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DIFFERENTIATION OF AN ADDITIVE INTERVAL MEASURE WITH VALUES IN A CONJUGATE BANACH SPACE

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Dedicated to Lech Drewnowski on
the occasion of his 70th birthday

Abstract: We present a complete characterization of finitely additive interval measures with values in conjugate Banach spaces which can be represented as Henstock-Kurzweil-Gelfand integrals. If the range space has the weak Radon-Nikodým property (WRNP), then we precisely describe when these integrals are in fact Henstock-Kurzweil-Pettis integrals.

Keywords: Kurzweil–Henstock integral, Pettis integral, variational measure.

1. Notations and preliminaries.

Let $[0,1]$ be the unit interval of the real line equipped with the usual topology and the Lebesgue measure λ. We denote by $\mathcal{I}$ the family of all nontrivial closed subintervals of $[0,1]$, by $\mathcal{L}$ the family of all Lebesgue measurable subsets of $[0,1]$ and by $\mathcal{L}^+$ the family of all Lebesgue measurable subsets of $[0,1]$ of positive measure.

If $E \subset [0,1]$, then its Lebesgue measure is denoted by $|E|$ or $\lambda(E)$. Throughout X is a Banach space with its dual X^*. The closed unit ball of X is denoted by $B(X)$. A mapping $\nu\colon \mathcal{L} \to X$ is said to be an X-*valued measure* if ν is countably additive in the norm topology of X. If μ is a positive measure on $\mathcal{L}$ or an X-valued measure, then by $\mu \ll \lambda$ we mean that $|E| = 0$ implies $\mu(E) = 0$. We say then that μ is λ-*continuous*. The *variation* of an X-valued measure ν is denoted by $|\nu|$.

$\tau(X^*, X)$ is the Mackey topology on X^* and $\tau_c(X^*, X)$ is the topology of uniform convergence on compact subsets of X. It is known (cf. [12]) that $\tau_c(X^*, X)$ coincides on $B(X^*)$ with the weak*-topology $\sigma(X^*, X)$.

A *partition in* $[0,1]$ is a finite collection of pairs $\mathcal{P} = \{(I_1, t_1), \dots, (I_p, t_p)\}$, where $I_1, \dots, I_p$ are non-overlapping subintervals of $[0,1]$ and $t_i \in I_i$, for all $i \leqslant p$. Given a subset E of $[0,1]$, we say that the partition $\mathcal{P}$ is *anchored on* E if $t_i \in E$

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2010 Mathematics Subject Classification: primary: 28B20; secondary: 26A39, 28B05, 46G10, 54C60 for each $i = 1, ..., p$. If $\cup_{i=1}^{p} I_i = [0,1]$ we say that $\mathcal{P}$ is a *partition of* $[0,1]$. A *gauge* on $E \subset [0,1]$ is a positive function on E. For a given gauge δ, we say that a partition $\{(I_1,t_1),\ldots,(I_p,t_p)\}$ is *δ-fine* if $I_i \subset (t_i - \delta(t_i), t_i + \delta(t_i))$, $i = 1,\ldots,p$.

Given two real numbers a, b, we denote by the symbol $< a, b >$ the interval $[\min\{a,b\}, \max\{a,b\}]$.

Definition 1.1. A function $f\colon [0,1] \to \mathbb{R}$ is said to be *Henstock-Kurzweil integrable*, or simply *HK-integrable*, on $[0,1]$ if there exists $w \in \mathbb{R}$ with the following property: for every $\epsilon > 0$ there exists a gauge δ on $[0,1]$ such that

$$\left| \sum_{i=1}^{p} f(t_i)|I_i| - w \right| < \varepsilon ,$$

for each δ-fine partition $\mathcal{P} = \{(I_1,t_1),\ldots,(I_p,t_p)\}$ of $[0,1]$.
We set $(HK)\int_0^1 f d\lambda := w$. By $HK[0,1]$ is denoted the set of all HK-integrable functions $f\colon [0,1] \to \mathbb{R}$.

It is well known that if $f \in HK[0,1]$ then f is HK-integrable on each $I \in \mathcal{I}$. We call the additive interval function $F(I) := (HK)\int_I f d\lambda$ the *HK-primitive* of f.

Definition 1.2. A function $f\colon [0,1] \to X$ is said to be *scalarly Henstock-Kurzweil integrable* if, for each $x^* \in X^*$, the function x^*f is Henstock-Kurzweil integrable. A scalarly Henstock-Kurzweil integrable function f is said to be *Henstock-Kurzweil-Pettis integrable* (or simply *HKP-integrable*) if for each $I \in \mathcal{I}$ there exists $w_I \in X$ such that

$$\langle x^*, w_I \rangle = \int_I \langle x^*, f(t) \rangle \, dt , \qquad \text{for every } \; x^* \in X^*.$$

We call w_I the *Henstock-Kurzweil-Pettis integral* of f over I and we write $w_I := (HKP)\int_a^b f(t)\, dt$.

