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Abstract
The classical sediment transport models based on shallow water equations (SWE) describes the hydro-morphodynamic process without horizontal velocity shear along the vertical and considers that the fluid velocity is equal to sediment velocity. The classical shear shallow water (SSW) with friction and topography source terms assumes that the fluid density is uniform in all the space. Nevertheless, for the coastal flows with sediment transport we are interested in it seems essential to consider these shear effects, the phase lag effect and the nonhomogenous ness of fluid density. In this paper, we develop new sediment transport models incorporating the shear velocity along the vertical, the phase lag effect and the spatial variation of the fluid density. The starting point is the 2D equations for the evolution of mixing quantities and sediment volume rate. These equations describe the motion of fluid mixing in a domain bounded by a dynamical water surface and water bed. Contrary to the classical sediment transport models, the second-order vertical fluctuations of the horizontal velocity are considered. Considering the kinematic conditions on the moving surfaces, we apply an average along the depth on the three-dimensional equations to obtain simplified equations. The resulting model has a wider range of validity and integrates the morphodynamic processes proposed in the literature. The proposed mathematical derivation is in the context of recent developments with the additional presence of sediment and a dynamic bed.

Keywords
Shear shallow water; Sediment transport, Morphological dynamic; Douala; Coastal city.

I INTRODUCTION

Douala is a coastal city that is subject to weighty rains that frequently cause flooding and lead to the transport of sediments and their deposit in the port channel, which hinders the optimal operation of the port of Douala. The flooding result from the interactions between oceanic tidal forcing, torrential rains with a constrained runoff, transport of sediments of variable characteristics: either mineral (sand, clays,...) or organic (plants, sewage residues,...). The currents here are powerful and result in significant shear effects which play a essential role in the realistic description of the dynamics of these flows. Mathematical models describing such flows often assume that there is a carrier fluid (water) and that the sediments are either suspended in the carrier fluid (Suspended-Layer) or constitute a bottom layer in motion or at rest (Bedload-Layer). The interactions between these two main layers take place through a third exchange layer that is, in general, so thin that its thickness can be neglected. In the suspended layer, both the primary fluid and the suspended particles can be assumed to be incompressible. Instead of following the individual evolution of each suspended particle in the fluid, many models adopt a continuum scale formulation of the particles that can be reduced to their volume concentration in the carrier fluid. Based on continuum scale approach, the father-model considered here is a 2D Multi-fluid formulation of the fluid-sediment mixture described as: [**van_Rijn_1987**] (Chap. 2)

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z} = 0 \tag{1}$$

$$\frac{\partial}{\partial t}(u)+\frac{\partial}{\partial x}\left(uu+\frac{p}{\rho}\right)+\frac{\partial}{\partial z}(uw) = \boldsymbol{\mathcal{F}}_x \tag{2}$$

$$\frac{\partial}{\partial z}(p) = \boldsymbol{\mathcal{F}}_z \tag{3}$$

$$\frac{\partial \alpha_s}{\partial t}+\frac{\partial}{\partial x}(u\alpha_s)+\frac{\partial}{\partial z}(w\alpha_s) = \nabla\cdot\left(\underline{\mathbf{D}}_s\nabla\alpha_s\right) \tag{4}$$

where x and z are respectively the horizontal and the vertical coordinates. The mixture density is defined as $\rho=\rho_f\left(1-\alpha_s\right)+\rho_s\alpha_s$, where the main fluid density ρ_f and the suspended particles density ρ_s are assumed to be constant in space and time. The components of the mixture velocity are u in the horizontal direction and w in the vertical direction. The external forces, including viscous forces are defined by the vector

$$\boldsymbol{\mathcal{F}}_x$$

. And the term $\boldsymbol{\mathcal{F}}_x=\rho g$. The pressure p will be defined according to the hydrostatic assumption. The volume fraction of the sediments is denoted by α_s. The diffusion contribution in the evolution of the sediments volume fraction α_s is there to take into account the deviations from the mixture velocity of the sediment velocity. The suspension is sufficiently dilute (Boussinesq approximation) to consider that the value of the kinematic viscosity of water/sediment ($\underline{\mathbf{D}}_s$) is equal to that of clear water ($\underline{\mathbf{D}}_f$): $\underline{\mathbf{D}}_s=\underline{\mathbf{D}}_f$. There is a counterpart in the evolution of the fluid volume fraction that will compensate to achieve the following evolution of the mixture density:

$$\frac{\partial \rho}{\partial t}+\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial z}(\rho w)=0 \tag{5}$$

The partial differential equations (1)-(4) describes the dynamics of the flow by the variables u, w, and α_s in a domain defined by two moving surfaces parameterized by $z=\xi(t,x)$ for the upper surface of the flow and by $z=\mathrm{b}(x)+\mathrm{z}_b(t,x)$ for the bedload layer. For coastal flow, it is now well known that shear stress plays an important role in the flow dynamic. These numerical approaches are based on the Reynolds procedure and offer a simple way to consider the reduction of the mixing coefficients induced by turbulence. On the other hand, simplified modeling is also used by introducing the depth-averaging process that usually neglects the fluctuation of the velocities in the vertical direction [**Simpson_and_Castelltort_2006**]. This paper proposes intermediate modeling by considering the shear and non-homogeneity in depth averaging framework. This work is an extension of the classical shear shallow water model based on homogeneous fluid, developed and studied in [**Teshukov_2007**, **Gavrilyuk_et_Al_2018**, **Praveen_et_Al_2020**]. The work integrates the mixture density (water/sediment density) and assumes that this density is variable along the vertical. The obtained model can be named the Generalized Shear Shallow Water(Generalized-SSW) model. We propose a coupling with sediment transport equation and the bed-load dynamics that is an alternative to the classical Exner equation describing the bedload transport.

II SEDIMENT TRANSPORT MODELS IN GENERALIZED SHEAR SHALLOW WATER EQUATIONS.

