Analysis of multi-degree-of-freedom systems with fractional derivative elements of rational order

Mario Di Paola†, Francesco P. Pinnola†, Pol D. Spanos*

†Università degli Studi di Palermo - DICAM, Viale delle Scienze Ed.8, I-90128 Palermo, Italy
mario.dipaola@unipa.it, francesco.pinnola@unipa.it
*Rice University - Department of Civil and Environmental Engineering, 6100 Main Street MS-519, 77005, Houston, Texas
spanos@rice.edu

***Abstract*—In this paper a novel method based on complex eigenanalysis in the state variables domain is proposed to uncouple the set of rational order fractional differential equations governing the dynamics of multi-degree-of-freedom system. The traditional complex eigenanalysis is appropriately modified to be applicable to the coupled fractional differential equations. This is done by expanding the dimension of the problem and solving the system in the state variable domain. Examples of applications are given pertaining to multi-degree-of-freedom systems under both deterministic and stochastic loads.**

***Keywords*—Multi-degree-of-freedom systems, complex eigenvalue analysis, fractional state variables, frequency domain analysis**

I. Introduction

The study of systems of linear differential equations of fractional order has been dealt with by numerous authors, particularly in recent years. In engineering filed the interest in this kind of problems is increasing because the fractional calculus and fractional differential equations can be used in diverse themes such as the mechanical behavior of materials, the electric laws of the capacitors, the thermal properties of real conductors, etc.. Several examples of applications of the fractional calculus concern the description of viscoelastic behavior of real materials. In fact, many materials used in various fields of engineering have a particular viscoelastic mechanical behavior that can be modeled by fractional constitutive laws [1]–[5]. This kind of law is characterized by the presence of derivatives and/or integrals of fractional order in the stress-strain relation. Obviously, the dynamic analysis of structural and mechanical systems, which are built by materials with fractional constitutive law, must involve fractional operators in the equations of motion. Usually, this problem is represented by a set of coupled fractional differential equations, and the solution can not readily be found in the general case. This kind of problem that is known as the analysis of fractional multi-degree-of-freedom systems (FMDOF) has been solved in the time domain by using the step-by-step numerical approach [6], and in the frequency domain by using the approximate modal superposition method [7].

The analysis of FMDOF systems has been considered in several articles in which the authors have investigated the improvements of fractional viscoelastic devices in the frame structures [8]–[10]. FMDOF systems may also appear in the study of continuous systems with fractional constitutive law [11,12]. In fact, this kind of problem leads to a fractional multi-degree-of-freedom system in the discretized form [13,14]. From these articles, it can be concluded that the analysis of FMDOF systems is more cumbersome with respect to the classical one.

In certain previous efforts the integration of the equation of motion with fractional terms has been carried by finite difference schemes [15] or by the modified Newmark algorithm [16] and/or by using modal analysis; these approaches often lead to considerable computational burden.

In this paper a novel method to perform the analysis of the FMDOF system is presented. The method is based on modifying the traditional complex modal analysis in the state variables domain to obtain a set of uncoupled fractional differential equations. The sole limitation of the presented method regards the involved fractional orders, since the state variable analysis can be performed for FMDOF system only in the case in which all the derivative terms are rational.

II. Fractional multi-degree-of-freedom systems

The equation of motion of a quiescent single-degree-of-freedom linear fractional system (FSDOF) is

$$m\ddot{x}(t) + c_\beta \left(\mathrm{D}^{\beta}_{0^+}x\right)(t) + kx(t) = f(t), \quad t \geqslant 0, \tag{1}$$

where $f(t)$ is the forcing function, $x(t)$ is the response of the system, m is the mass, c_β is the coefficient of fractional term, k is the stiffness, and $\left(\mathrm{D}^{\beta}_{0^+}x\right)(t)$ is the fractional derivative of $x(t)$, defined as

$$\left(\mathrm{D}^{\beta}_{0^+}x\right)(t) = \frac{1}{\Gamma(1-\beta)}\frac{d}{dt}\int_0^t x(\tau)(t-\tau)^{-\beta}d\tau, \quad 0 \le \beta \le 1. \tag{2}$$

The fractional r-degree-of-freedom system is described by a coupled system of fractional differential equations of various FSDOF. This kind of problem can be cast in the form

$$\mathbf{M}\ddot{\boldsymbol{x}}(t) + \sum_{i=1}^{l} \mathbf{C_i}\mathrm{D}^{\beta_i}\boldsymbol{x}(t) + \mathbf{K}\boldsymbol{x}(t) = \mathbf{f}(t), \tag{3}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and the stiffness matrices; $\mathbf{C_i}$ is the matrix of the coefficients c_{β_i} of the involved fractional terms of order β_i; $\boldsymbol{x}(t)$ is a vector which describes the response of the systems; and $\mathbf{f}(t)$ is the vector of loading.

Consider next the simpler quasi-static problem, in which the terms of the vector $\mathbf{f}(t)$ vary in time slowly to the extent that the inertial force may be neglected. Thus, assuming that the involved fractional terms are all of the same order of derivation β_1, the equation of motion becomes

$$\mathbf{C_1}\mathrm{D}^{\beta_1}\boldsymbol{x}(t)+\mathbf{K}\boldsymbol{x}(t)=\mathbf{f}(t). \tag{4}$$

This kind of system related to the eigenvectors $\boldsymbol{\phi}_j$ of the matrix

$$\mathbf{K}^{-1}\mathbf{C_1}=\boldsymbol{D}, \tag{5}$$

where $\boldsymbol{D}$ is the dynamical matrix. Assume that the modal matrix $\boldsymbol{\Phi}$, where columns are the eigenvectors $\boldsymbol{\phi}_j$, is also normalized with respect to $\mathbf{C_1}$. That is

$$\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{C_1}\boldsymbol{\Phi}=\mathbf{I},\qquad \boldsymbol{\Phi}^{\mathrm{T}}\mathbf{K}\boldsymbol{\Phi}=\mathbf{U}_{\mathrm{D}}, \tag{6}$$

where $\mathbf{I}$ is the identity matrix, and $\mathbf{U}_{\mathrm{D}}$ is a diagonal matrix whose elements are posistive, since $\mathbf{K}$ is positive definite.

