Non-existence of stable social groups in information-driven networks

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Abstract

We study a group-formation game on an undirected complete graph G with all edge-weights in a set $\mathcal{W} \subseteq \mathbb{R} \cup \{-\infty\}$. This work is motivated by a recent information-sharing model for social networks (Kleinberg and Ligett, *GEB*, 2013). Specifically, we consider partitions of the vertex-set of G into groups. The individual utility of any vertex v is the sum of the weights on the edges uv between v and the other vertices u in her group. – Informally, u and v represent social users, and the weight of uv quantifies the extent to which u and v (dis)agree on some fixed topic. – For a fixed integer $k \geq 1$, a *k-stable partition* is a partition in which no coalition of at most k vertices would increase their respective utilities by leaving their groups to join or create another *common* group. Before our work, it was known that such a partition always exists if $k = 1$ (Burani and Zwicker, *Math. Soc. Sci.*, 2003). We focus on the regime $k \geq 2$.

- Our first result is that when all the social users are either friends, enemies or indifferent to each other (*i.e.*, $\mathcal{W} = \{-\infty, 0, 1\}$), a partition as above always exists if $k \leq 2$, *but it may not exist if* $k \geq 3$. This is in sharp contrast with (Kleinberg and Ligett, *GEB*, 2013) who proved that k-stable partitions always exist, for any k, if $\mathcal{W} = \{-\infty, 1\}$.
- We further study the intriguing relationship between the existence of k-stable partitions and the allowed set of edge-weights $\mathcal{W}$. Specifically, we give sufficient conditions for the existence or the non existence of such partitions based on tools from Graph Theory. Doing so, we obtain for most sets $\mathcal{W}$ the largest $k(\mathcal{W})$ such that *all* graphs with edge-weights in $\mathcal{W}$ admit a $k(\mathcal{W})$-stable partition.
- From the computational point of view, we prove that for any $\mathcal{W}$ containing $-\infty$, the problem of deciding whether a k-stable partition exists is NP-complete for any $k > k(\mathcal{W})$.

Our work hints that the emergence of stable communities in a social network requires a trade-off between the level of collusion between social users, and the diversity of their opinions.

1 Introduction

In the line of [KL13, BCDM19, Duc16a], we continue the study of a *simplified* model of group formation within Online Social Networks (OSNs). The combinatorics behind this model is surprisingly intricate, and, we think, interesting to study on its own. As a starter, let us sketch the motivations behind the original model of Kleinberg and Ligett, from [KL13]. Users' behaviour in OSNs is driven by two conflicting objectives, namely: *sharing* with the other users while enforcing *privacy* of their personal information. Information is commonly shared between users within a same social group, also known as a community. However, communities are dynamic. As a user leaves one community for another, she may reveal the information learnt from her previous community to the new one, which results in privacy leakage. The model from [KL13] helps for understanding how strategic users can best choose their community in this context. We study the existence of stable outcomes in this model.

Specifically, a *generalized coloring game* is played on an edge-weighted graph $G = (V, E, w)$, with each vertex being an agent. Throughout this paper, the set of edge-weights is denoted by $\mathcal{W}$ (*i.e.*, $\mathcal{W} = w(E)$). We partition agents into groups. Then, the individual goal of each agent is to maximize the sum of the weights of the edges between herself and the other agents in her group. – Alternatively, we may see a partition as a graph coloring, and each group as a color class, hence the name of generalized "coloring game". – We refer to Figure 1 for an illustration (see also Section 2 for a formal definition of the game, and for any undefined terminology). Our main focus is on the existence, and the computation, of *k-stable partitions* for generalized coloring games. Roughly, these are partitions such that no k-subset of agents have an incentive to deviate from their current strategy, *i.e.*, from leaving their respective groups to join another (possibly, new) one. See Figure 3 for an illustration. Note that 1-stable partitions are exactly the Nash equilibria of the game. On a social network point of view, stable partitions ensure that no small coalition of users have an incentive to leave their current community for another one, thus preventing information leakage from a community to another.

Related work. Kleinberg and Ligett studied a uniform version of the game: where G must be a complete graph, and all edge-weights must be taken in $\mathcal{W} = \{-\infty, 1\}$ (the latter representing enmity and friendship between two users, respectively) [KL13]. Amongst many results, they proved that a k-stable partition always exists, for any k, but that it is NP-hard to compute one if k is part of the input. In two follow-up papers [BCDM19, Duc16b], we further studied the complexity of computing k-stable partitions, for small values of k, using parallel algorithms or better-response dynamics. In particular, better-response dynamics do not converge in polynomial time for any fixed $k \geq 4$ [BCDM19]. On the way, we uncovered an equivalence between the model of Kleinberg and Ligett and the earlier introduced *coloring games* [PS08, CKPS10, MW13, EGM12]. Angel et al. studied the sequential (non game-theoretic) complexity of a more general model where, informally speaking, there are different categories of information considered [ABK$^+$16]. We here propose a deeper study of the original model of Kleinberg and Ligett for *arbitrary* edge-weight sets $\mathcal{W}$.

The generalized coloring games that we study are equivalent to the so called *additively separable symmetric Hedonic games* [FMZ17]. We note that there have been several subclasses of Hedonic games proposed in the literature as a solution concept for group formation dynamics [DBHS06, BZ03, Bal04, SD10, HJ17, FMZ17, OBI$^+$17]. Additively separable symmetric Hedonic games are quite appealing in this context, as they always admit a Nash equilibrium [BZ03]. Equivalently with our terminology, every generalized coloring game admits a 1-stable partition. We focus in this paper on the existence of k-stable partitions, for $k \geq 2$. The particular case $k = 2$ can be seen as a variation of the well-known concept of pairwise stability [JW96], while larger values of k correspond to larger subsets of people coordinating their actions.

Finally, as noted by Kleinberg and Ligett, their model is also distantly related to the well-known stable marriage problem [GS62]. Recall that there always exists a solution for the stable marriage problem, which can be computed in polynomial time. In contrast, even for slight variations of this original problem (*e.g.*, stable roommates), solutions may not always exist, and this is sometimes

NP-complete to decide [MS17, CNS18, Ron90]. This brings us with two close questions, namely: can we find generalized coloring games for which no k-stable partition exists? and if so, what is the complexity to decide whether, given a generalized coloring game, there exists a k-stable partition? In what follows, we almost completely answer these two questions.

Our contributions. We give the first known relationships between the existence of k-stable partitions and the properties of G and its subgraphs. For instance, we prove that a k-stable partition always exists if k is less than the girth in the "friendship" subgraph G^+ that is induced by all edges with positive weights (Theorem 1). Doing so, we establish several positive and negative results on the existence of k-stable partitions:

- In Section 3, we study the case of a graph G with all edge-weights in $\mathcal{W} = \{-\infty, 0, 1\}$. We prove that, while a modest generalization of the uniform version of the game Kleinberg and Ligett studied in [KL13], the situation is quite different from the latter, as *k-stable partitions may not always exist*! Specifically, we prove on the positive side that a 2-stable partition always exists (Corollary 1). However, on the negative side, we found a surprising counter-example with only four null-weight edges for which no 3-stable partition can exist (Theorem 2). The latter result shows that the existence of neutral relationships between users can strongly impact the outcome of the game[1]. It also makes of pairwise stability (*a.k.a.*, 2-stable partition) the best stability result that one can hope for.

- Our counter-example for $k = 3$ and $\mathcal{W} = \{-\infty, 0, 1\}$ is derived from a larger class of very symmetric instances, all of which do not admit a k-stable partition for some small value of k (typically, $k = 2$). We use this larger class of counter-examples in order to obtain, for almost any possible edge-weight set $\mathcal{W}$, the largest value $k(\mathcal{W})$ such that any graph G with edge-weights in $\mathcal{W}$ admits a $k(\mathcal{W})$-stable partition (Table 1). We see our result as evidence of the tension between the collusion level between users (parameter k) and the diversity of their opinions (edge-weight set $\mathcal{W}$) in the existence of stable outcomes.

