STRAIN ENERGY EVALUATION IN STRUCTURES HAVING ZONE-WISE PHYSICAL- MECHANICAL QUANTITIES

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Abstract: Among the possible aims of structural analysis inside some engineering spheres it can be useful to know the strain energy stored in all or in a part of the structure caused by assigned external actions, like the boundary and domain quantities. This serves to evaluate globally whether an assigned portion of structure undergoes an excessive store of energy able to compromise the stability of all the structure. This evaluation can be carried out through boundary work obtained using appropriate boundary generalized quantities connected to the results of the analysis on the whole structure. The advantage consists in using a very restricted number of quantities which, because of the characteristics of the method, are only evaluated on the boundary. Some strategies used to evaluate the error made are introduced through the computation of the external direct work and of the reciprocal works involving quantities only connected to the boundary of the complementary domain and quantities connected to either the real boundary of the structure or the boundary of its complementary domain. A reduction of this error is suggested.

Introduction

This paper has as its objective the computation of the elastic strain energy stored in a body or in a limited portion of it, and this can happen also when zone-wise variable physical and mechanical characteristics are present. In this case the body is subdivided into substructures and the analysis is performed by the displacement method within the Symmetric Boundary Element Method (SGBEM), according to the Galerkin hypotheses. Fig. 1 shows a system having zone-wise variable physical characteristics, subjected to displacements imposed on the constraint and known forces on the free boundary. One wants to know the elastic strain energy stored in the substructures marked by 1, 2, 3, 4.

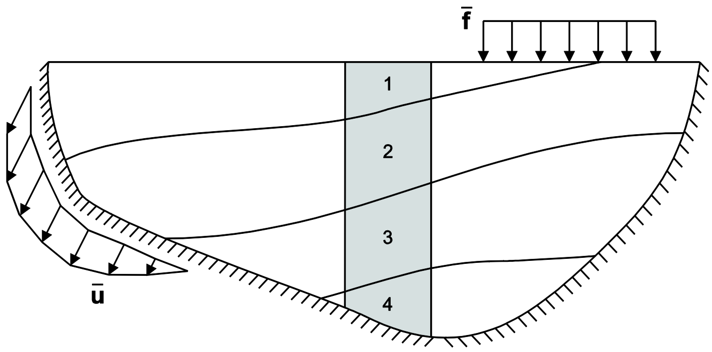

Figure 1: System subjected to imposed displacements and known forces; substructures 1, 2, 3, 4 where the elastic strain energy has to be evaluated.

Several researches have developed substructuring techniques [3-5]. Only in the last few years, in virtue of a particular approach for substructuring in [1], has the implementation of the SGBEM become possible, making this method competitive in comparison with other analysis methods.

Through this method the elastic strain energy can be computed using domain integrals after evaluating the stresses and strains by Somigliana Identities (SI), but this strategy shows considerable computational difficulties. Moreover, the energy balance between elastic energy stored and boundary work in terms of punctual quantities, also difficult to evaluate for the same reasons, suggested introducing a type of work here defined as generalized through interpretation of the mechanical and kinematical boundary quantities as

layered quantities. This form of work was obtained in [2] as a direct product between boundary quantities, referred to as generalized and directly obtainable in the phase of post-analysis and nodal values, thus avoiding tedious evaluation of the SI and the use of the boundary integrals involving continuous boundary functions. The numerical equality between the boundary work in terms of continuous functions and the generalized work, here shown, justifies the procedure followed and makes it easy implementable within the Karnak.sGbem program, utilized for the numerical trials. This work is based on the employment of appropriate boundary quantities to be computed when the analysis is concluded.
A first attempt at using these boundary quantities was made in [6,7] through the introduction of weighted residuals regarding the coefficients of Dirichlet and Neumann dual equations, relations not utilized to solve the analysis problem. But these quantities have not made it possible to apply the method based on computing the boundary work to the case of a system subdivided into substructures where the interface boundary also appears.
The problem was overcome by giving an appropriate interpretation of the boundary quantities.
Indeed, when the solid was embedded in the infinite domain, a boundary Γ^- of the Ω domain and a boundary Γ^+ of the complementary domain Ω_∞ / Ω are introduced and the boundary quantities, whether kinematical or mechanical, take on the meaning of layered quantities, that is to say of jumps between the actual boundaries Γ^- of the solid and the boundary Γ^+ of the complementary domain.
The use of these layered functions within the expressions of the work and the utilizations of the basic theory of the SGBEM led to the definition of a new type of generalized work in which, in addition to the direct work, the indirect works where the kinematical and mechanical quantities of the boundary Γ^- interact in energy meaning with the related quantities of Γ^+ appear.
This way of interpreting the work, together with a more cogent discussion of the boundary conditions leading to characterization of the algebraic operators of the single substructure within the displacement method, made it possible to evaluate in a fast and simple way the strain energy in all the infinite domain, the sum of the actual and complementary domains.
Further, embedding of the single substructure in the infinite domain allows one to assert that, depending on the boundary discretization, the solution is reached when the complementary domain is unstrained and unstressed, or when the boundary work is all in the solid as strain energy. This can happen when the direct work on Γ^+ and the reciprocal ones cancel each other out on all the boundary elements.
On the basis of these remarks it is possible to create a strategy which reduces the error of the solution through appropriate boundary discretization where the scattered energy quantities are greater.