We denote by $HKP([0,1],X)$ the set of all X-valued Henstock-Kurzweil-Pettis integrable functions on $[0,1]$ (functions that are scalarly equivalent are identified).

Definition 1.3. A function $f\colon [0,1] \to X^*$ is said to be *w^*-scalarly Henstock-Kurzweil integrable* if, for each $x \in X$, the function xf is Henstock-Kurzweil integrable. A w^*-scalarly Henstock-Kurzweil integrable function $f\colon [0,1] \to X^*$ is said to be *Henstock-Kurzweil-Gelfand integrable* (or simply *HKG-integrable*) if, for each interval $I \in \mathcal{I}$, there exists a vector $\Psi(I) \in X^*$ such that for every $x \in X$

$$\langle x, \Psi(I) \rangle = (HK)\int_I \langle x, f(t) \rangle \, dt .$$

We call $\Psi(I)$ the *Henstock-Kurzweil-Gelfand integral* of f over I and we write $\Psi(I) := (HKG)\int_I f(t)\, dt$. Ψ is called the *HKG-primitive* of f.

Following the proof of [9, Theorem 3] (with suitable changes), it is easy to see that a function $f\colon [0,1] \to X^*$ is *HKG*-integrable if and only if f is w^*-scalarly Henstock-Kurzweil integrable.

Throughout, we identify a function $\Psi : [0,1] \to X$ (resp. $\Psi : [0,1] \to X^*$) with the additive interval function $\Psi : \mathcal{I} \to X$ (resp. $\Psi : \mathcal{I} \to X^*$) defined by $\Psi(I) = \Psi(b) - \Psi(a)$, if $I = [a,b]$. And conversely, with each $\Psi : \mathcal{I} \to X$, (resp. $\Psi : \mathcal{I} \to X^*$) we associate $\Psi : [0,1] \to X$ (resp. $\Psi : [0,1] \to X^*$) by setting $\Psi(t) = \Psi([0,t])$.

Definition 1.4. A function $f\colon [0,1] \to X$ is said to be *scalarly* measurable (*scalarly* integrable) if, for each $x^* \in X^*$, the function x^*f is Lebesgue measurable (integrable). A scalarly integrable function $f\colon [0,1] \to X$ is said to be *Pettis integrable* if, for each set $A \in \mathcal{L}$ there exists a vector $\nu_f(A) \in X$ such that for every $x^* \in X^*$

$$\langle x^*, \nu_f(A)\rangle = \int_A \langle x^*, f(t)\rangle \, dt\,.$$

We call $\nu_f(A)$ the *Pettis integral of f* over A and we write $\nu_f(A) := (P)\int_A f(t)\, dt$. It is known (see [15]) that $\nu_f : \mathcal{L} \to X$ is a measure of σ-finite variation.

Definition 1.5. A function $f\colon [0,1] \to X^*$ is said to be w^*-*scalarly* measurable (resp. w^*-*scalarly* integrable) if, for each $x \in X$, the function xf is Lebesgue measurable (resp. integrable). It is well known that each w^*-scalarly integrable function $f\colon [0,1] \to X^*$ is *Gelfand integrable*, that is, for each set $A \in \mathcal{L}$, there exists a vector $\nu(A) \in X^*$ such that

$$\langle x, \nu(A)\rangle = \int_A \langle x, f(t)\rangle \, dt\,,$$

for every $x \in X$.

We call the set function $\nu\colon \mathcal{L} \to X^*$ the *Gelfand integral of f* on $[0,1]$ and we write $\nu(A) := (G)\int_A f(t)\, dt$.

Definition 1.6. A function $f\colon [0,1] \to X^*$ is said to be *weak*-scalarly bounded on E* if

$$\exists M > 0\ \forall x \in B(X)\ |\langle x, f\rangle| \leqslant M \qquad \text{a.e on } E.$$

A function $f\colon [0,1] \to X$ is said to be *scalarly bounded on E*, if it is weak*-scalarly bounded, when considered as an X^{**}-valued function.

Definition 1.7. Let $\Phi : [0,1] \to X$ be a function. If there is a function $\Phi'_p : [0,1] \to X$ such that for each $x^* \in X^*$

$$\lim_{h \to 0} \frac{x^*(\Phi < t, t+h >)}{|h|} = x^*(\Phi'_p(t))\ ,$$

for almost all $t \in [0,1]$ (the exceptional sets depend on x^*), then Φ is said to be *pseudo-differentiable on* $[0,1]$, with *pseudo-derivative* Φ'_p (see [16], p. 300).

Let $\Phi : [0,1] \to X^*$ be a function. If there is a function $\Phi'_p : [0,1] \to X^*$ such that for each $x \in X$

$$\lim_{h\to 0} \frac{x(\Phi < t, t+h >)}{|h|} = x(\Phi'_p(t)) \ ,$$

for almost all $t \in [0,1]$ (the exceptional sets depend on x^*), then Φ is said to be *w^*-pseudo-differentiable on* $[0,1]$, with *w^*-pseudo-derivative* Φ'_p.

