MACRO-ZONES SGBEM APPROACH FOR STATIC SHAKEDOWN ANALYSIS AS CONVEX OPTIMIZATION

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Abstract: A new strategy utilizing the Multidomain SGBEM for rapidly performing shakedown analysis as a convex optimization problem has been shown in this paper. The present multidomain approach, called displacement method, makes it possible to consider step-wise physically and geometrically non-homogeneous materials and to obtain a self-equilibrium stress equation regarding all the bem-elements of the structure. Since this equation includes influence coefficients, which characterize the input of the quadratic constraints, it provides a nonlinear optimization problem solved as a convex optimization problem. Furthermore, the strategy makes it possible to introduce a domain discretization exclusively of zones involved by plastic strain storage, leaving the rest of the structure as elastic macroelements, consequently governed by few boundary variables. It limits considerably the number of variables in the problem and makes the proposed strategy extremely advantageous. The implementation of the procedure by the Karnak.sGbem code, coupled with optimization toolbox Matlab 7.6.0, made it possible to perform some numerical tests showing the high performance of the algorithm due to solution accuracy and low computational cost.

Introduction

A reformulation of the static approach to evaluate directly the shakedown multiplier by using the Symmetric Galerkin Boundary Element Method (SGBEM) for multidomain type problems [1,2] is shown in this paper. The present formulation utilizes the self-equilibrium stress equation [3-5] connecting the stresses at the Gauss points of each substructure (bem-e) to plastic strains through a stiffness matrix (self stress matrix) involving all the bem-elements in the discretized system. The numerical method proposed is a direct approach because it permits to evaluate the multiplier directly as lower bound through the static approach. The analysis has been performed as a constrained optimization problem is rephrased in the canonical form as a convex optimization (CO) problem with quadratic constraints, in terms of discrete variables within the SGBEM, and here implemented using the Karnak.sGbem code [7] coupled with an optimization toolbox by MatLab.

In this procedure, through the use of the Multidomain SGBEM, it was possible to reduce the size of the problem, since this method allows one to introduce a domain discretization exclusively in the zones of potential storage of the plastic strains, the remaining part of the structure being considered as made up of macroelements having purely elastic behaviour, consequently governed by few boundary variables. This aspect makes the strategy proposed extremely advantageous by the point of view of the computational burdens. In order to prove the efficiency of the proposed approach, the response is compared with the elastoplastic iterative analysis via Multidomain SGBEM, an advantageous strategy developed by some of the present authors [5]. Finally, a numerical test where the limit multiplier value has been compared both with the value obtained by elastoplastic analysis and by literature available [4,5] is shown. The application shows a very important computational advantage to confine the domain discretization only in the potentially plastic zones and to leave the rest of the structure subdivided in elastic macroelements, these latter therefore governed by few boundary variables.

1. Shakedown analysis via SGBEM and convex optimization

The multidomain Symmetric Boundary Element Method (SGBEM), developed by some authors [1,2], is utilized to riformulate the static shakedown theorem [3,4], which represents a powerful tool for providing directly, by means of mathematical programming techniques, the safety condition of a structure. The proposed strategy uses the self-equilibrium stress equation [3,5] to define the self-equilibrium stress field employed in the classical shakedown approach. This equation connects stresses, computed at each bem-e Gauss point, to plastic strains through an influence matrix (self-stress matrix).
Then the shakedown multiplier is obtained through a constrained optimization problem within the canonical form of a CO in terms of discrete variables.

1.1 Self-equilibrium stress equation via multidomain SGBEM

The proposed strategy uses the stress equation [5], obtained by means of a displacement approach of the SGBEM, to define the self-equilibrium stress field $\boldsymbol{\sigma}^p = \mathbf{Z}\mathbf{p}$. Indeed, the following equation:

$$\boldsymbol{\sigma} = \mathbf{Z}\mathbf{p} + \beta\hat{\boldsymbol{\sigma}}^e \tag{1}$$

provides the stress at the strain points of each bem-e as a function of the volumetric plastic strain $\mathbf{p}$ and of the external actions $\hat{\boldsymbol{\sigma}}^e$, the latter amplified by β. The matrix $\mathbf{Z}$, defined as the self-stress influence matrix of the assembled system, is a square matrix having $3m$x$3m$ dimensions, with m bem-elements, fully-populated, non-symmetric and semi-definite negative. The evaluation of this matrix only involves the knowledge of the material elastic characteristics and of the structure geometry within a domain discretization process. The reader can refer to Zito et al. [5] for a more detailed discussion of the characteristics of this equation introduced for a multidomain SGBEM problem.

