Averaging method for the stability analysis of strongly nonlinear mechanical systems

Alexander Aleksandrov [a,b] Denis Efimov [c,d]

[a] *Saint Petersburg State University, 7-9 Universitetskaya nab., Saint Petersburg, 199034, Russia*

[b] *Institute for Problems of Mechanical Engineering, Russian Academy of Sciences, Saint Petersburg 199178, Russia*

[c] *Inria, Univ. Lille, CNRS, UMR 9189 - CRIStAL, F-59000 Lille, France*

[d] *ITMO University, 49 av. Kronverkskiy, 197101 Saint Petersburg, Russia*

Abstract

A mechanical system under strongly nonlinear potential and dissipative forces, with nonlinear nonstationary perturbations having zero mean values, is studied. Proposing a special construction of Lyapunov function, the conditions are found, under which the perturbations do not influence the asymptotic stability of the trivial equilibrium position of the system. These conditions include the requirements on asymptotic stability of the disturbance-free system and the relations of the nonlinearity orders between potential and dissipative forces. The developed approach is extended to the problem of monoaxial stabilization of a rigid body.

Key words: nonlinear mechanical system, nonstationary perturbations, averaging, asymptotic stability, Lyapunov function

1 Introduction

For stability analysis and control design, it is often necessary to include in the model the acting perturbations and to investigate their impact. Frequently, the disturbances are presented as exogenous and bounded (in certain sense) signals, or as parameter variations, but in numerous applications, these perturbations are periodic or almost periodic oscillations [Bogoliubov and Mitropolsky, 1961, Beletsky, 1966, Khapaev, 1993, Rivin, 2003]. An efficient tool for investigation of influence of such a type of perturbations on the dynamics of the systems is the averaging method [Bogoliubov and Mitropolsky, 1961, Khapaev, 1993, Khalil, 2002]. This approach is widely used in various problems of nonlinear mechanics (see, *e.g.*, [Mitropolsky and Martynyuk, 1980, Giri and Sinha, 2014, Aleksandrov and Tikhonov, 2018] and the references therein): it allows a nonstationary model to be reduced to a time-invariant one, whose properties ensure stability of the original system.

⋆ Section 5 was supported by the Ministry of Science and Higher Education of the Russian Federation (Project No. 075-15-2021-573). The rest of the paper was supported by the Ministry of Science and Higher Education of Russian Federation, passport of goszadanie no. 2019-0898.

In [Aleksandrov, 1996a,b], stability of a class of strongly nonlinear systems with time-varying perturbations was studied. A special approach to the construction of nonstationary Lyapunov functions was proposed, which explicitly incorporates the disturbances in the structure of Lyapunov function. It was proven that the asymptotic stability of the origin of the associated averaged dynamics implies the same property for zero solution of the original time-varying system. These results do not assume existence of a small parameter and demonstrate that strongly nonlinear systems can be more robust to nonstationary perturbations than linear ones. Further development of the approach of [Aleksandrov, 1996a,b] was obtained in [Aleksandrov and Zhabko, 2006, Peuteman and Aeyels, 2011, Aleksandrov et al., 2016, Aleksandrov and Aleksandrova, 2018].

In the present paper, a class of mechanical systems is studied containing strongly nonlinear potential and dissipative forces with nonlinear nonstationary perturbations having zero mean values. Comparing to [Aleksandrov, 1996b, Aleksandrov and Tikhonov, 2018], a broader type of nonstationary disturbances and more nonlinearity are allowed for the model. It is assumed that the origin of the corresponding unperturbed (averaged) system is asymptotically stable, then conditions on the order of nonlinearity are derived, under which the oscillating disturbances do not destroy the stability. The development of the obtained results to the problem of the monoaxial stabilization of a rigid body is presented.

2 Preliminaries

In the sequel, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^n$ is the n-dimensional Euclidean space with the associated norm $\|\cdot\|$ of a vector, the notation $\mathbb{R}^{n\times m}$ is used for the space of $n\times m$ matrices.

The definitions of the used stability concepts are standard [Khalil, 2002].

Definition 1 *(see [Efimov and Polyakov, 2021]) A function $f(\mathbf{x}):\mathbb{R}^n\mapsto\mathbb{R}$ is called homogeneous of the order $\sigma\in\mathbb{R}$ if $f(\lambda\mathbf{x})=\lambda^\sigma f(\mathbf{x})$ for any $\lambda>0$ and $\mathbf{x}\in\mathbb{R}^n$.*

Lemma 1 *[Efimov and Polyakov, 2021] Let $f:\mathbb{R}^n\mapsto\mathbb{R}$ be a continuous homogeneous function of order $\sigma\in\mathbb{R}$, then*

$$\underline{c}\|x\|^\sigma\le f(x)\le\overline{c}\|x\|^\sigma,\quad\forall x\in\mathbb{R}^n,$$
$$\underline{c}=\inf_{\|x\|=1}f(x),\ \overline{c}=\sup_{\|x\|=1}f(x).$$

