# Optimal Sequential Gossiping by Short Messages

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**Abstract**

Gossiping is the process of information diffusion in which each node of a network holds a block that must be communicated to all the other nodes in the network. We consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number $p$ of blocks during each call. We study the minimum numbers of call necessary to perform gossiping among $n$ processor for any arbitrary fixed upper bound on the message size $p$.

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# 1 Introduction

Gossiping (also called total exchange or all–to–all communication) in interconnection networks is the process in which initially each processor has an item of information that must be distributed to every other processor of the system.

The gossiping problem was originally introduced by the community of discrete mathematicians, to which it owes most of its terminology, as a combinatorial problem in graphs. Nonetheless, it was soon realized that, once cast in more realistic models of communication, gossiping is a fundamental primitive in multiprocessor systems. Gossiping arises in a large class of parallel computation problems, such as linear system solving, matrix manipulation, and sorting, where both input and output data are required to be distributed across the network [6, 8, 16]. Due to the considerable practical relevance in parallel and distributed computation and the related interesting theoretical issues, gossiping has been extensively studied in the literature [9, 13, 15].

The great majority of the previous work on gossiping has considered the case in which the items of information known to a processor at any given time during the execution of the gossiping protocol can be freely concatenated and the resulting (longer) message can be transmitted in a constant amount of time, that is, it has been assumed that the time required to transmit a message is independent from its length. While this assumption is reasonable for short messages, it is clearly unrealistic in case the size of the messages becomes large. Notice that most of the gossiping protocols proposed in the literature require the transmission of messages of size $\Theta(n)$, where $n$ is the number of nodes in the network. Therefore, it would be interesting to have gossiping protocols that require only the transmission of bounded length messages between processors. In this paper we consider the problem of gossiping in communication networks under the restriction that communicating nodes can exchange up to a fixed number $p$ of items of information during each call.

## 1.1 The Model

We assume that the network is modelled by the complete graph and that the processors are labelled with the integers in the set $\{0, \ldots, n-1\}$.

*Gossiping:* Each processor $i$, with $0 \leq i \leq n-1$, has a block of data $B(i)$. The goal is to disseminate these blocks so that each processor gets all the blocks $B(0), \cdots, B(n-1)$.

The process is accomplished by means of a sequence of *calls* between processors. During each call, communicating nodes can exchange blocks they know. We assume that each processor can participate in at most one call at time. This communication model is usually referred to as *telephone model* or *Full–Duplex 1–Port* ($F_1$) [16, 13, 9, 17]. Another popular communication model is the *mail model* or *Half–Duplex 1–Port* ($H_1$) [16, 13, 9, 17], in which during each call any node can either send a message to one of its neighbors or receive a message from it but not simultaneously.

Furthermore, we add the condition that during each call communicating nodes can exchange up to $p$ blocks, where $p$ is an *a priori* fixed integer. The problem of estimating number of calls necessary for gossiping in the $H_1$ model has been considered in [2]. The problem of minimizing the time (number of rounds) to complete the gossiping process in the $H_1$ and $F_1$ model, has been considered in [4] and [3], respectively; analogous problems have been studied in [11, 10, 14, 12, 7, 5]. Packet routing in interconnection networks in the $F_1$ model has been considered in [1].

In this paper we give a lower bound on the minimum possible number of calls $\mathrm{c}(n, p)$ necessary to complete gossiping among $n$ nodes under the condition that during each call communicating nodes can exchange at most $p$ blocks.

Notice that if $n \leq p + 1$ then the bound on $p$ does not impose any actual restriction to the size of the message exchanged during the gossiping process and $\mathrm{c}(n, p) = 2n - 4$ (see [13]).
We also notice that $\mathrm{c}(n, 1) = n(n-1)/2$, for each $n \geq 2$; indeed if during each call communicating nodes can exchange at most one block then it is necessary for any node to receive each of the blocks of the other nodes during $n - 1$ different calls.
Therefore, in the following we will consider $p \geq 2$ and $n > p + 1$.