As we have already pointed out, the sediment flow is confined in a domain bounded by an upper interface with the atmosphere and a bottom bedload interface. These interfaces are evolving in time and their positions are

$$z(t) = \xi(t,x) \qquad \text{and} \qquad z(t) = \mathrm{b}(x) + \mathrm{Z}_b(t,x) \tag{6}$$

where $\xi(t,x)$ and $\mathrm{b}(x)$ are the vertical positions of respectively the air-fluid interface and the non-erodible bedload (assumed independent of time). The thickness of the deposited sediment is $\mathrm{Z}_b(t,x)$ and variable in space and time, such that the interface of the fluid with the sedimentary bedload is at the vertical position $z = \mathrm{b}(x) + \mathrm{Z}_b(t,x)$. In order to model the dynamics of the mouving interfaces, let us considered a fluid particle located at one of that surfaces at the time t. The position in Lagrangian coordinates is given by $\mathcal{X}(x(t),t) = (x(t), z(t))^T$ where for the upper surface we have $z(t) = \xi(t,x)$. Then the time derivative of the function $\mathrm{F}_\xi(t) = z(t) - \xi(t, x(t))$ function will defined the net of particles that are detached and/or agglomerated to the interface. On the other hand the velocity is the time derivative of the Lagrangian coordinates. Therefore, the kinematic condition writes as

$$\frac{\partial \xi}{\partial t} + u(t,x,\xi)\frac{\partial \xi}{\partial x} - w(t,x,\xi) = \frac{d\mathrm{F}_\xi}{dt} \tag{7}$$

The material derivative $\frac{d\mathrm{F}_\xi}{dt}$ is the net water volume rate change per unit of time related to evaporation and/or rainfall. This exchange rate is one input datum, usually provided from the measuring stations covering the study area. In the same way, at the bedload interface, using the fact that the material derivative of $\mathrm{b}(x(t))$ is zero, we obtain with $\mathrm{F}_b(t) = z(t) - \mathrm{Z}_b^\star(t, x(t))$, the following kinematic condition

$$\frac{\partial \mathrm{Z}_b^\star}{\partial t} + u(t,x,\mathrm{Z}_b^\star)\frac{\partial \mathrm{Z}_b^\star}{\partial x} - w(t,x,\mathrm{Z}_b^\star) = \frac{d\mathrm{F}_b}{dt} \tag{8}$$

where $\mathrm{Z}_b^\star(t, x(t)) = \mathrm{b}(x(t)) + \mathrm{Z}_b(t, x(t))$ is the elevation of the bedload interface.

Bedload transport. The bed-load transport is hard to predict because of the mixing of antagonistic flow regimes: fast and slow, the non-equilibrium and noise-driven, temporal and spatial scalings, heterogeneities, and nonlinearities. Moreover, there are threshold effects, cascades of interacting processes, hysteresis, poor knowledge of initial and boundary conditions, difficulties in obtaining reliable measurements. The simple transport of sediment can be considered by assuming the bed-load layer as a uniform reservoir of independent particles [**Charru_et_Al_2004**, **Mouilleron_et_Al_2009**]. The mass conservation of moving particles is then applied to formulate the transport at the bed-load interface in terms of a transport discharge flux $q_b^\star$:

$$u\left(t, x, Z_b^\star\right) \frac{\partial Z_b^\star}{\partial x} - w\left(t, x, Z_b^\star\right) \simeq \frac{\partial q_b^\star}{\partial x} \tag{9}$$

Here, the motion at the bedload interface is balanced by the gradient of the horizontal mass sediment flux $q_b^\star$. Indeed, with an interface normal $\left(\frac{\partial Z_b^\star}{\partial x}, -1\right)$ the left-hand side(LHS) of the previous equation can be viewed as an asymptotic limit of a divergence formulation. The evolution of the bedload interface elevation is then given by the flux form of the Exner's equation [**Paola_and_Voller_2005**](Eq. 15, page 4)

$$\frac{\partial Z_b^\star}{\partial t} + \frac{\partial q_b^\star}{\partial x} = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s}. \tag{10}$$

In the considered context, the sediment transport flux is computed using the energetic-based formula of [**Bailard_1981**] where small values proportional to the bottom slope are neglected. A similar approximation is also used in [**Liu_et_Al_2015**] where

$$q_b^\star \simeq \frac{\mu \overline{u^2 u}}{1 - \phi_s} \tag{11}$$

where $\overline{u^2 u}$ is the depth-average of the horizontal speed to the cubic power, μ is a coefficient usually obtained experimentally by taking into account the grain diameter and the kinematic viscosity of the sediment mixture. In [**Liu_et_Al_2015**], the depth-averaged is approximated as $\overline{u^2 u} \simeq \overline{u}^2 \overline{u}$. In the coming sections, we will propose an extended version of this modeling, including shear fluctuations of the velocities in the vertical direction. Nevertheless, given the numerical difficulties (loss of hyperbolicity) encountered with this flux-formulation of the bed-load transport, another alternative that is numerically suitable is to formulate the bed-load velocities as a function of the depth-average of the above flow characteristics.

2.1 Depth averaged equations

We define the depth average $\overline{\phi} = \overline{\phi}\left(t, x\right)$ for any quantity $\phi\left(t, x, z\right)$ by

$$\overline{\phi} = \frac{1}{h} \int_{Z_b^\star}^{\xi} \phi dz \qquad \text{where} \qquad Z_b^\star = b + Z_b, \quad h\left(t, x\right) = \xi\left(t, x\right) - Z_b^\star\left(t, x\right) \tag{12}$$

The fluctuation with respect to the average value is $\phi' = \phi - \overline{\phi}$ and clearly we have

$$\overline{\phi'} = 0$$

Moreover, we have noted the following identities:

$$\int_{Z_b^\star}^{\xi} \frac{\partial \phi}{\partial x} dz = \frac{\partial}{\partial x}\left(h\overline{\phi}\right) - \frac{\partial \xi}{\partial x} \phi\left(t, x, \xi\right) + \frac{\partial Z_b^\star}{\partial x} \phi\left(t, x, Z_b^\star\right)$$

and

$$\int_{Z_b^\star}^{\xi} \frac{\partial \phi}{\partial t} dz = \frac{\partial}{\partial t}\left(h\overline{\phi}\right) - \frac{\partial \xi}{\partial t} \phi\left(t, x, \xi\right) + \frac{\partial Z_b^\star}{\partial t} \phi\left(t, x, Z_b^\star\right)$$

Integrating the divergence-free equation over the depth of water and using the previous relations for $\phi \equiv u$ yields

$$\frac{\partial}{\partial x}(h\overline{u}) - \frac{\partial \xi}{\partial x} u(t,x,\xi) + \frac{\partial \mathrm{Z}_b^\star}{\partial x} u(t,x,\mathrm{Z}_b^\star) + w(t,x,\xi) - w(t,x,\mathrm{Z}_b^\star) = 0$$

Then, using the kinematic condition (7) and (8), we derive that

$$-\frac{\partial \xi}{\partial x} u(t,x,\xi) + \frac{\partial \mathrm{Z}_b^\star}{\partial x} u(t,x,\mathrm{Z}_b^\star) + w(t,x,\xi) - w(t,x,\mathrm{Z}_b^\star) = \frac{\partial h}{\partial t} - \frac{d\mathrm{F}_\xi}{dt} + \frac{d\mathrm{F}_b}{dt}$$