Lemma 2 *[Efimov and Polyakov, 2021] Let $f:\mathbb{R}^n\mapsto\mathbb{R}$ be a continuously differentiable homogeneous function of order $\sigma\in\mathbb{R}$, then*

$$\sum_{i=1}^n\frac{\partial f(x)}{\partial x_i}x_i=\sigma f(x),\quad\forall x\in\mathbb{R}^n.$$

For any $\mathfrak{a},\mathfrak{b}\ge 0$ and $\gamma,\delta>0$ Young's inequality claims:

$$\mathfrak{a}^\gamma\mathfrak{b}^\delta\le\frac{1}{p}\mathfrak{a}^{\gamma p}+\frac{p-1}{p}\mathfrak{b}^{\frac{\delta p}{p-1}}$$

for any $p>1$.

3 Statement of the problem

Let motions of a mechanical system be modeled by the equations

$$\mathbf{A}\ddot{\mathbf{q}}+\frac{\partial R(\dot{\mathbf{q}})}{\partial\dot{\mathbf{q}}}+\frac{\partial\Pi(\mathbf{q})}{\partial\mathbf{q}}+\mathbf{B}_1(t)\mathbf{G}(\dot{\mathbf{q}})+\mathbf{B}_2(t)\mathbf{Q}(\mathbf{q})=\mathbf{0},\quad(1)$$

where $\mathbf{q}(t),\dot{\mathbf{q}}(t)\in\mathbb{R}^n$ are vectors of generalized coordinates and velocities, respectively; $\mathbf{A}\in\mathbb{R}^{n\times n}$ is a constant symmetric and positive definite matrix; $R(\dot{\mathbf{q}})\in\mathbb{R}$ is a continuously differentiable for $\dot{\mathbf{q}}\in\mathbb{R}^n$ homogeneous of the order $\nu+1$ Rayleigh dissipation function, $\nu>1$; the potential energy $\Pi(\mathbf{q})\in\mathbb{R}$ is a continuously differentiable for $\mathbf{q}\in\mathbb{R}^n$ homogeneous of the order $\mu+1$ function, $\mu>1$, the matrices $\mathbf{B}_1(t)\in\mathbb{R}^{n\times l}$ and $\mathbf{B}_2(t)\in\mathbb{R}^{n\times m}$ are continuous and bounded for $t\ge 0$; components of the vectors $\mathbf{G}(\dot{\mathbf{q}})\in\mathbb{R}^l$ and $\mathbf{Q}(\mathbf{q})\in\mathbb{R}^m$ are continuously differentiable for $\mathbf{q},\dot{\mathbf{q}}\in\mathbb{R}^n$ homogeneous of orders ν and μ, respectively, functions. The terms $-\mathbf{B}_1(t)\mathbf{G}(\dot{\mathbf{q}})$ and $-\mathbf{B}(t)\mathbf{Q}(\mathbf{q})$ may be interpreted as perturbations acting on the system.

Let us note that, in numerous models of mechanical systems, forces of viscous friction and restoring forces are approximated by homogeneous functions with homogeneity orders higher than one (see, *e.g.*, [Gendelman and Lamarque, 2005, Kovacic and Zukovic, 2017, Efimov and Polyakov, 2021] and the references therein).

Assumption 1 *Let*

$$\frac{1}{T}\int_t^{t+T}\mathbf{B}_i(\tau)d\tau\to\mathbf{0}\quad as\ \ T\to+\infty$$

uniformly with respect to $t\ge 0$, $i=1,2$.

Hence, perturbations in (1) are oscillatory functions of time with zero mean values, and the associated averaged system has the form

$$\mathbf{A}\ddot{\mathbf{q}}+\frac{\partial R(\dot{\mathbf{q}})}{\partial\dot{\mathbf{q}}}+\frac{\partial\Pi(\mathbf{q})}{\partial\mathbf{q}}=\mathbf{0}.\quad(2)$$

The systems (1) and (2) admit the trivial equilibrium position

$$\mathbf{q}=\dot{\mathbf{q}}=\mathbf{0}.\quad(3)$$

Assumption 2 *Functions $R(\dot{\mathbf{q}})$ and $\Pi(\mathbf{q})$ are positive definite.*

The objective of the present contribution is finding conditions of asymptotic stability for (1).

