Event-Triggered Control for Discrete-Time Systems Using a Positive Systems Approach

Frédéric Mazenc, Michael Malisoff, Corina Barbalata, and Zhong-Ping Jiang

***Abstract*— We provide an output feedback event-triggered controller for discrete-time linear systems. We make novel use of positive systems, interval observers, an event-triggered state estimator, and triggering times that are computed from estimator values. This provides a discrete-time analog of our recent positive systems approach for continuous-time systems. A key novel ingredient in our discrete-time event triggers is their use of vectors of absolute values, instead of the usual Euclidean norm. We illustrate the benefits of our method using a model for event-triggered BlueROV2 underwater vehicles.**

***Index Terms*— stabilization, event-triggered.**

I. Introduction

THIS paper continues our use of interval observers and positive systems to develop event-triggered control methods for systems where current state measurements may not be available; see [13]–[16]. The work [14] covered time invariant continuous-time linear systems. It was extended to linear time-varying continuous-time systems with uncertainties and outputs in [15]. Then, [13] and [16] provided continuous-time extensions of [14] under input and state delays. Here, we provide output feedback stabilization results for linear discrete-time systems with outputs.

Event-triggered control has the advantage that it can reduce computational burdens of implementing feedback controls, by only changing control values when a significant event occurs. This motivated significant research; see [3], [5], [7], [11], and [17]. Although computing methods can make it easier to recompute control values, the increasing use of shared networks strongly motivates reducing the frequency of control computations, to take computation, communication, or energy constraints into account [5]. During the same time, many works studied positive systems, which are systems for which the nonnegative orthant is positively invariant; see [10] for a general definition of positive systems. A key technique in the positive systems literature is interval observers, which produce intervals containing values of unknown states when the inequalities are interpreted to be componentwise; see [12]. Control theory based on interval observers and positive systems led to significant novel methods in aerospace engineering, mathematical biology, and other applications.

This motivates the event-triggered controls with outputs in this paper, which we believe are the first use of positive systems and interval observers for event-triggered control of discrete-time systems. The problems in this work cannot be solved by discretizing continuous-time event-triggered works (e.g., [14]), because of the continuous-time measurements in the event triggering rule in works like [14]. Instead, we use interval observer and positive systems methods for discrete-time systems, using new synergies of small-gain methods [6] and discrete-time systems [4], to prove input-to-state stability (or ISS). A key novel feature is our use of vectors of absolute values instead of standard Euclidean norms; see Section IV.

However, our stability proof uses interval observers for comparison systems, and so is inspired by works such as [12] insofar that it uses linear Lyapunov functions. One of the positive systems is the dynamics for $(\bar{X}, -\underline{X})$ where $\bar{X}$ and $\underline{X}$ are the upper and lower bounds for the observer state in the interval observer, respectively. The negative sign in front of the $\underline{X}$ will be required in order to produce dynamics with nonnegative coefficient matrices. On the other hand, our event-triggered controlled closed loop systems are not assumed to be, or transformed into, positive systems. Instead, we transform interval observers into positive systems. Hence, our work is not overlapping with notable works such as [1] and [9] on estimating reachability sets for (not necessarily positive) dynamics that are transformed into systems that are positive. Also, [1], [9], and [12] made no use of event-triggering. Hence, this work has the potential to reduce the computational burden as compared with discrete-time controls that are not event triggered. After stating and proving our theorem in Section III, Section IV has an application to a model of the BlueROV2 underwater vehicle that can be used for environmental surveys. This illustrates a benefit of our work in a example that is of compelling interest in robotics.

II. Preliminaries

We review notation that we also used in continuous-time cases in [14]. The dimensions of our Euclidean spaces are arbitrary unless we indicate otherwise. We omit arguments of functions when no confusion would arise. We set $\mathbb{Z}_0 = \{0, 1, 2, ...\}$. For a matrix $G = [g_{ij}] \in \mathbb{R}^{r\times s}$, we set $|G| =$

Manuscript received September 13, 2021; revised November 15, 2021; accepted December 4, 2021. The work was supported in part by NSF grants 2009659 (Malisoff and Barbalata) and 2009644 (Jiang). Recommended by Senior Editor J. Daafouz. (Corresponding author: *Michael Malisoff*.)

F. Mazenc is with Inria Saclay, L2S-CNRS-CentraleSupélec, 3 rue Joliot Curie, 91192, Gif-sur-Yvette, France (e-mail: frederic.mazenc@l2s.centralesupelec.fr)

M. Malisoff is with Department of Mathematics, Louisiana State University, Baton Rouge, LA 70803, USA (e-mail: malisoff@lsu.edu)

C. Barbalata is with Department of Mechanical and Industrial Engineering, Louisiana State University, Baton Rouge, LA 70803, USA (email: cbarbalata@lsu.edu)

Z.-P. Jiang is with Department of Electrical and Computer Engineering, Tandon School of Engineering, New York University, Brooklyn, NY 11201, USA (email: zjiang@nyu.edu)

$[|g_{ij}|]$, so the entries of $|G|$ are the absolute values of the corresponding entries g_{ij} of the matrix G. We also set $G^+ = [\max\{g_{ij}, 0\}]$ and $G^- = G^+ - G$. For a bounded sequence $M_k \in \mathbb{R}^{r\times s}$ where M_k has (i,j) entry m_{ijk} for $i = 1,\dots,r$ and $j = 1,\dots,s$, we define the matrix $\sup_k |M_k|$ to be the matrix whose (i,j) entry is s_{ij} where $s_{ij} = \sup_k |m_{ijk}|$ for $i = 1,\dots,r$ and $j = 1,\dots,s$. A square matrix is called Schur stable provided all of its eigenvalues are inside the open unit disk in the complex plane. For matrices $D = [d_{ij}]$ and $E = [e_{ij}]$ of the same size, we write $D < E$ (resp., $D \le E$) provided $d_{ij} < e_{ij}$ (resp., $d_{ij} \le e_{ij}$) for all i and j. We then also use $D \nleq E$ to mean that there is a pair (i,j) such that $d_{ij} > e_{ij}$. We adopt similar notation for vectors. We then call a matrix S positive (resp., nonnegative) provided $0 < S$ (resp., $0 \le S$), where 0 is the zero matrix. We also use standard notions of input-to-state stability (or ISS) for discrete-time systems [8]. We use $||\cdot||$ to denote the usual Euclidean norm. Also, I is the identity matrix. We consider the system