Making the modal transformation

$$\boldsymbol{x}(t)=\boldsymbol{\Phi}\boldsymbol{y}(t), \tag{7}$$

inserting Eq. (7) in Eq. (4), and premultiplying by $\boldsymbol{\Phi}^{\mathrm{T}}$, Eq. (4) yields

$$\mathbf{I}\,\mathrm{D}^{\beta_1}\boldsymbol{y}(t)+\mathbf{U}_{\mathrm{D}}\boldsymbol{y}(t)=\mathbf{g}(t), \tag{8}$$

where $\mathbf{g}(t)=\boldsymbol{\Phi}^{\mathrm{T}}\mathbf{f}(t)$ is the forcing vector in the modal space, and $\boldsymbol{y}(t)$ is the vector of the modal displacements that contains the terms $y_j(t)$. Clearly, Eq. (8) represents a set of uncoupled differential equations that may be readily solved.

Once $y_j(t)$ is found for $j=1,2,\ldots,r$, then $\boldsymbol{x}(t)$ can be obtained using Eq. (7). This kind of modal transformation can diagonalize the system in Eq. (3) in some particular cases in which all the matrices can be represented by a linear combination of the matrices $\mathbf{K}$ and $\mathbf{C_1}$.

The next section describes a new method to decouple the system in Eq. (3) in the case in which all the involved matrices are arbitrary. The method is based on a proper transformation/augmentation in the state variables domain of fractional MDOF system.

III. State variable analysis of FMDOF system with rational orders

There are various fractional terms in the Eq. (3). By assuming that all fractional orders are rational, it is possible to represent the generic fractional order in Eq. (3) as $\beta_i=a_i/b_i$ where $a_i,b_i\in\mathbb{N}$ with $i=1,2,\ldots,l$. Thus, the system in Eq. (3) can be rewritten as the following sequential linear differential equations of fractional orders:

$$\sum_{j=1}^{n}\mathbf{C_j}\mathrm{D}^{j\alpha}\boldsymbol{x}(t)+\mathbf{K}\boldsymbol{x}(t)=\mathbf{f}(t), \tag{9}$$

where α is chosen such that $n\alpha$ is equal to the maximum order that appears in the system of equations, and such that all involved orders in Eq. (3) can be represented as $\beta_i=d_i\alpha$ where $d_i\in\mathbb{N}$. In this case $n\alpha=2$ and the corresponding matrix $\mathbf{C_n}=\mathbf{M}$ is the matrix of the mass; all matrices in Eq. (9) have dimension $r\times r$. Introducing the vector of state variables

$$\mathbf{z}^{\mathrm{T}}(t)=\left[\boldsymbol{x}^{\mathrm{T}}(t)\;\;\mathrm{D}^{\alpha}\boldsymbol{x}^{\mathrm{T}}(t)\;\;\mathrm{D}^{2\alpha}\boldsymbol{x}^{\mathrm{T}}(t)\;\;\ldots\;\;\mathrm{D}^{\alpha(n-1)}\boldsymbol{x}^{\mathrm{T}}(t)\right], \tag{10}$$

and appending to Eq. (9) the $n-1$ identities

$$\begin{aligned}
&\sum_{j=1}^{n-1}\mathbf{C_{j+1}}\mathrm{D}^{\alpha}\mathrm{D}^{(j-1)\alpha}\boldsymbol{x}(t)=\sum_{j=1}^{n-1}\mathbf{C_{j+1}}\mathrm{D}^{j\alpha}\boldsymbol{x}(t),\\
&\sum_{j=1}^{n-2}\mathbf{C_{j+2}}\mathrm{D}^{\alpha}\mathrm{D}^{(j-1)\alpha}\boldsymbol{x}(t)=\sum_{j=1}^{n-2}\mathbf{C_{j+2}}\mathrm{D}^{j\alpha}\boldsymbol{x}(t),\\
&\quad\vdots\quad\vdots\quad\vdots\quad\vdots\\
&\mathbf{C_n}\mathrm{D}^{\alpha}\boldsymbol{x}(t)=\mathbf{C_n}\mathrm{D}^{\alpha}\boldsymbol{x}(t),
\end{aligned} \tag{11}$$

then a set of $r\times n$ coupled differential equations is readily cast in the form

$$\mathbf{A}\mathrm{D}^{\alpha}\mathbf{z}(t)+\mathbf{B}\mathbf{z}(t)=\mathbf{g}(t), \tag{12}$$

where $\mathbf{g}^{\mathrm{T}}(t)=\left[\mathbf{f}^{\mathrm{T}}(t)\,\mathbf{0}\,\ldots\,\mathbf{0}\right]$, $\mathbf{A}$ and $\mathbf{B}$ are symmetric matrices defined as