The evolution of the water depth finally writes as

$$\frac{\partial h}{\partial t} + \frac{\partial}{\partial x}(h\overline{u}) = \frac{d\mathrm{F}_\xi}{dt} - \frac{d\mathrm{F}_b}{dt} \tag{13}$$

In the context of hydrostatic approximation (long-wave approximation), the pressure is given by $p = p_0 - \rho g(z-\xi)$. where p_0 is the atmospheric pressure at the free surface, and the density ρ is variable in the vertical direction. Assuming that p_0 is constant in space, the gradient of the pressure is defined by

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \xi}{\partial x} + (\xi - z) g \frac{\partial \rho}{\partial x} \quad \text{and} \quad \int_{\mathrm{Z}_b^\star}^{\xi} \frac{1}{\rho}\frac{\partial p}{\partial x} dz = hg\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x}$$

Using the conservation of the mixture density and the divergence-free assumption, the first equation of the momentum writes as

$$\frac{\partial u}{\partial t} + 2\frac{\partial \mathrm{K}}{\partial x} + \frac{\partial}{\partial z}(uw) + \frac{1}{\rho}\frac{\partial p}{\partial x} = \boldsymbol{\mathcal{F}}_x \quad \text{where} \quad \boldsymbol{\mathcal{F}}_x = \frac{1}{\rho}\mathbf{F}_x + \frac{1}{\rho}\frac{\partial \tau_{xx}}{\partial x} + \frac{1}{\rho}\frac{\partial \tau_{xz}}{\partial z} \quad \text{and} \quad \mathrm{K} = \frac{uu}{2}$$

Averaging this equation, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}(h\overline{u}) + \frac{\partial}{\partial x}(2h\overline{\mathrm{K}}) + gh\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x} = h\overline{\boldsymbol{\mathcal{F}}}_x &+ \left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}u - w\right) u(t,x,\xi) \\ &- \left(\frac{\partial \mathrm{Z}_b^\star}{\partial t} + \frac{\partial \mathrm{Z}_b^\star}{\partial x}u - w\right) u(t,x,\mathrm{Z}_b^\star)\end{aligned}$$

Then, still using the kinematic conditions, we derive that

$$\frac{\partial}{\partial t}(h\overline{u}) + \frac{\partial}{\partial x}(2h\overline{\mathrm{K}}) + gh\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\frac{\partial \rho}{\partial x} = h\overline{\boldsymbol{\mathcal{F}}}_x + \frac{d\mathrm{F}_\xi}{dt}u(t,x,\xi) - \frac{d\mathrm{F}_b}{dt}u(t,x,\mathrm{Z}_b^\star) \tag{14}$$

where

$$\overline{\mathrm{K}} = \frac{\overline{u}\overline{u}}{2} + \frac{\mathrm{P}}{2} \quad \text{with} \quad \mathrm{P} = \overline{u'u'}$$

We can see that the average $\overline{\mathrm{K}}$ is not entirely defined for us, as we still need to deal with non-averaged components of the velocity. The classical shallow water model is only valid under the assumption that $\overline{u'u'}$ is negligible. To take into account some amount of vertical shear, we will now derive an equation for the average $\overline{\mathrm{K}}$. Starting from the momentum equation, we can derive the following set of equations,

$$\frac{\partial \mathrm{K}}{\partial t} + u\frac{\partial}{\partial x}(uu) + u\frac{\partial}{\partial z}(uw) + \frac{u}{\rho}\frac{\partial p}{\partial x} = u\boldsymbol{\mathcal{F}}_x$$

As the velocity is the divergence-free, and under hydrostatic assumption, this equation becomes

$$\frac{\partial \mathrm{K}}{\partial t} + \frac{\partial}{\partial x}(u\mathrm{K}) + \frac{\partial}{\partial z}(w\mathrm{K}) + gu\frac{\partial \xi}{\partial x} + \frac{\xi - z}{\rho}gu\frac{\partial \rho}{\partial x} = u\boldsymbol{\mathcal{F}}_x$$

Averaging this equation gives

$$\frac{\partial}{\partial t}(h\overline{\mathrm{K}}) + \frac{\partial}{\partial x}(h\overline{\mathrm{K}u}) + gh\overline{u}\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\overline{u}\frac{\partial \rho}{\partial x} = -2g\frac{\partial \rho}{\partial x}\overline{(\eta - x_3)u'} + h\overline{u\boldsymbol{\mathcal{F}}_x} + \frac{d\mathrm{F}_\xi}{dt}\mathrm{K}(t,x,\xi) - \frac{d\mathrm{F}_b}{dt}\mathrm{K}(t,x,\mathrm{Z}_b^\star)$$

where

$$2\overline{\mathrm{K}u} = 2\overline{\mathrm{K}}\overline{u} + 2\mathrm{P}\overline{u} + \overline{u'u'u'}$$

The third-order fluctuations is here assumed to be smaller than the second order fluctuations so it allows us to formulate the third-order fluctuations as a gradient, to produce dissipation of the depth average energy [**Gavrilyuk_et_Al_2018**, **Praveen_et_Al_2020**] :

$$\overline{u'u'u'} \simeq -2\kappa\frac{\partial \mathrm{P}}{\partial x} \qquad \Longrightarrow \qquad \overline{\mathrm{K}u} = \overline{\mathrm{K}}\overline{u} + \mathrm{P}\overline{u} - \kappa\frac{\partial \mathrm{P}}{\partial x} \tag{15}$$

The source term $\boldsymbol{\mathcal{F}}_x$ is related to bed friction (F) and the viscous tensor of the Navier-Stokes equations. In this paper for the simplicity, only the friction contribution is taken into accounts. Therefore we have assumed that $\overline{u\boldsymbol{\mathcal{F}}_x} \simeq \overline{u}\mathbf{F}_x$ and the evolution of the average energy writes as

$$\begin{aligned}\frac{\partial}{\partial t}(h\overline{\mathrm{K}}) + \frac{\partial}{\partial x}\left(h\overline{u}\left(\overline{\mathrm{K}} + \mathrm{P}\right)\right) + gh\overline{u}\frac{\partial \xi}{\partial x} + \frac{gh^2}{2\rho}\overline{u}\frac{\partial \rho}{\partial x} &= -2g\frac{\partial \rho}{\partial x}\overline{(\eta - x_3)u'} + h\overline{u}\mathrm{F} + \frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{P}}{\partial x}\right) \\ &+ \frac{d\mathrm{F}_\xi}{dt}\mathrm{K}_\xi - \frac{d\mathrm{F}_b}{dt}\mathrm{K}_{\mathrm{Z}_b^\star}\end{aligned} \tag{16}$$