$$\left\{\begin{array}{rcl} X_{k+1} & = & (A+\delta_k^A)X_k + (B+\delta_k^B)u_k + \delta_k^D \\ Y_k & = & CX_k \end{array}\right. \tag{1}$$

where the sequence X_k of state vectors is valued in $\mathbb{R}^n$, the input sequence u_k is valued in $\mathbb{R}^p$, $Y_k \in \mathbb{R}^m$ are the measured variables, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, and $C \in \mathbb{R}^{m\times n}$ are known constant matrices, the unknown sequences $\delta_k^A \in \mathbb{R}^{n\times n}$ and $\delta_k^B \in \mathbb{R}^{n\times p}$ represent model uncertainty, and the unknown sequence $\delta_k^D \in \mathbb{R}^n$ represents a disturbance. The uncertainties δ^A and δ^B represent uncertain model parameters, which are prevalent in underwater marine robotics. We assume:

Assumption 1: There is a matrix $K \in \mathbb{R}^{p\times n}$ such that

$$H = A + BK \tag{2}$$

is nonnegative.

Assumption 2: There is a matrix $\Gamma \in \mathbb{R}^{n\times n}$ such that $\Gamma > 0$ and such that the matrix

$$H + |BK|\Gamma \tag{3}$$

is Schur stable.

Assumption 3: The pair (A, C) is observable.

Assumptions 1-2 are not restrictive, insofar that they can be satisfied after a change of coordinates under standard controllability conditions on the pair (A, B), as follows.

When a pair (A_a, B_a) is controllable, we can choose a matrix K_a such that $A_a + B_aK_a$ is Schur stable, having distinct real eigenvalues on the interval $(0,1)$. This follows from the Pole Shifting Theorem (e.g., [19, p.186]), which ensures that for any complex values $\lambda_1,\dots,\lambda_n$ and any controllable pair $(A_a, B_a) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times p}$, we can construct a matrix K_a such that these λ_i's form the set of all eigenvalues of $A_a + B_aK_a$. Then, there is a $P \in \mathbb{R}^{n\times n}$ such that $P(A_a + B_aK_a)P^{-1}$ is both Schur stable and nonnegative (by diagonalizing $A_a + B_aK_a$). Hence, if we consider $Z_{k+1} = A_aZ_k + B_au + \delta_{a,k}$, then the change of coordinates $X_k = PZ_k$ gives

$$X_{k+1} = AX_k + Bu + P\delta_{a,k} \tag{4}$$

with $A = PA_aP^{-1}$ and $B = PB_a$. Then, in this case, (4) satisfies Assumptions 1-2, because $K = K_aP^{-1}$ gives $A + BK = PA_aP^{-1} + PB_aK_aP^{-1} = P(A_a + B_aK_a)P^{-1}$ which is Schur stable and nonnegative, and because when a matrix $H \ge 0$ is Schur stable, there always exists a positive matrix $\Gamma > 0$ such that $H + |BK|\Gamma$ is Schur stable and nonnegative. This follows because the continuity of eigenvalues as functions of the entries of a matrix ensures that, if all eigenvalues of H are in the open unit circle in the complex plane, then the same is true for $H + \mathcal{E}$ when $\mathcal{E} \in \mathbb{R}^{n\times n}$ has small enough entries, and because we can choose the entries of Γ to be small enough so that $\mathcal{E} = |BK|\Gamma$ has small enough entries. Hence, we can find our required K by combining the Pole-Shifting Theorem with a similarity transformation. Nonnegativity of H is essential for showing that our interval observers provide the required bounds for proving our exponential stability theorem. However, we will not assume that H is Schur stable.

Assumption 2 ensures that there are a positive vector $V \in \mathbb{R}^n$ and a constant $p \in (0,1)$ such that

$$V^\top(H + |BK|\Gamma) \le pV^\top \tag{5}$$

holds; see [4, Lemma 2.7, p.79]. By the same reasoning, Assumption 3 provides a matrix $L \in \mathbb{R}^{n\times m}$ and an invertible matrix $Q \in \mathbb{R}^{n\times n}$ such that, with the choices

$$R = A + LC \text{ and } QRQ^{-1} = M, \tag{6}$$

the matrix R is Schur stable and $M \ge 0$. Since $M \ge 0$ is Schur stable, we can again apply [4, Lemma 2.7] to find a vector $W > 0$ in $\mathbb{R}^n$ and a constant $q \in (0,1)$ such that

$$W^\top M \le qW^\top. \tag{7}$$

We fix K, L, Γ, p, q, V, and W satisfying the preceding requirements. Then in terms of the sequence

$$\mathcal{M}_k = |Q(\delta_k^A + \delta_k^B K)| + |Q\delta_k^B K|\Gamma, \tag{8}$$

our last assumption is:

Assumption 4: The inequalities

$$\begin{array}{l} \sup_k W^\top \left(M + \left|Q\delta_k^A Q^{-1}\right|\right) \le qW^\top \text{ and} \\ \left|\left|V^\top |LC|\,|Q^{-1}|\right|\right| \sup_k \left|\left|W^\top \mathcal{M}_k\right|\right| < 1 \end{array} \tag{9}$$

are satisfied.

