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# A mechanical picture of fractional-order Darcy equation

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ABSTRACT

In this paper the authors show that fractional-order force-flux relations are obtained considering the flux of a viscous fluid across an elastic porous media. Indeed the one-dimensional fluid mass transport in an unbounded porous media with power-law variation of geometrical and physical properties yields a fractional-order relation among the ingoing flux and the applied pressure to the control section. As a power-law decay of the physical properties from the control section is considered, then the flux is related to a Caputo fractional derivative of the pressure of order $0 \leqslant \beta \leq 1$. If, instead, the physical properties of the media show a power-law increase from the control section, then flux is related to a fractional-order integral of order $0 \leqslant \beta \leq 1$. These two different behaviors may be related to different states of the mass flow across the porous media.


## 1. Introduction

Diffusion in biophysics and medical sciences is a crucial mechanism of transport of chemical species and masses across biological structures and ultrastructures including cell membranes, epithelial tissues, and perfused organs parenchima. The basic relations ruling the transport of chemical species by diffusion is the so-called Fick equation that relates, linearly, the flux of chemical species moving in a fluid media to the concentration gradient in the direction of transport.

A similar phenomena is encountered in the mass transport of fluid particles across a porous media. In this regard the flux of fluid mass is related, linearly, to the gradient of hydraulic load along the flux, neglecting the contribution of the kinetic energy due to the small speed of the particle flow. The transport equation, formally analogous to the Fick relation is the so-called Darcy equation so that transport of chemical species and fluid transport across a porous media are both dubbed as diffusive problems.

The linear diffusive equations, either in the form of Fick or of Darcy, has shown, however, several discrepancies with observed experimental data [1–3]. Indeed the time evolution of the concentration and velocity profile predicted by Fick or Darcy is described by exponential-type solution and, several deviations from experimental results have been found in

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scientific literature regarding fluid flows in biological tissues [4,5] usually referred to long-tails of the diffusion processes as well as through biological membranes [6,7]. The difference among Fick prediction and experimental results have been captured, recently, considering particle transport at nanometric scale by means of molecular dynamics simulations [8–11].

In such cases the deviations have been modeled introducing an acceleration contribute to the transport equation in the form of a time-derivative of the flux vector [12] yielding an hyperbolic partial differential equation for the pressure field (Brinkman correction). As an alternative, some corrections to the classical Darcy and/or Fick equations have been proposed in scientific literature, resorting to non-linear, Forchheimer-type, corrective terms of the state variables of the problem leading to non-linear space–time PDE [12].

The choice of the state variables in the transport equation is related to the state of the matter either in solid, liquid or gas phase. However, biological soft matter, at cellular and/or tissue resolution is far from the distinction in one of the three fundamental state of the matter and intermediate behavior is to be expected as shown by the deviations of experiments from theoretical predictions ([13–15]). The main question about the physical model of the phenomenological Forchheimer or Brinkman-type correction is still an open problem and no studies in scientific literature have been provided.

Recent approaches involving the use of Continuous Time Random Walk (CTRW) models to describe the random path of contaminant flux plumes in heterogeneous porous media have shown that the statistical moments of the concentration decays in time with power-laws exponent different from $\beta = \frac{1}{2}$ that is expected for Brownian path of the contaminant (see e.g. papers by [16–18] and references cited therein). The approach followed in these studies relies on the consideration that the presence of a probability density function (pdf) of the mechanical features of the porous media, that is the presence of a stochastic porous solid, involves a material heterogeneity at any length scale that affects, significatively, the effective transport parameter across the porous solid.

A different approach to handle deviations from the classical diffusion has been framed, since the end of the last century, in the context of anomalous diffusions in terms of power-laws with real exponents. In this latter case a recently proposed analytic description of transport across a porous media has involved, after some experimental set-up [19], the use of fractional-order integrals and derivatives [20]. The main reason to use extended, real-order, operators in Fick/Darcy transport equation, beside the integer-order counterpart, relies in the *memory effect* induced by the interactions of fluid particles with pores of the considered media. In this setting the transport equations has been modified introducing convolution integrals of time with power-law kernels.

The main feature of these studies involves the replacement of classical integer-order differentials with their modern counterpart known as fractional-order derivatives, namely $\frac{df(t)}{dt} \to \frac{d^{(\beta)}f}{dt^{(\beta)}}$ with $\beta$ a real-order number as reported in ([21–24]) but no physical/mechanical model has ever been provided to justify such an assumption.

In this study the authors aim to provide a mechanical justification for the presence of anomalous diffusion in porous media as recently provided in scientific literature for the hereditary behavior of the matter [25,26] as well as in bone tissues viscoelasticity [27]. It will be shown that fractional-order operators arise in terms of a macroscopic transport equation as an 1D mass transport in a porous media with power-law variations of permeability and porosity is considered. The obtained force-flux relations will involve fractional operators with order $-1 \leqslant \beta \leq 1$ as it will be shown in the course of the paper. The proposed model may be considered as the macroscopic equivalent of the random models based on the CTRW assuming that the measure of the fluid particle motion is represented by the overall fluid mass instead than the pdf of the concentration of the moving molecules. In this regard the derivation of the anomalous transport parameters involved in the fractional-order generalization of the Darcy equation by means of the microscopic probabilistic features of the underlying random process is underway and it will be reported elsewhere.

