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IMPROVEMENTS TO THE NUMBER FIELD SIEVE FOR NON-PRIME FINITE FIELDS

(PRELIMINARY VERSION)

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Abstract. We propose various strategies for improving the computation of discrete logarithms in non-prime fields of medium to large characteristic using the Number Field Sieve. This includes new methods for selecting the polynomials; the use of explicit automorphisms; explicit computations in the number fields; and prediction that some units have a zero virtual logarithm. On the theoretical side, we obtain a new complexity bound of $L_{p^n}(1/3, \sqrt[3]{96/9})$ in the medium characteristic case. On the practical side, we computed discrete logarithms in $\mathbb{F}_{p^2}$ for a prime number p with 80 decimal digits.

Warning: This unpublished version contains some inexact statements.

1. Introduction

Discrete logarithm computations in finite fields is one of the important topics in algorithmic number theory, partly due to its relevance to public key cryptography. The complexity of discrete logarithm algorithms for finite fields $\mathbb{F}_{p^n}$ depends on the size of the characteristic p with respect to the cardinality $Q = p^n$. In order to classify the known methods, it is convenient to use the famous L function. If $\alpha \in [0, 1]$ and $c > 0$ are two constants, we set

$$L_Q(\alpha, c) = \exp\left((c + o(1))(\log Q)^\alpha (\log\log Q)^{1-\alpha}\right),$$

and sometimes we simply write $L_Q(\alpha)$ if the constant c is not made explicit. When we consider discrete logarithm computations, we treat separately families of finite fields for which the characteristic p can be written in the form $p = L_Q(\alpha)$ for a given range of values for α. We say that we are dealing with finite fields of *small characteristic* if the family is such that $\alpha < 1/3$; *medium characteristic* if we have $1/3 < \alpha < 2/3$; and *large characteristic* if $\alpha > 2/3$. In this article, we concentrate on the cases of medium and large characteristic. This covers also the situation where $p = L_Q(2/3)$, that we call the medium–large characteristic boundary case. We start with a brief overview of the general situation, including the small characteristic case for completeness (all the complexities mentioned here are based on unproven heuristics).

The case of small characteristic is the one that has been improved in the most dramatic way in the recent years. Before 2013, the best known complexity of $L_Q(1/3, \sqrt[3]{32/9})$ was obtained with the Function Field Sieve [Adl94, AH99, JL02, JL06] but a series of improvements [Jou13b, Jou14, BGJT14, GGMZ13, GKZ14b] has led to a quasi-polynomial complexity for fixed characteristic, and more generally to a complexity of $L_Q(\alpha + o(1))$ when $p = L_Q(\alpha)$, with $\alpha < 1/3$.

The case of large characteristic is covered by an algorithm called the Number Field Sieve (NFS) that is very close to the algorithm with the same name used for factoring integers [LL93, Gor93, Sch93, JL02, Sch05]. This is particularly true for prime fields, and it shares the same complexity of $L_Q(1/3, \sqrt[3]{64/9})$. In the case of small extension degrees, the main reference is a

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variant by Joux, Lercier, Smart and Vercauteren [JLSV06] who showed how to get the same complexity in the whole range of fields of large characteristic.

The case of medium characteristic was also tackled in the same article, thus getting a complexity of $L_Q(1/3, \sqrt[3]{128/9})$, with another variant of NFS.

The complexities listed above use versions of NFS where only two number fields are involved. It is however known that using more number fields can improve the complexity. For prime fields it has been done in [Mat03, CS06], while for large and medium characteristic, it has been recently studied in [BP14]. In all cases, the complexity remains of the form $L_Q(1/3, c)$, but the exponent constant c is improved: in the large characteristic case we have $c = \sqrt[3]{(92 + 26\sqrt{13})/27}$, like for prime fields, while in the medium characteristic case, we have $c = \sqrt[3]{2^{13}/3^6}$. For the moment, these multiple number field variants have not been used for practical record computations (they have not yet been used either for records in integer factorization).

In the medium–large characteristic boundary case, where $p = L_Q(2/3, c_p)$, the complexity given in [BP14] is also of the form $L_Q(1/3, c)$, where c varies between 16/9 and $\sqrt[3]{2^{13}/3^6}$ in a way that is non-monotonic with c_p. We also mention another variant of NFS that has been announced [BGK14] that seems to be better in some range of c_p, when using multiple number fields.

In terms of practical record computations, the case of prime fields has been well studied, with frequent announcements [JL05, Kle07, BGI+14]. In the case of medium characteristic, there were also some large computations performed to illustrate the new methods; see Table 8 in [JL+07] and [Zaj08, HAKT13]. However in the case of non-prime field of large characteristic, we are not aware of previous practical experiments, despite their potential interest in pairing-based cryptography.

Summary of contributions. Our two main contributions are, on one side, new complexity results for the finite fields of medium characteristic, and on the other side, a practical record computation in a finite field of the form $\mathbb{F}_{p^2}$.

Key tools for these results are two new methods for selecting the number fields; the first one is a generalization of the method by Joux and Lercier [JL03] and we call the second one the conjugation method. It turned out that both of them have practical and theoretical advantages.

On the theoretical side, the norms that must be tested for smoothness during NFS based on the conjugation method or the generalized Joux-Lercier method are smaller than the ones obtained with previous methods for certain kind of finite fields. Therefore, the probability of being smooth is higher, which translates into a better complexity. Depending on the type of finite fields, the gain is different:

- In the medium characteristic finite fields, NFS with the conjugation method has a complexity of $L_Q(1/3, \sqrt[3]{96/9})$. This is much better than the complexity of $L_Q(1/3, \sqrt[3]{128/9})$ obtained in [JLSV06] and also beats the $L_Q(1/3, \sqrt[3]{2^{13}/3^6})$ complexity of the multiple number field algorithm of [BP14].
- In the medium–large characteristic boundary case, the situation is more complicated, but there are also families of finite fields for which the best known complexity is obtained with the conjugation method or with the generalized Joux-Lercier method. The overall minimal complexity is obtained for fields with $p = L_Q(2/3, \sqrt[3]{12})$, where the complexity drops to $L_Q(1/3, \sqrt[3]{48/9})$ with the conjugation method.

On the practical side, the two polynomials generated by the conjugation method (and for one of the polynomials with the generalized Joux-Lercier construction) enjoy structural properties: it is often possible to use computations with explicit units (as was done in the early ages of NFS for factoring, before Adleman introduced the use of characters), thus saving the use of Schirokauer maps that have a non-negligible cost during the linear algebra phase. Furthermore, it is also often possible to impose the presence of field automorphisms which can be used to speed-up various stages of NFS, as shown in [JLSV06].

Finally, the presence of automorphisms can interact with the general NFS construction and lead to several units having zero virtual logarithms. This is again very interesting in practice, because some dense columns (explicit units or Schirokauer maps) can be erased in the matrix. A careful study of this phenomenon allowed us to predict precisely when it occurs.

All these practical improvements do not change the complexity but make the computations faster. In fact, even though the conjugation method is at its best for medium characteristic, it proved to be competitive even for quadratic extensions. It was therefore used in our record computation of discrete logarithm in the finite field $\mathbb{F}_{p^2}$ for a random-looking prime p of 80 decimal digits. The running time was much less than what is required to solve the discrete logarithm problem in a prime field of similar size, namely 160 decimal digits.

Outline. In Section 2 we make a quick presentation of NFS, and we insist on making precise the definitions of virtual logarithms in the case of explicit units and in the case of Schirokauer maps. In Section 3 we show how to obtain a practical improvement using field automorphisms, again taking care of the two ways of dealing with units. Then, in Section 4 we explain how to predict the cases where the virtual logarithm of a unit is zero, and in Section 5 we show how to use this knowledge to reduce the number of Schirokauer maps if we do not use explicit units. Finally, in Section 6 we present our two new methods for selecting polynomials, the complexities of which are analyzed in Section 7. We conclude in Section 8 with a report about our practical computation in $\mathbb{F}_{p^2}$.

2. The number field sieve and virtual logarithms

2.1. **Sketch of the number field sieve algorithm.** In a nutshell, the number field sieve for discrete logarithms in $\mathbb{F}_{p^n}$ is as follows. In the first stage, called polynomial selection, two polynomials f, g in $\mathbb{Z}[x]$ are constructed (we assume that $\deg f \geq \deg g$), such that their reductions modulo p have a common monic irreducible factor φ_0 of degree n. For simplicity, we assume that f and g are monic. We call φ a monic polynomial of $\mathbb{Z}[x]$ whose reduction modulo p equals φ_0. Let α and β be algebraic numbers such that $f(\alpha) = 0$ and $g(\beta) = 0$ and let m be a root of φ_0 in $\mathbb{F}_{p^n}$, allowing us to write $\mathbb{F}_{p^n} = \mathbb{F}_p(m)$. Let K_f and K_g be the number fields associated to f and g respectively, and $\mathcal{O}_f$ and $\mathcal{O}_g$ their rings of integers.

For the second stage of NFS, called relation collection or sieve, a smoothness bound B is chosen and we consider the associated factor base

$$\mathcal{F} = \{\text{prime ideals } \mathfrak{q} \text{ in } \mathcal{O}_f \text{ and } \mathcal{O}_g \text{ of norm less than } B\},$$

that we decompose into $\mathcal{F} = \mathcal{F}_f \cup \mathcal{F}_g$ according to the ring of integers to which the ideals belong. An integer is B-smooth if all its prime factors are less than B. For any polynomial $\phi(x) \in \mathbb{Z}[x]$, the algebraic integer $\phi(\alpha)$ (resp. $\phi(\beta)$)) in K_f (resp. K_g) is B-smooth if the corresponding principal ideal $\phi(\alpha)\mathcal{O}_f$ (resp. $\phi(\beta)\mathcal{O}_g$) factors into prime ideals that belong to $\mathcal{F}_f$ (resp. $\mathcal{F}_g$). This is almost, but not exactly equivalent to asking that the norm $\mathrm{Res}(\phi, f)$ (resp. $\mathrm{Res}(\phi, g)$) is B-smooth.

In the sieve stage, one collects $\#\mathcal{F}$ polynomials $\phi(x) \in \mathbb{Z}[x]$ with coprime coefficients and degree bounded by $t-1$, for a parameter $t \geq 2$ to be chosen, such that both $\phi(\alpha)$ and $\phi(\beta)$ are B-smooth, so that we get *relations* of the form:

$$\begin{cases} \phi(\alpha)\mathcal{O}_f = \prod_{\mathfrak{q} \in \mathcal{F}_f} \mathfrak{q}^{\mathrm{val}_\mathfrak{q}(\phi(\alpha))} \\ \phi(\beta)\mathcal{O}_g = \prod_{\mathfrak{r} \in \mathcal{F}_g} \mathfrak{r}^{\mathrm{val}_\mathfrak{r}(\phi(\beta))}. \end{cases} \tag{1}$$

The norm of $\phi(\alpha)$ (resp. of $\phi(\beta)$) is the product of the norms of the ideals in the right hand side and will be (crudely) bounded by the size of the finite field; therefore the number of ideals involved in a relation is less than $\log_2(p^n)$. One can also remark that the ideals that can occur in a relation have degrees that are at most equal to the degree of ϕ, that is $t-1$. Therefore, it makes sense to include in $\mathcal{F}$ only the ideals of degree at most $t-1$ (for a theoretical analysis of NFS one can consider the variant where only ideals of degree one are included in the factor base).

In order to estimate the probability to get a relation for a polynomial ϕ with given degree and size of coefficients, we make the common heuristic that the integer $\mathrm{Res}(\phi, f) \cdot \mathrm{Res}(\phi, g)$ has the same probability to be B-smooth as a random integer of the same size and that the bias due to powers is negligible. Therefore, reducing the expected size of this product of norms is the main criterion when selecting the polynomials f and g.

In the linear algebra stage, each relation is rewritten as a linear equation between the so-called virtual logarithms of the factor base elements. We recall this notion in Section 2.2. We make the usual heuristic that this system has a space of solutions of dimension one. Since the system is sparse, an iterative algorithm like Wiedemann's [Wie86] is used to compute a non-zero solution in quasi-quadratic time. This gives the (virtual) logarithms of all the factor base elements.

In principle, the coefficient ring of the matrix is $\mathbb{Z}/(p^n - 1)\mathbb{Z}$, but it is enough to solve it modulo each prime divisor ℓ of $p^n - 1$ and then to recombine the results using the Pohlig-Hellman algorithm [PH78]. Since one can use Pollard's method [Pol78] for small primes ℓ, we can suppose that ℓ is larger than $L_{p^n}(1/3)$. It allows us then to assume that ℓ is coprime to $\mathrm{Disc}(f)$, $\mathrm{Disc}(g)$, the class numbers of K_f and K_g, and the orders of the roots of unity in K_f and K_g. These assumptions are used in many places in the rest of the article, sometimes implicitly.

In the last stage of the algorithm, called individual logarithm, the discrete logarithm of any element $z = \sum_{i=0}^{n-1} z_i m^i$ of $\mathbb{F}_{p^n}$ in the finite field is computed. For this, we associate to z the algebraic number $\overline{z} = \sum_{i=0}^{n-1} z_i \alpha^i$ in K_f and check whether the corresponding principal ideal factors into prime ideals of norms bounded by a quantity B' larger than B. We also ask the prime ideals to be of degree at most $t - 1$. If $\overline{z}$ does not verify these smoothness assumptions, then we replace z by z^e for a randomly chosen integer e and try again. This allows to obtain a linear equation similar to those of the linear system, in which one of the unknowns is $\log z$. The second step of the individual logarithm stage consists in obtaining relations between a prime ideal and prime ideals of smaller norm, until all the ideals involved are in $\mathcal{F}$. This allows to backtrack and obtain $\log z$.

2.2. **Virtual logarithms.** In this section, we recall the definition of virtual logarithms, while keeping in mind that in the rest of the article, we are going to use either explicit unit computations or Schirokauer maps. The constructions work independently in each number field, so we explain them for the field K_f corresponding to the polynomial f. During NFS, this is also applied to K_g.

We start by fixing a notation for the "reduction modulo p" map that will be used in several places of the article.

Definition 2.1 (Reduction map)**.** *Let ρ_f be the map from $\mathcal{O}_f$ to $\mathbb{F}_{p^n}$ defined by the reduction modulo the prime ideal $\mathfrak{p}$ above p that corresponds to the factor φ of f modulo p. This is a ring homomorphism. Furthermore, if the norm of z is coprime to p, then $\rho_f(z)$ is non-zero in $\mathbb{F}_{p^n}$. We can therefore extend ρ_f to the set of elements of K_f whose norm has a non-negative valuation at p.*

Since in this article we will often consider the discrete logarithm of the images by ρ_f, we restrict its definition to the elements of K_f whose norm is coprime to p, for which the image is non-zero.

Let h be the class number K_f that we assume to be coprime to the prime ℓ modulo which the logarithms are computed. We also need to consider the group of units U_f in $\mathcal{O}_f$. By Dirichlet's theorem it is a finitely generated abelian group of the form

$$U_f \sim U_{tors} \times \mathbb{Z}^r,$$

where r is the unit rank given by $r = r_1 + r_2 - 1$ where r_1 is the number of real roots of f and $2r_2$ the number of complex roots, and U_{tors} is cyclic. Any unit $\eta \in U_f$ can be written

$$\eta = \varepsilon_0^{u_0} \prod_{j=1}^{r} \varepsilon_j^{u_j}$$

for *fundamental units* ε_j, $j \geq 1$, and ε_0 a root of unity.