- We end up studying in Section 4.2 the complexity of the recognition of generalized coloring games with a k-stable partition. Our main result in this part is an amusing dichotomy result that we obtain for every edge-weight set $\mathcal{W} \supset \{-\infty\}$. Specifically, for the graphs G with all edge-weights in $\mathcal{W}$, the problem of deciding whether G has a k-stable partition is: trivial if $k \leq k(\mathcal{W})$, and NP-complete if $k > k(\mathcal{W})$. We stress that *any* known G_0 with no k-stable partition can be used as a black-box in our hardness reduction. Therefore, the mere existence of any single counter-example to k-stability is sufficient in order to make the problem intractable.

2 Definitions

Preliminaries. The following notations and terminology are from [Duc16b]. For standard terminology of Game Theory, see [OR94, Mye13]. For standard terminology of Graph Theory, see [BM08]. Let $G = (V, E, w)$ be an edge-weighted graph with $w : E \to \mathbb{R} \cup \{-\infty\}$ be its weight function. Here, the symbol $-\infty$ represents an arbitrarily small quantity which stays unaffected when added to any other quantity. Up to replacing the nonedges by null-weight edges, *for the remainder of the*

[1] See also [OBI$^+$17], where similar results are proved for another Hedonic game (that is related to the one we study in this paper), but for a stronger notion of (core) stability.

paper, we assume the graph G to be a clique. In particular, we may omit the edge-set E and write $G = (V, w)$. Let $\mathcal{W} =^{def} w(E)$ be the set of all edge-weights.

Partitions and Utilities. A coloring $c : V \to \{1, 2, \ldots, |V|\}$ of G is a partition of its vertex-set with each class (or group) being assigned a distinct integer. For every vertex $v \in V$ we denote by $c(v)$ the integer corresponding to her group, also known as her color. Then, in the *generalized coloring game* that is played on G, the vertices of G are the agents of the game, and the strategy of an agent is her color. The group of all the vertices of color i is denoted by $X_i^{(c)}$, or X_i if c is clear from the context. For every $v \in V$, the utility of v is defined as $f_c(v) = \sum_{u \in X_{c(v)}} w_{uv}$ (by convention, we assume that $w_{vv} = 0$). Every agent aims at maximizing her utility function. We define the global utility of a coloring as $U(c) = \sum_{v \in V} f_c(v)$.

Deviations, Stability. We call a subset $S \subseteq V$, with $|S| \leq k$, a k-deviation w.r.t. c, if it satisfies the following property: there exists some color $i \in \mathbb{N}$ so that, for every $v \in S$, we have $c(v) \neq i$ and:

$$f_c(v) < \sum_{u \in X_i \cup S} w_{uv}.$$

We call a coloring c a k-stable partition if there is no k-deviation w.r.t. c. In particular, note that k-stability implies j-stability for every $j \in \{1, 2, \ldots, k\}$. We further have an equivalence between $|V|$-stability and k-stability for every $k \geq |V|$. The following *better-response dynamics* (see also Algorithm 1) are a classical approach in order to compute stable partitions. Specifically, we start from a trivial configuration where each agent has a different color. Then, as long as there exists a k-deviation, we pick any existing k-deviation S and we assign a same color i to all the agents in S so that they increase their respective utilities. Let us point out that i can be either a new color (we make of S a new group) or a color already assigned to some other agents not in S (*i.e.*, we add the agents in S to an existing group). Furthermore, if this above dynamics converges then it stops on a k-stable partition.

Better-Response Dynamics (Algorithm 1)

Input: a positive integer $k \geq 1$, and a graph $G = (V, w)$.

1: Let c_0 be s.t. all the vertices of V have a different color.
2: Set $i = 0$.
3: **while** there exists a k-deviation w.r.t. c_i **do**
4: Choose one k-deviation S and let j be s.t. $\forall v \in S,\ f_{c_i}(v) < \sum_{u \in X_j^{(c_i)} \cup S} w_{uv}$.
5: Define c_{i+1} as the coloring s.t.:

$$c_{i+1}(v) = \begin{cases} j \text{ if } v \in S \\ c_i(v) \text{ otherwise.} \end{cases}$$

6: Set $i = i + 1$.

Friendship and conflict graphs. For any set of weights $\mathcal{W}'$, the subgraph $G\langle\mathcal{W}'\rangle$ is the one induced by all the edges of G whose weight is in $\mathcal{W}'$. In particular:

- The set $\mathcal{W}^+$ contains all positive weights, and it induces the *friendship graph* $G^+ := G\langle\mathcal{W}^+\rangle$.
- Similarly, the set $\mathcal{W}^-$ contains all negative weights, and it induces the *conflict graph* $G^- := G\langle\mathcal{W}^-\rangle$. In particular, if $\mathcal{W}^- = \{-\infty\}$ (that was the case in [KL13]), then any stable partition must be a proper coloring of G^-.

As an example, in Figure 1, the friendship graph is a disjoint union of two triangles $2K_3$, and the conflict graph is a complete bipartite graph $K_{3,3}$.

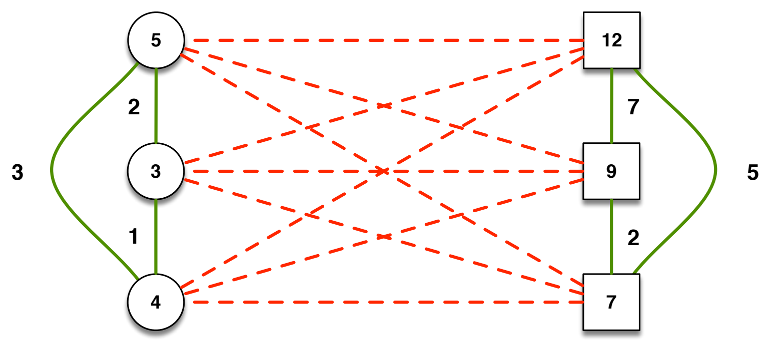

Figure 1: A bicoloring of a graph $G = (V, w)$. Agents that are represented by a circle (resp., by a square) have the same color. Red dashed edges have negative weight $-\infty$, while green continuous edges are labeled with their (positive) weight. Furthermore, each agent is labeled with her (positive) utility with respect to this bicoloring.

3 Games with a unique positive weight

3.1 Positive results

We relate some structural properties of the underlying graph G with the existence of stable partitions. In particular, we relate the existence of stable partitions with the girth (size of a smallest cycle) in the friendship graph. Throughout the paper, the notation $-\mathbb{N}$ stands for the set of negative integers.

Theorem 1. *Let $G = (V, w)$ has all its edge-weights in $\{-\infty, 0, 1\} \cup -\mathbb{N}$. If G^+ has girth at least $k+1$, then there exists a k-stable partition for the generalized coloring game that is played on G. Furthermore, better-response dynamics converge in $\mathcal{O}(|V|^2)$ steps to a k-stable partition.*

Before proving Theorem 1, we need to introduce a (very generic) potential function argument:

Lemma 1. *Let c be a coloring of $G = (V, w)$, let S be a k-deviation w.r.t. c, and let i be a color such that, for every $v \in S$, we have $f_c(v) < \sum_{u \in X_i \cup S} w_{uv}$. If c' is the coloring obtained from c by assigning color i to all the vertices in S, then we have:*

$$U(c') - U(c) \geq 2 \cdot \left[|S| - \sum_{v,v' \in S} w_{vv'} + \sum_{v,v' \in S | c(v) = c(v')} w_{vv'} \right].$$