Remark 1 *Note that the homogeneity orders of perturbations coincide with the related ones of potential or dissipative forces, and there is no restriction on the magnitudes of oscillation for $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$. If these orders are higher, then the analysis becomes simpler since the stabilizing terms $\frac{\partial R(\dot{\mathbf{q}})}{\partial\dot{\mathbf{q}}}$ and $\frac{\partial\Pi(\mathbf{q})}{\partial\mathbf{q}}$ naturally dominate the perturbations close to the origin (the stability can be analyzed directly by using the energy-based Lyapunov function of (2)).*

It is also worth mentioning that if (1) is linear (*i.e.*, $\nu=\mu=1$), then the perturbations satisfying Assumption 1 might destroy stability [Merkin, 1997]. In what follows we will show that if (1) is a strongly nonlinear system ($\nu>1$, $\mu>1$), then the preservation of the asymptotic stability is guaranteed under a mild constraint on ν and μ. In addition, we will demonstrate the applicability of the approach in the problem of the monoaxial stabilization of a rigid body.

4 Sufficient conditions of the asymptotic stability

For the stability analysis of (1), we will use the approaches of Lyapunov function design developed in [Mitropolsky and Martynyuk, 1980, Aleksandrov, 1996b, Aleksandrov and Zhabko, 2006].

Theorem 1 *Let assumptions 1 and 2 be fulfilled. If*

$$1 < \nu \leq \frac{2\mu}{\mu+1}, \qquad (4)$$

then the equilibrium (3) *of* (1) *is asymptotically stable.*

PROOF. In [Aleksandrov, 1996b], a strict Lyapunov function candidate for (2) was designed in the form

$$V(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{A}\dot{\mathbf{q}} + \Pi(\mathbf{q}) + \gamma\|\mathbf{q}\|^{\beta-1}\mathbf{q}^\top \mathbf{A}\dot{\mathbf{q}}, \qquad (5)$$

where $\gamma > 0$ and $\beta \geq 1$ are parameters to be chosen. The function (5) satisfies the estimates

$$c_1\|\dot{\mathbf{q}}\|^2 + c_2\|\mathbf{q}\|^{\mu+1} - \gamma c_3\|\dot{\mathbf{q}}\|\|\mathbf{q}\|^\beta \leq V(\mathbf{q},\dot{\mathbf{q}}) \\ \leq c_4\|\dot{\mathbf{q}}\|^2 + c_5\|\mathbf{q}\|^{\mu+1} + \gamma c_3\|\dot{\mathbf{q}}\|\|\mathbf{q}\|^\beta$$

for all $\mathbf{q},\dot{\mathbf{q}} \in \mathbb{R}^n$, where $c_1,\dots,c_5$ are some positive constants. Differentiating $V(\mathbf{q},\dot{\mathbf{q}})$ along the solutions of (1) and using lemmas 1 and 2, we obtain

$$\dot{V} = -\dot{\mathbf{q}}^\top\left(\frac{\partial R(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} + \mathbf{B}_1(t)\mathbf{G}(\dot{\mathbf{q}}) + \mathbf{B}_2(t)\mathbf{Q}(\mathbf{q})\right) \\ -\gamma\|\mathbf{q}\|^{\beta-1}\mathbf{q}^\top\left(\frac{\partial R(\dot{\mathbf{q}})}{\partial \dot{\mathbf{q}}} + \mathbf{B}_1(t)\mathbf{G}(\dot{\mathbf{q}}) + \mathbf{B}_2(t)\mathbf{Q}(\mathbf{q}) \right.\\ \left. + \frac{\partial \Pi(\mathbf{q})}{\partial \mathbf{q}}\right) + \gamma\dot{\mathbf{q}}^\top\mathbf{A}\frac{\partial}{\partial \mathbf{q}}\left(\|\mathbf{q}\|^{\beta-1}\mathbf{q}\right)\dot{\mathbf{q}} \\ \leq -c_6\|\dot{\mathbf{q}}\|^{\nu+1} - \gamma c_7\|\mathbf{q}\|^{\beta+\mu} + \gamma c_8\|\mathbf{q}\|^{\beta-1}\|\dot{\mathbf{q}}\|^2 \\ +\gamma c_9\|\mathbf{q}\|^\beta\|\dot{\mathbf{q}}\|^\nu - \dot{\mathbf{q}}^\top\left(\mathbf{B}_1(t)\mathbf{G}(\dot{\mathbf{q}}) + \mathbf{B}_2(t)\mathbf{Q}(\mathbf{q})\right) \\ -\gamma\|\mathbf{q}\|^{\beta-1}\mathbf{q}^\top\mathbf{B}_2(t)\mathbf{Q}(\mathbf{q})$$

for all $t \geq 0$, $\mathbf{q},\dot{\mathbf{q}} \in \mathbb{R}^n$, where c_j, $j = 6,7,8,9$ are other positive constants. With the aid of the Young's inequality and (4), it is straightforward to show that if

$$\beta \geq \max\left\{\frac{\mu}{\nu}; \frac{(\mu+1)(\nu+1)}{2} - \mu\right\}$$

and γ is sufficiently small, then there exists $\delta > 0$ such that

$$\frac{1}{2}\left(c_1\|\dot{\mathbf{q}}\|^2 + c_2\|\mathbf{q}\|^{\mu+1}\right) \leq V(\mathbf{q},\dot{\mathbf{q}}) \leq 2\left(c_4\|\dot{\mathbf{q}}\|^2 + c_5\|\mathbf{q}\|^{\mu+1}\right)$$