1.2 Shakedown analysis as CO problem

In order to evaluate the shakedown multiplier directly, the classic shakedown approach was rephrased by means of SGBEM for multidomain type problems. In the hypothesis of a von Mises yield function, which is a convex quadratic function, the static theorem leads to a numerical optimization problem of a linear objective function subjected to linear and quadratic constraints. Therefore the analysis was developed by solving a constrained nonlinear optimization problem using known mathematical programming methods.
The present formulation couples a multidomain SGBEM procedure with nonlinear optimization techniques through the introduction of the self-equilibrium stress field, defined in eq.(1).
According to the shakedown theorem, the safety condition for the structure is guaranteed by a stress state satisfying the yield condition, the latter rephrased in terms of discrete variables, i.e.:

$$F\left[\boldsymbol{\sigma}_i\right] \leq 0 \tag{2}$$

with $i = 1....v$ the basic load and

$$\boldsymbol{\sigma}_i = \hat{\boldsymbol{\sigma}}_i^e + \boldsymbol{\sigma}^p \tag{3}$$

representing the total stress as the sum of the elastic stress vector $\hat{\boldsymbol{\sigma}}_i^e$, due to external actions, and the self-equilibrium stress vector $\boldsymbol{\sigma}^p$.
The classical static approach makes it possible to obtain the shakedown factor β_{sh} as the maximum of the shakedown factors β for which the structure does not fail:

$$\begin{cases} \beta_{sh} = \max\limits_{(\beta,\boldsymbol{\sigma}^p)} \ \beta \\ \text{s.t.:} \\ F\left[\beta\hat{\boldsymbol{\sigma}}_i^e + \boldsymbol{\sigma}^p\right] \leq 0 \end{cases} \tag{4}$$

Since the self-equilibrium stress vector $\boldsymbol{\sigma}^p$ is a function of the volumetric plastic strain vector $\mathbf{p}$, through the following relation:

$$\boldsymbol{\sigma}^p = \mathbf{Z}\,\mathbf{p} \tag{5}$$

the optimization problem can be written as follows:

$$\begin{cases} \beta_{sh} = \max\limits_{(\beta,\mathbf{p})} \beta \\ \text{s.t.}: \\ \boldsymbol{F}\left[\beta\hat{\boldsymbol{\sigma}}_i^e + \mathbf{Z}\mathbf{p}\right] \le \mathbf{0} \end{cases} \tag{6}$$

or in explicit form:

$$\begin{cases} \beta_{sh} = \max\limits_{(\beta,\mathbf{p}_1,\cdots\mathbf{p}_m)} \beta \\ \text{s.t.}: \\ F_1\left[\beta\hat{\boldsymbol{\sigma}}_{i1}^e + \mathbf{Z}_{11}\mathbf{p}_1\cdots + \mathbf{Z}_{1m}\mathbf{p}_m\right] \le 0 \\ \vdots \\ F_m\left[\beta\hat{\boldsymbol{\sigma}}_{im}^e + \mathbf{Z}_{m1}\mathbf{p}_1\cdots + \mathbf{Z}_{mm}\mathbf{p}_m\right] \le 0 \end{cases} \tag{7}$$

where m is the bem-e number.
In the hypothesis of the von Mises yield law, the present approach allows one to write the problem through the optimization of an objective linear function subjected to quadratic constraints only:

$$\begin{cases} \beta_{sh} = \max\limits_{(\beta,\mathbf{p}_1,\ldots\mathbf{p}_m)} \beta \\ \text{s.t.}: \\ \frac{1}{2}\left(\beta\hat{\boldsymbol{\sigma}}_{i1}^e + \mathbf{Z}_{11}\mathbf{p}_1\cdots + \mathbf{Z}_{1m}\mathbf{p}_m\right)^T \boldsymbol{M}\left(\beta\hat{\boldsymbol{\sigma}}_{i1}^e + \mathbf{Z}_{11}\mathbf{p}_1\cdots + \mathbf{Z}_{1m}\mathbf{p}_m\right) - \boldsymbol{\sigma}_{1y}^2 \le 0 \\ \vdots \\ \frac{1}{2}\left(\beta\hat{\boldsymbol{\sigma}}_{im}^e + \mathbf{Z}_{m1}\mathbf{p}_1\cdots + \mathbf{Z}_{mm}\mathbf{p}_m\right)^T \boldsymbol{M}\left(\beta\hat{\boldsymbol{\sigma}}_{m1}^e + \mathbf{Z}_{m1}\mathbf{p}_1\cdots + \mathbf{Z}_{mm}\mathbf{p}_m\right) - \boldsymbol{\sigma}_{my}^2 \le 0 \end{cases} \tag{8}$$

where $\boldsymbol{M}$ is a constants matrix and σ_{iy} the uniaxial yield stress.
In order to solve the previous problem, the general form of a CO problem was rewritten as follows:

$$\begin{cases} \min\limits_{(\mathbf{y})} \mathbf{y} \\ \text{s.t.}: \\ \mathbf{y}^T\mathbf{B}\,\mathbf{y} \le 0 \end{cases} \tag{9}$$

where $\mathbf{B}$ is a symmetric positive matrix and $\mathbf{y}$ is the unknown quantity vector.
The canonical form (9) is obtained by collecting in the $\mathbf{B}$ matrix the constant terms of eq.(8), i.e. for the j-th bem-e:

$$F_{ij} = \underbrace{\left|\beta \;\; \mathbf{p}_1^T\cdots\mathbf{p}_m^T\right|}_{\mathbf{y}^T} \underbrace{\left|\hat{\boldsymbol{\sigma}}_{ij}^e \;\; \mathbf{Z}_{j1}\cdots\mathbf{Z}_{jm}\right|^T \frac{\boldsymbol{M}}{2\sigma_y^2}\left|\hat{\boldsymbol{\sigma}}_{ij}^e \;\; \mathbf{Z}_{j1}\cdots\mathbf{Z}_{jm}\right|}_{\mathbf{B}_{ij}} \underbrace{\left|\beta \;\; \mathbf{p}_1^T\cdots\mathbf{p}_m^T\right|^T}_{\mathbf{y}} - 1 \le 0 \tag{10}$$

and in compact form:

$$F_{ij} = \mathbf{y}^T\mathbf{B}_{ij}\,\mathbf{y} - 1 \le 0 \tag{11}$$

As this computational strategy has to be applied to practical engineering cases, it is appropriate to introduce suitable constraints on the plastic strains. Therefore the unknown vector $\mathbf{y}$ must also satisfy the following condition:

$$\mathbf{y} \le \|\overline{\mathbf{y}}\| \qquad with \quad \overline{\mathbf{y}} = \left|\infty \quad \overline{\mathbf{q}}_1^T \quad \cdots \quad \overline{\mathbf{q}}_m^T\right|^T \tag{12}$$

The values to be assigned as constraints on the strains have to be meaningfully analyzed from the point of view of practical engineering and obtained by experimental tests on the material ductility.
The shakedown problem can be rewritten as follows:

$$\begin{cases} \min\limits_{(\mathbf{y})} \ \mathbf{c}^T \mathbf{y} \\ \text{s.t.:} \\ \mathbf{y}^T \mathbf{B}_1 \mathbf{y} - 1 \le 0 \\ \vdots \qquad\qquad\qquad k = m * v \\ \mathbf{y}^T \mathbf{B}_k \mathbf{y} - 1 \le 0 \\ \mathbf{y} \le \|\bar{\mathbf{y}}\| \end{cases} \tag{13}$$

where the vector $\mathbf{c}^T = |-1 \ \ 0 \ \ \cdots \ \ 0|$ has been introduced.
Problem (6), in the form (13), was implemented by coupling the Karnak.sGbem code [7] with a Matlab 7.6.0 optimization toolbox.
In this procedure, using multidomain SGBEM, it was also possible to reduce the size of the problem. Indeed, since this method introduces a domain discretization exclusively in the zones of potential store of the plastic strains, the remaining part of the structure can be considered as made up of elastic macroelements, and therefore governed by few boundary variables. This aspect makes the strategy proposed computationally advantageous.

