Some relationships between the calculus of Newton, Bombelli's *Algebra* and Leibniz.

Nicla Palladino*

Abstract: In this paper we develop some relationships between the approximation method Rafael Bombelli used to find the square root of an integer number in his *Algebra* (1572), Leibniz's "hidden calculus" in infinitesimal algorithms (*Nova Methodus*, 1684) and Newton's procedures of extraction *more arithmetico* of the root of a binomial: these procedures lead to the series development of a binomial root that Newton used in integral calculus (*ca.* 1666).

Riassunto: Nell'articolo, vengono confrontati alcuni procedimenti di approssimazione, dovuti a Rafael Bombelli per calcolare la radice quadrata di un numero (*Algebra*, 1572) con il "calcolo nascosto" degli algoritmi infinitesimali di Leibniz (*Nova Methodus*, 1684) e, ancora, con procedure per l'estrazione della radice di un binomio concepite da Newton: queste ultime conducono agli sviluppi in serie di binomi che Newton adoperò per il calcolo integrale.

1. THE SQUARE ROOT AND METHODS OF APPROXIMATION.

It is possible to find *direct* and *recursive* methods for calculating square roots of integer numbers since the second century. Bombelli, in his *Algebra*, and Newton, in *Arithmetica Universalis,* exposed Claudius Ptolemy's method that has been used for calculating both *exact* and *non-exact* square roots.

Newton proposed a graphic scheme ("*a danda lunga"*, which means all passages included) for calculating the square root following Ptolemy's method[1]. The extraction of the square root of 22178791 is one of the two examples given by Newton [Newton 1707, pp. 32-33]:

*I Trav. Ianniello, 7. Frattamaggiore (NA). E-mail: nicla.palladino@unina.it

[1] Ptolemy's method is nearly equal to methods that we can read nowadays in arithmetic manuals for the first level middle school.

```
√22·17·87·91 | 4709,43 etc.
 16              87 × 7 = 609
   617            9409 × 9 = 84681
   609            94184 × 4 = 376736
     88791         941883 × 3 = 2825649
     84681
       4110.00
       376736
         34264.00
         2825649
           600751...
           ............
```

Bombelli also explained Ptolemy’s method in the first “libro” of his *Algebra* [Bombelli 1966, pp. 34-35].[2] He showed the extraction with an example: how to calculate the square root of 5678, by the “a danda lunga” procedure:

“If we would to find the side[3] of 5678, we have to do as follows. Let us draw the *a* line so far that it is possible to insert another number below, and over the 8 make a point. Then we leave a blank going to the left and over the 6 make another point. If the number is more large, we have to make other points, leaving blanks beetwen the ciphers. And then, going on from the left to the right, we have to take the ciphers until the first point, that are 56, and put then under the *a* line. Let us find the closest square number in 56, that is not smaller than 56, and that is 49; its side is 7 and put 7 under 6, where is the first point, over the *a* line and put two 7s at the side of the scheme. Let us draw the *e* line and make the sum, that is 14, and the product, that is 49; put 49 under 56, draw the *b* line and make di difference[4] that is 7; the first part is ended.

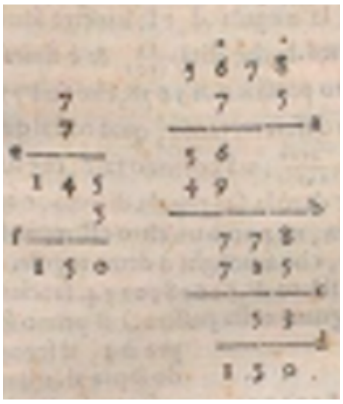
Going on, if we put 7, that is between the 6 and the 8, over the *a* line, and it becomes 77.

We have to find how many times 14, that is under the *e* line, is in 77, and that is 5; let us put 5 next to 14, and becomes 145, put another 5 under 145, and make the sum (drawing the *f* line) that is 150; and put another 5 under the 8, where is another point, and put the 8 under the *b* line next to 77, and becomes 778; under 778 put the product of 145 by 5 that is 725, and make the difference (drawing the *c* line) that is 53, and under 53 draw the *d* comma[5], and under that put 150, that is under the *f* line, and it becomes 53/150. The extraction is ended, that is the

2 This point is maybe less known because Bombelli’s *Algebra* is not easy to read (the first edition was published in Bologna, by G. Rossi, 1572, in three chapters): he used the *Millecinquecento Italian language* that is not always easy to understand.
3 That is the square root.
4 Bombelli used the verb “cavare”, that is “to take out”.
5 Bombelli used the term “virgula” that is “comma”, but let us read “line”.

approximate side of 5678, that is 75 and 53/150 that are different by the square of the "rotto"[6], that is 2809/22500*"*.[7]

Bombelli showed the corresponding graphic scheme and concluded saying that the approximate square root of 5678 is 75 plus 53/150 and the difference between 5678 and $\left(75+\frac{53}{150}\right)^2$ is 2809/22500;[8]

<table>
<tr><td></td><td>5678
75 a
7 56
e 7 49 b
145 778
f 5 725 c
150 53 d
150</td></tr>
<tr><td>Original Bombelli's scheme for the square root of 5678.</td><td>Reproduction of Bombelli's scheme.</td></tr>
</table>

[6] I.e. the square of the remainder (of the unit).