For each prime ideal $\mathfrak{q}$ in the factor base $\mathcal{F}_f$, the ideal $\mathfrak{q}^h$ is principal and therefore there exists a generator $\gamma_\mathfrak{q}$ for it. It is not at all unique, and the definition of the virtual logarithms will depend on the choice of the fundamental units and of the set of generators for all the ideals of $\mathcal{F}_f$. We denote by Γ this choice, and will use it as a subscript in our notations to remember the dependence in Γ. In particular, the notation $\log_\Gamma$ used just below means that the definition of the virtual logarithm depends on the choice of Γ, and does not mean that the logarithm is given in base Γ; in fact all along the article we do not make explicit the generator used as a basis for the logarithm in the finite field.

Definition 2.2 (Virtual logarithms – explicit version)**.** *Let $\mathfrak{q}$ be an ideal in the factor base $\mathcal{F}_f$, and $\gamma_\mathfrak{q}$ the generator for its h-th power, given by the choice Γ. Then the virtual logarithm of $\mathfrak{q}$ w.r.t. Γ is given by*

$$\log_\Gamma \mathfrak{q} \equiv h^{-1} \log(\rho_f(\gamma_\mathfrak{q})) \mod \ell,$$

where the log *notation on the right-hand side is the discrete logarithm function in $\mathbb{F}_{p^n}$.*

In the same manner, we define the virtual logarithms of the units by

$$\log_\Gamma \varepsilon_j \equiv h^{-1} \log(\rho_f(\varepsilon_j)) \mod \ell.$$

We now use this definition to show that for any polynomial ϕ yielding a relation, we can obtain a linear expression between the logarithm of $\rho_f(\phi(\alpha))$ in the finite field and the virtual logarithms of the ideals involved in the factorization of the ideal $\phi(\alpha)\mathcal{O}_f$:

$$\phi(\alpha)\mathcal{O}_f = \prod_{\mathfrak{q} \in \mathcal{F}_f} \mathfrak{q}^{\mathrm{val}_\mathfrak{q}(\phi(\alpha))}.$$

After raising this equation to the power h, we get an equation between principal ideals that can be rewritten as the following equation between field elements:

$$\phi(\alpha)^h = \varepsilon_0^{u_{\phi,0}} \prod_{j=1,r} \varepsilon_j^{u_{\phi,j}} \prod_{\mathfrak{q} \in \mathcal{F}_f} \gamma_\mathfrak{q}^{\mathrm{val}_\mathfrak{q}(\phi(\alpha))},$$

where the $u_{\phi,j}$ are integers used to express the unit that pops up in the process. We then apply the map ρ_f, and use the fact that it is an homomorphism. We obtain therefore

$$\rho_f(\phi)^h = \rho_f(\varepsilon_0)^{u_{\phi,0}} \prod_{j=1,r} \rho_f(\varepsilon_j)^{u_{\phi,j}} \prod_{\mathfrak{q} \in \mathcal{F}_f} \rho_f(\gamma_\mathfrak{q})^{\mathrm{val}_\mathfrak{q}(\phi(\alpha))},$$

from which we deduce our target equation by taking logarithms on both sides:

$$\log\left(\rho_f(\phi(\alpha))\right) \equiv \sum_{j=1}^{r} u_{\phi,j} \log_\Gamma \varepsilon_j + \sum_{\mathfrak{q} \in \mathcal{F}_f} \mathrm{val}_\mathfrak{q}\left(\phi(\alpha)\right) \log_\Gamma \mathfrak{q} \mod \ell. \tag{2}$$

In this last step, the contribution of the root of unity ε_0 has disappeared. Indeed, the following simple lemma states that its logarithm vanishes modulo ℓ.

Lemma 2.3. *Let ε_0 be a torsion unit of order r_0 and assume that $\gcd(hr_0, \ell) = 1$. Then we have $\log_\Gamma \varepsilon_0 \equiv 0 \mod \ell$.*

Proof. Since $\varepsilon_0^{r_0} = 1$ in K_f, we have $\rho_f(\varepsilon_0)^{r_0} = 1$ in $\mathbb{F}_{p^n}$ and we get $hr_0 \log_\Gamma \varepsilon_0 \equiv 0 \mod \ell$. □

In order to make the equation 2 explicit for a given ϕ that yields a relation, it is necessary to compute the class number h of K_f, to find the generators of all the $\mathfrak{q}^h$ and to compute a set of fundamental units. These are reknowned to be difficult problems except for polynomials f with tiny coefficients.

We now recall an alternate definition of virtual logarithms based on the so-called Schirokauer maps, for which none of the above have to be computed explicitly.

Definition 2.4 (Schirokauer maps)**.** *Let K_ℓ be the multiplicative subgroup of K_f^* of elements whose norms are coprime to ℓ.*

A Schirokauer map is an application $\Lambda : (K_\ell)/(K_\ell)^\ell \to (\mathbb{Z}/\ell\mathbb{Z})^r$ such that

- *$\Lambda(\gamma_1\gamma_2) = \Lambda(\gamma_1) + \Lambda(\gamma_2)$ (Λ is linear);*
- *$\Lambda(U_f)$ is surjective (Λ preserves the unit rank).*

Schirokauer [Sch93] proposed a fast-to-evaluate map satisfying these conditions that we recall now. Let us define first an integer, that is the LCM of the exponents required to apply Fermat's theorem in each residue field modulo ℓ:

$$\epsilon = \operatorname{lcm}\{\ell^\delta - 1, \text{ such that } f(x) \bmod \ell \text{ has an irreducible factor of degree } \delta\}.$$

Then, by construction, for any element γ in K_ℓ, we have γ^ϵ congruent to 1 in all the residue fields above ℓ. Therefore, the map

$$\gamma(\alpha) \mapsto \frac{\gamma(x)^\epsilon - 1}{\ell} \bmod (\ell, f(x)), \tag{3}$$

is well defined for $\gamma \in K_\ell$. Taking the coordinates of the image of this map in the basis $1, X, \ldots, X^{\deg f - 1}$, we can expect to find r independent linear combinations of these coordinates. They then form a Schirokauer map. In [Sch05], Schirokauer gave heuristic arguments for the existence of such independent linear combinations; and in practice, in most of the cases, taking the r first coordinates is enough.

From now on, we work with a fixed choice of Schirokauer map that we denote by Λ. We start by taking another set of r independent units: for each $j \in [1, r]$, we choose a unit ε_j such that

$$\Lambda(\varepsilon_j) = (0, \ldots, 0, h, 0, \ldots, 0),$$

where the coordinate h is in the j-th position. We can then refine the choice of the generators of the h-th power of the factor base ideals, so that we get another definition of the virtual logarithms.

Definition 2.5 (Virtual logarithms – Schirokauer's version)**.** *Let Λ be a Schirokauer map as described above. Let $\mathfrak{q}$ be an ideal in the factor base $\mathcal{F}_f$, and $\gamma_\mathfrak{q}$ an (implicit) generator for its h-th power, such that $\Lambda(\gamma_\mathfrak{q}) = 0$. Then the virtual logarithm of $\mathfrak{q}$ w.r.t. Λ is given by*

$$\log_\Lambda \mathfrak{q} \equiv h^{-1} \log(\rho_f(\gamma_\mathfrak{q})) \mod \ell.$$

The virtual logarithms of the units are defined in a similar manner:

$$\log_\Lambda \varepsilon_j \equiv h^{-1} \log(\rho_f(\varepsilon_j)) \mod \ell.$$

As shown in [Sch05], by an argument similar to the case of explicit generators, one can write

$$\log\left(\rho_f(\phi(\alpha))\right) \equiv \sum_{j=1}^{r} \lambda_j\left(\phi(\alpha)\right) \log_\Lambda \varepsilon_j + \sum_{\mathfrak{q} \in \mathcal{F}_f} \operatorname{val}_\mathfrak{q}\left(\phi(\alpha)\right) \log_\Lambda \mathfrak{q} \mod \ell, \tag{4}$$

where λ_j is the j-th coordinate of Λ.

2.3. **Explicit units or Schirokauer maps?** Equation (4) can be written for the two polynomials f and g and hence we obtain a linear equation relating only virtual logarithms. We remark that it is completely allowed to use the virtual logarithms in their explicit version for one of the polynomials if it is feasible, while using Schirokauer maps on the other side.

Using explicit units requires to compute a generator for each ideal in the factor base, and therefore the polynomial must have small coefficients (and small class number). A lot of techniques and algorithms are well described in [LL93]. These include generating units and generators in some box or ellipsoid of small lengths, and recovery of units using floating point computations. These are quite easy to implement and are fast in practice. We may do some

simplifications when K_f has non-trivial automorphisms, since in this case the generators of several ideals can be computed from one another using automorphisms (see Section 3).

In the general case, one uses Schirokauer maps whose coefficients are elements of $\mathbb{Z}/\ell\mathbb{Z}$ for a large prime ℓ. In our experiments, the values of the Schirokauer maps seem to spread in the full range $[0, \ell-1]$ and must be stored on $\log_2 \ell$ bits. In a recent record [BGI+14], each row of the matrix consisted in average of 100 non-zero entries in the interval $[-10, 10]$ and two values in $[0, \ell-1]$, for a prime ℓ of several machine words. It is then worth to make additional computations in order to reduce the number of Schirokauer maps. This motivated our study in Section 5.

3. Exploiting automorphisms

Using automorphisms of the fields involved in a discrete logarithm computation is far from being a new idea. It was already proposed by Joux, Lercier, Smart and Vercauteren [JLSV06] and was a key ingredient in many of the recent record computations in small characteristic [Jou13a, GKZ14a]. In this section we recall the basic idea and make explicit the interaction with both definitions of virtual logarithms, using or not Schirokauer maps.

3.1. **Writing Galois relations.** The results of this subsection apply potentially to both number fields K_f and K_g independently. Therefore, we will express all the statements with the notations corresponding to the polynomial f (that again, we assume to be monic for simplicity).

We assume that K_f has an automorphism σ, and we denote by A_σ and $A_{\sigma^{-1}}$ the polynomials of $\mathbb{Q}[x]$ such that $\sigma(\alpha) = A_\sigma(\alpha)$ and $\sigma^{-1}(\alpha) = A_{\sigma^{-1}}(\alpha)$. For any subset I of K_f, we denote by I^σ the set $\{\sigma(x) \mid x \in I\}$.

Proposition 3.1. *Let q be a rational prime not dividing the index $[\mathcal{O} : \mathbb{Z}[\alpha]]$ of the polynomial f. Then, any prime ideal above q of degree one can be generated by two elements of the form $I = \langle q, \alpha - r\rangle$ for some root r of f modulo q. If the denominators of the coefficients of A_σ and $A_{\sigma^{-1}}$ are not divisible by q, then we have*

$$I^\sigma = \langle q, \alpha - A_{\sigma^{-1}}(r)\rangle .$$

Proof. Since σ^{-1} is an automorphism, we have $f\left(A_{\sigma^{-1}}(\alpha)\right) = 0$. This is equivalent to $f(A_{\sigma^{-1}}(x)) \equiv 0 (\mathrm{mod} f(x))$ and then $f(A_{\sigma^{-1}}(x)) = u(x)f(x)$ for some polynomial $u \in \mathbb{Q}[x]$. By evaluating in r we obtain $f(A_{\sigma^{-1}}(r)) \equiv 0 \pmod q$. Then, by Dedekind's Theorem, $J = \langle q, \alpha - A_{\sigma^{-1}}(r)\rangle$ is a prime ideal of degree one.

Since q and $A_{\sigma^{-1}}(r)$ are rational, we have $J^{\sigma^{-1}} = \langle q, A_{\sigma^{-1}}(\alpha) - A_{\sigma^{-1}}(r)\rangle$. Since the polynomial $A_{\sigma^{-1}}(x) - A_{\sigma^{-1}}(r)$ is divisible by $x - r$, $J^{\sigma^{-1}}$ belongs to $\langle q, \alpha - r\rangle = I$. Therefore, J belongs to I^σ. But J is prime, so $J = I^\sigma$.

□

Before stating the main result on the action of σ on the virtual logarithms, we need the following result on the Schirokauer maps.

Lemma 3.2. *Let Λ be a Schirokauer map modulo ℓ associated to K_f and let σ be an automorphism of K_f. Assume in addition that this Schirokauer map is based on the construction of Equation 3. Then we have*

$$\ker \Lambda = \ker(\Lambda \circ \sigma).$$

Proof. Let $A_\sigma(x) \in \mathbb{Z}[x]$ be such that $A_\sigma(\alpha) = \sigma(\alpha)$. If $\gamma = P(\alpha)$ is in the kernel of Λ, then there exist $u, v \in \mathbb{Z}[x]$ such that

$$P(x)^\epsilon - 1 = \ell^2 u(x) + \ell v(x) f(x). \tag{5}$$

By substituting $A_\sigma(x)$ to x, we obtain

$$P(A_\sigma(x))^\epsilon - 1 = \ell^2 u(A_\sigma(x)) + \ell v(A_\sigma(x)) f(A_\sigma(x)). \tag{6}$$

Since σ is an automorphism of f, $f(A_\sigma(x))$ is a multiple of $f(x)$. Hence, we obtain that $\sigma(\gamma) = P(A_\sigma(\alpha))$ is in the kernel of Λ. □

Example 3.3. When f is an even polynomial, i.e. $f(-x) = f(x)$, the application $\sigma(x) = -x$ is an automorphism of the number field $K_f = \mathbb{Q}[x]/f(x)$. Consider the Schirokauer map as defined in Equation 3. We denote by $\Lambda = (\lambda_1, \ldots, \lambda_r)$ the r first coordinates in basis $1, X, \ldots, X^{\deg f - 1}$, and we assume that they are independent, so that Λ is indeed a Schirokauer map. Then applying the automorphism, we get $\Lambda \circ \sigma = (\lambda_1, -\lambda_2, \lambda_3, -\lambda_4, \ldots, (-1)^{r+1}\lambda_r)$, and we can check that its kernel coincides with the kernel of Λ.

The following counter-example shows that the condition that Λ is constructed from Equation 3 is necessary for Lemma 3.2 to hold.

Example 3.4. Let $\Lambda = (\lambda_1, \ldots, \lambda_r)$ be a Schirokauer map of K_f with respect to ℓ, σ an automorphism of K_f and $\mathfrak{q}$ a prime ideal. Then $\Lambda' = (\lambda_1 + \mathrm{val}_\mathfrak{q}(\cdot), \lambda_2, \lambda_3, \ldots, \lambda_r)$ does not satisfy $\ker \Lambda' = \ker \Lambda' \circ \sigma$. Indeed, let γ be a generator of $(\mathfrak{q}^{\sigma^{-1}})^h$ with $\Lambda(\gamma) = 0$. On the one hand we have $\Lambda'(\gamma) = 0$. On the other hand, the first coordinate of $\Lambda'(\sigma(\gamma))$ is the valuation in $\mathfrak{q}$ of $\sigma(\gamma)$, which is non zero because $\sigma(\gamma)$ is in $\mathfrak{q}$.