We can now use the relation (15) reformulating the bedload interface evolution (8) as

$$\frac{\partial \mathrm{Z}_b^\star}{\partial t} + \frac{2\mu}{1-\phi_s}\frac{\partial}{\partial x}\left(\overline{\mathrm{K}}\overline{u} + \mathrm{P}\overline{u}\right) = \frac{\mathcal{D} - \mathcal{E}}{1-\phi_s} + \frac{2\mu}{1-\phi_s}\frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{P}}{\partial x}\right) \tag{17}$$

There is an alternative formulation, using the relation (9), that writes as

$$\frac{\partial \mathrm{Z}_b^\star}{\partial t} + u(t,x,\mathrm{Z}_b^\star)\frac{\partial \mathrm{Z}_b^\star}{\partial x} = \frac{\mathcal{D} - \mathcal{E}}{1-\phi_s} + \frac{2\mu}{1-\phi_s}\frac{\partial}{\partial x}\left(\kappa\frac{\partial \mathrm{P}}{\partial x}\right) + w(t,x,\mathrm{Z}_b^\star) \tag{18}$$

In this context, $(u(t,x,\mathrm{Z}_b^\star))$ and vertical $(w(t,x,\mathrm{Z}_b^\star))$ velocities at the bed-load should be defined as functions of the averaged quantities characterizing the fluid flow.

The equation for volume concentration, when averaged, gives

$$\frac{\partial h\overline{\alpha_s}}{\partial t} + \frac{\partial}{\partial x}(h\overline{u\alpha_s}) = \left(\frac{\partial \xi}{\partial t} + \frac{\partial \xi}{\partial x}u - w\right)\alpha_s(t,x,\xi) - \left(\frac{\partial \mathrm{Z}_b^\star}{\partial t} + \frac{\partial \mathrm{Z}_b^\star}{\partial x}u - w\right)\alpha_s(t,x,\mathrm{Z}_b^\star)$$

we assume that $\alpha_s(t,x,\xi) = 0$ and $\alpha_s(t,x,\mathrm{Z}_b^\star) = 1 - \phi_s$. To compute the average $\overline{u\alpha_s}$, we use a Fick's law approximation written as

$$\overline{u\alpha_s} \simeq \overline{u}\,\overline{\alpha_s} - \gamma\frac{\partial \overline{\alpha_s}}{\partial x}$$

where γ is a positive coefficient. The final averaged equation writes as

$$\frac{\partial}{\partial t}\left(h\overline{\alpha_s}\right)+\frac{\partial}{\partial x}\left(h\overline{\alpha_s}\,\overline{u}\right)=\frac{\partial}{\partial x}\left(\gamma h\frac{\partial\overline{\alpha_s}}{\partial x}\right)-\left(\mathcal{D}-\mathcal{E}\right)$$

All together the depth-averaged model for sediment flows writes as

$$\frac{\partial h}{\partial t}+\frac{\partial}{\partial x}\left(h\overline{u}\right) = \frac{d\mathrm{F}_\xi}{dt}-\frac{d\mathrm{F}_b}{dt} \tag{19}$$

$$\frac{\partial}{\partial t}\left(h\overline{u}\right)+\frac{\partial}{\partial x}\left(2h\overline{\mathrm{K}}\right)+gh\frac{\partial\xi}{\partial x}+\frac{gh^2}{2\rho}\frac{\partial\rho}{\partial x} = h\mathrm{F}+\frac{d\mathrm{F}_\xi}{dt}u_\xi-\frac{d\mathrm{F}_b}{dt}u_{\mathrm{Z}_b^\star} \tag{20}$$

$$\frac{\partial}{\partial t}\left(h\overline{\mathrm{K}}\right)+\frac{\partial}{\partial x}\left(h\overline{u}\left(\overline{\mathrm{K}}+\mathrm{P}\right)\right)+gh\overline{u}\frac{\partial\xi}{\partial x}+\frac{gh^2}{2\rho}\overline{u}\frac{\partial\rho}{\partial x} = -2g\frac{\partial\rho}{\partial x}\overline{(\eta-x_3)u'}+h\overline{u}\mathbf{F}_x+\frac{d\mathrm{F}_\xi}{dt}\mathrm{K}_\xi-\frac{d\mathrm{F}_b}{dt}\mathrm{K}_{\mathrm{Z}_b^\star}+\frac{\partial}{\partial x}\left(\kappa\frac{\partial\mathrm{P}}{\partial x}\right) \tag{21}$$

$$\frac{\partial}{\partial t}\left(h\overline{\alpha_s}\right)+\frac{\partial}{\partial x}\left(h\overline{u}\,\overline{\alpha_s}\right) = -\phi_f\frac{d\mathrm{F}_b}{dt}+\frac{\partial}{\partial x}\left(\gamma h\frac{\partial\overline{\alpha}_s}{\partial x}\right) \tag{22}$$

$$\frac{\partial\mathrm{Z}_b^\star}{\partial t}+\frac{2\mu}{\phi_f}\frac{\partial}{\partial x}\left(\overline{\mathrm{K}}\overline{u}+\mathrm{P}\overline{u}\right) = \frac{d\mathrm{F}_b}{dt}+\frac{2\mu}{\phi_f}\frac{\partial}{\partial x}\left(\kappa\frac{\partial\mathrm{P}}{\partial x}\right) \tag{23}$$

where $\phi_f+\phi_s=1$.

$$\xi=\mathrm{Z}_b^\star+h,\quad \rho=\overline{\alpha_s}\rho_s+\left(1-\overline{\alpha_s}\right)\rho_f=\rho_f+\delta\rho\overline{\alpha_s}\quad\text{and}\quad\delta\rho=\rho_s-\rho_f$$

and where the bed friction term is given by $\mathbf{F}_x=-C_f|\overline{u}|\overline{u}$, with $C_f=n^2gh^{-1/3}$ and with n being the Manning roughness coefficient at the bed.

Note that, in the case where we constraint $\mathrm{P}=0$, when vertical fluctuations of the velocity are neglected, the equation for the energy $\overline{\mathrm{K}}$ is useless and we recover the model used in [**Liu_et_Al_2015**]. If in addition we take $\mu=0$, then the present model degenerates to the one used in [**Simpson_and_Castelltort_2006**].
Here we can for instance take $u\left(t,x,\mathrm{Z}_b^\star\right)$ as:

$$u\left(t,x,\mathrm{Z}_b^\star\right)=-\frac{1}{1-\phi_f}\frac{1}{1-Fr^2}\frac{\overrightarrow{u}_s}{|\overline{u}|}, \tag{24}$$

where u_s is the sediment velocity, Fr is the Froude number (computed for shear-shallow water). The characteristic velocity of advection of body sedimentary given by equation (24) allows to take into account a phase lag between sediment velocity and water velocity.
Therefore, the proposed model is an extension of classical (or reduced) averaged sediment transport model.

