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Asymptotics for Graded Capelli Polynomials

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Abstract The finite dimensional simple superalgebras play an important role in the theory of PI-algebras in characteristic zero. The main goal of this paper is to characterize the T_2-ideal of graded identities of any such algebra by considering the growth of the corresponding supervariety. We consider the T_2-ideal $\Gamma_{M+1,L+1}$ generated by the graded Capelli polynomials $Cap_{M+1}[Y, X]$ and $Cap_{L+1}[Z, X]$ alternanting on $M + 1$ even variables and $L + 1$ odd variables, respectively. We prove that the graded codimensions of a simple finite dimensional superalgebra are asymptotically equal to the graded codimensions of the T_2-ideal $\Gamma_{M+1,L+1}$, for some fixed natural numbers M and L. In particular

$$c_n^{sup}(\Gamma_{k^2+l^2+1,2kl+1}) \simeq c_n^{sup}(M_{k,l}(F))$$

and

$$c_n^{sup}(\Gamma_{s^2+1,s^2+1}) \simeq c_n^{sup}(M_s(F \oplus tF)).$$

These results extend to finite dimensional superalgebras a theorem of Giambruno and Zaicev [6] giving in the ordinary case the asymptotic equality

$$c_n^{sup}(\Gamma_{k^2+1,1}) \simeq c_n^{sup}(M_k(F))$$

between the codimensions of the Capelli polynomials and the codimensions of the matrix algebra $M_k(F)$.

Keywords Superalgebras · Polynomial identities · Codimensions · Growth

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1 Introduction

Let F be a field of characteristic zero, $X = \{x_1, x_2, \ldots\}$ a countable set and $F\langle X\rangle = F\langle x_1, x_2, \ldots\rangle$ the free associative algebra on X over F. Recall that an algebra A is a superalgebra (or $\mathbb{Z}_2$-graded algebra) with grading $(A^{(0)}, A^{(1)})$ if $A = A^{(0)} \oplus A^{(1)}$, where $A^{(0)}, A^{(1)}$ are subspaces of A satisfying:

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \text{ and } A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

If we write $X = Y \cup Z$ as the disjoint union of two countable sets, then $F\langle X\rangle = F\langle Y \cup Z\rangle$ has a natural structure of free superalgebra if we require that the variables from Y have degree zero and the variables from Z have degree one.

Recall that an element $f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ of $F\langle Y \cup Z\rangle$ is a graded identity or superidentity for A if $f(a_1, \ldots, a_n, b_1, \ldots, b_m) = 0$, for all $a_1, \ldots, a_n \in A^{(0)}$ and $b_1, \ldots, b_m \in A^{(1)}$. The set $Id^{sup}(A)$ of all graded identities of A is a T_2-ideal of $F\langle Y \cup Z\rangle$ i.e., an ideal invariant under all endomorphisms of $F\langle Y \cup Z\rangle$ preserving the grading. Moreover, every T_2-ideal Γ of $F\langle Y \cup Z\rangle$ is the ideal of graded identities of some superalgebra $A = A^{(0)} \oplus A^{(1)}$, $\Gamma = Id^{sup}(A)$.

For $\Gamma = Id^{sup}(A)$ a T_2-ideal of $F\langle Y \cup Z\rangle$, we denote by supvar(Γ) or supvar(A) the supervariety of superalgebras having the elements of Γ as graded identities.

As it was shown by Kemer (see [8, 9]), superalgebras and their graded identities play a basic role in the study of the structure of varieties of associative algebras over a field of characteristic zero. More precisely, Kemer showed that any variety is generated by the Grassmann envelope of a suitable finite dimensional superalgebra (see also Theorem 3.7.8 [7]).

Recall that, if F is an algebraically closed field of characteristic zero, then a simple finite dimensional superalgebra over F is isomorphic to one of the following algebras (see [9], [7]):

1. $M_k(F)$ with trivial grading $(M_k(F), 0)$;
2. $M_{k,l}(F)$ with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}\right)$, where $F_{11}, F_{12}, F_{21}, F_{22}$ are $k \times k$, $k \times l$, $l \times k$ and $l \times l$ matrices respectively, $k \geq 1$ and $l \geq 1$;
3. $M_s(F \oplus tF)$ with grading $(M_s(F), tM_s(F))$, where $t^2 = 1$.

Thus an important problem in the theory of PI-algebras is to describe the T_2-ideals of graded identities of these simple finite dimensional superalgebra, $Id^{sup}(M_k(F))$, $Id^{sup}(M_{k,l}(F))$, $Id^{sup}(M_s(F \oplus tF))$.

In case char$F$=0, it is well known that $Id^{sup}(A)$ is completely determined by its multilinear polynomials and an approach to the description of the graded identities of A is based on the study of the graded codimension sequence of this superalgebra.

If V_n^{sup} denotes the space of multilinear polynomials of degree n in the variables $y_1, z_1, \ldots, y_n, z_n$ (i.e., y_i or z_i appears in each monomial at degree 1), then the sequence of spaces $\{V_n^{sup} \cap Id^{sup}(A)\}_{n\geq 1}$ determines $Id^{sup}(A)$ and

$$c_n^{sup}(A) = \dim_F \left(\frac{V_n^{sup}}{V_n^{sup} \cap Id^{sup}(A)}\right)$$

is called the n-th graded codimension of A.

The asymptotic behaviour of the graded codimensions plays an important role in the PI-theory of graded algebras. It was shown in [4] that the sequence $\{c_n^{sup}(A)\}_{n\geq 1}$ is exponentially bounded if and only if A satisfies an ordinary polynomial identity.

In [2] it was proved that if A is a finitely generated superalgebra satisfying a polynomial identity, then $\lim_{n\to\infty} \sqrt[n]{c_n^{sup}(A)}$ exists and is a non negative integer. It is called superexponent (or $\mathbb{Z}_2$-exponent) of A and it is denoted by

$$supexp(A) = \lim_{n\to\infty} \sqrt[n]{c_n^{sup}(A)}.$$

Now, if $f \in F\langle Y \cup Z\rangle$ we denote by $\langle f\rangle_{T_2}$ the T_2-ideal generated by f. Also for a set of polynomials $V \subset F\langle Y \cup Z\rangle$ we write $\langle V\rangle_{T_2}$ to indicate the T_2-ideal generated by V.

In PI-theory a prominent role is played by the Capelli polynomial. If S_m is the symmetric group on $\{1,\dots,m\}$, the polynomial

$$Cap_m[T,X] = Cap_m(t_1,\dots,t_m;x_1,\dots,x_{m-1}) =$$
$$= \sum_{\sigma\in S_m} (\text{sgn}\sigma) t_{\sigma(1)}x_1t_{\sigma(2)}\cdots t_{\sigma(m-1)}x_{m-1}t_{\sigma(m)}$$

is the m-th graded Capelli polynomial in the homogeneous variables $t_1,\dots,t_m$ ($x_1,\dots,x_{m-1}$ are arbitrary variables). In particular $Cap_m[Y,X]$ and $Cap_m[Z,X]$ denote the m-th graded Capelli polynomial in the alternanting variables of homogeneous degree zero $y_1,\dots,y_m$ and of homogeneous degree one $z_1,\dots,z_m$, respectively.