Proof. By the hypothesis, the utility of each vertex in S increases by at least 1. So, we get as the variation of the utility for the whole k-set: $\sum_{v \in S}[\ f_{c'}(v)\ -\ f_c(v)\ ] \geq |S|$. For every $v \in S$, we then define $\delta_v = \sum_{u \notin S | c(u)=c(v)} w_{uv}$ and $\sigma_v = \sum_{u \in S | c(u)=c(v)} w_{uv}$. We define $\delta'_v = \sum_{u \notin S | c(u)=i} w_{uv}$ and $\sigma'_v = \sum_{u \in S} w_{uv}$ in a similar fashion. Then, $f_c(v) = \delta_v + \sigma_v$, while $f_{c'}(v) = \delta'_v + \sigma'_v$. So, we get by summation that:

$$\sum_{v \in S} f_c(v) = \sum_{v \in S} \delta_v + 2 \cdot \sum_{v,v' \in S | c(v)=c(v')} w_{vv'},$$
$$\sum_{v \in S} f_{c'}(v) = \sum_{v \in S} \delta'_v + 2 \cdot \sum_{v,v' \in S} w_{vv'}.$$

Note that for any $v, v' \in S$, we count $w_{vv'}$ twice: once for v, and another time for v'. Furthermore, the variation of the global utility includes that of the nodes of S, that of the nodes of $X_i^{(c)}$, plus that of the nodes in $X_{c(v)}^{(c)}$, $\forall v \in S$. In other words, we get by symmetry that:

$$\begin{aligned}
U(c') - U(c) &= \sum_{v \in S}[\ f_{c'}(v)\ -\ f_c(v)\ ]\ +\ \sum_{v \in S}[\ \delta'_v - \delta_v\ ] \\
&= 2 \cdot \sum_{v \in S}[\ f_{c'}(v)\ -\ f_c(v)\ ] + 2 \cdot \sum_{v,v' \in S | c(v)=c(v')} w_{vv'} - 2 \cdot \sum_{v,v' \in S} w_{vv'} \\
&\geq 2|S| + 2 \cdot \sum_{v,v' \in S | c(v)=c(v')} w_{vv'} - 2 \cdot \sum_{v,v' \in S} w_{vv'}.
\end{aligned}$$

□

We are now ready to prove the main result of this subsection:

Proof of Theorem 1. Let us consider any one phase of the better-response dynamics. That is, we have a color c, a k-deviation S w.r.t. c, and a color i s.t. $\forall v \in S,\ f_c(v) < \sum_{u \in X_i \cup S} w_{uv}$. Then, during this phase, we assign color i to all the vertices in S, and in doing so we get a new coloring c'. Let $e = |E(G^+[S])|$ be the number of edges in the friendship subgraph induced by S. Since the only positive weight is 1, we get by Lemma 1:

$$U(c') - U(c) \geq 2 \cdot \left[|S| - \sum_{v,v' \in S | c(v) \neq c(v')} w_{vv'} \right] \geq 2 \cdot (|S| - e).$$

Furthermore, since the girth of G^+ is at least $k+1$, S induces a forest in G^+, and so, $e \leq |S| - 1$. Hence, $U(c') - U(c) \geq 2$, and as a result the global utility increases at each step of the dynamics. The latter concludes the proof, because the global utility is upper bounded by an $\mathcal{O}(|V|^2)$. □

In particular, since any friendship graph has girth at least three, we obtain the following corollary:

Corollary 1. *Let $G = (V, w)$ has all its edge-weights in $\{-\infty, 0, 1\} \cup -\mathbb{N}$. Then, there exists a 2-stable partition for the generalized coloring game that is played on G. Furthermore, better-response dynamics converge in $\mathcal{O}(|V|^2)$ steps to a 2-stable partition.*

We remark that Theorem 1 (and so, Corollary 1) still holds true if one replaces 1 by any positive integer a. However, it is unclear to us whether Theorem 1 still holds true if one replaces $-\mathbb{N}$ by the larger set of all negative real numbers. Indeed, it is no more true that each vertex in a k-deviation increases her utility by at least 1 (we only have that each such vertex increases her utility by a positive real value, that can be made arbitrarily small by including some suitable subset of negative real edge-weights).

3.2 A counter-example for $k = 3$

As we prove next, the result of Corollary 1 cannot be improved, already for $\mathcal{W} = \{-\infty, 0, 1\}$.

Theorem 2. *There is a graph $G = (V, w)$ whose edge-weights are in $\mathcal{W} = \{-\infty, 0, 1\}$ and such that there does not exist a 3-stable partition for the generalized coloring game that is played on G.*

The proof of Theorem 2 relies on another structural result which we state as follows. Given an edge-weighted graph $G = (V, w)$, we say that u and u' are *quasi-twins* if $w_{uu'} > 0$ and for all nodes $v \in V \setminus \{u, u'\}$, $w_{uv} = w_{u'v}$ except maybe for one v_0 for which $|w_{uv_0} - w_{u'v_0}| \leq w_{uu'}$.

Lemma 2. *Let $G = (V, w)$ be a graph, and let c be a 1-stable partition. Then, for all quasi-twin vertices u, u' in G, we have $c(u) = c(u')$.*

Proof. Without loss of generality we have that for all vertices $v \in V \setminus \{u, u'\}$, $w_{uv} \geq w_{u'v}$. In particular, either $w_{uv} = w_{u'v}$ for all $v \in V \setminus \{u, u'\}$, or there is a unique v_0 such that $0 < w_{uv_0} - w_{u'v_0} \leq w_{uu'}$ and $w_{uv} = w_{u'v}$ for all $v \in V \setminus \{u, u', v_0\}$. Suppose by contradiction $c(u) \neq c(u')$. There are two cases to be considered.

- **Case $f_c(u) > f_c(u')$.** If u' changes her color for $c(u)$, then we get a new coloring c' such that $f_{c'}(u') \geq (f_c(u) - w_{uu'}) + w_{uu'} = f_c(u) > f_c(u')$.
- **Case $f_c(u) \leq f_c(u')$.** If u changes her color for $c(u')$, then we get a new coloring c' such that $f_{c'}(u) \geq f_c(u') + w_{uu'} > f_c(u') \geq f_c(u)$.

In both cases, there is a 1-deviation, that is contradiction, hence $c(u) = c(u')$. □

Proof of Theorem 2. The set of vertices consists of four sets A_i, $0 \leq i \leq 3$, each of equal size $h \geq 2$ and each with a special vertex a_i, plus four vertices b_i, $0 \leq i \leq 3$, and two vertices c_0 and c_1. In what follows, indices are taken modulo 2 for c_j, $j \in \{0, 1\}$, and they are taken modulo 4 everywhere else. Figure 2 represents the example with $h = 3$. The friendship graph G^+ here consists of all the edges with weight 1; it contains:

1. all the edges between nodes in A_i $(0 \leq i \leq 3)$;
2. edges between b_i and A_i $(0 \leq i \leq 3)$;
3. edges between b_i and $A_{i+1} \setminus \{a_{i+1}\}$ $(0 \leq i \leq 3)$;
4. edges between b_i and b_{i-1} and b_{i+1} $(0 \leq i \leq 3)$;
5. edges between c_0 and all the b_i, and edges between c_1 and all the b_i;
6. edges between c_0 and $A_0 \cup A_2$, and edges between c_1 and $A_1 \cup A_3$.

Moreover, there are four edges with weight 0, namely the edges $\{b_i, a_{i+1}\}$. All the other pairs of agents represent "*enemies*" (they are pairwise connected by an edge with negative weight $-\infty$). That is two nodes in different $A_i, A_{i'}$ are enemies; a user b_i is enemy of b_{i+2} and of the nodes in A_{i+2} and A_{i+3}; c_0 and c_1 are enemies; c_0 is enemy of the nodes in A_1 and A_3, and c_1 is enemy of the nodes in A_0 and A_2. We now assume by contradiction there exists a 3-stable partition c for the generalized coloring game which is played on $G = (V, w)$.