Remark: Generalized-SSW model. The first third equation of the proposed model given by equation (19) without sediment contribution coupled with mass conservation equation (or evolution of mixture density equation) given by equation (5) is named Generalized-SSW model.

2.2 Brief hyperbolicity study

This model can also be put in the following compact form

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \mathbf{F}}{\partial x} + gh\mathbf{B}_x^{\mathrm{K}}\frac{\partial h}{\partial x} + gh\mathbf{B}_x^{\star}\frac{\partial \mathrm{Z}_b^{\star}}{\partial x} + \frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^{\star}\frac{\partial \overline{\alpha_s}}{\partial x} = \mathbf{S} \tag{25}$$

where

$$\mathbf{W} = \begin{pmatrix} h \\ h\overline{u} \\ h\overline{\mathrm{K}} \\ h\overline{\alpha_s} \\ \mathrm{Z}_b^{\star} \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} h\overline{u} \\ 2h\overline{\mathrm{K}} + \frac{gh^2}{2} \\ h\overline{u}\left(\overline{\mathrm{K}} + \mathrm{P}\right) \\ h\overline{u}\,\overline{\alpha_s} \\ \frac{2\mu(\overline{\mathrm{K}}\overline{u} + \mathrm{P}\overline{u})}{\phi_f} \end{pmatrix}, \qquad \mathbf{B}_x^{\mathrm{K}} = \begin{pmatrix} 0 \\ 0 \\ \overline{u} \\ 0 \\ 0 \end{pmatrix}, \qquad \mathbf{B}_x^{\star} = \begin{pmatrix} 0 \\ 1 \\ \overline{u} \\ 0 \\ 0 \end{pmatrix}$$

and $\mathbf{S}$ is the vector of the remaining contributions including the effects of erosion/deposition, rainfall/evaporation, external forces and dissipations. The system (25) is genuinely non conservative and its numerical approximation, in the context of finite volume schemes, needs some specific treatments [**Praveen_et_Al_2020**]. The Jacobian matrix associated to the conservative flux $\mathbf{F}$ is diagonalizable (Hyperbolic system). When adding the non conservative contribution associated to the derivative of $\mathrm{Z}_b^{\star}$ the hyperbolicity of the left hand size is no more guaranteed. This will lead to further numerical complications. The system (25) was obtained using the formulation (17). When using the alternative formulation for the bedload dynamics (18), we get the following system

$$\frac{\partial \mathbf{W}}{\partial t} + \frac{\partial \widetilde{\mathbf{F}}}{\partial x} + gh\mathbf{B}_x^{\mathrm{K}}\frac{\partial h}{\partial x} + \widetilde{\mathbf{B}}_x^{\star}\frac{\partial \mathrm{Z}_b^{\star}}{\partial x} + \frac{gh^2\delta\rho}{2\rho}\mathbf{B}_x^{\star}\frac{\partial \overline{\alpha_s}}{\partial x} = \mathbf{S} \tag{26}$$

where

$$\mathbf{W} = \begin{pmatrix} h \\ h\overline{u} \\ h\overline{\mathrm{K}} \\ h\overline{\alpha_s} \\ \mathrm{Z}_b^{\star} \end{pmatrix}, \qquad \widetilde{\mathbf{F}} = \begin{pmatrix} h\overline{u} \\ 2h\overline{\mathrm{K}} + \frac{gh^2}{2} \\ h\overline{u}\left(\overline{\mathrm{K}} + \mathrm{P}\right) \\ h\overline{u}\,\overline{\alpha_s} \\ 0 \end{pmatrix}, \qquad \widetilde{\mathbf{B}}_x^{\star} = \begin{pmatrix} 0 \\ gh \\ gh\overline{u} \\ 0 \\ u_{\mathrm{b}}^{\star} \end{pmatrix}$$

with $u_{\mathrm{b}}^{\star} = u\left(t, x, \mathrm{Z}_b^{\star}\right)$. The terms on the left-hand side of the equation (26) define a hyperbolic non-conservative system whose eigenvalues are:

$$u, \;\; u_{\mathrm{b}}^{\star}, \text{`} \;\; u - \sqrt{gh + 3\mathrm{P}} \;\text{ and }\; u + \sqrt{gh + 3\mathrm{P}}. \tag{27}$$

We recover here the eigenvalues obtained in [**Gavrilyuk_et_Al_2018**] and two other waves associated with sediment concentration and the bed-load dynamic. The existence of a local solution leads to proposing an appropriate numerical scheme that is exposed in the next section.

III PATH-CONSERVATIVE CENTRAL-UPWIND (PCCU) AENO SCHEME WITH SI-RK3 TIME DISCRETIZATION METHOD.

The model put in form of equation (26) admits a discontinuous solution. The non-conservative terms present in equation (26) can be viewed as Borel measure [**DalMaso**]. This definition requires the choice of sufficiently smooth paths:

$$\Psi_{i+1/2}(s) = (\Psi_{i+1/2}^{(1)}, \Psi_{i+1/2}^{(2)}, ..., \Psi_{i+1/2}^{(N)}, \Psi_{i+1/2}^{(N+1)}) := \Psi(s, \mathbf{W}_{i+1/2}^{+}, \mathbf{W}_{i+1/2}^{-}). \tag{28}$$

Connecting the two states $\mathbf{W}^{+}_{i+1/2}$ $\mathbf{W}^{-}_{i+1/2}$ across the jump discontinuity at $\mathbf{x} = \mathbf{x}_0$ such that a local-Lipschitz application $\Psi : [0,\ 1] \times \Omega \times \Omega \to \Omega$ satisfies the following property:

$$\Psi(0, \mathbf{W}^{-}_{i+1/2}, \mathbf{W}^{+}_{i+1/2}) = \mathbf{W}^{-}_{i+1/2} and \Psi(1, \mathbf{W}^{-}_{i+1/2}, \mathbf{W}^{+}_{i+1/2}) = \mathbf{W}^{+}_{i+1/2}, \quad \text{for all } \mathbf{W}^{-}_{i+1/2}, \mathbf{W}^{+}_{i+1/2} \in \Omega. \tag{29}$$