Theorem 3.5 (Galois relations)**.** *We keep the same notations as above, where in particular φ is a degree-n irreducible factor of f modulo p. Let σ be an automorphism of K_f different from the identity such that*

$$\varphi(\rho_f(A_\sigma(\alpha))) = 0.$$

Then, there exists a constant $\kappa \in [1, \mathrm{ord}(\sigma) - 1]$ such that the following holds:

(1) *Let Γ be a choice of explicit generators that is compatible with σ, i.e. such that for any prime ideal $\mathfrak{q}$ the generators for the h-th powers of $\mathfrak{q}$ and $\sigma(\mathfrak{q})$ are conjugates:*

$$\gamma_{\sigma(\mathfrak{q})} = \sigma(\gamma_\mathfrak{q}).$$

Then we have for any prime ideal $\mathfrak{q}$:

$$\log_\Gamma \mathfrak{q}^\sigma \equiv p^\kappa \log_\Gamma \mathfrak{q} \pmod{\ell}.$$

(2) *For any Schirokauer map Λ which has a polynomial formula (as in Lemma 3.2) and for any prime ideal $\mathfrak{q}$, we have*

$$\log_\Lambda \mathfrak{q}^\sigma \equiv p^\kappa \log_\Lambda \mathfrak{q} \pmod{\ell}.$$

Proof. Since $\rho_f(\sigma(\alpha))$ is a root of φ other than $m = \rho_f(\alpha)$, the map $T(x) \mapsto T(A_\sigma(x))$ is an element of $\mathrm{Gal}(\mathbb{F}_{p^n}/\mathbb{F}_p)$ other than the identity. So, there exists a constant $\kappa \in [1, \mathrm{ord}(\sigma) - 1]$ such that $A_\sigma(x) = x^{p^\kappa}$ for all $x \in \mathbb{F}_{p^n}$. In particular, if $\mathfrak{q}$ is a prime ideal and $\gamma_\mathfrak{q}$ is a generator of $\mathfrak{q}^h$, we have

$$\log \rho_f(\sigma(\gamma_\mathfrak{q})) = p^\kappa \log(\rho_f(\gamma_\mathfrak{q})). \tag{7}$$

In the first assertion of the theorem, it is assumed that $\sigma(\gamma_\mathfrak{q})$ is precisely the generator used for $\sigma(\mathfrak{q})^h$, and therefore the relation between virtual logarithms follows from their definition.

For the second assertion, the compatibility of the generators is deduced from the definition of the virtual logarithms using Schirokauer maps. Indeed, for any prime ideal $\mathfrak{q}$, the generator used for the definition of $\log_\Lambda \mathfrak{q}$ is such that $\Lambda(\gamma_\mathfrak{q}) = 0$. By Lemma 3.2, $\gamma_\mathfrak{q}$ is also in the kernel of $\Lambda \circ \sigma$, that is $\Lambda(\sigma(\gamma_\mathfrak{q})) = 0$, so that the conjugate of the generator is a valid generator for the conjugate of the ideal. The conclusion follows. □

We give an immediate application of the preceding results, which is useful when K_f is an imaginary quadratic field.

Lemma 3.6. *Let q be a rational prime which is totally ramified in K_f, and write $q\,\mathcal{O}_f = \mathfrak{q}^n$. Assume that the unit rank of K_f is 0 and that n is coprime to ℓ. Then we have*

$$\log \mathfrak{q} \equiv 0 \bmod \ell.$$

Proof. Let h be the class number of K_f and $\gamma_{\mathfrak{q}}$ a generator of $\mathfrak{q}^h$ such that $\log \mathfrak{q} = h^{-1} \log \gamma_{\mathfrak{q}}$. Then one can write $q^h = u(\gamma_{\mathfrak{q}})^n$ for some root of unity u. By Lemma 2.3, $\log u \equiv 0 \mod \ell$, so

$$\log(q^h) \equiv \log((\gamma_{\mathfrak{q}})^n) \bmod \ell.$$

Since q belongs to the subgroup of $\mathbb{F}_{q^n}$ given by equation $x^{q-1} = 1$ and since $\gcd(q-1, \ell) = 1$, Lemma 2.3 gives $\log q = 0$. Then the results follows from the fact that n is coprime to ℓ. □

3.2. **Using Galois relations in NFS.** Let σ and τ be automorphisms of K_f and K_g, and let us assume that they verify the hypothesis of Theorem 3.5. We can split $\mathcal{F}_f$ and $\mathcal{F}_g$ respectively in orbits $(\mathfrak{q}, \mathfrak{q}^\sigma, \ldots)$ if $\mathfrak{q}$ is in $\mathcal{F}_f$ and $(\mathfrak{q}, \mathfrak{q}^\tau, \ldots)$ if $\mathfrak{q}$ is in $\mathcal{F}_g$.

This allows to reduce the number of unknowns in the linear algebra stage by a factor $\mathrm{ord}(\sigma)$ on the f-side and by a factor $\mathrm{ord}(\tau)$ on the g-side, at the price of having entries in the matrix that are roots of unity modulo ℓ instead of small integers. We collect as many relations as unknowns, hence reducing also the cost of the sieve.

Note that the case where σ or τ is the identity is not excluded in our discussion (in that case, the orbits are singletons on the corresponding side).

As an example, in Section 6 we will see how to construct polynomials f and g whose number fields have automorphisms σ and τ, both of order n. Then, the number of unknowns is reduced by n and the number of necessary relations is divided by n. Since the cost of the linear algebra stage is λN^2, where N is the size of the matrix and λ is its average weight per row, i.e. the number of non-zero entries per row, we obtain the following result.

Fact 3.7. *If f and g are two polynomials with automorphisms σ and τ of order n verifying the hypothesis of Theorem 3.5, then we have:*

- *a speed-up by a factor n in the sieve;*
- *a speed-up by a factor n^2 in the linear algebra stage.*

The particular case when $A_\sigma = A_\tau$. In Section 6.3, we will present a method to select polynomials f and g with automorphisms σ and τ; it that σ and τ are expressed by the same rational fraction $A_\sigma = A_\tau$. Moreover, the numerator and denominator are constant or linear polynomials. A typical example is when both polynomials are reciprocal and then $\sigma(\alpha) = 1/\alpha$ and $\tau(\beta) = 1/\beta$.

Let $\phi \in \mathbb{Z}[x]$ be a polynomial yielding a relation. When we apply σ and τ to the corresponding system of equations (1), we get:

$$\begin{cases} \phi(A_\sigma(\alpha))\mathcal{O}_f = \prod_{\mathfrak{q} \in \mathcal{F}_f} (\mathfrak{q}^\sigma)^{\mathrm{val}_{\mathfrak{q}}(\phi(\alpha))} \\ \phi(A_\tau(\tau))\mathcal{O}_g = \prod_{\mathfrak{r} \in \mathcal{F}_g} (\mathfrak{r}^\tau)^{\mathrm{val}_{\mathfrak{r}}(\phi(\beta))}, \end{cases} \tag{8}$$

Since $A_\sigma = A_\tau$ have a simple form, there is a chance that $\phi \circ A_\sigma$ has a numerator that is again a polynomial of the form that would be tested later. The relations being conjugates of each others, the second one brings no new information and should not be sieved.

Again, we illustrate this on the example of reciprocal polynomials, where $A_\sigma(x) = A_\tau(x) = 1/x$. For polynomials $\phi(x) = a - bx$ of degree 1, the numerator of $\phi \circ A_\sigma$ is $b - ax$. Therefore, it is interesting not to test the pair (b, a) for smoothness if the pair (a, b) has already been tested.

If the sieve is implemented using the lattice sieve, e.g. in CADO-NFS [BFG+09], one can collect precisely these polynomials ϕ such that $\phi(\alpha)$ is divisible by one of the ideals $\mathfrak{q}$ in a list given by the user. In this case, we make a list of ideals $\mathfrak{q}$ which contains exactly one ideal in each orbit $\{\mathfrak{q}, \sigma(\mathfrak{q}), \ldots, \sigma^{n-1}(\mathfrak{q})\}$. Hence, we do not collect at the same time ϕ and the numerator of $\phi \circ A_\sigma$ except if the decomposition of $\phi(\alpha)$ in ideals contains two ideals $\mathfrak{q}$ and $\mathfrak{q}'$ which are in our list of ideals or conjugated to such an ideal.

4. Vanishing of the logarithms of units

In this section, we are again in the case where we study the fields K_f and K_g independently. Therefore, we stick to the notations for the f-side, but we keep in mind that this could be applied to g. Furthermore, for easier reading, for this section we drop the subscript f, for structures related to f: $K = \mathbb{Q}(\alpha)$ is the number field of f, U the unit group whose rank is denoted by r, and ρ is the reduction map to $\mathbb{F}_{p^n}$.

Also, some of the results of this section depend on the fact that ℓ is a factor of $p^n - 1$ that is in the "new" part of the multiplicative group: we will therefore always assume that ℓ is a prime factor of $\Phi_n(p)$. The aim of this section is to give cases where the logarithms of some or all fundamental units are zero, more precisely units u for which $\log \rho(u) \equiv 0 \bmod \ell$.

4.1. **Units in subfields.** The main case where we can observe units with zero virtual logarithms is when the subfield fixed by an automorphism as in Section 3 has some units.

Theorem 4.1. *With the same notations as above, assume that $v_1, \ldots, v_r$ are units of K which form a basis modulo ℓ. Let σ be an automorphism of K and assume that there exists an integer A such that $A \not\equiv 1 \mod \ell$ and, for all $x \in K$ of norm coprime to p,*

$$\log \rho(\sigma(x)) \equiv A \log \rho(x) \mod \ell. \tag{9}$$

Let $K^{\langle\sigma\rangle}$ be the subfield fixed by σ and let r' be its unit rank. Let $u'_1, \ldots, u'_{r'}$ be a set of units of $K^{\langle\sigma\rangle}$ which form a basis modulo ℓ. Then, K admits a basis $u_1, \ldots, u_r$ modulo ℓ such that the discrete logarithms of $\rho(u_1), \ldots, \rho(u_{r'})$ are zero modulo ℓ.

Proof. For any $x \in K^{\langle\sigma\rangle}$ we have $\sigma(x) = x$, so, when ρ is defined, we have $\log(\rho(\sigma(x))) \equiv \log(\rho(x)) \mod \ell$. Using Equation (9) we obtain that $\log(\rho(x)) \equiv 0 \mod \ell$ for all x in $K^{\langle\sigma\rangle}$ of norm coprime to p. In particular, for $1 \le i \le r'$, we have $\log(\rho(u'_i)) \equiv 0 \mod \ell$.

One checks that $u'_1, \ldots, u'_{r'}$ are units in K. Since they form a basis modulo ℓ, there is no non-trivial product of powers of $u'_1, \ldots, u'_{r'}$ which is equal to an ℓth power. Then, one can select $r - r'$ units among $v_1, \ldots, v_r$ to extend $u'_1, \ldots, u'_{r'}$ to a basis modulo ℓ. □

Example 4.2. Consider the family of CM polynomials

$$\begin{aligned} &f = x^4 + bx^3 + ax^2 + bx + 1, \\ &|a| < 2, \qquad |b| < 2 + a/2. \end{aligned} \tag{10}$$

There is always the automorphism, $\forall T \in \mathbb{Z}[x], \sigma(T(x)) = T(1/x)$ of order 2, so that we have $A = p \equiv -1 \mod \ell$ for use in the Theorem. We claim that $r = r' = 1$. Let us call α a complex root of f. Since $\beta = \alpha + 1/\alpha$ is not rational and fixed by σ, we have $K^{\langle\sigma\rangle} = \mathbb{Q}(\alpha + 1/\alpha)$. Since β is a root of the equation $P(Y) = Y^2 + bY + (a-2) = 0$, whose discriminant $b^2 + 4(2-a)$ is positive, $K^{\langle\sigma\rangle}$ is real and we have $r' = 1$. The roots of f are roots of $x + 1/x = y_1$ or y_2 for $y_1 = -b/2 - \sqrt{b^2/4 + (2-a)}$ and $y_2 = -b/2 + \sqrt{b^2/4 + (2-a)}$. Since $|b| < 2 + a/2$, f has no real roots, so $r = 1$.

A second proof is as follows. Note that $f(X)$ factors over $\mathbb{Q}(\beta)$ as

$$(X^2 - \beta X + 1)(X^2 + (b+\beta)X + 1).$$

We put $\varphi(X) = X^2 - \beta X + 1$. Let p be a prime for which $P(Y)$ is reducible modulo p and φ is not. The following picture shows the characteristic 0 picture, as well as the one modulo p.

$$\begin{array}{ccccc} K = \mathbb{Q}(\alpha) = \mathbb{Q}[X]/(f(X)) & & & & \\ \Big| & \searrow & & & \\ K^{\langle\sigma\rangle} = \mathbb{Q}(\beta) = \mathbb{Q}[Y]/(P(Y)) & & \mathbb{F}_{p^2} = \mathbb{F}_p[X]/(\overline{\varphi}(X)) & & \\ \Big| & \searrow & \Big| & & \\ \mathbb{Q} & & \mathbb{F}_p & & \end{array}$$

Let $\ell \mid p + 1$. If ε_1 is the fundamental unit of $K^{\langle\sigma\rangle}$ (and also of K by construction), we have $\log \rho(\varepsilon_1) \equiv 0 \bmod \ell$.

4.2. **Extra vanishing due to $\mathbb{F}_\ell$-linear action.** In the previous section, we have just seen that with a careful choice of the basis of units, some of the basis elements can have a zero virtual logarithm. In general, there could be another choice for the basis that give more zero logarithms. We call $\mathcal{R}_{\text{opt}}$ the maximum number of units of K in a basis modulo ℓ that can have zero logarithm. With this notation, the result of Theorem 4.1 becomes $\mathcal{R}_{\text{opt}} \geq r'$.

The aim of this section it to prove a better lower bound for $\mathcal{R}_{\text{opt}}$. By studying the $\mathbb{F}_\ell$-linear action of σ on the units, we will be able to choose a basis for which the number $\mathcal{R}$ of independent units with zero logarithm is (often) larger than r'. Therefore, the notation $\mathcal{R}$ in this section is a lower bound on the maximal number of units of K in a basis modulo ℓ that can have zero logarithm; and we always have $\mathcal{R}_{\text{opt}} \geq \mathcal{R}$.

For the unit group U of K, consider the vector space U/U^ℓ over $\mathbb{F}_\ell$. We assume that ℓ is large enough so that K has no roots of unity of order ℓ; therefore the dimension of U/U^ℓ is equal to r.

We denote by $\overline{\sigma}$ the vector space homomorphism $U/U^\ell \to U/U^\ell$, $\overline{\sigma}(uU^\ell) = \sigma(u)U^\ell$. For simplicity, in the sequel, we drop the bar above $\overline{\sigma}$. Let $\mu_{\ell,\sigma}(x)$ be the minimal polynomial of σ; it is a divisor of $x^n - 1$, since σ has order n. Note however that, $\overline{\sigma}$ can have a smaller order than σ, as seen by Example 4.2 where σ has order two but its restriction to the unit group is the identity.

Since ℓ is a divisor of $\Phi_n(p)$, $\Phi_n(x)$ splits completely in $\mathbb{F}_\ell$. Then, $x^n - 1$ and $\mu_{\ell,\sigma}$ split completely in $\mathbb{F}_\ell$:

$$\mu_{\ell,\sigma}(x) = \prod_{i=1}^{\deg \mu_{\ell,\sigma}} (x - c_i), \tag{11}$$

where c_i are distinct elements of $\mathbb{F}_\ell$. We remark at this point that as an endomorphism of $\mathbb{F}_\ell$-vector spaces, σ is diagonalizable. For any eigenvalue $c \in \mathbb{F}_\ell$ of σ, we denote by E_c the eigenspace of c:

$$E_c = \left\{ u \in U \mid \exists v \in U, \sigma(u) = u^c v^\ell \right\}, \tag{12}$$

and since the endomorphism is diagonalizable, the whole vector space can be written as a direct sum of eigenspaces:

$$U/U^\ell = \prod_{i=1}^{\deg \mu_{\ell,\sigma}} E_{c_i}. \tag{13}$$

The case covered by Theorem 4.1 corresponds to the units that are fixed by σ, i.e. the units in the eigenspace E_1. The following lemma generalizes the result to other eigenvalues.

Lemma 4.3. *If $c \in \mathbb{F}_\ell$ is an eigenvalue distinct from A (as defined in (9)), then, for all units u such that the class of u in U/U^ℓ belongs to E_c, we have* $\log(\rho(u)) \equiv 0 \mod \ell$.