Assumption 4 holds when the suprema of the sequences δ_k^A and δ_k^B are small enough. For instance, since $|Q\delta_k^A Q^{-1}| \le ||Q\delta_k^A Q^{-1}||\mathbf{1}_{n\times n} \le ||Q||\,||Q^{-1}||\,||\delta_k^A||\,\mathbf{1}_{n\times n}$, and $||\mathcal{M}_k|| \le \mathcal{M}_* \max\{||\delta_k^A||, \delta_k^B||\}$ hold for all k when we choose $\mathcal{M}_* = ||Q||(1 + ||K|| + |||K|\Gamma||)$ and when we define $\mathbf{1}_{r\times s} \in \mathbb{R}^{r\times s}$ to be the matrix each of whose entries is 1, a set of sufficient conditions for Assumption 4 to hold is that

$$\begin{array}{l} ||\delta_k^A||\mathbf{1}_{1\times n} \le \frac{W^\top(qI - M)}{n||W||\,||Q||\,||Q^{-1}||} \text{ and} \\ \max\{||\delta_k^A||, \delta_k^B||\} < \frac{1}{\mathcal{M}_*||W||\,||V^\top|LC|\,|Q^{-1}|\,||} \end{array} \tag{10}$$

hold for all k, when the denominators in (10) are nonzero.

III. Main Theorem

A. Statement of Result and Remarks

We next provide our interval observer based output feedback control theorem. A key novel feature is its event trigger rule, which is expressed using vectors of absolute values instead of the standard Euclidean norm.

Theorem 1: Let Assumptions 1-4 hold, and K, L, and Γ satisfy the requirements of Section II. Consider the system

$$\begin{cases} X_{k+1} &= (A+\delta_k^A)X_k + (B+\delta_k^B)K\hat{X}_{\sigma(k)} + \delta_k^D \\ \hat{X}_{k+1} &= A\hat{X}_k + BK\hat{X}_{\sigma(k)} + L[C\hat{X}_k - Y_k] \end{cases} \tag{11}$$

where $\sigma : \mathbb{Z}_0 \to \mathbb{Z}_0$ is defined by the event trigger rule

$$\begin{aligned} &\sigma(0) = 0 \\ &\sigma(j+1) = j+1 \text{ if } |\hat{X}_{j+1} - \hat{X}_{\sigma(j)}| \not\leq \Gamma|\hat{X}_{j+1}| \\ &\sigma(j+1) = \sigma(j) \text{ if } |\hat{X}_{j+1} - \hat{X}_{\sigma(j)}| \leq \Gamma|\hat{X}_{j+1}|. \end{aligned} \tag{12}$$

Then (11) is globally exponentially ISS to 0 on $\mathbb{R}^{2n}$ with respect to δ^D.

Remark 1: Theorem 1 uses $u_k = K\hat{X}_{\sigma(k)}$, where (12) can be equivalently defined in the following recursive way. Setting $\sigma(0) = 0$, we check whether $|\hat{X}_1 - \hat{X}_{\sigma(0)}| \leq \Gamma|\hat{X}_1|$ holds. If it holds, then we set $\sigma(1) = \sigma(0) = 0$ and no change in the control value is made at time $k = 1$; otherwise, we set $\sigma(1) = 1$ and we update the control value to $u_1 = K\hat{X}_{\sigma(1)} = K\hat{X}_1$. Then, we repeat this process with $j = 0$ replaced by $j = 1$, and argue inductively to define the control values and $\sigma(j)$ for all $j \geq 0$. The motivation for defining σ in this way is that it will ensure that $|\hat{X}_j - \hat{X}_{\sigma(j)}| \leq \Gamma|\hat{X}_j|$ holds for all $j \in \mathbb{Z}_0$, as we will show in our proof of the theorem. See Remark 2 for discussions on global exponential ISS, and the special case where X is available for measurement. See also the end of the proof of our theorem, for an approach that can be used to find comparison functions for the ISS estimate, which show how to explicitly treat the uncertainties δ^D.

B. Proof of Theorem

The proof has three parts. First, we build interval observers for $\hat{X}_k$ and $Q\tilde{X}_k$, where $\tilde{X}_k = X_k - \hat{X}_k$ and Q is from Section II. In the second part, we provide stability analyses for both interval observers. In the third part, we combine the stability analyses to prove the ISS conclusion. First note that direct calculations give $\tilde{X}_{k+1} = AX_k - A\hat{X}_k - L[C\hat{X}_k - CX_k] + \delta_k^\sharp = R\tilde{X}_k + \delta_k^\sharp$ where $\delta_k^\sharp = \delta_k^A X_k + \delta_k^B K\hat{X}_{\sigma(k)} + \delta_k^D$, and $R = A + LC$ as before. Therefore, with the notation

$$\begin{aligned} &\hat{w}_k = \hat{X}_{\sigma(k)} - \hat{X}_k \text{ and} \\ &\delta_k^E = (\delta_k^A + \delta_k^B K)\hat{X}_k + \delta_k^B K\hat{w}_k + \delta_k^A\tilde{X}_k + \delta_k^D \end{aligned} \tag{13}$$

and $H = A + BK$ as defined before, we obtain the dynamics

$$\begin{cases} \hat{X}_{k+1} &= H\hat{X}_k + BK\hat{w}_k - LC\tilde{X}_k \\ \tilde{X}_{k+1} &= R\tilde{X}_k + \delta_k^E. \end{cases} \tag{14}$$

First Part: We first introduce the interval observer

$$\begin{cases} \overline{\widehat{X}}_{k+1} &= H\overline{\widehat{X}}_k + (BK)^+\widehat{w}_k^+ + (BK)^-\widehat{w}_k^- \\ &\quad + (-LC\tilde{X}_k)^- \\ \underline{\widehat{X}}_{k+1} &= H\underline{\widehat{X}}_k - (BK)^+\widehat{w}_k^- - (BK)^-\widehat{w}_k^+ \\ &\quad - (-LC\tilde{X}_k)^+ \end{cases} \tag{15}$$

where for any $k_0 \in \mathbb{Z}_0$, we choose initial conditions satisfying

$$\underline{\widehat{X}}_{k_0} \leq \widehat{X}_{k_0} \leq \overline{\widehat{X}}_{k_0}, \ \ \underline{\widehat{X}}_{k_0} \leq 0, \ \text{ and } \ \overline{\widehat{X}}_{k_0} \geq 0. \tag{16}$$