Let Cap_m^0 denote the set of 2^{m-1} polynomials obtained from $Cap_m[Y,X]$ by deleting any subset of variables x_i (by evalauting the variables x_i to 1 in all possible way). Similarly, we define by Cap_m^1 the set of 2^{m-1} polynomials obtained from $Cap_m[Z,X]$ by deleting any subset of variables x_i.

If L and M are two natural numbers, we denote the T_2-ideal generated by the polynomials Cap_{M+1}^0, Cap_{L+1}^1 by $\Gamma_{M+1,L+1} = \langle Cap_{M+1}^0, Cap_{L+1}^1\rangle_{T_2}$. We also write $\mathcal{U}_{M+1,L+1}^{sup} = \text{supvar}(\Gamma_{M+1,L+1})$. In [1] it was shown that

$$(M+L)-10 \le \text{supexp}(\mathcal{U}_{M+1,L+1}^{\text{sup}}) \le (M+L).$$

The following relations between the superexponent of the graded Capelli polynomials and the superexponent of the simple finite dimensional superalgebras are well known (see [1, 2, 5])

$$\text{supexp}(\mathcal{U}_{k^2+1,1}^{\text{sup}}) = k^2 = supexp(M_k(F))$$
$$\text{supexp}(\mathcal{U}_{k^2+l^2+1,2kl+1}^{\text{sup}}) = (k+l)^2 = supexp(M_{k,l}(F))$$
$$\text{supexp}(\mathcal{U}_{s^2+1,s^2+1}^{\text{sup}}) = 2s^2 = supexp(M_s(F\oplus tF)).$$

In this paper we try to find a close relation among the asymptotics of $\mathcal{U}_{k^2+l^2+1,2kl+1}^{sup}$ and $M_{k,l}(F)$ and the asymptotics of $\mathcal{U}_{s^2,s^2}^{sup}$ and $M_s(F\oplus tF)$. Recall that two sequences a_n, b_n, $n=1,2,\dots$, are asymptotically equal, $a_n \simeq b_n$, if

$$\lim_{n\to+\infty} \frac{a_n}{b_n} = 1.$$

This paper was inspired by the ordinary case (see [6]) where Giambruno and Zaicev proved that

$$c_n^{sup}(\Gamma_{k^2+1,1}) \simeq c_n^{sup}(M_k(F)).$$

Here we show that

$$\Gamma_{k^2+l^2+1,2kl+1} = Id^{sup}(M_{k,l}(F)\oplus D')$$

and

$$\Gamma_{s^2+1,s^2+1} = Id^{sup}(M_s(F\oplus tF)\oplus D)$$

where D' and D are finite dimensional superalgebra with $supexp(D') < (k+l)^2$ and $supexp(D) < 2s^2$. It follows that asymptotically

$$c_n^{sup}(\Gamma_{k^2+l^2+1,2kl+1}) \simeq c_n^{sup}(M_{k,l}(F))$$

and

$$c_n^{sup}(\Gamma_{s^2+1,s^2+1}) \simeq c_n^{sup}(M_s(F \oplus tF)).$$

2 Preliminaries

In [6, Definition 1] the notion of reduced superalgebra was introduced: let $A = A_1 \oplus \cdots \oplus A_r + J$ be a finite dimensional superalgebra where $A_1, \ldots, A_r$ are simple superalgebras and $J = J(A)$ is the Jacobson radical of A, A is called reduced if $A_1 J A_2 J \cdots J A_r \neq 0$. Giambruno and Zaicev (see [6]) showed, also, that these superalgebras can be used as building blocks of any proper variety. Here, in the next theorem, we obtain an analogous result for supervarieties generated by a finitely generated superalgebra. We first prove a lemma that will be used throughout the paper.

Lemma 1 *Let A and B be PI-superalgebras. Then*

$$c_n^{sup}(A), c_n^{sup}(B) \leq c_n^{sup}(A \oplus B) \leq c_n^{sup}(A) + c_n^{sup}(B).$$

Proof The proof of this result is the same of the proof of the [6, Lemma 1]. □

Corollary 2 *If A and B are finitely generated superalgebras, then*

$$supexp(A \oplus B) = max\{supexp(A), supexp(B)\}.$$

Theorem 3 *Let A be a finitely generated superalgebra satisfying an ordinary polynomial identity. Then there exists a finite number of reduced superalgebras $B_1, \ldots, B_t$ and a finite dimensional superalgebra D such that*

$$supvar(A) = supvar(B_1 \oplus \cdots \oplus B_t \oplus D)$$

with $supexp(A) = supexp(B_1) = \cdots = supexp(B_t)$ and $supexp(D) < supexp(A)$.

Proof The proof follows closely the proof given in [6, Theorem 1]. Let A be a finitely generated superalgebra satisfying an ordinary polynomial identity. By a theorem of Kemer (see [9, Theorem 2.2]), there exists a finite dimensional superalgebra B such that $Id^{sup}(A) = Id^{sup}(B)$. Therefore throughout we may assume that $A = A^{(0)} \oplus A^{(1)}$ is a finite dimensional superalgebra over F satisfying an ordinary polynomial identity and char$F = 0$. Also, by [9, pag. 21] we may assume that $A = A_1 \oplus \cdots \oplus A_s + J$, where $A_1, \ldots, A_s$ are simple superalgebras and $J = J(A)$ is the Jacobson radical of A. It is well known that $J = J^{(0)} \oplus J^{(1)}$ is a homogeneous ideal. Let $supexp(A) = d$. Then, as it was shown in [2], there exist distinct simple superalgebras $A_{j_1}, \ldots A_{j_k}$ such that

$$A_{j_1} J \cdots J A_{j_k} \neq 0 \quad and \quad dim_F(A_{j_1} \oplus \cdots \oplus A_{j_k}) = d.$$

Let $\Gamma_1, \ldots, \Gamma_t$ be all possible subset of $\{1, \ldots, s\}$ such that, if $\Gamma_j = \{j_1, \ldots, j_k\}$, then $dim_F(A_{j_1} \oplus \cdots \oplus A_{j_k}) = d$ and $A_{\sigma(j_1)} J \cdots J A_{\sigma(j_k)} \neq 0$ for some permutation $\sigma \in S_k$.