## 2. Remarks on fractional-order differential calculus

Fractional calculus may be considered the extension of the ordinary differential calculus to non-integer powers of derivation orders (e.g. see [21,22]). In this section we address some basic notions about this mathematical tool.

The Euler-Gamma function $\Gamma(z)$ may be considered as the generalization of the factorial function since, as $z$ assumes integer values as $\Gamma(z+1) = z!$ and it is defined as the result of the integral as follows:

$$\Gamma(z) = \int_0^\infty e^{-x} x^{z-1} dx. \tag{1}$$

The Riemann–Liouville fractional integrals and derivatives with $0 < \beta < 1$ of functions defined on the entire real axis have the following forms:

$$\left(\mathrm{I}_+^\beta f\right)(t) = \frac{1}{\Gamma(\beta)} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \tag{2a}$$

$$\left(\mathrm{D}_+^\beta f\right)(t) = \frac{1}{\Gamma(1-\beta)} \frac{d}{dt} \int_{-\infty}^t \frac{f(\tau)}{(t-\tau)^{\beta}} d\tau. \tag{2b}$$

The Riemann–Liouville fractional integrals and derivatives with $0 < \beta < 1$ of functions defined over intervals of the real axis, namely $f(t)$ such that $t \in [a, b] \subset R$, have the following forms:

$$\left(\mathrm{I}_a^\beta f\right)(t) = \frac{1}{\Gamma(\beta)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\beta}} d\tau, \tag{3a}$$

$$\left(\mathrm{D}_a^\beta f\right)(t) = \frac{f(a)}{\Gamma(1-\beta)(t-a)^\beta} + \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau. \tag{3b}$$

Beside Riemann–Liouville fractional operators defined above, another class of fractional derivative that is often used in the context of fractional viscoelasticity is represented by Caputo fractional derivatives defined as:

$$\left({}_C\mathrm{D}_{a^+}^\beta f\right)(t) := \mathrm{I}_{a^+}^{m-\beta}(\mathrm{D}_{a^+}^m f)(t) \quad m-1 < \beta < m \tag{4}$$

and whenever $0 < \beta < 1$ it reads as follows:

$$\left({}_C\mathrm{D}_{a^+}^\beta f\right)(t) = \frac{1}{\Gamma(1-\beta)} \int_a^t \frac{f'(\tau)}{(t-\tau)^\beta} d\tau. \tag{5}$$

A closer observation of Eq. (4) and Eq. (5) shows that Caputo fractional derivative coincides with the integral part of the Riemann–Liouville fractional derivative in bounded domain. We conclude that Caputo fractional operators, coincide with Riemann–Liouville fractional derivatives in Eq. (2) as $f(a) = 0$, that is for vanishing boundary conditions. Additionally, the definition in Eq. (6) implies that the function $f(t)$ has to be absolutely integrable of order $m$ (e.g. in (5) the order is $m = 1$). Whenever $f(a) = 0$ Caputo and Riemann–Liouville fractional derivatives coalesce.

Similar considerations hold true also for Caputo and Riemann–Liouville fractional derivatives defined on the entire real axis. Caputo fractional derivatives may be considered as the interpolation among the well-known, integer-order derivatives, operating over functions $f(\circ)$ that belong to the class of Lebesgue integrable functions ($f(\circ) \in L^1$) as a consequence, they are very useful in the mathematical description of complex system evolution.

It is worth introducing integral transforms for fractional operators. Similarly to classical calculus, the Laplace integral transform $\mathcal{L}(\circ)$ is defined in the following forms:

$$\mathcal{L}\left[\left(\mathrm{D}_{0^+}^\beta f\right)(t)\right] = s^\beta \mathcal{L}[f(t)] = s^\beta \tilde{f}(s), \tag{6a}$$

$$\mathcal{L}\left[\left(\mathrm{I}_{0^+}^\beta f\right)(t)\right] = s^{-\beta} \mathcal{L}[f(t)] = s^{-\beta} \tilde{f}(s). \tag{6b}$$

In the same way, the Fourier integral transform $\mathcal{F}(\circ)$ assumes the following forms:

$$\mathcal{F}\left[\left(\mathrm{D}_+^\beta f\right)(t)\right] = (-i\omega)^\beta \mathcal{F}[f(t)] = (-i\omega)^\beta \hat{f}(\omega), \tag{7a}$$

$$\mathcal{F}\left[\left(\mathrm{I}_+^\beta f\right)(t)\right] = (-i\omega)^{-\beta} \mathcal{F}[f(t)] = (-i\omega)^{-\beta} \hat{f}(\omega). \tag{7b}$$

We recall that the Laplace and Fourier integral transforms are defined as follows:

$$\mathcal{L}[f(t)] = \int_0^\infty f(t) \mathrm{e}^{-st} dt, \tag{8a}$$

$$\mathcal{F}[f(t)] = \int_{-\infty}^{+\infty} f(t) \mathrm{e}^{-i\omega t} dt. \tag{8b}$$