1. Boundary conditions

In any mechanics problem, the boundary conditions of the continuum are the following:

$\mathbf{t} = \overline{\mathbf{f}}$ (Dirichlet conditions) (1)

$\mathbf{u} = \overline{\mathbf{u}}$ (Neumann conditions) (2)

In the Boundary Element Method, the solid of the Ω domain is embedded in the infinite domain Ω_∞ having the same physical and mechanical characteristics. The infinite domain is characterized by a boundary $\Gamma^- \equiv \Gamma$ of Ω having external unit vector $\mathbf{n} = \mathbf{n}^-$ and by a boundary Γ^+ of $\Omega_\infty \setminus \Omega$ with $\mathbf{n}^+ = -\mathbf{n}$ (Fig. 2).

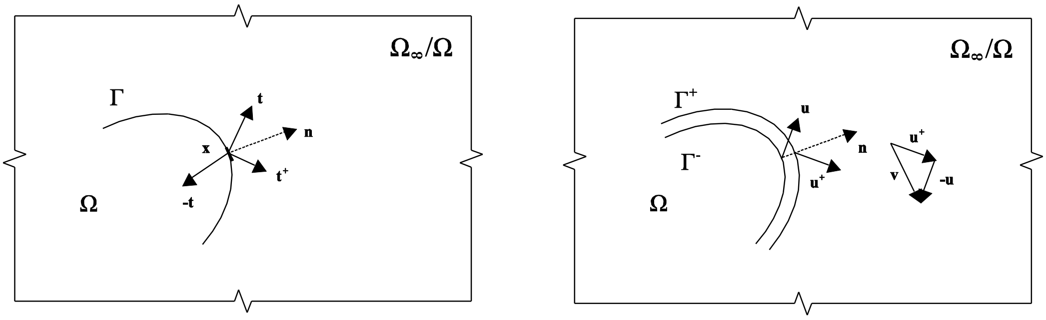

Figure 2. Body embedded in the infinite domain and layered quantities

The quantities characterizing the boundary take on the meaning of layered quantities and in particular:

- The forces acting on the boundary take on the meaning of single layer sources:

$$\mathbf{f(x) = -t^+ (x,n^+) + t^- (x,n)} \tag{3}$$

where $\mathbf{t}^+$ and $\mathbf{t}^-$ symbolize in Ω_∞ the response in terms of traction evaluated in $\mathbf{x}$ on the boundaries Γ^+ e Γ^-, respectively.

- The displacements of the boundary take on the meaning of double layer sources:

$$\mathbf{v(x) = u^+ (x) - u^- (x)} \tag{4}$$

where $\mathbf{u}^+$ and $\mathbf{u}^-$ symbolize in Ω_∞ the response in terms of displacements evaluated in $\mathbf{x}$ on the boundaries Γ^+ and Γ^-, respectively.
To obtain the solution of the elastic problem, it is necessary for the complementary domain Ω_∞ / Ω to prove to be unstressed and unstrained, and this can be obtained by imposing the condition that all the points of the boundary Γ^+ satisfy the following conditions:

$$\mathbf{t}^+ = \mathbf{0} \quad \rightarrow \quad \mathbf{f = t^-} \tag{5a,b}$$
$$\mathbf{u}^+ = \mathbf{0} \quad \rightarrow \quad \mathbf{v = -u^-} \tag{6a,b}$$

