Design of Finite/Fixed-time ISS-Lyapunov Functions for Mechanical Systems

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Alexander Aleksandrov†‡, Denis Efimov§¶, Sergey Dashkovskiy‖

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Abstract

For a canonical form of mechanical system defined through gradients of potential energy and dissipative terms, the conditions of finite-time and fixed-time (integral) input-to-state stability are derived by finding suitable Lyapunov functions. The proposed stability conditions are constructive, which is demonstrated in several applications.

1 Introduction

The theory of input-to-state stability (ISS) [27, 11] becomes nowadays one of the most popular frameworks for analysis of robust stability in the presence of external perturbations and modeling uncertainties. The conditions of ISS are numerous and well developed for different classes of models of the dynamical processes. Frequently, they are formulated with the use of Lyapunov functions (LFs), whose properties have to be verified globally (positive definiteness, differentiability, radial unboundedness) together with the properties of its time derivative on the trajectories of a nonlinear system of interest (negative definiteness, radial unboundedness). The principal difficulty in this way is to find a LF with desired properties for given dynamics. The design of LF in ISS analysis is even more challenging than for asymptotic stability in the disturbance-free case [17, 20], and a LF can be suitable for unperturbed system, but it may do not satisfy some restrictions needed for analysis of robustness.

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The LFs also can be used for other purposes, for example, to evaluate the rates of convergence to the origin (an equilibrium) in the dynamical systems [16]. Popular kinds of accelerated convergence rates include finite-time (exact settling at the origin in a finite time proportional to initial conditions), nearly fixed-time (a global reaching of any vicinity of the origin uniformly in time, or with the time independent on initial conditions) and fixed-time (uniform global settling at the origin in a bounded time) ones [14]. There exist corresponding equivalent or sufficient characterizations of these properties in terms of existence of LFs [26, 7, 22, 23, 25, 18, 14], which are even more restrictive than for conventional asymptotic stability. There exist also extensions of the ISS concept to the systems with accelerated convergences [15, 19, 14]. However, despite all mentioned theoretical findings and mature technical part, application of these results to a concrete system is still very difficult since there are not many constructive methods of design of LFs in applications.

Following [10, 2], in the present work the problem of finding LFs to check finite-time or (nearly) fixed-time convergence and (integral) ISS properties is considered for a class of mechanical systems with exogenous perturbations. Several designs of LFs are presented.

The outline of this work is as follows. After introduction of the preliminaries and the used concepts of stability in Section 2, the problem statement is given in Section 3. The conditions of finite-time, nearly fixed-time and fixed-time robust stability are presented in sections 4, 5 and 6, respectively, together with corresponding LFs. The results of applications are summarized in Section 7.

Notation

- $\mathbb{R}$ denotes the field of real numbers, and $\mathbb{R}^n$ is the n-dimensional Euclidean space, $\mathbb{R}_+ = \{x \in \mathbb{R} : x \geq 0\}$;
- the Euclidean norm $\|\cdot\|$ will be used for vectors;
- for a (Lebesgue) measurable function $d : \mathbb{R}_+ \to \mathbb{R}^m$ and $[t_0, t_1) \subseteq \mathbb{R}_+$ define the norm $\|d\|_{[t_0,t_1)} = \text{ess sup}_{t\in[t_0,t_1)}\|d(t)\|$, then $\|d\|_\infty = \|d\|_{[0,+\infty)}$ and the set of d with the property $\|d\|_\infty < +\infty$ we further denote as $\mathcal{L}_\infty^m$ (the set of essentially bounded measurable functions);
- a continuous function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}$ if $\alpha(0) = 0$ and the function is strictly increasing; the function $\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{K}_\infty$ if $\alpha \in \mathcal{K}$ and it is increasing to infinity; a continuous function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ belongs to the class $\mathcal{KL}$ if $\beta(\cdot, t) \in \mathcal{K}$ for each fixed $t \in \mathbb{R}_+$ and $\beta(s, \cdot)$ is decreasing to zero for each fixed $s \in \mathbb{R}_+$;
- $I \in \mathbb{R}^{n\times n}$ denotes the identity matrix, $\text{diag}(v) \in \mathbb{R}^{n\times n}$ is a diagonal matrix having $v \in \mathbb{R}^n$ on the main diagonal;
- for a symmetric matrix $P \in \mathbb{R}^{n\times n}$, the relations $P > 0$ or $P \geq 0$ mean that it is positive definite or semi-definite, respectively.

2 Preliminaries

We will use the following lemmas:

Lemma 1. [13] *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$,*

$$\mathcal{W}(x,y) = \|x\|^\alpha + \|y\|^\beta - c\|x\|^\gamma \|y\|^\delta,$$

where $c, \alpha, \beta, \gamma, \delta$ are positive constants. For any value of c, the function $\mathcal{W}(x,y)$ is positive definite if and only if $\gamma/\alpha + \delta/\beta > 1$; there exists $\rho > 0$ such that the function $\mathcal{W}(x,y)$ is positive for $||x\| + \|y|| \geq \rho$ if and only if $\gamma/\alpha + \delta/\beta < 1$. In the case where $\gamma/\alpha + \delta/\beta = 1$, function $\mathcal{W}(x,y)$ is globally positive definite for sufficiently small values of c.

Lemma 2. [1] *Let $x \in \mathbb{R}^n$, $y \in \mathbb{R}^n$,*

$$\mathcal{W}(x,y) = \|x\|^\alpha + \|y\|^\beta + c_1\|x\|^\eta \|y\|^\zeta - c_2\|x\|^\gamma \|y\|^\delta,$$

where $c_1, c_2, \alpha, \beta, \gamma, \delta, \eta, \zeta$ are positive constants. If

$$\frac{\eta}{\alpha} + \frac{\zeta}{\beta} < 1,$$

then function $\mathcal{W}(x,y)$ is positive definite for any values of c_1 and c_2 if and only if

$$\gamma + \delta\frac{\alpha - \eta}{\zeta} > \alpha, \quad \gamma\frac{\beta - \zeta}{\eta} + \delta > \beta.$$

For any $a, b \in \mathbb{R}^n$ and $q > 1$ the Young's inequality states:

$$a^\top b \leq \frac{\|a\|^q}{q} + (q-1)\frac{||b||^{\frac{q}{q-1}}}{q}.$$

The Gensen's inequality for a convex function $\phi : \mathbb{R} \to \mathbb{R}$ and $x_1, \ldots, x_n \in \mathbb{R}$ states:

$$\phi\left(\frac{\sum_{i=1}^n x_i}{n}\right) \leq \frac{\phi\left(\sum_{i=1}^n x_i\right)}{n},$$

and if ϕ is concave then the inequality sign has to be inverted.