These mathematical tools may be very useful to solve systems of fractional differential equations, which appear more and more frequently in various research areas and engineering applications [22]. An example of the application of Laplace transform to the solution of fractional-order differential equations is provided as we consider the following differential equation of order $\beta = 1/2$:

$$\left(\mathrm{D}_0^{\frac{1}{2}} f\right)(t) + af(t) = 0 \tag{9}$$

with the following initial condition

$$C = \left[\left(\mathrm{D}_0^{-\frac{1}{2}} f\right)(t)\right]_{t=0}, \tag{10}$$

that occurs very often in electrical engineering contexts. The use of the Laplace integral transform allows for writing the solution in the $s$-mapped domain as follows:

$$\tilde{f}(s) = \frac{C}{s^{1/2} + a}. \tag{11}$$

Whenever the time domain is restored, the solution has the following form:

$$f(t) = Ct^{-\frac{1}{2}} E_{\frac{1}{2},\frac{1}{2}}\left(-a\sqrt{t}\right), \tag{12}$$

where $E_{\alpha,\beta}(z)$ is the Mittag–Leffler function, defined as follows:

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\alpha k + \beta)} \quad \alpha > 0, \ \beta > 0. \tag{13}$$

In the textbook of Podlubny [22] (p. 21) an expression for the Laplace transform can be found in the following form

$$\mathcal{L}\left[t^{\frac{k-1}{2}} E^{(k)}_{\frac{1}{2},\frac{1}{2}}\left(a \pm \sqrt{t}\right)\right] = \frac{k!}{\left(\sqrt{s} \mp a\right)^{k+1}}, \tag{14}$$

where the notation $[\bullet]^k$ denotes the $k$th-derivative. We recognize that in Eq. (11) $k = 0$, henceforth the time domain solution reads has the form reported in Eq. (12) (see e.g. [21,22] for details on fractional-order calculus).

## 3. Anomalous fluid diffusion in porous media

The use of fractional-order calculus to model diffusion in porous media may be traced back to studies published by the end of the last century [19]. In these studies the force-flux relation provided by Darcy linear relation was modified to include changes in physical properties of the porous media during the flow. Indeed, the well-known diffusion problem in porous media is ruled by the equations (see e.g. Fig. 1(a) and (b) for 1D diffusion):

$$\begin{cases} \mathbf{q}(\mathbf{x},t) = -\rho_0 \lambda(\mathbf{x}) \nabla p(\mathbf{x},t), \\ \dfrac{\partial \rho(\mathbf{x},t)}{\partial t} = \dfrac{\partial \rho(\mathbf{x},t)}{\partial p} \dfrac{\partial p(\mathbf{x},t)}{\partial t} = \dfrac{\rho_0}{K(\mathbf{x})} \dfrac{\partial p(\mathbf{x},t)}{\partial t}, \\ \dfrac{\partial \rho(\mathbf{x},t)}{\partial t} + \nabla \bullet \mathbf{q}(\mathbf{x},t) = 0, \end{cases} \tag{15}$$

where $\nabla[\bullet] = \frac{\partial[\bullet]}{\partial x_1}\mathbf{i}_1 + \frac{\partial[\bullet]}{\partial x_2}\mathbf{i}_2 + \frac{\partial[\bullet]}{\partial x_3}\mathbf{i}_3$ denotes the Laplacian operator, $\mathbf{q}(x,t) = [q_1(\mathbf{x}) \ q_2(\mathbf{x}) \quad q_3(\mathbf{x})]^T$ is the vector of mass flux across a generic cross-section with $[\mathbf{q}] = \frac{FT}{L^3}$, $\lambda(\mathbf{x})$ is the Darcy conductivity coefficient with $[\lambda] = \frac{L^4}{FT}$ depending of the material permeability and of the viscosity of the embedded fluid, $[\rho(x,t)] = \frac{FT^2}{L^4}$ is the fluid mass density in the control volume considered, $[\rho_0] = \frac{FT^2}{L^4}$ is the fluid mass density in the reference configuration, $K(\mathbf{x})$ with $[K] = \frac{F}{L^2}$ is the bulk modulus of the porous material and $[p(x,t)] = \frac{F}{L^2}$ is the pressure field.