Proof. For such a unit u, we have

$$\log(\rho(\sigma(u))) \equiv c \log(\rho(u)) \mod \ell.$$

By assumption on A, we have

$$\log(\rho(\sigma(u))) \equiv A \log(\rho(u)) \mod \ell.$$

We conclude that the logarithm of $\rho(u)$ is zero. □

Corollary 4.4. *Using the notations above, we have $\mathcal{R} = r - \dim E_A$, where A is as in (9).*

At this stage, we get an expression that gives the units that have to be considered during the NFS algorithm. But this expression depends on ℓ, whereas in many cases it will be inherited from global notions and will be the same for any ℓ dividing $\Phi_n(p)$. Therefore we consider the linear action of σ on the group of non-torsion units.

Let U_{tor} be the torsion subgroup of U and ε_0 a generator of U_{tor}. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_r$ be a system of fundamental units. Let M_σ be the matrix of the endomorphism $\overline{\sigma}$ on U/U_{tor}, in basis $\varepsilon_1 U_{\text{tor}}, \ldots, \varepsilon_r U_{\text{tor}}$. Then M_σ belongs to $\mathrm{GL}_r(\mathbb{Z})$. Since M_σ cancels the monic polynomial $x^n - 1$, M_σ admits a minimal polynomial $\mu_{\mathbb{Z},\sigma}$ with integer coefficients. Note that $\mu_{\mathbb{Z},\sigma}$ does not depend on the system of fundamental units used. The following lemma shows that finding roots modulo ℓ of $\mu_{\mathbb{Z},\sigma}$ gives local information about the vanishing of the logarithms modulo ℓ.

Lemma 4.5. *For any root $c \in \mathbb{F}_\ell$ of $\mu_{\mathbb{Z},\sigma}(x)$ the dimension of the eigenspace $\dim(E_c)$ in U/U^ℓ is at least 1.*

Proof. Since M_σ has integer coefficients, its characteristic polynomial χ_{M_σ} is monic with integer coefficients. We deduce that $\mu_{\mathbb{Z},\sigma}$ is monic with integer coefficients. We claim that, for all primes ℓ,

$$\mu_{\mathbb{Z},\sigma} = \mu_{\ell,\sigma}. \tag{14}$$

On the one hand, $\mu_{\mathbb{Z},\sigma}$ has the same irreducible factors over $\mathbb{Q}$ as the characteristic polynomial χ_{M_σ} of M_σ. Since σ cancels $x^n - 1$, they occur with multiplicity one in $\mu_{\mathbb{Z},\sigma}$. Hence, $\mu_{\mathbb{Z},\sigma}$ is the product of irreducible factors of χ_{M_σ}, taken with multiplicity one.

On the other hand, $\mu_{\ell,\sigma}$ has the same irreducible factors as the characteristic polynomial of $\overline{\sigma}$, which is the reduction of χ_{M_σ} modulo ℓ. Since, $\overline{\sigma}$ cancels $x^n - 1$, $\mu_{\ell,\sigma}$ is product of the irreducible factors of $\chi_{\mathbb{Z},\sigma}$ modulo ℓ, taken with multiplicity one. We obtain equation (14).

Finally, it is a classic property of minimal polynomial that all its roots have nonzero eigenspaces. □

We already mentioned the link between the eigenspace of 1 and Theorem 4.1. We now make this more precise:

Lemma 4.6. *Using the previous notations, we have $\dim E_1 = r'$, except for a finite set of primes ℓ.*

Proof. Consider a system of fundamental units. By Theorem 4.1 there exists a basis (u_i), $1 \leq i \leq r$, of U/U_{tors} such that the first r' elements are fixed by σ and, no unit in the subgroup V generated by u_i, $r' + 1 \leq i \leq r$, is fixed by σ. After block-diagonalization, we can assume that V/V_{tors} is stable by σ and we let M_σ be the matrix of σ on V/V_{tors}. The determinant of $(M_\sigma - \mathrm{id})$ is an integer D. If ℓ is prime to D, the discriminant of σ on V/V^ℓ is non-zero. Hence, $\dim E_1 \leq r'$, which completes the proof. □

As a first application, we study the case of cyclic extensions of prime degree.

Proposition 4.7. *Let n be an odd prime and $K/\mathbb{Q}$ a cyclic Galois extension of degree n. Let p and ℓ be two primes such that $\Phi_n(p)$ is divisible by ℓ. Let ρ be a ring morphism which sends any element x of K with $\nu_p(x) \geq 0$ into the field $\mathbb{F}_{p^n}$. Let σ be an automorphism of K of order n for which there exists a constant κ such that, for all $x \in K$ of positive p-valuation, $\rho(\sigma(x)) = p^\kappa \rho(x)$. Then we have $\mathcal{R} = n - 2$.*

Proof. We want to compute $\mu_{\mathbb{Z},\sigma}$. Since σ has order n, $\mu_{\mathbb{Z},\sigma}$ is a divisor of $(x^n - 1) = (x-1)\Phi_n(x)$. By Lemma 4.6, $\dim \ker(\sigma - \mathrm{id}) = r'$, the unit rank of the subfield fixed by σ. In our case, this subfield is $\mathbb{Q}$, so $r' = 0$. Then, we have $\mu_{\mathbb{Z},\sigma} = \Phi_n(x)$.

Let f be a defining polynomial of K. Since f has odd degree, it has at least a real root α. Since K is Galois, $K = \mathbb{Q}(\alpha)$, so all roots of K are real, hence its unit rank is $n - 1$. By Lemma 4.5, since $\deg(\Phi_n) = n - 1 = \dim U/U^\ell$, all the eigenspaces of roots of Φ_n have dimension one. Using Corollary 4.4, we have $\mathcal{R} = n - 2$.

□

4.3. **Fields of small degree.** We are now in position to list possible cases for fields of small degree. As a warm-up, we start with the case of degree 2. The imaginary case is of course trivial, since the unit rank is 0. For the real quadratic case, the unit rank r is 1, and one could

$\deg(K)$	$\mathrm{ord}(\sigma)$	$\mathrm{sign}(K)$, $\mathrm{sign}(K^{\langle\sigma\rangle})$	$\mu_{\mathbb{Z},\sigma}$	$\mathcal{R}$	r	$r-\mathcal{R}$
4	2	(4,0), (2,0)	$(x-1)(x+1)$	1	3	2
		(2,1), (2,0)	$(x-1)(x+1)$	1	2	1
		(0,2), (0,1)	$x+1$	0	1	1
		(0,2), (2,0)	$x-1$	1	1	0
	4	(4,0), -	$(x+1)(x^2+1)$	2	3	1
		(0,2), -	$x+1$	1	1	0
6	2	(0,3), (1,1)	$(x-1)(x+1)$	1	2	1
		(0,3), (3,0)	$x-1$	2	2	0
	3	(6,0), (2,0)	$(x-1)(x^2+x+1)$	3	5	2
		(0,3), (0,1)	x^2+x+1	1	2	1
	6	(6,0), -	$(x+1)(x^2+x+1)(x^2-x+1)$	4	5	1
		(0,3), -	x^2+x+1	1	2	1

TABLE 1. Table of values of $\mathcal{R}$ for fields of degree 4 and 6.

wonder whether there are cases when we can tell in advance that the virtual logarithm of the unit is 0 modulo ℓ with our method.

In fact, this does not occur. Indeed, the automorphism σ is of order 2 and therefore the unit rank r' of the subfield is 0. By Lemma 4.6 the dimension of the eigenspace E_1 is therefore 0 as well. This is no surprise: the fundamental unit is not defined over $\mathbb{Q}$, so it is not fixed by σ. The next step is to study $\mu_{\mathbb{Z},\sigma}$. Since σ as order 2, $\mu_{\mathbb{Z},\sigma}$ divides x^2-1, and we have just seen that 1 is not an eigenvalue. Therefore we deduce that $\mu_{\mathbb{Z},\sigma} = x+1$. Hence the vector space U/U^ℓ is reduced to the eigenspace E_{-1}. Since -1 is precisely the value A as in (9), we can not conclude.

The cases of degree 3 and 5 are covered by Proposition 4.7. In Table 1, we list the cases for degree 4 and 6. In all cases, a classification according to the signatures of the field and of the fixed subfield is enough to conclude about the value of $\mathcal{R}$.

Theorem 4.8. *The values of $\mathcal{R}$ for $K/\mathbb{Q}$ of degree 4 or 6 having non-trivial automorphisms are as given in Table 1.*

Proof. Let us consider the various cases of Tab. 1. In each case, we use a strategy of proof that is not so different from the real quadratic case that we mentioned in introduction. In order to determine the minimal polynomial $\mu_{\mathbb{Z},\sigma}$, we consider the factors of x^n-1 in $\mathbb{Z}[x]$ and we use the fact that $\deg \mu_{\ell,\sigma}$ is at most $\dim(U/U^\ell) = r$.

Case $\deg(K) = 4$ and $\mathrm{ord}(\sigma) = 2$

- Case when $\mathrm{sign}(K)$=(4,0) and $\mathrm{sign}(K^{\langle\sigma\rangle})$=(2,0). Then, $r=3$ and $r'=1$. Further, $x-1$ divides $\mu_{\mathbb{Z},\sigma}$ with multiplicity one. Since σ cancels x^2-1, the minimal polynomial is $\mu_{\mathbb{Z},\sigma} = x^2-1$. Hence, we have $\dim E_{-1} = 2$. Since $A=-1$, we obtain $\mathcal{R} = r' = 1$.
- Case when $\mathrm{sign}(K)$=(2,1) and $\mathrm{sign}(K^{\langle\sigma\rangle})$=(2,0). Note first that (2,0) is the unique possibility for $\mathrm{sign}(K^{\langle\sigma\rangle})$. Indeed, if α is a real root of a defining polynomial of K, then $\mathbb{Q}(\alpha+\sigma(\alpha))$ is fixed by σ and has degree two, so it is $K^{\langle\sigma\rangle}$. Since this quadratic field is real, its signature is (2,0). As above we have that $\dim E_1 \geq 1$ and therefore $\mu_{\mathbb{Z},\sigma} = (x-1)(x+1)$, implying that $\mathcal{R}=1$.
- Case when $\mathrm{sign}(K)$=(0,2) and $\mathrm{sign}(K^{\langle\sigma\rangle})$=(0,1). Here $r=1$ and $r'=0$. For sufficiently large values of ℓ, by Lemma 4.6, the space E_1 of units fixed by σ has dimension $r'=0$. Since the minimal polynomial divides x^2-1, we have $\mu_{\mathbb{Z},\sigma} = x+1$. We deduce that $\mathcal{R}=0$.
- Case when $\mathrm{sign}(K)$=(0,2) and $\mathrm{sign}(K^{\langle\sigma\rangle})$=(2,0). Here we have $r=r'=1$. Then any unit is fixed by σ and $E_1 = U/U^\ell$, so $\mu_{\mathbb{Z},\sigma} = x-1$ and $\mathcal{R}=1$. We remark that the fields of Example 4.2 falls in this category.

Case $\deg(K) = 4$ and $\mathrm{ord}(\sigma) = 4$ Note that K is either totally real or its defining polynomial has no real root. Hence we have two cases:

- Case when sign(K)=(4,0). Here we can apply to $\tau = \sigma^2$ the results on the case of degree four polynomials with automorphisms of order two. Hence we have $\dim \ker(\sigma^2 - 1) = \dim \ker(\tau - 1) = 1$ and $\dim(\sigma^2 + 1) = \dim(\tau + 1) = 2$. The fixed field has degree 1, so $r' = 0$. Then, the minimal polynomial is $\mu_{\mathbb{Z},\sigma} = (x + 1)(x^2 + 1)$, and we have $\mathcal{R} = 2$.
- Case when sign(K)=(0,2). Here the unit rank of K is $r = 1$, so the minimal polynomial is linear. Since, $r' = 0$, we have $\dim E_1 = 0$, so $\mu_{\mathbb{Z},\sigma} = x + 1$. Since A is of order 4 it is not -1; hence, we obtain $\mathcal{R} = 0$. Note that here the group automorphism $\overline{\sigma}$ equals -1, so it has a smaller order that the field automorphism σ.

Case $\deg(K) = 6$ and $\mathrm{ord}(\sigma) = 2$
Here the signature of K can be $(6, 0)$, $(4, 1)$, $(2, 2)$ and $(0, 3)$. We only deal with the case $(0, 3)$ in the present version of our work.

The unit rank of K is $r = 2$ and the minimal polynomial is a factor of $x^2 - 1$. The value of $\mathcal{R}$ is determined by the signature of the subfield fixed by σ, which is cubic and can have signature (3,0) or (1,1).

- Case when sign($K^{\langle\sigma\rangle}$)=(1,1). The unit rank of the fixed subfield is $r' = 1$ and $\mathcal{R} = 1$.
- Case when sign($K^{\langle\sigma\rangle}$)=(3,0). Here we have $r' = 2$, so $\dim E_1 = r$ and $\mu_{\mathbb{Z},\sigma} = x - 1$. This shows that $\mathcal{R} = 2$.

Case $\deg(K) = 6$ and $\mathrm{ord}(\sigma) = 3$
Note that, the signature $(r_{\mathbb{R}},r_{\mathbb{C}})$ of K satisfies $r_{\mathbb{R}} \equiv 0 \mod 3$. Indeed, if a defining polynomial of K has a real root α, the roots $\sigma(\alpha)$ and $\sigma^2(\alpha)$ are also real. The two values for the signature are (6,0) and (0,3).

- Case when sign(K)=(6,0). Since K is real, $K^{\langle\sigma\rangle}$ is also real, so $r' = 1$. As in the previous cases, the polynomial $\mu_{\mathbb{Z},\sigma}$ is a factor of $(x - 1)(x^2 + x + 1)$. Since, $\dim E_1 = r' = 1$ is neither 0 nor r, we have $\mu_{\mathbb{Z},\sigma} = (x-1)(x^2+x+1)$. Since, the characteristic polynomial over $\mathbb{Q}$, of σ restricted to $V = \ker(\sigma^2 + \sigma + 1)$ has the same irreducible factors, we have $\chi_\sigma|_V = (x^2 + x + 1)^2$. Suppose ab absurdo that, for a root c of $\mu_{\mathbb{Z},\sigma}$ modulo ℓ, we have $\dim E_c \geq 3$. Then χ_σ modulo ℓ is divisible by $(x - c)^3$. It is impossible because it has two roots of multiplicity at least two, so, for the two roots of $x^2 + x + 1$ modulo ℓ we have $\dim E_c = 2$. It implies that $\mathcal{R} = 3$.
- Case when sign(K)=(0,3). The unit rank of K is $r = 2$, so the minimal polynomial is $x - 1$ or $x^2 + x + 1$. The fixed subgroup has degree 2, so we cannot have $r' = 2$. This shows that $\mu_{\mathbb{Z},\sigma} = x^2 + x + 1$. Then, $r' = 0$ and $\mathcal{R} = 1$.

Case $\deg(K) = 6$ and $\mathrm{ord}(\sigma) = 6$
As in the case of cyclic quartic Galois extensions, either K is real or has no real roots.