[7] "Se si haverà a trovare il lato (come sarebbe di 5678), facciasi come si vede qui da sotto. Tirisi la linea *a* tanto lontana, che sotto il numero ci capisca un altro ordine di caratteri, e sopra l'8 si faccia un punto, e poi venendo a man sinistra, lassando un carrattero nel mezo, e sopra il 6, si faccia un altro punto, e se il numero fosse maggiore, si seguitarà di fare li punti; ma interponendo un punto da un carrattero all'altro, e fatto questo, si ricomencia dall'altro capo a man sinistra andando verso la destra, e si pigliano gli caratteri, che sono fino al primo punto, e si pongono sotto la linea *a*, il qual'è 56. Fatto questo, si trova un numero quadrato, e il più prossimo, ma che non sia maggiore di 56, il quale sarà 49, che il suo lato è 7, il qual 7 si mette sotto il 6, sopra la linea *a*, sopra del quale è il primo punto, e dui altri 7 si pongono da canto, sotto li quali si tira la linea *e* poi si somma, che fanno 14, et il produtto delli detti dui 7 l'uno nell'altro è 49, il quale si mette sotto il 56, e si tira la linea *b* e si cava di 56 resta 7, et è finito fino al primo punto. E per seguire avanti; se gli aggionge il 7, che è sopra la linea *a* fra il 6, e l'8, e farà 77. Hora si veda quante volte entra il 14, ch'è sotto la linea *e* nel 77, che vi entra 5; il qual 5 si mette al pari del 14, e dirà 145, et un altro 5 si mette sotto quello, e si sommano (tirando la linea *f*) e fa 150, et il medesimo 5 si mette sotto l'8, sopra il qual è l'altro punto, e l'8 si mette sotto la linea *b* al pari del 77, e fa 778; sotto il quale se gli mette il produtto di 145 nel 5, che vi è sotto, ch'è 725, e si cava l'uno dell'altro (tirando la linea *c*) e resta 53, sotto il quale si tira la virgula *d*, e se li mette sotto il 150, che è sotto la linea *f* che dirà 53/150, et è finita la estrattione, over il lato prossimo di 5678, che sarà 75, e 53/150 che solo saranno differenti tanto, quanto è il quadrato del rotto, cioè 2809/22500.*"*

[8] In fact $\left(75+\frac{53}{150}\right)^2=(75)^2+2\cdot 75\cdot\frac{53}{150}+\left(\frac{53}{150}\right)^2=(75)^2+53+\left(\frac{53}{150}\right)^2=5625+53+\frac{2809}{22500}=5678+\frac{2809}{22500}$.

We can able to show the same example according to the modern schemes (as also Newton wrote):

$$
\begin{array}{l|l}
\sqrt{56{\cdot}78} & \underline{75,3 \text{ etc.}} \\
\underline{49} & 145 \times 5 = 725 \\
\quad 778 & 1503 \times 3 = 4509 \\
\underline{\quad 725} & \\
\qquad 5300... & \\
\qquad .... &
\end{array}
$$

Recursive methods (for *non-exact* square root) give an approximation for the square root of a integer number by using, just so, *recursive* algorithms which lead to the expected approximation of the value of the square root with a smaller number of operations (compared with Ptolemy's method): this is the method expressed by the *fundamental rule* – it could be called *mother rule*, preluding to modern recursive algorithm – which Bombelli himself explained in the first "libro" of the *Algebra* [Bombelli 1966, page 39].

By translating to modern notation the "paradigmatic example" of Bombelli for calculating the square root of 13, that rule works as follows:

The closest square to 13 is 9, whose root is 3 (he considered "the closest square number" which is not over the original number) and then he put

$$\sqrt{13} = 3 + x,$$

where x is the "*rotto*" – *that is*, *the remainder* – of the unit and therefore $0 < x < 1$; by squaring both the members of the equation, he obtained

$$13 = 9 + 6x + x^2,$$

from which

$$4 = 6x + x^2 \qquad (*)$$

and

$$4 = x\,(6 + x). \qquad (**)$$

By neglecting x^2 in (*), he had

$$x = \frac{4}{6} = \frac{2}{3}.$$

Therefore the first approximation expressed with a "*rotto*" was

$$\sqrt{13} = 3 + \frac{2}{3}.$$

Bombelli replaced x in the brackets in (**) with 4/6, looking for a better approximate value, and obtained

$$4 = x\left(6 + \frac{4}{6}\right);$$

from this

$$x = \frac{4}{6 + \frac{4}{6}} = \frac{3}{5};$$

therefore the new approximation was

$$\sqrt{13} = 3 + \frac{4}{6 + \frac{4}{6}} = 3 + \frac{3}{5}.$$

By using letters instead of numbers (passing from *logistica numerosa* – like Bombelli – to *logistica speciosa* – like Viète –), the procedure becomes:

$\sqrt{n} = \sqrt{a^2 + r} = a + x$
$a^2 + r = (a + x)^2$
$a^2 + r = a^2 + 2ax + x^2$
$r = 2ax + x^2 \qquad (*)$
$r = x(2a + x) \qquad (**)$
By neglecting x^2 in (*) $x = \frac{r}{2a}$
and: $\sqrt{a^2 + r} = a + \frac{r}{2a}$.
By replacing x in (**) with $x = \frac{r}{2a}$: $r = x\left(2a + \frac{r}{2a}\right)$
from which: $x = \frac{r}{2a + \frac{r}{2a}}$.
Therefore: $\sqrt{a^2 + r} = a + \frac{r}{2a + \frac{r}{2a}}$

If we again replace the last value of x in (**), the square root will be approximated how much we want, by repeating the same procedure as follows:

$$\sqrt{13} = 3 + \cfrac{4}{6 + \cfrac{4}{6 + \cfrac{4}{6}}} = 3 + \frac{20}{33}$$

and so on.

This procedure leads to the *infinite continued fraction method*. It inspired Pietro Antonio Cataldi (1552-1626) for the idea of the *continued fraction*, which is also useful to represent the square root of a number and to calculate its approximate value *ad libitum*[9].

Bombelli's idea, improved by Cataldi with full consciousness, gave *infinite continued fraction* the "ontological status" of equivalent entity by which it was possible to express a *non-exact* square root.