The anomalous force-flux relation in advection/diffusion has been obtained in terms of a modified transport equation as:

$$f_1(t) * \mathbf{q}(\mathbf{x},t) = f_2(t) * \nabla p(\mathbf{x},t), \tag{16}$$

where the symbol $*$ indicates convolution products among the functions $f_1(t)$, and $f_2(t)$ which is defined, for functions $f(t)$ and $g(t)$, as:

$$f(t) * q(t) = \int_0^t f(t-\tau) g(\tau) d\tau. \tag{17}$$

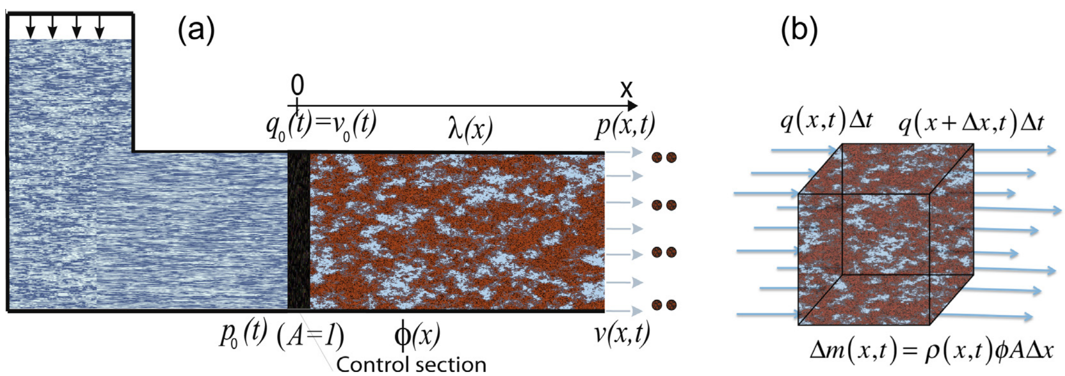


**Fig. 1.** 1D fluid transport in porous media: (a) the physical model of anomalous diffusion; (b) the mass balance equation in the elementary cell.

As we assume that functions $f_1(t)$ and $f_2(t)$ are fluid and solid dependent differential operators provided in the form:

$$f_1(t) = \gamma\delta(t) + \frac{\varepsilon\, t^{-n_1}}{\Gamma(1-n_1)}\frac{\partial}{\partial t}[\bullet], \tag{18a}$$

$$f_2(t) = c\delta(t) + \frac{d t^{-n_2}}{\Gamma(1-n_2)}\frac{\partial}{\partial t}[\bullet], \tag{18b}$$

where $\delta(t)$ is the Dirac-delta function, the real-order exponents $0 < n_1 < 1, 0 < n_2 < 1$ as well as $\gamma, \varepsilon, c$ and $d$ are model parameters. The substitution of Eq. (16) with the transport fluid-pressure relation in Eq. (15) has been dubbed *memory formalism* (see e.g. paper [20]). The introduction of the memory formalism contributions to the Darcy diffusion equation is used to capture changes in the chemical/physical properties of the pores as well as of the interactions among pore channels and fluid particles of the porous media during the transport process.

As we replace the expressions for functions $f_1(t)$ and $f_2(t)$ in the convolution products in Eq. (16) the transport equation is expressed as:

$$(\gamma + \varepsilon D^{n_1})\mathbf{q} = -(c + dD^{n_2})\nabla p, \tag{19}$$

where $\left(\mathrm{D}^{n_1}_{0^+}[\bullet]\right)(t)$ and $\left(\mathrm{D}^{n_2}_{0^+}[\bullet]\right)(t)$ are Caputo fractional derivatives. In this letter case the governing equation may be obtained introducing divergence operator of Eq. (19) as:

$$(\gamma + \varepsilon D^{n_1})\nabla\mathbf{q} = -(c + dD^{n_2})\nabla^2 p, \tag{20}$$

while the continuity equation, in term of pressure-flux variables reads:

$$\frac{1}{K}\frac{\partial p(x,t)}{\partial t} + \nabla\mathbf{q}(\mathbf{x},\mathbf{t}) = 0, \tag{21}$$

yielding the field equation in the form:

$$[\gamma + \varepsilon\left(\mathrm{D}^{n_1}_{0^+}[\bullet]\right)(t)]\frac{\partial p}{\partial t} = K[c + d\left(\mathrm{D}^{n_2}_{0^+}[\bullet]\right)(t)]\nabla^2 p. \tag{22}$$

Eq. (21) is the generalized, long memory pressure field equation that has been used in a slightly more general form to fit experimental data from different experimental set up showing an excellent agreement with measured temporal evolution of the flux [20].

Summing up, as we introduce the memory formalism, an alternative force-flux relation is introduced to account for the changes in the physical properties of the porous heterogeneous material. The changes may involve variation of the permeability of the media with time as well as microscopic changes of the fluid viscosity during the interactions with the solid particles of the porous material. In this setting the memory formalism has proved to be an efficient tool to predict the deviations of the measured outgoing flux from the exponential-type decay obtained with the use of Darcy transport relations in terms of the diffusion problem. However the presence of a convolution relation in the memory formulation of the Darcy equation has not been justified neither mechanically and geometrically.

This is analogous to the use of fractional-order calculus to other fields of mechanics [28–31] and thermodynamics [32,33]. In this regard, in the latter scientific literature appropriate representation of fractional-order operators have been introduced in the field of solid and material mechanics [34–38] as well as in thermodynamical setting (see papers by [39–41]).

In the next section the authors will show that the presence of fractional-order operators in the force flux relations correspond to a macroscopic relation of advection in a porous media with spatially-varying physical properties in presence of a one-dimensional mass transport.