2. Numerical results

To show the computational effectiveness of the proposal method the limit load multiplier ($v = 1$) has been computed by static theorem (Section 1) and compared with incremental approaches [5,6].
At this purpose the frame of Figure 1a, subjected to a uniform load $\mathrm{q} = 10\,\mathrm{KN/m}$, was considered as a bidimensional body. Its domain was discretized using 149 bem-elements and 3 macro-elements. The material characteristics are Young's modulus $\mathrm{E} = 210000\,\mathrm{MPa}$ and Poisson's ratio $n = 0.3$, whereas the uniaxial yield value is $\sigma_y = 250\,\mathrm{MPa}$.

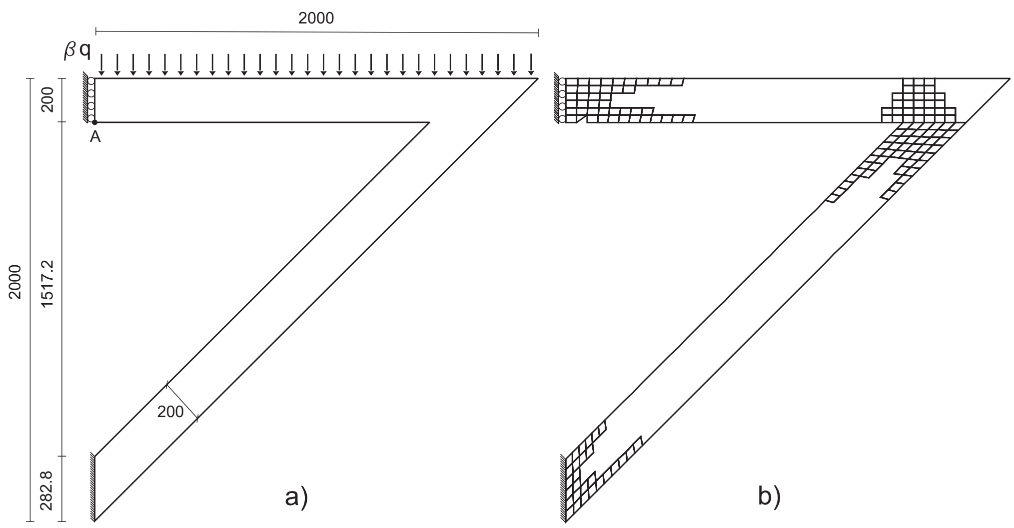

Figure 1. Frame subjected to a uniform load: a) geometric description, b) adopted meshes.

It is necessary to emphasize that only the potentially plastic zones were meshed and the rest was studied as made by elastic macro-elements, as shown in Figure 1b.
The numerical solutions were obtained using the strategies proposed in the present paper. Moreover, the results were compared with those obtained by another incremental method employed by some of the present authors [5,6]. This strategy works in the field of a multidomain SGBEM and it is characterized by strongly recursive elastoplastic analysis.

In detail, the computational work contemplates two direct approaches:
- the incremental elastoplastic analysis for active macrozones [5],
- the incremental elastoplastic analysis [6]);
and an indirect approach:
- the lower bound limit analysis as CO (see Section 1), considering different constraints on the plastic strains, that is $\overline{q}‰ = 1.0‰, 2.0‰, 3.0‰$.
In all these cases the tests were performed using as substructures, subjected to possible plastic strains, both bem-elements having four and eight nodes.

