1

PROPERTY (gb) THROUGH LOCAL SPECTRAL THEORY

PIETRO AIENA, JESÚS R. GUILLÉN, AND PEDRO PEÑA

Abstract. Property (gb) for a bounded linear operator $T \in L(X)$ on a Banach space X means that the points λ of the approximate point spectrum for which $\lambda I - T$ is upper semi B-Weyl are exactly the poles of the resolvent. In this paper we shall give several characterizations of property (gb). These characterizations are obtained by using typical tools from local spectral theory. We also show that property (gb) holds for large classes of operators and prove the stability of property (gb) under some commuting perturbations.

1. Introduction

Property (b) for bounded linear operators on Banach spaces, has been introduced by Berkani and Zariuoh ([20] and [21]), and this property may be thought, in a sense, as stronger versions of the classical a-Browder's theorem. In this paper we consider a stronger variant of property (b), the so-called property (gb), also introduced in [20] and [21], and studied in the more recent papers [19] and [10]. In particular, we show that this property may be characterized by means of typical tools of local spectral theory. Indeed, property (gb) for T holds if and only if its dual T^* has SVEP at the points of $\Delta_a^g(T)$, where $\Delta_a^g(T)$ denotes the set of all points λ of the approximate point spectrum for which $\lambda I - T$ is upper semi B-Weyl (in the sense of [18]), and this is equivalent to saying that all the points of $\Delta_a^g(T)$ are isolated in the spectrum. Furthermore, property (gb) for T may be characterized through the analytic core $K(\lambda I - T)$, as λ ranges at the points of $\Delta_a^g(T)$. We also show that property (gb) holds for large classes of operators. In the last part we consider the permanence of property (gb) under some commuting perturbations.

2. Definitions and preliminary results

Let $T \in L(X)$ be a bounded linear operator on an infinite-dimensional complex Banach space X, and denote by $\alpha(T)$ and $\beta(T)$, the dimension of the kernel $\ker T$ and the codimension of the range $R(T) := T(X)$, respectively. Let

$$\Phi_+(X) := \{T \in L(X) : \alpha(T) < \infty \text{ and } T(X) \text{ is closed}\}$$

denote the class of all *upper semi-Fredholm* operators, and let

$$\Phi_-(X) := \{T \in L(X) : \beta(T) < \infty\}$$

denote the class of all *lower semi-Fredholm* operators. If $T \in \Phi_\pm(X) := \Phi_+(X) \cup \Phi_-(X)$, the *index* of T is defined by ind $(T) := \alpha(T) - \beta(T)$. If $\Phi(X) := \Phi_+(X) \cap$

[1]1991 *Mathematics Reviews* Primary 47A10, 47A11. Secondary 47A53, 47A55.
Key words and phrases: Property (gb), Browder type theorems.
This research was supported by CDCHTA of Universidad de Los Andes-Venezuela, project I-1295-12-A.

$\Phi_-(X)$ denotes the set of all *Fredholm* operators, the set of *Weyl operators* is defined by

$$W(X) := \{T \in \Phi(X) : \operatorname{ind} T = 0\},$$

the class of *upper semi-Weyl operators* is defined by

$$W_+(X) := \{T \in \Phi_+(X) : \operatorname{ind} T \leq 0\},$$

and class of *lower semi-Weyl operators* is defined by

$$W_-(X) := \{T \in \Phi_-(X) : \operatorname{ind} T \geq 0\}.$$

Clearly, $W(X) = W_+(X) \cap W_(X)$. The classes of operators above defined generate the following spectra: the *Weyl spectrum*, defined by

$$\sigma_{\mathrm{w}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W(X)\};$$

and the *upper semi-Weyl spectrum*, defined by

$$\sigma_{\mathrm{uw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \notin W_+(X)\}.$$

Let $p := p(T)$ be the *ascent* of an operator T; i.e. the smallest non-negative integer p such that ker T^p = ker $T^{p+1}(X)$. If such integer does not exist we put $p(T) = \infty$. Analogously, let $q := q(T)$ be the *descent* of T; i.e the smallest non-negative integer q such that $T^q(X) = T^{q+1}(X)$, and if such integer does not exist we put $q(T) = \infty$. It is well known that if $p(T)$ and $q(T)$ are both finite then $p(T) = q(T)$, see [1, Theorem 3.3].

The class of all *Browder operators* is defined

$$B(X) := \{T \in \Phi(X) : p(T), q(T) < \infty\};$$

while the class of all *upper semi-Browder operators* is defined

$$B_+(X) := \{T \in \Phi_+(X) : p(T) < \infty\}.$$

Obviously, $B(X) \subseteq W(X)$ and $B_+(X) \subseteq W_+(X)$, see [1, Theorem 3.4].

Semi-Fredholm operators have been generalized by Berkani ([16], [18] and [17]) in the following way: for every $T \in L(X)$ and a nonnegative integer n let us denote by $T_{[n]}$ the restriction of T to $T^n(X)$, viewed as a map from the space $T^n(X)$ into itself (we set $T_{[0]} = T$). $T \in L(X)$ is said to be *semi B-Fredholm*, (resp. *B-Fredholm, upper semi B-Fredholm, lower semi B-Fredholm,*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is a semi-Fredholm operator (resp. Fredholm, upper semi-Fredholm, lower semi-Fredholm). In this case $T_{[m]}$ is a semi-Fredholm operator for all $m \geq n$ ([18]) with the same index of $T_{[n]}$. This enables one to define the index of a semi B-Fredholm as ind T = ind $T_{[n]}$.

A bounded operator $T \in L(X)$ is said to be *B-Weyl* (respectively, *upper semi B-Weyl*, *lower semi B-Weyl*) if for some integer $n \geq 0$ the range $T^n(X)$ is closed and $T_{[n]}$ is Weyl (respectively, upper semi-Weyl. lower semi-Weyl). The *B-Weyl spectrum* is defined by

$$\sigma_{\mathrm{bw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not B-Weyl}\},$$

and, analogously, the *upper semi B-Weyl spectrum* of T is defined by

$$\sigma_{\mathrm{ubw}}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not upper semi B-Weyl}\}.$$

The concept of Drazin invertibility has been introduced in a more abstract setting than operator theory. In the case of the Banach algebra $L(X)$, $T \in L(X)$

is said to be *Drazin invertible* (with a finite index) if $p(T) = q(T) < \infty$, and this is equivalent to saying that $T = T_0 \oplus T_1$, where T_0 is invertible and T_1 is nilpotent, see [25, Corollary 2.2] and [28, Prop. A]. Every B-Fredholm operator T admits the representation $T = T_0 \oplus T_1$, where T_0 is Fredholm and T_1 is nilpotent [17], so every Drazin invertible operator is B-Fredholm. Drazin invertibility for bounded operators suggests the following definition:

Definition 2.1. *$T \in L(X)$ is said to be* left Drazin invertible *if $p := p(T) < \infty$ and $T^{p+1}(X)$ is closed.*

The *Drazin spectrum* is then defined as

$$\sigma_{\rm d}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not Drazin invertible}\},$$

while the *left Drazin spectrum* is defined as

$$\sigma_{\rm ld}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not left Drazin invertible}\}.$$

In the sequel we denote by $\sigma_{\rm a}(T)$ the *approximate point spectrum*, defined by

$$\sigma_{\rm a}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\},$$

where an operator is said to be *bounded below* if it is injective and has closed range. The classical *surjective spectrum* of T is denoted by $\sigma_{\rm s}(T)$.

