Second-order cone programming for rolling friction contact mechanics

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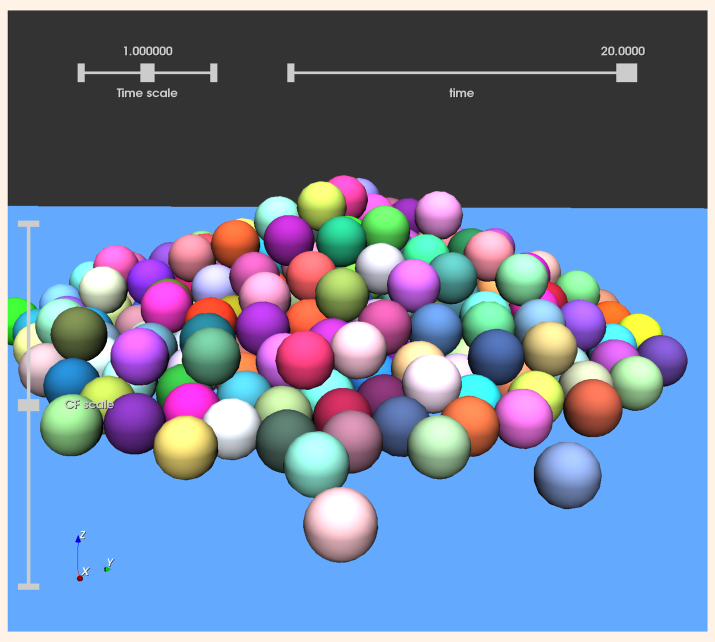
GENERAL FRICTION CONTACT PROBLEM



$$\begin{aligned} Mv + f - H^\top r &= 0 \\ Hv + w &= u \\ \mathcal{F}^* \ni u + \Phi(u) \perp r &\in \mathcal{F} \end{aligned}$$

where

- *Parameters:* n No. of contact points, m No. of degrees of freedom.
- *Given data:* $0 \prec M = M^\top \in \mathbb{R}^{m\times m}$, $H \in \mathbb{R}^{5n\times m}$, $f \in \mathbb{R}^m$, $w \in \mathbb{R}^{5n}$.
- *Convex problem:* $\Phi(u)$ is fixed.
- *Variables:* global velocity $v \in \mathbb{R}^m$, velocity $u \in \mathbb{R}^{5n}$, reaction $r \in \mathbb{R}^{5n}$.

ORIGINAL ROLLING FRICTION CONE

$$\mathcal{F} = \textstyle\prod_{i=1}^n \mathcal{R}_i := \{r_i = (r_{i0}, \bar{r}_i, \tilde{r}_i) \in \mathbb{R}^5 : \max\{\|\bar{r}_i\|_2, \|\tilde{r}_i\|_2\} \leq r_{i0}\}$$

$$\mathcal{F}^* = \textstyle\prod_{i=1}^n \mathcal{R}_i^* := \{u_i = (u_{i0}, \bar{u}_i, \tilde{u}_i) \in \mathbb{R}^5 : \|\bar{u}_i\|_2 + \|\tilde{u}_i\|_2 \leq u_{i0}\}$$

$$(\boldsymbol{P}) \quad \begin{aligned} &\min_{v,u} \tfrac{1}{2}v^\top Mv + f^\top v \\ &\text{s.t. } Hv + w = u \\ &\quad u \in \mathcal{F}^* \end{aligned} \qquad (\boldsymbol{D}) \quad \begin{aligned} &\max_{v,r} -\tfrac{1}{2}v^\top Mv - w^\top r \\ &\text{s.t. } Mv + f - H^\top r = 0 \\ &\quad r \in \mathcal{F} \end{aligned}$$

Since $\mathcal{F}$ is not self-dual, the Jordan algebra cannot be used as it is.