2. Boundary condition in weighted form

In accordance with the SGBEM formulation, the boundary conditions of problem (5a) and (6a) have to be imposed in weighted form.
Let us introduce the boundary discretization into boundary elements and the related modeling of the quantities through the introduction of the appropriate shape functions:

$$\mathbf{f} = \mathbf{\Psi}_\mathrm{t}\, \mathbf{F}, \qquad \mathbf{u} = \mathbf{\Psi}_\mathrm{u}\, \mathbf{U} \tag{7a,b}$$

where $\mathbf{F}$ and $\mathbf{U}$ symbolize the nodal value of the forces and displacements, respectively.
The weighting of the equilibrium (3) and compatibility (4) is achieved by using the shape functions as weighted functions, introduced in dual form according to the Galerkin hypotheses. One obtains:

$$\underbrace{\int_\Gamma \left(\mathbf{\Psi}_\mathrm{u}\right)^\mathrm{T} \mathbf{f}\, d\Gamma}_{\mathbf{P}} = -\underbrace{\int_\Gamma \left(\mathbf{\Psi}_\mathrm{u}\right)^\mathrm{T} \mathbf{t}^+\, d\Gamma}_{\mathbf{P}^+} + \underbrace{\int_\Gamma \left(\mathbf{\Psi}_\mathrm{u}\right)^\mathrm{T} \mathbf{t}^-\, d\Gamma}_{\mathbf{P}^-} \quad \rightarrow \quad \mathbf{P} = -\mathbf{P}^+ + \mathbf{P}^- \tag{8a,b}$$

$$\underbrace{\int_\Gamma \left(\mathbf{\Psi}_\mathrm{t}\right)^\mathrm{T} (-\mathbf{u})\, d\Gamma}_{-\mathbf{W}} = \underbrace{\int_\Gamma \left(\mathbf{\Psi}_\mathrm{t}\right)^\mathrm{T} \mathbf{u}^+\, d\Gamma}_{\mathbf{W}^+} - \underbrace{\int_\Gamma \left(\mathbf{\Psi}_\mathrm{t}\right)^\mathrm{T} \mathbf{u}^-\, d\Gamma}_{\mathbf{W}^-} \quad \rightarrow \quad -\mathbf{W} = \mathbf{W}^+ - \mathbf{W}^- \tag{9a,b}$$

Eqs.(8,9) are the equilibrium and compatibility equations written in terms of weighted tractions $\mathbf{P}^+$ and $\mathbf{P}^-$ and weighted displacements $\mathbf{W}^+$ and $\mathbf{W}^-$, respectively; on the analogy of the previous definitions, $\mathbf{P}$ and $\mathbf{W}$ symbolize layered generalized forces and distortions.
The previous relations can be utilized for the corresponding boundary conditions (5,6) in weighted form.

$$\mathbf{P}^+ = \mathbf{0} \quad \rightarrow \quad \mathbf{P} = \mathbf{P}^- \tag{10a,b}$$
$$\mathbf{W}^+ = \mathbf{0} \quad \rightarrow \quad \mathbf{W} = \mathbf{W}^- \tag{11a,b}$$

which can be interpreted by saying that, when the solution is obtained, the weighed traction and displacement on Γ^+ have to be equal to zero, whereas the solutions evaluated in terms of weighted quantities coincide with the weighted traction and displacement on Γ^-, respectively.
If in eqs.(8,9) one introduces the modeling (7), and utilizes the positions

$$\int_\Gamma \left(\mathbf{\Psi}_\mathrm{t}\right)^\mathrm{T} \mathbf{\Psi}_\mathrm{u} d\Gamma = \mathbf{C}_\mathrm{ut}, \quad \int_\Gamma \left(\mathbf{\Psi}_\mathrm{u}\right)^\mathrm{T} \mathbf{\Psi}_\mathrm{t} d\Gamma = \mathbf{C}_\mathrm{tu}, \qquad \mathbf{C}_\mathrm{tu} = \mathbf{C}_\mathrm{ut}^\mathrm{T} \tag{12a,b,c}$$

one obtains

$$\underbrace{\mathbf{C}_{tu}\mathbf{F}}_{\mathbf{P}} = -\underbrace{\mathbf{C}_{tu}\mathbf{T}^{+}}_{\mathbf{P}^{+}} + \underbrace{\mathbf{C}_{tu}\mathbf{T}^{-}}_{\mathbf{P}^{-}}$$

$$\mathbf{F} = -\mathbf{T}^{+} + \mathbf{T}^{-} \qquad \rightarrow \qquad \mathbf{P} = \mathbf{C}_{tu}(-\mathbf{T}^{+} + \mathbf{T}^{-}) = \mathbf{C}_{tu}\mathbf{F} \tag{13a,b,c}$$

$$\underbrace{\mathbf{C}_{ut}\mathbf{U}}_{\mathbf{W}} = -\underbrace{\mathbf{C}_{ut}\mathbf{U}^{+}}_{\mathbf{W}^{+}} + \underbrace{\mathbf{C}_{ut}\mathbf{U}^{-}}_{\mathbf{W}^{-}}$$

$$\mathbf{U} = -\mathbf{U}^{+} + \mathbf{U}^{-} \qquad \rightarrow \qquad \mathbf{W} = \mathbf{C}_{ut}(-\mathbf{U}^{+} + \mathbf{U}^{-}) = \mathbf{C}_{ut}\mathbf{U} \tag{14a,b,c}$$