$$\begin{aligned}
\mathbf{A}&=\begin{bmatrix}\mathbf{C_1}&\mathbf{C_2}&\ldots&\mathbf{C_{n-1}}&\mathbf{C_n}\\ \mathbf{C_2}&\mathbf{C_3}&\ldots&\mathbf{C_n}&\mathbf{0}\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \mathbf{C_{n-1}}&\mathbf{C_n}&\ldots&\mathbf{0}&\mathbf{0}\\ \mathbf{C_n}&\mathbf{0}&\ldots&\mathbf{0}&\mathbf{0}\end{bmatrix},\\
\mathbf{B}&=\begin{bmatrix}\mathbf{K}&\mathbf{0}&\ldots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&-\mathbf{C_2}&\ldots&-\mathbf{C_{n-1}}&-\mathbf{C_n}\\ \vdots&\vdots&\ddots&\vdots&\vdots\\ \mathbf{0}&-\mathbf{C_{n-1}}&\ldots&\mathbf{0}&\mathbf{0}\\ \mathbf{0}&-\mathbf{C_n}&\ldots&\mathbf{0}&\mathbf{0}\end{bmatrix}.
\end{aligned} \tag{13}$$

Next, it is possible to decompose $\mathbf{z}(t)$ in the orthogonal basis of the eigenvectors of $\mathbf{A}$ and $\mathbf{B}$. Specifically, consider the equations

$$\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{A}\boldsymbol{\Psi}=\mathbf{U_D},\qquad \boldsymbol{\Psi}^{\mathrm{T}}\mathbf{B}\boldsymbol{\Psi}=\mathbf{V_D}, \tag{14}$$

where $\mathbf{U_D}$ and $\mathbf{V_D}$ are diagonal matrices. Further, making the complex modal transformation

$$\mathbf{z}(t)=\boldsymbol{\Psi}\boldsymbol{y}(t), \tag{15}$$

a new set of decoupled fractional differential equation is derived in the form

$$\mathbf{U_D}\mathrm{D}^{\alpha}\boldsymbol{y}(t)+\mathbf{V_D}\boldsymbol{y}(t)=\boldsymbol{\mu}(t), \tag{16}$$

where $\boldsymbol{\mu}(t)=\boldsymbol{\Psi}^{\mathrm{T}}\mathbf{g}(t)$. Clearly, once the decoupled set is found, the fractional differential equations can be readily solved. It can be seen that the state variable problem in Eq. (12) has a greater number of involved variables with respect to the problem cast in terms of displacements in Eq. (9) of the nodal analysis. However, this apparent increase of the computational burden is balanced by the fact that the maximum involved order in the state variable domain is smaller than the maximum order in the nodal analysis. Further, the system in Eq. (9) is a set of coupled in the general case, vis-a-vis the system in Eq. (12) leads readily to the set of uncoupled equations in Eq. (16).

From this result it becomes clear that for application the method it is necessary to modify Eq. (3) to obtain the sequential differential form as in Eq. (9). To elucidate the procedure for obtaining a sequential differential from the given set of coupled fractional differential equations, consider the dynamical problem in which all of the involved fractional derivatives have the same order β. In this case, the Eq. (9) becomes

$$\mathbf{M}\ddot{\boldsymbol{x}}(t)+\mathbf{C}_\beta \mathrm{D}^\beta \boldsymbol{x}(t)+\mathbf{K}\boldsymbol{x}(t)=\mathbf{f}(t); \tag{17}$$

the number of terms and the minimum fractional order α in the summation of the Eq. (9) depend of the order β. Next, assume that $\beta = 0.5$. In this particular case, also the order $\alpha = \beta$, and the sequential form is given as

$$\sum_{j=1}^{4}\mathbf{C_j}\mathrm{D}^{j0.5}\boldsymbol{x}(t)+\mathbf{K}\boldsymbol{x}(t)=\mathbf{f}(t), \tag{18}$$

where $\mathbf{C_4} = \mathbf{M}$, $\mathbf{C_3} = \mathbf{C_2} = \mathbf{0}$ and $\mathbf{C_1} = \mathbf{C}_\beta$. Clearly, the smaller the order α is, the higher the number of terms is. However, since $\mathbf{C_3} = \mathbf{C_2} = \mathbf{0}$, the system (18) can be rewritten as

$$\mathbf{C_4}\mathrm{D}^2\boldsymbol{x}(t)+\mathbf{C_1}\mathrm{D}^{0.5}\boldsymbol{x}(t)+\mathbf{K}\boldsymbol{x}(t)=\mathbf{f}(t). \tag{19}$$

Further, taking as the state vector

$$\mathbf{z}^{\mathrm{T}}(t)=\left[\boldsymbol{x}^{\mathrm{T}}(t)\ \ \mathrm{D}^{0.5}\boldsymbol{x}^{\mathrm{T}}(t)\ \ \mathrm{D}^{1}\boldsymbol{x}^{\mathrm{T}}(t)\ \ \mathrm{D}^{1.5}\boldsymbol{x}^{\mathrm{T}}(t)\right], \tag{20}$$

and considering Eq. (19) and the identity relations in Eqs. (11), the following system of fractional differential equations in the state variables domain is found:

$$\mathbf{A}\mathrm{D}^\alpha\mathbf{z}(t)+\mathbf{B}\mathbf{z}(t)=\mathbf{g}(t), \tag{21}$$

where the matrices $\mathbf{A}$ and $\mathbf{B}$ become, for that particular case, as follows

$$\begin{aligned}\mathbf{A}&=\begin{bmatrix}\mathbf{C}_1 & \mathbf{0} & \mathbf{0} & \mathbf{M}\\ \mathbf{0} & \mathbf{0} & \mathbf{M} & \mathbf{0}\\ \mathbf{0} & \mathbf{M} & \mathbf{0} & \mathbf{0}\\ \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix},\\ \mathbf{B}&=\begin{bmatrix}\mathbf{K} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{M}\\ \mathbf{0} & \mathbf{0} & -\mathbf{M} & \mathbf{0}\\ \mathbf{0} & -\mathbf{M} & \mathbf{0} & \mathbf{0}\end{bmatrix}.\end{aligned} \tag{22}$$

Clearly, Eq. (21) can be solved by the complex modal analysis described in Eqs. (14-16) and the state variables vector $\mathbf{z}(t)$ can be determined. It is worth noting that if in Eq. (17) $\alpha = 1$, the vector of state variables is the classical one $\mathbf{z}^{\mathrm{T}}(t)=\left[\boldsymbol{x}^{\mathrm{T}}(t)\ \ \dot{\boldsymbol{x}}^{\mathrm{T}}(t)\right]$.