Hence we put $B_j = A_{j_1} \oplus \cdots \oplus A_{j_k} + J$ for any Γ_j, $j = 1, \ldots, t$. It follows, by the characterization of the superexponent, that

$$supexp(B_1) = \cdots = supexp(B_t) = d = supexp(A).$$

Let $D = D_1 \oplus \cdots \oplus D_p$, where $D_1, \ldots, D_p$ are all subsuperalgebras of A of the type $A_{i_1} \oplus \cdots \oplus A_{i_r} + J$, with $1 \leq i_1 < \cdots < i_r \leq s$ and $dim_F(A_{i_1} \oplus \cdots \oplus A_{i_r}) < d$. Then, by Corollary 2, we have $supexp(D) < supexp(A)$. Now, we want to prove that $supvar(B_1 \oplus \cdots \oplus B_t \oplus D) = supvar(A)$. The inclusion

$$supvar(B_1 \oplus \cdots \oplus B_t \oplus D) \subseteq supvar(A)$$

follows from $D, B_i \in supvar(A)$, $\forall i = 1, \ldots, t$. Now, let $f = f(y_1, \ldots, y_n, z_1, \ldots, z_m)$ be a multilinear polynomial such that $f \notin Id^{sup}(A)$. We shall prove that $f \notin Id^{sup}(B_1 \oplus \cdots \oplus B_t \oplus D)$. There exist $a_1, \ldots, a_n \in A^0$ and $b_1, \ldots, b_m \in A^1$ such that

$$f(a_1, \ldots, a_n, b_1, \ldots, b_m) \neq 0.$$

From the linearity of f we can put $a_1, \ldots, a_n \in A_1^0 \cup \cdots \cup A_s^0 \cup J^0$ and $b_1, \ldots, b_n \in A_1^1 \cup \cdots \cup A_s^1 \cup J^1$. Since $A_i A_j = 0$ for $i \neq j$, from the property of d we have

$$a_1, \ldots, a_n, b_1, \ldots, b_m \in A_{j_1} \oplus \cdots \oplus A_{j_k} + J$$

for some $A_{j_1}, \ldots, A_{j_k}$ such that $dim_F(A_{j_1} \oplus \cdots \oplus A_{j_k}) \leq d$. Thus f is not an identity for one of the algebras $B_1, \ldots, B_t, D$. Hence $f \notin Id^{sup}(B_1 \oplus \cdots \oplus B_t \oplus D)$. In conclusion

$$supvar(A) \subseteq supvar(B_1 \oplus \cdots \oplus B_t \oplus D)$$

and the proof is complete. □

Corollary 4 *Let A be a finitely generated superalgebra satisfying an ordinary polynomial identity. Then there exists a finite number of reduced superalgebras $B_1, \ldots, B_t$ such that*

$$c_n^{sup}(A) \simeq c_n^{sup}(B_1 \oplus \cdots \oplus B_t)$$

Proof By Theorem 3 there is a finite number of reduced superalgebras $B_1, \ldots, B_t$ such that

$$supvar(A) = supvar(B_1 \oplus \cdots \oplus B_t \oplus D)$$

with $supexp(A) = supexp(B_1) = \cdots = supexp(B_t)$ and $supexp(D) < supexp(A)$. Then, by Lemma 1

$$c_n^{sup}(B_1 \oplus \cdots \oplus B_t) \leq c_n^{sup}(B_1 \oplus \cdots \oplus B_t \oplus D) \leq c_n^{sup}(B_1 \oplus \cdots \oplus B_t) + c_n^{sup}(D).$$

Recalling that $supexp(D) < supexp(B_1) = supexp(B_1 \oplus \cdots \oplus B_t)$ we have that

$$c_n^{sup}(A) \simeq c_n^{sup}(B_1 \oplus \cdots \oplus B_t)$$

and the proof of the corollary is complete. □

3 Asymptotics for $\mathcal{U}^{sup}_{k^2+l^2+1,2kl+1}$ and $M_{k,l}(F)$

3.1 Evaluating polynomials

Throughout this section we assume that $A = M_{k,l}(F) + J$, where $J = J(A)$ is the Jacobson radical of the finite dimensional superalgebra A. Notice that $M_{k,l}(F)$ contains the unit and it belongs to the even part in the grading. It is also known that J is homogeneous under the grading of A [9]. We start with the following key lemmas.

Lemma 5 [3, Lemma 1] *The Jacobson radical J can be decomposed into the direct sum of four $M_{k,l}(F)$-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0, 1\}$, J_{pq} is a left faithful module or a 0-left module according to $p = 1$, or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according to $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0, 1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent superalgebra N such that $J_{11} \cong M_{k,l}(F) \otimes_F N$ (isomorphism of $M_{k,l}(F)$-bimodules and of superalgebras).

Lemma 6 *Let $M = k^2 + l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$, $k > l > 0$. Then the superalgebra $M_{k,l}(F)$ does not satisfy the graded Capelli polynomials $Cap_M[Y, X]$ and $Cap_L[Z, X]$.*

Proof In order to prove that $Cap_M[Y, X]$ does not vanish on $M_{k,l}(F)$ we need to find a non-zero valuation. Let $e^0_1, \ldots, e^0_M$ be a basis of $M_{k,l}(F)^{(0)}$ consisting of matrix units, $e^0_h \in \{e_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq k\} \cup \{e_{i,j} \mid k+1 \leq i \leq k+l, k+1 \leq j \leq k+l\}$. Then we evaluate

$$a_0 Cap_M(e^0_1, \ldots, e^0_M; a_1, \ldots, a_{M-1})a_M = e_{1,1} \neq 0,$$

where $a_0 = e_{1,1}$, $a_M = e_{k+l,1}$, for $y_1, \ldots, y_M$ we substituted all the e^0_h ordered according to the left lexicographic order of the indices and for all x_i's we made the unique substitution of elements of $M_{k,l}(F)$ making $y_1x_1y_2x_2 \cdots y_{M-1}x_{M-1}y_M$ the only monomial with non-zero evaluation, i.e. $a_1 = e_{1,1}, a_2 = e_{2,1}, \cdots, a_{M-1} = e_{k+l-1,k+l}$.

Now, we want to show that $Cap_L[Z, X]$ does not vanish in $M_{k,l}(F)$. Let $e^1_1, \ldots, e^1_L$ be a basis of $M_{k,l}(F)^{(1)}$ consisting of matrix units from the odd part of $M_{k,l}(F)$, $e^1_h \in \{e_{i,j} \mid 1 \leq i \leq k, k+1 \leq j \leq k+l\} \cup \{e_{i,j} \mid k+1 \leq i \leq k+l, 1 \leq j \leq k\}$. Then we evaluate

$$b_0 Cap_L(e^1_1, \ldots, e^1_L; b_1, \ldots, b_{L-1})b_L = e_{1,k+l} \neq 0,$$

where $b_0 = e_{1,1}$, $b_L = e_{k,k+l}$, for $z_1, \ldots, z_L$ we substituted all the e^1_h ordered according to the left lexicographic order of the indices and for all x_i's we made the unique substitution of elements of $M_{k,l}(F)$ making $z_1x_1z_2x_2 \cdots z_{L-1}x_{L-1}z_L$ the only monomial with non-zero evaluation. □

Lemma 7 *Let $M = k^2 + l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$, $k > l > 0$. If $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$, then $J_{10} = J_{01} = (0)$.*