We can defined the nonconservative product as the Borel measure [**DalMaso**]:

$$\mu_{\mathbf{x}_0} = \left[\int_0^1 \mathcal{A}(\Psi(s, \mathbf{W}^{-}_{i+1/2}, \mathbf{W}^{+}_{i+1/2})) \frac{d\Psi}{ds}(s, \mathbf{W}^{-}_{i+1/2}, \mathbf{W}^{+}_{i+1/2}) ds \right] \delta_{x_0}, \tag{30}$$

where δ_{x_0} is the fluctuation. This definition is similar to the one proposed by Volpert [**Volpert**] to define the non-conservative product. We take a particular example of the simplest linear segment path:

$$\Psi_{i+1/2}(s) = \mathbf{W}^{-}_{i+1/2} + s(\mathbf{W}^{+}_{i+1/2} - \mathbf{W}^{-}_{i+1/2}), \quad s \in [0, 1]. \tag{31}$$

The grid considered here is uniform that is $x_i = i\Delta x$, where Δx is the small spatial scale and the corresponding finite volume cells $K_i = [x_{i-1/2}, x_{i+1/2}]$. We assume that at certain time level t the solution realized in terms of its averages $\overline{\mathbf{W}}_i = \dfrac{1}{\Delta x} \int_{K_i} \mathbf{W}(\mathbf{x}, t) dx$ is available. $\mathbf{W}^{n}_{i+1/2}$ is an approximation of W at interface $i + 1/2$ and at time $t = t^n$. $\overline{\mathbf{W}}_i^{n+1}$ is the approximation of $\mathbf{W}$ averaged in the cell K_i at time $t = \Delta t + t^n$ where δt is the time step. Using the concept of path-conservative central-upwind scheme, we propose the following first order semi-discrete scheme:

$$\frac{d}{dt}\overline{\mathbf{W}}_i(t) = -\frac{1}{\Delta x}\left(\mathcal{F}_{i+1/2} - \mathcal{F}_{i-1/2}\right) - \frac{1}{\Delta x}\left(-B^{K}_{x,i} - B^{*}_{x,i} - \widetilde{B}^{*}_{x,i}\right) - \frac{1}{\Delta x}\mathcal{H}_{\Psi,i+1/2} + \mathbf{S}(\overline{\mathbf{W}}_i(t)) \tag{32}$$

Here, the discrete fluxes $\mathcal{F}_{i\pm1/2}$ are computed in CU sense. $\mathcal{H}_{\Psi,i+1/2}$ is the contribution of nonconservative terms:

$$\begin{aligned} \mathcal{H}_{\Psi,i+1/2} &= \frac{a^{+}_{i-1/2}}{a^{+}_{i-1/2} - a^{-}_{i-1/2}} \left[(B^{K}_{x})_{\Psi,i-1/2} + (\widetilde{B}^{*}_{x})_{\Psi,i-1/2} + (B^{*}_{x})_{\Psi,i-1/2} \right] \\ &- \frac{a^{-}_{i+1/2}}{a^{-}_{i+1/2} - a^{-}_{i+1/2}} \left[(B^{K}_{x})_{\Psi,i+1/2} + (\widetilde{B}^{*}_{x})_{\Psi,i+1/2} + (B^{*}_{x})_{\Psi,i+1/2} \right], \end{aligned} \tag{33}$$

where $a^{\pm}_{i\pm1/2}$ are the speeds propagation velocity computed using the eigenvalues given by equation (27) in CU sense, and where $B^{K}_{x,i}$, $\widetilde{B}^{*}_{x,i}$, $B^{*}_{x,i}$, $(B^{K}_{x})_{\Psi,i+1/2}$, $(\widetilde{B}^{*}_{x})_{\Psi,i+1/2}$ and $(B^{*}_{x})_{\Psi,i+1/2}$ are computed in sense of definition of linear path as in [**Ngatcha_et_Al_2022a**]. $\mathbf{S}(\overline{\mathbf{W}}_i(t)) = S_C(\overline{\mathbf{W}}_i(t)) + S_F(\overline{\mathbf{W}}_i(t)) + S_D(\overline{\mathbf{W}}_i(t)) + S_{F_b}(\overline{\mathbf{W}}_i(t)), S_w(\overline{\mathbf{W}}_i(t))$ where $S_C, S_F, S_D, S_w, S_{F_b}$ are respectively the diffusion of sediment source term, the friction source term, the dissipation of stress P source term, the evaporation/infiltration source term and the erosion/deposition exchange source term computed in well-balanced sense as in [**Ngatcha_et_al_2022b**].

Second order scheme. is obtained (to ameliorate the convergence) by using a reconstruction procedure based on a modified Averaging Essentially Non-oscillatory (AENO) method [**Ngatcha_et_al_2022b**](using a concept introduces by [**Toro_et_al_2021**]). The fully discrete scheme is obtained by using an variant of strong stability preserving (SSP) method also called third order semi-implicit Runge-Kutta (SI-RK3) method as in [**Ngatcha_et_Al_2022a**].

IV NUMERICAL TESTS

4.1 Accuracy test

In this test, Shallow Water Equations are solved using PCCU-AENO method developed here in a one-dimensional context without source terms. So only the hydrodynamical system with $\mathcal{D} - \mathcal{E} = 0$ is used. A dam-break performs the test. The fluid velocity $u(0, x) = 0$ and friction terms are not introduced; the final time is $T = 30s$. The domains of simulation are $\Omega = [0, 2000]$ for one-dimensional case. The initial discontinuity is in the middle of the domain Ω. The dam separates two initial water depth exhibits wet zone are $h(0, x) = 40m$ at the left side and $h(0, x) = 2m$ at for the right side of the domain. Numerical results are compared with the exact solutions see Fig.(1). In all the cases, they are good agreement and convergence to the exact solution.

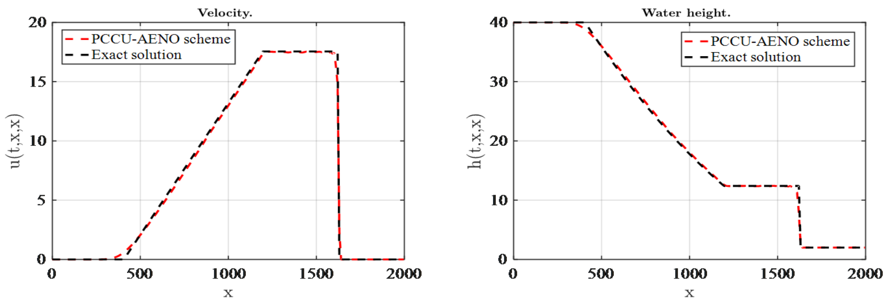

Figure 1: Comparison between exact solution and PCCU-AENO scheme.