We next prove that for all $k \geq k_0$, we have

$$\underline{\widehat{X}}_k \leq \widehat{X}_k \leq \overline{\widehat{X}}_k, \ \ \overline{\widehat{X}}_k \geq 0, \ \text{ and } \ \underline{\widehat{X}}_k \leq 0. \tag{17}$$

To this end, notice that since H is nonnegative, and since $(BK)^+\hat{w}_k^+ + (BK)^-\hat{w}_k^- + (-LC\tilde{X}_k)^- \geq 0$ and $-(BK)^+\hat{w}_k^- - (BK)^-\hat{w}_k^+ - (-LC\tilde{X}_k)^+ \leq 0$ hold for all $k \geq k_0$, we straightforwardly deduce that

$$\underline{\hat{X}}_k \leq 0 \text{ and } \overline{\hat{X}}_k \geq 0 \text{ for all } k \geq k_0, \tag{18}$$

e.g., by induction on k. This makes essential use of the nonnegativity of H. In particular, the Schur stability property of H is not sufficient.

Also, the $\hat{X}_k$ dynamics in (14) can be rewritten as

$$\begin{aligned} \hat{X}_{k+1} = H\hat{X}_k &+ (BK)^+\hat{w}_k^+ + (BK)^-\hat{w}_k^- - (BK)^+\hat{w}_k^- \\ &- (BK)^-\hat{w}_k^+ - (-LC\tilde{X}_k)^+ + (-LC\tilde{X}_k)^-. \end{aligned} \tag{19}$$

By combining (15) and (19), we get

$$\begin{cases} \overline{\hat{X}}_{k+1} - \hat{X}_{k+1} &= H(\overline{\hat{X}}_k - \hat{X}_k) + (-LC\tilde{X}_k)^+ \\ &\quad + (BK)^+\hat{w}_k^- + (BK)^-\hat{w}_k^+ \\ \hat{X}_{k+1} - \underline{\hat{X}}_{k+1} &= H(\hat{X}_k - \underline{\hat{X}}_k) + (-LC\tilde{X}_k)^- \\ &\quad + (BK)^+\hat{w}_k^+ + (BK)^-\hat{w}_k^- \end{cases} \tag{20}$$

Since $H \geq 0$, $(BK)^+\hat{w}_k^- + (BK)^-\hat{w}_k^+ + (-LC\tilde{X}_k)^+ \geq 0$, and $(BK)^+\hat{w}_k^+ + (BK)^-\hat{w}_k^- + (-LC\tilde{X}_k)^- \geq 0$ all hold for all $k \geq k_0$, one can prove by induction that

$$\overline{\hat{X}}_k - \hat{X}_k \geq 0 \text{ and } \hat{X}_k - \underline{\hat{X}}_k \geq 0 \tag{21}$$

for all $k \geq k_0$, which proves the first two inequalities of (17), and which also uses the nonnegativity of H in an essential way. Hence, (17) holds for all $k \geq k_0$, by (18).

To derive our interval observer for $Q\tilde{X}_k$, first note that (14) and the formula $Q^{-1}MQ = R$ from (6) give $Q\tilde{X}_{k+1} = MQ\tilde{X}_k + Q\delta_k^E$. We now introduce the interval observer

$$\overline{\xi}_{k+1} = M\overline{\xi}_k + (Q\delta_k^E)^-, \ \ \underline{\xi}_{k+1} = M\underline{\xi}_k - (Q\delta_k^E)^+ \tag{22}$$

with the initial conditions

$$\overline{\xi}_{k_0} = (Q\tilde{X}_{k_0})^+ \ \text{ and } \ \underline{\xi}_{k_0} = -(Q\tilde{X}_{k_0})^-. \tag{23}$$

Since $M \geq 0$, we can argue as we did for (15) (with $\hat{X}_k$ replaced by ξ_k, H replaced by M, $\hat{w}_k$ replaced by 0, and $-LC\tilde{X}_k$ replaced by $Q\delta_k^E$) to prove that

$$\underline{\xi}_k \leq Q\tilde{X}_k \leq \overline{\xi}_k, \ \ \underline{\xi}_k \leq 0, \ \text{ and } \ \overline{\xi}_k \geq 0 \tag{24}$$

for all $k \geq k_0$.

Second Part: Stability Analysis for Interval Observers (15) and (22). Using (15), elementary calculations give

$$\begin{aligned} \overline{\widehat{X}}_{k+1} - \underline{\widehat{X}}_{k+1} = &\ H\left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) \\ &+ (BK)^+\widehat{w}_k^+ + (BK)^+\widehat{w}_k^- \\ &+ (BK)^-\widehat{w}_k^- + (BK)^-\widehat{w}_k^+ \\ &+ (-LC\tilde{X}_k)^+ + (-LC\tilde{X}_k)^- \\ = &\ H\left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) + |BK||\widehat{w}_k| + |LC\tilde{X}_k|. \end{aligned} \tag{25}$$

Here and in the sequel, all equalities and inequalities should be understood to hold for all $k \geq k_0$ unless otherwise indicated. Also, by separately considering the cases $\sigma(k) = k$ and $\sigma(k) \neq k$, we can use (12) (with the choice $j = k - 1$) to conclude that the variable $\hat{w}_k$ defined in (13) satisfies

$$|\hat{w}_k| \leq \Gamma|\hat{X}_k| \tag{26}$$

for all $k \geq k_0$. By left multiplying (25) by $V^\top$, this gives

$$\begin{aligned} &V^\top \left(\overline{\widehat{X}}_{k+1} - \underline{\widehat{X}}_{k+1}\right) \\ &\leq V^\top H \left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) + V^\top|BK|\Gamma|\hat{X}_k| + V^\top|LC\tilde{X}_k|. \end{aligned} \tag{27}$$