## 2 Lower Bounds

In this section we give lower bounds on the minimum possible number of calls $\mathrm{c}(n, p)$ necessary to complete gossiping among $n$ nodes under the condition that during each call communicating nodes can exchange at most $p$ blocks.

Since each node must receive $n - 1$ blocks, we have that the number of blocks sent during the process is at least $n(n - 1)$. Moreover, since initially each node knows only its own block we have that at least $n$ of the messages sent during the gossiping protocol can carry at most one block. That means that at least $n$ calls between two nodes can carry at most $p + 1$ blocks in both directions while all the remaining call can carry up to $2p$ blocks. Therefore, we have that $c(n, p)$ must satisfy $2p(c(n, p) - n)p + n(p + 1) \geq n(n - 1)$ which implies the following lower bound on $c(n, p)$

$$c(n, p) \geq \frac{n^2}{2p} + \frac{p - 2}{2p} n. \tag{1}$$

We give now a lower bound which improves on (1) for most of the values of $n$ and $p$.

**Theorem 2.1** *For each $p \geq 2$ and $n \geq p+2$, if $n = hp+k$, for some $h \geq 1$ and $2 \leq k \leq p+1$, then*

$$\mathrm{c}(n,p) \geq \frac{n^2}{2p} + \left(1 - \frac{k}{2p} - \frac{1}{2(k-1)}\right) n. \tag{2}$$

**Proof.** Fix any gossiping protocol $\mathcal{A}$ on nodes $0,\ldots,n$-1 that uses messages of size at most $p$ and denote by $\mathrm{c}(\mathcal{A})$ the number of calls made by $\mathcal{A}$. Denote by $a_i(x)$ the number of calls during which node $x$ receives exactly $i$ blocks, for each $0 \leq x \leq n-1$ and $1 \leq i \leq p$. Since the number of calls involving node $x$ is $\sum_{i=1}^{p} a_i(x)$ the total number of calls made by $\mathcal{A}$ is

$$2\mathrm{c}(\mathcal{A}) = \sum_{x=0}^{n-1} \sum_{i=1}^{p} a_i(x) \tag{3}$$

We want therefore to bound the sum in (3). The fact that each node must receive the blocks of each of the other $n-1$ nodes implies that

$$n-1 \leq \sum_{i=1}^{p} i a_i(x) \leq a_1(x) + p \sum_{i=2}^{p} a_i(x), \tag{4}$$

which implies

$$\sum_{i=1}^{p} a_i(x) \geq \left\lceil \frac{n-1-a_1(x)}{p} \right\rceil + a_1(x). \tag{5}$$

Let us now write for each node $x$

$$a_1(x) = t(x)p + j(x), \quad \text{with } t(x) \geq 0,\ 0 \leq j(x) < p. \tag{6}$$

Writing $n = hp+k$, for some $h \geq 1$ and $2 \leq k \leq p+1$, inequalities (5) and (6) give

$$\sum_{i=1}^{p} a_i(x) \geq \left\lceil \frac{hp+k-1-t(x)p-j(x)}{p} \right\rceil + a_1(x) \geq h + a_1(x) - t(x) + \left\lceil \frac{k-1-j(x)}{p} \right\rceil. \tag{7}$$

Recalling (6) we have

$$\sum_{i=1}^{p} a_i(x) \geq h + \frac{p-1}{p} a_1(x) + \frac{1}{p} j(x) + \left\lceil \frac{k-1-j(x)}{p} \right\rceil. \tag{8}$$