Proof By Lemma 6, $M_{k,l}(F)$ does not satisfy the graded Capelli polynomial $Cap_M[Y, X]$. Then, there exist elements $a^{(0)}_1, \ldots, a^{(0)}_M \in M_{k,l}(F)^{(0)}$ and $b_1, \ldots, b_{M-1} \in M_{k,l}(F)$ such that

$$Cap_M(a^{(0)}_1, \ldots, a^{(0)}_M; b_1, \ldots, b_{M-1}) = e_{1,k+l},$$

where the $e_{i,j}$'s are the usual matrix units. We write $J_{10} = J^{(0)}_{10} \oplus J^{(1)}_{10}$ and $J_{01} = J^{(0)}_{01} \oplus J^{(1)}_{01}$. Now, we consider $d^{(0)} \in J^{(0)}_{10}$. Since $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$ we have

$$0 = Cap_{M+1}(a^{(0)}_1, \ldots, a^{(0)}_M, d^{(0)}; b_1, \ldots, b_{M-1}, e_{k+l,k+l}) = e_{1,k+l}d^{(0)}.$$

Hence $e_{1,k+l}d^{(0)} = 0$ and, so, $d^{(0)} = 0$, for all $d^{(0)} \in J_{10}^{(0)}$. Thus $J_{10}^{(0)} = (0)$. Analogously $J_{01}^{(0)} = (0)$.

Now, by a similar proof, we want to show that $J_{10}^{(1)} = (0)$ and $J_{01}^{(1)} = (0)$. By Lemma 6, the graded Capelli polynomial $Cap_L[Z, X]$ does not vanish in $M_{k,l}(F)$. Thus we can choose a suitable substitution $a_1^{(1)}, \dots, a_L^{(1)} \in M_{k,l}(F)^{(1)}$ and $c_1, \dots, c_{L-1} \in M_{k,l}(F)$ such that

$$Cap_L(a_1^{(1)}, \dots, a_L^{(1)}; c_1, \dots, c_{L-1}) = e_{1,l}.$$

Now, we compute

$$Cap_{L+1}(a_1^{(1)}, \dots, a_L^{(1)}, d^{(1)}; c_1, \dots, c_{L-1}, e_{l,l}) = e_{1,l}d^{(1)},$$

where $d^{(1)} \in J_{10}^{(1)}$. Since $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$ we have $e_{1,l}d^{(1)} = 0$. Thus $d^{(1)} = 0$, for all $d^{(1)} \in J_{10}^{(1)}$ and then $J_{10}^{(1)} = (0)$. Analogously $J_{01}^{(1)} = (0)$ and the lemma is proved. □

Lemma 8 *Let $M = k^2+l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$, $k > l > 0$. Let $J_{11} \cong M_{k,l}(F) \otimes_F N$, where $N = N^{(0)} \oplus N^{(1)}$, as in Lemma 5. If $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$, then N is commutative.*

Proof Let $e_1^0, \dots, e_M^0$ be an ordered basis of $M_{k,l}(F)^{(0)}$ consisting of matrix units, $e_h^0 \in \{e_{i,j} \mid 1 \leq i \leq k, 1 \leq j \leq k\} \cup \{e_{i,j} \mid k+1 \leq i \leq k+l, k+1 \leq j \leq k+l\}$ such that $e_1^0 = e_{1,1}$ and let $a_0, a_1, \dots, a_M \in M_{k,l}(F)$ be such that

$$a_0 e_1^0 a_1 \cdots a_{M-1} e_M^0 a_M = e_{1,1}$$

and

$$a_0 e_{\sigma(1)}^0 a_1 \cdots a_{M-1} e_{\sigma(M)}^0 a_M = 0$$

for any $\sigma \in S_M$, $\sigma \neq 1$. Consider $d_1^{(0)}, d_2^{(0)} \in N^{(0)}$ and set $\bar{d}_1^{(0)} = e_{1,1}d_1^{(0)}$ and $\bar{d}_2^{(0)} = e_{1,1}d_2^{(0)}$. Notice that $\bar{d}_1^{(0)}, \bar{d}_2^{(0)} \in A^{(0)}$. Now, recalling that N commutes with $M_{k,l}(F)$, we compute

$$\begin{gathered} Cap_{M+2}(\bar{d}_1^{(0)}, e_1^0, \dots, e_M^0, \bar{d}_2^{(0)}; a_0, \dots, a_M) = \\ \bar{d}_1^{(0)} e_{1,1} \bar{d}_2^{(0)} - \bar{d}_2^{(0)} e_{1,1} \bar{d}_1^{(0)} - e_{1,1} \bar{d}_1^{(0)} \bar{d}_2^{(0)} + \\ \bar{d}_2^{(0)} \bar{d}_1^{(0)} e_{1,1} + e_{1,1} \bar{d}_2^{(0)} \bar{d}_1^{(0)} - \bar{d}_1^{(0)} \bar{d}_2^{(0)} e_{1,1} = \\ [\bar{d}_2^{(0)}, \bar{d}_1^{(0)}] e_{1,1} = [d_2^{(0)}, d_1^{(0)}] e_{1,1}. \end{gathered}$$

Since $Cap_{M+1}[Y; X] \subseteq Id^{sup}(A)$ we have $[d_1^{(0)}, d_2^{(0)}] = 0$. Thus $d_1^{(0)} d_2^{(0)} = d_2^{(0)} d_1^{(0)}$, for all $d_1^{(0)}, d_2^{(0)} \in N^{(0)}$.

Now, let $e_1^1, \dots, e_L^1$ be an ordered basis of $M_{k,l}(F)^{(1)}$ consisting of matrix units from the odd part of $M_{k,l}(F)$, $e_h^1 \in \{e_{i,j} \mid 1 \leq i \leq k, k+1 \leq j \leq k+l\} \cup \{e_{i,j} \mid k+1 \leq i \leq k+l, 1 \leq j \leq k\}$ such that $e_1^1 = e_{1,k+1}$. We consider $b_0, \dots, b_L \in M_{k,l}(F)$ such that

$$b_0 e_1^1 b_1 \cdots b_{L-1} e_L^1 b_L = e_{1,1}$$

and

$$b_0 e_{\tau(1)}^1 b_1 \cdots b_{L-1} e_{\tau(L)}^1 b_L = 0$$

for all $\tau \in S_L$, $\tau \neq 1$. Let $d_1^{(1)}, d_2^{(1)} \in N^{(1)}$. We set $\bar{d}_1^{(1)} = e_{1,1}d_1^{(1)}$, $\bar{d}_2^{(1)} = e_{1,1}d_2^{(1)} \in A^{(1)}$. As above we compute

$$\begin{gathered} Cap_{L+2}(\bar{d}_1^{(1)}, e_1^1, \dots, e_L^1, \bar{d}_2^{(1)}; b_0, \dots, b_L) = \\ \bar{d}_1^{(1)} e_{1,1} \bar{d}_2^{(1)} - \bar{d}_2^{(1)} e_{1,1} \bar{d}_1^{(1)} = [d_1^{(1)}, d_2^{(1)}] e_{1,1}. \end{gathered}$$

Since $Cap_{L+1}[Z; X] \subseteq Id^{sup}(A)$ we get that $d_1^{(1)} d_2^{(1)} = d_2^{(1)} d_1^{(1)}$, for all $d_1^{(1)}, d_2^{(1)} \in N^{(1)}$.