In the following we need the following elementary lemma:

Lemma 2.2. *If $T \in L(X)$ is injective and upper semi B-Fredholm then T is bounded below.*

Proof. If T is upper semi B-Fredholm then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed. By assumption $\alpha(T) < \infty$, and this implies that $\alpha(T^n) < \infty$, so T^n is upper semi-Fredholm and by the classical Fredholm theory we deduce that T is upper semi-Fredholm. Consequently, $T(X)$ is closed and hence T is bounded below. ∎

It is known that if T is a Weyl operator and either $p(T)$ or $q(T)$ is finite then T is Browder, see [1, Theorem 3.4, part (iv)]. This result, in the framework of B-Fredholm theory, may be generalized as follows:

Theorem 2.3. *For an operator $T \in L(X)$ the following statements hold:*

(i) *If T is upper semi B-Weyl and $q(T) < \infty$ then T is Drazin invertible.*

(ii) *If T is lower semi B-Weyl and $p(T) < \infty$ then T is Drazin invertible.*

(iii) *If T is B-Weyl and has either ascent or descent finite then T is Drazin invertible.*

Proof. (i) Suppose that T is upper semi B-Weyl and $q(T) < \infty$. Then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is upper semi-Weyl, i.e. $T_{[n]}$ is upper semi-Fredholm with $\operatorname{ind} T_{[n]} \leq 0$. Moreover, $T_{[m]}$ is upper semi-Weyl for all $m \geq n$. Suppose that $q := q(T) < \infty$. Consider the operator $T_{[m]} : T^m(X) \to T^m(X)$. It is evident that

$$R(T_{[m]}) = T^{m+1}(X) = T^m(X), \quad \text{for all } m \geq q,$$

thus $T_{[m]}$ is onto, i.e $q(T_{[m]}) = 0$. Now, choose $m \geq \max\{n, q\}$ then $T_{[m]}$ is both onto and upper semi- Weyl. Then $\operatorname{ind} T_{[m]} = \alpha(T_{[m]}) \geq 0$, from which we obtain

$\operatorname{ind} T_{[m]} = 0$ and hence $\alpha(T_{[m]} = \beta(T_{[m]}) = 0$, i.e. $T_{[m]}$ is invertible. Consequently, $T^k_{[m]}$ is invertible for all $k \in \mathbb{N}$. Therefore,

$$\ker T^k \cap T^m(X) = \ker T^k_{[m]} = \{0\} \quad \text{for all } k \in \mathbb{N},$$

which implies that $p(T) < \infty$, by [1, Lemma 3.2, part (i)].

(ii) Assume that T is lower semi B-Weyl and $p := p(T) < \infty$. Then there exists $n \in \mathbb{N}$ such that $T^n(X)$ is closed and $T_{[n]}$ is lower semi-Fredholm and $\operatorname{ind} T_{[n]} \geq 0$. Moreover, $T_{[m]}$ is lower semi- Weyl for all $m \geq n$. By [1, Lemma 3.2, part (i)] the assumption $p := p(T) < \infty$ entails that

$$\ker\, T_{[m]} = \ker\, T \cap T^m(X) = \{0\} \quad \text{for all natural } m \geq p.$$

Choosing $m \geq \max\{n, p\}$ then $T_{[m]}$ is both injective and lower semi- Weyl, hence $\operatorname{ind} T_{[m]} = -\beta(T_{[m]}) \leq 0$, so $\operatorname{ind} T_{[m]} = 0$, and hence $T_{[m]}$ is invertible, since $\alpha(T_{[m]} = \beta(T_{[m]}) = 0$. Consequently,

$$T^{m+1}(X) = R(T_{[m]}) = T^m(X),$$

and hence $q(T) < \infty$.

(iii) It is evident from part (i) and (ii), since T is both upper and lower semi B-Weyl. ∎

Let $T \in L(X)$ and $d \in \mathbb{N}$. According Grabiner [23] T is said to have *uniform descent for* $n \geq d$ if $T(X) + \ker\, T^n = T(X) + \ker T^d$ for all $n \geq d$. If, in addition, $T(X) + \ker T^d$ is closed then T is said to have *topological uniform descent for* $n \geq d$. Note that if either of the quantities $\alpha(T)$, $\beta(T)$, $p(T)$, $q(T)$ is finite then T has uniform descent. Define

$$\Delta(T) := \{n \in \mathbb{N} : m \geq n, m \in \mathbb{N} \Rightarrow T^n(X) \cap \ker\, T \subseteq T^m(X) \cap \ker\, T\}.$$

The *degree of stable iteration* is defined as $\operatorname{dis}(T) := \inf \Delta(T)$ if $\Delta(T) \neq \emptyset$, while $\operatorname{dis}(T) = \infty$ if $\Delta(T) = \emptyset$.

Definition 2.4. $T \in L(X)$ *is said to be* quasi-Fredholm of degree d, *if there exists* $d \in \mathbb{N}$ *such that:*
(a) $dis(T) = d$,
(b) $T^n(X)$ *is a closed subspace of* X *for each* $n \geq d$,
(c) $T(X) + \ker\, T^d$ *is a closed subspace of* X.

Clearly, quasi-Fredholm operators are precisely all operators $T \in L(X)$ that has topological uniform descent $n \geq d$ and such that $T^{d+1}(X)$ is closed, see for details [26]. It is easily seen that if $T \in QF(d)$ then $T^* \in QF(d)$.

