 + \delta \bar{w}_2'$$

and we get

$$g(x_1, x_2) \equiv (-1)^{2m+1} g(x_2, x_1) = \alpha \bar{w}_1' + \beta \bar{w}_1 + \gamma \bar{w}_2' + \delta \bar{w}_2.$$

Hence

$$g \equiv \frac{\alpha - \beta}{2}(w_1 - w_1') + \frac{\gamma - \delta}{2}(w_2 - w_2') \ (\text{mod.} \ Id(A)).$$

As above this says that $m_{(k,k)}^Q \leq 2$, and the proof is complete.   ∎

Putting together the results so far obtained, we have the following

THEOREM 2:   (1) $\chi_{2k+1}^Q(A) = 2\chi_{(k+1,k)}$.
(2) $\chi_{2k}^Q(A) = a\chi_{(k,k)} + \chi_{(k+1,k-1)}$ with $a = 1$ or $a = 2$.

In the next proposition we shall determine the relation between the sequence of codimensions $c_n^Q(A)$ and $c_n(A), n = 1, 2, \ldots$.

PROPOSITION 4: For all $n \geq 1$ we have that $c_{n+1}(A) = (n + 1) \cdot c_n^Q(A)$.

Proof. For every $i \in \{0, \ldots, n\}$ let us denote by Q_n^i the vector space spanned by the left normed monomials in $x_0, x_1, \ldots, x_n$ starting with x_i. Hence in this terminology $Q_n = Q_n^0$. Let us also denote by $P_{n+1}(A)$ the space of multilinear left normed polynomials in $x_0, x_1, \ldots, x_n$ (not in $x_1, x_2, \ldots, x_{n+1}$ as before). It is obvious that

$$P_{n+1}(A) = Q_n^0(A) + \cdots + Q_n^n(A),$$

and so $c_{n+1}(A) \le (n+1) \cdot c_n^Q(A)$.

We claim that

$$P_{n+1}(A) = Q_n^0(A) \oplus \cdots \oplus Q_n^n(A).$$

This is clear by the structure of the algebra A. We only note that if $x_0 f_1(x_1, \ldots, x_n), \ldots, \ x_0 f_m(x_1, \ldots, x_n) \ \in \ Q_n^0$ are linearly independent (mod. $Id(A)$), then the $(n+1) \cdot m$ left-normed polynomials of P_{n+1}

$$x_0 f_i(\widehat{x_0}, x_1, \ldots, x_n), \ldots, x_n f_i(x_0, x_1, \ldots, \widehat{x_n}), \quad i = 1, \ldots, m,$$

are linearly independent (mod. $Id(A)$), where $\widehat{}$ means that the corresponding element is omitted. ∎

Next we want to study the variety of algebras $\mathcal{V} = \mathrm{var}(A)$, generated by A. Our aim is to prove that $\mathcal{V}$ is almost nilpotent. We start by proving the following

PROPOSITION 5: *Let $\mathcal{W} \subsetneq \mathcal{V}$ be a proper subvariety of $\mathcal{V}$. If, for some $n \ge 1$,*
$$x_0 \underbrace{x_1 x_1 x_2 x_2 x_1 x_1 \cdots x_2 x_2 x_1 x_1}_{2n} \in Id(\mathcal{W}), \text{ then } c_N^Q(\mathcal{W}) = 0, \text{ for all } N \ge 2n+2.$$

Proof. Clearly $c_N^Q(\mathcal{W}) = 0$ means that $Q_N \subseteq Id(\mathcal{W})$. Now since $\mathcal{W} \subsetneq \mathcal{V}$, recalling the decomposition given in Theorem 2, we get that $\chi_n^Q(\mathcal{W}) < \chi_n^Q(\mathcal{V})$, i.e., $\chi_{2k+1}^Q(\mathcal{W}) = a\chi_{(k+1,k)}$ and $\chi_{2k}^Q(\mathcal{W}) = b\chi_{(k,k)} + c\chi_{(k+1,k-1)}$, with $a \le 2$, $b \le 2, c \le 1$. In particular, this says that in order to prove that $c_N^Q(\mathcal{W}) = 0$ it is enough to show that $x_0 H_{h,l} \subseteq Id(\mathcal{W})$, where $H_{h,l}$ is the space of homogeneous polynomials of degree h in x_1 and l in x_2. Here $l = k$ or $l = k-1$ or $l = k-2$. Let $v = x_0 w(x_1, x_2) \in x_0 H_{k,l} \pmod{Id(\mathcal{W})}$.

We claim that $v = x_0 w(x_1, x_2) \equiv x_0 x_1 x_1 w'$ or $x_0 x_2 x_2 w'$ or $x_0 x_1 x_2 x_2 w'$ or $x_0 x_2 x_1 x_1 w'$, for some w'.

In fact, if $v = x_0 x_1 x_1 v'$ or $x_0 x_2 x_2 w'$ for some v', w', we are done.

Otherwise, recalling that, by Remark 3, $x_0 x_1 x_2 x_1 \equiv -x_0 x_2 x_1 x_1 - x_0 x_1 x_1 x_2$, we get the claim. Now, by eventually exchanging the role of x_1 and x_2, we may

assume that $v = ux_1x_1w'$ where $u = x_0$ or $u = x_0x_2$ and w' is a polynomial, not necessarily homogeneous, in x_1 and x_2. Then since $x_0x_1x_1x_1 \equiv 0$, it follows that

$$v = ux_1x_1x_2x_2x_1w''$$

or

$$v = ux_1x_1x_2x_1x_2w''' \equiv -ux_1x_1x_2x_2x_1w'''.$$

Now the proof is complete by induction on n. Notice that in this construction we have eventually one extra variable at the beginning after x_0 and eventually one extra variable at the right-end side.   ∎

We can now prove the main result of the paper.