- Case when sign(K)=(6,0). The unit rank of K is 5, so the minimal polynomial is equal to a factor of $(x - 1)(x + 1)(x^2 + x + 1)(x^2 - x + 1)$, having degree less than or equal to 5. Since the fixed subgroup is $\mathbb{Q}$, we have $r' = 0$, so $\dim E_1 = 0$. The fixed subfield of σ^3 is a cubic cyclic Galois extension, so its unit rank is two. Hence $\dim \ker(\sigma^3 - 1) = 2$. Since $\dim E_1 = 0$, $x^2 + x + 1$ divides $\mu_{\mathbb{Z},\sigma}$. We also deduce that $\dim \ker(\sigma^3 + 1) = 3$. The subfield fixed by σ^2 has degree two, so its unit rank is at most one. It implies that $\dim(\ker(\sigma + 1)) \neq 3$, so $x^2 - x + 1$ divides $\mu_{\mathbb{Z},\sigma}$. Since the dimension of its kernel is even, $\dim E_{-1} \neq 0$, so $(x + 1)$ divides $\mu_{\mathbb{Z},\sigma}$. We conclude that $\mu_{\mathbb{Z},\sigma} = (x + 1)(x^2 + x + 1)(x^2 - x + 1)$ and $\mathcal{R} = 4$.
- Case when sign(K)=(0,3). The subfield fixed by σ^3 is a cyclic cubic extension of $\mathbb{Q}$, so its unit rank is 2. This means that $\dim \ker(\sigma^3 - 1) = 2$, so the minimal polynomial divides $x^3 - 1$. The fixed subgroup of σ is $\mathbb{Q}$, so $\dim E_1 = 0$ and $\mu_{\mathbb{Z},\sigma} = x^2 + x + 1$. We obtain that $\mathcal{R} = 1$.

□

4.4. **Effective computations.** Theorem 4.8 tells us that we do not need to consider the logarithms of all units in many cases. If we have a system of units which form a basis modulo ℓ, we can make the theorem effective by solving linear systems. This is a less stronger condition than computing a system of fundamental units. One can investigate the use of Schirokauer maps to avoid any requirement of effective computations of units. Let us see a series of examples which illustrate Theorem 4.8.

4.4.1. *Minkowski units.* A *Minkowski unit* for K, if it exists, is a unit ε such that a subset of the conjugates of ε forms a system of fundamental units. Some results on the classification of such fields exist, we will come back to them in the final version of this work.

As an example, when K is totally cyclic of degree 3, it is real and there exists always a Minkowski unit as shown by Hasse [Has48, p. 20]. In that case, using the proof in 3.5, we see that

$$\log \rho(\varepsilon^\sigma) \equiv p^\kappa \log \rho(\varepsilon) \bmod \ell,$$

so that we need to find $\log \rho(\varepsilon) \bmod \ell$ only. It matches Table 1, where we read that only $r - \mathcal{R} = 3 - 2 = 1$ (well chosen) Schirokauer map is required.

4.4.2. *The degree 4 cases.* When the signature is $(4, 0)$ and the Galois group is C_4, we can precise the structure of U_K, see [Has48, Gra79]. The first case is when K admits a Minkowski unit, that is ε such that $U_K = \langle -1, \varepsilon, \varepsilon^\sigma, \varepsilon^{\sigma^2} \rangle$. And we use the same reasoning as in Section 4.4.1 to reduce the number of logarithms needed to 1.

In the second case, $U_K = \langle -1, \epsilon_1, \epsilon_\chi, \epsilon_\chi^\sigma \rangle$, where ϵ_1 is the fundamental unit of the quadratic subfield and ϵ_χ is a generator of the group of relative units, that is $\eta \in U_K$ such that $\mathrm{N}_{K/K_2}(\eta) = \pm 1$. We gain two logarithms since we can use the Galois action for $\log \rho(\epsilon_\chi^\sigma)$, and we know $\log \rho(\epsilon_1)$.

Note that Table 1 predicts that the two cases above, with or without relative units, lead to the same number of Schirokauer maps: $r - \mathcal{R} = 1$.

For signature $(0, 2)$, the rank of K is 1, and the fundamental unit is that from the real quadratic subfield, so we don't need any logarithm at all. To be more precise, let us detail the case of our favorite example: $f(X) = X^4 + 1$ which defines the 8-th roots of unity, say $K = \mathbb{Q}(\zeta_8)$. The Galois group of f is V_4 and K has two automorphisms $\sigma_1 : x \mapsto -x$, $\sigma_2 : x \mapsto 1/x$. We compute that

$$K^{\langle \sigma_1 \rangle} = \mathbb{Q}(i), K^{\langle \sigma_2 \rangle} = \mathbb{Q}(\sqrt{2}), K^{\langle \sigma_1 \sigma_2 \rangle} = \mathbb{Q}(\sqrt{-2}).$$

The corresponding factorizations of $f(X)$ are

$$f(X) = (X^2 + i)(X^2 - i),$$
$$f(X) = (X^2 - \sqrt{2}X + 1)(X^2 + \sqrt{2}X + 1),$$
$$f(X) = (X^2 - \sqrt{-2}X - 1)(X^2 + \sqrt{-2}X - 1).$$

Since f has signature $(0, 2)$, we have $U_K = \langle \zeta_8 \rangle \times \langle \varepsilon \rangle$, where ε comes from $K^{\langle \sigma_2 \rangle}$, the only real quadratic subfield. By Theorem 4.8, we do not need any logarithm of units for use in NFS.

5. Reducing the number of Schirokauer maps

In this section, we use the preceding Section to conclude that we can reduce the number of Schirokauer maps needed in NFS-DL.

We use the notations of Section 4. A system of r units is a basis modulo ℓ if its image in U/U^ℓ is a basis. Let p be a prime and n an integer such that the reduction of g modulo p has an irreducible factor of degree n. Let ℓ be a prime factor of $p^n - 1$, coprime to $p - 1$. In order to reduce the number of Schirokauer maps associated to g, we follow the steps below:

(1) We find a system of r elements in U which form a basis modulo ℓ.
(2) We compute an integer $\mathcal{R} \leq r$, as large as possible, and a system of r elements $u_1, \ldots, u_r$ in U which form a basis modulo ℓ, such that the discrete logarithms of $\rho(u_1), \ldots, \rho(u_\mathcal{R})$ are zero modulo ℓ.

(3) Using any set of r Schirokauer maps and the system of fundamental units above, we compute a set of Schirokauer maps $\lambda_1, \ldots, \lambda_r$ such that the NFS algorithm can be run using only the last $r - \mathcal{R}$ Schirokauer maps $\lambda_{\mathcal{R}+1}, \ldots, \lambda_r$.

We do not discuss the first point here. Point (2) was studied in Section 4. Point (3) is solved by the corollary of the following theorem.

Theorem 5.1. *Let $\lambda_1, \ldots, \lambda_r$ be a set of Schirokauer maps and let $u_1, \ldots, u_r$ a system of effectively computed units in U, whose image in U/U^ℓ form a basis. Then there exits a set of effectively computable Schirokauer maps $\lambda'_1, \ldots, \lambda'_r$ such that, for $1 \leq i, j \leq r$,*

$$\lambda'_i(u_j) = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases} \tag{15}$$

Proof. Let $L = (l_{i,j})$ be the $r \times r$ matrix of entries $l_{i,j} = \lambda_i(u_j)$. Let $C = (c_{i,j})$ be the inverse of L. Then, we put

$$\lambda'_i = \sum_{n=1}^{r} c_{i,n} \lambda_n. \tag{16}$$

We have $\lambda'_i(u_j) = (CL)_{i,j}$, so the maps λ'_i verify the condition in Equation (15). □

Corollary 5.2. *Let $u_1, \ldots, u_r$ be a set of units, effectively computed, which form a basis modulo ℓ. Assume that for some integer $\mathcal{R}$, $1 \leq \mathcal{R} \leq r$, the first $\mathcal{R}$ units $u_1, \ldots, u_{\mathcal{R}}$ are such that $\log(\rho(u_i)) \equiv 0 \mod \ell$. Then, there exists a set of r effectively computable Schirokauer maps $\lambda'_1, \ldots, \lambda'_r$ such that NFS can be run with the last $r - \mathcal{R}$ maps instead of the complete set of r maps.*

Proof. Using any set of Schirokauer maps, we compute $\lambda'_1, \ldots, \lambda'_r$ such that Equation (15) holds.

By Equation (4) in Section 2.2, when running NFS with the maps $\lambda'_1, \ldots, \lambda'_r$, the linear algebra stage computes the virtual logarithms of the ideals in the factor base together with r constants χ_i, $1 \leq i \leq r$ such that

$$\log(\rho(\gamma)) \equiv \sum_{\mathfrak{q} \in \mathcal{F}} \log \mathfrak{q} \, \mathrm{val}_{\mathfrak{q}}(\gamma) + \sum_{i=1}^{r} \chi_i \lambda_i(\gamma) \mod \ell. \tag{17}$$

For $1 \leq i \leq r$, when injecting $\gamma = u_i$ in Equation (17) we obtain that $\chi_i = \log(\rho(u_i))$. For $1 \leq i \leq \mathcal{R}$ we have $\log(\rho(u_i) \equiv 0 \mod \ell$, and therefore $\chi_i \equiv 0 \mod \ell$. Hence, Equation (17) can be rewritten with $r - \mathcal{R}$ Schirokauer maps:

$$\log(\rho(\gamma)) \equiv \sum_{\mathfrak{q} \in \mathcal{F}} \log \mathfrak{q} \, \mathrm{val}_{\mathfrak{q}}(\gamma) + \sum_{i=\mathcal{R}+1}^{r} \chi_i \lambda_i(\gamma) \mod \ell. \tag{18}$$

□

Example 5.3. (continued) The corollary above states that the polynomials in the family described in Example 4.2 do not require any Schirokauer map. Moreover, note that if f_1 and f_2 are two polynomials in this family and μ_1, μ_2 are two positive rationals such that $\mu_1 + \mu_2 = 1$, then $\mu_1 f_1 + \mu_2 f_2$ also belongs to this family.

A more important example is that of cubic polynomials with an automorphism of order three. Then, we can effectively compute a linear combination $\Lambda_{1,2}$ of any two Schirokauer maps Λ_1 and Λ_2 so that NFS can be run with $\Lambda_{1,2}$ as unique Schirokauer map.

6. Two new methods of polynomial selection

In this section, we propose two new methods to select the polynomials f and g, in the case of finite fields that are low degree extensions of prime fields. The first one is an extension to non-prime fields of the method used by Joux and Lercier [JL03] for prime fields. The second one, which relies heavily on rational reconstruction, insists on having coefficients of size $O(\sqrt{p})$ for g. For both methods, f has very small coefficients, of size $O(\log p)$.

6.1. **The state-of-art methods of polynomial selection.** Joux, Lercier, Smart and Vercauteren [JLSV06] introduced two methods of polynomial selection, one which is the only option for medium characteristic finite fields and one which is the only known for the non-prime large characteristic fields.

6.1.1. *The first method of JLSV.* Described in [JLSV06, §2.3], this method is best adapted to the medium characteristic case. It produces two polynomials f, g of same degree n, which have coefficients of size $\sqrt{p}$ each.

One starts by selecting a polynomial f of the form $f = f_v + af_u$ with a parameter a to be chosen. Then one computes a rational reconstruction (u, v) of a modulo p and one defines $g = vf_v + uf_u$. Note that, by construction, we have $f = v \cdot g \bmod p$. Also note that both polynomials have coefficients of size $\sqrt{p}$.

Example 6.1. Take $p = 1000001447$ and $a = 44723 \geq \lceil\sqrt{p}\rceil$. One has $f = x^4 - 44723x^3 - 6x^2 + 44723x + 1$ and $g = 22360x^4 - 4833x^3 - 134160x^2 + 4833x + 22360$ with $u/v = 4833/22360$ a rational reconstruction of a modulo p.

The norm product is $N_f N_g = E^{2n}p = E^{2n}Q^{1/n}$.

If one wants to use automorphisms as in Section 6, then one chooses f in a family of polynomials which admit automorphisms. For example when $n = 4$, one can take f in the family presented in Tab. 4, formed of degree 4 polynomials with cyclic Galois group of order four, having an explicit automorphism: $f = (x^4 - 6x^2 + 1) + a(x^3 - x) = f_v + af_u$. Note that the second polynomial g belongs to the same family and has the same automorphisms.

6.1.2. *The second method of JLSV.* The second method is described in [JLSV06, §3.2]. It starts by computing g of degree n then it computes f of degree $\deg f \geq n$. First one selects g_0 of degree n and small coefficients. Then one chooses an integer $W \sim p^{1/(d+1)}$, but slightly larger, and set $g = g_0(x + W)$. The smallest degree coefficient of g has size W^n. We need to take into account the skewness of the coefficients. The polynomial f is computed by reducing the lattice of polynomials of degree at most d, divisible by g modulo p. We do this by defining the matrix M in Sec. 6.2, eq. (20), with $\varphi = g$. We obtain a polynomial f with coefficients of size $p^{n/(d+1)} = Q^{1/(d+1)}$.

Example 6.2. Consider again the case of $p = 1125899906842783$ and $n = 4$. We take g_0 a polynomial of degree four and small coefficients, for example $g_0 = x^4 - x^3 - 6x^2 + x + 1$. We can have $\deg(f) = d$ for any value of $d \geq n$, we take $d = 7$ for the example. We use $W = 77 \geq p^{1/(d+1)}$, where we emphasize that we do not use $Q^{1/(d+1)}$, and we set

$$g = g_0(x + W) = x^4 + 307x^3 + 35337x^2 + 1807422x + 34661012.$$

We construct the lattice of polynomials of degree at most 7 which are divisible by g modulo p. We obtain

$$\begin{aligned} f = {} & 12132118x^7 + 11818855x^6 + 2154686x^5 \\ & -7076039x^4 + 7796873x^3 + 7685308x^2 + 4129660x - 14538916. \end{aligned}$$

Note that f and g have coefficients of size $Q^{1/8}$.

For comparison, we compute the norms' product: $E^{d+n}Q^{2/(d+1)}$. However, one might obtain a better norms product using the skewness notion introduced by Murphy [Mur99]. Without entering into details, we use as a lower bound for the norms product the quantity $E^{d+n}Q^{3/2(d+1)}$. Indeed, the coefficients of f have size $Q^{1/(d+1)}$ and the coefficients of g have size $Q^{1/(d+1)}$, which cannot be improved more than $Q^{1/2(d+1)}$ using skewness. This bound is optimistic, but even so the new methods will offer better performances.

6.2. **The generalized Joux-Lercier method.** In the context of prime fields, Joux and Lercier proposed a method in [JL03] to select polynomials using lattice reduction. They start with a polynomial f of degree $d+1$ with small coefficients, such that f admits a root m modulo p. Then, a matrix M is constructed with rows that generate the lattice of polynomials of degree at most d with integer coefficients, that also admits m as a root modulo p. We denote by $\mathrm{LLL}(M)$ the matrix obtained by applying the LLL algorithm to the rows of M:

$$M = \begin{bmatrix} p & 0 & \cdots & 0 \\ -m & 1 & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ -m^d & \cdots & 0 & 1 \end{bmatrix}, \quad \mathrm{LLL}(M) = \begin{bmatrix} g_0 & g_1 & \cdots & g_d \\ * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \cdots & * \end{bmatrix}. \tag{19}$$

The first row gives a polynomial $g = g_0 + g_1 x + \ldots + g_d x^d$, that has a common root m with f, and the pair of polynomials (f, g) can be used for computing discrete logarithm in $\mathbb{F}_p$ with the NFS algorithm.