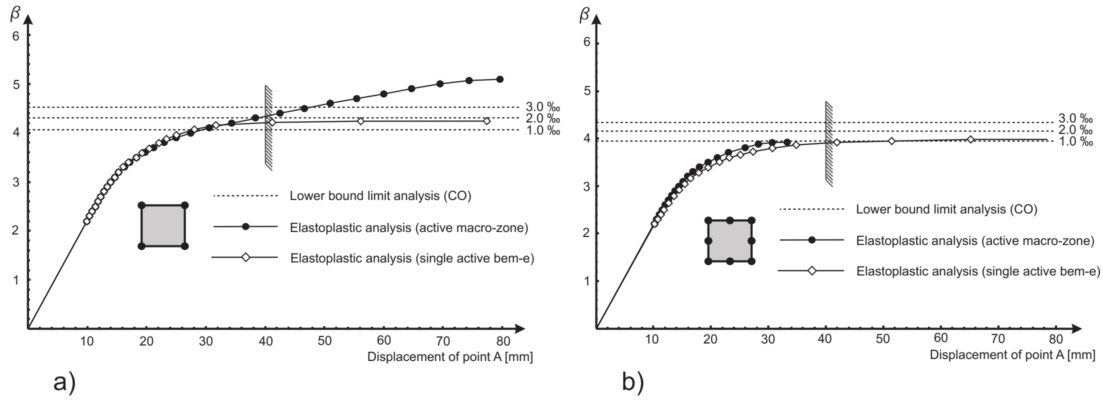

Figure 2. Load factor-displacement curves by elastoplastic analysis compared with lower bound via CO: a) four nodes bem-e, b) eight nodes bem-e.

	Load factor β	
Method	4 nodes bem-e	8 nodes bem-e
Limit analysis (CO, $\overline{q}‰ = 3.0‰$)*	4.531	4.342
Limit analysis (CO, $\overline{q}‰ = 2.0‰$)*	4.310	4.164
Limit analysis (CO, $\overline{q}‰ = 1.0‰$)*	4.060	3.951
Elastoplastic analysis (active macro-zones)	4.4	3.9
Elastoplastic analysis (single active bem-e)	4.2	3.9

Table 1: Collapse load factor, obtained by present approaches (*) compared to other formulation.

In Figure 2 the curves characterizing the load multipliers $\beta = \beta(u)$, as functions of the displacements of the point A, are shown. In the step-by-step analyses the increment of the load is equal to $\Delta\beta = 0.1$. In Figure 2 the dropped line shows the lower bounds obtained through a CO analysis by imposing three different constraints on all the plastic strains, that is $\overline{q}‰ = 1.0‰, 2.0‰, 3.0‰$.
The load multipliers are shown in Table 1, obtained both through direct analysis (using the active macro-zone or the single active bem-e) and indirect analysis (CO). In the direct analyses the multipliers were taken by choosing those load values before attaining the displacement characterizing the structure as not usable. This parameter was fixed as the ratio between the maximum displacement u of the point A and the length of the horizontal beam, that is $u/L = 2\%$, therefore choosing as maximum displacement $u = 40$ mm.
It is to be noted that the structure partially discretized by 4-node bem-elements appears more rigid and the load multiplier proves to be higher in comparison with the values obtained by the same discretization with 8-node bem-elements in both the direct and indirect analyses, that is step-by-step elastoplastic analysis and CO analysis.
Further, the incremental analyses performed on the system discretized by using 8-node bem-elements through either the active macro-zones or the single active bem-e proves to have the same value, whereas both

prove to be lower in comparison with that obtained within the CO analysis performed assuming a constraint on the plastic strains equal to $\overline{q}\permil = 1.0\permil$.

In conclusion, this application proves the high performance and effectiveness of the 8-node bem-e discretization, used in zones where the plasticity is potentially active. Its use allows one to obtain, using the direct or indirect approach, a useful solution regarding the safety of the structure.

3. Conclusions

The static shakedown approach of the classical plasticity theory is rephrased using the Multidomain Symmetric Galerkin Boundary Element Method, under conditions of plane and initial strains, ideal plasticity and associated flow rule. The new formulation couples a multidomain procedure with nonlinear programming techniques and defines the self-equilibrium stress field by an equation involving all the substructures (bem-elements) of the discretized system. The analysis is performed in a canonical form as a convex optimization problem with quadratic constraints, in terms of discrete variables, and implemented using the Karnak.sGbem code coupled with the optimization toolbox by MatLab. The numerical tests, compared with the iterative elastoplastic analysis via the Multidomain Symmetric Galerkin Boundary Element Method, developed by some of the present authors prove the computational advantages of the proposed algorithm.

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