2. Variational measures.

Definition 2.1. Given an additive interval function $\Phi \colon \mathcal{I} \to X$, a gauge δ and a set $E \subset [0,1]$ we define

$$\mathrm{Var}(\Phi, \delta, E) = \sup \left\{ \sum_{i=1}^p \|\Phi(I_i)\| : \begin{array}{c} \{(I_i,t_i) : \ i = 1,...,p\} \ \delta\text{−fine} \\ \text{partition anchored on } E \end{array} \right\}$$

if $E \neq \emptyset$ and $\mathrm{Var}(\Phi, \delta, \emptyset) = 0$. Then we set

$$\mathrm{V}_\Phi(E) = \inf\{\mathrm{Var}(\Phi, \delta, E) : \delta \text{ is a gauge on } E\}$$

if $E \neq \emptyset$ and $\mathrm{V}_\Phi(\emptyset) = 0$.

We call V_Φ the *variational measure generated by* Φ. V_Φ is known to be a metric outer measure in $[0,1]$ (see [17]). In particular, V_Φ restricted to Borel subsets of $[0,1]$ is a measure. We say that V_Φ is absolutely continuous with respect to λ (we write then $V_\Phi \ll \lambda$), if $\lambda(E) = 0$ yields $V_\Phi(E) = 0$, for all $E \in \mathcal{L}$. Notice that if $V_\Phi \ll \lambda$, then given $\varepsilon > 0$ and $\emptyset \neq E \in \mathcal{L}$ with $|E| = 0$, there exists a gauge δ such that $Var(\Phi, \delta', E) < \varepsilon$, for every $\delta' \leqslant \delta$.

If Φ is continuous, then $V_\Phi(I) \leqslant |\Phi|(I)$ for every $I \in \mathcal{I}$, where

$$|\Phi|(I) = \sup \left\{ \sum_{i=1}^p \|\Phi(I_i)\| : I_i \quad \text{are non-overlapping subintervals of } I \right\} .$$

We would like to remark that if Φ is discontinuous the inequality $V_\Phi(I) \leqslant |\Phi|(I)$ may fail. As an example consider Φ on $[0,1]$ defined in the following way: $\Phi(t) = 1$ for $t \in [0,1/2)$, $\Phi(t) = 0$ for $t \in [1/2,1]$. Φ is not continuous, and $V_\Phi([1/2,1]) = 1 > |\Phi|([1/2,1]) = 0$.

Moreover we say that a variational measure V_Φ is *σ-finite* if there is a sequence of (pairwise disjoint) sets F_n covering $[0,1]$ and such that $V_\Phi(F_n) < \infty$, for every $n \in \mathbb{N}$.

By a result of Thomson (see [17, Theorem 3.15]) it follows that the sets F_n in the previous definition can be taken from $\mathcal{L}$.

We recall that a function $\Phi : [0,1] \to X$ is said to be BV_* on a set $E \subseteq [0,1]$ if $\sup \sum_{i=1}^n \omega(\Phi(J_i)) < +\infty$, where the supremum is taken over all finite collections $\{J_1, ..., J_n\}$ of non overlapping intervals in $\mathcal{I}$ with end-points in E, and the symbol $\omega(\Phi(J))$ stands for $\sup\{\|\Phi(u) - \Phi(z)\| : \ u, \ z \in J\}$. The function Φ is said to be BVG_* on $[0,1]$ if $[0,1] = \bigcup_n E_n$ and Φ is BV_* on each E_n.

In the following we will use the following results proved in [2].

Proposition 2.2. *Let $\Phi\colon \mathcal{I} \to X$ be an additive interval function.*

1. *If $V_\Phi \ll \lambda$, then Φ is continuous on $[0,1]$ and V_Φ is σ-finite.*
2. *V_Φ is σ-finite if and only if Φ is BVG_* on $[0,1]$.*

In case of a separable Banach space X and Φ being an HKP-integral we are able to describe the variational measure V_Φ more precisely. Our result generalizes a well known fact for real valued functions.

Proposition 2.3. *Assume that X is a separable Banach space, $\Phi : \mathcal{I} \to X$ is additive and*

$$\Phi(I) = (HKP) \int_I f(t)\, dt\,.$$

If $V_\Phi \ll \lambda$, then

$$V_\Phi(E) = \int_E ||f||\, dt\,, \qquad \textit{for every} \;\; E \in \mathcal{L}.$$