In order to tackle discrete logarithms in $\mathbb{F}_{p^n}$, we generalize this to polynomials (f, g) that share an irreducible common factor φ of degree n modulo p. Let d' be an integer parameter that we choose below. We select f an irreducible polynomial in $\mathbb{Z}[x]$ of degree $d^{'} + 1 \geq n$, with small coefficients, good sieving properties and an irreducible degree n factor $\varphi = \sum_{i=0}^{n} \varphi_i x^i$ modulo p, that we force to be monic. We define a $(d'+1) \times (d'+1)$ matrix M whose rows generate the lattice of polynomials of degree at most $d'+1$ for which φ is also a factor modulo p. Then, running the LLL algorithm on this matrix gives a matrix whose rows are generators with smaller coefficients. A possible choice for the matrix M is as follows, where the missing coefficients understand to be zero.

$$M = \begin{bmatrix} p & & & & & & \\ & \ddots & & & & & \\ & & p & & & & \\ \varphi_0 & \varphi_1 & \cdots & \varphi_n & & & \\ & \ddots & \ddots & & \ddots & & \\ & & \varphi_0 & \varphi_1 & \cdots & \varphi_n \end{bmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \deg\varphi = n \text{ rows} \\ \left.\vphantom{\begin{matrix} a \\ a \\ a \end{matrix}}\right\} \begin{matrix} \deg g + 1 - \deg\varphi \\ = d' + 1 - n \text{ rows} \end{matrix} \end{matrix} \quad \mathrm{LLL}(M) = \begin{bmatrix} g_0 & g_1 & \cdots & g_{d'} \\ & & & \\ & & * & \\ & & & \end{bmatrix}. \tag{20}$$

One can remark that since φ has been made monic, the determinant of M is $\det(M) = p^n$. The first row of $\mathrm{LLL}(M)$ gives a polynomial g of degree at most $d^{'}$ that shares the common factor φ of degree n modulo p with f. The coefficients of g have approximately a size $(\det M)^{1/(d^{'}+1)} = p^{n/(d^{'}+1)}$ if we assume that the dimension stays small.

Note that, when $n = 1$, this method produces the same pair (f, g) as the method of Joux-Lercier. Indeed, in this case $\varphi = x - m$ and the rows of the matrix M in Equation (19) generate the same lattice as the rows of matrix M in Equation (20).

Remark 6.3. By considering a smaller matrix, it is possible to produce a polynomial g whose degree is smaller than $d' = \deg f - 1$. This does not seem to be a good idea. Indeed, the size of the coefficients of g would be the same as the coefficients of a polynomial obtained starting with a polynomial f with coefficients of the same size but of a smaller degree (d' or less).

We now discuss the criteria to select the parameter $d' = \deg f - 1$ with respect to the bitsize of p. The most important quantity to minimize is the size of the product of the norms $\mathrm{Res}(\phi, f)\,\mathrm{Res}(\phi, g)$, for the typical polynomials ϕ that will be used. In this setting, the best choice is to stick to polynomials ϕ of degree 1, and we denote by E a bound on its two coefficients that we will tune later. For any polynomial P, let us denote by $|P|_\infty$ the maximum absolute value of the coefficients of P. Since f has been selected to have small coefficients, we obtain the following estimate for the product of the norms:

$$|\,\mathrm{Res}(\phi, f)\,\mathrm{Res}(\phi, g)| \approx \left(E^{\deg(f)}\right)\left(||g||_\infty E^{\deg(g)}\right), \tag{21}$$

where we did not write factors that contribute in a negligible way. In Table 2, we list the possible choices for the degrees, that we expect to be practically relevant for discrete logarithms in $\mathbb{F}_{p^2}$ and $\mathbb{F}_{p^3}$.

TABLE 2. Size of the product of the norms, for various choices of parameters with the generalized Joux-Lercier method, in $\mathbb{F}_{p^2}$ and $\mathbb{F}_{p^3}$.

Field	$\deg\varphi$	$\deg f$	$\deg g$	$\|\|g\|\|_\infty$	$E^{\deg f}E^{\deg g}\|\|g\|\|_\infty$
$\mathbb{F}_Q = \mathbb{F}_{p^2}$	2	4	3	$p^{1/2} = Q^{1/4}$	$Q^{1/4}E^7$
	2	3	2	$p^{2/3} = Q^{1/3}$	$Q^{1/3}E^5$
$\mathbb{F}_Q = \mathbb{F}_{p^3}$	3	6	5	$p^{3/6} = Q^{1/6}$	$Q^{1/6}E^{11}$
	3	5	4	$p^{3/5} = Q^{1/5}$	$Q^{1/5}E^9$
	3	4	3	$p^{3/4} = Q^{1/4}$	$Q^{1/4}E^7$

As for the value of the parameter E, although the asymptotic complexity analysis can give hints about its value, it is usually not reliable for fixed values. Therefore we prefer to use a rough approximation of E using the values of the same parameter in the factoring variant of NFS as implemented in CADO-NFS. These values of E w.r.t. Q are collected in Table 3 and we will use them together with Table 2 in order to plot the estimate of the running time in Figure 1 to compare with other methods. Note that, a posteriori, the norms product in our case is smaller than in the factoring variant of NFS. Hence, one can take a slightly smaller values for E.

TABLE 3. Practical values of E for Q from 60 to 220 decimal digits.

Q(dd)	60	80	100	120	140	160	180	204	220
Q(bits)	200	266	333	399	466	532	598	678	731
E(bits)	19	20	21	23	25	27	28	29	30

6.3. **The conjugation method.** We propose another method to select polynomials for solving discrete logarithms in $\mathbb{F}_{p^n}$ with the following features: the resulting polynomial f has degree $2n$ and small coefficients, while the polynomial g has degree n and coefficients of size bounded by about $\sqrt{p}$. In the next section, an asymptotic analysis shows that there are cases where this is more interesting than the generalized Joux-Lercier method; furthermore, it is also well suited for small degree extension that can be reached with the current implementations.

Let us take an example.

Example 6.4. We take the case of $n = 11$ and $p = 134217931$, which is a random prime congruent to 1 modulo n. The method is very general, this case is the simplest. We enumerate the integers $a = 1, 2, \ldots$ until $\sqrt{a}$ is irrational but exists in $\mathbb{F}_p$, i.e. the polynomial $x^2 - a$ splits in $\mathbb{F}_p$. We call λ a square root of a in $\mathbb{F}_p$ and test if $x^n - \lambda$ is irreducible modulo p. If it is not the case, we continue and try the next value of a. For example $a = 5$ works.

Then we set $\lambda = 108634777 = \mathbb{F}_p(\sqrt{5})$ and we put $\varphi = x^{11} - \lambda$. Next, we do a rational reconstruction of λ, i.e. we find two integers of size $O(\sqrt{p})$ such that $u/v \equiv \lambda \mod p$. We find $u = 1789$ and $v = 10393$. The conjugation method consists in setting:

(1) $f = (x^{11} - \sqrt{5})(x^{11} + \sqrt{5}) = x^{22} - 5$;
(2) $g = vx^{11} - u = 10393x^{11} - 1789$.

By construction f and g are divisible by φ modulo p.

We continue with a construction that works for $\mathbb{F}_{p^2}$ when p is congruent to 7 modulo 8.

Example 6.5. Let $p \equiv 7 \bmod 8$ and let $f = x^4 + 1$ that is irreducible modulo p. From the results in Section 4.4.2, we use the factorization $(x^2 + \sqrt{2}x + 1)(x^2 - \sqrt{2}x + 1)$ of $f(x)$. Since 2 is a square modulo p, we take $\varphi = x^2 + \sqrt{2}x + 1 \in \mathbb{F}_p[x]$. Now, by rational reconstruction of $\sqrt{2}$ in $\mathbb{F}_p$, we can obtain two integers $u, v \in \mathbb{Z}$ such that $\frac{u}{v} \equiv \sqrt{2} \mod p$, and u and v have size similar to $\sqrt{p}$. We define $g = vx^2 + ux + v$. Then f and g share a common irreducible factor of degree 2 modulo p, and verify the degree and size properties that we announced.

This construction can be made general: first, it is possible to obtain pairs of polynomials f and g with the claimed degree and size properties for any extension field $\mathbb{F}_{p^n}$; and second, in many small cases that are of practical interest, it is also possible to enforce the presence of automorphisms. The general construction is based on Algorithm 1.

Algorithm 1: Polynomial selection with the conjugation method

Input: p prime and n a small exponent
Output: $f, g \in \mathbb{Z}[x]$ suitable for discrete logarithm computation with NFS in $\mathbb{F}_{p^n}$

Select $g_u(x), g_v(x)$, two polynomials with small integer coefficients, $\deg g_u < \deg g_v = n$;
repeat
  Select $\mu(x)$ a quadratic, monic, irreducible polynomial over $\mathbb{Z}$ with small coefficients ;
until $\mu(x)$ *has a root* λ *in* $\mathbb{F}_p$ *and* $g_v + \lambda g_u$ *is irreducible in* $\mathbb{F}_p$;
$(u, v) \leftarrow$ a rational reconstruction of λ ;
$f \leftarrow \mathrm{Res}_Y(\mu(Y), g_v(x) + Y g_u(x))$;
$g \leftarrow v g_v + u g_u$;
return (f, g)

Fact 6.6 (Properties of the conjugation method)**.** *The polynomials (f, g) returned by Algorithm 1 verify:*

(1) *f and g have integer coefficients and degrees $2n$ and n respectively;*
(2) *the coefficients of f have size $O(1)$ and the coefficients of g are bounded by $O(\sqrt{p})$.*
(3) *f and g have a common irreducible factor φ of degree n over $\mathbb{F}_p$.*

Proof. The fact that g has integer coefficients and is of degree n is immediate by construction. As for f, since it is the resultant of two bivariate polynomials with integer coefficients, it is also with integer coefficients. Using classical properties of the resultant, f can be seen as the product of the polynomial $g_v(x) + Y g_u(x)$ evaluated in Y at the two roots of $\mu(Y)$, therefore its degree is $2n$. Also, since all the coefficients of the polynomials involved in the definition of f have size $O(1)$, and the degree n is assumed to be "small", then the coefficients of f are also $O(1)$.

For the size of the coefficients of g, it follows from the output of the rational reconstruction of λ in $\mathbb{F}_p$, which is expected to have sizes in $O(\sqrt{p})$ (in theory, we can not exclude that we are in a rare case where the sizes are larger, though).

The polynomials f and g are suitable for NFS in $\mathbb{F}_{p^n}$, because both are divisible by $\varphi = g_v + \lambda g_u$ modulo p, and by construction it is irreducible of degree n. □

In the example above, for $\mathbb{F}_{p^2}$ with $p \equiv 7 \mod 8$, Algorithm 1 was applied with $g_u = x$, $g_v = x^2 + 1$ and $\mu = x^2 - 2$. One can check that $f = \mathrm{Res}_Y(Y^2 - 2, (x^2 + 1) + Yx) = x^4 + 1$, as can be seen from Section 4.4.2.

In Algorithm 1, there is some freedom in the choices of g_u and g_v. The key idea to exploit this opportunity is to base the choice on one-parameter families of polynomials for which an automorphism with a nice form is guaranteed to exist, in order to use the improvements of Section 3.

In Table 4, we list possible choices for g_u and g_v in degree 2, 3, 4 and 6, such that for any integer λ, $g_v + \lambda g_u$ as a simple explicit cyclic automorphism. The families for 3, 4 and 6 are taken from [Gra79, Gra87] (see also [Fos11] references for larger degrees).

TABLE 4. Families of polynomials of degree 2, 3, 4 and 6 with cyclic Galois group.

n	coeffs of $g_v + ag_u$	g_v	g_u	automorphism: $\theta \mapsto$
2	$(1, a, 1)$	$x^2 + 1$	x	$1/\theta$
	$(-1, a, 1)$	$x^2 - 1$	x	$-1/\theta$
	$(a, 0, 1)$	x^2	1	$-\theta$
3	$(1, -a-3, -a, 1)$	$x^3 - 3x - 1$	$-(x^2 + x)$	$-(\theta + 1)/\theta$
4	$(1, -a, -6, a, 1)$	$x^4 - 6x^2 + 1$	$x^3 - x$	$-(\theta + 1)/(\theta - 1)$
6	$(1, -2a, -5a - 15,$ $-20, 5a, 2a + 6, 1)$	$x^6 + 6x^5 -$ $20x^3 - 15x^2 + 1$	$2x^5 + 5x^4 -$ $5x^2 - 2x$	$-(2\theta + 1)/(\theta - 1)$

Theorem 6.7. *For any prime p and n in $\{2, 3, 4, 6\}$, the polynomials f and g obtained by the conjugation method using g_u and g_v as in Table 4 generate number fields with two automorphisms σ and τ of order n that verify the hypothesis of Theorem 3.5.*

Proof. The polynomial g belongs to a family of Table 4, so its number field K_g has an automorphism of order n given by the formula in the last column.

Let ω be a root of $\mu(x)$. The polynomial $g_v + \omega g_u$ defines a number field that is an extension of degree n of $\mathbb{Q}(\omega)$ and that admits an automorphism of order n, which fixes $\mathbb{Q}(\omega)$. Since f and $g_v + \omega g_u$ generate the same number field, this shows that the number field K_f has an automorphism of order n.

The polynomial φ is given by $g_v + \lambda g_u$. Therefore, it belongs to the same family as g hence it has the same automorphism of order n as f and g. This shows that modulo p, the automorphism sends a root of φ to another root of φ, as required in the hypothesis of Theorem 3.5. □

Example 6.8. Let us apply the conjugation method for $\mathbb{F}_{p^3}$, where $p = 2^{31} + 11$. Running Algorithm 1 with $g_u = -x^2 - x$ and $g_v = x^3 - 3x - 1$, one sees that $\mu = x^2 - x + 1$ has a root $\lambda = 2021977950$ in $\mathbb{F}_p$ and that $g_v + \lambda g_u$ is irreducible in $\mathbb{F}_p[x]$. We obtain $f = x^6 - x^5 - 6x^4 + 3x^3 + 14x^2 + 7x + 1$ and $g = 20413x^3 + 32630x^2 - 28609x + 20413$. With $\varphi = x^3 + 125505709x^2 + 125505706x + 2147483658$ as their GCD modulo p, we can check that the three polynomials f, g and φ admit $\theta \mapsto -(\theta + 1)/\theta$ as an automorphism of order 3.

6.4. **Estimation and comparison of the methods.** We have four methods of polynomial selection which apply to NFS in non-prime fields:

- the two methods of JLSV, presented in 6.1.1 and 6.1.2, denoted JLSV_1 and, respectively, JLSV_2;
- the generalized Joux-Lercier method, presented in 6.2, denoted GJL;
- the conjugation method, presented in 6.3, denoted bu Conj.

We take the size of the product of the norms as the main quantity to minimize, and we estimate its value as

$$E^{\deg f} ||f||_\infty E^{\deg g} ||g||_\infty \ . \tag{22}$$

The starting point are the properties of the polynomials obtained with the various methods in Tab. 5.

TABLE 5. Theoretical complexities for polynomial selection methods, n is the extension degree ($\mathbb{F}_{p^n}$), $Q = p^n$

method	$\deg g$	$\deg f$	$\|\|g\|\|_\infty$	$\|\|f\|\|_\infty$	$E^{\deg f + \deg g} \|\|f\|\|_\infty \|\|g\|\|_\infty$
Conj	n	$2n$	$Q^{1/(2n)}$	$O(1)$	$E^{3n} Q^{1/(2n)}$
GJL	$\geq n$	$> \deg g$	$Q^{1/(\deg g + 1)}$	$O(1)$	$E^{\deg f + \deg g} Q^{1/(\deg g + 1)}$
JLSV_1	n	n	$Q^{1/(2n)}$	$Q^{1/(2n)}$	$E^{2n} Q^{1/n}$
JLSV_2	n	$\geq \deg g$	$Q^{1/(2(\deg f + 1))}$	$Q^{1/(\deg f + 1)}$	$E^{\deg f + n} Q^{(3/2)1/(\deg f + 1)}$

When the best method depends on the size of the finite field in consideration, we use rough estimates of E taken from Table 3.

In Table 6 we summarize all the sizes that we can get for reasonable choices of parameters for $\mathbb{F}_{p^n}$ with $n \in \{2,3,4,5,6\}$, with all the methods at our disposition.