THEOREM 3: *The variety $\mathcal{V} = var(A)$ is almost nilpotent and $exp(\mathcal{V}) = exp(A) = 2$.*

Proof. We first show that $exp(A) = 2$.

In fact, from the decomposition of the cocharacter of $Q_n(A)$ and Proposition 4 we have the asymptotic equality $\sqrt[n]{c_n(A)} \simeq \sqrt[n]{d_{(m,m)}}$, where $n = 2m$ or $2m + 1$. Since $\lim_{n \to \infty} \sqrt[n]{d_{(m,m)}} = 2$, we get that $exp(A) = 2$.

Next we show that $\mathcal{V} = var(A)$ is almost nilpotent.

Let $\mathcal{W} \subsetneq \mathcal{V}$ be a proper subvariety. Then, there exists a multilinear polynomial f, of degree l, such that $f \in Id(\mathcal{W})$ and $f \notin Id(\mathcal{V})$. We claim that we may assume that $f = x_0w$ where $w = w(R_{x_1}, R_{x_2})$. In fact, since $x(yz) \in Id(\mathcal{V})$, we have that $f = \sum x_s w_s$ with $w_s = w_s(R_{x_1}, \ldots, \widehat{R_{x_s}}, \ldots, R_{x_l})$ (here again, the symbol ^ means that the corresponding element is omitted).

Let t be such that $x_t w_t \notin Id(\mathcal{V})$ and let φ be a non-zero evaluation in A. Since f is linear, by Lemma 1, we may assume that $\varphi(x_t) = z(R_aR_b)^k$ or $z(R_aR_b)^kR_a$ or $z(R_aR_b)^kR_b$ and $k \geq 1$. If we now make the substitution $x_t = x_0x_{l+1}x_{l+2}$ in f, then we obtain $x_0x_{l+1}x_{l+2}w_t \notin Id(\mathcal{V})$ and $x_0x_{l+1}x_{l+2}w_t \in Id(\mathcal{W})$.

In fact if $\varphi(x_t) = z(R_aR_b)^k$, it is sufficient to set $x_0 = \varphi(x_t) = z(R_aR_b)^{k-1}$, $x_{l+1} = R_a, x_{l+2} = R_b$, similarly for the other cases. Since $\chi_n^Q(A)$ lies in a strip of height 2, we get the claim, i.e., we may assume that $f = x_0w$ and w is in only two variables x_1 and x_2.

By multiplying on the right by x_1 and x_2 in a suitable way, we may assume that $f = x_0w$ and $\deg w = 4m+1$. By Proposition 2 or its proof we may assume that

$$f = \alpha x_0 x_1 \underbrace{\tilde{x}_1\tilde{x}_2 \cdots \tilde{x}_1\tilde{x}_2}_{2m \text{ pairs}} + \beta x_0 \underbrace{\tilde{x}_1\tilde{x}_2 \cdots \tilde{x}_1\tilde{x}_2}_{2m \text{ pairs}} x_1 \in Id(\mathcal{W}),$$

with $\alpha\beta \neq 0$. If we now substitute x_0x_1 for x_0, we get

$$(2) \qquad \alpha x_0 x_1 x_1 \bar{x_1} \bar{x_2} \cdots \tilde{x_1} \tilde{x_2} + \beta x_0 x_1 \bar{x_1} \bar{x_2} \cdots \tilde{x_1} \tilde{x_2} x_1 \ \in Id(\mathcal{W}).$$

Multiplying on the right by x_1 we obtain

$$(3) \qquad \alpha x_0 x_1 \bar{x_1} \bar{x_2} \cdots \tilde{x_1} \tilde{x_2} x_1 + \beta x_0 \bar{x_1} \bar{x_2} \cdots \tilde{x_1} \tilde{x_2} x_1 x_1 \ \in Id(\mathcal{W}).$$

Set

$$g_0 = x_0 \underbrace{x_1 x_1 x_2 x_2 \ldots x_1 x_1 x_2 x_2}_{4m} x_1 x_1, \quad g_1 = x_0 x_1 x_1 \underbrace{\bar{x_1}\bar{x_2}\cdots\tilde{x_1}\tilde{x_2}}_{2m \text{ pairs}}$$

and

$$g_2 = x_0 x_1 \underbrace{\bar{x_1}\bar{x_2}\cdots\tilde{x_1}\tilde{x_2}}_{2m \text{ pairs}} x_1.$$

By using Lemma 3 and Lemma 4, from (2) we obtain that $g_1 \equiv (-1)^m g_0$ and $g_2 = 2^k(-1)^m g_0$. This implies $\alpha g_0 + \beta^k(-1)^m g_0 \equiv 0$.

From (3), by applying Lemma 3 and Lemma 4, we obtain $2^k \alpha g_0 + \beta g_0 \equiv 0$. Since $\alpha\beta \neq 0$ it follows that $g_0 \equiv 0 \pmod{Id(\mathcal{W})}$. By Proposition 5, $c_n^Q(\mathcal{W}) = 0$, for all $n \geq 4m + 2$. Moreover, as in the proof of Proposition 4 it follows that $c_{n+1}(\mathcal{W}) \leq (n+1)c_n^Q(\mathcal{W})$. Hence $c_{n+1}(\mathcal{W}) = 0$ and so $\mathcal{W}$ is nilpotent. ∎

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