## 4. A mechanical picture of the anomalous transport equations

In the previous section we reported the main assumptions of the anomalous transport obtained by means of the generalization of the Darcy relation. The need for an alternative formulation of the Darcy transport relies on the consideration that any variation of the pressure gradient $\nabla p(\mathbf{x}, t)$ is instantaneously transferred in the flux $q(\mathbf{x}, t)$, and consequently, instantaneous change in the outgoing flux is evidenced in the pressure gradient $\nabla p(x, t)$. This consideration holds true, only, for moderate time-changes in the flux and/or in driving pressure gradients, but, as well as, the macroscopic flux is the result of a smaller scale flux in a porous media, the instantaneous dependence of the flux by the driving pressure gradient is no more acceptable. In this regards appropriate changes in the transport equations introducing acceleration terms and proper relaxation times of the flux field have been proposed in the literature (see e.g. [9]). The generalization of these relations by means of fractional-order calculus represents, indeed, an attempt to capture intermediate effects that are neither diffusive-and convective-types. The mechanics beyond fractional generalization, is, however still uncovered and, in this paper, we provide a first attempt to the topic. In this regard we consider an 1D fluid motion in a porous media under different conditions: (i) The case of uniform physical properties of the fluid and solid phase and; (ii) The case of a spatially varying properties, either of solid and fluid phase in terms of material compressibility and fluid conductivity.

The formulation will be discussed, without loss of generality for a 1D fluid velocity field in a saturated porous media as it will be reported in the next section.

### 4.1. Uniform fluid diffusion in homogeneous porous media

Let us consider a 1D fluid motion across an uniform porous media in the horizontal direction. Let us assume, moreover, that the incoming flux at the origin of the coordinate axis $q_x(0,t) = q_0(t)$ is measured, that is the cross-section at $x = 0$ is the control section (see Fig. 1(a)). Let us assume, moreover, that the pressure field, measured at $x \to \infty$, is vanishing so that the asymptotic boundary condition reads:

$$\lim_{x\to\infty} p(x,t) = 0 \quad \forall t. \tag{23}$$

In this context the pressure field,obtained by combination of the Eqs. (10) is provided as:

$$C\frac{\partial p(x,t)}{\partial t} = \lambda \frac{\partial^2 p(x,t)}{\partial x^2}, \tag{24}$$

where $C = \frac{1}{K}$ is the bulk compliance of the saturated porous media and the associated boundary condition (for $x \to 0$) read:

$$\lim_{x\to 0} p(x,t) = p_0(t), \tag{25}$$

where $p_0(t)$ is the applied pressure at the control section.

The evaluation of the pressure field in the porous media domain is obtained with Laplace transforms as:

$$Cs\widehat{p}(x,s) = \lambda \frac{d^2\widehat{p}(x,s)}{dx^2}. \tag{26}$$

Solution of Eq. (26) may be obtained as a linear combination of exponential functions as:

$$\widehat{p}(x,s) = Ae^{-\sqrt{\frac{C}{\lambda}s}x} + Be^{\sqrt{\frac{C}{\lambda}s}x}. \tag{27}$$

Position of the boundary condition in Eq. (23) and (25) yields $B = 0$ $A = \widehat{p}_0(s)$ in this regard the pressure field $\widehat{p}(x,s)$ in the porous domain is obtained in the functional class of the spatial exponential decay as:

$$\widehat{p}(x,s) = \widehat{p}_0(s)e^{-\sqrt{\frac{C}{\lambda}s}x}, \tag{28}$$

that may be reported to a relation among the ingoing flux $q_0(t)$ across the control section and the applied pressure $p_0(t)$ as:

$$\widehat{q}_0(s) = -\rho_0\lambda\left(\frac{\partial p}{\partial x}\right)_{x=0} = \rho_0\sqrt{C\lambda}\,\widehat{p}_0(s)s^{\frac{1}{2}}. \tag{29}$$

Introducing inverse Laplace transform a fractional-order derivative with Caputo fractional operator among the ingoing measured flux and the applied pressure is obtained as:

$$q_0(t) = \rho_0\sqrt{C\lambda}\left({}_C\mathrm{D}^{\frac{1}{2}}_{0^+}p_0\right)(t) = R_{1/2}\left({}_C\mathrm{D}^{\frac{1}{2}}_{0^+}p_0\right)(t), \tag{30}$$

where $[R_{1/2}] = T^{3/2}/L$ is the anomalous mass diffusivity. Introducing the inverse Laplace transform a fractional-order relation among the obtained pressure and the applied ingoing flux is obtained as:

$$p_0(t) = \frac{1}{\rho_0\sqrt{C\lambda}}\left(\mathrm{I}^{\frac{1}{2}}_{0^+}q_0\right)(t) = \frac{1}{R_{1/2}}\left(\mathrm{I}^{\frac{1}{2}}_{0^+}q_0\right)(t), \tag{31}$$

that corresponds to a long-memory relation among flux $q_0(t)$ and the measured pressure $p_0(t)$. In passing we observe that the inverse relation in Eq. (31) is obtained under the assumption of vanishing initial flux $q_0(0) = 0$ and $p_0(0) = 0$. In this regard the fractional-order generalization of Darcy equation may involve either the Riemann–Liouville or the Caputo fractional derivative.