To choose the best method, we now need to compare the values in the last column of Tab. 6. For that we consider in Tab. 3 practical values of E and Q for Q from 60 to 220 decimal digits (dd). We note that $\log E = 0.095 \log Q$ for Q of 60 dd and $\log E = 0.041 \log Q$ for Q of 220 dd. We can now eliminate a few other methods in Tab. 6:

- $\boldsymbol{n = 2}$**:** We discard the JLSV_1 and JLSV_2 methods because their complexities are worse than GJL complexity: $E^4 Q^{1/2} > E^5 Q^{1/3}$ since $Q^{1/6} > E$ (indeed, $Q^{0.1} > E$).
- $\boldsymbol{n = 3}$**:** A second time we discard the JLSV_1 method because the GJL method is better. Indeed $E^6 Q^{1/3} < E^7 Q^{1/4}$ while $E > Q^{1/12}$ i.e. when Q is less or around 60 dd.
- $\boldsymbol{n = 4}$**:** This time we discard GJL method with $(\deg g, \deg f) = (5,6)$ because it is less efficient than GJL with $(4,5)$ whenever $E > Q^{1/60}$. We also discard the Conj method because we are not in the case $E < Q^{1/40}$.
- $\boldsymbol{n = 5}$**:** We discard the Conj method ($E^{15} Q^{1/10}$) which is worse than GJL with $(5,6)$ while $E > Q^{1/60}$. We also discard GJL method with $(6,7)$ because the same method with $(5,6)$ is more efficient whenever $Q^{1/84} < E$.
- $\boldsymbol{n = 6}$**:** As for $n = 5$, the Conj method is not competitive because we are not in the case $E < Q^{1/84}$. We also discard the GJL method with $(7,8)$ compared with $(6,7)$ because we don't have $E < Q^{1/112}$.

We represent the results in Fig. 1. We can clearly see that when $Q = p^2$ is more than 70 decimal digits long (200 bits, i.e. $\log_2 p = 100$), it is much better to use the construction with $\deg f = 4$ and $\deg g = 2$ for computing discrete logarithms in $\mathbb{F}_Q = \mathbb{F}_{p^2}$. For Q of more than 220 dd, $(\deg f, \deg g) = (3,4)$ starts to be a better choice than $(2,3)$ but our new method with $(2,4)$ is even better, the value in (22) is about 20 bits smaller. For $Q = p^3$ from 60 to 220 dd (i.e. p from 20 to 73 dd), the choice $(3,4)$ gives a lower value of (22). Then for Q of more than 220 dd, the method with $(3,6)$ is better. For Q of 220 dd, (22) takes the same value with $(\deg g, \deg f) = (3,6)$ as with $(3,4)$.

6.5. **Improving the selected polynomials.** We explained in Sec. 6.2 our generalized Joux-Lercier method and in Sec. 6.3 our method of conjugated polynomials. In both cases when $\deg \varphi \geq 2$ one obtains two distinct reduced polynomials g_1 and $g_2 \in \mathbb{Z}[x]$ such that $g_1 \equiv g_2 \equiv \varphi \bmod p$ up to a coefficient in $\mathbb{F}_p$. We propose to search for a polynomial $g = \lambda_1 g_1 + \lambda_2 g_2$ with $\lambda_1, \lambda_2 \in \mathbb{Z}$ small, e.g. $|\lambda_1|, |\lambda_2| < 200$ that maximises the Murphy E value of the pair (f,g).

The Murphy $\mathbb{E}$ value is explained in [Mur99, Sec. 5.2.1, Eq. 5.7 p. 86]. This is an estimation of the smoothness properties of the values taken by either a single polynomial f of a pair (f,g). First one homogenizes f and g and defines

$$u_f(\theta_i) = \frac{\log |f(\cos\theta_i, \sin\theta_i)| + \alpha(f)}{\log B_f}$$

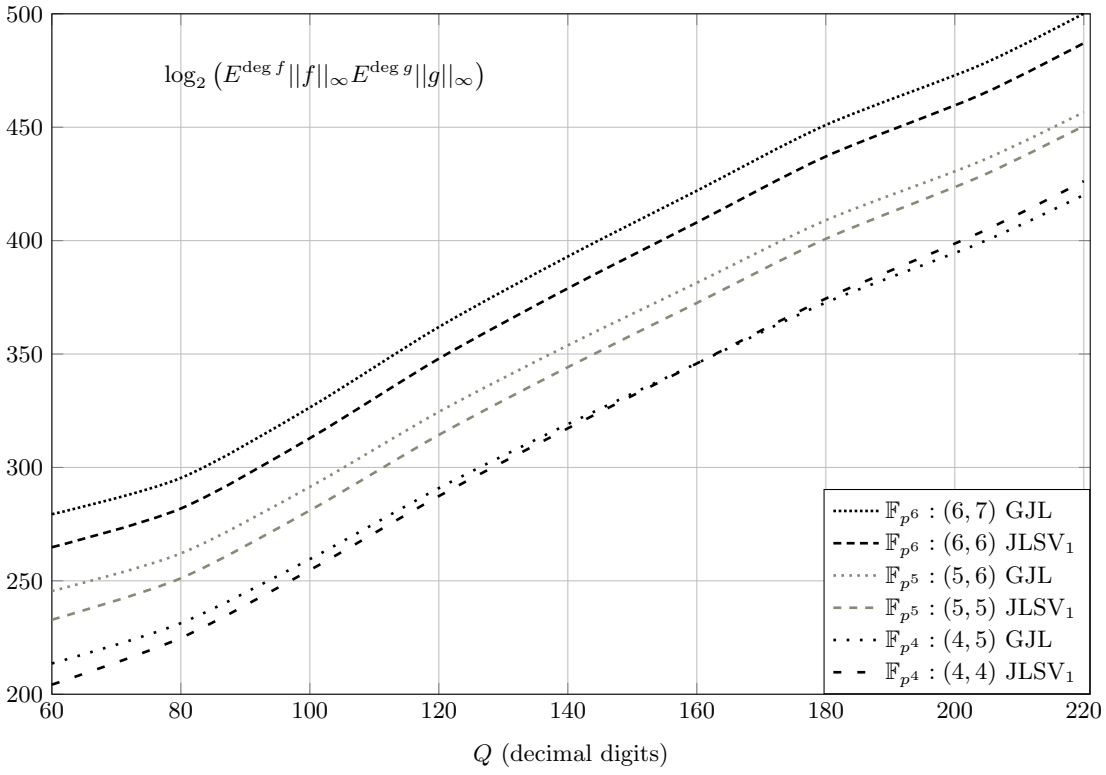
with $\theta_i \in [0,\pi]$, more precisely, $\theta_i = \frac{\pi}{K}\left(i - \frac{1}{2}\right)$ (with e.g. $K = 2000$ and $i \in \{1,\ldots,K\}$), $\alpha(f)$ defined in [Mur99, Sec. 3.2.3] and B_f a smoothness bound set according to f. Murphy advises to take $B_f = 1\,\mathrm{e}\,7$ and $B_g = 5\,\mathrm{e}\,6$. Finally

$$\mathbb{E}(f,g) = \sum_{i=1}^{K} \rho(u_f(\theta_i))\rho(u_g(\theta_i)) \ .$$

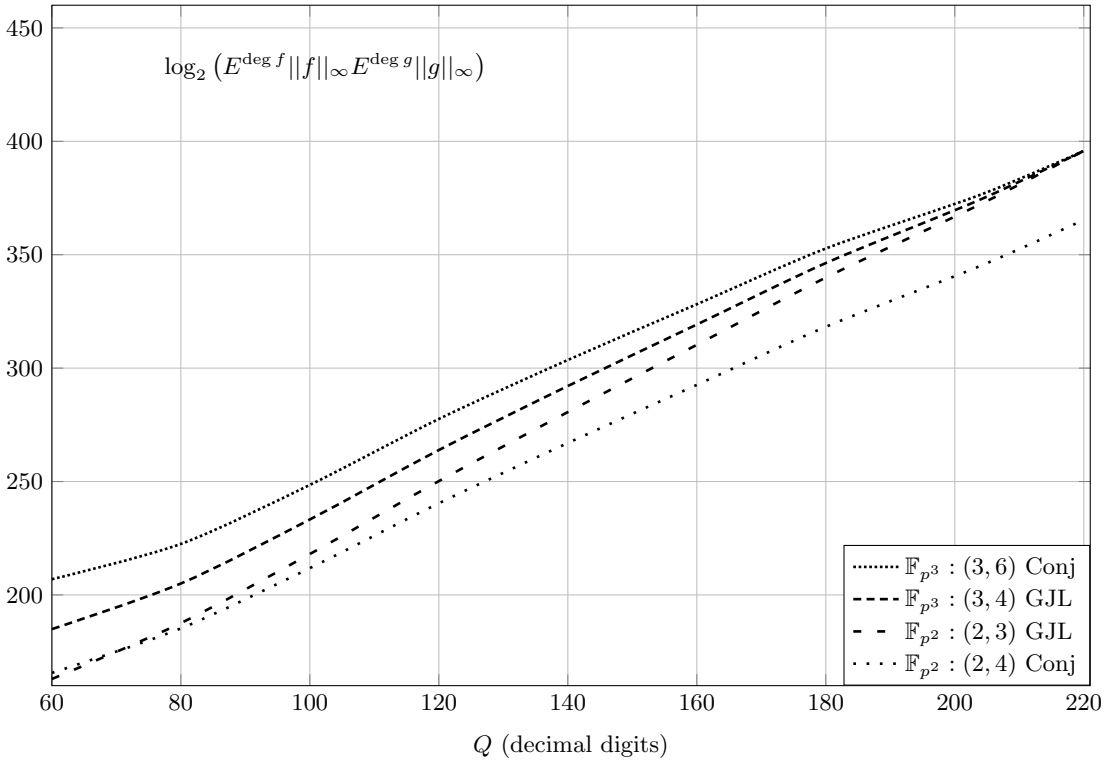
We propose to search for $g = \lambda_1 g_1 + \lambda_2 g_2$ with $|\lambda_i| < 200$ and such that $\mathbb{E}(f,g)$ is maximal. In practice we obtain g with $\alpha(g) \leq -1.5$ and $\mathbb{E}(f,g)$ improved of 2% up to 30 %.

TABLE 6. Size of the product of norms for various choices of parameters and algorithms. We discard ($\otimes$) the methods which offer sizes of norms product which are clearly not competitive compared to some other one, assuming that $0.04 \log Q \leq \log E \leq 0.1 \log Q$ (Tab. 3)

$\deg g, \deg f$	Q	$\mathbb{F}_{p^n}$	$\|\|f\|\|_\infty$	g	$\|\|g\|\|_\infty$	$E^{\deg f}\|\|f\|\|_\infty E^{\deg g}\|\|g\|\|_\infty$	
$(2,3)$	p^2	$\mathbb{F}_{p^2}$	$O(1)$	GJL	$Q^{1/3}$	$E^5Q^{1/3}$	
$(3,4)$				GJL	$Q^{1/4}$	$E^7Q^{1/4}$	$\otimes$
$(2,4)$				Conj	$Q^{1/4}$	$E^6Q^{1/4}$	
$(2,2)$			$Q^{1/4}$	JLSV_1	$Q^{1/4}$	$E^4Q^{1/2}$	$\otimes$
$(2,2)$			$Q^{1/6}$	JLSV_2	$Q^{1/3}$	$E^4Q^{1/2}$	$\otimes$
$(2,3)$			$Q^{1/8}$	JLSV_2	$Q^{1/4}$	$E^5Q^{3/8}$	$\otimes$
$(2,4)$			$Q^{1/5}$	JLSV_2	$Q^{1/10}$	$E^6Q^{3/10}$	$\otimes$
$(2,5)$			$Q^{1/6}$	JLSV_2	$Q^{1/12}$	$E^7Q^{1/4}$	$\otimes$
$(3,4)$	p^3	$\mathbb{F}_{p^3}$	$O(1)$	GJL	$Q^{1/4}$	$E^7Q^{1/4}$	
$(4,5)$				GJL	$Q^{1/5}$	$E^9Q^{1/5}$	$\otimes$
$(3,6)$				Conj	$Q^{1/6}$	$E^9Q^{1/6}$	
$(3,3)$			$Q^{1/6}$	JLSV_1	$Q^{1/6}$	$E^6Q^{1/3}$	$\otimes$
$(3,3)$			$Q^{1/8}$	JLSV_2	$Q^{1/4}$	$E^6Q^{3/8}$	$\otimes$
$(3,4)$			$Q^{1/10}$	JLSV_2	$Q^{1/5}$	$E^7Q^{3/10}$	$\otimes$
$(3,5)$			$Q^{1/12}$	JLSV_2	$Q^{1/6}$	$E^8Q^{1/4}$	$\otimes$
$(3,6)$			$Q^{1/7}$	JLSV_2	$Q^{1/14}$	$E^9Q^{3/14}$	$\otimes$
$(4,5)$	p^4	$\mathbb{F}_{p^4}$	$O(1)$	GJL	$Q^{1/5}$	$E^9Q^{1/5}$	
$(5,6)$				GJL	$Q^{1/6}$	$E^{11}Q^{1/6}$	$\otimes$
$(4,8)$				Conj	$Q^{1/8}$	$E^{12}Q^{1/8}$	$\otimes$
$(4,4)$			$Q^{1/8}$	JLSV_1	$Q^{1/8}$	$E^8Q^{1/4}$	
$(4,4)$			$Q^{1/10}$	JLSV_2	$Q^{1/5}$	$E^8Q^{3/10}$	$\otimes$
$(4,5)$			$Q^{1/12}$	JLSV_2	$Q^{1/6}$	$E^9Q^{1/4}$	$\otimes$
$(4,6)$			$Q^{1/14}$	JLSV_2	$Q^{1/7}$	$E^{10}Q^{3/14}$	$\otimes$
$(4,7)$			$Q^{1/8}$	JLSV_2	$Q^{1/16}$	$E^{11}Q^{3/16}$	$\otimes$
$(5,6)$	p^5	$\mathbb{F}_{p^5}$	$O(1)$	GJL	$Q^{1/6}$	$E^{11}Q^{1/6}$	
$(6,7)$				GJL	$Q^{1/7}$	$E^{13}Q^{1/7}$	$\otimes$
$(5,10)$				Conj	$Q^{1/10}$	$E^{15}Q^{1/10}$	$\otimes$
$(5,5)$			$Q^{1/10}$	JLSV_1	$Q^{1/10}$	$E^{10}Q^{1/5}$	
$(5,5)$			$Q^{1/12}$	JLSV_2	$Q^{1/6}$	$E^{10}Q^{1/4}$	$\otimes$
$(5,6)$			$Q^{1/14}$	JLSV_2	$Q^{1/7}$	$E^{11}Q^{3/14}$	$\otimes$
$(5,7)$			$Q^{1/8}$	JLSV_2	$Q^{1/16}$	$E^{12}Q^{3/16}$	$\otimes$
$(5,8)$			$Q^{1/18}$	JLSV_2	$Q^{1/9}$	$E^{13}Q^{1/6}$	$\otimes$
$(6,7)$	p^6	$\mathbb{F}_{p^6}$	$O(1)$	GJL	$Q^{1/7}$	$E^{13}Q^{1/7}$	
$(7,8)$				GJL	$Q^{1/8}$	$E^{15}Q^{1/8}$	$\otimes$
$(6,12)$				Conj	$Q^{1/12}$	$E^{18}Q^{1/12}$	$\otimes$
$(6,6)$			$Q^{1/12}$	JLSV_1	$Q^{1/12}$	$E^{12}Q^{1/6}$	
$(6,6)$			$Q^{1/14}$	JLSV_2	$Q^{1/7}$	$E^{12}Q^{3/14}$	$\otimes$
$(6,7)$			$Q^{1/16}$	JLSV_2	$Q^{1/8}$	$E^{13}Q^{3/16}$	$\otimes$
$(6,8)$			$Q^{1/18}$	JLSV_2	$Q^{1/9}$	$E^{14}Q^{1/6}$	$\otimes$
$(6,9)$			$Q^{1/20}$	JLSV_2	$Q^{1/10}$	$E^{15}Q^{3/20}$	$\otimes$



(a) $Q = p^4, p^5, p^6$: JLSV$_1$ or GJL method



(b) $Q = p^2$ or p^3: Conjugation or GJL method

FIGURE 1. Estimation of (22) for various pairs $(\deg f, \deg g)$ selected with our two methods for computing discrete logarithms in $\mathbb{F}_{p^n}$ with $n \in \{2, 3, 4, 5, 6\}$.