Obviously, the method is applicable exactly when the involved orders are rational. However, even if in the general case $\alpha \in \mathbb{R}$, it could yield reasonable results by approximating α as fraction. To demonstrate the applicability of the proposed method some numerical results are discussed next.

IV. Numerical Applications of FSDOF

First, consider the single-degree-of-freedom system described in Eq. (1) forced by deterministic load. This case has been already discussed by Bonilla et al. [22] by using the fractional Wronskian method. Alternatively, Eq. (1) may also be solved by using the Green function $G(t)$

$$G(t)=\frac{1}{m}\sum_{j=0}^{\infty}-\frac{1}{j!}\left(\frac{k}{m}\right)^{j}t^{2j+1}E^{(j)}_{2-\beta,2+j\beta}\left(-\frac{c_\beta}{m}t^{2-\beta}\right), \tag{23}$$

where $E^{(j)}_{\lambda,\mu}(z)$ is the derivative of order j of the two-parameters Mittag-Leffler function that is defined as

$$E^{(j)}_{\lambda,\mu}(z)=\frac{d^j}{dz^j}E_{\lambda,\mu}(z)=\sum_{l=0}^{\infty}\frac{(l+j)!z^l}{l!\Gamma(\lambda l+\lambda j+\mu)}. \tag{24}$$

Specifically, assuming that the system in Eq. (1) is quiescent at $t = 0$, the solution $x(t)$ is given by integral

$$x(t)=\int_0^t G(t-\tau)f(\tau)d\tau. \tag{25}$$

Note that Eq. (25) may be computationally demanding since there are two summation with infinity terms as kernel in the convolution integral.

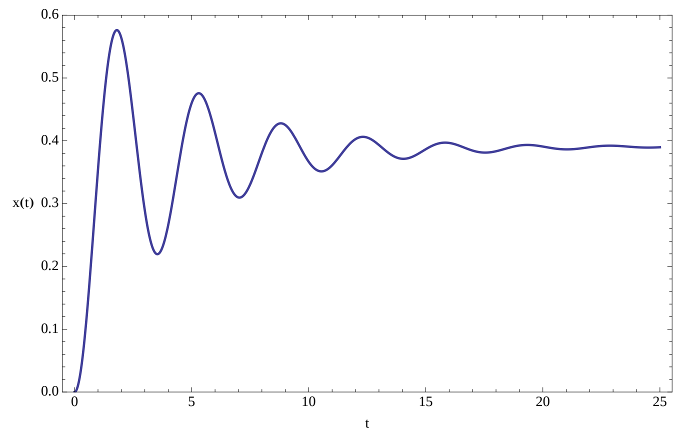

Figure 1. Unit step response of the fractional single-degree-of-freedom system

Using the state variable analysis proposed in this paper the solution becomes straightforward. To show this, suppose that the involved fractional order is $\beta = 1/2$. Then the minimum fractional order that is involved in the state variable analysis is $\alpha = \beta = 1/2$ and the state variable vector is

$$\mathbf{z}^{\mathrm{T}}(t)=\left[x(t),\ \left(\mathrm{D}^{\frac{1}{2}}x\right)(t),\ \left(\mathrm{D}^{1}x\right)(t),\ \left(\mathrm{D}^{\frac{3}{2}}x\right)(t)\right]. \tag{26}$$

Further, the matrices involved in the state variable equations in the Eq. (21) become

$$\mathbf{A}=\begin{bmatrix}c_\beta & 0 & 0 & m\\ 0 & 0 & m & 0\\ 0 & m & 0 & 0\\ m & 0 & 0 & 0\end{bmatrix},\quad \mathbf{B}=\begin{bmatrix}k & 0 & 0 & 0\\ 0 & 0 & 0 & -m\\ 0 & 0 & -m & 0\\ 0 & -m & 0 & 0\end{bmatrix}. \tag{27}$$

Furthermore, performing the state variable analysis, the set of four uncoupled equations

$$u_j\left(D^{\frac{1}{2}}_{0+}y_j\right)(t)+v_j y(t)=\mu_j(t),\qquad j=1,2,3,4, \tag{28}$$

is derived. Note that the unit step response of the Eq. (28) involves a Mittag-Leffler function, but with only one parameter $E_\beta(\cdot)$. That is,

$$J_j(t) = \frac{1}{v_j}\left[1 - E_\beta\left(-\frac{v_j}{u_j}t^\beta\right)\right] = -\frac{1}{v_j}\sum_{k=1}^{\infty}\frac{\left(-v_j/u_j t^\beta\right)^k}{\Gamma(\beta k+1)}. \tag{29}$$

Clearly, the solution in terms of the modal displacements $y_j(t)$ can be obtained by the Boltzman superposition integral as

$$y_j(t) = \int_0^t J_j(t-\tau)\dot{\mu}_j(\tau)d\tau, \tag{30}$$

and the displacement $x(t)$ is given as the first term of the state variable vector $\mathbf{z}(t)$ defined in Eq. (15). It can be seen that the solution of Eq. (30) is more convenient to obtain vis-a-vis Eq. (25). In the Figure 1 the unit step response of the FSDOF in which the chosen coefficients are $m = 1$, $c_\beta = k = \pi/4$ is shown. Clearly, if the order β is equal to one, the state variable analysis becomes identical to the classic dynamic case, and the state variables become the displacement $x(t)$ and its first derivative $\dot{x}(t)$.