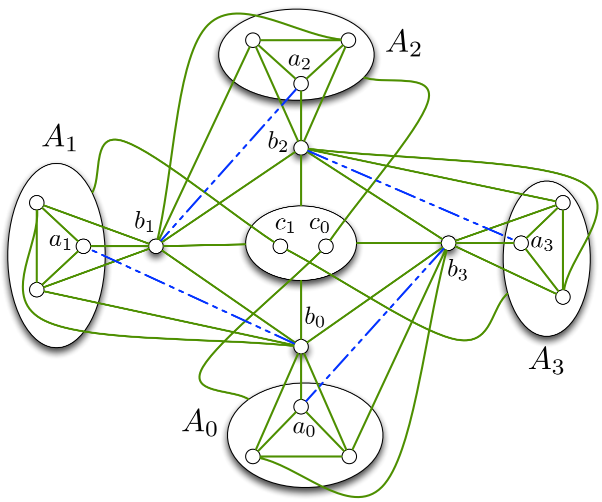

Figure 2: A graph $G = (V, w)$ with edge-weights in $\mathcal{W} = \{-\infty, 0, 1\}$. The coloring game played on G does not admit a 3-stable partition. To keep the graph readable, we use conventions. (1) Some sets of nodes are grouped within a circle; an edge from another node to that circle denotes an edge to *all* elements of this set. (2) Edges of the conflict graph are not represented. In particular, all nodes that are not connected by an edge on the figure are connected by an edge with negative weight $-\infty$. (3) Green solid edges represent edges with weight 1, whereas blue dashed edges represent edges with weight 0.

Claim 1. *Every agent in A_i picks the same color.*

Proof. Since the vertices of A_i are pairwise quasi-twins, this is a direct corollary of Lemma 2. ◇

Claim 2. *b_i picks the same color as the agents in A_i or the agents in A_{i+1}.*

Proof. Suppose that it is not the case. Then $X_{c(b_i)}$ contains at most two other nodes: one of b_{i-1} and b_{i+1} (together enemies), and one of c_0 and c_1 (enemies). In particular, $|X_{c(b_i)}| \leq 3$. If $|X_{c(b_i)}| \leq 2$ or the group X_j containing A_i has size at least 3, then b_i can increase her utility by choosing color j. Therefore, since c is 1-stable, we assume from now on $|X_{c(b_i)}| = 3$, $h = 2$ and $A_i = X_j$. There are two cases.

- If $c(b_i) = c(c_i)$ then, $S = \{b_i, c_i\}$ is a 2-deviation. Indeed, both vertices can increase their respective utilities by choosing color j.
- Else, $c(b_i) = c(c_{i-1})$. Recall that $X_{c(b_i)} \cap \{b_{i-1}, b_{i+1}\} \neq \emptyset$. In particular, if $b_{i-1} \in X_{c(b_i)}$, then $S = \{c_{i-1}, b_{i-1}\}$ is a 2-deviation. Indeed, all the vertices of S can increase their respective utilities by choosing the same color j' as all the vertices of A_{i-1}. Similarly, if $b_{i+1} \in X_{c(b_i)}$, then $S = \{c_{i-1}, b_{i+1}\}$ is a 2-deviation. Indeed, all the vertices of S can increase their respective utilities by choosing the same color j' as all the vertices of A_{i+1}.

In both cases, we derive a contradiction, because c is 2-stable. ◇

Claim 3. *There is an i such that agents in A_i, b_i and b_{i-1} pick the same color.*

Proof. We show the claim by contradiction. For that, we distinguish two cases:

- Case 1: b_{i-1} is with A_i, but not b_i. So, as the claim is supposed to be false, b_i is with A_{i+1}, b_{i+1} is with A_{i+2}, and b_{i+2} is with A_{i+3}. If $c(b_{i-1}) = c(c_i)$, then b_i would increase her utility by choosing color $c(b_{i-1})$. Else, $c(b_{i-1}) \neq c(c_i)$, but then $S = \{b_i, c_i\}$ is a 2-deviation. Indeed, both vertices can increase their respective utilities by choosing color $c(b_{i-1})$.

- Case 2: b_i is with A_i, but not b_{i-1}. So, as the claim is supposed to be false, b_{i-1} is with A_{i-1}, b_{i+1} is with A_{i+1}, and b_{i+2} is with A_{i+2}. We observe that either $c(c_i) = c(b_i)$ or $c(c_i) = c(b_{i+2})$. W.l.o.g., suppose $c(c_i) = c(b_{i+2})$. If $c(b_{i+1}) = c(c_{i+1})$, then b_i would increase her utility by choosing color $c(b_{i+1})$. Else, $c(b_{i+1}) \neq c(c_{i+1})$, but then $S = \{b_i, c_{i+1}\}$ is a 2-deviation. Indeed, both vertices would increase their respective utilities by choosing color $c(b_{i+1})$.

In both cases, we derive a contradiction, because c is 2-stable. ◇

Let i be s.t. the agents in A_i, b_i, b_{i-1}, c_i all pick the same color. Such an i exists by Claim 3, and it is necessarily unique. By symmetry, we assume $i = 0$, and we now consider the group $X_{c(a_0)} = \{b_0, b_3, c_0\} \cup A_0$. There are several cases to distinguish.

- Case 1: $c(a_2) = c(b_1) = c(b_2)$. In particular, by Claims 1 and 2, $X_{c(a_2)} = A_2 \cup \{b_1, b_2\}$.

  Then, there are two subcases. Suppose that $c(a_1) = c(c_1)$, in which case we have $X_{c(c_1)} = A_1 \cup \{c_1\}$. In this situation, the agent b_1 would increase her utility from $1+(|A_2|-1) = |A_2| = h$ to $1+|A_1| = h+1$ by choosing the same color as a_1 and c_1. So, there is a 1-deviation. Otherwise, $c(a_1) \neq c(c_1)$, and so, $X_{c(a_1)} = A_1$, while $X_{c(c_1)}$ is equal to either $\{c_1\}$ or $A_3 \cup \{c_1\}$. But then, the agents b_1 and c_1 would increase their respective utilities from $1 + (|A_2| - 1) = |A_2| = h$ and $\leq |A_3| = h$ to $1 + |A_1| = h + 1$, by choosing the same color as a_1. So, there is a 2-deviation.

- Case 2: $c(a_2) = c(b_2) \neq c(b_1)$. In particular, by Claims 1 and 2, $X_{c(a_2)} = A_2 \cup \{b_2\}$, and $X_{c(b_1)}$ is equal to either $A_1 \cup \{b_1\}$ or $A_1 \cup \{b_1, c_1\}$.

  Then, there are two subcases. Suppose that $c(a_3) = c(c_1)$, in which case we have $X_{c(c_1)} = A_3 \cup \{c_1\}$. Then, if b_2 and b_3 pick the color of a_3, they would increase their respective utilities from $|A_2| = h$ and $2+(|A_0|-1) = 1+|A_0| = h+1$ to, respectively, $2+(|A_3|-1) = |A_3|+1 = h+1$ and $2 + |A_3| = h + 2$. Otherwise, $c(a_3) \neq c(c_1)$, in which case $X_{c(c_1)}$ is equal to either $\{c_1\}$ or $A_1 \cup \{b_1, c_1\}$. But then, if the three of b_2, b_3, c_1 pick the same color as a_3, they would increase their respective utilities from $|A_2| = h,\ 2+(|A_0|-1) = 1+|A_0| = h+1,$ and $\leq 1+|A_1| = h+1$ to, respectively, $2 + (|A_3| - 1) = 1 + |A_3| = h + 1,\ 2 + |A_3| = h + 2,$ and $2 + |A_3| = h + 2$.

- Case 3: $c(a_2) = c(b_1) \neq c(b_2)$. In particular, by Claim 2, $X_{c(a_2)} = A_2 \cup \{b_1\}$, and $X_{c(b_2)}$ is equal to either $A_3 \cup \{b_2\}$ or $A_3 \cup \{b_2, c_1\}$. In that case, b_1 would increase her utility from $|A_2| - 1 = h - 1$ to $\geq |A_1| = h$ by choosing the color of a_1. Therefore, there is a 1-deviation.

- Case 4: $c(a_2) \notin \{c(b_1), c(b_2)\}$. In particular, $c(a_2), c(b_1)$ and $c(b_2)$ are pairwise different, and we have: $X_{c(a_2)} = A_2$; $X_{c(b_1)}$ is equal to either $A_1 \cup \{b_1\}$ or $A_1 \cup \{b_1, c_1\}$; $X_{c(b_2)}$ is equal to either $A_3 \cup \{b_2\}$ or $A_3 \cup \{b_2, c_1\}$. We observe that in this case, $c(b_2) = c(a_3)$.