Recall that a function $h : \mathbb{R}^n \to \mathbb{R}^m$ is called homogeneous with respect to the standard dilation [6, 14] if

$$h(\lambda x) = \lambda^\nu h(x)$$

for all $x \in \mathbb{R}^n$ and all $\lambda > 0$, for some $\nu \in \mathbb{R}$ that is called the degree of homogeneity of h, then for a differentiable h the Euler's theorem on homogeneous functions claims:

$$x^\top \frac{\partial h(x)}{\partial x} = \nu h(x), \ \forall x \in \mathbb{R}^n.$$

2.1 Definitions of stability

Consider a nonlinear system:

$$\dot{x}(t) = f(x(t), d(t)),\ t \geq 0, \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state, $d(t) \in \mathbb{R}^m$ is the external input, $d \in \mathcal{L}_\infty^m$, and $f : \mathbb{R}^{n+m} \to \mathbb{R}^n$ is a locally Lipschitz (or Hölder) continuous function, $f(0,0) = 0$. For an initial condition $x_0 \in \mathbb{R}^n$ and input $d \in \mathcal{L}_\infty^m$, define the corresponding solutions by $X(t, x_0, d)$ for any $t \geq 0$ for which the solution exists.

Definition 1. [14, 16, Chapter 4] At the steady state $x = 0$ the system (1) for $d = 0$ is said to be *globally*

(a) *Lyapunov stable* if there is $\delta \in \mathcal{K}$ such that $\|X(t, x_0, 0)\| \leq \delta(\|x_0\|)$ for all $t \geq 0$ for any $x_0 \in \mathbb{R}^n$;

(b) *asymptotically stable* if it is Lyapunov stable and $\lim_{t\to+\infty} \|X(t, x_0, 0)\| = 0$ for any $x_0 \in \mathbb{R}^n$;

(c) *finite-time stable* (FTS) if it is Lyapunov stable and finite-time converging, *i.e.*, for any $x_0 \in \mathbb{R}^n$ there exists $0 \leq T < +\infty$ such that $X(t, x_0, 0) = 0$ for all $t \geq T$. The function $T_0(x_0) = \inf\{T \geq 0 : X(t, x_0, 0) = 0\ \forall t \geq T\}$ is called the settling time of the system (1);

(d) *nearly fixed-time stable* (nFxTS) if it is Lyapunov stable and for any $\rho > 0$ there is $T_\rho \geq 0$ such that $\sup_{x_0 \in \mathbb{R}^n} \|X(t, x_0, 0)\| \leq \rho$ for all $t \geq T_\rho$;

(e) *fixed-time stable* (FxTS) if it is FTS and nFxTS (*i.e.*, $\sup_{x_0 \in \mathbb{R}^n} T_0(x_0) < +\infty$).

Definition 2. [16] A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called a *LF* for the system (1) if there are $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$\frac{\partial V(x)}{\partial x} f(x, 0) \leq 0$$

for all $x \in \mathbb{R}^n$. Such a LF is called *strict* if additionally

$$\frac{\partial V(x)}{\partial x} f(x, 0) \leq -\alpha_3(\|x\|) \tag{2}$$

for some $\alpha_3 \in \mathcal{K}$ for all $x \in \mathbb{R}^n$.

Theorem 1. [16, 14] *The system* (1) *with* $d = 0$ *at the origin is globally*

- *Lyapunov stable if and only if it admits a LF;*
- *asymptotically stable if and only if it has a strict LF;*
- *FTS if it admits a strict LF and there exist $r > 0$, $a > 0$ and $\gamma \in (0, 1)$ such that $\alpha_3 \circ \alpha_2^{-1}(s) \geq as^\gamma$ for all $s \in [0, r]$;*
- *nFxTS if it admits a strict LF and there exist $R > 0$, $a > 0$ and $\beta > 1$ such that $\alpha_3 \circ \alpha_2^{-1}(s) \geq as^\beta$ for all $s \geq R$.*
- *FxTS if it admits a strict LF and there exist $R \geq r > 0$, $a > 0$, $\gamma \in (0, 1)$ and $\beta > 1$ such that $\alpha_3 \circ \alpha_2^{-1}(s) \geq as^\gamma$ for all $s \in [0, r]$ and $\alpha_3 \circ \alpha_2^{-1}(s) \geq as^\beta$ for all $s \geq R$.*

It is demonstrated that the given restrictions on LF are also necessary for FTS and (n)FxTS under auxiliary constraints in [7, 21] (continuity at the origin for the settling-time function is imposed) and [18, 14] (existence of the solutions in the backward time out the origin is additionally required), respectively.

2.2 Input-to-state stability

The theory of ISS is well developed [27, 11]:

Definition 3. [27] The system (1) is called *ISS*, if there are functions $\beta \in \mathcal{KL}$, $\gamma \in \mathcal{K}$ such that

$$\|X(t, x_0, d)\| \leq \beta(\|x_0\|, t) + \gamma(\|d\|_{[0,t)}) \quad \forall t \geq 0$$

for any input $d \in \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$. The function γ is called nonlinear *asymptotic gain*.

The system (1) is called *locally ISS* if the above inequality is satisfied for $\|d\|_\infty \leq \rho$ and $\|x_0\| \leq \rho$ for some $\rho \in \mathbb{R}_+$.

Definition 4. [27] The system (1) is called *integral ISS* (iISS), if there are functions $\beta \in \mathcal{KL}$, $\alpha \in \mathcal{K}_\infty$, $\gamma \in \mathcal{K}$ such that

$$\alpha(\|X(t, x_0, d)\|) \leq \beta(\|x_0\|, t) + \int_0^t \gamma(\|d(s)\|)ds \quad \forall t \geq 0$$

for any input $d \in \mathcal{L}_\infty^m$ and any $x_0 \in \mathbb{R}^n$.

Definition 5. [14] The system (1) is called *finite-time/(nearly)fixed-time (i)ISS* (FT/(n)FxT-(i)ISS) if it is (i)ISS and it admits FTS/(n)FxTS properties for the case $\|d\|_\infty = 0$.

Definition 6. [27] A smooth function $V : \mathbb{R}^n \to \mathbb{R}_+$ is called *ISS-LF* for the system (1) if there are $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ and $\theta \in \mathcal{K}$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|),$$
$$\frac{\partial V(x)}{\partial x} f(x, d) \leq \theta(\|d\|) - \alpha_3(\|x\|)$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$. Such a function V is called *local ISS-LF* if the above inequalities are verified only for $\|d\| \leq \rho$ and $\|x\| \leq \rho$ for some $\rho \in \mathbb{R}_+$.

