Journal of Russian Laser Research, Volume 35, Number 1, January, 2014

Dedicated to the memory of our friend and colleague Allan Solomon,
Emeritus Professor at Sorbonne University, Paris
and Emeritus Professor of Mathematical Physics
at the Open University, Milton Keynes, UK.

BALANCE EQUATIONS-BASED PROPERTIES OF THE RABI HAMILTONIAN

Antonino Messina,[1]* Anna Napoli,[1] Margarita A. Man'ko,[2] and Vladimir I. Man'ko[2,3]

[1]*Dipartimento di Fisica e Chimica, University of Palermo*
Via Archirafi 36, Palermo 90123, Italy

[2]*P. N. Lebedev Physical Institute, Russian Academy of Sciences*
Leninskii Prospect 53, Moscow 119991, Russia

[3]*Moscow Institute of Physics and Technology (State University)*
Institutskií per. 9, Dolgoprudnyí, Moscow Region 141700, Russia

*Corresponding author e-mail: antonino.messina@unipa.it

Abstract

A stationary physical system satisfies peculiar balance conditions involving mean values of appropriate observables. In this paper, we show how to deduce such quantitative links, named balance equations, demonstrating as well their usefulness in bringing to light physical properties of the system without solving the Schrödinger equation. The knowledge of such properties in the case of the Rabi Hamiltonian is exploited to provide arguments to make easier the variational engineering of the ground state of this model.

Keywords: ground state, variational approach, Wigner function, Rabi model, balance equations.

1. Introduction

Let $H(\eta)$ be the time-independent Hamiltonian of a binary physical system S, whose states $\{\mid \phi\rangle\}$ live and evolve in the Hilbert space $\mathcal{H}$. We assume that such a Hermitian operator depends on a real (set of) parameter(s) η. We denote by $\mid \psi_E(\eta)\rangle$ a normalized eigenvector of $H(\eta)$ of eigenvalue $E(\eta)$. We recall that in the Schrödinger representation the time derivative of an operator A (generally depending both on η and explicitly on t), denoted by $\dfrac{dA}{dt}$, is defined as ($\hbar = 1$)

$$\langle\phi(t) \mid \frac{d}{dt}A \mid \phi(t)\rangle \equiv \frac{d}{dt}\langle\phi(0) \mid e^{iHt}Ae^{-iHt} \mid \phi(0)\rangle. \tag{1}$$

Manuscript submitted by the authors in English on December 18, 2013.
1071-2836/14/3501-0101

It is well known that such a definition leads to

$$\frac{d}{dt}A = \frac{\partial}{\partial t}A + i[H, A], \tag{2}$$

which, on the one hand, implies that A is a constant of motion, if either $[A, H] = 0$ or the partial time-derivative contribution is compensated by the commutator term (see, e.g., [1]). On the other hand, Eqs. (1) and (2) guarantee that the expectation value of any time-independent operator A taken on any stationary state $\mid \psi_E(\eta)\rangle$ of $H(\eta)$ vanishes. Since, in general, A does not commute with $H(\eta)$, Eq. (2) originates in each stationary state a link between the mean values of the operators additively stemming from the commutator $[A, H]$. This algebraic relation is an identity with respect to η and, when A is an observable, it reflects, in any stationary state of $H(\eta)$, the existence of a necessary quantitative relationship among the η-dependent expectation values of appropriate operators which, being Hermitian like $i[A, H]$, are amenable to a physical interpretation.

Following [2–6], we call these equations balance equations associated to $H(\eta)$.

The interest in the balance equations is at least threefold.

First of all, they are the fruit of a simple calculation and might help to highlight physical properties common to all the stationary states of S, in particular, its ground state, without solving the time-independent Schrödinger equation, which might indeed be difficult to handle. On the other hand, considered that, when this is the case, one generally looks for approximate expressions of the eigensolutions of the Hamiltonian model under scrutiny, the balance equations might provide an exact tool to check, a posteriori, the accuracy of the approximated solution found. As the third and final remark, we, in addition, emphasize that the knowledge of a set of easily constructed and physically transparent balance equations associated to a given Hamiltonian might put at our disposal arguments/constraints to control a priori the quality of the approximation route. For example, we might be asked, on physical grounds, to express $H(\eta)$ in a selected basis where it might be diagonalized with the help of a convincing truncation protocol, or, within a variational approach aimed at finding the ground state of the system under study, we might wisely guess a "lucky" class of trial states.