Therefore, it follows from (17) and (5) that

$$\begin{aligned} &V^\top \left(\overline{\widehat{X}}_{k+1} - \underline{\widehat{X}}_{k+1}\right) \\ &\leq V^\top(H + |BK|\Gamma)\left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) + V^\top|LC\tilde{X}_k| \\ &\leq pV^\top \left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) + V^\top|LC||\tilde{X}_k| \\ &\leq pV^\top \left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) + V^\top|LC||Q^{-1}||Q\tilde{X}_k|, \end{aligned} \tag{28}$$

because $|\hat{X}_k| \leq \overline{\widehat{X}}_k - \underline{\widehat{X}}_k$. Since (24) gives $|Q\tilde{X}_k| \leq \bar{\xi}_k - \underline{\xi}_k$ for all $k \geq k_0$, we can use (28) to get

$$\begin{aligned} &V^\top \left(\overline{\widehat{X}}_{k+1} - \underline{\widehat{X}}_{k+1}\right) \\ &\leq pV^\top \left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) + V^\top|LC||Q^{-1}|(\bar{\xi}_k - \underline{\xi}_k), \end{aligned} \tag{29}$$

which is our desired estimate for the first interval observer.

To obtain our stability estimate for the second interval observer (22), notice that (22) gives

$$\bar{\xi}_{k+1} - \underline{\xi}_{k+1} = M(\bar{\xi}_k - \underline{\xi}_k) + |Q\delta_k^E|. \tag{30}$$

Also, since our conditions (17) and (24) give $|\hat{X}_k| \leq \bar{\hat{X}}_k - \underline{\hat{X}}_k$ and $|Q\tilde{X}_k| \leq \bar{\xi}_k - \underline{\xi}_k$ for all $k \geq k_0$, we can combine (13), (17), and (26) to obtain

$$\begin{aligned} |Q\delta_k^E| \;\leq\; &\left(|Q(\delta_k^A + \delta_k^B K)| + |Q\delta_k^B K|\Gamma\right)\left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) \\ &+ |Q\delta_k^D| + |Q\delta_k^A Q^{-1}|(\bar{\xi}_k - \underline{\xi}_k). \end{aligned} \tag{31}$$

From Assumption 4, it follows from (30)-(31) that

$$\begin{aligned} W^\top(\bar{\xi}_{k+1} - \underline{\xi}_{k+1}) \;=\; &W^\top M(\bar{\xi}_k - \underline{\xi}_k) + W^\top|Q\delta_k^E| \\ \leq\; &qW^\top(\bar{\xi}_k - \underline{\xi}_k) + W^\top|Q\delta_k^D| \\ &+ W^\top\mathcal{M}_k\left(\overline{\widehat{X}}_k - \underline{\widehat{X}}_k\right) \end{aligned} \tag{32}$$

for all $k \geq k_0$, where $\mathcal{M}_k$ is from (8).

Third Part. Setting $V^a(q) = V^\top q$, $V^B(q) = W^\top q$, $q^a = \bar{\hat{X}} - \underline{\hat{X}}$, $q^B = \bar{\xi} - \underline{\xi}$, $\gamma^a(s) = ||V^\top|LC||Q^{-1}|||s$, $\gamma^B(s) = \sup_k ||W^\top\mathcal{M}_k||s$, $\gamma^D(s) = ||W||\,||Q||s$, $c^a = 1 - p$, and $c^B = 1 - q$, we can combine (29) and (32) to get

$$\begin{aligned} V^a(q^a_{k+1}) - V^a(q^a_k) \;&\leq\; -c_a V^a(q^a_k) + \gamma^a(||q^B_k||) \text{ and} \\ V^B(q^B_{k+1}) - V^B(q^B_k) \;&\leq\; -c_b V^B(q^B_k) + \gamma^B(||q^a_k||) \\ &\quad + \gamma^D(||\delta^D_k||) \end{aligned} \tag{33}$$

for all $k \geq k_0$. It now follows from the second inequality in (9) (which implies that the small-gain condition $\gamma^a \circ \gamma^B(s) < s$ holds for all $s > 0$) and the proof of [6, Theorem 1] that the dynamics for the joint variable

$$X^\sharp = \left(\overline{\widehat{X}} - \underline{\widehat{X}}, \bar{\xi} - \underline{\xi}\right) \tag{34}$$

is globally exponentially ISS on $\mathbb{R}^n$ with respect to δ^D. If we now assume that the initial conditions are

$$\overline{\widehat{X}}_{k_0} = \widehat{X}^+_{k_0} \text{ and } \underline{\widehat{X}}_{k_0} = -\widehat{X}^-_{k_0}, \tag{35}$$

then we can use (17), (23), and (24) to get $|\hat{X}_k| \leq \bar{\hat{X}}_k - \underline{\hat{X}}_k$ and $|Q\tilde{X}_k| \leq \bar{\xi}_k - \underline{\xi}_k$ for all $k \geq k_0$, and

$$\bar{\hat{X}}_{k_0} - \underline{\hat{X}}_{k_0} = |\hat{X}_{k_0}| \text{ and } \bar{\xi}_{k_0} - \underline{\xi}_{k_0} = |Q\tilde{X}_{k_0}|, \tag{36}$$

and therefore also

$$\begin{aligned} ||(\hat{X}_k, X_k)|| \;&\leq\; ||\overline{\hat{X}}_k - \underline{\hat{X}}_k|| + ||\tilde{X}_k + \hat{X}_k|| \\ &\leq\; 2||\overline{\hat{X}}_k - \underline{\hat{X}}_k|| + ||Q^{-1}||||Q\tilde{X}_k|| \\ &\leq\; 2||\overline{\hat{X}}_k - \underline{\hat{X}}_k|| + ||Q^{-1}||||\bar{\xi}_k - \underline{\xi}_k|| \\ &\leq\; 2\max\{2, ||Q^{-1}||\}||X^\sharp_k|| \end{aligned} \tag{37}$$

for all $k \geq k_0$ and

$$\begin{aligned} ||X^\sharp_{k_0}|| \;&\leq\; ||\hat{X}_{k_0}|| + ||Q\tilde{X}_{k_0}|| \\ &\leq\; ||\hat{X}_{k_0}|| + ||Q||(||X_{k_0}|| + ||\hat{X}_{k_0}||) \\ &=\; (1 + ||Q||)||\hat{X}_{k_0}|| + ||Q||\,||X_{k_0}|| \\ &\leq\; (1 + 2||Q||)||(X_{k_0}, \hat{X}_{k_0})|| \end{aligned} \tag{38}$$

for all $k_0 \geq 0$. We can use (37)-(38) to convert the ISS estimate for $X^\sharp$ into the required ISS estimate of the theorem.