Such a function V is called *iISS-LF* if these inequalities hold for a positive definite function $\alpha_3 : \mathbb{R}_+ \to \mathbb{R}_+$.

Note that an ISS-LF can also satisfy the following equivalent condition for some $\chi \in \mathcal{K}$:

$$\|x\| > \chi(\|d\|) \Rightarrow \frac{\partial V(x)}{\partial x} f(x, d) \leq -\alpha_3(\|x\|)$$

for all $x \in \mathbb{R}^n$ and all $d \in \mathbb{R}^m$. For $d = 0$ an ISS-LF becomes a strict LF.

Theorem 2. [16, 15, 19, 14] *The system* (1) *is (locally) (i)ISS if and only if it admits a (local) (i)ISS-LF.*

The system (1) *is (n)F(x)T-(i)ISS if it admits a (i)ISS-LF with the function* $\alpha_3 \circ \alpha_2^{-1}$ *satisfying the respective conditions of Theorem 1.*

The converse Lyapunov results for FT-ISS are proposed in [19] under additional mild restrictions.

Clearly, (n)F(x)T-(i)ISS properties imply the respective global (n)F(x)TS.

3 Statement of the problem

In [10], a nonlinear oscillator is studied:

$$\ddot{x} + \omega(\dot{x}) + k(x) = d,$$

where $x, \dot{x}, d \in \mathbb{R}$ are scalar position, velocity and input variables, respectively, $d \in \mathcal{L}_\infty^1$; the nonlinearities are assumed to be passive: $r\omega(r) > 0$, $rk(r) > 0$ for $r \neq 0$. Several designs of strict LFs for the case $d = 0$ are presented, and conditions of ISS are introduced for $d \neq 0$ using the synthesized LFs. In [2], a similar problem is investigated for a more general model of mechanical systems:

$$\ddot{x} + \frac{\partial W(\dot{x})}{\partial \dot{x}} + \frac{\partial G(x)}{\partial x} = d, \tag{3}$$

where $x, \dot{x} \in \mathbb{R}^n$ are vectors of positions and velocities; $d \in \mathbb{R}^n$ is the external input, $d \in \mathcal{L}_\infty^n$; $W : \mathbb{R}^n \to \mathbb{R}_+$ and $G : \mathbb{R}^n \to \mathbb{R}_+$ are continuously differentiable functions of their arguments (then G represents the potential energy, while W characterizes the shape of friction). Extending [10], several different designs of strict LFs are developed in [2] for the case $d = 0$ under distinct additional hypotheses imposed on the functions W and G, and next the conditions of ISS with respect to the input d are formulated considering the same LFs. In particular, the following hypotheses have been used:

Assumption 1. *Let* $W(x) = \frac{1}{2}\dot{x}^\top \mathbb{W} \dot{x}$, *where* $0 < \mathbb{W} = \mathbb{W}^\top \in \mathbb{R}^{n\times n}$, *while* G *be continuously differentiable,* $G(0) = 0$ *and such that*

$$G(x) > 0,\ x^\top \frac{\partial G(x)}{\partial x} > 0,\ \forall x \in \mathbb{R}^n \setminus \{0\}.$$

Then using the approach proposed in [3], a LF can be selected in the form:

$$U(x, \dot{x}) = h\left(\frac{1}{2}\dot{x}^\top \dot{x} + G(x)\right) + \frac{1}{2}x^\top \mathbb{W} x + x^\top \dot{x}, \tag{4}$$

where $h > 0$ is a design parameter, then for (3) with $d = 0$:

$$\dot{U} = -\dot{x}^\top \left(h\mathbb{W} - I\right)\dot{x} - x^\top \frac{\partial G(x)}{\partial x}.$$

Hence, (4) is a strict LF for a sufficiently big value of h for the system (3) under Assumption 1, moreover, a mild modification is needed to get ISS in the case $d \neq 0$:

Theorem 3. [2] *Consider the system* (3) *under Assumption 1, then it is locally ISS. Let additionally*

$$x^\top \frac{\partial G(x)}{\partial x} \geq 2\eta(\|x\|)\|x\|, \ \forall x \in \mathbb{R}^n,$$

for some $\eta \in \mathcal{K}_\infty$*, then the system is ISS.*

Existence of a strict LF allows the rates of convergence in the system to be evaluated. Hence, in this note, our goal is to design such LFs providing the conditions of (n)F(x)TS for (3) with $d = 0$. Next, eligibility of these LFs will be tested in (n)F(x)T-(i)ISS studies.

4 FTS analysis

Consider the system (3) under the following restrictions:

Assumption 2. *The functions* $W(\dot{x}), G(x)$ *are continuously differentiable for all* $\dot{x}, x \in \mathbb{R}^n$*, positive definite and homogeneous (with respect to the standard dilation) of degrees* $\mu + 1$ *and* $\lambda + 1$*, respectively, where* $\lambda, \mu \in (0, 1)$*.*

Denote

$$G_{\min} = \min_{\|x\|=1} G(x), \ G_{\max} = \max_{\|x\|=1} G(x),$$
$$W_{\min} = \min_{\|\dot{x}\|=1} W(\dot{x}), \ W_{\max} = \max_{\|\dot{x}\|=1} \left\| \frac{\partial W(\dot{x})}{\partial \dot{x}} \right\|,$$

then under Assumption 2 all these constants are strictly positive. In this section we will design (in an explicit form) for the system (3) a continuously differentiable LF, which guarantees FTS (locally).

Remark 1. In the scalar case ($n = 1$), and under additional restriction that the system (3) itself is homogeneous with respect to a weighted dilation (*i.e.*, when $\mu = \frac{2\lambda}{\lambda+1}$ and $[1 \ \frac{\lambda+1}{2}]^\top$ is the vector of weights), a homogeneous LF is constructed in the paper [9].

In this work the following LF candidate is proposed:

$$V(x, \dot{x}) = \left(\frac{1}{2}\dot{x}^\top \dot{x} + G(x)\right)^{r+1} + \gamma \|x\|^{\beta-1} x^\top \dot{x}, \tag{5}$$

where $\gamma > 0, \beta \geq 1$ and $r > 0$ are the parameters selected below.

Remark 2. If a LF candidate is chosen as a sum of "full energy" with a "cross-term", which corresponds to (5) with $r = 0$:

$$V = \frac{1}{2}\dot{x}^\top \dot{x} + G(x) + \gamma \|x\|^{\beta-1} x^\top \dot{x} \tag{6}$$

for some $\gamma, \beta > 0$, then to ensure its continuous differentiability we have to assume that $\beta \geq 1$, and it will be impossible next to get the required in Theorem 1 upper estimate (2). With additional tuning parameter $r > 0$ such a problem can be avoided, as it is shown below.