$$\dot{V} \leq -\frac{1}{2}\left(c_6\|\dot{\mathbf{q}}\|^{\nu+1} + \gamma c_7\|\mathbf{q}\|^{\beta+\mu}\right) \\ -\dot{\mathbf{q}}^\top\left(\mathbf{B}_1(t)\mathbf{G}(\dot{\mathbf{q}}) + \mathbf{B}_2(t)\mathbf{Q}(\mathbf{q})\right) - \gamma\|\mathbf{q}\|^{\beta-1}\mathbf{q}^\top\mathbf{B}_2(t)\mathbf{Q}(\mathbf{q})$$

for all $t \geq 0$ and $\|\mathbf{q}\| < \delta$, $\|\dot{\mathbf{q}}\| < \delta$.

Next, using the approach of [Morin and Samson, 1997, Aleksandrov and Zhabko, 2006, Aleksandrov and Tikhonov, 2018], let us construct a nonstationary Lyapunov function in the form

$$\widetilde{V}(t,\mathbf{q},\dot{\mathbf{q}}) = V(\mathbf{q},\dot{\mathbf{q}}) \\ +\gamma\|\mathbf{q}\|^{\beta-1}\mathbf{q}^\top\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\mathbf{Q}(\mathbf{q}) \qquad (6) \\ +\dot{\mathbf{q}}^\top\left(\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_1(\tau)d\tau\mathbf{G}(\dot{\mathbf{q}}) + \int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\mathbf{Q}(\mathbf{q})\right),$$

where $V(\mathbf{q},\dot{\mathbf{q}})$ is defined in (5) and α is a positive parameter (in the estimate for $\dot{V}$ there are terms proportional to the nonstationary perturbations, which cannot be compensated, then to resolve this issue, $\widetilde{V}$ is introduced explicitly dependent on them). Taking into account the boundedness of matrices $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$, we obtain

$$\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_i(\tau)d\tau\right\| \leq \rho_i\left|\int_0^t e^{\alpha(\tau-t)}d\tau\right| \leq \frac{\rho_i}{\alpha}$$

for all $t \geq 0$ with some $\rho_i > 0$, $i = 1,2$. Hence, the function (6) and its derivative along the solutions of (1) satisfy the inequalities

$$\frac{1}{2}\left(c_1\|\dot{\mathbf{q}}\|^2 + c_2\|\mathbf{q}\|^{\mu+1}\right) - \frac{1}{\alpha}c_{10}\chi(\mathbf{q},\dot{\mathbf{q}}) \leq \widetilde{\mathbf{V}}(\mathbf{t},\mathbf{q},\dot{\mathbf{q}}) \\ \leq 2\left(c_4\|\dot{\mathbf{q}}\|^2 + c_5\|\mathbf{q}\|^{\mu+1}\right) + \frac{1}{\alpha}c_{10}\chi(\mathbf{q},\dot{\mathbf{q}}), \\ \chi(\mathbf{q},\dot{\mathbf{q}}) = \|\dot{\mathbf{q}}\|^{\nu+\mathbf{1}} + \gamma\|\mathbf{q}\|^{\mu+\beta} + \|\dot{\mathbf{q}}\|\|\mathbf{q}\|^\mu,$$

$$\dot{\widetilde{V}} \leq -\frac{1}{2}\left(c_6\|\dot{\mathbf{q}}\|^{\nu+1} + \gamma c_7\|\mathbf{q}\|^{\beta+\mu}\right) \\ +c_{11}\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_1(\tau)d\tau\right\|\|\dot{\mathbf{q}}\|^{\nu+1} \\ +c_{12}\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\right\|\left(\|\dot{\mathbf{q}}\|\|\mathbf{q}\|^\mu + \gamma\|\mathbf{q}\|^{\mu+\beta}\right) \\ +\gamma c_{13}\|\mathbf{q}\|^{\beta+\mu-1}\|\dot{\mathbf{q}}\|\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\right\| \\ +c_{14}\|\dot{\mathbf{q}}\|^\nu(\|\dot{\mathbf{q}}\|^\nu + \|\mathbf{q}\|^\mu)\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_1(\tau)d\tau\right\| \\ +c_{15}(\|\mathbf{q}\|^{\mu-1}\|\dot{\mathbf{q}}\|^2 + \|\mathbf{q}\|^\mu(\|\dot{\mathbf{q}}\|^\nu + \|\mathbf{q}\|^\mu))\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\right\|$$