REFORMULATION WITH PRODUCT OF LORENTZ CONES

To simplify, set $n = 1$. By letting $u_0 = t + t'$, we obtain

$$(\boldsymbol{P'}) \quad \begin{aligned} &\min_{v,z} \tfrac{1}{2}v^\top Mv + f^\top v \\ &\text{s.t. } Hv + w = Jz \\ &\quad z \in \mathcal{L}^2 \end{aligned} \qquad (\boldsymbol{D'}) \quad \begin{aligned} &\max_{v,y} -\tfrac{1}{2}v^\top Mv - w^\top Ky \\ &\text{s.t. } Mv + f - H^\top Ky = 0 \\ &\quad y \in \mathcal{L}^2 \end{aligned}$$

where

$$J = \begin{pmatrix} 1 & 1 & \\ & I_2 & \\ & & I_2 \end{pmatrix}, \quad K = \begin{pmatrix} 1 & 0 & \\ & I_2 & \\ & & I_2 \end{pmatrix}, \quad z = \begin{pmatrix} t \\ \bar{u} \\ t' \\ \tilde{u} \end{pmatrix} \quad \text{and} \quad y = \begin{pmatrix} r_0 \\ \bar{r} \\ r_0 \\ \tilde{r} \end{pmatrix}.$$

Note that

- $Jz = u, \quad Ky = r$ and $KJ^\top = I$,
- $\mathcal{L}^2$ is self-dual, then Jordan algebra can be used.

NESTEROV & TODD (NT) SCALING

To get a symmetric and non-singular system, we need a change of variables at each iteration of the **Interior Point algorithm**. Define a vector p such that

$$\hat{z} := Q_p z = Q_{p^{-1}} y =: \check{y}$$

where Q_p is quadratic representation of p.

$$(\boldsymbol{P}_{NT}) \quad \begin{aligned} &\min_{v,\hat{z}} \tfrac{1}{2}v^\top Mv + f^\top v \\ &\text{s.t. } Hv + w = JQ_{p^{-1}}\hat{z} \\ &\quad \hat{z} \in \mathcal{L}^{2n} \end{aligned} \qquad (\boldsymbol{D}_{NT}) \quad \begin{aligned} &\max_{v,\check{y}} -\tfrac{1}{2}v^\top Mv - w^\top (KQ_p\check{y}) \\ &\text{s.t. } Mv + f - H^\top KQ_p\check{y} = 0 \\ &\quad \check{y} \in \mathcal{L}^{2n} \end{aligned}$$

SOLVER

The **pertubed KKT system** is

$$\begin{aligned} Mv + f - H^\top KQ_p\check{y} &= 0, \\ JQ_{p^{-1}}\hat{z} - Hv - w &= 0, \\ \hat{z} \circ \check{y} &= 2\mu e, \\ (\hat{z}, \check{y}) &\in \text{int}(\mathcal{L}^4). \end{aligned}$$

The **linear system** to solve at each iteration is of the form

$$\begin{bmatrix} M & -H^\top & 0 \\ -H & 0 & J \\ 0 & J^\top & Q_{p^2} \end{bmatrix} \begin{bmatrix} d^v \\ d^r \\ d^z \end{bmatrix} = - \begin{bmatrix} Mv + f - H^\top r \\ Jz - Hv - w \\ J^\top r - 2\mu z^{-1} \end{bmatrix}.$$

One difficulty is that Q_{p^2} is badly conditionned near the optimal solution.