For (5), using homogeneity of G, the following inequalities can be derived:

$$a_1(\|\dot{x}\|^{2(r+1)} + \|x\|^{(\lambda+1)(r+1)}) - \gamma\|x\|^{\beta}\|\dot{x}\| \leq V \leq a_2(\|\dot{x}\|^{2(r+1)} + \|x\|^{(\lambda+1)(r+1)}) + \gamma\|x\|^{\beta}\|\dot{x}\|,$$

where

$$a_1 = \min\{2^{-r-1}, G_{\min}^{r+1}\},\ a_2 = \max\{\frac{1}{2}, 2^r G_{\max}^{r+1}\}.$$

Applying Lemma 1, we obtain that if

$$\beta > (r + 1/2)(\lambda + 1)$$

or

$$\beta = (r + 1/2)(\lambda + 1)$$

and the value of γ is sufficiently small, then there exists $H > 0$ such that

$$\frac{a_1}{2}(\|\dot{x}\|^{2(r+1)} + \|x\|^{(\lambda+1)(r+1)}) \leq V \leq 2a_2(\|\dot{x}\|^{2(r+1)} + \|x\|^{(\lambda+1)(r+1)}) \tag{7}$$

for $\max\{\|x\|, \|\dot{x}\|\} < H$.

Differentiating V with respect to the system (3), the following estimate can be derived

$$\begin{aligned}\dot{V} \leq -a_3\|\dot{x}\|^{\mu+1}\left(\frac{1}{2}\dot{x}^T\dot{x} + G(x)\right)^r - \gamma a_4\|x\|^{\beta+\lambda} + \gamma(W_{\max}\|x\|^{\beta}\|\dot{x}\|^{\mu} + \beta\|x\|^{\beta-1}\|\dot{x}\|^2) + a_5\|d\| \\ \leq -a_6\|\dot{x}\|^{2r+\mu+1} - a_7\|\dot{x}\|^{\mu+1}\|x\|^{r(\lambda+1)} - \gamma a_4\|x\|^{\beta+\lambda} \\ +\gamma(W_{\max}\|x\|^{\beta}\|\dot{x}\|^{\mu} + \beta\|x\|^{\beta-1}\|\dot{x}\|^2) + a_5\|d\|,\end{aligned}$$

where Gensen's inequality and Euler's theorem on homogeneous functions were used, and

$$a_3 = W_{\min}(r+1)(\mu+1),\ a_4 = G_{\min}(\lambda+1),$$
$$a_5 = \max_{\|x\|+\|\dot{x}\|<H}\left\{(r+1)\left(\frac{1}{2}\dot{x}^{\top}\dot{x} + G(x)\right)^r\|\dot{x}\| + \gamma\|x\|^{\beta}\right\},$$
$$a_6 = 2^{-r}\min\{1, 2^{r-1}\}a_3,\ a_7 = G_{\min}^r\min\{1, 2^{r-1}\}a_3.$$

Consider the case when

$$\frac{r(\lambda+1)}{\beta+\lambda} + \frac{\mu+1}{2r+\mu+1} > 1,$$

and let us establish conditions of negative definiteness with respect to x and $\dot{x}$ of the right-hand side of the calculated estimate on $\dot{V}$, which results in application of Lemma 1, leading to a contradiction of this inequality for the term $\|x\|^{\beta-1}\|\dot{x}\|^2$. Conversely, assume that

$$\frac{r(\lambda+1)}{\beta+\lambda} + \frac{\mu+1}{2r+\mu+1} \leq 1,$$

or equivalently,

$$\beta \geq \frac{\lambda+1}{2}(2r+\mu+1) - \lambda,$$

then with the aid of Lemma 2, it can be shown that there exist sufficiently small values of γ (in the case of the equality above), $\tilde{H} \in (0, H)$ and

$$\beta \geq \max\left\{1, r(\lambda+1) + \max\left\{\frac{\lambda}{\mu}, \frac{\lambda+1}{2}(\mu+1) - \lambda\right\}\right\}, \tag{8}$$

such that the estimate

$$\dot{V} \leq -a_8(\|\dot{x}\|^{2r+\mu+1} + \|x\|^{\beta+\lambda}) + a_5\|d\| \tag{9}$$

holds for $\max\{\|x\|, \|\dot{x}\|\} < \tilde{H}$, where $a_8 = 0.5\min\{\gamma a_4, a_6\}$. Note that the fulfillment of (8) implies that $\beta \geq (r+1/2)(\lambda+1)$ (the condition needed for positive definiteness of V).

Using the estimates (7) and (9), we arrive at the differential inequality of the desired form (applying again Gensen's inequality):

$$\dot{V} \leq -a_9 V^{\rho} + a_5\|d\|$$

for $\max\{\|x\|, \|\dot{x}\|\} < \min\{1, \tilde{H}\}$, where $a_9 = \frac{a_8(2a_2)^{-\rho}}{\max\{1, 2^{\rho-1}\}}$ and $\rho \geq \frac{\beta+\lambda}{(\lambda+1)(r+1)}$. This estimate guarantees FTS of (3) with $d = 0$ only in the case of $\rho < 1$, which follows the condition

$$\lambda < \mu. \tag{10}$$

Therefore, V in (5) is a strict LF for the system (3) for any $\lambda, \mu \in (0, 1)$, but it implies the FTS property only under the restriction (10). The following result is obtained:

Theorem 4. *Consider the system* (3) *under Assumption 2 with* (10)*, then it is FT-iISS and locally FT-ISS.*

Proof. For $d = 0$, the global asymptotic stability of the system can be proven considering its full energy:

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^{\top}\dot{x} + G(x),\ \dot{E} = -(\mu+1)W(\dot{x}) + \dot{x}^{\top}d,$$

and applying LaSalle invariance principle, then global FTS property follows from local estimates in (7), (9) under (10). Local FT-ISS has been obtained above. Moreover, using Young's inequality:

$$\begin{aligned}\dot{E} &\leq -(\mu+1)W_{\min}\|\dot{x}\|^{\mu+1} + \dot{x}^{\top}d \\ &\leq -\frac{(\mu+1)W_{\min}}{2}\|\dot{x}\|^{\mu+1} + \frac{\mu}{\mu+1}\left(\frac{(\mu+1)^2 W_{\min}}{2}\right)^{-\frac{1}{\mu}}\|d\|^{\frac{\mu+1}{\mu}},\end{aligned}$$

which implies that the system (3) with respect to the output $\dot{x}$ is dissipative and weakly zero-detectable [5], and these characteristics are equivalent to iISS property. □

Since under conditions of this theorem the system (3) is simultaneously FTS, iISS and locally ISS, we can call it *strongly* finite-time iISS in the sense of [8] (this kind of iISS is preserved under cascade interconnections).