In this paper, we highlight the usefulness and advantages of such balance equations-based approach by considering, as an application, the variational engineering of the ground state of the following ever appealing and always fashionable quantum Rabi model [7–9]

$$H = \omega\alpha^\dagger\alpha + \lambda(\alpha + \alpha^\dagger)\sigma_x + \frac{\omega_0}{2}\sigma_z \equiv H(\eta) \tag{3}$$

describing the linear coupling of strength λ between a quantum harmonic oscillator (or a single bosonic mode) of angular frequency ω and an effective two-level system (or a spin 1/2) with the Bohr frequency ω_0. The dynamical variables of the quantum mode are the annihilation and creation operators α and $\alpha^\dagger$, whereas the two-level system is completely represented by the Pauli matrices σ_x, σ_y, and σ_z.

Over the last 70 years, this paradigmatic model has been investigated in a myriad of papers, even in its multimode version useful to treat the two-level system as an open quantum system [10]. Many facets of its static, dynamical, and thermodynamical behavior have been theoretically disclosed and experimentally revealed in a lot of quite different physical contexts as, for example, cavity, circuit, solid-state quantum electrodynamics, quantum information, and so on [11–16]. Quite recently advancements on the exact analytical representation of the eigensolutions of H, motivated by the experimental realization of ultra strong coupling regimes, have been reported [8, 17–19].

The Rabi model depends on two independent effective real parameters defining a bidimensional space S, which contains a region of experimental interest where, however, the physical behavior of the system is analytically less characterized with respect to that which the system exhibits in the complementary region of its parameter space. In this challenging region, the relative weight of the three parameters ω, λ, and ω_0 does not legitimatize any obvious perturbative treatment of H, so that the entanglement that is established between the two parties in such a condition reflects the occurrence of a mutual influence higher than that exhibited by the system out of this region. Such a situation singles out the so-called intermediate coupling regime between the weak and the strong regimes realized by the system when $\lambda^2 \ll \omega\omega_0$ and $\lambda^2 \gg \omega\omega_0$, respectively.

We stress from the very beginning that we do not intend here to improve the quality of the variational ground state of the Rabi model as reported in the literature. Rather we wish to provide a concrete example of how the knowledge of an appropriate set of balance equations associated to the Rabi Hamiltonian allows one to understand the failure of an optimized specific trial state (for example, coherent state) in some regions of the parameter space and at the same time how to improve the class of trial state in order to get a new optimized solution closer to the exact ground state in a larger region of S. In other words, the balance equations-based approach, exploited within the variational framework, might provide arguments useful to justify a specific choice of the trial state in accordance with our expectations (that is, disposable balance equations) concerning the exact ground state.

This paper is organized as follows.

In Sec. 2, we report some useful general properties possessed by the Rabi system in its ground state, while the construction of a set of exact balance equations is presented in Sec. 3. The knowledge of these exact constraints is exploited in Sec. 4 to engineer a class of trial variational ground states and find an analytical optimized expression of the ground state. Some concluding remarks are pointed out in Sec. 5, where possible developments based on the novel approach reported in this paper are briefly discussed.

2. Some General Properties of the Rabi Hamiltonian Ground State

The derivation of balance equations associated with the Rabi Hamiltonian is postponed to the next section. Here, instead, we wish to resume and/or to derive some exact properties of the ground state of this model. Such properties, conjugated with the balance equations, play an interesting role since they reveal in a transparent way the nature of η-dependent constraints in the structure of the exact ground state of the Rabi model, which then must be taken into account when tailoring the analytical form of a variational trial state. It is easy to prove that the hermitian and unitary parity operator $P = -\sigma_z \cos(\pi\alpha^\dagger\alpha)$ commutes with H, so that the normalized stationary states $\mid \psi_{E,p}\rangle$ of H of definite parity $p = \pm 1$, belonging to the energy eigenvalue E, may be represented as follows:

$$\mid \psi_{E,p}\rangle = \sum_{n=0}^{\infty} a_n^{(p)} \mid n\rangle \mid \sigma = (-1)^{n+1} p\rangle \tag{4}$$

provided that the bosonic state

$$\mid \phi_{E,p}\rangle = \sum_{n=0}^{\infty} a_n^{(p)} \mid n\rangle \tag{5}$$

is normalized.