By (3) and (8) we obtain

$$2\mathrm{c}(\mathcal{A}) \geq \sum_{x=0}^{n-1} \sum_{i=1}^{p} a_i(x) \geq nh + \frac{p-1}{p} \sum_{x=0}^{n-1} a_1(x) + \frac{1}{p} \sum_{x=0}^{n-1} j(x) + \sum_{x=0}^{n-1} \left\lceil \frac{k-1-j(x)}{p} \right\rceil \tag{9}$$

with

$$\left\lceil \frac{k-1-j(x)}{p} \right\rceil = \begin{cases} 1 & \text{if } 0 \leq j(x) \leq k-2 \\ 0 & \text{if } k-1 \leq j(x) < p. \end{cases} \tag{10}$$

Denoting by $Z$ the set of nodes for which the left–end side of (10) is 0, that is,

$$Z = \left\{ x \mid 0 \leq x \leq n-1,\ \left\lceil \frac{k-1-j(x)}{p} \right\rceil = 0 \right\} = \{x \mid 0 \leq x \leq n-1,\ j(x) \geq k-1\} \tag{11}$$

from (9) we obtain that

$$2c(\mathcal{A}) \geq nh + \frac{p-1}{p}\sum_{x=0}^{n-1} a_1(x) + \frac{1}{p}\sum_{x=0}^{n-1} j(x) + n - |Z|.$$

Since $\sum_{x=0}^{n-1} j(x) \geq \sum_{x\in Z} j(x) \geq |Z|(k-1)$ we get

$$2c(\mathcal{A}) \geq nh + n + \frac{p-1}{p}\sum_{x=0}^{n-1} a_1(x) - \frac{p-k+1}{p}|Z|. \quad (12)$$

We want now a lower bound for $\sum_{x=0}^{n-1} a_1(x)$. Let us first observe that the first time each node is involved in a call it can send only its own block, therefore

$$\sum_{x=0}^{n-1} a_1(x) \geq n. \quad (13)$$

Moreover,

$$\sum_{x=0}^{n-1} a_1(x) \geq \sum_{x=0}^{n-1} j(x) \geq \sum_{x\in Z} j(x) \geq |Z|(k-1). \quad (14)$$

¿From (12), using (13) and (14) we get

$$2c(\mathcal{A}) \geq nh + n + \frac{1}{p}\max\{(p-1)n - (p-k+1)|Z|,\ |Z|(k-1)(p-1) - (p-k+1)|Z|\}. \quad (15)$$

Therefore,

$$2c(\mathcal{A}) \geq nh + n + \frac{1}{p}\min_{0\leq z\leq n}\max\{(p-1)n - (p-k+1)z,\ ((k-1)(p-1) - p + k - 1)z\}. \quad (16)$$

An easy computation gives

$$\frac{1}{p}\min_{0\leq z\leq n}\max\{(p-1)n - (p-k+1)z,\ ((k-1)(p-1) - p + k - 1)z\} = \left(1 - \frac{1}{k-1}\right)n$$

and this value is attained when $z = n/(k-1)$, for each $2 \leq k \leq p+1$. Therefore, by (16) we get

$$2c(\mathcal{A}) \geq nh + 2n - \left\lfloor \frac{n}{k-1} \right\rfloor.$$

Recalling that $h = (n-k)/p$ we get the desired bound. □

Notice that bounds (1) and (2) coincide for $k = 2$ and $k = p+1$.

## 3 Upper Bound

In this section we give a protocol to perform complete gossiping among $n$ nodes under the condition that during each call communicating nodes can exchange at most $p$ blocks.

The main result of this section will be the following theorem.

**Theorem 3.1** *There exist a protocol that performs gossiping among $n$ nodes with packets of size up to $k$ with at most*

$$\frac{n^2}{2p} + \left(1 - \frac{k}{2p} - \frac{1}{2(k-1)}\right) n + p$$

*calls.*

Notice that the difference between the number of calls required by the proposed protocol and the lower bound given in (2) is less than $p$, a value not depending on $n$, so the protocol is asymptotically optimal.