A symbolically compacted representation of $\sqrt{13}$, in terms of continued fraction, can be provided with modern notation by the general formula:

$$\sqrt{13} = 3 + x_n$$

where

$$x_n = \frac{4}{6 + x_{n-1}} \text{ with } x_0 = 0,\ n = 1, 2, \ldots$$

Bombelli "sealed" his method, which has been explained previously, and classified it with the significance of *mother rule,* by the phrase:

> "[…] so it is possible to see where the others rules were born".[10]

He formerly had given some early recursive rules for calculating approximate values of the square root of the numbers 13 (a number which is "very far" from being a perfect square) and 8 (a number which is "almost" a perfect square). The rules differ each other for some details, as it is possible to see from his quotes and from the schemes shown. The way to explain the rules used by Bombelli does not let immediately see the possibility to get to a continued fraction. It is very important that Bombelli declared to have "found with basis" their justification. The "basis" is the *mother rule.*

---

[9] Cataldi explained the *infinite continued fraction method* in his *Trattato del modo brevissimo di trovare la radice quadra delli numeri, et regole da approssimarsi di continuo al vero nelle radici de' numeri non quadrati*, published in Bologna by Bartolomeo Cochi in 1613.

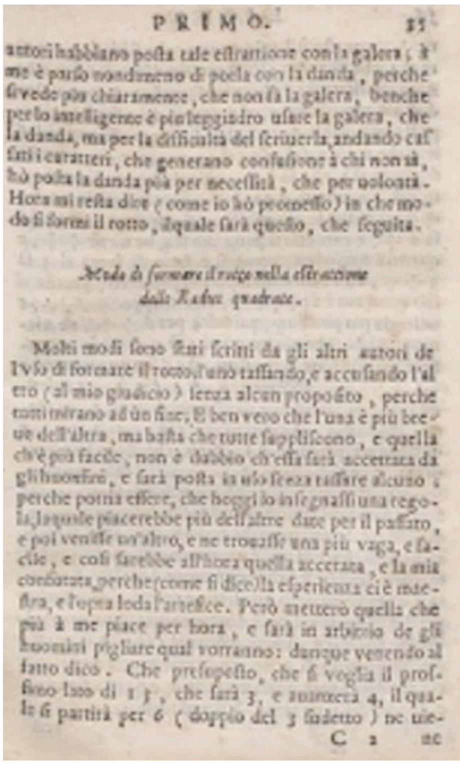
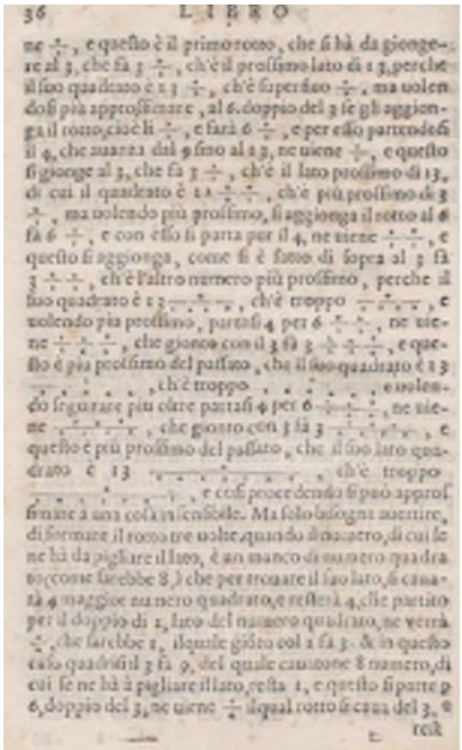
[10] "[...] sicché si vede donde nascano le [altre] regole dette di sopra [sul modo di formare il rotto]".

The first quote follows; it shows how calculate the rule for $\sqrt{13}$ with respective schemes expressed in modern notation (where $n = a^2 + r$ and a^2 is the greatest square number held in n and r the difference between n and a^2):

[…] presuposto che si voglia il prossimo lato di 13,	[...] if we want to find the approximate side of 13,	$\sqrt{13}$	$\sqrt{n}$
che sarà 3, e avanzerà 4,	that is 3, and 4 remains,	$\sqrt{3^2+4}=3$	$\sqrt{a^2+r}=a$
il quale si partirà per 6 (doppio del 3 suddetto) ne viene 2/3, e questo è il primo rotto, che si ha da giongere al 3, che fa $3\frac{2}{3}$, ch'è il prossimo lato di 13, perché il suo quadrato è $13\frac{4}{9}$, ch'è superfluo 4/9,	that we divide for 6 (dual 3) and the result is 2/3 that is the first rotto, that we have to add to 3 and the result is $3\frac{2}{3}$, that is the approximate side of 13, because is square is $13\frac{4}{9}$ that is larger for 4/9,	$\sqrt{3^2+4}=$ $=3+\frac{4}{6}=3+\frac{2}{3}$	$\sqrt{a^2+r}=a+\frac{r}{2a}$
ma volendosi più approssimare, al 6 doppio del 3 se gli aggionga il rotto, cioè li 2/3, e farà $6\frac{2}{3}$, e per esso partendosi il quattro, che avanza dal 9 fino al 13, ne viene 3/5, e questo si giunge al 3, che fa $3\frac{3}{5}$, ch'è il lato prossimo di 13, di cui il quadrato è $12\frac{24}{25}$, ch'è più prossimo di $3\frac{2}{3}$,	if we want better approximate, we have to add the rotto to 6, i.e. 2/3, and it is $6\frac{2}{3}$, ad we divide 4 by $6\frac{2}{3}$ and remains 3/5 that we add to $3\frac{3}{5}$, that is the approximate side of 13. His square is $12\frac{24}{25}$, a better approximation,	$\sqrt{3^2+4}=$ $=3+\cfrac{4}{6+\cfrac{4}{6}}=3+\frac{3}{5}$	$\sqrt{a^2+r}=$ $=a+\cfrac{r}{2a+\cfrac{r}{2a}}$
ma volendo più prossimo, si aggionga il rotto al 6 fa $6\frac{3}{5}$, e con esso si parta pur il 4, ne viene 20/33, e questo si aggionga, come si è fatto di sopra al 3 fa $3\frac{20}{33}$, ch'è l'altro numero più prossimo, perché il suo quadrato è $13\frac{4}{1089}$, ch'è è troppo 4/1089, e volendo più prossimo … [Bombelli 1966, p. 38].	but if we want to go on, we have to add the rotto to 6 and it is $6\frac{3}{5}$, then we divide 4 by $6\frac{3}{5}$ and the result is 20/33 that we add to 3. The result is $3\frac{20}{33}$ that is a better approximation because is square is $13\frac{4}{1089}$ that is larger for 4/1089, and if we want approximate...	$\sqrt{3^2+4}=$ $=3+\cfrac{4}{6+\cfrac{4}{6+\cfrac{4}{6}}}=3+\frac{20}{33}$	$\sqrt{a^2+r}=$ $=a+\cfrac{r}{2a+\cfrac{r}{2a+\cfrac{r}{2a}}}$
		…	…

Bombelli well described his rule into the sixth application and wrote this fondamental observation:

"And so going on, it is possible to approximate to imperceptible thing".[11]

The two pages of Bombelli's Algebra with his method for $\sqrt{13}$.