Since the knowledge of $\mid \phi_{E,p}\rangle$ univocally determines $\mid \psi_{E,p}\rangle$, it is not surprising that, as a consequence of the peculiar spin–boson entanglement induced by P, the Rabi Hamiltonian may be unitarily traced back to the following p-dependent bosonic Hamiltonian

$$\tilde{H}_p = \omega\alpha^\dagger\alpha + \lambda(\alpha + \alpha^\dagger) - \frac{\omega_0}{2} p \cos(\pi\alpha^\dagger\alpha), \tag{6}$$

where $p = \pm 1$ is an eigenvalue of P.

It has been rigorously demonstrated [5, 6] that the ground state of $\tilde{H}_+$ generates the ground state $\mid g_{+1}\rangle$ of H everywhere in S, and that there exist regions of S where the ground state of $\tilde{H}_{-1}$ is degenerate with that found in $\tilde{H}_+$. This happens, for example, when $\omega_0 = 0$ and, in such a case, ground states of H of no definite parity exist. It is easy to convince oneself that the probability amplitudes of the ground state of $\tilde{H}_p$ (and then of H) may be chosen all real without loss of generality.

It is useful to write $\mid \psi_{E,p}\rangle$ exploiting the eigenstates of σ_x instead of those of σ_z. It is not difficult to show that

$$\mid \Psi_{E,p}\rangle = \frac{1}{\sqrt{2}}\big\{ \mid \phi_{E,p}\rangle \mid +\rangle_x - p\cos(\pi\alpha^\dagger\alpha) \mid \phi_{E,p}\rangle \mid -\rangle_x \big\}. \tag{7}$$

Let E_g be the exact ground state energy of the Rabi model. Since, occasionally, E_g may result in degeneracy, we simply denote by $\mid g_p\rangle$ a (the) solution of $H \mid g\rangle = E_g \mid g\rangle$ having a definite parity p. It is possible to show the validity of the following properties everywhere in S:

$$-\frac{\omega_0}{2} - \frac{\lambda^2}{\omega} \le E_g \le -\frac{\omega_0}{2}, \tag{8}$$

$$\langle g_p \mid \sigma_z \mid g_p\rangle = -p\langle g_p \mid \cos(\pi\alpha^\dagger\alpha) \mid g_p\rangle \le 0, \tag{9}$$

$$\langle g_p \mid (\alpha + \alpha^\dagger)\sigma_x \mid g_p\rangle \le 0, \tag{10}$$

$$-\omega_0 \le \omega\langle g_p \mid \alpha^\dagger\alpha\cos(\pi\alpha^\dagger\alpha) \mid g_p\rangle = -\omega\langle g_p \mid \alpha^\dagger\alpha\sigma_z \mid g_p\rangle \le \omega_0. \tag{11}$$

Equations (8)–(11) are certainly valid for $\mid g_{+1}\rangle$ (the ground state of H of parity $p = +1$ exists in all points of S) and also for $\mid g_{-1}\rangle$ in the case of degeneration. Equation (8) stems from elementary considerations based on the position of E_g in the energy spectrum of H. Equation (9) reflects the property $P \mid g_p\rangle = p \mid g_p\rangle$ (then valid also outside the minimum energy subspace), as well as that the probability of finding the oscillator in its ground state $\mid g_{+1}\rangle$ ($\mid g_{-1}\rangle$) with an even number of excitations exceeds (is less than) that of finding the oscillator with an odd number of excitations. Equation (10) means that the covariance of the dimensional coordinate of the quantum oscillator and the "coordinate" of the two-level system is always negative in S, since $\langle g_p \mid (\alpha+\alpha^\dagger) \mid g_p\rangle = \langle g_p \mid \sigma_x \mid g_p\rangle = 0$ for symmetry reasons. Moreover, this equation says that the interaction energy counters the nonnegative contribution of the free energy of the quantum oscillator on $\mid g_p\rangle$ in accordance with the requirement for E_g prescribed by Eq. (8). This last comment is also valid for the expectation value of σ_z on $\mid g_p\rangle$.