We describe now the protocol.
Again we write $n = hp + k$, for some $h > 0$ and $2 \leq k \leq p + 1$. Moreover, all the operations on the nodes $0, \ldots, n-1$ are intended modulo $n$. Finally, by saying that a node knows/sends $[a, b]$ we will mean that the node knows/sends the blocks of all the nodes $a, a+1, \ldots, b$ if $a \leq b$ and of the nodes $a, \ldots, n-1, 0, \ldots, b$ if $a > b$.

We design an almost optimal protocol as follows. In a first phase consisting of $n - 1$ calls each node is involved in at least one call. We design the calls in such a way that each node $i = p, \ldots, n-1$ knows either $[i-p, i]$ or $[i-p, i+k-1]$. Moreover the nodes knowing $p + k$ blocks are of the form $p - 1 + j(k-1)$ for some $j$.
In a second phase consisting of $p$ calls we extend the above property to all the nodes.
In a third phase the above property allows to make calls so that each node learns $p$ new blocks at each call. Finally, in a last phase we complete the protocol.

**First Phase.** For $t = 1, \ldots, p-1$ during the $t$–th call nodes $t-1$ and $t$ communicate exchanging all the blocks they know, that is, $t-1$ sends $[0, t-1]$ and receives $[t]$.
Therefore, at the end of call $p-1$ each node $i$, for $i = 0, \ldots, p-2$, knows $[0, i+1]$ and node $p-1$ knows $[0, p-1]$.

Let now $\Delta = \lfloor (n-1-p)/(k-1) \rfloor$ and consider integers $\delta$ and $\lambda$ such that

$$p + \delta(k-1) + \lambda \leq n - 1 \quad \text{with} \qquad 0 \leq \lambda \leq k-2; \tag{17}$$

this implies $0 \leq \delta \leq \Delta$.
During the call $p + \delta(k-1) + \lambda$ the node $p - 1 + \delta(k-1)$ communicates with node $p + \delta(k-1) + \lambda$ receiving the block $[p + \delta(k-1) + \lambda]$ and sending $[\delta(k-1) + \lambda, p + \delta(k-1) + \lambda - 1]$.

According to (17) this first phase consists of $n - 1$ calls. At the end of this phase we have:

- for $i = 0, \ldots, p-2$, node $i$ knows $[0, i+1]$;
- node $p-1$ knows $[0, p+k-2]$;

- node $p-1+\delta(k-1)$, with $0<\delta<\Delta$ knows $[\delta(k-1)-1, p+\delta(k-1)+k-2]$

  (namely it knows the $p$ blocks $[\delta(k-1)-1, p-1+\delta(k-1)-1]$ learned from $p-1+\delta(k-1)$, its own block, and the $k-1$ blocks learned during the calls with $p+\delta(k-1)+\lambda$, for $0 \leq \lambda \leq k-2$);

- node $p-1+\Delta(k-1)$, knows between $p+2$ and $p+k$ blocks according to the values of $k$ and $n$, namely $[\Delta(k-1)-1, n-1]$.

- node $p+\delta(k-1)+\lambda$, with $0 \leq \lambda \leq k-2$ and $p+\delta(k-1)+\lambda \leq n-1$, knows the $p+1$ blocks $[\delta(k-1)+\lambda, p+\delta(k-1)+\lambda]$.

Notice that for each $i \geq p$ we have two types of nodes:

- nodes of type 1 that know at least $[i-p, i]$

- nodes of type k that know $[i-p, i+k-1]$;

  at the end of Phase 1 the nodes of type k are all the nodes $p-1+\delta(k-1)$, for $1 \leq \delta < \Delta$.