  Then, there are two subcases. Suppose that $c(c_1) = c(a_3)$. In this situation, we have $X_{c(a_3)} = A_3 \cup \{b_2, c_1\}$. But then, the agent b_3 would increase her utility from $2+(|A_0|-1) = h+1$

to $2+|A_3| = h+2$ by choosing this color, so, there is a 1-deviation. Otherwise, $c(c_1) \neq c(a_3)$, in which situation we have: $X_{c(c_1)}$ is equal to either $\{c_1\}$ or $A_1 \cup \{b_1, c_1\}$; and $X_{c(a_3)} = A_3 \cup \{b_2\}$. But then, both b_3 and c_1 would increase their respective utilities from $\leq h+1$ to $h+2$ by choosing the color of a_3.

Finally, since in all cases there is a 3-deviation, there does not exist a 3-stable partition for the generalized coloring game that is played on G. □

4 The general case

4.1 The threshold for stability

Let us define, for every fixed set $\mathcal{W}$, $k(\mathcal{W})$ to be the largest k such that, for every graph with edge-weights in $\mathcal{W}$, there exists a k-stable partition. We prove sharp bounds on $k(\mathcal{W})$ for almost all sets $\mathcal{W}$ (see Table 1). Note that the two first lines of Table 1 were only proved for $a = 1$. However, it follows from the definition of k-stability that scaling all edge-weights by some positive value a has no impact on the (non)existence of a k-stable partition. As for the third line of Table 1, see our brief discussion at the end of Sec. 3.1. The constructions of our counter-examples are presented next. Remarkably, our results show that, along with the trivial cases where all the weights are either non negative or non positive, the uniform version of the game Kleinberg and Ligett studied in [KL13] is the only set $\mathcal{W}$ s.t. we have $k(\mathcal{W}) = \infty$.

$\mathcal{W}$	$k(\mathcal{W})$	
$\{-\infty, a\}, a > 0$	∞	[KL13]
$\{-\infty, 0, a\}, a > 0$	2	Theorem 2, Corollary 1
$\{-\infty, 0, a\} \cup -\mathbb{N}, a \in \mathbb{N} \setminus \{0\}$	2	Theorem 2, Corollary 1
$\{-\infty, a, b\}, b > a > 0$	1	Lemma 3
$\{-a, b\}, a > 0, b > 0$	$\leq 2 \cdot \lceil \frac{a+1}{b} \rceil + 1$	Lemma 4

Table 1: Values of $k(\mathcal{W})$ for different $\mathcal{W}$.

Starters. Consider the graph G of Figure 3. There are three non negative weights, namely:

- The edges v_1v_2, v_2v_3, v_3v_1 have the same weight w_1 (on the Figure, $w_1 = 2$);
- The edges v_1u_3, v_2u_1, v_3u_2 have the same weight w_2 (on the Figure, $w_2 = 3$);
- The edges v_1u_2, v_2u_3, v_3u_1 have the same weight w_3 (on the Figure, $w_3 = 4$).

All the other edges have weight $-\infty$. Moreover, if $w_1 < w_2 < w_3$ and $w_1 + w_2 > w_3$, then we claim that there does not exist a 2-stable partition. Indeed, the partition with groups $\{v_1, v_2, v_3\}, \{u_1\}, \{u_2\}, \{u_3\}$ is 1-stable but not 2-stable. All the other 1-stable partitions for the generalized coloring game that is played on G are isomorphic to the one we drew in Figure 3. Therefore, none of them is 2-stable. Next, we generalize this construction.

Lemma 3. *Let a, b be two positive integers such that $a < b$. There exists a graph $G = (V, w)$ s.t. all the edge-weights are in $\mathcal{W} = \{-\infty, a, b\}$, and there does not exist a 2-stable partition.*

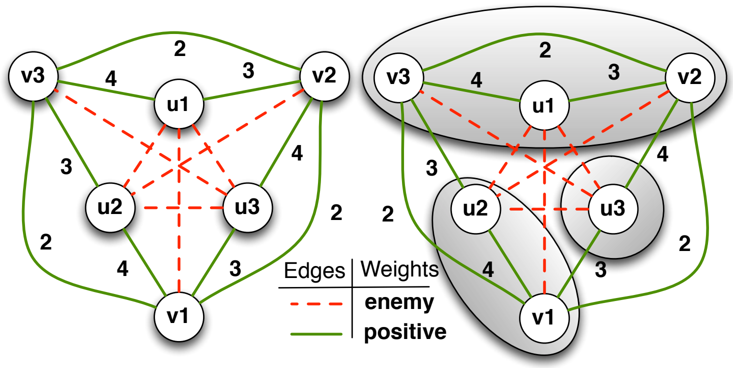

Figure 3: A coloring game that does not admit a 2-stable partition: edges in green have the positive weight indicated, whereas the red edges have negative weight $-\infty$. The partition on the right is not 2-stable. As an example, nodes v_1 and v_2 can decide to simultaneously move into the group containing u_3, which effectively "rotates" the groups in the partition.

Proof. Users are partitioned into four sets $U_1 = \{x_1, x_2, x_3\}$, $U_2 = \{y_1, y_2, y_3\}$, $U_3 = \{z_1, z_2, z_3\}$ and $\{v_1, v_2, v_3\}$. Each of these sets induces a triangle whose three edges are weighted b. In addition:

- each edge between a node in U_i and another node in $U_{i'}, i \neq i'$ has negative weight $-\infty$.
- the edges between v_1 and U_2 are weighted b, whereas the edges between v_1 and U_1 have negative weight $-\infty$. Similarly, the edges between v_2 and U_3 (resp., between v_3 and U_1) are weighted b, whereas the edges between v_2 and U_2 (resp., between v_3 and U_3) have negative weight $-\infty$.
- Finally, we set:
  - $w_{v_1 z_1} = w_{v_1 z_2} = b$, $w_{v_1 z_3} = a$;
  - $w_{v_2 x_1} = w_{v_2 x_2} = b$, $w_{v_2 x_3} = a$;
  - $w_{v_3 y_1} = w_{v_3 y_2} = b$, $w_{v_3 y_3} = a$.

Let us assume by contradiction that there exists a 2-stable partition c.

Claim 4. *For any $1 \leq i \leq 3$, the nodes in U_i pick the same color.*

Proof. By symmetry, it suffices to show the claim for U_1. First, since the nodes x_1, x_2 are quasi-twins, they pick the same color by Lemma 2. Furthermore, by construction all the edges between a node of $X_{c(x_1)}$ and x_3 have a positive weight (*i.e.*, because the nodes of U_1 are twins in the conflict graph G^-). Suppose for the sake of contradiction $x_3 \notin X_{c(x_1)}$. Then, the utility of x_3 is at most $w_{v_2 x_3} + w_{v_1 x_3} = a + b < 2b$. However, this implies that by choosing color $c(x_1)$, the node x_3 would increase her utility to at least $2b$. ⋄

By replacing each subset U_i by a single node u_i, we get the counter-example of Figure 3 for the choices of weights $w_1 = b, w_2 = 2b + a$ and $w_3 = 3b$. In particular, by Claim 4, we may associate to any 1-stable partition c of G a partition c' of the graph of Figure 3. Suppose by contradiction that

c' is *not* 1-stable. We can check that there always exists a 1-deviation whose node is taken from $\{v_1, v_2, v_3\}$. Therefore, c is not a 1-stable partition of G, that is a contradiction. From now on, we assume that c' is 1-stable. However, in any 1-stable partition c', there exists a 2-deviation which only contains nodes amongst $\{v_1, v_2, v_3\}$. In particular, this is also a 2-deviation w.r.t. c. $\square$

Lemma 4. *Let a, b be positive integers (not necessarily distinct). There exists a graph $G = (V, w)$ s.t. all the edge-weights are in $\mathcal{W} = \{-a, b\}$, and there does not exist a $2 \cdot \left(1 + \lceil \frac{a+1}{b} \rceil\right)$-stable partition.*

Proof. Let x, y be non negative integers such that $bx - ay = \gcd\{a, b\} = d$. The vertex-set is partitioned in nine subsets $V_1, V_2, V_3, U_1^+, U_2^+, U_3^+, U_1^-, U_2^-, U_3^-$ plus three vertices $u_1^=, u_2^=, u_3^=$. In particular, there exist large enough constants k_1, k_2, k_3 s.t.:

- The subsets V_i have respective size $1 + \lceil \frac{a+1}{b} \rceil$;
- The subsets U_i^+ have respective size $\left\lceil \frac{1 + b \cdot \left(1 + \lceil \frac{a+1}{b} \rceil\right)}{d} \right\rceil \cdot x + k_i \cdot a$.
- The subsets U_i^- have respective size $\left\lceil \frac{1 + b \cdot \left(1 + \lceil \frac{a+1}{b} \rceil\right)}{d} \right\rceil \cdot y + k_i \cdot b - 1$.