Proof. By Proposition 2.2, V_Φ is σ-finite and so Φ is a BVG^* function. Moreover, by [13, Theorem 9], for each measurable set E, we have

$$V_\Phi(E) = \int_E |\overline{D}|\Phi(t)\, dt$$

where the symbol $|\overline{D}|\Phi(t)$ denotes the upper absolute derivative of Φ in t, that is

$$|\overline{D}|\Phi(t) = \limsup_{h \to 0} \frac{||\Phi < t, t+h > ||}{|h|}.$$

Let us observe that since Φ is the HKP-primitive of f, then f is a pseudo-derivative of Φ. Now, since X is separable, then by a result in an unpublished paper of Gordon [11] (see also [13]), Φ is differentiable a. e. on $[0,1]$ with derivative f. So $|\overline{D}|\Phi(t) = ||f||$ a.e. on $[0,1]$ and this completes the proof. ■

Question 2.4. Do we have always $V_\Phi(E) = \int_E \|f(t)\|\, dt$ or $V_\Phi(E) \leqslant \int_E \|f\|\, d\lambda$, for every $E \in \mathcal{L}$, if the function $\|f\|$ is measurable?

Besides the above variational measure we define the following two outer measures, introduced for technical reasons only:

$$W^w_\Phi(E) = \sup_{x^* \in B(X^*)} V_{x^*\Phi}(E), \qquad \text{if } \; \Phi : \mathcal{I} \to X$$

and

$$W^*_\Phi(E) = \sup_{x \in B(X)} V_{x\Phi}(E), \qquad \text{if } \; \Phi : \mathcal{I} \to X^*.$$

In general, the two outer measures are not metric and not all Borel subsets of $[0,1]$ are measurable with respect to them.

Let us observe that if $\Phi : \mathcal{I} \to X^*$ is an additive interval function, then by the definitions of variational measures we have:

$$W^*_\Phi(E) \leqslant W^w_\Phi(E) \leqslant V_\Phi(E) \tag{1}$$

for every $E \subset [0,1]$. In fact, for every $I \in \mathcal{I}$, $x^{**} \in B(X^{**})$ and $x \in B(X)$, we have: $|x^{**}\Phi(I)| \leqslant ||\Phi(I)||$ and $|x\Phi(I)| \leqslant ||\Phi(I)||$. So $V_{x^{**}\Phi}(E) \leqslant V_\Phi(E)$, $V_{x\Phi}(E) \leqslant V_\Phi(E)$ and inequalities (1) follow.

Definition 2.5. Let V be one of the above introduced outer measures and let $AV := \{\frac{V(E)}{|E|} : |E| > 0\}$ be the *average range of* V. We say that AV is *locally bounded* if there are sets $E_n \in \mathcal{L}$ such that $|\bigcup_n E_n| = 1$ and $V(E_n \cap E) \leqslant n|E_n \cap E|$, for every $n \in \mathbb{N}$ and $E \in \mathcal{L}$.

Proposition 2.6. *Let $\Phi\colon \mathcal{I} \to X$. If $V_\Phi \ll \lambda$, then AV_Φ is locally bounded.*

Proof. By Proposition 2.2 we have that V_Φ is σ-finite. Since $V_\Phi|_\mathcal{L}$ is a measure, applying the Radon-Nikodým Theorem, we conclude that AV_Φ is locally bounded. ■

Remark 2.7. Assume that

$$\Phi(I) = (HKP)\int_I f(t)\,dt\,.$$

In general V_Φ is neither σ-finite nor absolutely continuous. In fact, if V_Φ is σ-finite, then by Proposition 2.2, Φ is a BVG_* function. So, if X has the RNP, then Φ is a.e. differentiable (see [1, Theorem 3.6]). But by a result in [5] we know that in each infinite dimensional Banach space (in particular in a conjugate space with the RNP) there exist strongly measurable Pettis (and then Henstok-Kurzweil-Pettis) integrable functions whose Pettis integrals are nowhere differentiable. Each such a function is HKP-integrable and induces a non-σ-finite variational measure V_Φ.

In the general case the following characterization holds.

Proposition 2.8. *A function $\Phi : [0,1] \to X$ is an HKP-primitive (of a function f) if and only if $W^w_\Phi \ll \lambda$ and Φ is pseudo-differentiable (with pseudo-derivative f).*

Proof. The proof follows at once from the characterization of the primitives of real valued *HK*-integrable functions (see [3]). ■

3. Henstock-Kurzweil-Gelfand integral.

The following result gives a full description of X^*-valued additive interval measures that can be represented as an HKG-integral.

Theorem 3.1. *An additive function $\Phi : \mathcal{I} \to X^*$ is an HKG-primitive if and only if $W^*_\Phi \ll \lambda$ and AW^*_Φ is locally bounded.*

Proof. Assume first that $f\colon [0,1] \to X^*$ is HKG-integrable and let $\Phi(I) = (HKG)\int_I f(t)\,dt$, for every $I \in \mathcal{I}$. Since $xf \in HK[0,1]$ for every $x \in X$, we have $V_{x\Phi} \ll \lambda$, and so also $W_\Phi^* \ll \lambda$. Moreover, according [14, Corollary 3.1] there are pairwise disjoint sets $E_n \in \mathcal{L}$ such that $\bigcup_n E_n = [0,1]$ and $|xf\chi_{E_n}| \leqslant n$ a.e., for each $x \in B(X)$ (the exceptional sets depend on x). It follows that every $f\chi_{E_n}$ is Gelfand integrable.