The observation of Eq. (31) shows that as for as we control the pressure at $x = 0$, the ingoing flux dependents an the histories of the pressure field with a fractional derivative order $\beta = 1/2$. On the other hand, if we control the flux across the cross section at $x = 0$ the measured pressure depends on the flux histories with a fractional-order integral of order $\beta = 1/2$.

The generalization of the force-flux relations in Eqs. (30) and (31) is provided in the next section for values of the differentiation order $\beta \in \mathbb{R}$ yielding, on physical basis, appropriate bounds to the differentiation order. Indeed it may be observed that the relation among the flux velocity field and the gradient of pressure may be written as:

$$\frac{\partial^{(0)}q(x,t)}{\partial t^{(0)}} = -\rho_0\lambda\frac{\partial^{(0)}}{\partial t^{(0)}}\frac{\partial p}{\partial x}(x,t), \tag{32}$$

where we denoted $\frac{\partial^{(0)}f(x,t)}{\partial t^{(0)}} = f(x,t)$. The transport relation in the form in Eq. (32) yields to conclude that the 0th-order time derivative of the flux is related to the 0th-order time derivative of the pressure gradient. However the relation obtained in Eqs. (30) and (31) yields that, as the fluid transport occurs in a proper media, then the flux is related to the fractional-order derivative of order $\beta = \frac{1}{2}$ of the pressure applied in the control section. Appropriate generalization of the differentiation order must account for two main physical behavior: (i) A purely inertial fluid yielding a relation involving first order time derivative of the flux and the pressure and (ii) A purely elastic fluid yielding a relation among the pressure and the first-order integral of the velocity field. Bounds of the fractional differentiation order $\beta \in [-1, 1]$ satisfies those requirements yielding appropriate bounds on the decaying coefficient $\alpha$ as reported in the following section.

### 4.2. One-dimensional fluid diffusion in non-homogeneous porous media

In this section the authors will introduce the generalized version of the force-flux relation proposed in Eqs. (30) and (31) for $\beta = 1/2$ to arbitrary values of fractional differentiation order $\beta$. To this aim let us consider that the flux $q(x,t)$ occurs in an heterogeneous porous media with compressibility coefficient $C(x) = \frac{1}{K(x)}$ and diffusion coefficient $\lambda(x)$. in this setting the governing equations of the problem reads:

$$\begin{cases} q(x,t) = -\rho_0 \lambda(x) \dfrac{\partial p(x,t)}{\partial x}, \\ \dfrac{\partial p(x,t)}{\partial t} = \dfrac{K(x)}{\rho_0} \dfrac{\partial \rho(x,t)}{\partial t} = \dfrac{1}{\rho_0 C(x)} \dfrac{\partial \rho(x,t)}{\partial t}, \\ \dfrac{\partial \rho(x,t)}{\partial t} + \dfrac{\partial q(x,t)}{\partial x} = 0, \end{cases} \tag{33a}$$

that after proper substitution yields the field equation for the pressure field in the porous domain as:

$$C(x) \frac{\partial p(x,t)}{\partial t} = \frac{\partial}{\partial x} \left[ \lambda(x) \frac{\partial p(x,t)}{\partial x} \right], \tag{34}$$

that is a partial differential equation for the pressure field $p(x,t)$ with variable coefficients $C(x)$ and $\lambda(x)$. Let us assume that the variation of the diffusive and mechanical properties of the two phases variate with the power law from the origin of the coordinate system as:

$$C(x) = \frac{C_\alpha x^{-\alpha}}{\Gamma(3-\alpha)}, \tag{35a}$$

$$\lambda(x) = \frac{\lambda_\alpha x^{-\alpha}}{\Gamma(3+\alpha)}, \tag{35b}$$

where $[\lambda_\alpha] = (FT)^{-1} L^{4+\alpha}$ and $[C_\alpha] = F^{-1} L^{2+\alpha}$ are, respectively, the anomalous diffusivity, the compressibility modulus of the saturated porous material and $\alpha$ is a real exponent that must belong to a specific subset of the real axis as it will be reported in the following discussion.

In this regard, the pressure field may be obtained introducing Laplace transform of Eq. (34), and referring to a pressure gradient for unitarly length as $\hat{p}(x,s) = \mathcal{L}[p(x,s)]$ in Eq. (35), an ordinary differential equation in Laplace domain is obtained in the form:

$$\frac{d}{dx} \left[ \lambda(x) \frac{d\hat{p}(x,s)}{dx} \right] = sC(x)\hat{p}(x,s), \tag{36}$$

that may be cast, after some straightforward manipulation as:

$$\frac{d^2 \hat{p}(s,x)}{dx^2} + \frac{\lambda'(x)}{\lambda(x)} \frac{d\hat{p}(s,x)}{dx} - \frac{C(x)}{\lambda(x)} s\hat{p}(s,x) = 0. \tag{37}$$