The edge-set is as follows:

- $V_1 \cup V_2 \cup V_3$ induces a clique in the friendship graph G^+ (*i.e.*, all the edges between these vertices are weighted b).
- The subsets $U_i = U_i^+ \cup U_i^- \cup \{u_i^=\}$ induce cliques in the friendship graph G^+.
- For every $i \neq j$, the edges between V_i and U_j^+ have weight b (*i.e.*, they induce a complete bipartite subgraph in G^+).
- Finally, all the edges between $u_1^=$ and V_3 have weight b, and in the same way all the edges between $u_2^=$ and V_1 (resp., between $u_3^=$ and V_2) have weight b.

Every remaining edge has weight $-a$.

Suppose by contradiction that there exists a $2\left(1 + \lceil \frac{a+1}{b} \rceil\right)$-stable partition c. By construction, $V_1, V_2, V_3, U_1^+, U_2^+, U_3^+, U_1^-, U_2^-, U_3^-$ are quasi-twin sets. In particular, by Lemma 2, all the nodes in any one of these sets pick the same color. For simplicity, let us write $p = 1 + \left\lceil \frac{a+1}{b} \right\rceil$, and $q = \left\lceil \frac{1+b \cdot p}{d} \right\rceil$. W.l.o.g., let us assume $k_1 \leq k_2 \leq k_3$. If a vertex $v \in V_3$ picks the same color as those nodes in U_3^+, then her maximum utility would be:

$$
\begin{aligned}
f_c(v) = b \cdot & \Big[|V_1| + |V_2| + (|V_3| - 1) + |U_1^+| + |U_2^+| + 1\Big] \\
& - a \cdot |U_3^+| \\
= b \cdot & [3p + 2qx + (k_1 + k_2) \cdot a] \\
& - a \cdot (qx + k_3 \cdot a)
\end{aligned}
$$

that is negative for large enough k_3 (with respect to k_1, k_2). Therefore, the vertices of V_3 pick a different color than those in U_3^+ (otherwise, the partition is not 1-stable). Similarly, if a vertex $u \in U_1 \cup U_2$ picks the same color as those nodes in U_3^+, then her maximum utility would be:

$$\begin{aligned} f_c(v) &= b \cdot [|V_1| + |V_2| + |U_2| - 1] \\ &\quad - a \cdot |U_3^+| \\ &= b \cdot [2p + qx + k_2 \cdot a - 1] \\ &\quad - a \cdot (qx + k_3 \cdot a) \\ &< 0. \end{aligned}$$

As a result, the vertices of $U_1 \cup U_2$ pick different colors than those of U_3^+. We can prove in the exact same way that the vertices of $V_3 \cup U_1 \cup U_2$ pick different colors than those of U_3^-. Then, the group of U_3^+, resp. of U_3^-, is a subset of $U_3 \cup V_1 \cup V_2$. In particular, since $|U_3^+|, |U_3^-| \gg |V_1| + |V_2| = 2p$, we obtain that all the vertices of U_3 must pick the same color (otherwise, there would exist 1-deviations).

By similar arguments as above, we may choose k_1, k_2 large enough so that all the vertices of U_2 (resp., of U_1) pick the same color. Moreover, the vertices of V_i pick a different color than those of U_i, and for any $i \neq j$ the vertices of U_j pick a different color than those of U_i.

Let us replace the subsets V_i, U_i by fresh new vertices v_i, u_i. Doing so, we get an isomorphic copy of the counter-example of Figure 3. In order to fix the weights w_1, w_2, w_3, we fix a subset V_j, and then we consider the contribution of any other subset to the utility of the nodes of V_j. We obtain that:

- $w_1 = b \cdot |V_i| = b \cdot p = b \cdot \left(1 + \left\lceil \frac{a+1}{b} \right\rceil\right) \geq b + a + 1$;
- $w_2 = b \cdot |U_i^+| - a \cdot |U_i^- \cup \{u_i^=\}| = bqx + bk_ia - aqy - ak_ib = (bx - ay)q = d \cdot q \geq 1 + b \cdot p$;
- $w_3 = b \cdot |U_i^+ \cup \{u_i^=\}| - a \cdot |U_i^-| = bqx + bk_ia + b - aqy - ak_ib + a = d \cdot q + (b + a)$.

In particular, $w_1 < w_2 < w_3 < w_1 + w_2$. This implies that we can map c to a 1-stable partition c' of the graph of Figure 3. For the latter, there exists a 2-deviation whose nodes are taken amongst v_1, v_2, v_3. In turn, there exists a $2p$-deviation w.r.t. c, that is a contradiction. □

4.2 Hardness results

Surprisingly, under the additional assumption $-\infty \in \mathcal{W}$, the threshold $k(\mathcal{W})$ *fully characterizes* the complexity of recognizing generalized coloring games with a k-stable partition. Specifically, we prove the following dichotomy result:

Theorem 3. *Let $\mathcal{W}$ contain $-\infty$, and let $k \geq 1$ be fixed. Then, the problem of deciding whether a given generalized coloring game, played on a graph with edge-weights in $\mathcal{W}$, admits a k-stable partition is either:*

- *trivial if $k \leq k(\mathcal{W})$;*
- *or NP-complete if $k > k(\mathcal{W})$.*

Under the assumption $-\infty \in \mathcal{W}$, our results from Sections 3.2 and 4.1 show that in most cases $k(\mathcal{W}) \leq 2$. Then, we observe that whenever $k \leq k(\mathcal{W})$, we can also *compute* a k-stable partition in polynomial time, *e.g.*, by using better-response dynamics. This approach does not work anymore for the uniform case [KL13], thereby reinforcing the specificity of the latter compared to the other possible sets of weights.

The remaining of this subsection is devoted to the proof of Theorem 3. The problem is clearly in NP because, for any fixed k, we can decide whether a k-deviation exists in polynomial-time $n^{O(k)}$. Informally, in order to prove the NP-hardness we will assume the existence of a counter-example, and we will build a supergraph of it that is arbitrarily large. We will characterize the k-stable partitions for the generalized coloring game that is played on this supergraph. In particular, we will prove that a necessary and sufficient condition for having a k-stable partition is that one user from the counter-example picks the same color as a large independent set from the supergraph. By doing so, we will be able to reduce the well-known Maximum Independent Set to our problem.

Intermediate reductions. For technical reasons, we need a constant lower bound on the utility of a user. Intuitively, we use this lower bound in order to ensure that if there exists a k-stable partition, then there is some user from the counter-example that picks the same color as a large independent set from the supergraph. In what follows, we introduce two reductions for enforcing this constant lower bound.

Reduction 1. *Let t be a positive integer, and let $\mathcal{W}$ be a finite set such that $\mathcal{W}^+ \neq \emptyset$. We set $w_p = \max \mathcal{W}$. For a given graph $G = (V, w)$ whose weight in $\mathcal{W}$, and $n' \geq |V|$, we construct a new graph $\widetilde{G_{t,n'}}$ as follows:*

- *We create n' distinct copies of the complete graph K_t, whose edges are all weighted w_p.*
- *Then, we add an edge of weight $-\infty$ between any two nodes in two distinct copies of K_t.*
- *Finally, we add an edge of weight w_p between any node in V and any node belonging to some copy of K_t.*

Intuitively, Reduction 1 increases the minimum utility of the nodes to $w_p t$.