Theorem 2.5. ([2]) *If* $T \in QF(d)$ *then following statements are equivalent:*
(i) $p(T) < \infty$*;*
(ii) $\sigma_{\mathrm{a}}(T)$ *does not cluster at* 0*;*

Moreover, if these equivalent conditions are satisfied then T *is upper semi* B*-Fredholm operator with index* $indT \leq 0$.

The next result is dual to the result of Theorem 2.5.

Theorem 2.6. ([2]) *If $T \in QF(d)$ then following statements are equivalent:*

(i) $q(T) < \infty$;

(ii) $\sigma_{\rm s}(T)$ *does not cluster at* 0;

Moreover, if these equivalent conditions are satisfied then T is lower semi B-Fredholm operator with index $ind T \geq 0$.

3. PROPERTY (gb)

The concept of pole of the resolvent suggests the following definition:

Definition 3.1. *Let $T \in L(X)$, X a Banach space. If $\lambda I - T$ is left Drazin invertible and $\lambda \in \sigma_{\rm a}(T)$ then λ is said to be a* left pole. *A left pole λ is said to have finite rank if $\alpha(\lambda I - T) < \infty$. If $\lambda I - T$ is right Drazin invertible and $\lambda \in \sigma_{\rm s}(T)$ then λ is said to be a* right pole. *A right pole λ is said to have finite rank if $\beta(\lambda I - T) < \infty$.*

Denote by $\Pi(T)$ and $\Pi^a(T)$ the set of all poles and the set of left poles of T, respectively. Clearly, $\Pi^a(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ld}(T)$. Obviously, $\Pi(T) \subseteq \operatorname{iso}\sigma(T)$, and analogously we have

$$\Pi^a(T) \subseteq \operatorname{iso}\sigma_{\rm a}(T) \quad \text{for all } T \in L(X). \tag{1}$$

In fact, if $\lambda_0 \in \Pi^a(T)$ then $\lambda I - T$ is left Drazin invertible and hence $\lambda I - T \in QF(d)$. This implies that $\lambda I - T$ has topological uniform descent and since $p(\lambda_0 I - T) < \infty$ it then follows, from Corollary 4.8 of [23], that $\lambda I - T$ is bounded below in a punctured disc centered at λ_0.

Define

$$\Delta_a^g(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm ubw}(T).$$

It should be noted that the set $\Delta_a^g(T)$ may be empty. This is, for instance, the case of a right shift R on $\ell^2(\mathbb{N})$. In this case $\sigma_{\rm a}(R)$ is the unit circle, so $\operatorname{iso}\sigma_{\rm a}(R) = \emptyset$. Furthermore, R has SVEP and hence, by [12, Corolary 2.12], $\sigma_{\rm a}(R) = \sigma_{\rm ubw}(R)$.

Lemma 3.2. *If $T \in L(X)$ then*

$$\Delta_a^g(T) = \{\lambda \in \mathbb{C} : \lambda I - T \text{ is upper semi B-Weyl and } 0 < \alpha(\lambda I - T)\}. \tag{2}$$

Furthermore, $\Pi^a(T) \subseteq \Delta_a^g(T)$.

Proof. The inclusion $\supseteq$ in (2) is obvious. To show the opposite inclusion, suppose that $\lambda \in \Delta_a^g(T)$. There is no harm if we assume $\lambda = 0$. Then T is upper semi B-Weyl and $0 \in \sigma_{\rm a}(T)$. Both conditions entail that $\alpha(T) > 0$ (otherwise if $\alpha(T) = 0$, and hence $p(T) = 0$, then, by [26, Lemma 12], we have $T(X)$ closed. Thus $0 \notin \sigma_{\rm a}(T)$, a contradiction). Therefore the equality (2) holds.

To show the inclusion $\Pi^a(T) \subseteq \Delta_{\rm a}(T)$, let assume that $\lambda \in \Pi^a(T) = \sigma_{\rm a}(T) \setminus \sigma_{\rm ld}(T)$. Since $\lambda I - T$ is left pole then $\lambda \in \sigma_{\rm a}(T)$, and $\lambda I - T$ is left Drazin invertible, hence, upper semi B-Weyl. Therefore $\Pi^a(T) \subseteq \Delta_a^g(T)$. ∎

The following class of operators has been introduced in [20].

Definition 3.3. *$T \in L(X)$ satisfies* property (gb) *if $\Delta_a^g(T) = \Pi(T)$.*

In order to give some characterizations of property (gb) by means of local spectral theory, we now give some definitions. Let X be a complex Banach space and $T \in L(X)$. The operator T is said to have *the single valued extension property* at $\lambda_0 \in \mathbb{C}$ (abbreviated SVEP at λ_0), if for every open disc U of λ_0, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$.
An operator $T \in L(X)$ is said to have SVEP if T has SVEP at every point $\lambda \in \mathbb{C}$. Evidently, an operator $T \in L(X)$ has SVEP at every point of the resolvent $\rho(T) := \mathbf{C} \setminus \sigma(T)$, and both T and T^* have SVEP at the isolated points of the spectrum. From definition of SVEP we easily obtain:

$$\sigma_{\rm a}(T) \text{ does not cluster at } \lambda \Rightarrow T \text{ has SVEP at } \lambda,$$

and, by duality,

$$\sigma_{\rm s}(T) \text{ does not cluster at } \lambda \Rightarrow T^* \text{ has SVEP at } \lambda.$$

Note that if T is quasi-Fredholm then the above implications are equivalence, see [2]. Property (gb) may be characterized by means of localized SVEP as follows.

Theorem 3.4. *For a bounded operator $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies property (gb);*
(ii) $\Delta_a^g(T) \subseteq \operatorname{iso} \sigma(T)$*;*
(iii) $\Delta_a^g(T) \subseteq \operatorname{iso} \sigma_{\rm s}(T)$*;*
(iv) *T^* has SVEP at every $\lambda \in \Delta_a^g(T)$;*
(v) *$q(\lambda I - T) < \infty$ for all $\lambda \in \Delta_a^g(T)$.*

Proof. The implication (i) $\Rightarrow$ (ii) is obvious, since $\Pi(T) \subseteq \operatorname{iso} \sigma(T)$.

The implication (ii) $\Rightarrow$ (iii) easily follows, once observed that the isolated points of the spectrum belong to $\sigma_{\rm s}(T)$.

The implication (iii) $\Rightarrow$ (iv) is clear: for every operator T, its dual T^* has SVEP at the points $\lambda \in \operatorname{iso} \sigma_{\rm s}(T)$.

(iv) $\Rightarrow$ (v) If $\lambda \in \Delta_a^g(T)$, then $\lambda I - T$ is upper semi B-Weyl, in particular quasi-Fredholm. By Theorem 2.5 we then have $q(\lambda I - T) < \infty$.