$$\leq -\frac{1}{2}\left(c_6\|\dot{\mathbf{q}}\|^{\nu+1} + \gamma c_7\|\mathbf{q}\|^{\beta+\mu}\right) \\ +c_{11}\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_1(\tau)d\tau\right\|\|\dot{\mathbf{q}}\|^{\nu+1} \\ +c_{12}\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\right\|\left(\|\dot{\mathbf{q}}\|\|\mathbf{q}\|^\mu + \gamma\|\mathbf{q}\|^{\mu+\beta}\right) \\ +\frac{1}{\alpha}c_{16}(\|\dot{\mathbf{q}}\|^{2\nu} + \|\dot{\mathbf{q}}\|^\nu\|\mathbf{q}\|^\mu \\ +\|\dot{\mathbf{q}}\|^2\|\mathbf{q}\|^{\mu-1} + \|\mathbf{q}\|^{2\mu} + \gamma\|\mathbf{q}\|^{\mu+\beta-1}\|\dot{\mathbf{q}}\|)$$

for all $t \geq 0$, $\|\mathbf{q}\| < \delta$, $\|\dot{\mathbf{q}}\| < \delta$, where c_j, $j = 10,\dots,16$ are suitable positive constants (again lemmas 1 and 2 were used). It is known [Bogoliubov and Mitropolsky, 1961] that

$$\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_i(\tau)d\tau\right\| \to 0 \text{ as } \alpha \to 0, \quad i = 1,2,$$

uniformly with respect to $t \geq 0$. Taking into account this limit relation and once again applying the Young's inequality, we obtain that there exist positive numbers $\delta_0,\gamma_0,\alpha_0$ such that if $\beta = \frac{\mu}{\nu}$, $\alpha < \alpha_0$, $\gamma < \gamma_0$, $\delta < \delta_0$ and the condition (4) is valid, then

$$\frac{c_1\|\dot{\mathbf{q}}\|^2 + c_2\|\mathbf{q}\|^{\mu+1}}{3} \leq \widetilde{V}(t,\mathbf{q},\dot{\mathbf{q}}) \leq 3\left(c_4\|\dot{\mathbf{q}}\|^2 + c_5\|\mathbf{q}\|^{\mu+1}\right),$$

$$\dot{\widetilde{V}} \leq -\frac{1}{3}\left(c_6\|\dot{\mathbf{q}}\|^{\nu+1} + \gamma c_7\|\mathbf{q}\|^{\beta+\mu}\right)$$

for all $t \geq 0$, $\|\mathbf{q}\| < \delta$, $\|\dot{\mathbf{q}}\| < \delta$. This completes the proof.

Remark 2 *It is worth mentioning that in the case where $\nu = \frac{2\mu}{\mu+1}$ the system* (1) *is homogeneous of the order $\frac{\mu-1}{2}$ with respect to the dilation* $(1; \frac{\mu+1}{2})$ *(see [Efimov and Polyakov, 2021]).*

5 Monoaxial stabilization of a rigid body under time-varying perturbations

Let a rigid body rotate around its mass center C with an angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$ under the action of a control torque $\mathbf{M} \in \mathbb{R}^3$, and the principal central axes of inertia of the body be $Cxyz$. Then (see [Beletsky, 1966]) the attitude motion is defined by the Euler dynamical equations:

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) = \mathbf{M}, \tag{7}$$

where $\mathbf{J} = \operatorname{diag}\{A_1, A_2, A_3\} \in \mathbb{R}^{3\times 3}$ is the body inertia tensor in the axes $Cxyz$.

Assume that two unit vectors $\mathbf{s} \in \mathbb{R}^3$ and $\mathbf{r} \in \mathbb{R}^3$ are given, where $\mathbf{r}$ is constant in the body-fixed frame and $\mathbf{s}$ is constant in the inertial frame. Hence, $\mathbf{s}$ rotates with respect to $Cxyz$ having the angular velocity $-\boldsymbol{\omega}$, and the Poisson kinematic equations admits the conventional form

$$\dot{\mathbf{s}} = -\boldsymbol{\omega} \times \mathbf{s}. \tag{8}$$

The monoaxial stabilization of the body consists of designing a control torque $\mathbf{M} = \mathbf{M}(\boldsymbol{\omega}, \mathbf{s}, \mathbf{r})$ for which the closed-loop system (7), (8) admits the equilibrium

$$\boldsymbol{\omega} = \mathbf{0}, \qquad \mathbf{s} = \mathbf{r} \tag{9}$$

that is asymptotically stable. This problem is of practical importance for Earth-pointing of satellites, space missions with telescopes, remote sensing, *etc.* (see, for instance, [Beletsky, 1966, Giri and Sinha, 2014]). Following [Zubov, 1975], the torque $\mathbf{M}$ can be chosen in the form

$$\mathbf{M} = -\frac{\partial R(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} - \lambda\|\mathbf{s} - \mathbf{r}\|^{\mu-1}\mathbf{s} \times \mathbf{r}, \tag{10}$$

where $R(\boldsymbol{\omega}) \in \mathbb{R}$ is a continuously differentiable for all $\boldsymbol{\omega} \in \mathbb{R}^3$ positive definite homogeneous of the order $\nu + 1$ function, $\nu \geq 1$; λ is a positive coefficient, $\mu \geq 1$.