Remark 2: The preceding proof shows that in the special case where the state X is available for measurement, the theorem remains true if we replace the estimator $\hat{X}$ by X in the event trigger. Its global exponential ISS conclusion means that there are positive constants c_1 and c_2 such that the following estimate holds for all solutions of (11), all choices of δ^A and δ^B satisfying Assumption 4, all choices of δ^D, and all initial times $k_0 \in \mathbb{Z}_0$: $||(X_k, \hat{X}_k)|| \leq c_1 e^{-c_2(k-k_0)}||(X_{k_0}, \hat{X}_{k_0})|| + c_1 \sup\{||\delta^D_r|| : k_0 \leq r \leq k\}$ for all $k \geq k_0$. See also Remark 3 for a comparison with event triggers that instead use the usual Euclidean norm.

Remark 3: The triggering rule (12) calls for choosing the event triggering times such that $|\hat{X}_j - \hat{X}_{\sigma(j)}| \leq \Gamma|\hat{X}_j|$ holds for all indices j; see (26). Due to our $|\cdot|$ notation, this is equivalent to requiring that

$$|\hat{X}_{j,i} - \hat{X}_{\sigma(j),i}| \leq \sum_{k=1}^{n} \Gamma_{ik}|\hat{X}_{j,k}| \text{ for } i = 1, \ldots, n, \tag{39}$$

where $\Gamma = [\Gamma_{ij}]$ and $\hat{X}_{r,s}$ is the sth component of $\hat{X}_r$ for all $r \in \mathbb{Z}_0$ and $s = 1, \ldots, n$. By contrast, if we had instead used the standard Euclidean norm to define our triggering criterion (as one would do, e.g., when using quadratic Lyapunov functions), then we would have selected a constant $\sigma_* > 0$ such that with the σ notation from Theorem 1, we have

$$||\hat{X}_j - \hat{X}_{\sigma(j)}|| \leq \sigma_*||\hat{X}_j|| \tag{40}$$

for all $j \in \mathbb{Z}_0$. To better understand the effects of using vectors of absolute values in (12) instead of the Euclidean norm in the event triggers, it is therefore useful to find the largest possible σ_* such that the requirement from (39) holds for all $\hat{X}$ and j choices such that (40) holds. Let us now show that this largest value is $\sigma_* = \min_{ij}\Gamma_{ij}$, i.e., the smallest of the entries of the positive matrix Γ. To this end, first note that with this choice of σ_*, and for any v and w in $\mathbb{R}^n$ such that $||v - w|| \leq \sigma_*||v||$ and any $i \in \{1, \ldots, n\}$, we can use the subadditivity of the square root to get

$$|v_i - w_i| \leq ||v - w|| \leq \sigma_*||v|| \leq \sum_{k=1}^{n} \Gamma_{ik}|v_k|. \tag{41}$$

Hence, with this σ_*, the condition from (40) implies (39). Next choose a pair (p,q) such that $\sigma_* = \Gamma_{pq}$. First consider the case where $p \neq q$, and any constant $\ell_* > \sigma_*$ and the choices of $v \in \mathbb{R}^n$ and $w \in \mathbb{R}^n$ whose components are defined by $v_i = 1$ if $i = q$ and $v_i = 0$ if $i \neq q$, and by

$$w_i = \begin{cases} 1, & \text{if } i = q \\ \ell_*, & \text{if } i = p \\ 0, & \text{if } i \neq p \text{ and } i \neq q \end{cases} \tag{42}$$

for $i = 1, \dots, n$. With the preceding choices, we have $||v - w|| = \ell_* = \ell_* ||v||$. The preceding choices also give

$$|v_p - w_p| = \ell_* > \sigma_* = \sum_{k=1}^{n} \Gamma_{pk} |v_k| \tag{43}$$

so the condition from (39) would not be satisfied for the index $i = p$. The same reasoning applies in the case where $p = q$, by instead defining the components of v and w to be $v_p = 1$, $w_p = 1 + \ell_*$, and $v_i = w_i = 0$ for all $i \neq p$. This shows how the largest (i.e., least conservative) σ_* we could have allowed if we had instead used the standard Euclidean norm would have been $\sigma_* = \min_{ij} \Gamma_{ij}$. In Section IV, we illustrate an advantage of using our triggers based on vectors of absolute values instead of using Euclidean norms.

IV. Illustration

We consider a dynamics for the control of the depth and pitch degrees-of-freedom (or DOF) of an autonomous underwater vehicle (or AUV), e.g., the BlueROV2 vehicle which is used for environmental surveys such as the study of corals. The vehicle is assumed to be equipped with a Doppler Velocity Logger (or DVL) that estimates its velocity. When working close to the sea floor, the DVL experiences bottom lock, which makes it impractical to continuously change the control values. Hence, we design a control system for the depth plane, using a more practical sample data approach.