(v) $\Rightarrow$ (i) If $\lambda \in \Delta_a^g(T)$ then $\lambda I - T$ is upper semi B-Weyl and, by Theorem 2.3, the condition $q(\lambda I - T) < \infty$ entails that λ is a pole of the resolvent, thus $\Delta_a^g(T) \subseteq \Pi(T)$. The opposite inclusion is always true for any operator, since for a pole λ we have that $\lambda I - T$ is Drazin invertible, and in particular $\lambda I - T$ is upper semi B-Weyl. Therefore, $\Delta_a^g(T) = \Pi(T)$. ∎

Corollary 3.5. *If T^* has SVEP then T satisfies property (gb).*

Remark 3.6. Set $\Delta_a(T) := \sigma_{\rm a}(T) \setminus \sigma_{\rm uw}(T)$. Recall that $T \in L(X)$ is said to satisfy *property* (b) if $\Delta_a(T) = p_{00}(T)$, where $p_{00}(T) = \sigma(T) \setminus \sigma_{\rm b}(T)$, i.e., $p_{00}(T)$ is the set of all poles having finite rank. Since $\sigma_{\rm ubw}(T) \subseteq \sigma_{\rm uw}(T)$, we have $\Delta_a(T) \subseteq \Delta_a^g(T)$, hence, by Theorem 3.4, property (gb) entails that $\Delta_a(T) \subseteq \operatorname{iso} \sigma(T)$, and the last inclusion holds precisely when T satisfies property (b), see [10, Theorem 2.1]. Therefore,

$$\text{property } (gb) \Rightarrow \text{property } (b).$$

Property (b) for T implies *a-Browder's theorem* for T, i.e. $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{ub}}(T)$, see [10] for details. Furthermore, a-Browder's theorem is equivalent to *generalized a-Browder's theorem*, i.e., $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$, see [6, Theorem 2.5].

The next example shows that the SVEP for T does not ensure, in general, that property (gb) holds for T.

Example 3.7. Let $R \in L(\ell_2(\mathbb{N}))$ denote the right shift and let $P \in L(\ell_2(\mathbb{N}))$ be the idempotent operator defined by $P(x) := (0, x_2, x_3, \dots)$ for all $x = (x_1, x_2, \dots) \in \ell_2(\mathbb{N})$. It is easily seen that if $T := R \oplus P$ then $\sigma(T) = D(0,1)$, where $D(0,1)$ denotes the closed unit disc, and $\sigma_{\mathrm{a}}(T) = \Gamma \cup \{0\}$, where Γ denotes the unit circle. Since $\sigma_{\mathrm{ubw}}(T) = \Gamma$, we then have $\Delta_a^g(T) = \{0\}$, while since $\sigma(T)$ has no isolated points, we have $\Pi(T) = \emptyset$. Therefore, T does not have property (gb), while T inherits SVEP from R and P.

The *quasi-nilpotent part* of T, is defined as follows:

$$H_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

It is easily seen that $ker\, T^n \subseteq H_0(T)$ for every $n \in \mathbb{N}$, so $\mathcal{N}^\infty(T) \subseteq H_0(T)$, where $\mathcal{N}^\infty(T) := \bigcup_{n=1}^\infty ker\, T^n$ denotes the *hyper-kernel* of T.

An other important subspace in local spectral theory is given by the *analytic core* $K(T)$. To define this subspace, recall that the *local resolvent set* of T at the point $x \in X$, is the set $\rho_T(x)$ defined as the union of all open subsets U of $\mathbf{C}$ for which there exists an analytic function $f : U \to X$ with the property that $(\lambda I - T)f(\lambda) = x$ for all $\lambda \in U$. The *local spectrum* $\sigma_T(x)$ of T at x is the set defined by $\sigma_T(x) := \mathbf{C} \setminus \rho_T(x)$. For every subset F of $\mathbf{C}$, then the *analytic spectral subspace* of T associated with F is defined as the set $X_T(F) := \{x \in X : \sigma_T(x) \subseteq F\}$. Note that $K(T) \subseteq T^\infty(X) \subseteq T^n(X)$, where $T^\infty(X) := \bigcap_{n=0}^\infty T^n(X)$ denotes the *hyper-range* of T, and $T(K(T)) = K(T)$, see [1, Theorem 1.21].

The two subspaces $H_0(T)$ and $K(T)$ are, in general, not closed, and, by [1, Theorem 2.31]

$$H_0(\lambda I - T) \text{ closed} \Rightarrow T \text{ has SVEP at } \lambda. \tag{3}$$

Furthermore, if $\lambda \in \operatorname{iso} \sigma(T)$ then $X = H_0(\lambda I - T) \oplus K(\lambda I - T)$, see [1, Theorem 3.74]. Consequently, if T has property (gb) then

$$X = H_0(\lambda I - T) \oplus K(\lambda I - T) \quad \text{for all } \lambda \in \Delta_a^g(T) \tag{4}$$

since $\Delta_a^g(T) \subseteq \Pi(T)$. The following results show that property (gb) may be characterized by some conditions that are formally weaker than the one expressed by the decomposition (4).

Theorem 3.8. *For an operators $T \in L(X)$ the following statements are equivalent:*

(i) *T satisfies property (gb);*

(ii) *$X = H_0(\lambda I - T) + K(\lambda I - T)$ for all $\lambda \in \Delta_a^g(T)$;*

(iii) *there exists a natural $\nu := \nu(\lambda)$ such that $K(\lambda I - T) = (\lambda I - T)^\nu(X)$ for all $\lambda \in \Delta_a^g(T)$;*

(iv) *$X = \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X)$ for all $\lambda \in \Delta_a^g(T)$;*

(v) *there exists a natural* $\nu := \nu(\lambda)$ *such that* $(\lambda I - T)^\infty(X) = (\lambda I - T)^\nu(X)$ *for all* $\lambda \in \Delta_a^g(T)$.

Proof. (i) $\Rightarrow$ (ii) Clear, as observed in (4).

(ii) $\Rightarrow$ (i) The condition $X = H_0(\lambda I - T) + K(\lambda I - T)$ is equivalent to the inclusion $\lambda \in \operatorname{iso}\sigma_{\rm s}(T)$, see [22, Theorem 5]. Hence $\Delta_a^g(T) \subseteq \operatorname{iso}\sigma_{\rm s}(T)$ and from Theorem 3.4 it immediately follows that T satisfies property (gb).