Next we show that $N^{(0)}$ commutes with $N^{(1)}$. Take $e_1^0, \dots, e_M^0$ an ordered basis of $M_{k,l}(F)^{(0)}$ consisting of matrix units such that $e_1^0 = e_{1,1}$ and let $a_1, \dots, a_M \in M_{k,l}(F)$ be such that

$$a_1 e_1^0 a_2 \cdots a_M e_M^0 = e_{1,k+1}$$

and

$$a_1 e_{\rho(1)}^0 a_2 \cdots a_M e_{\rho(M)}^0 = 0$$

for any $\rho \in S_M$, $\rho \neq 1$. Notice that $a_2 = e_{1,1}$. Let $d_1^{(0)} \in N^{(0)}$ and $d_2^{(1)} \in N^{(1)}$. We set $\bar{d}_1^{(0)} = e_{2,1} d_1^{(0)}$ and $\bar{a}_2 = e_{1,1} d_2^{(1)}$. Notice that $\bar{d}_1^{(0)} \in A^{(0)}$. Then, since $Cap_{M+1}[Y; X] \subseteq Id^{sup}(A)$, we obtained

$$0 = Cap_{M+1}(\bar{d}_1^{(0)}, e_1^0, \dots, e_M^0; a_1, \bar{a}_2, a_3, \dots, a_M) =$$

$$[d_1^{(0)}, d_2^{(1)}] e_{2,k+l}.$$

Thus $d_1^{(0)} d_2^{(1)} = d_2^{(1)} d_1^{(0)}$, for all $d_1^{(0)} \in N^{(0)}$, $d_2^{(1)} \in N^{(1)}$ and we are done. □

3.2 The main result for $M_{k,l}(F)$

In this section we prove our main result about the T_2-ideal $\Gamma_{k^2+l^2+1,2kl+1}$ generated by the graded Capelli polynomials $Cap_{k^2+l^2+1}[Y, X]$, $Cap_{2kl+1}[Z, X]$ and the T_2-ideal $Id^{sup}(M_{k,l}(F))$.

Theorem 9 *Let $M = k^2 + l^2$ and $L = 2kl$ with $k, l \in \mathbb{N}$, $k > l > 0$. Then $\mathcal{U}^{sup}_{M+1,L+1} = supvar(\Gamma_{M+1,L+1}) = supvar(M_{k,l}(F) \oplus D)$, where D' is a finite dimensional superalgebra such that $supexp(D') < M + L$. In particular*

$$c_n^{sup}(\Gamma_{M+1,L+1}) \simeq c_n^{sup}(M_{k,l}(F)).$$

Proof In [1] we proved that

$$supexp(\mathcal{U}^{sup}_{M+1,L+1}) = M + L.$$

Moreover, by [7, Lemma 11.4.1] and [7, Theorem 11.4.3], there exists a finitely generated superalgebra satisfying an ordinary polynomial identity A such that $\mathcal{U}^{sup}_{M+1,L+1} = supvar(A)$. Thus, by Theorem 3, there exists a finite number of superalgebras $B_1, \dots, B_s, D$ such that

$$\mathcal{U}^{sup}_{M+1,L+1} = supvar(A) = supvar(B_1 \oplus \cdots \oplus B_s \oplus D) \tag{1}$$

where $B_1, \dots, B_s$ are reduced, D is finite dimensional with $supexp(B_1) = \cdots = supexp(B_s) = supexp(\mathcal{U}^{sup}_{M+1,L+1}) = M + L$ and $supexp(D) < supexp(\mathcal{U}^{sup}_{M+1,L+1}) = M + L$. By a theorem of Kemer (see [9, Theorem 2.2]), if A is a finitely generated superalgebra, then there exists a finite dimensional superalgebra C such that $Id^{sup}(A) = Id^{sup}(C)$. Therefore throughout we may assume that $A = A^{(0)} \oplus A^{(1)}$ is a finite dimensional superalgebra over F satisfying an ordinary polynomial identity and char$F = 0$. Next, we analyze the structure of a finite dimensional reduced superalgebra R which satisfies $\Gamma_{M+1,L+1}$.

Let R be a finite dimensional reduced superalgebra such that $\text{supexp}(R) = M + L = \text{supexp}(\mathcal{U}^{sup}_{M+1,L+1})$ and $\Gamma_{M+1,L+1} \subseteq Id^{sup}(R)$. We can write $R = R_1 \oplus \cdots \oplus R_q + J$, where R_i are simple subsuperalgebras of R and $J = J(R)$ is the Jacobson radical of R.

Recall that a simple finite dimensional superalgebra R_i over F is isomorphic to one of the following algebras (see [9]):

1. $M_{d_i}(F)$, with trivial grading $(M_{d_i}(F), 0)$;
2. $M_{s_i}(F \oplus tF)$, where $t^2 = 1$, with grading $(M_{s_i}(F), tM_{s_i}(F))$;
3. $M_{k_i,l_i}(F)$ with grading $\left(\begin{pmatrix} F_{11} & 0 \\ 0 & F_{22} \end{pmatrix}, \begin{pmatrix} 0 & F_{12} \\ F_{21} & 0 \end{pmatrix}\right)$, where F_{11}, F_{12}, F_{21}, F_{22} are $k_i \times k_i$, $k_i \times l_i$, $l_i \times k_i$ and $l_i \times l_i$ matrices respectively, $k_i > 0$ and $l_i > 0$.

Let t_1 be the number of superalgebras R_i of the first type, let t_2 be the number of superalgebras R_i of the second type and finally let t_3 be the number of R_i of the third type, with $t_1 + t_2 + t_3 = q$.