The method has an iterative form:

$$r(0) = 0;$$

$$r(i+1) = \frac{n - a^2}{2a + r(i)};$$

for $i=0, 1, \ldots$

and $a+r(i)$ is the result.

The second quote, where Bombelli shows the rule for $\sqrt{8}$, and respective schemes follow:

[…] per trovare il suo lato,	[...] if we want to find the approximate side of 8,	$\sqrt{8}$	$\sqrt{n}$

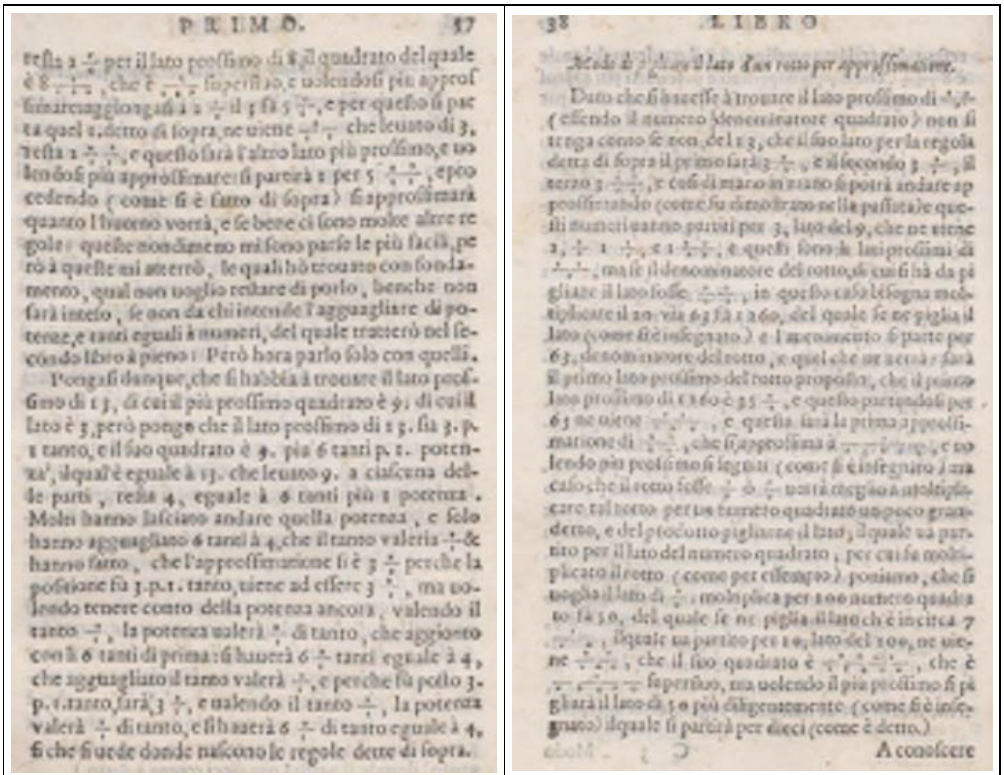
[11] "E così procedendo si può approssimare a una cosa insensibile".

si cavarà 4 maggior numero quadrato, e resterà 4,	let us subtract 4, the largest square number, and 4 remains,	$\sqrt{8} = \sqrt{2^2+4} = 2$	$\sqrt{n} = \sqrt{a^2+r} = a$
che partito per il doppio di 2, lato del numero quadrato, ne verrà 4/4, che sarebbe 1, il quale gionto col 2 fa 3.	then we divide 4 by 2 twice, side of the square number, and the result is 4/4, i. e. 1, that added to 2, is 3.	$\sqrt{2^2+4} = 2+\frac{4}{4} =$ $= 2+1 = 3$	$\sqrt{a^2+r} = a+\frac{r}{2a}$
Et in questo caso quadrisi il 3 fa 9, del quale cavatone 8 numero di cui se ne ha a pigliare il lato, resta 1, e questo si parte per 6, doppio del 3, ne viene 1/6 il qual rotto si cava del 3, e resta $2\frac{5}{6}$ per il lato prossimo di 8, il quadrato del quale è $8\frac{1}{36}$, ch'è 1/36 superfluo,	Let us find the square of 3 that is 9; from 9 subtract 8 and the result is 1 and divide 1 by 6, twice 3, and the result is 1/6 and its rotto we subtract from 3 and $2\frac{5}{6}$ remains, and its square is $\frac{1}{36}$ that is bigger for 1/36,	$\sqrt{2^2+4} =$ $= 3-\frac{3^2-8}{6} = 3-\frac{1}{6} =$ $= 2+\frac{5}{6}$	$\sqrt{a^2+r} =$ $= \left(a+\frac{r}{2a}\right)+$ $-\frac{\left(a+\frac{r}{2a}\right)^2-\left(a^2+r\right)}{2\left(a+\frac{r}{2a}\right)}$
e volendosi più approssimare: aggiongasi a $2\frac{5}{6}$ il 3 fa $5\frac{5}{6}$, e per questo si parta quel 1 detto di sopra, ne viene 6/35 che levato di 3 resta $2\frac{29}{35}$, e questo sarà l'altro lato più prossimo, e volendosi più approssimare … [Bombelli 1966, p. 38].	and if we want to find a better approximation, we have to add 3 to $2\frac{5}{6}$ and the result is $5\frac{5}{6}$, and then divide 1 by it, and we have 6/35. If we subtract 3, it remains $2\frac{29}{35}$ and it is the closest side, and if we eant to find a better approximation...	$\sqrt{2^2+4} =$ $= 3-\frac{3^2-8}{3+2+\frac{5}{6}} =$ $= 3-\frac{1}{\frac{35}{6}} =$ $= 3-\frac{6}{35} = 2+\frac{29}{35}$	$\sqrt{a^2+r} =$ $= \left(a+\frac{r}{2a}\right)+$ $-\frac{\left(a+\frac{r}{2a}\right)^2-\left(a^2+r\right)}{2\left(a+\frac{r}{2a}\right)-\frac{\left(a+\frac{r}{2a}\right)^2-\left(a^2+r\right)}{2\left(a+\frac{r}{2a}\right)}}$
		…	…