4.2 Dam break without sediment transport.

In this example, the initial condition of the associated Riemann problem is given in the table (2). Here, there is no sediments ($\mathcal{D} - \mathcal{E} = 0$) the bed evolution and sediment concentration

	h	u	P
$x < 0.5$	0.02	0	4×10^{-2}
$x > 0.5$	0.01	0	4×10^{-2}

Table 1: Initial conditions for 1D dam-break.

assumed constant $Z_b^*(x, t) = 0, \quad \alpha_s(x, t) = 0$. It is observed that all the property of PCCU scheme is conserved along of simulation even at low numerical resolution, as seen in Fig. (2).

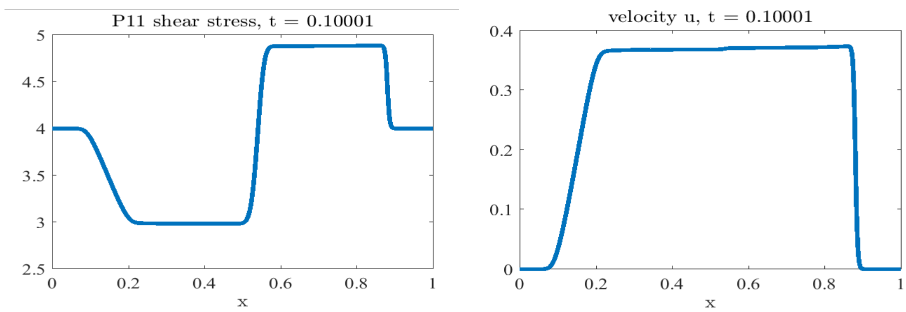

Figure 2: Dam break with first order approximations. Stress component P and horizontal velocity u.

4.3 Exact shock stationary

We consider here a test where the exact shock is stationary [**Nkonga_Praveen_2021**]. The corresponding Riemann data is given by: This problem is solved using the PCCU scheme pre-

	h	u	P
$x < 0.5$	0.02	0.66509783218609	4×10^{-2}
$x > 0.5$	0.03	0.44339598554790727	4×10^{-2}

Table 2: Initial conditions for exact shock stationary test.

sented above with the wave speed obtained by the Jacobian matrix. The solution is computed and plotted in Fig. (3) on a mesh of 2000 cells.

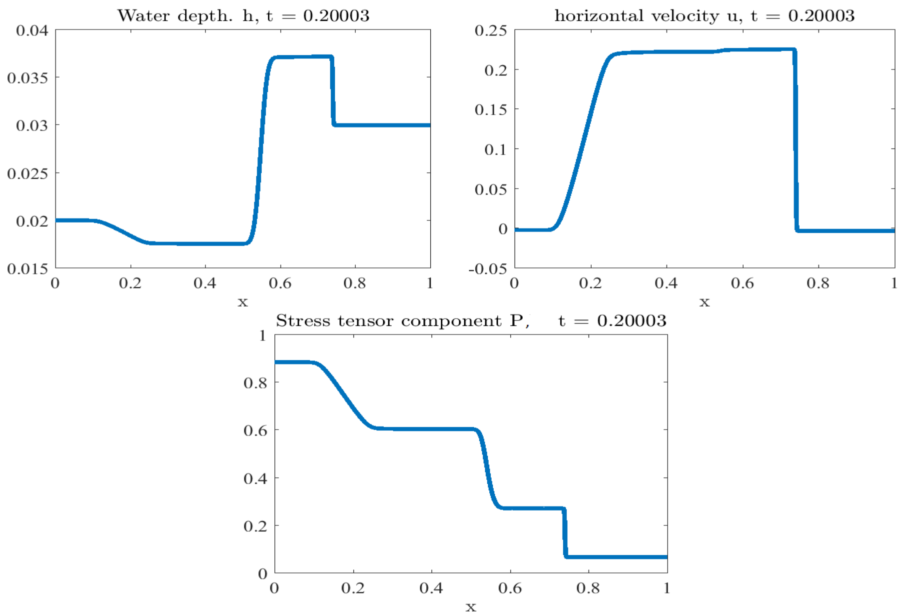

Figure 3: Shock stationary test using first order PCCU scheme

We observe that the proposed method can solve static shock using a first-order approximation of the one-sided local speeds of propagation $a^-_{j+1/2}$, $a^+_{j+1/2}$. The results do no present any dispersive shock.

4.4 Dam-break with sediment transport.

Here, the classical test shows that the scheme can reproduce sediment transport processes in agreement with observation in nature. For this test, the shear effect is considered in the scheme. The kinetic energy is, $h\overline{K} = 2$, at the left side and $h\overline{K} = 3$, at the right side. The sediment load is $Z_b^*(0, x) = 0$, fluid velocity $u(0, x) = 0$ and friction term is introduced, the final time is $t = 1$. The domain of simulation is $\Omega = [0, 10]$ with a dam located at the middle of Ω. The dam separates two initial water depth exhibits wet zone are $h(0, x) = 2m$ at the left side and $h(0, x) = 1m$ at for the right side of the domain. The initial concentration is $\alpha_s(x, t) = 0.0001$ the bed porosity is $\phi_s = 0.28$. Here the source term given by $S_w(\mathbf{W})$ is considered with $\dfrac{dFu}{dt} = E_w|\mathbf{u}|$ where E_w is a calibration coefficient given by $E_w = \dfrac{0.00153}{0.0204 + R_\rho}$ with $R_\rho = \dfrac{\rho_w g h}{u^2}$. The bed evolution, sediment concentration, shear stress, and water discharge are plotted using $N = 200$ grids in Fig. (4)

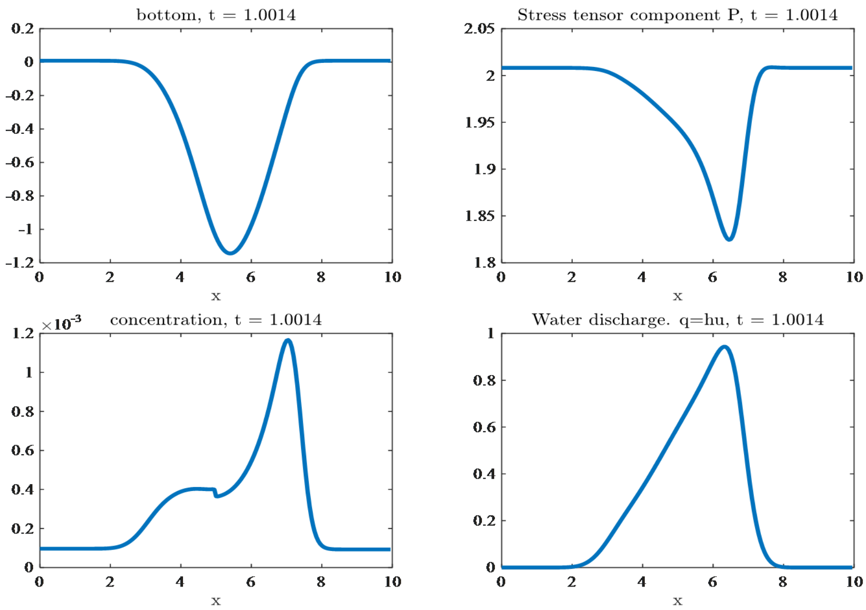

Figure 4: Computational solutions of Dam-break over erodible bed . The calibration parameter is $C_s = 0.5$