V. Numerical Applications of FMDOF

Consider the two-degree-of-freedom system in Figure 2, which is typical of shear type structures with fractional viscoelastic terms. Next, the preceding concepts of the new complex modal analysis in the state variables domain are used to determine the solution of the fractional multi-degree-of-freedom system. The fractional terms are represented by spring-pots in the figure. The equations which govern motion

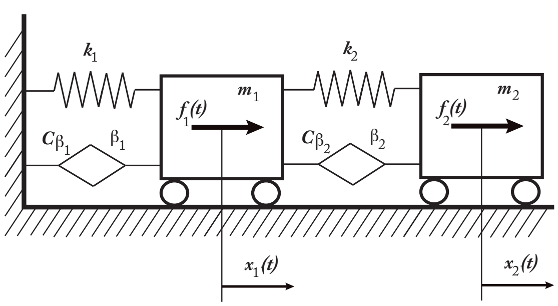

Figure 2. Fractional two-degree-of-freedom system

of given system in Figure 2 are

$$\begin{cases} m_1\ddot{x}_1(t) + c_{\beta_1}\left(\mathrm{D}^{\beta_1}x_1\right)(t) - c_{\beta_2}\left(\mathrm{D}^{\beta_2}x_2 - x_1\right)(t) + \\ \qquad + k_1x_1(t) - k_2(x_2 - x_1)(t) = f_1(t), \\ m_2\ddot{x}_2(t) + c_{\beta_2}\left(\mathrm{D}^{\beta_2}x_2 - x_1\right)(t) + k_2(x_2 - x_1)(t) = f_2(t), \end{cases} \tag{31}$$

where the lumped parameters of the system are the stiffness of the springs k_j; the coefficients of the spring-pots c_{β_j} and their related orders β_j; and the mass of the layers m_j with $j = 1, 2$.

A. FMDOF under deterministic load

Consider the case in which $\beta_1 = 3/4$ and $\beta_2 = 1/2$. In this case the coupled equations of motion are the same as expressed in Eq. (31), and the involved matrices in the nodal space are

$$\mathbf{K} = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}, \quad \mathbf{C_2} = \begin{bmatrix} c_{\beta_2} & -c_{\beta_2} \\ -c_{\beta_2} & c_{\beta_2} \end{bmatrix}, \\ \mathbf{C_3} = \begin{bmatrix} c_{\beta_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad \mathbf{C_8} = \mathbf{M} = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}. \tag{32}$$

Other matrices involved are $\mathbf{C_1} = \mathbf{C_4} = \mathbf{C_5} = \mathbf{C_6} = \mathbf{C_7} = \mathbf{0}$; the order $\alpha = 1/4$, and the vector of state variables becomes

$$\mathbf{z}^{\mathrm{T}}(t) = \Big[\boldsymbol{x}^{\mathrm{T}}(t)\,\mathrm{D}^{\frac{1}{4}}\boldsymbol{x}^{\mathrm{T}}(t)\,\mathrm{D}^{\frac{1}{2}}\boldsymbol{x}^{\mathrm{T}}(t)\,\mathrm{D}^{1}\boldsymbol{x}^{\mathrm{T}}(t)\,\mathrm{D}^{\frac{3}{4}}\boldsymbol{x}^{\mathrm{T}}(t) \\ \mathrm{D}^{\frac{5}{4}}\boldsymbol{x}^{\mathrm{T}}(t)\,\mathrm{D}^{\frac{3}{2}}\boldsymbol{x}^{\mathrm{T}}(t)\,\mathrm{D}^{\frac{7}{4}}\boldsymbol{x}^{\mathrm{T}}(t)\Big]. \tag{33}$$

The dimension of the problem in the state variable domain is $r \times n = 16$. For simplicity, considered that $f_1(t) = 0$ and $f_2(t)$ is

$$f_2(t) = \begin{cases} \sin(t), & 0 < t < 2\pi, \\ 0, & \text{otherwise}. \end{cases} \tag{34}$$

Further, the chosen parameters of the two-degree-of-freedom system in Figure 2 are $m_1 = 1$, $m_2 = 3/4$, $c_{\beta_1} = c_{\beta_2} = 1$, $k_1 = 3/2$ and $k_2 = 1/2$. One of the approaches to solve the uncoupled fractional differential equations relies on the Mellin transform. This method has been recently introduced [23,24]. For this example the following parameters of discretized Mellin transform have been chosen: $\rho = 1$, $\bar{\eta} = 100$ and $\Delta\eta = 0.5$ (see the cited articles pertaining to the method). In this manner, the displacements $x_1(t)$ and $x_2(t)$ are readily determined. The displacements of the two layers obtained by the complex modal analysis are shown in the Figure 3.