According to [4] and [6] we have also

$$V_{x\Phi}(E \cap E_n) = \int_{E\cap E_n} |xf(t)|\,dt \leqslant n|E \cap E_n|\|x\|$$

for every $E \in \mathcal{L}$ and $n \in \mathbb{N}$. Hence $W_\Phi^*(E \cap E_n) \leqslant n|E \cap E_n|$ and consequently AW_Φ^* is locally bounded.

Assume now that $W_\Phi^* \ll \lambda$ and AW_Φ^* is locally bounded. Then $V_{x\Phi} \ll \lambda$ for every $x \in X$. According to [3], for every $x \in B(X)$, let $f_x \in HK[0,1]$ be such that

$$\langle x, \Phi(I)\rangle = (HK)\int_I f_x(t)\,dt \qquad \text{for every} \;\; I \in \mathcal{I}\,.$$

Let ρ be a lifting on $L_\infty[0,1]$. Since AW_Φ^* is locally bounded, there are pairwise disjoint sets $E_n = \rho(E_n) \in \mathcal{L}$ such that $|\bigcup_n E_n| = 1$ and

$$W_\Phi^*(E_n \cap E) \leqslant n|E_n \cap E|, \qquad \text{for every} \;\; n \in \mathbb{N} \;\; \text{and} \;\; E \in \mathcal{L}. \tag{2}$$

According to [4] and [6], then

$$V_{x\Phi}(E) = \int_E |f_x(t)|\,dt \qquad \text{for every} \;\; E \in \mathcal{L} \;\; \text{and} \;\; x \in X. \tag{3}$$

In particular (3) holds true for measurable $E \subseteq E_n$. It follows from (2) and (3) that for every $n \in \mathbb{N}$ and $x \in B(X)$ we have $|f_x|\chi_{E_n} \leqslant n\chi_{E_n}$, a.e. In particular

$$|\rho(f_x)|(t)\chi_{E_n}(t) = \rho(|f_x|)(t)\chi_{E_n}(t) \leqslant n \quad \text{for every } t \in [0,1]\,, x \in B(X) \text{ and } n \in \mathbb{N}.$$

Define now a function $f : [0,1] \to X^*$ by setting for each $x \in X$

$$\langle x, f(t)\rangle = \begin{cases} \rho(f_x)(t)\chi_{E_n}(t) & \text{if} \;\; t \in E_n \\ 0 & \text{if} \;\; t \notin \bigcup_n E_n \end{cases}$$

For each $t \in E_n$ the function $x \longrightarrow \langle x, f(t)\rangle$ is linear and $|\langle x, f(t)\rangle| \leqslant n\|x\|$. If $t \notin \bigcup_n E_n$, then $f(t) = 0$. It follows that $f(t) \in X^*$, for every t.

Since $\langle x, f\rangle \overset{a.e.}{=} f_x \in HK[0,1]$, we get the representation

$$\langle x, \Phi(I)\rangle = (HK)\int_I \langle x, f(t)\rangle\,dt \qquad \text{for every} \;\; I \in \mathcal{I}\,. \tag{4}$$

of Φ as an HKG-integral of f. ■

It follows from the construction of f that it is w^*-scalarly bounded, hence Gelfand integrable on every E_n. It is a consequence of lifting measurability properties that $\|f\|$ is measurable on every E_n, and so on $[0,1]$.

If X^* has the WRNP, then according to [14, Proposition 12.3] and [14, Corollary 3.1.], f is Pettis integrable and scalarly bounded on each E_n. Thus, we can formulate the following consequence of the proof of Theorem 3.1:

Corollary 3.2. *Assume that $\Phi : \mathcal{I} \to X^*$ is an HKG-primitive. Then there exists a function $f : [0,1] \to X^*$ such that f is a weak*-pseudo-derivative of Φ and there exists a sequence of pairwise disjoint sets $E_n \in \mathcal{L}$ such that $\bigcup_n E_n = [0,1]$, f is weak*-scalarly bounded and Gelfand integrable on every E_n, $n \in \mathbb{N}$, $AW^*_\Phi(E_n) < \infty$ and $\|f\|$ is measurable.*

If X^* has the WRNP, then f and the sets E_n $n \in \mathbb{N}$ can be taken in such a way that f is Pettis integrable and scalarly bounded on each E_n.

If $V_\Phi \ll \lambda$, then by Proposition 2.6, AV_Φ is locally bounded. Consequently, in view of (1), AW^*_Φ is locally bounded. Thus, the following result is a direct consequence of Theorem 3.1.

Proposition 3.3. *Let $\Phi : \mathcal{I} \to X^*$ be additive and such that $V_\Phi \ll \lambda$. Then Φ is an HKG-primitive.*

4. Henstock-Kurzweil-Pettis integral.

We begin with the following characterization of Pettis integrability that holds true in case of an arbitrary perfect measure in place of the Lebesgue one.