Bombelli observed, for concluding the exposition of this other rule:

"And going on, one is able to approximate how much he wants".[12]

[12] "E procedendo si approssimarà quanto l'huomo vorrà".



PRIMO. 57

58 LIBRO

The two pages of Bombelli's Algebra with his method for $\sqrt{8}$.

The method also has an iterative form:

$$r(0) = 0;$$

$$r(i+1) = \frac{\left(a + \frac{r}{2a}\right)^2 - n}{2\left(a + \frac{r}{2a}\right) - r(i)}$$

for $i=0, 1,\ldots$

where $a+r(i)$ is the result.

It is useful looking, starting from the first steps, for a perfect correspondence among the approximate values at any step, by means of each of three rules (*mother rule* and the last two rules explained). Bombelli, as able calculator, would have noticed this fact but, as able mathematician, would have understood soon the importance that in these cases we have to consider the *definitive character* of this succession of steps (where the calculations lead to do an indefinite number of steps; "quanto l'huomo vorrà" he wrote).

It is useful to do another important consideration: in the *mother rule,* for the approximation (included in the procedure here explained) of $\sqrt{13}$ and $\sqrt{8}$, Bombelli used an empirical observation – its origin is very ancient: the first approximate value, 3+4/6 (for $\sqrt{13}$), is also given by the arithmetic mean between 3 (a) and 13/3 (n/a).[13]

This empirical observation dates back to the Babylon, even if, in the historical-mathematician literature, it is always attributed to the Pythagorean Archytas of Tarentum (428-365 B.C.) or to Heron of Alexandria (ca. 10-85 A.D.) or, even, to Newton (but it appears nowhere in the *Arithmetica Universalis* where it seems reasonable looking for it). This latter ascription is in fact misleading, because, as the reader can see afterwards, it seems to be based on the fact that in other works (which will be mentioned afterwards), Newton suggested an algorithm for calculating the square root of a binomial of $\left(a^2+x^2\right)$-form in order to obtain the development in series of $\sqrt{a^2+x^2}$ (where, as it could be noticed, at first two terms of development in series we could arrive also through the calculation of the arithmetical mean) to make then the necessary termwise integration in order to calculate the surface included "under the curve" of $y=\sqrt{a^2+x^2}$ form.

We know that Babylonian mathematicians used a method of approximation in order to find the *non-exact* square root of a natural number; a confirmation of their method comes out by studying the *YBC 7289* tablet of the *Yale University Babylonian Collection* (see the following figure). This tablet dates back to the pre-Babylonian period of the Hammurabi dynasty (1800-1600 B.C.); a square with its diagonals is drawn and some numbers in cuneiform characters are impressed on the table.

The *YBC 7289* tablet.

This method is connected to the investigation of the relation between the diagonal and the side of a same square, on "Pythagorean theorem" basis and

[13] The arithmetic mean between a and n/a, i.e. between a and $\frac{a^2+r}{a}$, is $\frac{1}{2}\left(a+\frac{a^2+r}{a}\right)=\frac{2a^2+r}{2a}=a+\frac{r}{2a}$.

consists of calculating two rounded values of the square, one by excess and the other one by defect, and taking the arithmetic mean between these two values as subsequent approximation.

It is possible to calculate, for example, an approximate value of the root of 2, by using the Babylonian method.[14]

For $\sqrt{2}$, the value 1 (the square of 1 is the closest to 2) is chosen as first approximation and the consequent relation is:

$$1 \cdot (2:1) = 2$$

which expresses the product between the rounded down value of the square root (i.e. 1) and the rounded up one (i.e. 2:1). The mean between these two values is the second approximation:

$$\sqrt{2} = \frac{1}{2}(1+2) = 1 + \frac{1}{2} = \frac{3}{2}.$$

Being

$$\left(\frac{3}{2}\right)^2 = \frac{9}{4} = 2 + \frac{1}{4} > 2,$$

the approximation found, 3/2, is a value by excess of $\sqrt{2}$.

By repeating the same procedure, we have:

$$\frac{3}{2} \cdot \left(2 : \frac{3}{2}\right) = 2$$

that is

$$\frac{3}{2} \cdot \frac{4}{3} = 2;$$

in this way we obtain 4/3. It is the rounded down value of $\sqrt{2}$: let us match it with 3/2, that is the rounded up value. It follows that a subsequent approximation is given by the arithmetic mean of the preceding values, 3/2 and 4/3, and therefore:

$$\sqrt{2} = \frac{1}{2}\left(\frac{3}{2} + \frac{4}{3}\right) = \frac{3}{4} + \frac{2}{3} = \frac{17}{12} = 1{,}416.$$

It could be noticed that repeating only two times the mean procedure of the Babylonian comes out a good approximation of the square root of 2.

---

[14] The Babylonian method is explained in Heron's *Metrica*, as Paul Tannery (1843-1904) revealed in [Tannery 1894]. Here the calculation is expressed by decimal fractions, although the Babylonian used sexagesimal fractions (see [Neugebauer 1974, pp. 52-53]): this was their ordinary numeration system.