Substitution for the diffusivity coefficient $\lambda(x)$ and the compressibility $C(x)$ the corresponding power-laws reported in Eqs. (34 a,b) the differential equation ruling the pressure field reads:

$$\frac{d^2 \hat{p}(x,s)}{dx^2} - \frac{\alpha}{x} \frac{d\hat{p}(x,s)}{dx} - \delta_\alpha s\hat{p}(x,s) = 0 \tag{38}$$

with $\delta_\alpha = C_\alpha \Gamma(3+\alpha) / \lambda_\alpha \Gamma(3-\alpha)$ is a characteristic time rate of change of the flux across the porous media. The governing equation of the pressure field may be reverted into a Bessel equation of the second kind introducing an auxiliary function: $\overline{p}(x,s)$ related to the unknown function $\hat{p}(x,s)$ by means of the non-linear mapping $\hat{p}(x,s) = x^\alpha \overline{p}(x,s)$ so that first and second-order derivatives involved in Eq. (38) read:

$$\frac{d\hat{p}(x,s)}{dx} = \alpha x^{\alpha-1} \overline{p}(x,s) + x^\alpha \frac{d\overline{p}(x,s)}{dx}, \tag{39a}$$

$$\frac{d^2 \hat{p}(x,s)}{dx^2} = \frac{d}{dx} \left[ \alpha x^{\alpha-1} \overline{p}(x,s) + x^\alpha \frac{d\overline{p}(x,s)}{dx} \right] = \alpha(\alpha-1) x^{\alpha-2} \overline{p}(x,s) + 2\alpha x^{\alpha-1} \frac{d\overline{p}(x,s)}{dx} + x^\alpha \frac{d^2 \overline{p}(x,s)}{dx^2} \tag{39b}$$

and substitutions into Eq. (38) yields a modified Bessel equation for function $\overline{p}(x,s)$ as:

$$x^2\frac{d^2\overline{p}(x,s)}{dx^2}+\alpha x\frac{d\overline{p}(x,s)}{dx}-(x^2\delta_\alpha s+\alpha)\overline{p}(x,s)=0. \tag{40}$$

Eq. (40) may be solved in terms of the first and the second modified Bessel functions denoted, respectively $Y_\beta(x\sqrt{\delta_\alpha s})$ and $K_\beta(x\sqrt{\delta_\alpha s})$ defined as:

$$Y_\beta(x)=\sum_{k=0}^{\infty}\frac{(x/2)^{\beta+2k}}{k!\Gamma(k+\beta+1)}, \tag{41a}$$

$$K_\beta(x)=\frac{\pi}{2sin(2\pi\beta)}\left[Y_{-\beta}(x)-Y_\beta(x)\right], \tag{41b}$$

yielding a solution of the modified Bessel function in the form:

$$\hat{p}(x,s)=x^\beta\left(B_1Y_\beta\left(x\sqrt{\delta_\alpha s}\right)+B_2K_\beta\left(x\sqrt{\delta_\alpha s}\right)\right), \tag{42}$$

where we introduced the $\alpha$-dependent relaxation time and the index $\beta$ that are defined, respectively, as:

$$\beta=\frac{(1+\alpha)}{2}. \tag{43}$$

Integration constants $B_1$ and $B_2$ in Eq. (38) are defined as we impose the relevant boundary conditions that are defined in Laplace domain as:

$$\lim_{x\to}\hat{p}(x,s)=\widehat{p}_0(s), \tag{44a}$$

$$\lim_{x\to\infty}\hat{p}(x,s)=0, \tag{44b}$$

yielding the integration constants:

$$B_1=0;\quad B_2=\frac{\hat{p}_0(s)}{\Gamma(\beta)2^{\beta-1}}(\delta_\alpha s)^{\beta/2}. \tag{45}$$

The pressure field of the fluid phase reads:

$$\hat{p}(x,s)=\frac{\widehat{p}_0(s)}{\Gamma(\beta)2^{\beta-1}}(\delta_\alpha)^{\beta/2}x^{1-\beta}K_\beta\left(x\sqrt{\delta_\alpha s}\right). \tag{46}$$

The force-flux relation at $x=0$ is then provided at the limit

$$\lim_{x\to 0}-\rho_0\lambda(x)\frac{\partial\hat{p}(x,s)}{\partial x}=\hat{q}_0(s), \tag{47}$$

yielding the flux in the control section $\widehat{q}_0(s)$ as function of the applied pressure $\widehat{p}_0(s)$

$$q_0(t)=R_\beta\left({}_CD^\beta_{0^+}p_0\right)(t), \tag{48}$$

where we denoted $\rho_0\lambda_\beta(\delta_\alpha)^\beta=R_\beta$ the apparent anomalous resistivity of the porous media to the mass flow and the coefficient $\lambda_\beta$ reads:

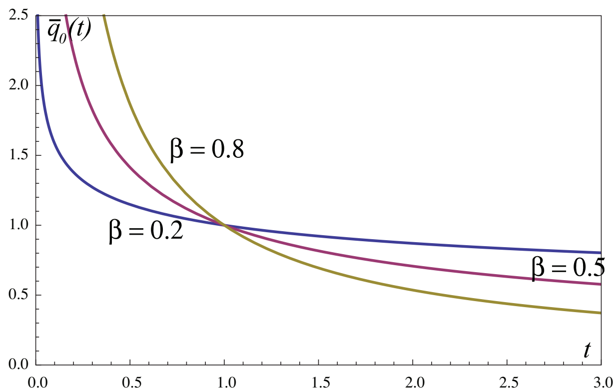


**Fig. 2.** Time evolution of non-dimensional flow across control section for different values of differentiation order: visco-inertial fluids.