Remark 3. As mentioned in Remark 1, for $n = 1$ the system (3) is homogeneous with respect to a weighted dilation for the vector of weights $[1\ \frac{\lambda+1}{2}]^\top$ and $\mu = \frac{2\lambda}{\lambda+1}$ (of degree $\frac{\lambda-1}{2}$). In such a case the condition (10) is verified. The function V can also be selected homogeneous by choosing

$$\beta = r(\lambda+1) + \frac{1}{2}(\lambda+1), \quad r \geq \frac{1}{2}\frac{1-\lambda}{\lambda+1},$$

then the estimates on V and its derivative, (7) and (9), will be valid globally, which complements the result of [9].

Inline with (3) we can analyze a perturbed system (whose approximation is the system (3)):

$$\ddot{x} + \frac{\partial W(\dot{x})}{\partial \dot{x}} + \frac{\partial G(x)}{\partial x} = \Psi(t, x, \dot{x}) + d, \tag{11}$$

where $\Psi(t, x, \dot{x})$ is a continuous function of its arguments that satisfies the condition

$$\|\Psi(t, x, \dot{x})\| \leq c(\|x\|^\nu + \|\dot{x}\|^\sigma) \tag{12}$$

for $t \geq 0$, $\max\{\|x\|, \|\dot{x}\|\} < H'$, where c, ν, σ, H' are positive constants. Using the Lyapunov function (5) and applying Lemma 2 we obtain a straightforward result:

Corollary 1. *Let all restrictions of Theorem 4 be satisfied,*

$$\nu > \max\left\{\lambda; \frac{\lambda+1}{2}\mu\right\}, \ \sigma > \mu,$$

then the system (11), (12) *is locally FT-ISS.*

If any of ISS-LF designs from [2] can be applied (or any other practical ISS-LFs can be found), then it is possible to strengthen the results of Theorem 4 to FT-ISS of (3).

5 nFxTS analysis

Consider the system (3) under slightly modified restrictions:

Assumption 3. *The functions $W(\dot{x}), G(x)$ are continuously differentiable for all $\dot{x}, x \in \mathbb{R}^n$, positive definite and homogeneous (with respect to the standard dilation) of degrees $\mu + 1$ and $\lambda + 1$, respectively, where $\lambda, \mu > 1$.*

In this section we will design for the system (3) a continuously differentiable LF, which guarantees a fixed-time rate of convergence with respect to a compact set containing the origin (*i.e.*, a local version of nFxTS property, where locality comes from indefiniteness of the properties of LF close to the origin). To this end we will analyze the same LF candidate as before, given in (6) for $r = 0$, where $\gamma > 0$ and $\beta \geq 1$ are the parameters chosen below.

For (6), using homogeneity of G and applying Lemma 1, we obtain that if

$$\beta < \frac{\lambda+1}{2}$$

(or $\beta = \frac{\lambda+1}{2}$ and the value of γ is sufficiently small), then there exists $H > 0$ such that the relations

$$\frac{a_{10}}{2}(\|\dot{x}\|^2 + \|x\|^{\lambda+1}) \leq V \leq 2a_{11}(\|\dot{x}\|^2 + \|x\|^{\lambda+1}) \quad (13)$$

are verified for $\max\{\|x\|, \|\dot{x}\|\} > H$, where

$$a_{10} = \min\{\frac{1}{2}, G_{\min}\},\ a_{11} = \max\{\frac{1}{2}, G_{\max}\},$$

i.e., the function V in (6) is positive for all sufficiently big values of x and $\dot{x}$.

Differentiating V with respect to the system (3) and performing similar derivations (*e.g.*, with Euler's theorem on homogeneous functions), the following estimate can be derived

$$\dot{V} \leq -a_3\|\dot{x}\|^{\mu+1} - \gamma a_4\|x\|^{\beta+\lambda} + \gamma(W_{\max}\|x\|^{\beta}\|\dot{x}\|^{\mu} \\ +\beta\|x\|^{\beta-1}\|\dot{x}\|^2) + (\|\dot{x}\| + \gamma\|x\|^{\beta})\|d\|.$$

Further, with the aid of Lemma 1 and Young's inequality, it can be shown that for

$$\beta \leq \min\left\{\frac{\lambda}{\mu}, \frac{\mu(1+\lambda)+1-\lambda}{2}\right\}, \quad (14)$$

there exist sufficiently small value of γ (in the case of the equality above) and $\tilde{H} \geq H$ such that the estimate

$$\dot{V} \leq -a_{12}(\|\dot{x}\|^{\mu+1} + \|x\|^{\beta+\lambda}) + a_{13}(\|d\|^{\frac{\mu+1}{\mu}} + \|d\|^{\frac{\beta+\lambda}{\lambda}}) \quad (15)$$

holds for $\max\{\|x\|, \|\dot{x}\|\} > \tilde{H}$, where $a_{12} = 0.5\min\{\gamma a_4, a_3\}$ and

$$a_{13} = \max\left\{\frac{\mu}{\mu+1}\left(\frac{a_3}{4}(\mu+1)\right)^{-\frac{1}{\mu}}, \frac{\lambda}{\beta+\lambda}\left(\frac{a_4}{4}\frac{\beta+\lambda}{\beta}\right)^{-\frac{\beta}{\lambda}}\right\}.$$

Using the estimates (13) and (15), the differential inequality of the desired form (applying again Gensen's inequality) can be derived:

$$\dot{V} \leq -a_{14}V^{\varrho} + a_{13}(\|d\|^{\frac{\mu+1}{\mu}} + \|d\|^{\frac{\beta+\lambda}{\lambda}})$$

for $\max\{\|x\|, \|\dot{x}\|\} > \hat{H} = \max\{1, \tilde{H}\}$, where $a_{14} = \frac{a_{12}(2a_{11})^{-\varrho}}{\max\{2, 2^{\varrho}\}}$ and $\varrho \leq \min\left\{\frac{\mu+1}{2}, \frac{\beta+\lambda}{\lambda+1}\right\}$. And this estimate can be rewritten in a global fashion as follows:

$$\max\{\|x\|, \|\dot{x}\|\} \geq \max\{\hat{H}, \psi(\|d\|)\} \Rightarrow \dot{V} \leq -\frac{a_{14}}{2}V^{\varrho}$$

for all $x \in \mathbb{R}^n$, $\dot{x} \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, where

$$\psi(s) = \theta^{-1}\left(\frac{2}{a_{10}}\left(\frac{2a_{13}}{a_{14}}(s^{\frac{\mu+1}{\mu}} + s^{\frac{\beta+\lambda}{\lambda}})\right)^{\frac{1}{\varrho}}\right),\ \theta(s) = \max\{s^{\lambda+1}, s^2\}.$$