Proposition 4.1. *For a scalarly integrable function $f : [0,1] \to X$ the following conditions are equivalent:*

1. *f is Pettis integrable;*
2. *the mapping $X^* \ni x^* \longrightarrow x^*f \in L_1[0,1]$ is $\tau_c(X^*,X)$-norm continuous;*
3. *the mapping $X^* \ni x^* \longrightarrow x^*f \in L_1[0,1]$ is $\tau(X^*,X)$-norm continuous.*

Proof. $(i) \Rightarrow (ii)$ Since f is Pettis integrable, the functional $x^* \longrightarrow \int_E \langle x^*, f(t)\rangle\, dt$ is, for each $E \in \mathcal{L}$, weak*-continuous (cf. [14]). Due to Stegall's result [8], the set $\nu_f(\mathcal{L})$ is norm relatively compact. Hence, if $x^*_\alpha \overset{\tau_c(X^*,X)}{\longrightarrow} x^*_0$, then $x^*_\alpha \longrightarrow x^*_0$ uniformly on $\nu_f(\mathcal{L})$. It follows that $\lim_\alpha \int_0^1 |x^*_\alpha f(t) - x^*_0 f(t)|\, dt = 0$.

$(i) \Rightarrow (iii)$ The proof is almost the same.

$(iii) \Rightarrow (i)$ If $x^*_\alpha \overset{\tau(X^*,X)}{\longrightarrow} x^*_0$, then $\int_E \langle x^*_\alpha, f(t)\rangle\, dt \longrightarrow \int_E \langle x^*_0, f(t)\rangle\, dt$ for each $E \in \mathcal{L}$. Thus, the functional $x^* \longrightarrow \int_E \langle x^*, f(t)\rangle\, dt$ is, for each $E \in \mathcal{L}$, weak*-continuous. Consequently, f is Pettis integrable (see [14]).

$(ii) \Rightarrow (i)$ The proof is the same, but now we assume that $B(X^*) \ni x^*_\alpha \overset{\sigma(X^*,X)}{\longrightarrow} x^*_0$. We obtain now the weak* continuity of the functionals $x^* \longrightarrow \int_E \langle x^*, f(t)\rangle\, dt$ on $B(X^*)$, but due to the Banach-Dieudonné Theorem (see [12, p. 154]) this yields its weak* continuity. Consequently, f is Pettis integrable (see [14]). ∎

In order to obtain a complete characterization of the HKP-primitive of functions taking values in a dual space with the WRNP, we need some preliminary results.

Proposition 4.2. *Assume that* $\Phi : \mathcal{I} \to X$ *is of the form*

$$\Phi(I) = (HKP)\int_I f(t)\,dt\,, \qquad \text{for each } I \in \mathcal{I}.$$

Then, for each $I \in \mathcal{I}$*, the mapping* $x^* \longrightarrow \int_I \langle x^*, f(t)\rangle\,dt$ *is weak***-continuous. Moreover, there exists a partition* $[0,1] = \bigcup_k H_k$ *such that, for every* $k \in \mathbb{N}$*,* f *is Pettis integrable and scalarly bounded on* H_k*,* $AW^w_\Phi(H_k) < \infty$ *and the functional* $x^* \longrightarrow V_{x^*\Phi}(H_k)$ *is* $\tau_c(X^*, X)$*-continuous.*

Proof. The first continuity fact has been proven in [7]. Exactly as in the proof of Theorem 3.1 one can obtain a sequence of pairwise disjoint sets $E_n \in \mathcal{L}$ such that $AW^w_\Phi(E_n) < \infty$, for each $n \in \mathbb{N}$. It follows also from [7, Corollary 1] that there exists a decomposition $[0,1] = \bigcup_k F_k$ into sets of positive measure such that f is Pettis integrable and scalarly bounded on each F_k. Denote by $\{H_k : k \in \mathbb{N}\}$ the collection of all intersections $E_n \cap F_m$ of positive measure. Then, by Proposition 4.1, for each k, the function $x^* \longrightarrow x^* f|_{H_k}$ is $\tau_c(X^*, X)$-norm continuous as a map from X^* to $L_1(\lambda|_{H_k})$, because f is Pettis integrable on H_k. Consequently, if $x^*_\alpha \stackrel{\tau_c(X^*,X)}{\longrightarrow} x^*_0$, then according to [4] and [6] we have

$$\lim_\alpha V_{(x^*_\alpha - x^*_0)\Phi}(H_k) = \lim_\alpha \int_{H_k} |x^*_\alpha f(t) - x^*_0 f(t)|\,dt = 0. \qquad \blacksquare$$