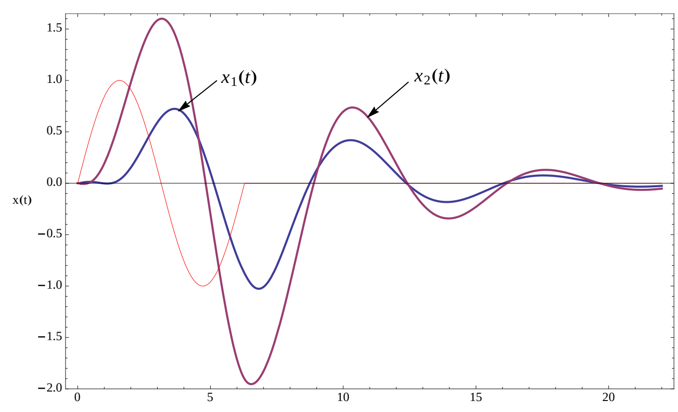

Figure 3. Displacements $x_1(t)$ and $x_2(t)$ of the dynamical system

B. FMDOF under stochastic input

Next, assume that the fractional two-degree-of-freedom system shown in Figure 2 is forced by a stochastic input. In particular, consider that the involved fractional orders are the same in the preciding case and that the $f_1(t) = 0$ and $f_2(t) = w(t)$, where $w(t)$ denotes the zero mean Gaussian white noise. Obviously, the frequency domain solution is readily obtainable. Specifically, the Fourier transforms of the Eqs. (31) yields

$$\hat{\mathbf{x}}(\omega) = \mathbf{H}(\omega)\hat{\mathbf{f}}(\omega) \tag{35}$$

where $\hat{\mathbf{x}}(\omega)$ and $\hat{\mathbf{f}}(\omega)$ are the Fourier transforms of the vectors $\boldsymbol{x}(t)$ and $\mathbf{f}(t)$, respectively. Further, the non-diagonal transfer matrix $\mathbf{H}(\omega)$ is defined by its inverse as

$$\begin{aligned}\mathbf{H}^{-1}(\omega) =& \mathbb{K}(\omega) = \begin{bmatrix} \mathrm{k}_{11} & \mathrm{k}_{12} \\ \mathrm{k}_{21} & \mathrm{k}_{22} \end{bmatrix}, \\ & \mathrm{k}_{11} = k_1 + k_2 + c_{\beta_1}(i\omega)^{\beta_1} + c_{\beta_2}(i\omega)^{\beta_2} - m_1\omega^2, \\ & \mathrm{k}_{21} = \mathrm{k}_{12} = -k_2 - c_{\beta_2}(i\omega)^{\beta_2}, \\ & \mathrm{k}_{22} = k_2 + c_{\beta_2}(i\omega)^{\beta_2} - m_2\omega^2,\end{aligned} \tag{36}$$

where i is the imaginary unit. The terms $(i\omega)^{\beta_j}$ with $j = 1, 2$ in Eq. (36) are due to the Fourier transform of the fractional operators that are involved in the Eq. (31). By this transformation it is clear that the fractional derivative term introduces in the system dynamics both effective damping and effective stiffness frequency dependent terms. In fact, the following relation holds

$$(i\omega)^\beta = |\omega|^\beta \left[\cos\left(\frac{\beta\pi}{2}\right) + i\operatorname{sgn}(\omega)\sin\left(\frac{\beta\pi}{2}\right)\right]. \tag{37}$$

The spectral matrix of the inputs is denoted with $\mathbf{S}_f(\omega)$, and based on the preceding assumption it becomes

$$\mathbf{S}_f(\omega) = \begin{bmatrix} 0 & 0 \\ 0 & S_0 \end{bmatrix}, \tag{38}$$

where

$$S_0 = \lim_{\mathrm{T}\to\infty} \frac{1}{2\pi\mathrm{T}} \mathrm{E}\left[\hat{w}^*(\omega)\hat{w}(\omega)\right], \tag{39}$$

with $\hat{w}(\omega)$ denoting the Fourier transform of the Gaussian white noise; the asterisk representing complex conjugation; and $\mathrm{E}[\cdot]$ being the expectation operator.

Based on the above considerations, the spectral matrix $\mathbf{S}_x(\omega)$ of the response can be expressed as

$$\mathbf{S}_x(\omega) = \mathbf{H}^*(\omega)\mathbf{S}_f(\omega)\mathbf{H}^{\mathrm{T}}(\omega), \tag{40}$$

where

$$\mathbf{S}_x(\omega) = \begin{bmatrix} S_{x_1}(\omega) & S_{x_1 x_2}(\omega) \\ S_{x_2 x_1}(\omega) & S_{x_2}(\omega) \end{bmatrix}, \tag{41}$$

with the elements of the matrix $\mathbf{S}_x(\omega)$ expressed as

$$S_{x_j x_k}(\omega) = \lim_{\mathrm{T}\to\infty} \frac{1}{2\pi\mathrm{T}} \mathrm{E}\left[\hat{x}_j^*(\omega)\hat{x}_k(\omega)\right], \quad j, k = 1, 2. \tag{42}$$