Note that $\varrho > 1$ provided that $\beta > 1$ in addition to the restriction (14), which all can be satisfied for

$$\lambda > \mu \tag{16}$$

only (with the chosen LF). Hence, again V in (6) is a LF for the system (3) for any $\lambda, \mu > 1$ showing global attractiveness of a set containing origin or global practical ISS, but it implies the nFxTS property only under the restriction (16):

Theorem 5. *Consider the system* (3) *under Assumption 3 with* (16)*, then it is nFxT-ISS.*

Proof. The global asymptotic stability of the system for $d = 0$ and iISS can be proven considering the full energy E of the system as in the proof of Theorem 4. Together with the established properties of V from (6) and under the conditions of the theorem, the system is also globally nFxTS for the case $d = 0$. It is left to show ISS property.

Using the converse Lyapunov theorems for globally asymptotically stable systems [16], in this case there is a continuously differentiable LF $U : \mathbb{R}^{2n} \to \mathbb{R}_+$ such that

$$\alpha_1(\max\{\|x\|, \|\dot{x}\|\}) \le U(x, \dot{x}) \le \alpha_2(\max\{\|x\|, \|\dot{x}\|\}), \tag{17}$$
$$\dot{U} \le -\alpha_3(U(x, \dot{x})) + \frac{\partial U(x, \dot{x})}{\partial \dot{x}} d$$

for all $x \in \mathbb{R}^n$, $\dot{x} \in \mathbb{R}^n$ and $d \in \mathbb{R}^n$, for some $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ (here the derivative is calculated on solutions of (3)). Or equivalently,

$$\max\{\|x\|, \|\dot{x}\|\} \ge \eta(\|d\|) \Rightarrow \dot{U} \le -\frac{1}{2}\alpha_3(U(x, \dot{x}))$$

for all $(x, \dot{x}) \in \Omega = \{(x, \dot{x}) \in \mathbb{R}^n \times \mathbb{R}^n : U(x, \dot{x}) \le \alpha_1(2\hat{H})\} \subseteq \{(x, \dot{x}) \in \mathbb{R}^n \times \mathbb{R}^n : \max\{\|x\|, \|\dot{x}\|\} \le 2\hat{H}\}$, where

$$\eta(s) = \alpha_1^{-1} \circ \alpha_3^{-1}(2a_{15}s),\ a_{15} = \sup_{(x,\dot{x})\in\Omega} \left\|\frac{\partial U(x, \dot{x})}{\partial \dot{x}}\right\|.$$

Following [19], define a new global nFxT and ISS-LF candidate

$$\tilde{V}(x, \dot{x}) = s(U(x, \dot{x}))V(x, \dot{x}) + (1 - s(U(x, \dot{x})))U(x, \dot{x}),$$

where a continuously differentiable function $s : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies:

$$s(r) = \begin{cases} 1 & r \ge \alpha_1(2\hat{H}) \\ 0 & r \le \alpha_1(\hat{H}) \end{cases}$$

and $\frac{ds(r)}{dr} > 0$ for all $r \in (\alpha_1(\hat{H}), \alpha_1(2\hat{H}))$. Moreover, we assume that $U(x, \dot{x}) \le V(x, \dot{x})$ for all $(x, \dot{x}) \in \Upsilon = \{(x, \dot{x}) \in \mathbb{R}^{2n} : \alpha_1(\hat{H}) \le U(x, \dot{x}) \le \alpha_1(2\hat{H})\}$, which can be provided on a compact set Υ through a simple scaling of the function U by a constant multiplier. Obviously, $\tilde{V}$ is continuously differentiable, positive definite and radially unbounded since the functions V and U possess these properties. Next, for (3) we obtain for a suitably defined $\alpha_4 \in \mathcal{K}_\infty$:

$$\begin{aligned}\dot{\tilde{V}} &= s(U)\dot{V} + (1 - s(U))\dot{U} + \frac{ds(U)}{dr}\dot{U}\,(V - U)\\ &\le s(U)\dot{V} + (1 - s(U))\dot{U} \le -\alpha_4(\tilde{V})\end{aligned}$$

provided that $\max\{\|x\|, \|\dot{x}\|\} \ge \max\{\eta(\|d\|), \psi(\|d\|)\}$. Clearly, all conditions of Theorem 2 are satisfied ($\alpha_4(s) = a_{14}s^{\varrho}$ for $s > 0$ high enough with $\varrho > 1$) and the system is nFxT-ISS. □

Note that in the proof of the last theorem no explicit construction of nFxTS ISS-LF is given, since the function U was obtained by converse Lyapunov arguments. An example of a possible choice of U, under Assumption 1, is given in (4), and the related restrictions are formulated in Theorem 3.

Similarly to Corollary 1, this result can be extended to the case of the system (11), (12):

Corollary 2. *Let all restrictions of Theorem 5 be satisfied,*

$$\nu < \min\left\{\lambda, \frac{\lambda + 1}{2}\mu\right\}, \ \sigma < \mu,$$

then the system (11) *admitting* (12) *for* $\max\{\|x\|, \|\dot{x}\|\} > H'$ *is nFxT-ISS for* $\max\{\|x\|, \|\dot{x}\|\} > \hat{H}' \ge \max\{\hat{H}, H'\}$.

In other words, for (11) a practical nFxTS-ISS property is substantiated. Another interpretation is that in corollaries 1 and 2 the asymptotic gain function for (3) is evaluated (the form of admissible additive nonlinearities that can be tolerated by the system is qualified).

6 FxTS analysis

To establish FxTS consider the system (3) under combined restrictions from assumptions 2 and 3:

Assumption 4. *Let* $W(\dot{x}) = W_1(\dot{x}) + W_2(\dot{x})$ *and* $G(x) = G_1(x) + G_2(x)$*, with the functions* $W_1(\dot{x}), W_2(\dot{x})$ *and* $G_1(x), G_2(x)$ *being continuously differentiable for all* $\dot{x}, x \in \mathbb{R}^n$*, positive definite and homogeneous (with respect to the standard dilation) of degrees* $\mu_1 + 1, \mu_2 + 1$ *and* $\lambda_1 + 1, \lambda_2 + 1$*, respectively, where* $\lambda_1, \mu_1 \in (0, 1)$ $\lambda_2, \mu_2 > 1$.