Lemma 4.3. *(see [1, Lemma 3.3]) Let* X *be a Banach space and let* $\nu : \mathcal{L} \to Y$ *be a* λ*-continuous measure of finite variation. If* $\Phi\colon \mathcal{I} \to X$ *is defined by* $\Phi(I) := \nu(I)$*, for all* $I \in \mathcal{I}$*, then* V_Φ *is finite,* $V_\Phi \ll \lambda$ *and* $V_\Phi(E) \leqslant |\nu|(E)$*, whenever* $E \in \mathcal{L}$*.*

Theorem 4.4. *Let* X *be a Banach space. Consider the following two properties of an additive interval function* $\Phi : \mathcal{I} \to X$*:*

- (k) $W^w_\Phi \ll \lambda$ *and there exists a decomposition* $[0,1] = \bigcup_k H_k$ *of* $[0,1]$ *into sets of positive measure such that for every* $k \in \mathbb{N}$ *the function* $x^* \longrightarrow V_{x^*\Phi}(H_k)$ *is* $\tau(X^*, X)$*-continuous and* $AW^w_\Phi(H_k) < \infty$*.*
- (kk) *There is an HKP-integrable function* $f : [0,1] \to X$ *such that*

$$\langle x^*, \Phi(I)\rangle = (HK)\int_I \langle x^*, f(t)\rangle\,dt \qquad \text{for every } I \in \mathcal{I}\,.$$

If (k) $\Rightarrow$ *(kk) for every additive* $\Phi : \mathcal{I} \to X$*, then* X *has the WRNP.*

Proof. Let $\nu : \mathcal{L} \to X$ be a λ-continuous measure of finite variation. Define $\Phi\colon \mathcal{I} \to X$ by $\Phi(I) := \nu(I)$. It follows from Lemma 4.3 that $V_\Phi \ll \lambda$ and V_Φ is finite. So $\Phi : \mathcal{I} \to X$ is an additive interval measure such that $V_{x^*\Phi} \ll \lambda$ for every $x^* \in X^*$. Moreover, $V_{x^*\Phi}(E) \leqslant |x^*\nu|(E)$, for every $E \in \mathcal{L}$. Let $\langle x^*_\alpha\rangle \subset B(X^*)$ be a net of functionals that is $\tau(X^*, X)$-convergent to 0. Since $\nu(\mathcal{L})$ is a weakly relatively compact subset of X, the net $\langle x^*_\alpha \nu \rangle$ is uniformly convergent to zero on $\mathcal{L}$. Hence, $\lim_\alpha |x^*_\alpha \nu|[0,1] = 0$. By the inequality $V_{x^*_\alpha \Phi}(E) \leqslant |x^*_\alpha \nu|(E)$, for every $E \in \mathcal{L}$, we have also $\lim_\alpha V_{x^*_\alpha \Phi}[0,1] = 0$, what proves the weak*-continuity of the map $x^* \to V_{x^*\Phi}[0,1]$.

We are going to prove yet the local boundedness of W^w_Φ. To do it notice that the classical Radon-Nikodým Theorem yields the existence of a decomposition $[0,1] = \bigcup_k H_k$ such that $|\nu|(E) \leqslant k|E|$, for every measurable $E \subset H_k$. It follows that

$$\frac{V_{x^*\Phi}(E)}{|E|} \leqslant \frac{|x^*\nu|(E)}{|E|} \leqslant k$$

and hence $AW^w_\Phi(H_k) < \infty$.

Thus, condition (k) is satisfied. Hence, there is a Henstock-Kurzweil-Pettis integrable function $f : [0,1] \to X$ such that

$$\Phi(I) = (HKP) \int_I f(t)\, dt\,, \qquad \text{for every } \ I \in \mathcal{I}.$$

Proceeding as in the proof of [2, Theorem 4.5] we see that f is also Pettis integrable and ν is its indefinite Pettis integral. ∎

Proposition 4.5. *Let X be an arbitrary Banach space and $\Phi : \mathcal{I} \to X$ be an additive interval function such that $W^w_\Phi \ll \lambda$. Assume that there is a decomposition $[0,1] = \bigcup_k H_k$ into measurable sets of positive measure such that $V_{x^*\Phi}(H_k) < \infty$ for every $k \in \mathbb{N}$ and every $x^* \in X^*$ and, for every $k \in \mathbb{N}$, the function $x^* \to V_{x^*\Phi}(H_k)$ is sequentially weak*-continuous.*

If $f\colon [0,1] \to X$ is a scalarly measurable function, then the set

$$K = \left\{ x^* \in X^* : x^*f \in HK[0,1] \ \text{ and } \ x^*\Phi(I) = (HK) \int_I \langle x^*, f(t)\rangle\, dt,\ \forall\ I \in \mathcal{I} \right\}$$

is sequentially weak-closed.*

If for every $k \in \mathbb{N}$, the function $x^* \to V_{x^*\Phi}(H_k)$ is $\tau(X^*, X)$-continuous and f is Pettis integrable on H_k, then K is weak-closed.*

Proof. It is obvious that $K \neq \emptyset$ and K is convex. Notice first that if $x^* \in K$, then $(x^*\Phi)' = x^*f$ a.e. (see [10]). Let $\{x^*_n\} \subset K$ be such that $x^*_n \to x^*_0$ in the w^*-topology. We may assume, without loss of generality, that all x^*_n, $n = 0, 1, 2, ...$ belong to $B(X^*)$. By hypothesis $V_{x^*_0\Phi} \ll \lambda$, and so there exists $g \in HK[0,1]$ such that $x^*_0\Phi(I) = (HK) \int_I g(t)\, dt$, for all $I \in \mathcal{I}$ (cf. [3]).