where $\mathbf{T}^{+}$ and $\mathbf{T}^{-}$ are the nodal forces acting respectively on the nodes of the boundaries Γ^{+} and Γ^{-} whereas $\mathbf{U}^{+}$ and $\mathbf{U}^{-}$ are the corresponding displacements. Eqs. (13b) and (14b) are the equilibrium and compatibility equations (3,4) written in terms of nodal variables. Eqs. (13c) and (14c) are respectively equations in which the matrix $\mathbf{C}_{tu}$ transforms the nodal forces into corresponding weighted tractions associated with the nodes, whereas the matrix $\mathbf{C}_{ut}$ transforms the nodal displacements into the corresponding weighted displacements associated with the nodes.

3. Generalized work

In the infinite domain Ω_{∞} the classic energy balance is valid

$$L_{\Gamma}^{+} + L_{\Gamma}^{-} = U_{(\Omega_{\infty}\setminus\Omega)} + U_{\Omega} = \frac{1}{2}\int_{(\Omega_{\infty}\setminus\Omega)} \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}\, d(\Omega_{\infty}\setminus\Omega) + \frac{1}{2}\int_{\Omega} \boldsymbol{\sigma}^{\mathrm{T}}\boldsymbol{\varepsilon}\, d\Omega \tag{17}$$

where L_{Γ}^{+} and L_{Γ}^{-} are respectively the work performed by the boundary quantities Γ^{+} of $\Omega_{\infty}\setminus\Omega$ and Γ^{-} of Ω, whereas $U = U_{(\Omega_{\infty}-\Omega)} + U_{\Omega}$ is the total energy stored in Ω_{∞}.
Since in the infinite domain the forces and the displacements take on the meaning of layered quantities (3,4), taking into account modeling (7) and the definition of the weighted quantities (8,9), the work can be expressed in the following alternative forms

I

$$L_{\Gamma} = \frac{1}{2}\int_{\Gamma} \mathbf{f}^{\mathrm{T}}\mathbf{u}\, d\Gamma = \frac{1}{2}\int_{\Gamma} (-\mathbf{t}^{+} + \mathbf{t}^{-})^{\mathrm{T}}(-\mathbf{u}^{+} + \mathbf{u}^{-})\, d\Gamma = \frac{1}{2}\left[(\mathbf{T}^{+})^{\mathbf{T}}\mathbf{W}^{+} - (\mathbf{T}^{+})^{\mathbf{T}}\mathbf{W}^{-} - (\mathbf{T}^{-})^{\mathbf{T}}\mathbf{W}^{+} + (\mathbf{T}^{-})^{\mathbf{T}}\mathbf{W}^{-}\right] = \frac{1}{2}\mathbf{F}^{\mathbf{T}}\mathbf{W} \tag{18}$$

II

$$L_{\Gamma} = \frac{1}{2}\int_{\Gamma} \mathbf{u}^{\mathrm{T}}\mathbf{f}\, d\Gamma = \frac{1}{2}\int_{\Gamma} (-\mathbf{u}^{+} + \mathbf{u}^{-})^{\mathrm{T}}(-\mathbf{t}^{+} + \mathbf{t}^{-})\, d\Gamma = \frac{1}{2}\left[(\mathbf{U}^{+})^{\mathbf{T}}\mathbf{P}^{+} - (\mathbf{U}^{+})^{\mathbf{T}}\mathbf{P}^{-} - (\mathbf{U}^{-})^{\mathbf{T}}\mathbf{P}^{+} + (\mathbf{U}^{-})^{\mathbf{T}}\mathbf{P}^{-}\right] = \frac{1}{2}\mathbf{U}^{\mathbf{T}}\mathbf{P} \tag{19}$$

These are the same expressions as provided by Denda [8] in terms of nodal and weighted layered quantities. The previous expressions will be utilized depending on the goals to be reached.

4. Displacement method: algebraic operators

The response on the boundary Γ of the solid, caused by known and unknown actions distributed on the same boundary, is given by the Somigliana Identities (SI) of the displacements and tractions. Introducing modeling (7) in the SI and executing the weighting second Galerkin hypotheses, one obtains an equation system in weighted form, which, through the introduction of the boundary condition and a reordering of the equations, provides the following block system