Since R is reduced, $R_1 J \cdots J R_q \neq 0$. Then, by [7, Lemma 8.1.4], there exists a minimal superalgebra $\overline{R}$ contained in R with semisimple part $R_1 \oplus \cdots \oplus R_q$. Hence R contains a superalgebra isomorphic to the upper block triangular matrix algebra

$$\begin{pmatrix} \overline{R}_1 & & * \\ 0 & \ddots & \\ \vdots & & \\ 0 & \ldots\ 0 & \overline{R}_q \end{pmatrix},$$

where $\overline{R}_i$ is one of the following: $M_{d_i}(F)$, $M_{s_i}(F \oplus tF)$, $M_{k_i,l_i}(F)$ and

$$(k+l)^2 = M + L = supexp(\overline{R}) = dim_F \overline{R}_1 + \cdots + dim_F \overline{R}_q =$$

$$= d_1^2 + \cdots + d_{t_1}^2 + 2s_1^2 + \cdots + 2s_{t_2}^2 + (k_1 + l_1)^2 + \cdots + (k_{t_3} + l_{t_3})^2.$$

If $t_1 \geq 1$, by [1, Lemma 1], $\overline{R}$ does not satisfy the polynomials $Cap_{d^0+t_2+t_3+\bar{t}_1}[Y; X]$ and $Cap_{d^1+t_2+t_3+\tilde{t}_1}[Z; X]$, where $d^j = dim_F(R_1 \oplus \cdots \oplus R_q)^{(j)}$, $d^0 + d^1 = dim_F(R_1 \oplus \cdots \oplus R_q) = supexp(\overline{R}) = M + L$ and $\bar{t}_1 + \tilde{t}_1 = t_1 - 1$. But $\overline{R}$ satisfies $Cap_{M+1}[Y; X]$ and $Cap_{L+1}[Z; X]$. Thus $d^0 + t_2 + t_3 + \bar{t}_1 \leq M$ and also $d^1 + t_2 + t_3 + \tilde{t}_1 \leq L$. Hence $d^0 + d^1 + 2t_2 + 2t_3 + \bar{t}_1 + \tilde{t}_1 \leq M + L$. Then, since $d^0 + d^1 = M + L$, $M + L \leq d^0 + d^1 + 2t_2 + 2t_3 + \bar{t}_1 + \tilde{t}_1 \leq M + L$, thus $d^0 + d^1 + 2t_2 + 2t_3 + \bar{t}_1 + \tilde{t}_1 = M + L$, then, since $d^0 + d^1 = M + L$, $2t_2 + 2t_3 + \bar{t}_1 + \tilde{t}_1 = 0$. Since $t_2, t_3, \bar{t}_1$ and $\tilde{t}_1$ are nonnegative integers, we have $t_2 = t_3 = \bar{t}_1 = \tilde{t}_1 = 0$. Then $t_1 - 1 = \bar{t}_1 + \tilde{t}_1 = 0$ and so $t_1 = 1$. In conclusion, if $t_1 \geq 1$, then $t_1 = 1$, $t_2 = 0$ and $t_3 = 0$. In this case $\overline{R} \simeq M_{d_i}(F) + J$. Hence $supexp(\overline{R}) = d_i^2$. Since $supexp(\overline{R}) = M + L = (k+l)^2$, we have that $d_i = k + l$. Then $\overline{R}$ satisfies $Cap_{M+1}[Y; X]$ and $Cap_{L+1}[Z; X]$ and $\overline{R}$ does not satisfy $Cap_{d^0}[Y; X]$ where $d^0 = dim_F(R_1 \oplus \cdots \oplus R_q)^0 = d_i^2$. It is impossible since $M + 1 = k^2 + l^2 + 1 < k^2 + l^2 + 2kl = (k+l)^2 = d_i^2$.

If $t_1 = 0$, then, also by [1, Lemma 1], $\overline{R}$ does not satisfy $Cap_{d^0+t_2+t_3-1}[Y; X]$ and $Cap_{d^1+t_2+t_3-1}[Z; X]$. But $\overline{R}$ satisfies $Cap_{M+1}[Y; X]$ and $Cap_{L+1}[Z; X]$. Then $d^0 + t_2 + t_3 - 1 \leq M$ and $d^1 + t_2 + t_3 - 1 \leq L$. Thus $d^0 + d^1 + 2t_2 + 2t_3 - 2 \leq M + L$. Since $d^0 + d^1 = M + L$, we have $M + L + 2(t_2 + t_3 - 1) \leq M + L$. Then $t_2 + t_3 - 1 \leq 0$. So $t_2 + t_3 \leq 1$. Since t_2 and t_3 are non negative integers we have only two possibilities $t_2 = 0$ and $t_3 = 1$ or $t_2 = 1$ and $t_3 = 0$. If $t_2 = 1$ and $t_3 = 0$, then $\overline{R} \simeq M_{s_i}(F \oplus tF) + J$. Thus

$(k+l)^2 = M + L = supexp(\overline{R}) = 2s_i^2$. It is impossible. Let $t_2 = 0$ and $t_3 = 1$. Then we have $k_i = k$, $l_i = l$ and

$$\overline{R} \simeq M_{k,l}(F) + J.$$

From Lemmas 5, 7, 8 we have

$$R \cong (M_{k,l}(F) + J_{11}) \oplus J_{00} \cong (M_{k,l}(F) \otimes N^{\sharp}) \oplus J_{00}.$$

where $N^{\sharp}$ is the algebra obtained from N by adjoining a unit element. Since $N^{\sharp}$ is commutative, it follows that $M_{k,l}(F) + J_{11}$ and $M_{k,l}(F)$ satisfy the same identities. Thus $supvar(R) = supvar(M_{k,l}(F) \oplus J_{00})$ with J_{00} a finite dimensional nilpotent algebra. Hence, recalling the decomposition given in (1), we get

$$\mathcal{U}^{sup}_{M+1,L+1} = supvar(\Gamma_{M+1,L+1}) = supvar(M_{k,l}(F) \oplus D'),$$

where D' is a finite dimensional superalgebra with $supexp(D') < M + L$. Then, from Corollary 4 we have

$$c_n^{sup}(\Gamma_{M+1,L+1}) \simeq c_n^{sup}(M_{k,l}(F))$$

and the theorem is proved. □

4 Asymptotics for $\mathcal{U}^{sup}_{s^2,s^2}$ and $M_s(F \oplus tF)$

4.1 Evaluating polynomials

Throughout this section we assume that $A = M_s(F \oplus tF) + J$, where $J = J(A)$ is the Jacobson radical of the finite dimensional superalgebra A. We start with the following lemma which establishes a result similar to Lemma 5 (see [3, Lemma 6]).

Lemma 10 *The Jacobson radical J can be decomposed into the direct sum of four $M_s(F \oplus tF)$-bimodules*

$$J = J_{00} \oplus J_{01} \oplus J_{10} \oplus J_{11}$$

where, for $p, q \in \{0, 1\}$, J_{pq} is a left faithful module or a 0-left module according as $p = 1$, or $p = 0$, respectively. Similarly, J_{pq} is a right faithful module or a 0-right module according as $q = 1$ or $q = 0$, respectively. Moreover, for $p, q, i, l \in \{0, 1\}$, $J_{pq}J_{ql} \subseteq J_{pl}$, $J_{pq}J_{il} = 0$ for $q \neq i$ and there exists a finite dimensional nilpotent superalgebra N such that $J_{11} \cong M_s(F \oplus tF) \otimes_F N$ (isomorphism of $M_s(F \oplus tF)$-bimodules and of superalgebras).