We observe that the proposed model solved by the PCCU scheme can solve the dam-break problem over an erodible bed and with sediment transport. The profiles obtained present a good agreement with reality observed in nature. The shear effect is well observed in the concentration profile. It demonstrate that the PCCU scheme can predict a stable bed evolution dynamic and leads to solution according to the observation in nature. In a dilute flow, the dissipation takes place due to the presence of particles which increases the viscosity of the fluid and the collisions between the grains.

V CONCLUSION

We have performed in this work a mathematical derivation of a physical model for sediment transport in shear shallow flow. This model, inspired by recent works, is an extension of the classical averaged sediment transport models and considers the effects of horizontal velocity shear. The work proposes also an extension of the classical shear shallow water based models recently developed by incorporating nonhomogenous ness of fluid density.

As a result, the validity regime of the present model is extended, which gives a more appropriate framework for the study of coastal flows. The model described here assumes quasi-2D flows and, after averaging, gives 1D equations. There are no particular difficulties in extending this approach to 3D flows, and this derivation is in progress. This last step does not pose any particular difficulties and will be necessary for concrete applications. The proposed models will then be the subject of numerical approximations in the context of finite volume, then numerical simulations associated with the target watershed of Douala.

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Appendices

A MATERIAL DERIVATIVE: CLOSURE MODELS.

The material derivative $\frac{d\mathrm{F}_b}{dt}$ is now a function of the balance between the particles that are eroded and/or deposited on the surface and of the local bedload porosity ϕ_s with $0 \leq \phi_s < 1$. This the morphological change of the bathymetry is the volume rate exchange per unit of time. The volume increment $d\mathrm{F}_b$ can be decomposed as the sum of the increment of sediments volume plus a fraction of void (porosity ϕ_s) filled by the clean water. Moreover, the volume increment of sediment is balance between the amount of sediment left behind by the current (deposed) and the amount of sediment carried away (eroded) from the bedload interface

$$d\mathrm{F}_b = d\vartheta_s^{deposed} - d\vartheta_s^{eroded} + \phi_s d\mathrm{F}_b$$

Therefore, denoting $\mathcal{D} = \frac{d\vartheta_s}{dt}\Big|^{deposed}$ the deposition rate and $\mathcal{E} = \frac{d\vartheta_s}{dt}\Big|^{eroded}$ the erosion rate, we obtain

$$\frac{d\mathrm{F}_b}{dt} = \frac{\mathcal{D} - \mathcal{E}}{1 - \phi_s}$$

To define the morphological change, we need to give explicit expressions for the erosion $\mathcal{E}$, the deposition $\mathcal{D}$ and the bedload porosity ϕ_s. Proper modeling of volume rate change is not obvious and many strategies are available in the literature [**Einstein_1950**, **Luque_and_Beek_1976**, **Yalin_1977**, **Nagakawa_Tsujimoto_1980**, **van_Rijn_1984c**]. The deposition rate of sediments $\mathcal{D}$ is almost equal to the vertical flux of particle at the boundary [**Fang_and_Rodi_2003**](page 381):

$$\mathcal{D} \simeq \mathrm{w}_s \alpha_s \left(t, x, \mathrm{Z}_b^\star\right) \tag{34}$$

where w_s is the sediment settling velocity and $\alpha_s\left(t, x, \mathrm{Z}_b^\star\right)$ is the volume concentration of sediment at the vicinity above the bedload [**van_Rijn_1987**] :

$$\mathrm{w}_s = g\frac{(\rho_s - \rho_f)\mathrm{d}_s^2}{18\mu_f\rho_f}$$

The number of grains eroded from the bed-load per unit area and time, also defined as the pick-up rate, for small particles sediments at low flow velocity is estimated as [**van_Rijn_1984c**] and later modified in [**van_Rijn_et_All_2019**]:

$$\mathcal{E} = \zeta \mathrm{f}_d \rho_s \mathrm{d}_\star^{0.3} \sqrt{g\mathrm{d}_s\left(\frac{\rho_s - \rho_f}{\rho_f}\right)} \left(\max\left(0, \frac{\theta' - \theta_{\mathrm{cr}}}{\theta_{\mathrm{cr}}}\right)\right)^{\frac{3}{2}}$$

where $\zeta = 3.310^{-4}$, $\mathrm{f}_d = \min\left(1, \frac{1}{\theta'}\right)$ a damping factor,

$$\mathrm{d}_\star = \mathrm{d}_s\left(g\frac{\rho_s - \rho_f}{\rho_f\mu_f^2}\right)^{\frac{1}{3}}, \quad \theta' = \frac{\tau_b}{g\mathrm{d}_s(\rho_s - \rho_f)}, \quad \tau_b = \rho_f g\left(\frac{\|\overline{\boldsymbol{u}}\|}{\mathrm{C}_z}\right)^2, \quad \mathrm{C}_z = 5.75\sqrt{g}\ln\left(\frac{12\mathrm{R}}{3\mathrm{d}_l}\right)$$

The erosion pick-up rate $\mathcal{E}$ is given in mass per unit area and time (kg/m^2/s), where $\mathrm{d}_\star$ the dimensionless grain size parameter, d_s median grain size (m), μ_f the kinematic viscosity coefficient of fluid (m^2/s), ρ_s the sediment density (kg/m^3), ρ_f the fluid density (kg/m^3), θ' grain-related parameter or particle mobility parameter that is the ratio of the hydrodynamic forces by the submerged particle weight, τ_b the average grain-related bedload-shear stress (N/m^2) due to currents and waves, $\overline{\boldsymbol{u}}$ the depth-averaged flow velocity (m/s), C_z the Chézy-coefficient ($\sqrt{m}$/s), R the hydraulic radius (m), (assumed to be equal to water depth), θ_{cr} the Shields value at initiation of motion, g the gravity acceleration (m/s^2). The sediment particles will leave the interface when the Shields parameter θ' exceeds the critical value θ_{cr}.