$$
\left[\begin{array}{c}
\mathbf{W}_2^+ \\ \hline
\mathbf{W}_1^+ \equiv 0 \\
\mathbf{W}_0^+ \equiv 0 \\
\mathbf{P}_2^+ \equiv 0 \\ \hdashline
\mathbf{P}_0^- \\ \hline
\mathbf{P}_1^+
\end{array}\right]
=
\left[\begin{array}{c|ccc:c|c}
\mathbf{A}_{\mathrm{u2u2}} & \mathbf{A}_{\mathrm{u2u1}} & \mathbf{A}_{\mathrm{u2u0}} & \mathbf{H}_{\mathrm{u2f2}} & \mathbf{A}_{\mathrm{u2f0}} & \mathbf{A}_{\mathrm{u2f1}} \\ \hline
\mathbf{A}_{\mathrm{u1u2}} & \mathbf{A}_{\mathrm{u1u1}} & \mathbf{A}_{\mathrm{u1u0}} & \mathbf{A}_{\mathrm{u1f2}} & \mathbf{A}_{\mathrm{u1f0}} & \mathbf{H}_{\mathrm{u1f1}} \\
\mathbf{A}_{\mathrm{u0u2}} & \mathbf{A}_{\mathrm{u0u1}} & \mathbf{A}_{\mathrm{u0u0}} & \mathbf{A}_{\mathrm{u0f2}} & \mathbf{H}_{\mathrm{u0f0}} & \mathbf{A}_{\mathrm{u0f1}} \\
\mathbf{E}_{\mathrm{f2u2}} & \mathbf{A}_{\mathrm{f2u1}} & \mathbf{A}_{\mathrm{f2u0}} & \mathbf{A}_{\mathrm{f2f2}} & \mathbf{A}_{\mathrm{f2f0}} & \mathbf{A}_{\mathrm{f2f1}} \\ \hdashline
\mathbf{A}_{\mathrm{f0u2}} & \mathbf{A}_{\mathrm{f0u1}} & \mathbf{E}_{\mathrm{f0u0}} & \mathbf{A}_{\mathrm{f0f2}} & \mathbf{A}_{\mathrm{f0f0}} & \mathbf{A}_{\mathrm{f0f1}} \\ \hline
\mathbf{A}_{\mathrm{f1u2}} & \mathbf{E}_{\mathrm{f1u1}} & \mathbf{A}_{\mathrm{f1u0}} & \mathbf{A}_{\mathrm{f1f2}} & \mathbf{A}_{\mathrm{f1f0}} & \mathbf{A}_{\mathrm{f1f1}}
\end{array}\right]
\left[\begin{array}{c}
\bar{\mathbf{F}}_2 \\ \hline
\mathbf{F}_1 \\
\mathbf{F}_0 \\
-\mathbf{U}_2 \\ \hdashline
-\mathbf{U}_0 \\ \hline
-\bar{\mathbf{U}}_1
\end{array}\right]
\tag{20}
$$

Making use of this block matrix equation, through appropriate block subdivisions [1] the weighed force-nodal displacement characteristic elasticity equation of the single substructure takes on the following form:

$$
\mathbf{P}_0^- = \mathbf{D}_0\,\mathbf{U}_0 + \hat{\mathbf{P}}_0 \tag{21}
$$

5. Galerkin residuals and boundary work

In the previous section the structure of the algebraic relations leading to the writing of the characteristic equation of the elasticity of the single substructure utilized within the displacement method of the SGBEM formulation was analyzed.

On the basis of the boundary conditions imposed one notes that:

- The weighted displacements $\mathbf{W}_2^+$ and the weighted tractions $\mathbf{P}_1^+$ are not utilized in the relations of the solving system; as a consequence the solution of the analysis problem will give $\mathbf{W}_2^+ \neq \mathbf{0}$ and $\mathbf{P}_1^+ \neq \mathbf{0}$.

These boundary quantities are defined by Paulino [6,7] as Galerkin weighted residuals and their values have to be null depending on the convergence of the solution of the discrete toward the solution of the continuum.

- Further, one notes that to write the characteristic elasticity equation the following conditions were imposed $\mathbf{W}_1^+ = \mathbf{0}$, $\mathbf{W}_0^+ = \mathbf{0}$, $\mathbf{P}_2^+ = \mathbf{0}$; these relations will always be verified when the solution is obtained.

5. 1 Boundary work evaluation

The generalized work L_Γ on the boundary will be computed taking into account, among the expressions defining the alternative forms (I,II), those best suited to examining the boundary type.

5.1.1 Free boundary $\boldsymbol{\Gamma}_2$

Through the relation $\mathbf{P}_2^+ = \mathbf{C}_{\mathrm{t2u2}}\mathbf{T}_2^+$, the position $\mathbf{P}_2^+ = \mathbf{0}$ does not necessarily involve $\mathbf{T}_2^+ = \mathbf{0}$, because it is possible to find on the boundary Γ_2^+ a system of auto-stressed nodal forces, whose weighted resultant is null. In [2] through an appropriate system the vector $\mathbf{T}_2^-$ was computed which by using the relation $\mathbf{T}_2^+ = -\bar{\mathbf{F}}_2 + \mathbf{T}_2^-$ allows one to obtain a system of nodal force such that $\mathbf{P}_2^+ = \mathbf{C}_{\mathrm{t2u2}}\mathbf{T}_2^+ = \mathbf{0}$ is satisfied.