The described procedure can be expressed in modern terms by using the following formula (which nowadays, according to different considerations, several scholars ascribe to Newton):

$$\sqrt{n} = \frac{1}{2}\left(\frac{n}{a} + a\right)$$

where a is, at the first step, the closest approximation of the value of the square root of n; it is necessary replace in the following steps each time a with the result obtained calculating the second member of the preceding formula. The procedure ends when a result coinciding with its immediately previous comes out.

The preceding formula could be indexed in order to better underline the iterative nature of the procedure. Let us define the rounded generic value of the square root of n as x_i; we obtain:

$$\sqrt{n} = x_i = \frac{1}{2}\left(\frac{n}{x_{i-1}} + x_{i-1}\right)$$

where $i = 1, 2, 3, \dots$ and x_0 is the first chosen approximation.

It could be noticed that it is possible to get to the same formula by arguing as follows: if we want to calculate $\sqrt{n}$, let us put

$$x = \sqrt{n}\ ,$$

that can be written as

$$x^2 = n,$$

or else

$$x \cdot x = n,$$

that is

$$x = \frac{n}{x}.$$

Let us suppose that x_0 is a first approximation of the value of $\sqrt{n}$; from the previous relation follows that $\frac{n}{x_0}$ is also an approximation. Therefore the arithmetic mean of these two approximations gives a value x_1 that is closer to $\sqrt{n}$ than x_0:

$$x_1 = \frac{1}{2}\left(x_0 + \frac{n}{x_0}\right).$$

By using the value x_1 just obtained, and by repeating the same reasoning, it is possible write:

$$x_2 = \frac{1}{2}\left(x_1 + \frac{n}{x_1}\right)$$

and so on, until the desired approximation is obtained:

$$x_i = \frac{1}{2}\left(x_{i-1} + \frac{n}{x_{i-1}}\right),$$

for $i = 1, 2, \ldots$, where x_0 is the first chosen approximation.

An intermediate observation can be just made: *by applying the arithmetic mean used in the Babylonian age, for Bombelli was helpful resigning himself of the scientific dignity of the "mother rule"*.

By reasoning according to Bombelli's method, we have:

$$\sqrt{n} = a + x$$

where a is the value whose square is the closest number to n, without going over it; by squaring both members we obtain:

$$n = a^2 + 2ax + x^2;$$

by neglecting x^2 at the first approximation, we have:

$$x = \frac{n - a^2}{2a};$$

and then:

$$\sqrt{n} = a + x = a + \frac{n - a^2}{2a} = \frac{a}{2} + \frac{n}{2a} = \frac{1}{2}\left(a + \frac{n}{a}\right).$$

2. NEWTON AND THE EXTRACTION OF ROOT.

Sometimes happens to read the misleading assertion that the algorithm of extraction of square root, usually ascribed to Newton, is in the *Analysis per Quantitatum Series etc.* [Newton 1723] and in his work *La Méthode des Fluxions* [Newton 1740], which are works imagined in the first decades of the second half of the XVII century.[15]

Newton, in the *De Analysi* and in the treatise *La Méthode des Fluxions,* developed in power series particular functions like $y = \frac{a^2}{b+x}$ and $y = \sqrt{a^2 + x^2}$ (see [Newton 1723, pp. 6-7] and [Newton 1740, pp. 5-6]). He

[15] The first one was published around the 1669 with the temporary title *De Analysi per aequationes numero terminorum infinitas* and was printed for the first time in 1711 in *Analysis per Quantitatum Series, Fluxiones ac Differentias*; the second one was published around the 1671 with the temporary title *Methodus fluxionum et serierum infinitorum* and was printed for the first time in 1736, in English, with the title *The Method of Fluxions and Infinite Series* and, then, in French – 1740 – and in Latin – 1742

executed the so-called *long division* or “Mercator’s division”, for functions of the first kind, which Mercator made public in his *Logarithmotechnia* in 1668; Newton used the method that he called of extraction of the root for functions of the second kind.

Newton however wanted to underline the structural connection between these two “symbolic” methods. In the *Epistola prior* (13th June 1676 [Gerhardt 1971, 1, p. 100]), which he sent to Leibniz via Oldenburg (secretary to the Royal Society of London), Newton wrote:

> *Fractiones in infinitas series reducuntur per divisionem et quantitates radicales per extractionem radicum, perinde instituendo operationes istas in speciebus istis ac institui solent in decimalibus numeris.*[16]

He made a calculation for the extraction of the square root of $\left(a^2 + x^2\right)$ – an algorithm *more arithmetico* (*solent in decimalibus numeris*; i.e. in the same way through which operating with decimal numbers is usual) – represented in the scheme that follows (we added only the indication of I, II, etc. partial difference):

$$\sqrt{a^2+x^2}\ \Big(\, a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^8}{128a^7} + \frac{7x^{10}}{256a^9} \text{ etc.}$$

$$\begin{array}{ll}
 & a^2 \\
 & \overline{0 + x^2} \\
\underline{\text{I part. diff.}} & \\
 & x^2 + \dfrac{x^4}{4a^2} \\
 & \overline{0 - \dfrac{x^4}{4a^2}} \\
\underline{\text{II part. diff.}} & \\
 & -\dfrac{x^4}{4a^2} - \dfrac{x^6}{8a^4} + \dfrac{x^8}{64a^6} \\
\underline{\text{III part. diff.}} & \overline{0 \quad + \dfrac{x^6}{8a^4} + \dfrac{x^8}{64a^6}} \\
 & +\dfrac{x^6}{8a^4} + \dfrac{x^8}{16a^6} - \dfrac{x^{10}}{64a^8} + \dfrac{x^{12}}{256a^{10}} \\
\underline{\text{IV part. diff.}} & \overline{0 - \dfrac{5x^8}{648a^6} + \dfrac{x^{10}}{64a^8} - \dfrac{x^{12}}{256a^{10}}} \\
 & \text{etc.}
\end{array}$$

[16] “Fractions are reduced to infinite series by division and radical quantitaties by extraction of the roots, by carrying out those operations in the symbols just as they are commonly carried out in decimal numbers”.