Lemma 11 *Let $M = L = s^2$ with $s \in \mathbb{N}$. Then the superalgebra $M_s(F \oplus tF)$ does not satisfy the graded Capelli polynomials $Cap_M[Y, X]$ and $Cap_L[Z, X]$.*

Proof As in Lemma 6, let $e_1^0, \ldots, e_M^0$ be a basis of $M_s(F \oplus tF)^{(0)}$ consisting of matrix units. Then

$$a_0 Cap_M(e_1^0, \ldots, e_M^0; a_1, \ldots, a_{M-1}) a_M = e_{1,1} \neq 0,$$

where $a_0 = e_{1,1}$, $a_M = e_{s,1}$, for $y_1, \ldots, y_M$ we substituted all the e_h^0 ordered according to the left lexicographic order of the indices and for all x_i's we made the unique substitution of elements of $M_s(F \oplus tF)^{(0)}$ making $y_1 x_1 y_2 x_2 \cdots y_{M-1} x_{M-1} y_M$ the only monomial with non-zero evaluation.

Now, let $e_1^1, \ldots, e_L^1$ be a basis of $M_s(F \oplus tF)^{(1)}$ consisting of matrix units from the odd part of $M_s(F \oplus tF)$. Then we evaluate

$$b_0 Cap_L(e_1^1, \ldots, e_L^1; b_1, \ldots, b_{L-1}) b_L = t^L e_{1,1} \neq 0,$$

where $b_0 = e_{1,1}$, $b_L = e_{s,1}$, for $z_1, \ldots, z_L$ we substituted all the e_h^1 ordered according to the left lexicographic order of the indices and for all x_i's we made the unique substitution of elements of $M_s(F \oplus tF)^{(0)}$ making $z_1 x_1 z_2 x_2 \cdots z_{L-1} x_{L-1} z_L$ the only monomial with non-zero evaluation. □

Lemma 12 *Let $M = L = s^2$ with $s \in \mathbb{N}$. If $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$, then $J_{10} = J_{01} = (0)$.*

Proof By Lemma 11, we have that $Cap_M[Y; X]$ is not an identity for $M_s(F \oplus tF)$. Hence, as in Lemma 7, there exist elements $a_1^{(0)}, \ldots, a_M^{(0)} \in M_s(F \oplus tF)^{(0)}$ and $b_1, \ldots, b_{M-1} \in M_s(F \oplus tF)^{(0)} \subseteq M_s(F \oplus tF)$ (see the proof of the Lemma 11) such that

$$Cap_M(a_1^{(0)}, \ldots, a_M^{(0)}; b_1, \ldots, b_{M-1}) = e_{s,s}.$$

We write $J_{10} = J_{10}^{(0)} \oplus J_{10}^{(1)}$ and $J_{01} = J_{01}^{(0)} \oplus J_{01}^{(1)}$. Let $d^{(0)} \in J_{10}^{(0)}$. Now, we compute

$$0 = Cap_{M+1}(a_1^{(0)}, \ldots, a_M^{(0)}, d^{(0)}; b_1, \ldots, b_{M-1}, e_{k+l,k+l}) = e_{s,s} d^{(0)}.$$

Since $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$, we have $e_{s,s} d^{(0)} = 0$. In particular $d^{(0)} = 0$, for all $d^{(0)} \in J_{10}^{(0)}$. Thus $J_{10}^{(0)} = (0)$. Similarly $J_{01}^{(0)} = (0)$.

Now, we want to prove that $J_{10}^{(1)} = (0)$ and $J_{01}^{(1)} = (0)$. By Lemma 11, $M_s(F \oplus tF)$ does not satisfy $Cap_L[Z; X]$, then we can take an opportune substitution $a_1^{(1)}, \ldots, a_L^{(1)} \in M_s(F \oplus tF)^{(1)}$ and $c_1, \ldots, c_{L-1} \in M_s(F \oplus tF)^{(0)} \subseteq M_s(F \oplus tF)$ (see the proof of the Lemma 11) such that

$$Cap_L(a_1^{(1)}, \ldots, a_L^{(1)}; c_1, \ldots, c_{L-1}) = t^L e_{s,s}.$$

By the hypothesis we obtain

$$Cap_{L+1}(a_1^{(1)}, \ldots, a_L^{(1)}, d^{(1)}; c_1, \ldots, c_{L-1}, e_{s,s}) = t^L e_{s,s} d^{(1)}.$$

where $d^{(1)} \in J_{10}^{(1)}$. Thus $t^L e_{s,s} d^{(1)} = 0$ and $d^{(1)} = 0$ for all $d^{(1)} \in J_{10}^{(1)}$. In conclusion $J_{10}^{(1)} = (0)$. Similarly $J_{01}^{(1)} = 0$ and the lemma is proved. □

Lemma 13 *Let $M = L = s^2$ with $s \in \mathbb{N}$. Let $J_{11} \cong M_s(F \oplus tF) \otimes N$, where $N = N^{(0)} \oplus N^{(1)}$, as in Lemma 10. If $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$, then N is commutative.*

Proof As in Lemma 8, we consider an ordered basis of $M_s(F \oplus tF)^{(0)}$ consisting of all matrix units $e_1^0, \ldots, e_M^0$ such that $e_1^0 = e_{1,1}$ and also we take $a_0, a_1, \ldots, a_M \in M_s(F \oplus tF)^{(0)} \subseteq M_s(F \oplus tF)$ such that

$$a_0 e_1^0 a_1 \cdots a_{M-1} e_M^0 a_M = e_{1,1}$$

and

$$a_0 e_{\sigma(1)}^0 a_1 \cdots a_{M-1} e_{\sigma(M)}^0 a_M = 0$$

for all $\sigma \in S_M$, $\sigma \neq 1$. Now, as in Lemma 8, we consider $d_1^{(0)}, d_2^{(0)} \in N^{(0)}$ and we take $\tilde{d}_1^{(0)} = e_{1,1} d_1^{(0)}$, $\tilde{d}_2^{(0)} = e_{1,1} d_2^{(0)}$. Notice that $\tilde{d}_1^{(0)}, \tilde{d}_2^{(0)} \in A^{(0)}$. Since N commutes with

$M_s(F \oplus tF)$ and $\Gamma_{M+1,L+1} \subseteq Id^{sup}(A)$, we have

$$0 = Cap_{M+2}(\widetilde{d}_1^{(0)}, e_1^0, \ldots, e_M^0, \widetilde{d}_2^{(0)}; a_0, \ldots, a_M) =$$
$$\widetilde{d}_1^{(0)} e_{1,1} \widetilde{d}_2^{(0)} - \widetilde{d}_2^{(0)} e_{1,1} \widetilde{d}_1^{(0)} - e_{1,1} \widetilde{d}_1^{(0)} \widetilde{d}_2^{(0)} +$$
$$\widetilde{d}_2^{(0)} \widetilde{d}_1^{(0)} e_{1,1} + e_{1,1} \widetilde{d}_2^{(0)} \widetilde{d}_1^{(0)} - \widetilde{d}_1^{(0)} \widetilde{d}_2^{(0)} e_{1,1} =$$
$$[\widetilde{d}_2^{(0)}, \widetilde{d}_1^{(0)}] e_{1,1} = [d_2^{(0)}, d_1^{(0)}] e_{1,1}.$$

Hence $[d_2^{(0)}, d_1^{(0)}] = 0$, for all $d_1^{(0)}, d_2^{(0)} \in N^{(0)}$.