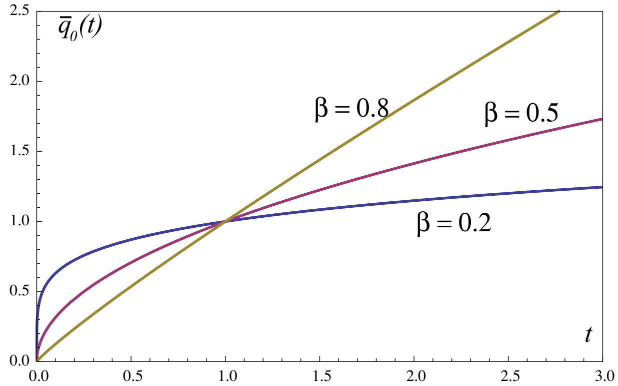


**Fig. 3.** Time evolution of non-dimensional flow across control section for different values of differentiation order: visco-Elastic fluid.

$$\lambda_\beta = \frac{\lambda_\alpha \Gamma(1-\beta)}{\Gamma(2(1-\beta))\Gamma(\beta)2^{1-2\beta}}. \tag{49}$$

It follows that, the mechanical picture of a viscous flow in a porous media with non-homogeneous mechanical features varying with power-laws correspond, exactly, to a force-flux relation in terms of Caputo' fractional derivatives and integrals. (see Fig. 2)

The mechanical model introduced in Fig. 3 corresponds, to values of the fractional differentiation order $\beta \in [-1, 1]$ yielding a decaying exponent of diffusivity and the compressibility coefficient $\alpha \in [-3, 1]$ (see Eq. (43)).

Boundaries of exponent $\alpha$ have been introduced for two mechanical reasons: (i) values of $\alpha > 1$ cannot be accepted since, in this case $\beta > 1$ that is the value corresponding to a time integration of order one of the flux across the control section, that represents a displacement from a physical perspective; (ii) Values of the exponent $\alpha < -3$ cannot be accepted since they corresponds to $\beta \leqslant -1$ that is the value corresponding to force-flux relations involving order of time differentiation of the velocity field higher than that corresponding to inertial forces and, therefore, not acceptable in a physical framework. We conclude that, as we consider two different variation ranges of the differentiation order $\beta$ as $0 \leqslant \beta \leq 1$ and $-1 \leqslant \beta \leq 0$ we are dealing with two different macroscopic behaviors. In the former case, we are dealing with a fluid with a constitutive equation interpolating among a purely viscous and a purely elastic behavior and, therefore we may define it as a ViscoElastic (VE) fluid. In the latter case, instead, we are considering a fluid with an intermediate behavior among a pure inertial and pure viscous case, and, therefore, we define it as a ViscoInertial (VI) fluid.

The derivations of the fractional-order generalization of Darcy diffusion equations presented up to this point involves mass transport across an unbounded porous media. However, real-type transport phenomena occur, usually, in bounded models so that, as general comment a fractional generalization of the transport equation is obtained. In this case it may be expected that the outgoing flux $\bar{q}_0(t)$ follows with a good approximation a power-law $\propto t^\beta$ up to a cut-off time $\bar{t}$. For $t \geqslant \bar{t}$ the outgoing flux $\bar{q}_0(t)$ reaches a stationary value depending upon the physical and geometrical properties of the porous media as well as on the rheological properties of the circulating fluid. Such a behavior have already been observed for the rheological models of power-law memory functions involved in fractional-order hereditariness (see [22,25–27]).

## 5. Conclusions

In this paper the authors discuss the origin of anomalous fluid diffusion across a porous media. In this regard it has been observed that the presence of an 1D mass transport in a unbounded porous media and subjected to an uniform pressure in the control section yields a relation among the applied pressure field and the ingoing flux in terms of a fractional-order derivative of order 1/2. As the physical and geometrical coefficients of the porous solid vary with power law, a generalized relation among flux and pressure in terms of fractional-order integrals and derivatives is obtained. This result represent a first step toward the use of fractional-order calculus for the mechanical description of anomalous diffusion in biological environment. Indeed, it is well-known that diffusion across cell membrane, as well as across perfused tissues as liver, rines as well as pancreatic tissue deviates from predictions of Darcy/Fick relations and more realistic relations have been reported in terms of extension of classical transport relations involving non-linear terms. In this paper those deviations have been predicted in terms of the fractional-order calculus introducing a fluid-mechanics description of the generalized Darcy diffusion equation.

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