EQUAL STEPLENGTHS

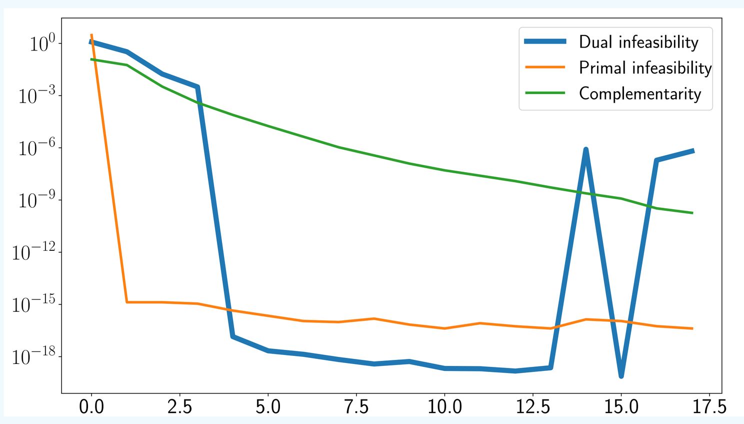


Figure 1. $n = 70$, *steplengths* $\alpha_{\text{P}} \neq \alpha_{\text{D}}$

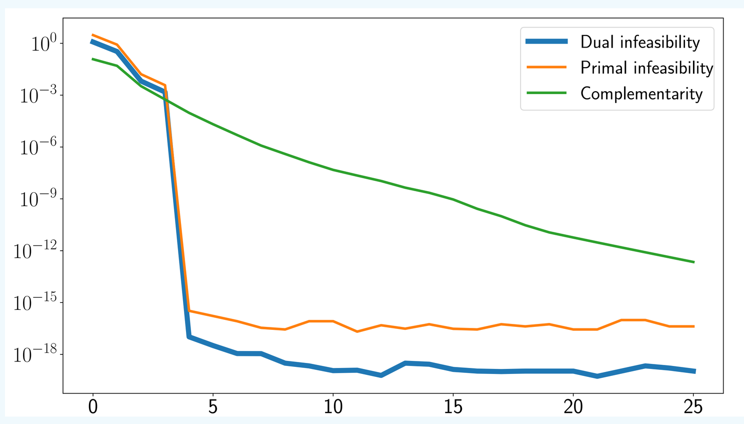


Figure 2. $n = 70$, *steplengths* $\alpha_{\text{P}} = \alpha_{\text{D}}$

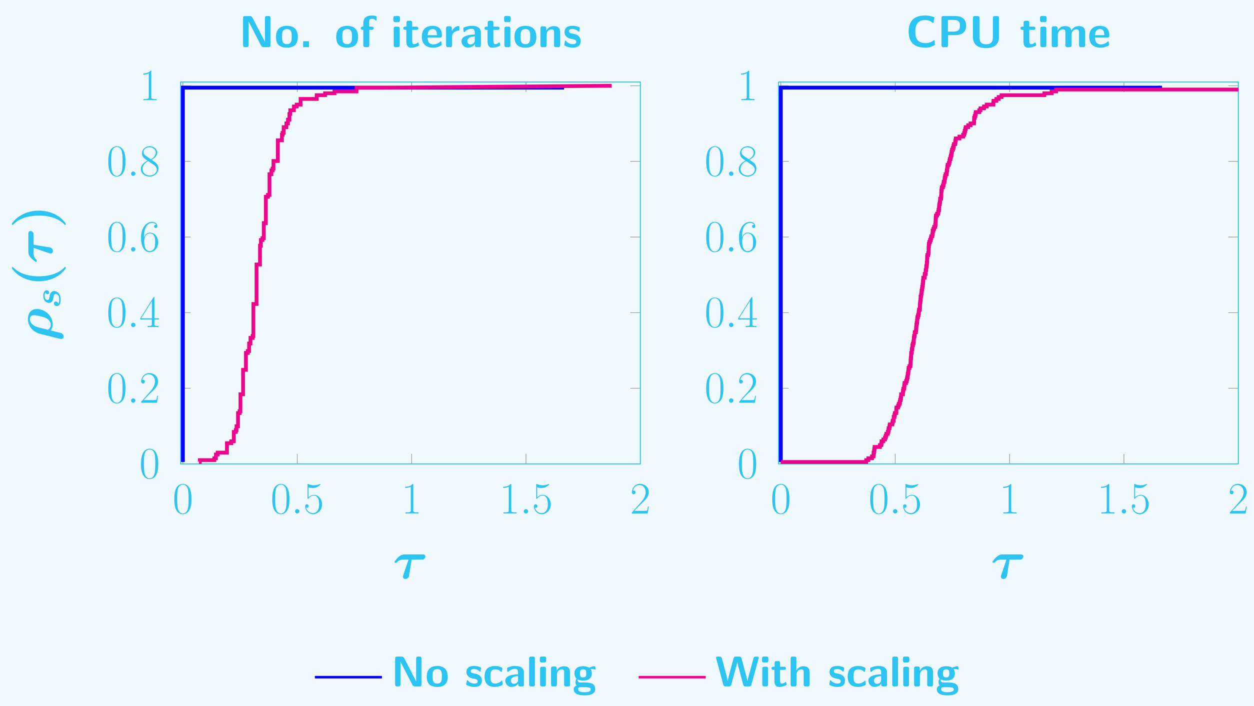
Primal-dual infeasibilities

$$\text{pinfeas}^+ = (1 - \alpha_{\text{P}})\text{pinfeas} + \alpha_{\text{P}}\xi_{\text{P}}$$

$$\text{dinfeas}^+ = (1 - \alpha_{\text{D}})\text{dinfeas} + \alpha_{\text{D}}\xi_{\text{D}} + (\alpha_{\text{D}} - \alpha_{\text{P}})Md^v$$

where $\xi_{\text{P}}, \xi_{\text{D}}$ are residual vectors.

EFFICIENCY & ROBUSTNESS



- 200 test problems
- $2 \leq n \leq 542$
- $24 \leq m \leq 1500$
- **tol** $= 10^{-12}$

Figure 3. *Performance profile*

THEORETICAL RESULT FOR FRICTION CONTACT MECHANICS WITHOUT ROLLING FRICTION