7. Asymptotic complexity

The two new methods of polynomial selection require a dedicated analysis of complexity. First, we show that the generalized Joux-Lercier method offers an alternative to the existing method of polynomial selection in large characteristic [JLSV06] and determine the range of applicability in the boundary case. When getting close to the limit, it provides the best known complexity. Second, we analyze the conjugation method and obtain the result announced in the introduction, namely the existence of a family of finite fields for which the complexity of computing discrete logarithms is in $L_Q(1/3, \sqrt[3]{48/9})$.

7.1. **The generalized Joux-Lercier method.** Using the generalized Joux-Lercier method, one constructs two polynomials f and g such that, for a parameter $d \geq n$, we have $\deg f = d+1$, $\deg g = d$, $|g|_\infty = Q^{1/d}$ and $|f|_\infty$ is very small, say $O(\log Q)$.

We consider the variant of NFS in which one sieves on linear polynomials $a - bx$ such that $|a|, |b| \leq E$ for a sieve parameter E, in order to collect the pairs such that the norms $\mathrm{Res}(f, a-bx)$ and $\mathrm{Res}(g, a - bx)$ are B-smooth.

Since the cost of the sieve is $E^{2+o(1)}$ and the cost of the linear algebra stage is $B^{2+o(1)}$, we impose $E = B$. We set $E = B = L_Q(1/3, \beta)$ for a parameter β to be chosen. We write $d = \frac{\delta}{2} (\log Q / \log\log Q)^{1/3}$, for a parameter δ to be chosen.

Since the size of the sieving domain must be large enough so that we collect B pairs (a, b), we must have $\mathcal{P}^{-1} = B$, where $\mathcal{P}$ is the probability that a random pair (a, b) in the sieving domain has B-smooth norms. We make the usual assumption that the product of the norms of any pair (a, b) has the same probability to be B-smooth as a random integer of the same size. We upper-bound the norms product by

$$|\mathrm{Res}(f, a - bx)\,\mathrm{Res}(g, a - bx)| \leq (\deg f)|f|_\infty E^{\deg f} (\deg g)|g|_\infty E^{\deg g}, \tag{23}$$

and further, with the L-notation, we obtain

$$|\mathrm{Res}(f, a - bx)\,\mathrm{Res}(g, a - bx)| \leq L_Q\left(2/3, \delta\beta + \frac{2}{\delta}\right). \tag{24}$$

Using the Canfield-Erdös-Pomerance theorem, we obtain

$$\mathcal{P} = 1/L_Q\left(1/3, \frac{\delta}{3} + \frac{2}{3\beta\delta}\right). \tag{25}$$

The equality $\mathcal{P}^{-1} = B$ imposes

$$\beta = \frac{\delta}{3} + \frac{2}{3\beta\delta}. \tag{26}$$

The optimal value of δ is the one which minimizes the expression in the right hand member, so we take $\delta = \sqrt{2/\beta}$ and we obtain $\beta = 2/3\sqrt{2/\beta}$, or equivalently $\beta = \sqrt[3]{8/9}$. Since the complexity of NFS is $E^2 + B^2 = L_Q(1/3, 2\beta)$, we obtain

$$\text{complexity(NFS with Generalized Joux-Lercier)} = L_Q\left(1/3, \sqrt[3]{64/9}\right). \tag{27}$$

The method requires $n \leq d$. Since $d = \delta/2 \left(\frac{\log Q}{\log\log Q}\right)^{1/3}$ with $\delta = \sqrt{2/\beta} = \sqrt[3]{3}$, the method applies only to fields $\mathbb{F}_{p^n}$ such that

$$p \geq L_Q\left(2/3, \sqrt[3]{8/3}\right). \tag{28}$$

7.2. **The conjugation method.** The conjugation method allows us to construct two polynomials f and g such that $\deg f = 2n$, $\deg g = n$, $|g|_\infty \approx p^{1/2}$ and $|f|_\infty$ is very small, say $O(\log Q)$. We study first the case of medium characteristic and then the boundary case between medium and large characteristic. We start with those computations which are common for the two cases.

7.2.1. *Common computations.* We consider the higher degree variant of NFS of parameter t, i.e. one sieves on polynomials ϕ of degree $t-1$, with coefficients of absolute value less than $E^{2/t}$, where E is called the sieve parameter. The cost of the sieve is then $E^{2+o(1)}$. Since cost of the linear algebra stage is $B^{2+o(1)}$, where B is the smoothness bound, we impose $E = B$ and we write $E = B = L_Q(1/3, \beta)$, for some parameter β to be chosen. Then the product of the norms of $\phi(\alpha)$ and $\phi(\beta)$ for any polynomial ϕ in the sieve domain is

$$|\operatorname{Res}(\phi, f) \operatorname{Res}(\phi, g)| \leq (\deg f + t)!(\deg g + t)! E^{4n/t} |f|_\infty^{t-1} E^{2n/t} |g|_\infty^{t-1}.$$

Since $(\deg f + t)!(\deg f + t)! \leq L_Q(2/3, o(1))$, this factor's contribution will be negligible compared to the main term which is in $L_Q(2/3)$. Therefore we have

$$|\operatorname{Res}(\phi, f) \operatorname{Res}(\phi, g)| \leq \left(E^{6n/t} Q^{(t-1)/2n}\right)^{1+o(1)}.$$

We make the usual assumption that the norms product has the same probability to be B-smooth as a random integer of the same size.

7.2.2. *The medium characteristic case.* Let us set the value of the number of terms in the sieve:

$$t = c_t n \left(\frac{\log Q}{\log\log Q}\right)^{-1/3}. \tag{29}$$

The probability that a polynomial ϕ in the sieving domain has B-smooth norms is

$$\mathcal{P} = 1/L_Q\left(1/3, \frac{2\beta}{c_t} + \frac{c_t}{6}\right). \tag{30}$$

We choose $c_t = 2\sqrt{3\beta}$ in order to minimize the right hand member:

$$\mathcal{P} = 1/L_Q\left(1/3, 2\sqrt{\beta/3}\right). \tag{31}$$

In an optimal choice of parameters, the sieve produces just enough relations, so we require that $\mathcal{P}^{-1} = B$, and equivalently $\beta = \sqrt[3]{4/3}$. We obtain

$$\text{complexity}(NFS \text{ with medium char.}) = L_Q\left(1/3, \sqrt[3]{96/9}\right). \tag{32}$$

7.2.3. *The boundary case.* For every constant $c_p > 0$, we consider the family of finite fields $\mathbb{F}_{p^n}$ such that

$$p = L_{p^n}(2/3, c_p)^{1+o(1)}. \tag{33}$$

The parameter t is a constant, or equivalently we have a different algorithm for each value $t = 2, 3, \ldots$.

Then the probability that a polynomial ϕ in the sieving domain has B-smooth norms is

$$\mathcal{P} = 1/L_Q\left(1/3, \frac{2}{c_p t} + \frac{c_p(t-1)}{6\beta}\right). \tag{34}$$

If the parameters are tuned to have just enough relations in the sieve, then one has $\mathcal{P}^{-1} = B$. This leads to $\frac{2}{c_p t} + \frac{c_p(t-1)}{6\beta} = \beta$, or $\beta = \frac{1}{c_p t} + \sqrt{\frac{1}{(c_p t)^2} + \frac{1}{6} c_p(t-1)}$. Hence, the complexity of NFS with the conjugation method is:

$$\text{complexity}(\text{NFS with the conjugation method}) = L_Q\left(1/3, \frac{2}{c_p t} + \sqrt{\frac{4}{(c_p t)^2} + \frac{2}{3} c_p(t-1)}\right). \tag{35}$$

In Figure 2, we have plotted the complexities of various methods, including the Multiple number field sieve variant of [BP14]. There are some ranges of the parameter c_p where our conjugation method is the fastest and a range where the generalized Joux-Lercier method is the

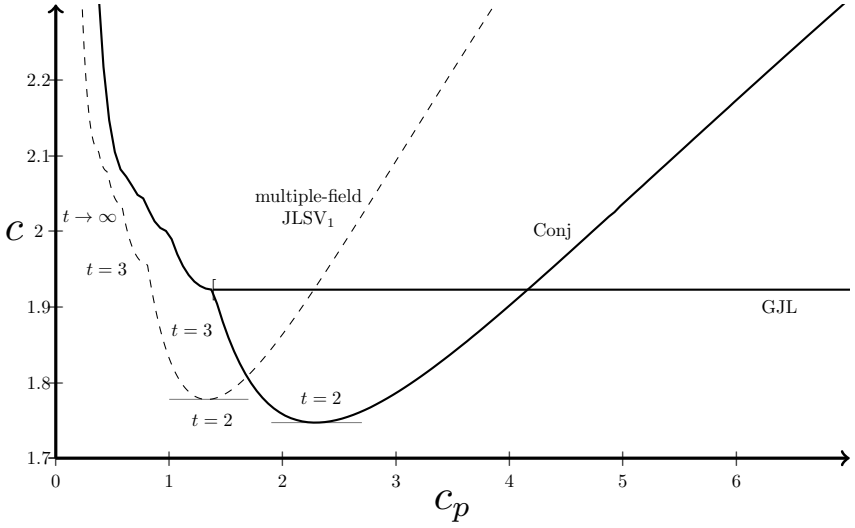


FIGURE 2. The complexity of NFS for fields $\mathbb{F}_{p^n}$ with $p = L_Q(2/3, c_p)$ is $L_Q(1/3, c)$. The blue curve corresponds to the multiple number field sieve of [BP14], the green semi-line to the generalized Joux-Lercier method and the red thick curve to the conjugation method.

fastest. The best case for our new method corresponds to the case where $c_p = 12^{1/3} \approx 2.29$ and $t = 2$. In that case we get:

$$\text{complexity(best case for the conjugation method)} = L_Q\left(1/3, \sqrt[3]{\frac{48}{9}}\right). \tag{36}$$

8. Effective computations of discrete logarithms

In order to test how our ideas perform in practice, we did a medium-sized practical experiment in a field of the form $\mathbb{F}_{p^2}$. Since we could not find any publicly announced computation for this type of field, we have decided to choose a prime number p of 80 decimal digits so that $\mathbb{F}_{p^2}$ has size 160 digits. To demonstrate that our approach is not specific to a particular form of the prime, we took the first 80 decimal digits of π. Our prime number p is the next prime such that $p \equiv 7 \bmod 8$ and both $p+1$ and $p-1$ have a large prime factor: $p = \lfloor \pi \cdot 10^{79} \rfloor + 217518$.

$$\begin{aligned}
p &= 31415926535897932384626433832795028841971693993751058209749445923078164063079607\\
\ell &= 39269908169872415480783042290993786052464617492188822762186807403847705078849 51\\
p-1 &= 6 \cdot h_0 \text{ with } h_0 \text{ a 79 digit prime}\\
p+1 &= 8 \cdot \ell
\end{aligned}$$

We tried to solve the discrete logarithm problem in the order ℓ subgroup. We imposed p to be congruent to -1 modulo 8, so that the polynomial $x^4 + 1$ could be used, as in Section 4.4.2, so that no Schirokauer map is needed. The conjugation method yields a polynomial g of degree 2 and negative discriminant, a particular case that requires no Schirokauer map either:

$$\begin{aligned}
f &= x^4 + 1\\
g &= 22253888644283440595423136557267278406930\ x^2\\
&\quad + 41388856349384521065766679356490536297931\ \ x\\
&\quad + 22253888644283440595423136557267278406930\ .
\end{aligned}$$

Since p is 80 digits long, the coefficients of g have almost 40 digits (precisely 41 digits). The polynomials f and g have the irreducible factor

$$\varphi(t) = t^2 + 882784365956656290081700417360106466084364666244465292158128917413749504096699 0\, t + 1$$

in common modulo p, and $\mathbb{F}_{p^2}$ will be taken as $\mathbb{F}_p[X]/(\varphi)$.

The relation collection step was then done using the sieving software of CADO [BFG+09]. More precisely, we used the special-$\mathfrak{q}$ technique for ideals $\mathfrak{q}$ on the g-side, since it produces norms that are larger than on the f-side. We sieved all the special-$\mathfrak{q}$ larger than $40,000,000$ and smaller than 2^{27}, keeping only one in each pair of conjugates, as explained in Section 3. In total, they produced about 15M relations. The main parameters in the sieve were the following: we sieved all primes below 40M, and we allowed two large primes less than 2^{27} on each side. The search space for each special-$\mathfrak{q}$ was set to $2^{15} \times 2^{14}$ (the parameter `I` in CADO was set to 15).

The total CPU time for this relation collection step is equivalent to 68 days on one core of an Intel Xeon E5-2650 at 2 GHz. This was run in parallel on a few nodes, each with 16 cores, so that the elapsed time for this step was a few days, and could easily be made arbitrary small with enough nodes.

The filtering step was run as usual, but we modified it to take into account the Galois action on the ideals: we selected a representative ideal in each orbit under the action $x \mapsto x^{-1}$, and rewrote all the relations in terms of these representatives only. This amounts just to keep track of sign-change, that has to be reminded when combining two relations during the filtering, and when preparing the sparse matrix for the sparse linear algebra step. The output of the filtering step was a matrix with $839,244$ rows and columns, having on average 83.6 non-zero entries per row.

Thanks to our choice of f and g, it was not necessary to add columns with Schirokauer maps. We used Jeljeli's implementation of Block Wiedemann's algorithm for GPUs [Jel14]. In fact, this was a small enough computation so that we did not distribute it on several cards: we used a non-blocked version. The total running time for this step was around 30.3 hours on an NVidia GTX 680 graphic card.

At the end of the linear algebra we know the virtual logarithms of almost all prime ideals of degree one above primes of at most 26 bits, and of some of those above primes of 27 bits. At this point we could test that the logs on the f-side were correct.

The last step is that of computing some individual logarithms. We used $G = t + 2$ as a generator for $\mathbb{F}_{p^2}$ and the following "random" element:

$$s = \lfloor(\pi(2^{264})/4)\rfloor t + \lfloor(\gamma \cdot 2^{264})\rfloor.$$

We started by looking for an integer e such that $z = s^e$, seen as an element of the number field of f, is smooth. After a few core-hours, we found a value of e such that $z = z_1/z_2$ with z_1 and z_2 splitting completely into prime ideals of at most 60 bits. With the lattice-sieving software of CADO-NFS, we then performed a "special-q descent" for each of these prime ideals. We remark that one of the prime ideals in z_1 was an ideal of degree 2 above 43, that had to be descended in a specific way, starting with a polynomial of degree 2 instead of 1. The total time for descending all the prime ideals was a few minutes. Finally, we found

$$\log_G(s) = 431724646474717499532141432099069517832607980262114471597315861099398586114668 \bmod \ell.$$

Verification scripts in various mathematical software are given in the NMBRTHRY announcement.

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