- Utilizing the first form of work one finds that:
  $\mathbf{W}_2^+ \neq \mathbf{0}$ with the additional condition $\mathbf{P}_2^+ = \mathbf{0}$ and $\mathbf{T}_2^+ \neq \mathbf{0}$ provides

$$
\text{I)}\quad L_{\Gamma_2} = \frac{1}{2}\left[\underbrace{\left(-\mathbf{T}_2^+ + \mathbf{T}_2^-\right)^{\mathrm{T}}}_{\bar{\mathbf{F}}_2}\underbrace{\left(-\mathbf{W}_2^+ + \mathbf{W}_2^-\right)}_{\mathbf{W}_2}\right] \tag{22}
$$

where $(1/2)(\mathbf{T}_2^-)^{\mathrm{T}}\mathbf{W}_2^-$ is the direct work on Γ_2^- with $\mathbf{T}_2^-$ obtained in [2] and $\mathbf{W}_2^- = \mathbf{W}_2^+ + \mathbf{C}_{\mathrm{u2t2}}\mathbf{U}_2$.

In order for the residual quantities on Γ_2^+ to be null, the terms on which it is necessary to work are $(1/2)(\mathbf{T}_2^+)^{\mathrm{T}}(\mathbf{W}_2^+)$, $(1/2)(\mathbf{T}_2^+)^{\mathrm{T}}(\mathbf{W}_2^-)$, $(1/2)(\mathbf{T}_2^-)^{\mathrm{T}}(\mathbf{W}_2^+)$.

5.1.2 Constrained boundary $\boldsymbol{\Gamma}_1$

Through the relation $\mathbf{W}_1^+ = \mathbf{C}_{\mathrm{u1t1}}\mathbf{U}_1^+$, the position $\mathbf{W}_1^+ = \mathbf{0}$ does not necessarily involve $\mathbf{U}_1^+ = \mathbf{0}$, $\mathbf{T}_2^+ = \mathbf{0}$, because it is possible to find on the boundary Γ_1^+ a system of nodal displacements, whose weighted resultant is null. In [2] through an appropriate system the vector $\mathbf{U}_1^-$ was computed which by using the relation

$\mathbf{U}_1^+ = -\bar{\mathbf{U}}_1 + \mathbf{U}_1^-$ allows one to obtain a system of nodal displacements such that $\mathbf{W}_1^+ = \mathbf{C}_{\mathrm{u1t1}}\mathbf{U}_1^+ = \mathbf{0}$ is satisfied.

- Utilizing the second form of work one finds that:

$\mathbf{P}_1^+ \neq \mathbf{0}$ with the additional condition $\mathbf{W}_1^+ = \mathbf{0}$ and $\mathbf{U}_1^+ \neq \mathbf{0}$ provides

$$\text{II)} \quad L_{\Gamma_1} = \frac{1}{2}\left[\underbrace{(-\mathbf{U}_1^+ + \mathbf{U}_1^-)}_{\bar{\mathbf{U}}_1}{}^{\mathbf{T}} \underbrace{(-\mathbf{P}_1^+ + \mathbf{P}_1^-)}_{\mathbf{P}_1}\right] \tag{23}$$

where $(1/2)(\mathbf{U}_1^-)^{\mathrm{T}}\mathbf{P}_1^-$ is the direct work on Γ_1^- with $\mathbf{U}_1^-$ obtained in [2] and $\mathbf{P}_1^- = \mathbf{P}_1^+ + \mathbf{C}_{\mathrm{t1u1}}\mathbf{F}_1$.
In order that the residual quantities on Γ_1^+ are null, the terms on which it is necessary to work are $(1/2)(\mathbf{U}_1^+)^T\mathbf{P}_1^+$, $(1/2)(\mathbf{U}_1^+)^{\mathrm{T}}\mathbf{P}_1^-$, $(1/2)(\mathbf{U}_1^-)^{\mathrm{T}}\mathbf{P}_1^+$ with $\mathbf{U}_1^+ = -\bar{\mathbf{U}}_1 + \mathbf{U}_1^-$

5.1.3 Interface boundary $\boldsymbol{\Gamma}_0$

The boundary conditions on Γ_0 are: $\mathbf{W}_0^+ = \mathbf{0}$, $\mathbf{P}_0^- = \mathbf{D}_0\,\mathbf{U}_0 + \hat{\mathbf{P}}_0 \neq \mathbf{0}$