By the assumption and by [6, Corollary 3] we have, for each $k \in \mathbb{N}$,

$$\lim_n \int_{H_k} |x^*_n f(t) - g(t)|\, dt = \lim_n V_{(x^*_n - x^*_0)\Phi}(H_k) = 0.$$

Hence, there is a subsequence $\{x^*_{k,n_m}\}_m$ of $\{x^*_n\}$ with $\lim_m x^*_{k,n_m} f = g$, a.e. on H_k. It follows that $g = x^*_0 f$ a.e. and so $x^*_0 f \in HK[0,1]$. Moreover

$$\lim_m \int_I \langle x^*_{k,n_m}, f(t)\rangle\, dt = \lim_m \langle x^*_{k,n_m}, \Phi(I)\rangle = \langle x^*_0, \Phi(I)\rangle = \int_I \langle x^*_0, f(t)\rangle\, dt\,.$$

This yields $x^*_0 \in K$ and so K is weak* sequentially closed.

Assume now that f is Pettis integrable on every H_k. We are going to prove that K is weak*-closed. We know that for each $k \in \mathbb{N}$ the function $x^* \longrightarrow x^* g|_{H_k}$ is $\tau(X^*, X)$-norm continuous as a map from X^* to $L_1(\lambda|_{H_k})$. Consequently, if $x^*_\alpha \overset{\tau(X^*,X)}{\to} x^*_0$, then

$$\lim_\alpha \int_{H_k} |x^*_\alpha f(t) - x^*_0 f(t)|\, dt = 0.$$

By hypothesis $V_{x^*_0 \Phi} \ll \lambda$, and so there exists $g \in HK[0,1]$ such that $x^*_0 \Phi(I) = (HK) \int_I g(t)\, dt$, for all $I \in \mathcal{I}$ and so [6, Corollary 3] we have

$$\lim_\alpha \int_{H_k} |x^*_\alpha f(t) - g(t)|\, dt = \lim_\alpha V_{(x^*_\alpha - x^*_0)\Phi}(H_k) = 0.$$

It follows that $x^*_0 f = g \in HK[0,1]$. Moreover

$$\lim_\alpha \int_I \langle x^*_\alpha, f(t)\rangle\, dt = \lim_\alpha \langle x^*_\alpha, \Phi(I)\rangle = \langle x^*_0, \Phi(I)\rangle = \int_I \langle x^*_0, f(t)\rangle\, dt$$

and so $x^*_0 \in K$. Thus, K is $\tau(X^*, X)$-closed, and as it is convex, it is also weak*-closed. ■

Now we are ready to prove the main result of this section.

Theorem 4.6. *Let X be a Banach space such that X^* has the WRNP and let $\Phi : \mathcal{I} \to X^*$ be an additive interval measure. Then the following two conditions are equivalent:*

(j) *$W^w_\Phi \ll \lambda$ and there exists a decomposition $[0,1] = \bigcup_k H_k$ of $[0,1]$ into sets of positive measure such that for every $k \in \mathbb{N}$ the function $x^{**} \longrightarrow V_{x^{**}\Phi}(H_k)$ is weak*-continuous and $AW^*_\Phi(H_k) < \infty$.*

(jj) *There is an HKP-integrable function $f : [0,1] \to X^*$ such that*

$$\langle x^{**}, \Phi(I)\rangle = (HK) \int_I \langle x^{**}, f(t)\rangle\, dt \qquad \text{for every } I \in \mathcal{I}.$$

Moreover, f can be chosen in such a way that $\|f\|$ is a measurable function.

Proof. The implication $(jj) \Rightarrow (j)$ is a particular case of Proposition 4.2. In order to prove the implication $(j) \Rightarrow (jj)$, we may apply Theorem 3.1 to conclude that there exists a function $f : [0,1] \to X^*$ that is HKG-integrable on $[0,1]$ and Pettis integrable on each H_k, $k \in \mathbb{N}$. Proposition 4.5 yields the HKP-integrability of f on $[0,1]$. ■

Remark 4.7. According to Remark 2.7 each strongly measurable Pettis integrable (and hence also Henstok-Kurzweil-Pettis integrable) function with nowhere differentiable Pettis integral satifies the conditions (j) and (jj) of Theorem 4.6 and has non-σ-finite variational measure V_Φ.

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