(i) $\Leftrightarrow$ (iii) If T satisfies property (gb) then, by Theorem 3.4, $q := q(\lambda I-T) < \infty$ for all $\lambda \in \Delta_a^g(T)$, so $(\lambda I - T)^\infty(X) = (\lambda I - T)^q(X)$. Since $\lambda I - T$ is upper semi B-Fredholm, then there exists $\nu \in \mathbb{N}$ such that $(\lambda I-T)^n(X)$ is closed for all $n \geq \nu$, so $(\lambda I - T)^\infty(X)$ is closed. Furthermore, by Theorem 3.4 of [23], the restriction $\lambda I - T|(\lambda I - T)^\infty(X)$ is onto, so $\lambda I - T)((\lambda I - T)^\infty(X)) = (\lambda I - T)^\infty(X)$. By [1, Theorem 1.22] it then follows that $(\lambda I-T)^\infty(X) \subseteq K(\lambda I-T)$, and, since the reverse inclusion holds for every operator, we then conclude that $(\lambda I-T)^\infty(X) = K(\lambda I - T) = (\lambda I - T)^q(X)$.

Conversely, let $\lambda \in \Delta_a^g(T)$ be arbitrary given and suppose that there exists a natural $\nu := \nu(\lambda)$ such that $K(\lambda I - T) = (\lambda I - T)^\nu(X)$. Then we have

$$(\lambda I - T)^\nu(X) = K(\lambda I - T) = (\lambda I - T)(K(\lambda I - T)) = (\lambda I - T)^{\nu+1}(X),$$

thus $q(\lambda I - T) \leq \nu$. By Theorem 3.4 we the conclude that property (gb) holds for T.

(ii) $\Leftrightarrow$ (iv) Every semi B-Fredholm operator has topological uniform ascent ([18]), so, by [27, Corollary 2.8],

$$H_0(\lambda I - T) + K(\lambda I - T) = \mathcal{N}^\infty(\lambda I - T) + (\lambda I - T)^\infty(X),$$

for every $\lambda \in \Delta_a^g(T)$.

(i) $\Leftrightarrow$ (v) Suppose that T satisfies property (gb). By Theorem 3.4 then $q := q(\lambda I - T) < \infty$ for all $\lambda \in \Delta_a^g(T)$, and hence $(\lambda I - T)^\infty(X) = (\lambda I - T)^q(X)$ for all $\lambda \in \Delta_a^g(T)$. Conversely, suppose that (v) holds and $\lambda \in \Delta_a^g(T)$. Then

$$(\lambda I - T)^\nu(X) = (\lambda I - T)^\infty(X) \subseteq (\lambda I - T)^{\nu+1}(X),$$

and since $(\lambda I - T)^{n+1}(X) \subseteq (\lambda I - T)^n(X)$ for all $n \in \mathbb{N}$, we then obtain that $(\lambda I - T)^\nu(X) = (\lambda I - T)^{\nu+1}(X)$. Hence $q(\lambda I - T) \leq \nu$, so T satisfies property (gb) by Theorem 3.4. ∎

Property (gb) is related to Browder type theorems as follows:

Theorem 3.9. *If* $T \in L(X)$ *the following statements are equivalent:*

(i) T *satisfies property* (gb);

(ii) T *satisfies a-Browder's theorem and* $\sigma_{\rm bw}(T) \cap \Delta_a^g(T) = \emptyset$;

(iii) T *satisfies Browder's theorem and* $\sigma_{\rm bw}(T) \cap \Delta_a^g(T) = \emptyset$.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) has been proved in [20, Theorem 2.15]. *a*-Browder's theorem entails Browder's theorem, so (ii) $\Rightarrow$ (iii). Suppose (iii). If $\lambda \in \Delta_a^g(T)$, then $\lambda I - T$ is B-Weyl, hence $\lambda \notin \sigma_{\rm bw}(T) = \sigma_{\rm d}(T)$, since Browder's theorem holds for T. Therefore λ is a pole and hence is an isolated point of $\sigma(T)$. By Theorem 3.4 we conclude that T has property (gb). ∎

An operator $T \in L(X)$ is said to be *a-polaroid* is every isolated point of $\sigma_{\rm a}(T)$ is a pole, while $T \in L(X)$ is said to be *polaroid* is every isolated point of $\sigma(T)$ is a pole. Since $\operatorname{iso}\sigma(T) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T)$ (it is known that every isolated point of $\sigma(T)$ belongs to $\sigma_{\mathrm{a}}(T)$), then every a-polaroid operator is polaroid, while the converse, in general, is not true. Note that $\in L(X)$ is polaroid if and only if T^* is polaroid.

Theorem 3.10. *If $T \in L(X)$ is a-polaroid then the properties (b), (gb), and a-Browder's theorem for T, are equivalent.*

Proof. We know that for every $T \in L(X)$ we have $(gb) \Rightarrow (b) \Rightarrow a$-Browder's theorem, so we have only to prove that a-Browder's theorem for T implies property (gb) for T. Since a-Browder's theorem and generalized a-Browder's theorem are equivalent we have $\sigma_{\mathrm{ubw}}(T) = \sigma_{\mathrm{ld}}(T)$. Therefore $\Delta_a^g(T) = \sigma_{\mathrm{a}}(T)\backslash\sigma_{\mathrm{ld}}(T) = \Pi^a(T)$. From the inclusion (3.15) we know that every left pole λ is an isolated point of $\sigma_{\mathrm{a}}(T)$. Our assumption that T is a-polaroid entails that $\lambda \in \Pi(T)$, and hence $\Delta_a^g(T) \subseteq \Pi(T)$, from which we conclude that property (gb) holds for T. ∎

Corollary 3.11. *Suppose that $T \in L(X)$ is a-polaroid and has SVEP. Then property (gb) holds for T.*

Proof. T satisfies a-Browder's theorem. ∎

The result of Theorem 3.10 cannot be extended to polaroid operators. For instance, if T is defined as in Example 3.7, then T is polaroid and satisfies a-Browder's theorem, since T has SVEP, while property (gb) does not hold for T.

Property (gb) is related to a *Weyl type theorem*, the so-called *property* (gw), introduced in [15] (other Weyl type theorems are *Weyl's theorem*, and *a-Weyl's theorem*, in their form generalized or not, see [5]). Property (gw) means that $\Delta_a^g(T) = E(T)$, where $E(T) := \{\lambda \in \operatorname{iso}\sigma(T) : \alpha(\lambda I - T) > 0\}$. In such a case we have $\Delta_a^g(T) \subseteq \operatorname{iso}\sigma(T)$, so property (gw) for T trivially entails property (gb). This result has been proved in [20], but our argument is very simple. It should be noted that the SVEP for T entails that $\sigma(T^*) = \sigma_{\mathrm{a}}(T^*)$.