Equations (11) and (9) reveal the occurrence of a limited variability for $\big|\langle g_p \mid (\alpha^\dagger\alpha)x \mid g_p\rangle\big|$, with $x = \cos(\pi\alpha^\dagger\alpha)$ or $x = \sigma_z$, traceable back to the negativity of E_g everywhere in S. The link between $\langle g_p \mid \alpha^\dagger\alpha\cos(\pi\alpha^\dagger\alpha) \mid g_p\rangle$ and $\langle g_p \mid \alpha^\dagger\alpha\sigma_z \mid g_p\rangle$ is once again a consequence of the entanglement introduced in the binary Rabi system by the parity constraint. In principle, other identities and inequalities in S, as in Eqs. (8) and (11), may be systematically constructed exploiting Eq. (5) and the relations deducible by taking the mean value in the ground state $\mid g_p\rangle$ of the operator equation expressing the anticommutator between H and appropriate observables. From the mathematical point of view, the only hypothesis used to derive Eqs. (8)–(11) is the assumed existence of all the expectation values involved, since, in accordance with our procedure, we are not making use of the analytical form of $\mid g_p\rangle$.

3. Balance Equations

In this section, we demonstrate that all the stationary states of H share the occurrence of analytical links valid everywhere in S among the expectation values of selected and physically interpretable observables as anticipated in the Introduction. We call such relations balance equations even if, strictly speaking, they are identities in S. What makes the balance equations of theoretical interest is the fact that they may be systematically generated as necessary conditions of the stationarity, without any a priori knowledge of the analytical form of the eigensolutions of H.

To make easier capturing the physical meaning of the results to be derived in this section, we write the Rabi hamiltonian describing the bosonic mode as that for a quantum oscillator in its phase space

$$H = \frac{p^2}{2m} + \frac{1}{2}m\omega_2 q^2 + F_0 q\sigma_x + \frac{\omega_0}{2}\sigma_z - \frac{\omega}{2}\,, \tag{12}$$

where

$$q = \sqrt{\frac{1}{2m\omega}}(\alpha + \alpha^\dagger), \qquad p = i\sqrt{\frac{m\omega}{2}}(\alpha^\dagger - \alpha), \qquad F_0 = \sqrt{2m\omega}\lambda. \tag{13}$$

Since, in view of Eq. (2),

$$m\frac{d^2q}{dt^2} = -m\omega^2 q - F_0\sigma_x \equiv F_q + F_e, \tag{14}$$

we immediately derive

$$\langle\psi_E(\eta) \mid F_q \mid \psi_E(\eta)\rangle = -\langle\psi_E(\eta) \mid F_e \mid \psi_E(\eta)\rangle. \tag{15}$$

Thus, in the ground state of the Rabi Hamiltonian the elastic force F_q and external force F_e due to the two-level subsystem are, on the average, opposite. When the ground state has a definite parity, both vanish, and Eq. (10) tells us that the two forces are anticorrelated since the expectation values of q and σ_x vanish on $\mid g_p\rangle$.

To appreciate further how a balance equation may contribute to bringing light to peculiar properties of the ground state of the Rabi system, we exploit the fact that the mean value on $\mid \Psi_{E,p}\rangle$ of the operator $\dfrac{d^2(q\sigma_x)}{dt^2}$ vanishes. The resulting balance equation on $\mid g_p\rangle$, in particular, becomes

$$\langle g_p \mid p^2/2m \mid g_p\rangle = \frac{F_0}{2}\langle g_p \mid q\sigma_x \mid g_p\rangle + \langle g_p \mid m\omega^2 q^2/2 \mid g_p\rangle, \tag{16}$$

where the Fock states in the expression of $\mid g_p\rangle$ are to be considered in the q-representation.