The final result of the development in series of the square root of $\left(a^2 + x^2\right)$ is reported on the right (after the round bracket) of the scheme.

We think that the *first term of development*, that is a, is obtained simply by extracting the square root of a^2; the first partial difference between the radicand $a^2 + x^2$ and a^2 is x^2.

Newton could have obtained the *second term of development* ($\frac{x^2}{2a}$), according to an intuition: by dividing x^2 (I part. diff.) by a and multiplying the result by 1/2.

The second partial difference is obtained by subtracting the product of $\frac{x^2}{a}$ (appeared in the preceding step) by a (*I term of development*) plus the square of $\frac{x^2}{2a}$ (*II term of development*) from x^2. The result is $-\frac{x^4}{4a^2}$.

Newton could have obtained again the *third term of development*, that is $-\frac{x^4}{8a^3}$, by dividing $-\frac{x^4}{4a^2}$ (II part. diff.) by a and then multiplying the result by 1/2. The third partial difference is obtained by subtracting the product of $-\frac{x^4}{4a^3}$ (appeared in the preceding step) by a (*I term of development*) plus the product of the same $-\frac{x^4}{4a^3}$ by $\frac{x^2}{2a}$ (*II term of development*) plus the square of the *third term of development*, from $-\frac{x^4}{4a^2}$. The result is $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$.

The *fourth term of development,* that is $\frac{x^6}{16a^5}$, would be obtained by Newton by dividing $\frac{x^6}{8a^4}$ (let us pay attention: he considered only the first term of the third partial difference and neglected the other ones) by a and then multiplying the result by 1/2.

The fourth partial difference_is obtained by subtracting the product of $\frac{x^6}{8a^5}$ (appeared in the preceding step) by a (*I term of development*) plus the product of the same $\frac{x^6}{8a^5}$ by $\frac{x^2}{2a}$ (*II term of development*) plus the product of the same $\frac{x^6}{8a^5}$ by $-\frac{x^4}{8a^3}$ (*III term of development*) plus the square of the *fourth term of development*, from $\frac{x^6}{8a^4} - \frac{x^8}{64a^6}$. The result is $-\frac{5x^8}{64a^6} + \frac{x^{10}}{64a^8} - \frac{x^{12}}{256a^{10}}$.

The *fifth term of development*, that is $-\frac{5x^8}{128a^7}$, would have been obtained by Newton by dividing $-\frac{5x^8}{64a^6}$ (he considered also here only the first term of the fourth partial difference and neglected the other two) by a, and it results $-\frac{5x^8}{64a^7}$; then he multiplied this result by 1/2.

And so on.

It is possible to read it according to this other manner in order to make the recursive nature of this algorithm more explicit:

Given

$$y = \sqrt{a^2 + x^2},$$

the recursive rule becomes:

$$a_i = \frac{I°(D_{i-1})}{2a_0}$$

$$D_i = D_{i-1} - \left[\frac{I°(D_{i-1})}{a_o} \cdot a_o + \frac{I°(D_{i-1})}{a_o} \cdot a_1 + \ldots + \frac{I°(D_{i-1})}{a_o} \cdot a_{i-1} + a_i^2 \right]$$

with

$$a_0 = \sqrt{a^2} = a$$
$$D_0 = a^2 + x^2 - a_0^2 = x^2$$

and where $I°$ is a function that sets only the first term of the sum.

Newton, or else anyone before him has got to this algorithm of the square root, probably has been encouraged in pushing forward the steps of the algorithm (thing that it is possible that will have happened to Bombelli too, convincing him of the scientific dignity of the *mother rule*) by the observation that *we could arrive at the first two terms of development in series also through the calculation of the arithmetical mean* (calculation which had been done by the Babylonian, as just said), if $\left(a^2 + x^2\right)$ is explained under the form $\left(a^2 + r\right)$.

The peculiarity of this algorithm is represented by the fact that "some quantities are neglected", and *it is scientifically "allowed" to do it*: in this aspect there is the substantial difference between the calculation leading to the extraction of the *non-exact* square root before and after the appearance of the method adopted by Newton.

It seems that Newton did not underline that some quantities, definable as "infinitesimals of superior order", were neglected in the symbolic procedure. Neglecting such quantities had been noticed clearly by Bombelli while Leibniz, in conceiving the rules of differentiation (proposed in the article *Nova Methodus…*, [Gerhardt 1971, 5, pp. 220-226] published in 1684), revealed later "to have neglected" anything, in his correspondence with philosophers and mathematicians [Palladino 1995].

It is perhaps useful to remember that from the epistolar correspondence of Leibniz derives what can be called a "hidden calculation" which supports his rules of differentiation stated without any comment in the *Nova Methodus*. It is possible to make Leibniz's procedure clear (which is structurally equal to Bombelli's procedure adopted in the *mother rule* – and Bombelli's algebra is well known by Leibniz, as we can see by glancing at *Matematische Schriften* of this latter).

Leibniz considered an equation like

$$y = x^2$$

and substituted y with $y + dy$ and x with $x + dx$, where dx and dy are "infinitesimally little" increments; he obtained:

$$y + dy = (x + dx)^2$$

$$y + dy = x^2 + 2xdx + (dx)^2.$$

He annulled y and x^2 because they compensate each other ($y = x^2$); therefore:

$$dy = 2xdx + (dx)^2$$

and subsequently he neglected $(dx)^2$ because it was "still more little" than dx, obtaining:

$$dy = 2xdx.$$

Therefore the differential ratio was

$$\frac{dx}{dy} = \frac{1}{2x}$$

that was useful for him for calculating the subtangent t in a point of the curve $y = x^2$.

Starting from Bombelli and getting to Newton and Leibniz, the procedures of approximation, used for a limited number of steps and connoted by an “alchemistical spirit”, assumed more modern scientific dignity: the approximation become “controlled” and “unified” because a new idea of the infinite in mathematics and his correspondent control were growing.