Here the functions $W_1(\dot{x}), G_1(x)$ respect the conditions of Assumption 2, while $W_2(\dot{x}), G_2(x)$ verify the hypotheses of Assumption 3. Moreover, the restrictions of corollaries 1 and 2 between λ_1, λ_2 and μ_1, μ_2 are also true. Hence, the required conclusion on FxTS-ISS is a direct consequence of the previously proven results:

Theorem 6. *Consider the system* (3) *under Assumption 4 with* $\lambda_1 < \mu_1$ *and* $\lambda_2 > \mu_2$, *then it is FxT-ISS.*

Proof. For $d = 0$, the global asymptotic stability of the system can be proven in the same way by considering the full energy of the system:

$$E(x, \dot{x}) = \frac{1}{2}\dot{x}^\top \dot{x} + G(x), \ \dot{E} = -(\mu_1 + 1)W_1(\dot{x}) - (\mu_2 + 1)W_2(\dot{x}),$$

and applying LaSalle invariance principle. By Corollary 1,

$$V_1(x, \dot{x}) = \left(\frac{1}{2}\dot{x}^\top \dot{x} + G_1(x)\right)^{r+1} + \gamma_1 \|x\|^{\beta_1 - 1} x^\top \dot{x},$$

where $\gamma_1 > 0, \beta_1 \geq 1$ and $r > 0$ are the parameters respecting the derived previously relations, is a local FT ISS-LF for $\max\{\|x\|, \|\dot{x}\|\} < H_1$. By Corollary 2,

$$V_2(x, \dot{x}) = \frac{1}{2}\dot{x}^\top \dot{x} + G_2(x) + \gamma_2 \|x\|^{\beta_2 - 1} x^\top \dot{x},$$

where $\gamma_2 > 0$ and $\beta_2 \geq 1$ are the parameters satisfying the respective established relations, is a local nFxT ISS-LF for $\max\{\|x\|, \|\dot{x}\|\} > H_2$. Using the LF (17) obtained for (3) through global asymptotic stability converse arguments, and applying the same patching technique as in the proof of Theorem 5, a global FxTS ISS-LF can be designed by combining V_1, V_2 and U, which will verify all conditions of Theorem 2. $\square$

Note that the obtained results are conceptually different from existing in the literature [4], where homogeneity arguments are used and existence of local homogeneous approximations near the origin and at infinity are assumed. The above results include the homogeneous case as a particular example. Moreover, the LFs used close to the origin and at infinity are given explicitly.

7 Applications

Let us demonstrate applicability and the ways of development of the proposed theory in three practical cases.

7.1 FTS of Lagrange systems

The result of Theorem 4 can be adapted to other canonical forms of mechanical models.

For example, consider a Lagrange system

$$\frac{d}{dt}\frac{\partial T}{\partial \dot q} - \frac{\partial T}{\partial q} = -\frac{\partial W(\dot q)}{\partial \dot q} - \frac{\partial G(q)}{\partial q} + d,$$

where $q \in \mathbb{R}^n$ and $\dot q \in \mathbb{R}^n$ are vectors of generalized coordinates and velocities, respectively; $d \in \mathbb{R}^n$ is a perturbation as above; the kinetic energy $T = T(q, \dot q)$ of the system is of the form $T(q, \dot q) = \frac{1}{2}\dot q^T A(q)\dot q$, where $A(q)$ is a symmetric and continuously differentiable matrix function for $q \in \mathbb{R}^n$; $W(\dot q), G(q)$ are continuously differentiable and positive definite functions for $\dot q, q \in \mathbb{R}^n$, which are homogeneous with respect to the standard dilation of degrees $\mu + 1$ and $\lambda + 1$, respectively, $\lambda, \mu \in (0, 1)$ as before in Assumption 2. Assume that the estimates

$$k_1\|\dot q\|^2 \le T(q, \dot q) \le k_2\|\dot q\|^2,$$
$$\left\|\frac{\partial T(q, \dot q)}{\partial \dot q}\right\| \le k_3\|\dot q\|, \quad \left\|\frac{\partial T(q, \dot q)}{\partial q}\right\| \le k_4\|\dot q\|^2$$

hold for $\max\{\|q\|, \|\dot q\|\} < k_5$, where $k_i, i = 1, \dots, 5$ are positive constants. Construct a LF candidate in the form

$$V(q, \dot q) = (T(q, \dot q) + G(q))^{r+1} + \gamma\|q\|^{\beta-1}\left(\frac{\partial T(q, \dot q)}{\partial \dot q}\right)^\top q,$$

where $\gamma > 0, \beta \ge 1$, $r \ge 0$ are parameters (this LF is a modification of (5)). In a similar way, under the same hypotheses we can substantiate local ISS and FTS-iISS of this system.

7.2 Finite-time monoaxial stabilization of a rigid body

Consider a rigid body rotating about its mass center O with angular velocity $\boldsymbol{\omega} \in \mathbb{R}^3$. Assume that the axes $Oxyz$ are principal central axes of inertia of the body. Differential equations governing the attitude motion of the body under control torque $\mathbf{M} \in \mathbb{R}^3$ have the following form

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = \mathbf{M}, \tag{18}$$

where $\mathbf{J} = \mathrm{diag}\{A, B, C\}$ is a body inertia tensor in the axes $Oxyz$, A, B, C are positive parameters.

Let unit vectors $\mathbf{s} \in \mathbb{R}^3$ and $\mathbf{r} \in \mathbb{R}^3$ be given, the vector $\mathbf{s}$ be constant in the inertial space and the vector $\mathbf{r}$ be constant in the body-fixed frame. Then the vector $\mathbf{s}$ rotates with respect to the coordinate system $Oxyz$ with angular velocity $-\boldsymbol{\omega}$. Hence,

$$\dot{\mathbf{s}} = -\boldsymbol{\omega} \times \mathbf{s}. \tag{19}$$

Thus, we will consider the differential system consisting of the Euler dynamic equations (18) and the Poisson kinematic equations (19). Let the torque $\mathbf{M}$ be a sum of the dissipative component $\mathbf{M}_d$ and the restoring one $\mathbf{M}_r$:

$$\mathbf{M} = \mathbf{M}_d + \mathbf{M}_r.$$

The restoring torque $\mathbf{M}_r$ should be chosen such that the torque $\mathbf{M}$ ensures monoaxial stabilization of a rigid body: the system of equations (18), (19) should admit the asymptotically stable equilibrium position:

$$\boldsymbol{\omega} = \mathbf{0}, \qquad \mathbf{s} = \mathbf{r}. \tag{20}$$

We will assume that the dissipative torque is defined by the formula $\mathbf{M}_d = -\partial W(\boldsymbol{\omega})/\partial\boldsymbol{\omega}$, where $W(\boldsymbol{\omega})$ is a continuously differentiable for $\boldsymbol{\omega} \in \mathbb{R}^3$ positive definite homogeneous function of the degree $\mu + 1$, $0 < \mu < 1$, and

$$\mathbf{M}_r = -a\|\mathbf{s} - \mathbf{r}\|^{\lambda-1}\mathbf{s} \times \mathbf{r},$$

where $a > 0$, $0 < \lambda < 1$. Then the system (18) takes the form

$$\mathbf{J}\dot{\boldsymbol{\omega}} + \boldsymbol{\omega} \times \mathbf{J}\boldsymbol{\omega} = -\partial W(\boldsymbol{\omega})/\partial\boldsymbol{\omega} - a\|\mathbf{s} - \mathbf{r}\|^{\lambda-1}\mathbf{s} \times \mathbf{r}.$$

Let

$$\bar{V}(\boldsymbol{\omega}, \mathbf{s}) = \left(\frac{1}{2}\boldsymbol{\omega}^\top\mathbf{J}\boldsymbol{\omega} + \frac{a}{\lambda+1}\|\mathbf{s} - \mathbf{r}\|^{\lambda+1}\right)^{p+1} + \gamma\|\mathbf{s} \times \mathbf{r}\|^{\beta-1}\boldsymbol{\omega}^\top\mathbf{J}(\mathbf{s} \times \mathbf{r}),$$

where $\gamma > 0$, $\beta \geq 1$, $p \geq 0$. In a similar way as in Section 4, we obtain that the LF $\bar{V}$ with a special choice of parameters γ, β, p and (10) guarantees local FTS of the equilibrium position (20) (it is not the unique equilibrium of the system (18), (19), hence, a global result cannot be derived).

Remark 4. Developing these results and using $\bar{V}$, conditions of finite-time triaxial stabilization of a rigid body can be obtained. Note that previously such a problem was considered with application of homogeneity and sliding mode control, and without construction of a strict LF [12].

7.3 Formation control

Consider a group of n mobile agents on a line. The agents are treated as numbered points with coordinates $x_i(t) \in \mathbb{R}$, where $i = 1, \dots, n$; $x(t) = (x_1(t), \dots, x_n(t))^\top$. Assume that a segment $[a, b]$ of the line is given. It is required to provide the equidistant fast distribution of the agents on this segment.

Assume that the agents are modeled by the double integrators with additive perturbations:

$$\ddot{x}_i(t) = u_i(t) + d_i(t), \quad i = 1, \dots, n,$$

where $u_i(t) \in \mathbb{R}$ is a control input, $u(t) = (u_1(t), \dots, u_n(t))^\top$, and each agent receives information about the distances and relative velocities between itself and its nearest left and right neighbors (neighbors are understood in terms of agents' numbers, and in such a case the points a and b are interpreted as static agents: $x_0(t) = a$, $x_{n+1}(t) = b$ for $t \geq 0$), $d_i(t) \in \mathbb{R}$ is a disturbance for an agent, $d(t) = (d_1(t), \ldots, d_n(t))^\top$. Let it be required to design a decentralized control protocol. To apply the developed theoretical findings, the FxT-ISS of the equilibrium position

$$\tilde{x} = a \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} + \frac{b-a}{n+1} \begin{pmatrix} 1 \\ 2 \\ \vdots \\ n \end{pmatrix}$$

will be guaranteed by a choice of u that provides the equidistant distribution of the agents. For the case where agent's dynamics is described by the first-order integrators, a similar problem was solved in [24].

Let

$$z_i(t) = \frac{1}{2}(x_{i-1}(t) - x_i(t)) + \frac{1}{2}(x_{i+1}(t) - x_i(t)), \quad i = 1, \ldots, n,$$

$$u_i = \dot{z}_i^\alpha + \dot{z}_i^\beta + z_i^\mu + z_i^\nu,$$

α, β, μ, ν are rationals with odd numerators and denominators, $0 < \mu < \alpha < 1$ and $\nu > \beta > 1$. For $z(t) = (z_1(t), \ldots, z_n(t))^\top$ we obtain

$$z = Ax + B,$$

where

$$A = \begin{pmatrix} -1 & 0.5 & 0 & 0 & \cdots & 0 & 0 \\ 0.5 & -1 & 0.5 & 0 & \cdots & 0 & 0 \\ 0 & 0.5 & -1 & 0.5 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -1 & 0.5 \\ 0 & 0 & 0 & 0 & \cdots & 0.5 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 0.5a \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0.5b \end{pmatrix}.$$

Using this vector representation, we arrive at the system

$$D\ddot{z} = -\frac{\partial W_1(\dot{z})}{\partial \dot{z}} - \frac{\partial G_1(z)}{\partial z} - \frac{\partial W_2(\dot{z})}{\partial \dot{z}} - \frac{\partial G_2(z)}{\partial z} - d,$$

where $D = -A^{-1}$ is a positive definite matrix,

$$W_1(\dot{z}) = \frac{1}{\alpha+1} \sum_{i=1}^{n} \dot{z}_i^{\alpha+1}, \quad G_1(z) = \frac{1}{\mu+1} \sum_{i=1}^{n} z_i^{\mu+1},$$

$$W_2(\dot{z}) = \frac{1}{\beta+1} \sum_{i=1}^{n} \dot{z}_i^{\beta+1}, \quad G_2(z) = \frac{1}{\nu+1} \sum_{i=1}^{n} z_i^{\nu+1}.$$

Thus, we obtain a variant of the system (3) and using a mild modification of LF in (5):

$$V(z,\dot{z})=\left(\frac{1}{2}\dot{z}^{\top}D\dot{z}+G(z)\right)^{r+1}+\gamma\|z\|^{\beta-1}z^{\top}D\dot{z},$$

where $\gamma>0,\beta\geq 1$ and $r>0$ are as before, it is possible to demonstrate FxTS-ISS property of this system (all conditions of Theorem 6 are verified), which corresponds to the required robust and fast equidistant distribution of the agents on the interval.

8 Conclusion

For a canonical representation of mechanical systems, (3), the conditions of (n)F(x)TS-(i)ISS are investigated, and the respective LFs are proposed. It is demonstrated in the examples that these conditions can be verified/developed in different practical scenarios. The main restrictions on the values of powers in the system are given in (10) and (16), and an open question is existence of (n)F(x)TS if these conditions are violated, which can be studied in future works.

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