**Second Phase.** The first call is between $p-1$ and $p-1+\Delta(k-1)$; node $p-1$ receives the block $[n-1]$ and sends $[0, \ldots, p-1+(\Delta+1)(k-1)-n]$; this is possible since $p+\Delta(k-1) \leq n-1$ and $k \leq p+1$. After this call node $p-1$ and $p-1+\Delta(k-1)$ are both of type k.
Now node $n-1$ will communicate with each of the nodes $i=0,\ldots,p-2$ as follows:

**Case 1** If $i \not\equiv p-2 \pmod{(k-1)}$, then $n-1$ sends $[i-p+1, -1]$ to $i$ and so $i$ knows $[i-p+1, i+1]$;

**Case 2** If $i \equiv p-2 \pmod{(k-1)}$ and $i \geq k-2$, then $n-1$ sends $[i-(k+p-2), -1]$ to $i$ and so $i$ knows $[i-(k+p-2), i+1]$;

**Case 3** If $i \equiv p-2 \pmod{(k-1)}$ and $i < k-2$, then $n-1$ sends $[-p, -1]$ to $i$ and so $i$ knows at least $[-p, 0]$.

Node $n-1$ receives all the blocks of $i=0,\ldots,p-2$. Therefore, after the above calls also the node $n-1$ will know $p+k$ blocks and it will be of type k.

We can now relabel the nodes $i=0,\ldots,p-2$ in the following way

If $i$ satisfies Case 1, the it is relabeled $i+1$;

If $i$ satisfies Case 2, the it is relabeled $i+1-(k-1)$;

If $i$ satisfies Case 3, the it is relabeled $0$;

Therefore, at the end of the second phase, with the above relabeling of the nodes we have $\lceil n/(k-1)\rceil$ nodes of type $k$, that is all the nodes $i$ with $i \equiv p-1 \pmod{(k-1)}$ and the node $n-1$. Moreover,

each node of type 1 knows the $p+1$ blocks $[i-p, i]$;

each node of type k knows the $p+k$ blocks $[i-p, i+k-1]$.

**Third Phase.** It consists of $h' = \lfloor (h-1)/2 \rfloor$ steps. During each step each node $i$, $0 \leq i \leq n-1$, communicates with $i+p+1$. Therefore, during each step there are $n$ calls and each node is involved in 2 of them.

We organize the calls in such a way that at the end of each step $s$:

1) each node of type 1 knows the $(2s+1)p+1$ blocks $[i-(s+1)p, i+sp]$;

2) each node of type k knows the $(2s+1)p+k$ blocks $[i-(s+1)p, i+sp+k-1]$.

We prove now by induction that it is possible to keep 1) and 2) at each step. The property is trivially true at the beginning of third phase for $s=0$. Suppose now that it is true for some $s < h'$. Consider step $s+1$ and let $i$ and $i+p+1$ communicate. Always $i$ sends $[i-(s+1)p+1, i-sp]$. Since $s+1 \leq h'$ these $p$ blocks are unknown to $i+p+1$.
Notice that we perform the calls doing first, in any order, the calls for which $i$ is of type 1. In that case $i$ can receive $[i+sp+1, i+(s+1)p]$ from $i+p+1$; by inductive hypothesis these $p$ blocks are known to $i+p+1$ but unknown to $i$.
After we do, in any order, the calls for which $i$ is of type k. In that case $i$ can receive $[i+sp+k\ , i+(s+1)p+k-1]$ from $i+p+1$. By inductive hypothesis and as result of the previous calls of step $s+1$, these $p$ blocks are known to $i+p+1$ but unknown to $i$. Therefore, 1) and 2) hold.

Considering the above properties 1) and 2) for $s = h'$ we get that at the end of the third phase

each node of type 1 knows the $(2h'+1)p+1$ blocks $[i-(h'+1)p, i+h'p]$;

each node of type k knows the $(2h'+1)p+k$ blocks $[i-(h'+1)p, i+h'p+k-1]$.