By the relation of compatibility and equilibrium in weighted form one writes:

$\mathbf{W}_0^- = \cancel{\mathbf{W}_0^+} + \mathbf{C}_{\mathrm{u0t0}}\mathbf{U}_0 \quad \rightarrow \quad \mathbf{U}_0^- = \mathbf{U}_0$ and $\mathbf{U}_0 = -\mathbf{U}_0^+ + \mathbf{U}_0^- \quad \rightarrow \quad \mathbf{U}_0^+ = \mathbf{0}$

$\mathbf{P}_0^+ = \mathbf{P}_0^- - \mathbf{C}_{\mathrm{t0u0}}\mathbf{F}_0$

- Utilizing the second form of work one finds that:

$\mathbf{P}_0^- \neq \mathbf{0}$ with the additional condition $\mathbf{W}_0^+ = \mathbf{0}$ and $\mathbf{U}_0^+ = \mathbf{0}$ provides

$$\text{II)} \quad L_{\Gamma_0} = \frac{1}{2}\left[\underbrace{(-\cancel{\mathbf{U}_0^+} + \mathbf{U}_0^-)}_{\mathbf{U}_0}{}^{\mathbf{T}} \underbrace{(-\mathbf{P}_0^+ + \mathbf{P}_0^-)}_{\mathbf{P}_0}\right] \tag{24}$$

where $(1/2)(\mathbf{U}_0^-)^{\mathrm{T}}\mathbf{P}_0^-$ is the direct work on Γ_0^- obtained as a product between $\mathbf{U}_0^- = \mathbf{U}_0$ and $\mathbf{P}_0^- = \mathbf{D}_0\,\mathbf{U}_0 + \hat{\mathbf{P}}_0$.
In order for the residual quantities on Γ_0^+ to be null, the terms on which it is necessary to perform the convergence is the reciprocal work $(1/2)(\mathbf{U}_0^-)^T\mathbf{P}_0^+$ with $\mathbf{U}_0^- = \mathbf{U}_0$ and $\mathbf{P}_0^+ = \mathbf{P}_0^- - \mathbf{C}_{\mathrm{t0u0}}\mathbf{F}_0$.
On the basis of the strategy employed, the summary table is shown. These expressions are alternative to the integrals containing the punctual functions and are obtainable by the coefficients of the block matrix (20), utilized in the analysis process.

Boundary type	**Boundary work type**
$\boldsymbol{\Gamma}_2$	I) $L_{\Gamma_2} = \frac{1}{2}\left[\underbrace{\left(-\mathbf{T}_2^+ + \mathbf{T}_2^-\right)}_{\bar{\mathbf{F}}_2}{}^{\mathrm{T}} \underbrace{\left(-\mathbf{W}_2^+ + \mathbf{W}_2^-\right)}_{\mathbf{W}_2}\right]$
$\boldsymbol{\Gamma}_1$	II) $L_{\Gamma_1} = \frac{1}{2}\left[\underbrace{(-\mathbf{U}_1^+ + \mathbf{U}_1^-)}_{\bar{\mathbf{U}}_1}{}^{\mathbf{T}} \underbrace{(-\mathbf{P}_1^+ + \mathbf{P}_1^-)}_{\mathbf{P}_1}\right]$
$\boldsymbol{\Gamma}_0$	II) $L_{\Gamma_0} = \frac{1}{2}\left[\underbrace{(-\cancel{\mathbf{U}_0^+} + \mathbf{U}_0^-)}_{\mathbf{U}_0}{}^{\mathbf{T}} \underbrace{(-\mathbf{P}_0^+ + \mathbf{P}_0^-)}_{\mathbf{P}_0}\right]$

Table 1: Type contour and boundary work type adopted for every boundary type.

6. Numerical example

Let us consider the rectangular plate in Fig. 2 having dimension 2×4 of thickness $s = 1\,$cm, constrained at the lower side and subjected to the constant vertical load $q = 1$. The plate is divided into two substructures A, B having dimension 2×2. Each is discretized into 8 boundary elements having unit length with $\Gamma_A=\Gamma_2\cup\Gamma_0$ ed $\Gamma_B=\Gamma_2\cup\Gamma_1\cup\Gamma_0$. The boundary elements of each substructure are denoted by small letters $a,b,....h$ and the nodes are numbered clockwise. The physical characteristics of both the substructures are $E=1$, $\nu=0.3$.

The analysis is performed by the displacement method using the Karnak.sGbem program [9].