Getting rid of $\langle g_p \mid p^2/2m \mid g_p\rangle$ between Eq. (16) and the expression of E_g formally deducible from Eq. (12), we can use the resulting equation from such elimination to cast the double limitation on the lowest energy eigenvalue of H, as given by Eq. (14), in the following form:

$$\frac{1}{2m\omega} - \frac{\lambda^2}{m\omega^3} + C \leq \triangle^2(q\sigma_x) \leq \frac{1}{2m\omega} + C, \tag{17}$$

where $\triangle^2(q\sigma_x) \equiv \langle g_p \mid (q\sigma_x)^2 \mid g_p\rangle - \langle g_p \mid q\sigma_x \mid g_p\rangle^2$ and

$$m\omega^2 C = -(1 + \langle g_p \mid \sigma_z \mid g_p\rangle) - 3F_0\langle g_p \mid q\sigma_x \mid g_p\rangle - \langle g_p \mid q\sigma_x \mid g_p\rangle^2. \tag{18}$$

Considering that, in view of Eq. (7), $\mid g_p\rangle$ may be formally written as

$$\mid g_p\rangle = \frac{1}{\sqrt{2}}\{\mid \phi_{E_g,p}\rangle \mid +\rangle_x - p\cos(\pi\alpha^\dagger\alpha) \mid \phi_{E_g,p}\rangle \mid -\rangle_x\}, \tag{19}$$

we immediately see that

$$\langle g_p \mid q\sigma_x \mid g_p\rangle = \langle \phi_{E_g,p} \mid q \mid \phi_{E_g,p}\rangle, \tag{20}$$

and then

$$\triangle^2(q\sigma_x) = \triangle^2_\phi(q), \quad \text{where} \quad \triangle^2_\phi(q) = \langle \phi_{E_g,p} \mid q^2 \mid \phi_{E_g,p}\rangle - \langle \phi_{E_g,p} \mid q \mid \phi_{E_g,p}\rangle. \tag{21}$$

Since the fluctuations of q in a coherent bosonic state is $1/2m\omega$, independent of its amplitude, the exact inequality (17) suggests that the fluctuation of q on $\mid \phi_{E_g,p}\rangle$ might exhibit values different from $1/2m\omega$ in selected domains of the parameter space S. This observation is of relevance since it explains why an optimized coherent state fails in representing $\mid \psi_{g,+1}\rangle$ everywhere in S, as indeed found in the literature, without any attempt to go beyond [2, 3]. Thus, we are interested in finding the argument strengthening the hypothesis that, to overcome such a failure, we must introduce a variational trial state exhibiting flexibility in the fluctuation of q in different coupling regimes.

To this end, we now deduce another balance equation based on the second derivative of $\omega\alpha^\dagger\alpha$. The final result may be expressed in the following suggestive form when restricted to the ground state $\mid g_p\rangle$:

$$\langle g_p \mid F_q F_e \mid g_p\rangle + \langle g_p \mid p\,\frac{dF_e}{dt} \mid g_p\rangle + F_0^2 = 0, \quad \text{with} \quad \frac{dF_e}{dt} = F_0\omega_0\sigma_y. \tag{22}$$

This balance equation discloses the existence in each point of S of a link between the covariance of the two forces F_q and F_e on the oscillator and the covariance between the oscillator momentum and the rapidity of variation of F_e. In particular, it says that coupling regimes, where the correlations between the two forces are almost vanishing, are characterized by an anticorrelation between p and $\dfrac{dF_e}{dt}$. The relevance of this comment may be elucidated by the consideration that for $\omega_0 < \omega$, on the one hand, whatever η is, $\mid \phi_{E_g,+1}\rangle$ may be well approximated by an appropriate coherent state exhibiting an effective displacement $\bar{q}$.

When instead $\omega_0 \gg \omega$, in the region of S where $\lambda^2 \ll \omega_0\omega$, F_q and F_e decorrelate more and more approaching zero, when $\lambda^2/\omega_0\omega$ tends to zero. In such a condition, the correlation between p and $\dfrac{dF_e}{dt}$, in view of Eq. (22), grows in absolute value approaching its minimum negative value $-F_0^2$. Thus, in this region of S, the anticorrelation between p and $\dfrac{dF_e}{dt}$ leads to the diminution of the displacement of the oscillator with respect to $\bar{q}$. This means that, when $\omega_0 \gg \omega$ and λ is such as to guarantee a weak coupling regime, the ground state of the Rabi Hamiltonian exhibits an almost vanishing mean value of q, but the spread of the same observable is greater than that associated to the coherent state occurring when $\omega_0 < \omega$. Mathematically this fact stems from the inevitable presence in $\mid \phi_{E_g,+1}\rangle$ of the odd Fock states necessary to comply with the condition $\langle g_p \mid p\dfrac{dF_e}{dt} \mid g_p\rangle < 0$.