Reduction 2. *Let α be a positive integer, and let $\mathcal{W}$ be a finite set such that $\mathcal{W}^+ \neq \emptyset$. We set $w_p = \max \mathcal{W}$. For a given graph $G = (V, w)$ whose weight in $\mathcal{W}$, we construct KG_α as follows:*

- *We replace every node $u \in V$ with a clique of α nodes $K_\alpha(u) \subseteq V(KG_\alpha)$.*
- *For every $u_i, u_j \in K_\alpha(u)$, the edge $u_i u_j$ has weight w_p.*
- *For every $u_i \in K_\alpha(u)$ and $v_j \in K_\alpha(v)$, the edge $u_i v_j$ has weight w_{uv}.*

Overall, our reductions ensure the following properties:

Lemma 5. *Let $\mathcal{W}$ be a finite set such that $\mathcal{W}^+ \neq \emptyset$, and let $G = (V, w)$ have all its edge-weights restricted to $\mathcal{W}$. Then, for any $n' \geq |V|$ and $t > |V|$, there exists a k-stable partition for the game played on G if, and only if, there exists a k-stable partition for the game played on $\widetilde{G_{t,n'}}$.*

Proof. We remind that one obtains $\widetilde{G_{t,n'}}$ from G by adding n' distinct copies of the complete graph K_t, that we will denote by $K_t^1, \ldots, K_t^{n'}$.

First, let c be a k-stable partition for the game played on G. W.l.o.g., the colors used in c are $\{1, 2, \ldots, p\}$, for some $p \leq |V|$. Let c' be the coloring $\widetilde{G_{t,n'}}$ s.t.:

$$\begin{cases} c'(v) = c(v) \text{ if } v \in V \\ c'(v) = j \text{ if } v \in K_t^j. \end{cases}$$

We claim that c' is a k-stable partition for the game played on $\widetilde{G_{t,n'}}$. By contradiction, let S be a k-deviation w.r.t. c', and let a color j s.t., $\forall v \in S, f_{c'}(v) < \sum_{u \in X_j^{(c')} \cup S} w_{uv}$. If $S \subseteq K_t^i$ then the only possibility for the vertices of S is to create a new group. However, that would result in their utility being $k \cdot w_p \leq (|V| - 1) \cdot w_p < (t-1) \cdot w_p$. Since $\forall v \in K_t^i, f_{c'}(v) \geq (t-1) \cdot w_p$, this is impossible. Therefore, we must have $S' = S \cap V \neq \emptyset$. However, this implies:

$$\forall v \in S',\ f_{c'}(v) = f_c(v) + w_p \cdot t < \sum_{u \in X_j^{(c')} \cup S} w_{uv} \leq \left(\sum_{u \in (X_j^{(c')} \cap V) \cup S'} w_{uv} \right) + w_p \cdot t.$$

In particular, S' is a k-deviation w.r.t. c.

Conversely, let c' be a k-stable partition for the game played on $\widetilde{G_{t,n'}}$. By Lemma 2, for any i, the nodes in K_t^i pick the same color. Furthermore, we claim that every node of V must pick the same color as some clique K_t^i. Indeed, suppose that it is not the case for $u \in V$. Since $n' \geq n$, there exists an i such that no vertex of V picks the same color as K_t^i. In particular, K_t^i is a group of the partition. But then, $f_c(u) < \sum_{v \in V} w_{uv} < w_p \cdot t$, and so, there would exist a 1-deviation. Therefore, we proved as claimed that every node of V picks the same color as some clique K_t^i. Now, let c be such that, $\forall v \in V,\ c(v) = c'(v)$. We claim that c is a k-stable partition for the coloring game played on G. Indeed, suppose by contradiction that there exists a k-subset $S \subseteq V$, and a color j such that:

$$\forall v \in S,\ f_c(v) < \sum_{u \in X_j^{(c)} \cup S} w_{uv}.$$

Again, since $n' \geq n$, we may assume w.l.o.g. that $X_j^{(c')} \neq \emptyset$. Furthermore,

$$\forall v \in S,\ f_{c'}(v) = f_c(v) + w_p \cdot t < \sum_{u \in X_j^{(c)} \cup S} w_{uv} + w_p \cdot t = \sum_{u \in X_j^{(c')} \cup S} w_{uv}.$$

A contradiction. □

We are now able to prove Theorem 3:

Proof of Theorem 3. Let $G_0 = (V_0, w^0)$ be restricted to $\mathcal{W}$ and such that the game which is played on it does not admit a k-stable partition. W.l.o.g., there exists some $x_0 \in V_0$ whose removal makes the existence of a k-stable partition for the gotten subgraph. Indeed, otherwise, we remove nodes sequentially until obtaining this property. Moreover, we may substitute $\mathcal{W}$ by the subset of all the weights on the edges of G_0, that is finite. Since there is no k-stable partition, we have $\mathcal{W}^+ \neq \emptyset$. In what follows, let $w_p = \max \mathcal{W}$.

Let c_0 be a k-stable partition for the coloring game that is played on $G_0 - x_0$. Let c_0' be obtained from c_0 by adding a new group equal to the singleton $\{x_0\}$. By the hypothesis, c_0' is not k-stable.

Amongst all the k-deviations S w.r.t. c_0', and all the colors j s.t. $\forall v \in S,\ f_{c_0'}(v) < \sum_{u \in X_j^{(c_0')} \cup S} w_{uv}$, we choose one such a pair (S, j) such that, in the new coloring obtained from c_0' by assigning color j to all the nodes in S, the utility of x_0 is *maximized*. Denote by f_0 this maximum value of the utility function for x_0. Up to replacing G_0 with $\widetilde{(G_0)}_{t',n_0'}$ for some large enough constants t', n_0' (that can be done w.l.o.g. by Lemma 5), we may assume $f_0 > 0$. We also define the two constants $\alpha = \lceil \frac{f_0}{w_p} \rceil$ and $\beta_0 = 2n_0 + 1$, with $n_0 = |V_0|$.

We can now prove the NP-hardness of our problem by using a polynomial reduction for MAXIMUM INDEPENDENT SET. Specifically, let $G = (V, E)$ be an undirected unweighted graph, and let $\beta \geq \beta_0$ be an integer. We define $D_G = (V, w_G)$ such that $\forall uv \in E, w_{uv} = -\infty$ and $\forall uv \notin E, w_{uv} = w_p$ (note that the conflict graph D_G^- is equal to G). Furthermore: let $t = \alpha \cdot (\beta - 1) + 1$; let $G_1 = \widetilde{(G_0)}_{t,n_0}$; and let $G_2 = K(D_G)_\alpha$. Here, it is important to observe that we have $t > \alpha\beta - \frac{f_0}{w_p} - 1 \geq \alpha\beta - n_0 - 1 \geq \beta - n_0 - 1 \geq n_0$, because $f_0 \leq n_0 w_p$. Finally, we build the graph H_G from G_1 and G_2 as follows:

- For every edge between $G_1 - x_0$ and G_2, we assign a negative weight $-\infty$.
- For every edge between x_0 and G_2, we assign a positive weight w_p.

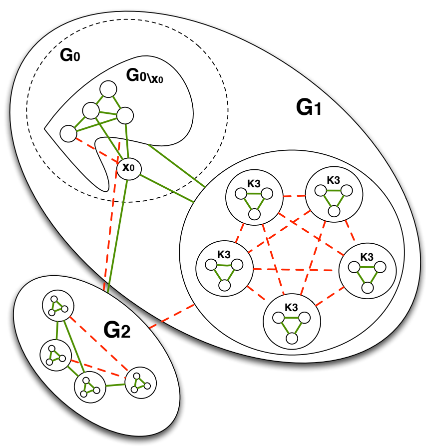

Figure 4: The NP-hardness reduction of Theorem 3.