In order to justify and numerically validate the expressions of the work shown in Table 2, for each substructure only the vertical forces and displacements are considered. Specifically, for substructure A the loaded side (*a*, *b*) and the interface one (*e*, *f*) are considered; similarly, for substructure B the interface side (*a*, *b*) and the constrained one (*e*, *f*).

It is to note that the values of the work obtained by the integrals containing the function evaluated on the boundary through the SI prove to be equal to that obtained through the layered generalized quantities of the matrix (20). This particular aspect allows one to perform all the numerical assessments, as for example the check of the error in the structural solution, by using the generalized quantities.

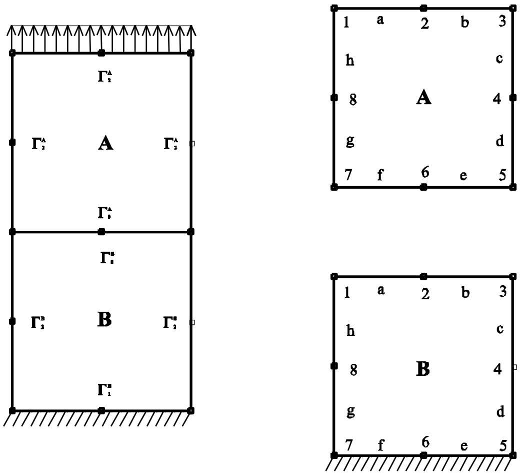

Figure 3. Body subjected to a uniform load: a) geometric description and substucturing, b) meshes adopted.

Boundary work: by punctual quantities and by generalized quantities	
Substructure A, a and b on Γ_2	$\frac{1}{2}\int_{\Gamma_{a,b}}\underbrace{(-t_2^+ + t_2^-)^t}_{f_2}\underbrace{(-u_2^+ + u_2^-)}_{u_2}d\Gamma=\frac{1}{2}\underbrace{(-\mathbf{T}_2^- + \mathbf{T}_2^-)^t}_{\bar{\mathbf{F}}_2}\underbrace{(-\mathbf{W}_2^- + \mathbf{W}_2^-)}_{\mathbf{W}_2=\mathbf{C}_{u2t2}\mathbf{U}_2}=3.58829$
Substructure A, e and f on Γ_0	$\frac{1}{2}\int_{\Gamma_{e,f}}\underbrace{(-u_0^+ + u_0^-)^t}_{u_0}\underbrace{(-t_0^+ + t_0^-)}_{f_0}d\Gamma=\frac{1}{2}\underbrace{(-\cancel{\mathbf{U}}_0^+ + \mathbf{U}_0^-)^t}_{\mathbf{U}_0^-\equiv\mathbf{U}_0}\underbrace{(-\mathbf{P}_0^+ + \mathbf{P}_0^-)}_{\mathbf{P}_0=\mathbf{C}_{t0u0}\mathbf{F}_0}=-1.76838$
Substructure B, a and b on Γ_0	$\frac{1}{2}\int_{\Gamma_{e,f}}\underbrace{(-u_0^+ + u_0^-)^t}_{u_0}\underbrace{(-t_0^+ + t_0^-)}_{f_0}d\Gamma=\frac{1}{2}\underbrace{(-\cancel{\mathbf{U}}_0^+ + \mathbf{U}_0^-)^t}_{\mathbf{U}_0^-\equiv\mathbf{U}_0}\underbrace{(-\mathbf{P}_0^+ + \mathbf{P}_0^-)}_{\mathbf{P}_0=\mathbf{C}_{t0u0}\mathbf{F}_0}=1.79103$
Substructure B, e and f on Γ_1	$\frac{1}{2}\int_{\Gamma_{e,f}}\underbrace{(-u_1^+ + u_1^-)^t}_{u_1}\underbrace{(-t_1^+ + t_1^-)}_{f_1}d\Gamma=\frac{1}{2}\underbrace{(-\mathbf{U}_1^+ + \mathbf{U}_1^-)^t}_{\bar{\mathbf{U}}_1}\underbrace{(-\mathbf{P}_1^+ + \mathbf{P}_1^-)}_{\mathbf{P}_1=\mathbf{C}_{t1u1}\mathbf{F}_1}=0$

Table 2: Energy evaluated on the boundary computed by the traction and displacement functions and by the generalized quantities.

7. Conclusions

The example used shows the effectiveness and the remarkable simplicity of the method in order to compute the elastic strain energy through the boundary work in terms of generalized layered quantities. Moreover, the procedure introduces remarkable potentialities for developing an energy criterion to evaluate the error of the solution obtained in the analysis phase.

The presence of the direct and indirect terms, constituting the generalized work, offers within the sphere of substructuring the possibility of formulating a strategy having as aim to limit the error of the solution through an appropriate discretization checking in what boundary elements the energy scattered is greater.

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