This heuristic analysis is qualitatively compatible with the double inequality (17) and provides an example of a region of S where certainly the trial choice of $\mid \phi_{E_g,+1}\rangle$ in the form of the coherent state is not legitimate. On the basis of the suggestions stemming from the arguments developed in this section, we can construct a proposal for $\mid g_{+1}\rangle$ flexible enough to comply with all the balance equations, exact necessary conditions, and to recover its coherent state-based description when $\omega_0 < \omega$.

4. Engineering a Variational Ground State

On the basis of what we have learned and highlighted about the properties of the ground state of parity $+1$ in S, we must go beyond the trial choice of $\mid \Phi_{E_g,+1}\rangle$, as given by Eq. (5), since the coherent state $D(\beta) \mid 0\rangle$ of the optimizable amplitude β essentially becomes incompatible with the double inequality (17) everywhere in S. The operator $D(\beta)$ is the unitary displacement operator defined as

$$D(\beta) = \exp\{\beta\alpha^\dagger - \beta^*\alpha\}. \tag{23}$$

To gain more flexibility in the fluctuations of $(\alpha + \alpha^\dagger)$ while not renouncing as well a coherent recover of $\mid \phi_{E_g,+1}\rangle$, where appropriate in S, we propose the following two real-parameter squeezed and displaced states:

$$\mid \phi_{E_g,+1}(\beta,\gamma)\rangle = S(\gamma)D(\beta) \mid 0\rangle, \tag{24}$$

the squeezing unitary operator $S(\gamma)$ being

$$S(\gamma) = \exp\left\{\gamma\left(\alpha^{\dagger 2} - \alpha^2\right)/2\right\}. \tag{25}$$

One can demonstrate that $\mid \phi_{E_g,+1}(\beta,\gamma)\rangle$ identically satisfies Eqs. (8)–(11) and that the two equations in γ and β obtained from the balance equations (16) and (22) coincide with the two variational equations determined from the optimization of the energy functional of the system in the class of states (24) with respect to the two variational parameters γ and β. Moreover, Eq. (24) is compatible with Eq. (17) since, when $\gamma = 0$, the squeezed displaced state gives back the coherent state $\mid \beta\rangle$ and, in general, meets all the requirements on the ground state built so far in this paper. To find the optimized dependence of γ and β on the model parameters, we must evaluate the energy $E(\beta,\gamma)$, that is,

$$E(\beta,\gamma) = \langle 0 \mid S^\dagger(\gamma)D^\dagger(\beta)\tilde{H}_+ D(\beta)S(\gamma) \mid 0\rangle, \tag{26}$$

where from Eq. (6)

$$\tilde{H}_+ = \omega\alpha^\dagger\alpha + \lambda(\alpha + \alpha^\dagger) - \frac{\omega_0}{2}\cos(\pi\alpha^\dagger\alpha) \tag{27}$$

is a nonlinear restriction of H in the parity-invariant subspace with $p = +1$, where the ground state certainly is.

To this end, in the following we deduce an interesting link between the mean value E of the Rabi reduced Hamiltonian $\tilde{H}_+$ in any arbitrary pure state of the quantum bosonic mode and the value $W(0,0)$ assumed by its Wigner function $W(p,q)$ [20] in the same state. It is well known that the expectation value of the parity operator $\cos(\pi\alpha^\dagger\alpha)$ is related to

$$W(0,0) = 2\int \langle x \mid \phi\rangle\langle\phi \mid -x\rangle\, dx \tag{28}$$

in a generic state $\mid \phi\rangle$ given in the Fock representation by

$$\langle\phi \mid \cos(\pi\alpha^\dagger\alpha) \mid \phi\rangle = W(0,0)/2 = \mathrm{Tr}\left(\rho\cos(\pi\alpha^\dagger\alpha)\right), \tag{29}$$