Now, we take $d_1^{(1)}, d_2^{(1)} \in N^{(1)}$ and we put $\widetilde{d}_1^{(0)} = te_{1,1} d_1^{(1)}, \widetilde{d}_2^{(0)} = te_{1,1} d_2^{(1)}$. Notice that $\widetilde{d}_1^{(0)}, \widetilde{d}_2^{(0)} \in A^{(0)}$. Then, as above, we have

$$0 = Cap_{M+2}(\widetilde{d}_1^{(0)}, e_1^0, \ldots, e_M^0, \widetilde{d}_2^{(0)}; a_0, \ldots, a_M) =$$
$$[\widetilde{d}_2^{(0)}, \widetilde{d}_1^{(0)}] e_{1,1} = t^2 [d_2^{(1)}, d_1^{(1)}] e_{1,1} = [d_2^{(1)}, d_1^{(1)}].$$

In conclusion $[d_2^{(1)}, d_1^{(1)}] = 0$, for all $d_1^{(1)}, d_2^{(1)} \in N^{(1)}$.

Finally, we consider $d_1^{(0)} \in N^{(0)}$ and $d_2^{(1)} \in N^{(1)}$. Hence we take $\widetilde{d}_1^{(0)} = e_{1,1} d_1^{(0)}$ and $\widetilde{d}_2^{(0)} = te_{1,1} d_2^{(1)}$. Also in this case $\widetilde{d}_1^{(0)}, \widetilde{d}_2^{(0)} \in A^{(0)}$. Then

$$0 = Cap_{M+2}(\widetilde{d}_1^{(0)}, e_1^0, \ldots, e_M^0, \widetilde{d}_2^{(0)}; a_0, \ldots, a_M) =$$
$$[\widetilde{d}_2^{(0)}, \widetilde{d}_1^{(0)}] e_{1,1} = t [d_2^{(1)}, d_1^{(0)}] e_{1,1}.$$

Hence $[d_2^{(1)}, d_1^{(0)}] = 0$ for all $d_1^{(0)} \in N^{(0)}$ and $d_2^{(1)} \in N^{(1)}$. The lemma is proved. □

4.2 The main result for $M_s(F \oplus tF)$

In this section we shall prove that the codimensions of Γ_{s^2,s^2} are asymptotically equal to the codimensions of the superalgebra $M_s(F \oplus tF)$, $s \in \mathbb{N}$

Theorem 14 *Let $M = L = s^2$ with $s \in \mathbb{N}$. Then $\mathcal{U}^{sup}_{M+1,L+1} = supvar(\Gamma_{M+1,L+1}) = supvar(M_s(F \oplus tF) \oplus D')$, where D is a finite dimensional superalgebra such that $supexp(D) < M + L$. In particular*

$$c_n^{sup}(\Gamma_{M+1,L+1}) \simeq c_n^{sup}(M_s(F \oplus tF)).$$

Proof The first part of the proof follows step by step that of Theorem 9 and we obtain a finite dimensional reduced superalgebra R with $supexp(R) = M + L = supexp(\mathcal{U}^{sup}_{M+1,L+1})$ and $\Gamma_{M+1,L+1} \subseteq Id^{sup}(R)$. We can write $R = R_1 \oplus \cdots \oplus R_q + J$, where R_i are simple subsuperalgebras of R and $J = J(R)$ is the Jacobson radical of R. Let t_1 be the number of superalgebras R_i isomorphic to $M_{d_i}(F)$, t_2 the number of superalgebras R_i isomorphic to $M_{s_i}(F \oplus tF)$ and finally let t_3 be the number of R_i isomorphic to $M_{k_i,l_i}(F)$, with $t_1 + t_2 + t_3 = q$. Hence, as in Theorem 9, R contains a superalgebra $\overline{R}$ isomorphic to the following upper block triangular matrix algebra

$$\begin{pmatrix} \overline{R}_1 & & & * \\ 0 & \ddots & & \\ \vdots & & & \\ 0 & \ldots & 0 & \overline{R}_q \end{pmatrix},$$

where $\overline{R}_i$ is one of the following: $M_{d_i}(F)$, $M_{s_i}(F \oplus tF)$, $M_{k_i,l_i}(F)$ and

$supexp(\overline{R}) = d_1^2 + \cdots + d_{t_1}^2 + 2s_1^2 + \cdots + 2s_{t_2}^2 + (k_1 + l_1)^2 + \cdots + (k_{t_3} + l_{t_3})^2 = 2s^2 = M + L.$

If $t_1 \geq 1$, then as in Theorem 9 we obtain $t_1 = 1$. Hence $\overline{R} = M_{d_i}(F) + J$, $2s^2 = M + L = supexp(\overline{R}) = d_i^2$ and we have a contradiction. Now, let $t_1 = 0$. Then by [1, Lemma 1], $\overline{R}$ does not satisfy $Cap_{d^0+t_2+t_3-1}[Y; X]$ and $Cap_{d^1+t_2+t_3-1}[Z; X]$. But $\overline{R}$ satisfies $Cap_{M+1}[Y; X]$ and $Cap_{L+1}[Z; X]$. Then $d^0+t_2+t_3-1 \leq M$ and $d^1+t_2+t_3-1 \leq L$. As in the proof of Theorem 9 we obtain $t_2 + t_3 \leq 1$. Also in this case we have only two possibilities $t_2 = 1$ and $t_3 = 0$ or $t_3 = 0$ and $t_2 = 1$. Let $t_2 = 0$ and $t_3 = 1$. Then $\overline{R} \simeq M_{k_i,l_i}(F) + J$ and $2s^2 = supexp(\overline{R}) = (k_i + l_i)^2$, a contraddiction. Finally, let $t_2 = 1$ and $t_3 = 0$. Then $s_i = s$ and $\overline{R} \simeq M_s(F \oplus tF) + J$. As in Theorem 9, by Lemmas 10, 12, 13 we obtain

$$R \cong (M_s(F \oplus tF) + J_{11}) \oplus J_{00} \cong (M_s(F \oplus tF) \otimes N^\sharp) \oplus J_{00}.$$

where $N^\sharp$ is the algebra obtained from N by adjoining a unit element. Since $N^\sharp$ is commutative, it follows that $R + J_{11}$ and R satisfy the same identities. Thus $supvar(R) = supvar(M_s(F \oplus tF) \oplus J_{00})$ with J_{00} a finite dimensional nilpotent algebra. As in Theorem 9 we get

$$\mathcal{U}_{M+1,L+1}^{sup} = supvar(\Gamma_{M+1,L+1}) = supvar(M_s(F \oplus tF) \oplus D),$$

where D is a finite dimensional superalgebra with $supexp(D) < M + L$. From Corollary 4 we have

$$c_n^{sup}(\Gamma_{M+1,L+1}) \simeq c_n^{sup}(M_s(F \oplus tF))$$

and the theorem is proved.

□

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