Assume that, in addition to (10), torques produced by time-varying perturbing forces affect the body:

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega}) = -\frac{\partial R(\boldsymbol{\omega})}{\partial \boldsymbol{\omega}} - \lambda\|\mathbf{s} - \mathbf{r}\|^{\mu-1}\mathbf{s} \times \mathbf{r} + \mathbf{B}_1(t)\mathbf{G}(\boldsymbol{\omega}) + \mathbf{B}_2(t)\mathbf{Q}(\mathbf{s} - \mathbf{r}), \tag{11}$$

where the matrices $\mathbf{B}_1(t) \in \mathbb{R}^{3\times l}$ and $\mathbf{B}_2(t) \in \mathbb{R}^{3\times m}$ are continuous and bounded for all $t \geq 0$, the components of the vectors $\mathbf{G}(\boldsymbol{\omega}) \in \mathbb{R}^l$ and $\mathbf{Q}(\mathbf{u}) \in \mathbb{R}^m$ are continuously differentiable for $\boldsymbol{\omega}, \mathbf{u} \in \mathbb{R}^3$ homogeneous of the order ν and μ, respectively, functions. As in the previous section, we will suppose that Assumption 1 is satisfied for the matrices $\mathbf{B}_1(t)$ and $\mathbf{B}_2(t)$, *i.e.*, the mean values of the perturbed torques are zero. Note that torques of such a type occur in numerous models of satellites moving on circular or elliptic orbits (see [Hughes, 1986, Erdong and Zhaowei, 2010, Aleksandrov and Tikhonov, 2017]).

We will look for conditions ensuring that the perturbations do not destroy the asymptotic stability of the equilibrium position (9) for (8), (11).

Remark 3 *If $\nu = 1$ and $\mathbf{B}_1(t)\mathbf{G}(\boldsymbol{\omega}) \equiv \mathbf{0}$, then such stability conditions were derived in [Aleksandrov and Tikhonov, 2018]. In the present paper, we will assume that the system is strongly nonlinear ($\nu > 1$, $\mu > 1$) and the above constraint is dropped.*

Note that in [Zubov, 1975], the stability analysis was based on the application of a weak Lyapunov function and the LaSalle's invariance principle. Such an approach cannot be effectively extended to the perturbed system [Malisoff and Mazenc, 2009]. In addition, the Lyapunov function constructed in [Aleksandrov and Tikhonov, 2018] is not applicable for the case of $\nu > 1$. Therefore, we need to propose a new strong Lyapunov function for (8), (11).

Remark 4 *Note that Theorem 1 cannot be applied directly to* (8), (11) *by several reasons: a) Lagrange equations in* (2) *represent other kind of dynamics; b)* (7) *contains the gyroscopic force $\boldsymbol{\omega} \times (\mathbf{J}\boldsymbol{\omega})$, which is absent in* (2)*; c) the system* (2) *has no state constraint, but $\|\mathbf{s}\| = 1$ in* (8)*; d)* (2) *has the unique equilibrium* (3)*, while* (7), (8) *has a steady-state* (9) *and another one for $\boldsymbol{\omega} = \mathbf{0}$ and $\mathbf{s} = -\mathbf{r}$.*

Theorem 2 *Let Assumption 1 and* (4) *be fulfilled. Then the equilibrium* (9) *of* (8), (11) *is asymptotically stable.*

PROOF. Using the approach proposed in the proof of Theorem 1, and recalling the expression of (6), choose a Lyapunov function candidate for (8), (11) as follows

$$\begin{aligned}
\widetilde{V}(t, \boldsymbol{\omega}, \mathbf{s}) &= \frac{1}{2}\boldsymbol{\omega}^\top \mathbf{J}\boldsymbol{\omega} + \frac{\lambda}{\mu+1}\|\mathbf{s} - \mathbf{r}\|^{\mu+1} \\
&+ \gamma\|\mathbf{s} \times \mathbf{r}\|^{\beta-1}\boldsymbol{\omega}^\top \mathbf{J}(\mathbf{s} \times \mathbf{r}) \\
&- \gamma\|\mathbf{s} \times \mathbf{r}\|^{\beta-1}(\mathbf{s} \times \mathbf{r})^\top \int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\mathbf{Q}(\mathbf{s} - \mathbf{r}) \\
&- \boldsymbol{\omega}^\top \int_0^t e^{\alpha(\tau-t)}\left(\mathbf{B}_1(\tau)\mathbf{G}(\boldsymbol{\omega}) + \mathbf{B}_2(\tau)\mathbf{Q}(\mathbf{s} - \mathbf{r})\right)d\tau,
\end{aligned}$$