This above transformation is illustrated in Figure 4. In what follows, we prove that there exists a k-stable partition for the game played on H_G if and only if there exists a maximum independent set of size at least β in G. For that, let us first assume that every independent set of G has a size less than β. Suppose for the sake of contradiction that there exists a k-stable partition for the coloring game played on H_G. By construction, no group can intersect both $V(G_1) \setminus \{x_0\}$ and $V(G_2)$. In particular, the group of x_0 is either fully contained in G_1, or in $G_2 + x_0$. Furthermore, there can be no group with more than $\alpha \cdot (\beta - 1)$ vertices of $V(G_2)$. But then, since $\alpha \cdot (\beta - 1) < t$, the group of x_0 must be in $V(G_1)$ (otherwise, we could increase the utility of x_0 from $\leq \alpha \cdot (\beta - 1) \cdot w_p$ to, at least $t \cdot w_p$, by joining some group in $V(G_1)$). As a result, we can bipartition the k-stable partition in, respectively, a k-stable partition for G_1, and a k-stable partition for G_2. By Lemma 5, the fact that there does not exist a k-stable partition for the game played on G_0 implies the same for the game played on G_1. A contradiction. Therefore, there is no k-stable partition for H_G.

Conversely, let us assume that there exists an independent set of G with size at least β. We observe that, up to replacing w_p by 1 in $G_2 + x_0$, we fall in the uniform version of the game studied by Kleinberg and Ligett in [KL13]. In particular, by [KL13], there exists a k-stable partition c' whose largest group $X_j^{(c')}$ is a maximum independent set of the conflict graph $G + x_0$. Then, $c'(x_0) = j$ because it is an isolated vertex in $G + x_0$. Also, recall that there exists a k-stable partition c_0 for $G_0 - x_0$. Therefore, by Lemma 5, there exists a k-stable partition c_1 for $G_1 - x_0$. Let c be s.t.:

$$\begin{cases} c(v) = c'(v) \text{ if } v \in V(G_2) \cup \{x_0\} \\ c(v) = c_1(v) \text{ if } v \in V(G_1) \setminus \{x_0\}. \end{cases}$$

We claim that c is a k-stable partition for the gamed played on H_G. Indeed, suppose for the sake of contradiction that there exists a k-deviation S. By definition of c', c_0, we must have $S \cap (V(G_1) \setminus \{x_0\}) \neq \emptyset$, and in the same way $S \cap (V(G_2) \cup \{x_0\}) \neq \emptyset$. This implies $x_0 \in S \subseteq V(G_1)$. But then, let us choose a color j' s.t. $\forall v \in S,\ f_c(v) < \sum_{u \in X_{j'}^{(c')} \cup S} w_{uv}$. We must have $X_{j'}^{(c')} \cup S \subseteq V(G_1)$, and therefore, by assigning color j' to all the vertices in S the utility of x_0 becomes at most $f_0 + w_p \cdot t$. However, the former utility of x_0 was at least $w_p \cdot \alpha \cdot \beta = w_p \cdot (t + \alpha) \geq w_p \cdot (t + \frac{f_0}{w_p}) = w_p \cdot t + f_0$. In particular, the utility of x_0 has not increased, that is a contradiction because $x_0 \in S$. This concludes the NP-hardness proof, as our transformation is polynomial, and Maximum Independent Set is NP-complete [Dai80]. □

5 Conclusion

We have obtained several new results on the (non)existence of k-stable partitions, by identifying some relevant structural properties of the subgraphs induced by some subsets of weights, *e.g.*: girth, quasi-twin sets, and the (non)existence of some highly symmetric patterns. It would be interesting to study these problems in a different setting, where we allow any set of weights but we restrict ourselves to some well-structured class of graphs. Also, we would find it interesting to study whether our techniques could apply to the model of Angel et al. [ABK+16].

References

[ABK+16] E. Angel, E. Bampis, A. Kononov, D. Paparas, E. Pountourakis, and V. Zissimopoulos. Clustering on k-edge-colored graphs. *Discrete Applied Mathematics*, 2016.

[Bal04] C. Ballester. NP-completeness in hedonic games. *Games and Economic Behavior*, 49(1):1–30, 2004.

[BCDM19] J.-C. Bermond, A. Chaintreau, G. Ducoffe, and D. Mazauric. How long does it take for all users in a social network to choose their communities? *Discrete Applied Mathematics*, 270:37–57, 2019.

[BM08] J. A. Bondy and U. S. R. Murty. *Graph theory.* Grad. Texts in Math., 2008.

[BZ03] N. Burani and W. S. Zwicker. Coalition formation games with separable preferences. *Mathematical Social Sciences*, 45(1):27–52, 2003.

[CKPS10] I. Chatzigiannakis, C. Koninis, P. N. Panagopoulou, and P. G. Spirakis. Distributed game-theoretic vertex coloring. In *OPODIS'10*, pages 103–118, 2010.

[CNS18] J. Chen, R. Niedermeier, and P. Skowron. Stable marriage with multi-modal preferences. In *Proceedings of the 2018 ACM Conference on Economics and Computation*, EC '18, pages 269–286. ACM, 2018.

[Dai80] D. P. Dailey. Uniqueness of colorability and colorability of planar 4-regular graphs are NP-complete. *Discrete Mathematics*, 30(3):289 – 293, 1980.

[DBHS06] D. Dimitrov, P. Borm, R. Hendrickx, and S-C. Sung. Simple priorities and core stability in hedonic games. *Social Choice and Welfare*, 26(2):421–433, 2006.

[Duc16a] G. Ducoffe. The parallel complexity of coloring games. In *International Symposium on Algorithmic Game Theory*, pages 27–39. Springer, 2016.

[Duc16b] G. Ducoffe. *Propriétés métriques des grands graphes*. PhD thesis, Université Côte d'Azur, December 2016.

[EGM12] B. Escoffier, L. Gourvès, and J. Monnot. Strategic coloring of a graph. *Internet Mathematics*, 8(4):424–455, 2012.

[FMZ17] M. Flammini, G. Monaco, and Q. Zhang. Strategyproof mechanisms for additively separable hedonic games and fractional hedonic games. In *WAOA*, pages 301–316, 2017.

[GS62] D. Gale and L. Shapley. College admissions and the stability of marriage. *The American Mathematical Monthly*, 69(1):9–15, 1962.

[HJ17] M. Hoefer and W. Jiamjitrak. On proportional allocation in hedonic games. In *SAGT*, pages 307–319. Springer, 2017.

[JW96] M. Jackson and A. Wolinsky. A strategic model of social and economic networks. *Journal of economic theory*, 71(1):44–74, 1996.

[KL13] J. Kleinberg and K. Ligett. Information-sharing in social networks. *Games and Economic Behavior*, 82:702–716, 2013.

[MS17] M. Mnich and I. Schlotter. Stable marriage with covering constraints–a complete computational trichotomy. In *SAGT*, pages 320–332. Springer, 2017.

[MW13] J. R. Marden and A. Wierman. Distributed welfare games. *Operations Research*, 61(1):155–168, 2013.

[Mye13] R. B. Myerson. *Game theory*. Harvard university press, 2013.

[OBI+17] K. Ohta, N. Barrot, A. Ismaili, Y. Sakurai, and M. Yokoo. Core stability in hedonic games among friends and enemies: impact of neutrals. In *Proceedings of the Twenty-Sixth International Joint Conference on Artificial Intelligence, IJCAI-17*, pages 359–365, 2017.

[OR94] M. J. Osborne and A. Rubinstein. *A course in game theory.* MIT press, 1994.

[PS08] P. N. Panagopoulou and P. G. Spirakis. A game theoretic approach for efficient graph coloring. In *ISAAC'08*, pages 183–195, 2008.

[Ron90] E. Ronn. NP-complete stable matching problems. *Journal of Algorithms*, 11(2):285–304, 1990.

[SD10] S.-C. Sung and D. Dimitrov. Computational complexity in additive hedonic games. *European Journal of Operational Research*, 203(3):635–639, 2010.