We have seen, in Example 3.7, that the SVEP for T does not ensure that property (gb) holds for T. However, the following result shows that in the case of polaroid operators, property (gb) holds for T^*.

Theorem 3.12. *Suppose that $T \in L(X)$ is polaroid. Then we have:*

(i) *T satisfies property (gw) if and only if T satisfies property (gb).*

(ii) *If T has SVEP then T^* satisfies property (gb), and in this case all Weyl type theorem hold for T^*.*

Proof. To show the equivalence (i) it suffices to prove that property (gb) implies property (gw). Clearly, $\Pi(T) \subseteq E(T)$. Conversely, the polaroid condition entails that every $\lambda \in E(T)$ is a pole of the resolvent, hence $\Pi(T) = E(T)$, from which the equivalence (i) follows.

(ii) The SVEP for T entails a-Browder's theorem for T^*. Moreover, T^* is polaroid and the SVEP for T implies that $\sigma(T^*) = \sigma(T) = \sigma_{\mathrm{s}}(T) = \sigma(T^*)$, see [1, Corollary 2.45]. Hence T^* is a-polaroid, so by Theorem 3.10 T^* has property (gb). From part (i) it then follows that T^* satisfies property (gw), and this property, see [5, Theorem 3.10], is equivalent to all Weyl type theorems. ∎

In the sequel we give some results concerning the stability of property (gb) under some commuting perturbations.

A bounded operator is said to be *finitely polaroid* if every isolated point of the spectrum is a pole of finite rank.

Theorem 3.13. *Suppose that T is finite polaroid which has SVEP. Then T^*+Q^* satisfies property (gb) for every quasi-nilpotent operator Q commuting with T.*

Proof. We prove first that $T+Q$ is polaroid. Let $\lambda \in \operatorname{iso}\sigma(T+Q)$. It is well-known that the spectrum is invariant under commuting quasi-nilpotent perturbations, thus $\lambda \in \operatorname{iso}\sigma(T)$ and hence is a pole of the resolvent of T (consequently, an eigenvalue of T). Therefore, $p := p(\lambda I-T) = q(\lambda I-T) < \infty$ and since by assumption $\alpha(\lambda I-T) < \infty$ we then have $\alpha(\lambda I-T) = \beta(\lambda I-T)$, see [1, Theorem 3.4], so $\lambda I-T$ is Browder. By a result of Rakočević ([31]) we know that the class of Browder operators is stable under commuting Riesz perturbations, in particular under quasi-nilpotent commuting perturbations, so $\lambda I-(T+Q)$ is Browder, and hence λ is a pole of the resolvent of $T+Q$, which shows that $T+Q$ is polaroid.

Now, the SVEP from T is transmitted to $T+Q$, see [13], and this implies, as observed in the proof of Theorem 3.12, that T^*+Q^* is a-polaroid. Moreover, the SVEP for $T+Q$ implies that T^*+Q^* satisfies a-Browder's theorem, see [7, Theorem 2.4]. By Theorem 3.10, we then conclude that property (gb) holds for T^*+Q^*. ∎

Theorem 3.14. *Suppose that $T \in L(X)$ has SVEP and $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$. Then we have*

(i) *$T+Q$ satisfies property (gb) for every commuting quasi-nilpotent operator Q.*

(ii) *$T+K$ satisfies property (gb) for every commuting finite rank operator K.*

Proof. (i) It is well known that $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T+Q)$ for every commuting quasi-nilpotent operator Q. Therefore $\operatorname{iso}\sigma_{\mathrm{a}}(T+Q) = \emptyset$, from which we conclude that $\Pi(T+Q)$ is empty, since a pole of the resolvent is an isolated point of the spectrum which belongs to the approximate point spectrum.

It remains only to show that $\Delta_a^g(T+Q) = \emptyset$. Suppose that $\Delta_a^g(T+Q)$ is nonempty and let $\lambda \in \Delta_a^g(T+Q) = \sigma_{\mathrm{a}}(T+Q) \setminus \sigma_{\mathrm{ubw}}(T+Q)$. The SVEP for T is inherited by $T+Q$, and since $\lambda I-(T+Q)$ is upper semi B-Weyl, the SVEP of $T+Q$ at λ implies, by Theorem 2.5, that $\sigma_{\mathrm{a}}(T+Q)$ does not cluster at λ. But $\lambda \in \sigma_{\mathrm{a}}(T+Q)$, so λ is an isolated point of $\sigma_{\mathrm{a}}(T+Q)$ and this is impossible. Therefore,

$$\Pi(T+Q) = \Delta_a^g(T+Q) = \emptyset.$$

(ii) Observe first that, by [3, Theorem 2.5], we have $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T+K)$, so $\sigma_{\mathrm{a}}(T+K)$ has no isolated points. Furthermore, the SVEP for T is inherited by $T+K$, see [3, Lemma 2.8]. The statement may be proved by using the same arguments of the proof of part (i). ∎

It should be noted that both the conditions $\operatorname{iso}\sigma_{\mathrm{a}}(T) = \emptyset$ and T has SVEP are satisfied by every not quasi-nilpotent unilateral right shift T on $\ell^p(\mathbb{N})$, with $1 \le p < \infty$, see [30, Proposition 1.6.15].

Let $\mathcal{H}(\sigma(T))$ be the set of all analytic functions defined on a neighborhood of $\sigma(T)$, and, for every $f \in \mathcal{H}(\sigma(T))$., let $f(T)$ be defined by means of the classical functional calculus.

Theorem 3.15. *Let $T \in L(X)$ be such that there exists $\lambda_0 \in \mathbb{C}$ such that*

$$K(\lambda_0 I - T) = \{0\} \quad \text{and} \quad \ker(\lambda_0 I - T) = \{0\}. \tag{5}$$

Then property (gb) holds for $f(T)$ for all $f \in \mathcal{H}(\sigma(f(T))$.

Proof. For all complex $\lambda \neq \lambda_0$ we have $\ker(\lambda I - T) \subseteq K(\lambda_0 I - T)$, so that $\ker(\lambda I - T) = \{0\}$ for all $\lambda \in \mathbb{C}$. Therefore, the puntual spectrum $\sigma_p(T)$ is empty.