As shown in [18, Equation (9.28)], after linearization and assuming that the vehicle is neutrally buoyant, the linearized dynamics in the depth plane are given by

$$\begin{aligned} &(m - X_{\dot{w}(t)})\dot{w}(t) - (mx_g + Z_{\dot{q}})\dot{q}(t) \\ &-Z_w w(t) - (mU(t) + z_q)q(t) = Z_{\gamma_s} u_Z \\ &\text{and } (mx_g + M_{\dot{w}}(t))\dot{w}(t) + (I_{yy} - M_{\dot{q}})\dot{q}(t) \\ &-M_w w(t) + (mx_g U - M_q)q(t) - M_\theta \theta = M_{\gamma_s} u_M \end{aligned} \tag{44}$$

whose parameters were experimentally computed and presented in [18]. We assume that the surge nominal velocity is $U = 0.1$m/s. The states are the depth and pitch velocity $x = [w,\ q]^\top$ and the control inputs u_Z and u_M represent the force and moment required to produce motion of the vehicle. With the parameter values and controller from [18], the system (44) has the form $\dot{x}(t) = Fx(t) + Gu$ with

$$F = \begin{bmatrix} -0.17742 & -0.3027 \\ 0.5394 & -1.4685 \end{bmatrix} \text{ and } G = \begin{bmatrix} -0.2063 \\ -0.7629 \end{bmatrix}. \tag{45}$$

Assuming that the control is piecewise constant with a constant sample rate $\bar{s}$, we now convert the dynamics into a discrete-time system of the form $Z_{k+1} = A_a Z_k + B_a u$ with

$$A_a = e^{\bar{s}F} \text{ and } B_a = \int_0^{\bar{s}} e^{F(\bar{s}-\ell)} \mathrm{d}\ell G, \tag{46}$$

by using a variation of parameters on $[i\bar{s}, (i+1)\bar{s})$ for each $i \in \mathbb{Z}_0$ with a constant control on each interval. This conversion to a discrete time system is strongly motivated by the fact that when implementing robotic controllers using the Robotic Operating System (ROS) for any robot, the implementation must be done in discrete time. It also illustrates the applicability of our method to continuous time plants, after a discretization.

We next show how the assumptions of Theorem 1 can be satisfied after a change of coordinates, where we first assume that $C = [1,\ 0]$. We choose $\bar{s} = 0.5$, and a matrix K_a such that $H_a = A_a + B_a K_a$ has the eigenvalues 0.25 and 0.5 to get the required Schur stability condition on H_a, using the command `StateFeedbackGains` in Mathematica. Then we diagonalize H_a to obtain a new matrix $P(A_a + B_a K_a)P^{-1} = H$ that is Schur stable and nonnegative. Then, with $A = PA_aP^{-1}$, $B = PB_a$, $V = W = [1,\ 2]^\top$, all entries of Γ being 0.25, $p = 0.95$, $q = 0.94$, $L = -[1.5, 0]^\top$, $K = [-0.0529002, -0.920342]$, and $Q = I$, the requirements of Theorem 1 all hold when the suprema of the uncertainties δ_k^A and δ_k^B are small enough to satisfy (9) from Assumption 4. For instance, for the constant matrices $\delta_k^A = [d_{ij}^A]$ with $d_{12}^A = c_*$ for some constant $c_* > 0$ and all other entries of the δ_k^A's being zero and $\delta_k^B = c_*[1,1]^\top$ for all k, Assumption 4 is satisfied for all $c_* \in [0, 0.085]$.

Moreover, we found that our use of vectors of absolute values in our triggering rule (12) instead of the usual norm had the desired benefit of reducing the number of triggering times in our numerical experiments. For instance, in the special case where the entire state is available for measurement, and with the initial state $X_0 = [1, -2]^\top$ and the constant perturbation $\delta_k^D = [0.2, 0.1]^\top$ and the time horizon $[0, 5]$ (with A, B, K, Γ, and $\bar{s}$ chosen as in the previous paragraphs, and with δ^A and δ^B both being zero), our event trigger (12) (with $\hat{X}$ replaced by X in (12)) produced $\sigma(1) = 1$, $\sigma(2) = 2$, and $\sigma(3) = \sigma(4) = \sigma(5) = 2$ and therefore only 2 triggering times for the preceding underwater dynamics. This compares favorably with standard zero-order hold controls which would have required 5 changes of control values on this time horizon. On the other hand, the corresponding standard event triggering criterion $||X_j - X_{\sigma(j)}|| \leq \sigma_* ||X_j||$ based on the usual Euclidean norm, with $\sigma_* = 0.25$ being the smallest element of Γ (and all other parameters the same as in the preceding paragraph), produced 3 event trigger times on the interval $[0, 5]$. This illustrates the reduction in computational burden from our novel event trigger (12). In this example, we ended at time $k = 5$ because $||X(k)||$ remained below 0.1 after time $k = 5$ with both triggers, while subjected to the preceding constant external disturbance, so $k = 5$ was a settling time.

We also found that the event triggers from our theorem provided fewer control recomputation times t_i (as compared with the event-trigger $||\hat{X}_j - \hat{X}_{\sigma(j)}|| \leq \sigma_* ||\hat{X}_j||$ with σ_* being the smallest entry of Γ that is based on the usual Euclidean norm) in cases where $C = [1, 0]$, i.e., when only the first component is available for measurement. We illustrate this point in Fig. 1, which shows our MATLAB simulations for the depth-pitch controller of the AUV using the positive system approach from our theorem. Our figure shows the results for three different cases for the initial states, and with the uncertainties δ^A, δ^B, and δ^D taken to be zero.

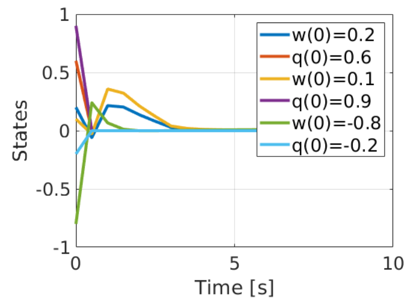

Fig. 1. Depth-pitch control simulation results

Our simulations in Fig. 1 used the K and L from the preceding paragraphs, and the choice

$$\Gamma = \begin{bmatrix} 0.015 & 0.045 \\ 0.15 & 0.15 \end{bmatrix}, \tag{47}$$

which satisfied the requirements of Theorem 1 with the preceding choices of K and L. For our simulations for the time horizon of $[0,\ 10]$ seconds, the event trigger from Theorem 1 produced an average of 8 trigger times t_i. By contrast, using the standard Euclidean norm in the event trigger as in Remark 3 with the same initial conditions, over the same time horizon and with $\sigma_* = 0.015$ being the smallest element of Γ, the event triggered an average of 12 times in our simulations. The settling times from using our event triggers from Theorem 1 and from using the one based on the Euclidean norm, and from using periodic sampling with period 1, were similar. Since the control from Theorem 1 resulted in a 33% average reduction in the number of triggering times without increasing the settling times, this helps illustrate the potential benefits of our method for reducing the numbers of control recomputation times.