**Last Phase.** We distinguish two cases.
<u>Case 1</u>. $h$ is odd and $h' = (h-1)/2$.
In such a case the nodes of type k know $hp+k=n$ blocks, that is, all the blocks.
A node $i$ of type 1 knows $hp+1$ blocks, in particular it knows all blocs but $[i+\frac{h-1}{2}p+1, i+\frac{h-1}{2}p+k-1]$. Therefore if two nodes $i$ and $j$ such that $\max\{i,j\}-\min\{i,j\} \geq k+1$ communicate

they can send each other the missing blocks. Therefore, since $n \geq p + k$, we can organize calls among the the nodes of type 1 so to end the protocol with $\lceil (n - \lceil n/(k-1) \rceil)/2 \rceil$ calls.

Case 2. $h$ is even and $h' = h/2 - 1$.

We first organize calls between nodes of type k. A node $i$ knows all the blocks but $[i + \frac{h-2}{2}p + k + 1, i + \frac{h}{2}p + k - 1]$. Therefore if two nodes $i$ and $j$ such that $\max\{i,j\} - \min\{i,j\} \geq p$ communicate they can send each other the missing blocks. Therefore, since $n \geq 2p + 2$ we can organize calls among the the nodes of type k so to let each of them know all the blocks with $\lceil \lceil n/(k-1) \rceil /2 \rceil$ calls.

For the nodes of type 1 let us label them from 0 to $n - \lceil n/(k-1) \rceil - 1$ and let node $i$ call node $i + p - \lfloor (p+1)/(k-1) \rfloor$. Each node is involved in 2 calls. This implies each node can learn the $p + k - 1$ blocks it does know yet. To this aim we need $n - \lceil n/(k-1) \rceil$ calls to end the protocol.

We count now the total number of calls made by the above protocol. We have

- $n - 1$ calls during the first phase
- $p$ calls during the second phase
- $h'n = \lfloor (h-1)/2 \rfloor n$ calls during the third phase
- $\lceil (n - \lceil n/(k-1) \rceil)/2 \rceil$ or $\lceil \lceil n/(k-1) \rceil /2 \rceil + n - \lceil n/(k-1) \rceil$ calls during the last phase according to $h$ odd or even.

Therefore for the third and last phase we have

$$\frac{h-1}{2}n + \lceil (n - \lceil n/(k-1) \rceil)/2 \rceil \leq \frac{h-1}{2}n + \frac{n}{2} - \frac{n}{2(k-1)} + 1 = \frac{h}{2}n - \frac{n}{2(k-1)} + 1$$

if $h$ is odd and

$$\frac{h-2}{2}n + n - \lfloor \lceil n/(k-1) \rceil /2 \rfloor \leq \frac{h-2}{2}n + n - \frac{n}{2(k-1)} + 1 = \frac{h}{2}n - \frac{n}{2(k-1)} + 1$$

if $h$ is even.

Therefore, recalling that $h = (n-k)/p$, the number of calls made by the protocol is at most

$$n - 1 + p + \frac{h}{2}n - \frac{n}{2(k-1)} + 1 = \frac{n^2}{2p} + \left(1 - \frac{k}{2p} - \frac{1}{2(k-1)}\right) n + p.$$

**Example 3.1** *Let us consider $p = 4$ and $n = 16$; this implies $k = 4$. The following table gives the knowledge of each node after the first phase. The $*$ indicates the nodes of type k.*

| | |
|---|---|
| 0 | [0,1] |
| 1 | [0,2] |
| 2 | [0,3] |
| 3 | [0,6] |
| 4 | [0,4] |
| 5 | [1,5] |
| 6* | [2,9] |
| 7 | [3,7] |
| 8 | [4,8] |
| 9* | [5,12] |
| 10 | [6,10] |
| 11 | [7,11] |
| 12* | [8,15] |
| 13 | [9,13] |
| 14 | [10,14] |
| 15 | [11,15] |