We show that also $\sigma_p(f(T)) = \emptyset$. To see this, let $\mu \in \sigma(f(T))$ and write $\mu - f(\lambda) = p(\lambda)g(\lambda)$, where g is analytic on an open neighborhood $\mathcal{U}$ containing $\sigma(T)$ and without zeros in $\sigma(T)$, p a polynomial of the form $p(\lambda) = \Pi_{k=1}^n(\lambda_k - \lambda)^{\nu_k}$, with distinct roots $\lambda_1, \ldots, \lambda_n$ lying in $\sigma(T)$. Then

$$\mu I - f(T) = \Pi_{k=1}^n(\lambda_k I - T)^{\nu_k} g(T).$$

Since $g(T)$ is invertible, $\sigma_{\mathrm{p}}(T) = \emptyset$ implies that $\ker(\mu I - f(T)) = \{0\}$ for all $\mu \in \mathbb{C}$, so $\sigma_p(f(T)) = \emptyset$.

To prove that property (gb) holds for $f(T)$, observe first that $\Pi(f(T))$ is empty, since each pole is an eigenvalue. So we need only to prove that $\Delta_a^g(f(T))$ is empty. Suppose that there exists $\lambda \in \Delta_a^g(f(T))$. Then $\lambda \in \sigma_{\mathrm{a}}(f(T))$ and $\lambda I - f(T)$ is upper semi B-Weyl. Since $\sigma_p(f(T)) = \emptyset$ we have that $\lambda I - f(T)$ is injective, hence, by Lemma 2.2, $\lambda I - f(T)$ is bounded below, i.e. $\lambda \notin \sigma_{\mathrm{a}}(f(T))$, a contradiction. Therefore $f(T)$ satisfies property (gb). ∎

The conditions of Theorem 3.15 are satisfied by any injective operator for which the hyperrange $T^\infty(X)$ is $\{0\}$, since $K(T) \subseteq T^\infty(X)$ for all $T \in L(X)$. In particular, the conditions of Theorem 3.15 are satisfied by a *semi-shift* T, i.e. T is an isometry for which $T^\infty(X) = \{0\}$, see [30] for details on this class of operators. Clearly, a semi-shift T on a non-trivial Banach space is a non-invertible isometry.

It is known that, in general, the equality $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + K)$, where K is a finite rank operator which commutes with T, does not hold. Actually, we have

$$\operatorname{acc}\sigma_{\mathrm{a}}(T) = \operatorname{acc}\sigma_{\mathrm{a}}(T + K),$$

while the isolated points of $\sigma_{\mathrm{a}}(T)$ and $\sigma_{\mathrm{a}}(T + K)$ may be different, see [3].

Theorem 3.16. *Let $T \in L(X)$ and let $K \in L(X)$ be a finite rank operator which commutes with T such that $\sigma_{\mathrm{a}}(T) = \sigma_{\mathrm{a}}(T + K)$. If T has property (gb) then also $T + K$ has property (gb).*

Proof. Property (gb) entails a-Browder's theorem for T, and from the equalities $\sigma_{\mathrm{uw}}(T) = \sigma_{\mathrm{uw}}(T + K)$ and $\sigma_{\mathrm{ub}}(T) = \sigma_{\mathrm{ub}}(T + K)$ we deduce that a-Browder's theorem holds for $T + K$, or equivalently generalized a-Browder's theorem holds for $T + K$. From [12, Theorem 2.11] it the follows that

$$\Delta_a^g(T + K) \subseteq \operatorname{iso}\sigma_{\mathrm{a}}(T + K) = \operatorname{iso}\sigma_{\mathrm{a}}(T).$$

Let $\lambda \in \Delta_a^g(T + K)$. Clearly, T has SVEP at every λ, since $\lambda \in \operatorname{iso}\sigma_{\mathrm{a}}(T)$. Since $\lambda I - (T + K)$ is upper semi B-Fredholm, from [18] we know that also $\lambda I - (T + K) + K = \lambda I - T$ is upper semi B-Fredholm. and the SVEP of T at λ

implies, by Theorem 2.6, that $\operatorname{ind}(\lambda I - T) \leq 0$. Therefore, $\lambda \in \Delta_a^g(T) = \Pi(T)$. Since $\Pi(T) = \Pi(T+K)$, see [17], then we have $\Delta_a^g(T+K) \subseteq \Pi(T+K)$. The reverse inclusion is satisfied for every operator, so $\Delta_a^g(T+K) = \Pi(T+K)$. ∎

A bounded operator $T \in L(X)$ is said to be *hereditarily polaroid* if the restriction $T|M$ of T to any closed invariant subspace M is polaroid. Recall that a bounded operator $T \in L(X)$ is said to be *algebraic* if there exists a non-constant polynomial h such that $h(T) = 0$. Trivially, every nilpotent operator is algebraic and it is well-known that every finite-dimensional operator is algebraic.

Let $\mathcal{H}_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain.

Theorem 3.17. *Suppose that $T \in L(X)$ is hereditarily polaroid operator which satisfies SVEP, $K \in LX)$ an algebraic operator which commutes with T. Then $f(T^* + K^*)$ satisfies property (gb) for every $f \in \mathcal{H}_{nc}(\sigma(T))$.*

Proof. Note first that $T + K$ satisfies SVEP, by [9, Theorem 2.14] and hence, by [1, Theorem 2.40], also $f(T+K)$ satisfies SVEP. If T is hereditarily polaroid then $T^* + K^*$ is polaroid, see [4, Theorem 2.15]. By Lemma 3.11 of [5] it then follows that $f(T^*+K^*)$ is polaroid. Therefore, the result of Theorem 3.12 applies to $f(T^* + K^*)$. ∎

The result of Theorem 3.17 applies to many classes of operators. For instance to the class of $H(p)$-operators introduced by Oudghiri in [29], where $T \in L(X)$ is said to belong to the class $H(p)$ if there exists a natural $p := p(\lambda)$ such that:

$$H_0(\lambda I - T) = \ker(\lambda I - T)^p \quad \text{for all } \lambda \in \mathbb{C}. \tag{6}$$

The implication (3) entails that T has SVEP and T is polaroid, by [8, Theorem 2.2]. Since the restriction of $H(p)$-operators are still $H(p)$ then these operators are hereditarily polaroid. Many other classes of Hilbert space operators are hereditarily polaroid and have SVEP, Recall that an operator $T \in L(X)$ is said to be *hereditarily normaloid*, $T \in \mathcal{H}N$, if the restriction $T|M$ of T, to any closed T-invariant subspace M, is normaloid, (i.e the norm of $T|M$ is equal to its spectral radius $r(T|M)$). An operator $T \in L(X)$ is said to be *totally hereditarily normaloid*, $T \in \mathcal{THN}$, if $T \in \mathcal{H}N$ and every invertible restriction $T|M$ has a normaloid inverse. A *quasi totally hereditarily normaloid operator* is an operator $T \in L(X)$ for which the restriction $T|\overline{T^k(X)}$, where $\overline{T^k(X)}$ denotes the closure of $T^k(X)$, is totally hereditarily normaloid for some $k \in \mathbb{N}$.