where $\beta \geq 1$, $\gamma > 0$, $\alpha > 0$ are tuning parameters. Performing computations, we obtain

$$\begin{aligned}
&c_1\|\boldsymbol{\omega}\|^2 + \frac{\lambda}{\mu+1}\|\mathbf{s} - \mathbf{r}\|^{\mu+1} - \gamma c_2\|\boldsymbol{\omega}\|\|\mathbf{s} - \mathbf{r}\|^{\beta} \\
&- \frac{1}{\alpha}c_3\left(\gamma\|\mathbf{s} - \mathbf{r}\|^{\beta+\mu} + \|\boldsymbol{\omega}\|^{\nu+1} + \|\boldsymbol{\omega}\|\|\mathbf{s} - \mathbf{r}\|^{\mu}\right) \\
&\leq \widetilde{V}(t, \boldsymbol{\omega}, \mathbf{s}) \leq c_4\|\boldsymbol{\omega}\|^2 + \frac{\lambda}{\mu+1}\|\mathbf{s} + \mathbf{r}\|^{\mu+1} + \gamma c_2\|\boldsymbol{\omega}\|\|\mathbf{s} - \mathbf{r}\|^{\beta}
\end{aligned}$$

$$
+\frac{1}{\alpha}c_3\left(\gamma\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu}+\|\boldsymbol{\omega}\|^{\nu+1}+\|\boldsymbol{\omega}\|\|\mathbf{s}-\mathbf{r}\|^{\mu}\right),
$$

$$
\dot{\widetilde{V}} \le -c_5\|\boldsymbol{\omega}\|^{\nu+1}-\gamma\lambda\|\mathbf{s}-\mathbf{r}\|^{\mu-1}\|\mathbf{s}\times\mathbf{r}\|^{\beta+1}
+\gamma c_6\left(\|\boldsymbol{\omega}\|^2+\|\boldsymbol{\omega}\|^{\nu}\right)\|\mathbf{s}-\mathbf{r}\|^{\beta}+\gamma c_7\|\boldsymbol{\omega}\|^2\|\mathbf{s}-\mathbf{r}\|^{\beta-1}
+\gamma c_8\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\right\|\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu}
+c_9\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_1(\tau)d\tau\right\|\|\boldsymbol{\omega}\|^{\nu+1}
+c_{10}\alpha\left\|\int_0^t e^{\alpha(\tau-t)}\mathbf{B}_2(\tau)d\tau\right\|\|\boldsymbol{\omega}\|\|\mathbf{s}-\mathbf{r}\|^{\mu}
+\frac{1}{\alpha}c_{11}\Big(\gamma\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu-1}\|\boldsymbol{\omega}\|+\|\boldsymbol{\omega}\|^{\nu+2}+\|\boldsymbol{\omega}\|^{2\nu}
+\|\boldsymbol{\omega}\|^{\nu}\|\mathbf{s}-\mathbf{r}\|^{\mu}+\|\mathbf{s}-\mathbf{r}\|^{2\mu}+\|\boldsymbol{\omega}\|^2\|\mathbf{s}-\mathbf{r}\|^{\mu-1}\Big),
$$

where $c_j > 0$, $j = 1, \dots, 11$. It should be noted that, for any $\varepsilon \in (0,1)$ there exists $\delta > 0$ such that

$$
\|\mathbf{s}-\mathbf{r}\|^{\beta+1}\|\mathbf{s}\times\mathbf{r}\|^{\mu-1} \ge \varepsilon\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu}
$$

for all $\|\mathbf{s}-\mathbf{r}\| < \delta$. Now using arguments similar to the proof of Theorem 1, it can be shown that if $\beta = \frac{\mu}{\nu}$, values of δ, γ, α are sufficiently small and (4) holds, then

$$
\frac{1}{2}\left(c_1\|\boldsymbol{\omega}\|^2+\frac{\lambda}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1}\right) \le \widetilde{V}(t,\boldsymbol{\omega},\mathbf{s})
\le 2\left(c_4\|\boldsymbol{\omega}\|^2+\frac{\lambda}{\mu+1}\|\mathbf{s}-\mathbf{r}\|^{\mu+1}\right),
$$

$$
\dot{\widetilde{V}} \le -\frac{1}{2}\left(c_5\|\boldsymbol{\omega}\|^{\nu+1}+\gamma\lambda\|\mathbf{s}-\mathbf{r}\|^{\beta+\mu}\right)
$$

for all $t \ge 0$, $\|\boldsymbol{\omega}\| < \delta$, $\|\mathbf{s}-\mathbf{r}\| < \delta$. That was necessary to prove.