*The following table gives the interval of blocks known by each node after the second phase before and after the relabeling, respectively.*

| | |
|---|---|
| 0 | [0,1][13,15] |
| 1 | [0,2][14,15] |
| 2 | [0,3][12,15] |
| 3 | [0,6][15] |
| 4 | [0,4] |
| 5 | [1,5] |
| 6 | [2,9] |
| 7 | [3,7] |
| 8 | [4,8] |
| 9 | [5,12] |
| 10 | [6,10] |
| 11 | [7,11] |
| 12 | [8,15] |
| 13 | [9,13] |
| 14 | [10,14] |
| 15 | [11,15][0,2] |

| | |
|---|---|
| 0* | [12,3] |
| 1 | [13,1] |
| 2 | [14,2] |
| 3* | [15,6] |
| 4 | [0,4] |
| 5 | [1,5] |
| 6* | [2,9] |
| 7 | [3,7] |
| 8 | [4,8] |
| 9* | [5,12] |
| 10 | [6,10] |
| 11 | [7,11] |
| 12* | [8,15] |
| 13 | [9,13] |
| 14 | [10,14] |
| 15* | [11,2] |

*The following table gives the interval of blocks known by each node after each step of the third phase after the call of the nodes of type 1 and of type k.*

| | |
|---|---|
| 0* | [12,3][8,11] |
| 1 | [13,1][2,5] |
| 2 | [14,2][3,6][10,13] |
| 3* | [15,6][11,5] |
| 4 | [0,4][5,8] |
| 5 | [1,5][6,9] |
| 6* | [2,9][14,1] |
| 7 | [3,7][8,11][15,2] |
| 8 | [4,8][9,12] |
| 9* | [5,12][1,4] |
| 10 | [6,10][11,14][2,5] |
| 11 | [7,11][12,15] |
| 12* | [8,15][4,7] |
| 13 | [9,13][14,1][5,8] |
| 14 | [10,14][15,2] |
| 15* | [11,2][7,10] |

| | |
|---|---|
| 0* | [12,3][8,11][4,7] |
| 1 | [13,1][2,5][9,12] |
| 2 | [14,2][3,6][10,13] |
| 3* | [15,6][11,5][7,10] |
| 4 | [0,4][5,8][12,15] |
| 5 | [1,5][6,9][13,0] |
| 6* | [2,9][14,1][10,13] |
| 7 | [3,7][8,11][15,2] |
| 8 | [4,8][9,12][0,3] |
| 9* | [5,12][1,4][13,0] |
| 10 | [6,10][11,14][2,5] |
| 11 | [7,11][12,15][3,6] |
| 12* | [8,15][4,7][0,3] |
| 13 | [9,13][14,1][5,8] |
| 14 | [10,14][15,2][6,9] |
| 15* | [11,2][7,10][3,6] |

=

| | |
|---|---|
| 0* | ALL |
| 1 | [9,5] |
| 2 | [10,6] |
| 3* | ALL |
| 4 | [12,8] |
| 5 | [13,9] |
| 6* | ALL |
| 7 | [15,11] |
| 8 | [0,12] |
| 9* | ALL |
| 10 | [2,14] |
| 11 | [3,15] |
| 12* | ALL |
| 13 | [5,1] |
| 14 | [6,2] |
| 15* | ALL |

*In the last phase the calls that complete the protocol are between nodes*

*1 and 8, 2 and 10, 4 and 11, 5 and 13, 7 and 14.*

**Example 3.2** *Let us consider* $p = 9$ *and* $n = 22$*; this implies* $k = 4$*. The following table gives the knowledge of each node after the first phase. The* $*$ *indicates the nodes of type* $k$*.*

| | |
|---|---|
| 0 | [0,1] |
| 1 | [0,2] |
| 2 | [0,3] |
| 3 | [0,4] |
| 4 | [0,5] |
| 5 | [0,6] |
| 6 | [0,7] |
| 7 | [0,8] |
| 8 | [0,11] |
| 9 | [0,9] |
| 10 | [1,10] |
| 11* | [2,14] |
| 12 | [3,12] |
| 13 | [4,13] |
| 14* | [5,17] |
| 15 | [6,15] |
| 16 | [7,16] |
| 17* | [8,20] |
| 18 | [9,18] |
| 19 | [10,19] |
| 20 | [11,21] |
| 21 | [12,21] |