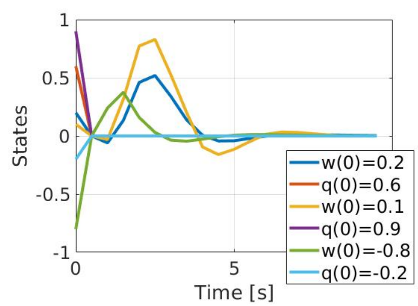

Fig. 2. Depth-pitch control simulations for large uncertainties

We found similar reductions in the numbers of event trigger times under larger uncertainties, for many uncertainty values. For instance, in Fig. 2, we present MATLAB simulations using the same parameter values we used for Fig. 1, except under larger uncertainties δ^A and δ^B in the state and input matrices. The uncertainty was modeled using a Gaussian representation, with a standard deviation of 0.25. The event was triggered 14 times using the control from Theorem 1, while with the standard 2-norm, it was triggered 19 times. Even under these large uncertainties, the system stabilizes, although oscillations are present. We also found that the settling times and oscillations using the controls from Theorem 1 were similar to what we obtained from instead using periodic controls (i.e., control recomputations at the each time k). Moreover, although the controls from Theorem 1 produced trigger times even in the absence of disturbances, and so do not satisfy the consistency condition from works such as [2], our novel event triggers provide benefits by reducing triggering instances.

V. Conclusion

We used interval observer and positive systems methods to prove global exponential ISS of event-triggered output feedback controlled linear discrete-time systems. A key feature was our new class of discrete-time event triggers, using matrices of absolute values instead of the usual Euclidean norm. We illustrated the benefits of using our approach, in an example that has implications for underwater environmental surveys. In future work, we aim to prove generalizations to time-varying systems with state, input, or sensor delays.

References

[1] Abate, M., and S. Coogan. Improving the fidelity of mixed-monotone reachable set approximations via state transformations. In *Proc. 2021 American Control Conf.*, pp. 4664-4669.

[2] Antunes, D., and M. Balaghi. Consistent event-triggered control for discrete-time linear systems with partial state information. *IEEE Control Systems Letters*, 4(1):181-186, 2020.

[3] Brunner, F., W.P.M.H. Heemels, and F. Allgower. Robust event-triggered MPC with guaranteed asymptotic bound and average sampling rate. *IEEE Trans. Autom. Control*, 62(11):5694-5709, 2017.

[4] Haddad, W., V. Chellaboina, and Q. Hui. *Nonnegative and Compartmental Dynamical Systems*. Princeton University Press, Princeton, NJ, 2010.

[5] Heemels, W., K. Johansson, and P. Tabuada. An introduction to event-triggered and self-triggered control. In *Proc. 51st IEEE Conf. Decision and Control (Maui, Hawaii, 10-13 December 2012)*, pp. 3270-3285.

[6] Jiang, Z.-P., Y. Lin, and Y. Wang. Nonlinear small-gain theorems for discrete-time feedback systems and applications. *Automatica*, 40(12): 2129-2136, 2004.

[7] Jiang, Z.-P., and T. Liu. Small-gain theory for stability and control of dynamical networks: A survey. *Annual Reviews in Control*, 46:58-79, 2018.

[8] Jiang, Z.-P., and Y. Wang. Input-to-state stability for discrete-time nonlinear systems, *Automatica*, 37(6):857-869, 2001.

[9] Khajenejad, M., and S.-Z. Yong. Tight remainder-form decomposition functions with applications to constrained reachability and interval observer design. *arXiv:2103.08638*.

[10] Krause, U. *Positive Dynamical Systems in Discrete Time*. De Gruyter, Boston, MA, 2015.

[11] Liu, D., and G.-H. Yang. Dynamic event-triggered control for linear time-invariant systems with L_2-gain performance. *International J. Robust and Nonlinear Control*, 29(2):507-518, 2019.

[12] Mazenc, F., and M. Malisoff. Stability analysis for time-varying systems with delay using linear Lyapunov functionals and a positive systems approach. *IEEE Trans. Autom. Control*, 61(3):771-776, 2016.

[13] Mazenc, F., M. Malisoff, and C. Barbalata. Event-triggered control for continuous-time linear systems with a delay in the input. *Systems and Control Letters*, 105075, 10pp., 2022.

[14] Mazenc, F., M. Malisoff, C. Barbalata, and Z-P. Jiang. Event-triggered control using a positive systems approach. *European J. Control*, 62:63-68, 2021.

[15] Mazenc, F., M. Malisoff, C. Barbalata, and Z-P. Jiang. Event-triggered control for linear time-varying systems using a positive systems approach. *Submitted*, http://www.math.lsu.edu/∼malisoff/.

[16] Mazenc, F., M. Malisoff, C. Barbalata, and Z.-P. Jiang. Event-triggered control for systems with state delays using a positive systems approach. In *Proc. 60th IEEE Conf. Decision and Control*, to be published, http://www.math.lsu.edu/∼malisoff/.

[17] Peralez, J., V. Andrieu, M. Nadri, and U. Serres. Event-triggered output feedback stabilization via dynamic high-gain scaling. *IEEE Trans. Autom. Control*, 63(8):2537-2549, 2018.

[18] Prestero, T. *Verification of a Six-Degree of Freedom Simulation Model for the REMUS Autonomous Underwater Vehicle*. MS Thesis, MIT, Cambridge, MA, 2001.

[19] Sontag, E. *Mathematical Control Theory, Second Edition*. Springer, New York, NY, 1998.