The solution in Eq. (42) can be used as a benchmark for the proposed method. In fact, the Fourier transform in the state variable domain yields

$$\begin{aligned}\mathbf{S}_z(\omega) &= \lim_{\mathrm{T}\to\infty} \frac{1}{2\pi\mathrm{T}} \mathrm{E}\left[\hat{\mathbf{z}}^*(\omega)\hat{\mathbf{z}}^{\mathrm{T}}(\omega)\right] \\ &= \lim_{\mathrm{T}\to\infty} \frac{1}{2\pi\mathrm{T}} \mathrm{E}\left[\boldsymbol{\Psi}^*\hat{\mathbf{y}}^*(\omega)\left(\boldsymbol{\Psi}\hat{\mathbf{y}}(\omega)\right)^{\mathrm{T}}\right] \\ &= \lim_{\mathrm{T}\to\infty} \frac{1}{2\pi\mathrm{T}} \mathrm{E}\left[\boldsymbol{\Psi}^*\hat{\mathbf{y}}^*(\omega)\hat{\mathbf{y}}^{\mathrm{T}}(\omega)\boldsymbol{\Psi}^{\mathrm{T}}\right] \\ &= \boldsymbol{\Psi}^*\mathbf{S}_y(\omega)\boldsymbol{\Psi}^{\mathrm{T}},\end{aligned} \tag{43}$$

where the vector $\hat{\mathbf{y}}(\omega)$ contains the Fourier transforms of all displacements $y_j(t)$ in the complex modal space. These terms can be readily determined by performing the Fourier transform of the set of equations in Eq. (16); the j-th term of the vector is

$$\hat{y}_j(\omega) = \frac{\hat{f}_1(\omega)\psi_{j1} + \hat{f}_2(\omega)\psi_{j2} \cdots + \hat{f}_r\psi_{jr}}{u_j(i\omega)^\alpha + v_j}, \tag{44}$$

where $\hat{f}_k(\omega)$ is the Fourier transform of the k-th external force $f_k(t)$, and ψ_{jr} is the r-th term of j-th eigenvector. In Eq. (43) the matrix $\mathbf{S}_z(\omega)$ is of order $(n \times r)^2$ (n is the number of state variables, and r is the number of degrees of freedom); and the matrix $\mathbf{S}_x(\omega)$ is contained in the first $r \times r$ block matrix of $\mathbf{S}_z(\omega)$. The matrix $\mathbf{S}_z(\omega)$ has dimension 16×16 and it is expressed in the form

$$\mathbf{S}_z(\omega) = \begin{bmatrix} \mathbf{S}_x(\omega) & \mathbf{S}_{xx^{(1/4)}}(\omega) & \dots & \mathbf{S}_{xx^{(7/4)}} \\ \mathbf{S}_{x^{(1/4)}x}(\omega) & \mathbf{S}_{x^{(1/4)}}(\omega) & \dots & \mathbf{S}_{x^{(1/4)}x^{(7/4)}} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{S}_{x^{(7/4)}x}(\omega) & \mathbf{S}_{x^{(7/4)}x^{(1/4)}}(\omega) & \dots & \mathbf{S}_{x^{(7/4)}}(\omega) \end{bmatrix}. \tag{45}$$

The first block of matrix in Eq. (45) represents the matrix $\mathbf{S}_x(\omega)$ that has dimension 2×2.

In Figure 4 are shown the power spectra of the responses $x_1(t)$ and $x_2(t)$ (the terms of the diagonal of the matrix $\mathbf{S}_x(\omega)$), while in the Figure 5 is shown the cross spectrum of the responses $x_1(t)$ and $x_2(t)$. The chosen parameters for the conducted frequency analysis are $m_1 = 5/4$, $m_2 = 1$, $c_{\beta_1} = 3/4$, $c_{\beta_2} = 1/3$, $\beta_1 = 3/4$, $\beta_2 = 1/2$, $k_1 = 1.1$, $k_2 = 1$ and $S_0 = 1$. The continuos lines represent the spectra and cross-spectra obtained using Eq. (40), while the black dots represent the spectra obtained using Eq. (43). These figures demonstrate a perfect agreement of the results obtained by the two methods and, thus, confirm the reliability of the presented state variable analysis.

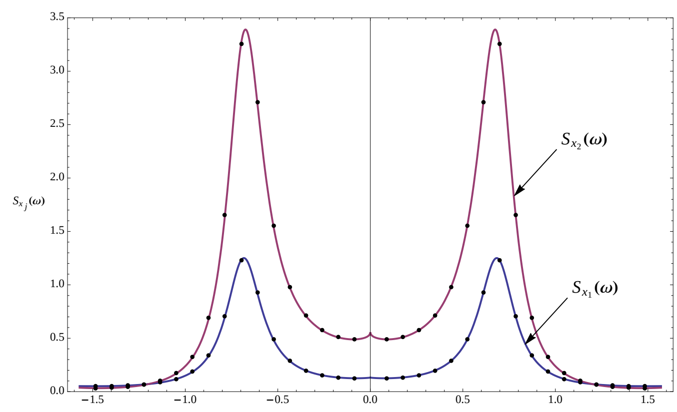

Figure 4. Power spectra of the displacements $x_1(t)$ and $x_2(t)$

VI. Concluding Remarks

An analysis of multi-degree-of-freedom systems with fractional viscoelastic elements in the equation of motion has been pursued in this paper. This kind of problem is characterized for the presence of a set of coupled fractional differential equations that govern the motion of the system. It has been pointed out that this kind of problem is more complex to solve vis-a-vis the analysis of multi-degree-of-freedom system with integer order derivatives.

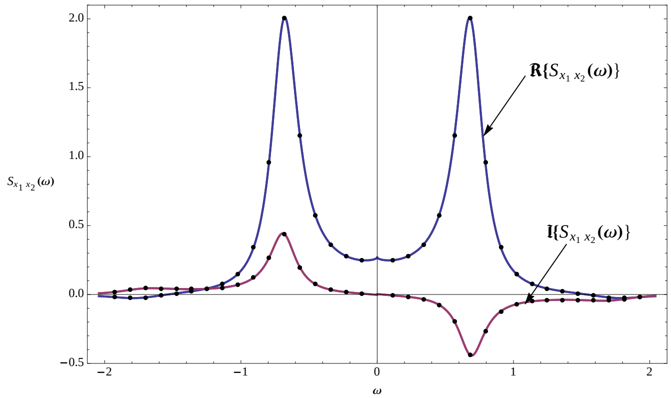

Figure 5. Cross-spectrum of the displacements $x_1(t)$ and $x_2(t)$

The analysis has been based on a novel approach using an augmented state variables transformation. The method, based on complex modal analysis in the state variable domain, is applicable if the involved fractional terms have rational order. This assumption is needed since only in this case it is possible to find an appropriate state variables vector. Note, however, that for engineering applications all real orders can be reasonably approximated by rational orders.