- For the case without rolling friction, the original friction cone $\mathcal{F}$ is the Lorentz cone $\mathcal{L}$. If $(\boldsymbol{P})$ is feasible, then it has a unique optimal solution (v^*, u^*). It is not the case for $(\boldsymbol{D})$, because H is rank deficient.
- If the **Slater's conditions** and the **strict complementarity** hold, there are 3 possible cases:

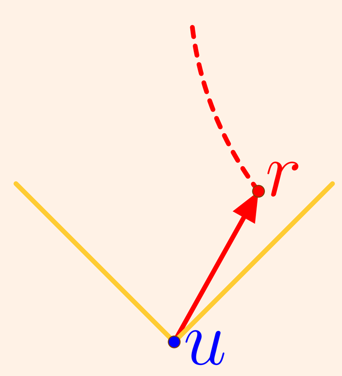


Figure 4. $u = 0$, $r \in$ *int*$(\mathcal{L})$

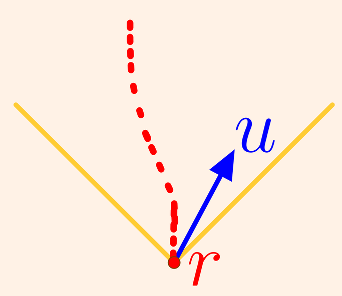


Figure 5. $u \in$ *int*$(\mathcal{L})$, $r = 0$

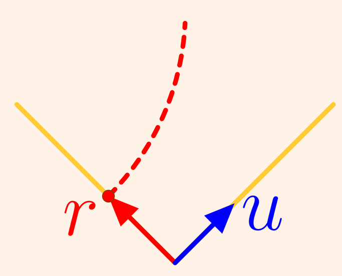


Figure 6. $u \in$ *bd*$(\mathcal{L})$, $r \in$ *bd*$(\mathcal{L})$

Then, the optimal solution r will converge to the **analytic center** of the optimal set $\widehat{R}$ which is the unique optimal solution of the problem

$$\min_{r \in \text{ri}(\widehat{\text{R}})} \psi(r) := -\sum_{i \in \text{B}} \log\det r_i - \sum_{i \in \text{R}} \log r_{i,0},$$

where $(\text{B}, \text{N}, \text{R})$ is a partition of $\{1, \ldots, n\}$.

- This result can be extended to the case of rolling friction.