where $\rho = \mid \phi\rangle\langle\phi \mid$. Then

$$\begin{aligned} E &\equiv \mathrm{Tr}(\rho\tilde{H}_+) = \mathrm{Tr}\left(\rho[\omega\alpha^\dagger\alpha + \lambda(\alpha+\alpha^\dagger)]\right) - \mathrm{Tr}\left(\rho[\omega_0/2]\cos(\pi\alpha^\dagger\alpha)\right) \\ &= \mathrm{Tr}\left(\tilde{\rho}[\omega\alpha^\dagger\alpha - \lambda^2/2\omega]\right) - (\omega_0/4)W(0,0) = \omega\langle\tilde{n}\rangle - (\lambda^2/2\omega) - (\omega_0/4)W(0,0), \end{aligned} \tag{30}$$

where $\tilde{\rho} = D^\dagger(-\lambda/\omega)\rho D(-\lambda/\omega)$, with $D(-\lambda/\omega)$ being the displacement operator accomplishing the exact diagonalization of $\tilde{H}_+$ when $\omega_0 = 0$. The mean value of $\alpha^\dagger\alpha$ in the state $\tilde{\rho}$ is here denoted by $\langle\tilde{n}\rangle$. Considering that $|W(q,p)| \leq 2$ [21] in the oscillator phase space, we immediately arrive at

$$-\frac{\omega_0}{2} - \frac{2\lambda^2}{\omega} \leq E(\beta,\gamma) - \omega\langle\tilde{n}\rangle \leq \frac{\omega_0}{2} - \frac{2\lambda^2}{\omega}\,. \tag{31}$$

When $|\,\phi\rangle$ belongs to the class of trial states given by Eq. (24), $W(0,0) = e^{-2\beta^2}$, and after evaluating $\langle\tilde{n}\rangle$, we obtain

$$E(\beta,\gamma) = \omega\left[\beta^2 e^{2\gamma} + \sinh(\gamma^2)\right] + 2\lambda\beta e^{\gamma} - (\omega_0/2)e^{-2\beta^2}. \tag{32}$$

Developing this variational approach, we minimize $E(\beta,\gamma)$ finding $\bar{\beta}$ and $\bar{\gamma}$, variable in S and such that the unitary operator $S(\bar{\gamma})D(\bar{\beta})$ transforms $\tilde{H}_+$ into the sum of a diagonal contribution, whose ground state is the vacuum state and another one that may legitimately be considered as perturbative with respect to the diagonal one. This means that the ground state found within the class of trial states given by Eq. (24) is reasonably close to the exact ground state and, as a consequence, that we are in a position to investigate properties, different from its energy, of the Rabi system in its ground state, using the variationally optimized fundamental state.

It is interesting to observe that other classes of trial states may be proposed, all fulfilling the balance equations, so that a comparative investigation of their reliability with the class here proposed might lead to the construction of new more reliable accurate proposals. This task will be faced with in a successive paper.

5. Concluding Remarks

In this paper, we presented and applied the novel idea of balance equations, which is a quantitative link existing among mean values of observables that necessarily hold in each stationary state of a physical system. Generally speaking, the balance equations are infinitely many and may be seen as a class of constraints making the system stationary. This circumstance has led the idea of exploiting the knowledge of even a finite set of such balance equations to introduce a systematic approach to bring to light physical properties that the system possesses in stationary conditions. To show the concreteness of such a point of view, we conjugated the balance equations with the variational protocol considering in detail the Rabi model, that is, one bosonic mode interacting with one qubit. For this model, we demonstrated the usefulness of the balance equations to choose the probe function of its ground state. To study the constraints for the system's ground state energy, we used, in particular, the known inequalities for the Wigner functions [20, 22]. Since there exists the probability representation of quantum states, where the wave functions and energies of stationary states are determined by the tomographic-probability distributions (see, e.g, [23] and the recent review [24]) obeying the corresponding equations [25], the balance-equations approach can be extended to study the properties of the probability distributions of the ground state satisfying the quantum equations. Such an extension will be considered in a future publication.

Acknowledgments

M.A.M. and V.I.M. thank the University of Palermo for the invitation and kind hospitality.

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