By the proposed method the set of coupled fractional differential equations has been decoupled in a proper fractional state variable domain. This transformation has led to a new set of fractional equations which are uncoupled and more convenient to solve. Therefore, a drastic reduction of the computational effort is achieved. To elucidate the applicability of the method several numerical examples have been considered which have confirmed the effectiveness of the method both for deterministic and stochastic system excitations.

REFERENCES

[1] Nutting, P. G., *A new general law of deformation*, Journal of the Franklin Institute, 191 (1921) pp. 679-685.

[2] Gemant, A., *A method of analyzing experimental results obtained from elasto-viscous bodies*, Physics, 7 (1936) pp. 311317.

[3] Di Paola, M., Pirrotta, A., Valenza, A., *Visco-elastic behavior through fractional calculus: an easier method for best fitting experimental results*, Mechanics of Materials, 43 (2011) pp. 799-806.

[4] Müller, S., Kästner, M., Brummund, J., Ulbricht, V., *A nonlinear fractional viscoelastic material model for polymers*, Computational Materials Science, 50 (2011) pp. 2938-2949.

[5] Di Paola, M., Fiore, V., Pinnola, F. P., Valenza, A., *On the influence of the initial ramp for a correct definition of the parameters of fractional viscoelastic materials*, Mechanics of Materials, 69 (2014) pp. 63-70.

[6] Singh, M.P., Chang, T.-S., Nandan, H., *Algorithms for seismic analysis of MDOF systems with fractional derivatives*, Engineering Structures, 33 (2011) pp. 2371-2381.

[7] Li, L., Hu, Y., Wang, X, *Improved approximate methods for calculating frequency response function matrix and response of MDOF systems with viscoelastic hereditary terms*, Journal of Sound and Vibration, 332 (2013) pp. 3945-3956.

[8] Escobedo-Torres, J., Ricles, J. M., *The fractional order elastic-viscoelastic equations of motion: Formulation and solution method*, Journal of Intelligent Matrial Systems and Structures, 29 (1998) pp. 489-502.

[9] Lewandowski, R.,Pawlak, Z., *Dynamic analysis of frames with viscoelastic dampers modelled by rheological models with fractional derivatives*, 330 (2011) pp. 923-936.

[10] Pawlak, Z., Lewandowski, R., *The continuation method for the eigenvalue problem of structures with viscoelastic dampers*, 125 (2013) pp. 53-61.

[11] Di Paola, M., Heuer, R., Pirrotta, A., *Fractional visco-elastic EulerBernoulli beam*, International Journal of Solids and Structures, 50 (2013) pp. 3505-3510.

[12] Di Lorenzo, S., Di Paola, M., Pinnola, F. P., Pirrotta, A., *Stochastic response of fractionally damped beams*, Probabilistic Engineering Mechanics, 35 (2013) pp. 37-43.

[13] Rossikhin, Y. A., Shitikova, M. V., *A new method for solving dynamic problems of fractional derivative viscoelasticicty*, International Journal of Engineering Science, 39 (2001) pp. 149-176.

[14] Schmidt, A., Gaul, L., *On the numerical evaluation of fractional derivatives in multi-degree-of-freedom systems*, Signal Processing, 86 (2006) pp. 2592-2601.

[15] Oldham, K. B., Spanier, J., *The Fractional Calculus: Theory and Applications of Differentiation and Integration to Arbitrary Order*, New York: Academic Press, 1974.

[16] Spanos, P. D., Evangelatos, G. I., *Response of a non-linear system with restoring forces governed by fractional derivatives–Time domain simulation and statistical linearization solution*, Soil Dynamics and Earthquake Engineering, 30 (2010) pp. 811-821.

[17] Podlubny, I., *Fractional differential equations*, San Diego, Academic Press, 1999.

[18] Miller, K. S., Ross, B., An Introduction to the Fractional Calculus and Fractional Differential Equations, Wiley-InterScience, New York, 1993.

[19] Samko, G. S., Kilbas, A. A., Marichev, O. I., 1993, Fractional Integrals and Derivatives: Theory and Applications, New York (NY) Gordon and Breach, 1993.

[20] Kilbas, A. A., Srivastava, H. M., Trujillo, J. J., *Theory and Applications of Fractional Differential Equations*, Elsevier, Amsterdam, 2006.

[21] Hilfer, R., Application of Fractional Calculus in Physics, World Scientific, Singapore, 2000.

[22] Bonilla, B., Rivero, M., Trujillo, J. J., *On systems of linear fractional differential equations with constant coefficients*, Applied Mathematics and Computation, 187 (2007) pp. 68-78.

[23] Di Paola, M. *Complex fractional moments and their use for the solution of the Fokker Planck equation*, Vienna Congress on Recent Advances in Earthquake Engineering and Structural Dynamics 2013 (VEESD 2013), Paper No. 463.

[24] Butera, S., Di Paola, M.,*Fractional differential equations solved by using Mellin transform*, Communications in Nonlinear Science and Numerical Simulation, 19 (2013) pp. 2220-2227.