This class of operators has been introduced in [11] and examples of quasi totally hereditarily normaloid operators on Hilbert spaces are (n,k)-quasiparanormal operators, in particular paranormal operators, k-quasiclass A operators, k-quasi *-paranormal operators, and (p,k)-quasihyponormal operators, see [11] for definitions and details. Every quasi totally hereditarily normaloid operator T is hereditarily polaroid ([11]). Moreover, all these classes of operators have SVEP, so Theorem 3.17 applies to them, and in particular, if T belongs to one of these classes, then $T^* + K^*$ satisfies property (gb) for all algebraic operators K which commute with T.

References

[1] P. Aiena: *Fredholm and local spectral theory, with application to multipliers.* Kluwer Academic Press (2004).

[2] P. Aiena: *Quasi Fredholm operators and localized SVEP.* Acta Scientiarum Mathematicarum (Szeged) **73** (2007), 251-263.

[3] P. Aiena: *Property* (w) *and perturbations II.* Journal Mathematical Analysis and Applications **342**, (2008), 830-837.

[4] P. Aiena, E. Aponte: *Polaroid type operators under perturbations*, (2013), to appear in Studia Mathematica.

[5] P. Aiena, E. Aponte, E. Bazan: *Weyl type theorems for left and right polaroid operators.* Integral Equations and Operator Theory 66 (2010), 1-20.

[6] P. Aiena, M. T. Biondi, C. Carpintero :*On Drazin invertibility*, Proceding American Mathematical Society **136** (2008), no. 8, 2839-2848

[7] P. Aiena, C. Carpintero, E. Rosas: *Some characterization of operators satisfying a-Browder theorem.* Journal Mathematical Analysis and Applications **311**, (2005), 530-544.

[8] P. Aiena, M. Chō, M. González: *Polaroid type operator under quasi-affinities.* Journal Mathematical Analysis and Applications 371 (2010), no. 2, 485-495.

[9] P. Aiena, J. Guillen, P. Peña : *Property* (w) *for perturbation of polaroid operators.* Linear Algebra and Applications **4284** (2008), 1791-1802.

[10] P. Aiena, J. Guillen, P. Peña: *Localized SVEP, property* (b) *and property* (ab), Mediterranean Journal of Mathematics (2013), DOI 10.1007/s00009-013-0257-1.

[11] P. Aiena, J. Guillen, P. Peña: *Quasi totally hereditarily normaloid operatos*, (2013), preprint.

[12] P. Aiena, T. L. Miller: *On generalized a-Browder's theorem.* Studia Mathematica 180 (2007), no. 3, 285300.

[13] P. Aiena, M. M. Neumann : *On the stability of the localized single-valued extension property under commuting perturbations* (2012), to appear in Proceding American Mathematical Society.

[14] M. Amouch, H. Zguitti *On the equivalence of Browder's and generalized Browder's theorem.* Glasgow Mathematical Journal **48**, (2006), 179-185.

[15] M. Amouch, M. Berkani: *On the property* (gw). Mediterranean Journal of Mathematics **5**, (2008), 371-378.

[16] M. Berkani : *On a class of quasi-Fredholm operators.* Integral Equations and Operator Theory **34** (1), (1999), 244-249.

[17] M. Berkani: *Index of B-Fredholm operators and generalization of a Weyl's theorem*, Proceding American Mathematical Society, vol. 130, **6**, (2001), 1717-1723.

[18] M. Berkani, M. Sarih: *On semi B-Fredholm operators.* Glasgow Mathematics Journal **43**, No. 4, (2001), 457-465.

[19] M. Berkani, M. Sarih, H. Zariuoh: *Browder-type theorems and SVEP.* Mediterranean Journal of Mathematics **8**, No. 4, (2011), 399-409.

[20] M. Berkani, H. Zariuoh: *Extended Weyl type theorems.* Mathematica Bohemica **134**, No. 4, (2009), 369-378.

[21] M. Berkani, H. Zariuoh: *New extended Weyl type theorems.* Matematichki Vesnik 62 (2010), no. 2, 145154.

[22] M. González,M. Mbekhta and M. Oudghiri: *On the isolated points of the surjective spectrum of a bounded operator* Proceding American Mathematical Society 136 (2008), no. 10, 3521-3528.

[23] S. Grabiner: *Uniform ascent and descent of bounded operators* Journal of Mathematical Society of Japan **34** (1982), 317-337.

[24] H. Heuser: *Functional Analysis.* (1982), Marcel Dekker, New York.

[25] D. C. Lay: *Spectral analysis using ascent, descent, nullity and defect.* Mathematiche Annalen **184** (1970), 197-214.

[26] M. Mbekhta, V. Müller: *On the axiomatic theory of the spectrum II.* Studia Mathematica **119** (1996), 129-147.

[27] Q. Jiang, H. Zhong, S. Zhang :*Components of topological uniform descent resolvent set and local spectral theory*, Linear Algebra and Applications **438** (2013), no. 3, 11491158.

[28] J. J. Koliha *Isolated spectral points*: Proceding American Mathematical Society **124** (1996), 3417-3424.

[29] M. Oudghiri: *Weyl's and Browder's theorem for operators satisfying the SVEP* Studia Mathematica **163**, 1, (2004), 85-101.

[30] K. B. Laursen, M. M. Neumann *Introduction to local spectral theory.*, Clarendon Press, Oxford 2000.

[31] V. Rakočević: *Semi-Browder's operators and perturbations*. Studia Mathematica **122** (1966), 131-137.

[32] Q. Zeng, H. Zhong: *On the property* (*gb*) *and perturbations* (2012), to appear in Applied and Abstract Analysis.

DIEETCAM, Facoltà di Ingegneria,, Viale delle Scienze, I-90128 Palermo (Italy), E-mail paiena@unipa.it

Departamento de Matemáticas, Facultad de Ciencias, ULA; Merida (Venezuela), E-mail rguillen@ula.ve

Departamento de Física y Matemáticas, NURR, ULA; Trujillo (Venezuela), E-mail pedrop@ula.ve