6 Conclusion

For a mechanical system possessing strongly nonlinear potential and dissipative forces (modeled by homogeneous functions of degrees higher than 1), the problem of uniform asymptotic stability was considered with respect to nonlinear nonstationary disturbances of the same homogeneity degrees admitting the zero asymptotic mean values. The analysis was based on design of a special Lyapunov function, which allowed the constraints on homogeneity degrees to be formulated guaranteeing the robust stability result. The advantages of the obtained conditions come from the fact that a strongly nonlinear system was studied (it is well-known that for the case of homogeneity degree one, which corresponds to linear dynamics, similar perturbations may destroy the stability). The proposed method was extended to the problem of the monoaxial stabilization of a rigid body. Future directions of research include development of the approach to time-delay systems.

References

A. Yu. Aleksandrov. On the asymptotical stability of solutions of nonstationary differential equation systems with homogeneous right hand sides. *Dokl. Akad. Nauk Rossii. (in Russian)*, 349(3):295–296, 1996a.

A. Yu. Aleksandrov. On the stability of equilibrium of unsteady systems. *Journal of Applied Mathematics and Mechanics. (in Russian)*, 60(2):205–209, 1996b.

A. Yu. Aleksandrov and E. B. Aleksandrova. Delay-independent stability conditionsfor a class of nonlinear difference systems. *J. Franklin Instit.*, 355:3367–3380, 2018.

A. Yu. Aleksandrov and A. A. Tikhonov. Asymptotic stability of a satellite with electrodynamic attitude control in the orbital frame. *Acta Astronautica*, 139:122–129, 2017. doi: 10.1016/j.actaastro.2017.06.033.

A. Yu. Aleksandrov and A. A. Tikhonov. Rigid body stabilization under time-varying perturbations with zero mean values. *Cybernetics and Physics*, 7(1):5–10, 2018.

A. Yu. Aleksandrov and A. P. Zhabko. On stability of the solutions of a class of nonlinear delay systems. *Automation and Remote Control*, 67(9):1355–1365, 2006.

A.Y. Aleksandrov, E.B. Aleksandrova, and Y. Chen. Partial stability analysis of nonlinear nonstationary systems via averaging. *Nonlinear Dynamics*, 86:153–163, 2016. URL `https://doi.org/10.1007/s11071-016-2878-y`.

V. V. Beletsky. *Motion of an Artificial Satellite about its Center of Mass.* Israel Program for Scientific Translation, Jerusalem, 1966.

N. N. Bogoliubov and Y. A. Mitropolsky. *Asymptotic Methods in the Theory of Non-Linear Oscillations.* Gordon and Breach, New York, 1961.

D. Efimov and A. Polyakov. Finite-time stability tools for control and estimation. *Foundations and Trends in Systems and Control*, 9(2-3):171–364, 2021. doi: 10.1561/2600000026.

J. Erdong and S. Zhaowei. Passivity-based control for a flexible spacecraft in the presence of disturbances. *International Journal of Non-Linear Mechanics*, 45(4): 348–356, 2010. doi: 10.1016/j.ijnonlinmec.2009.12.008.

O.V. Gendelman and C.H. Lamarque. Dynamics of linear oscillator coupled to strongly nonlinear attachment with multiple states of equilibrium. *Chaos, Solitons and Fractals*, 24:501–509, 2005.

D. K. Giri and M. Sinha. Magneto-Coulombic Attitude Control of Earth-Pointing Satellites. *Journal of Guidance, Control, and Dynamics*, 37(6):1946–1960, 2014. doi: 10.2514/1.G000030.

P.C. Hughes. *Spacecraft Attitude Dynamics.* Wiley, New York, 1986.

H. K. Khalil. *Nonlinear Systems.* Prentice-Hall, Upper Saddle River NJ, 2002.

M. M. Khapaev. *Averaging in Stability Theory.* Kluwer, Dordrecht, 1993.

I. Kovacic and M. Zukovic. Coupled purely nonlinear oscillators: normal modes and exact solutions for free and forced responses. *Nonlinear Dynamics*, 87:713–726,

2017. doi: https://doi.org/10.1007/s11071-016-3070-0.
M. Malisoff and F. Mazenc. *Constructions of Strict Lyapunov Functions.* Springer-Verlag, London, 2009.
D. R. Merkin. *Introduction to the Theory of Stability.* Springer, New York, 1997.
Yu. A. Mitropolsky and A. A. Martynyuk. Development of the Lyapunov functions method in non-linear mechanics. *International Journal of Non-Linear Mechanics*, 15:377–386, 1980.
P. Morin and C. Samson. Time-varying exponential stabilization of a rigid spacecraft with two control torques. *IEEE Transactions on Automatic Control*, 42 (4):528–534, 1997. doi: 10.1109/9.566663.
J. Peuteman and D. Aeyels. Averaging techniques without requiring a fast time-varying differential equation. *Automatica*, 47:192–200, 2011.
E. I. Rivin. *Passive Vibration Isolation.* Asme Press, New York, 2003.
V. I. Zubov. *Lectures on Control Theory. (Russian).* Nauka, Moscow, 1975.