*The following table gives the interval of blocks known by each node after the second phase before and after the relabeling, respectively. The relabeling gives to node* $i = 0, \ldots, 7$ *the new label* $\ell(i)$ *with* $\ell(0) = 1$, $\ell(1) = 0$, $\ell(2) = 3$, $\ell(3) = 4$, $\ell(4) = 2$, $\ell(5) = 6$, $\ell(6) = 7$, $\ell(7) = 5$.

| | |
|---|---|
| 0 | [0,1][14,21] |
| 1 | [0,2][13,21] |
| 2 | [0,3][16,21] |
| 3 | [0,4][17,21] |
| 4 | [0,5][15,21] |
| 5 | [0,6][19,21] |
| 6 | [0,7][20,21] |
| 7 | [0,8][18,21] |
| 8 | [0,11][21] |
| 9 | [0,9] |
| 10 | [1,10] |
| 11* | [2,14] |
| 12 | [3,12] |
| 13 | [4,13] |
| 14* | [5,17] |
| 15 | [6,15] |
| 16 | [7,16] |
| 17* | [8,20] |
| 18 | [9,18] |
| 19 | [10,19] |
| 20 | [11,21] |
| 21 | [12,21] |

| | |
|---|---|
| 0 | [0,2][13,21] |
| 1 | [0,1][14,21] |
| 2 | [0,5][15,21] |
| 3 | [0,3][16,21] |
| 4 | [0,4][17,21] |
| 5 | [0,8][18,21] |
| 6 | [0,6][19,21] |
| 7 | [0,7][20,21] |
| 8 | [0,11][21] |
| 9 | [0,9] |
| 10 | [1,10] |
| 11 | [2,14] |
| 12 | [3,12] |
| 13 | [4,13] |
| 14 | [5,17] |
| 15 | [6,15] |
| 16 | [7,16] |
| 17 | [8,20] |
| 18 | [9,18] |
| 19 | [10,19] |
| 20 | [11,21] |
| 21 | [12,21][0,2] |

→

| | |
|---|---|
| 0 | [13,0] |
| 1 | [14,1] |
| 2* | [15,5] |
| 3 | [16,3] |
| 4 | [17,4] |
| 5* | [18,8] |
| 6 | [19,6] |
| 7 | [20,7] |
| 8* | [21,11] |
| 9 | [0,9] |
| 10 | [1,10] |
| 11* | [2,14] |
| 12 | [3,12] |
| 13 | [4,13] |
| 14* | [5,17] |
| 15 | [6,15] |
| 16 | [7,16] |
| 17* | [8,20] |
| 18 | [9,18] |
| 19 | [10,19] |
| 20* | [11,21] |
| 21* | [12,2] |

*The following table gives the interval of blocks known by each node after each step of the third phase after the call of the nodes of type 1 and after the calls of nodes of type k. The calls between nodes of type k are between nodes*

*2 and 14, 5 and 17, 8 and 20, 14 and 21.*

*Relabelling the nodes of type 1 with labels from 0 to 13, we can organize calls among them so that we have*

| | |
|---|---|
| 0 | [13,0][1,9][4,12] |
| 1 | [14,1][2,10][5,13] |
| 2 | [16,3][4,12][7,15] |
| 3 | [17,4][5,13][6,16] |
| 4 | [19,6][7,15][10,18] |
| 5 | [20,7][8,16][11,19] |
| 6 | [0,9][10,18][13,21] |
| 7 | [1,10][11,19][0,14] |
| 8 | [3,12][13,21][16,2] |
| 9 | [4,13][14,0][17,3] |
| 10 | [6,15][16,2][19,5] |
| 11 | [7,16][17,3][20,6] |
| 12 | [9,18][19,5][6,8] |
| 13 | [10,19][20,6][7,9] |

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