3. NEWTON BETWEEN THE ALGORITHM OF EXTRACTION OF THE ROOT AND THE FORMULA OF THE BINOMIAL.

We could underline an interesting relationship between the method of Newton for searching $\sqrt{a^2+x^2}$, and the “binomial formula” which allows to obtain the development in series of the expression $\left(a^2+x^2\right)^r$ with r rational number (in particular with r=1/2). What kind of interactions, if are they? And, if interactions are, which of two algorithms did support the other one? Newton, in the *Epistola posterior* (of the 24th October 1676 [Gerhardt 1971, 1, p. 124]) that he sent to Leibniz, wrote that the general rule for reducing the root in infinite power series was thought before having formalized the rule for the extraction of the root of a binomial of the $\left(a^2+x^2\right)$ form:

> Sic itaque innotuit mihi generalis reductio radicalium in infinitas series per regulam illam, quam posui initio epistolae prioris, antequam scirem extractiones radicum[17].

We want to report a quote of the *Epistola prior* [Gerhardt 1971, 1, pp. 100-101]) where Newton revealed his theorem which had a fundamental role in all his treatises of infinitesimal Calculus of the XVIII century:

> Quamquam Dni. Leibnitii modestia, in excerptis, quae ex Epistola ejus ad me nuper misisti, nostratibus multum tribuat circa speculationem quandam infinitarum serierum, de qua jam coepit esse rumor: nullus dubito tamen quin ille, non tantum (quod asserit) methodum reducendi quantitates quascunque in ejusmodi series, sed et varia compendia, fortè nostris similia, si non et meliora, adinvenerit. Quoniam tamen

[17] “So then the general reduction of radicals into infinite series by that rule, wich I laid down at the beginning of my earlier letter became known to me, and that before I was acquainted with the extraction of the roots”.

ea scire pervelit, quae ab Anglis ea in re inventa sunt, et ipse ante annos aliquot in hanc speculationem inciderim: ut votis ejus aliqua saltem ex parte satisfacerem, nonnulla eorum quae mihi occurrerunt, ad te transmisi.

Fractiones in infinitas series reducuntur per divisionem, et quantitates radicales per extractionem radicum, perinde instituendo operationes istas in speciebus istis ac institui solent in decimalibus numeris. Haec sunt fundamenta harum reductionum; sed extractiones radicum multum abbreviantur per hoc Teorema:

$$\overline{P+PQ}\Big\}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + etc.$$

Ubi $P + PQ$ significat quantitatem, cujus radix vel etiam dimensio quaevis vel radix dimensionis investiganda est, P primum terminum quantitatis ejus, Q reliquos terminos divisos per primum, et m/n numeralem indicem dimensionis ipsius $P + PQ$ sive dimensio illa integra sit, sive (ut ita loquar) fracta, sive affirmativa, sive negativa[18].

By reading the two *Epistolae* and Newton's mathematical manuscripts, we thought that the steps approaching to the idea of the binomial formula have been the following ones.

As Newton himself underlined in his *Epistola posterior*, he wanted to generalize the development in series of powers of binomials like $\left(1-x^2\right)^0$, $\left(1-x^2\right)^1$, $\left(1-x^2\right)^2$, … These series are useful to find the surfaces included between the arcs of the curves represented by the functions $y=\left(1-x^2\right)^0$, $y=\left(1-x^2\right)^1$, $y=\left(1-x^2\right)^2$, ... and the x axis. Newton desired to find similar

[18] "Though the good sense of Mr Leibniz, in the extracts from his letter you have lately sent me, pays great tribute to our countrymen [English mathematicians] for their researches on infinite series which we are discussing about: yet I have no doubt that he has discovered not only a method for reducing any quantities whatever to such series, as he asserts, but also various shortened forms, perhaps like our own, if not even better. Since, however, he very much wants to know what the English have discovered in this subject, and since I myself fell upon this theory some years ago: I have sent you some of those things which occurred to me in order to satisfy his wishes, at any rate in part.
Fractions are reduced to infinite series by division and radical quantities by extraction of the roots, by carrying out those operations in the symbols just as they are commonly carried out in decimal numbers. These are the foundations of these reductions; but extractions of square roots are shortened very much by this theorem:

$$\overline{P+PQ}\Big\}^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ + \frac{m-3n}{4n}DQ + etc.$$

where $P + PQ$ is the quantity whose root, or even any power or the root of a power, is to be found; P is the first term of that quantity, Q is the remaining terms divided by the surfaces for functions where the same binomial was raised to a rational exponent as follows:

$$y=\left(1-x^2\right)^{\frac{1}{2}},\ y=\left(1-x^2\right)^{\frac{3}{2}},\ y=\left(1-x^2\right)^{\frac{5}{2}},\text{ etc.}$$

Concerning this, he noted that by integrating termwise the developments related to the exponents 0, 1, 2, etc., he obtained, respectively,

$$x;\quad x-\frac{1}{3}x^3;\quad x-\frac{2}{3}x^3+\frac{1}{5}x^5;\quad x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7;\ \text{ etc}$$

or also:

$$x-\frac{0}{3}x^3;\quad x-\frac{1}{3}x^3;\quad x-\frac{2}{3}x^3+\frac{1}{5}x^5;\quad x-\frac{3}{3}x^3+\frac{3}{5}x^5-\frac{1}{7}x^7;\ \text{ etc}$$

In order to find the series corresponding to the integrations of binomials, with rational exponent, he thought to *insert* or *interpolate* new series between the first and the second series, between the second and the third series, between the third and the fourth series, etc.; he gave the first two terms of every new series:

$$x-\frac{\frac{1}{2}x^3}{3},\ x-\frac{\frac{3}{2}x^3}{3},\ x-\frac{\frac{5}{2}x^3}{3},\text{ etc..}$$

These terms were created on the basis of a "first analogy": Newton detected that all the series calculated by starting from binomials with integer exponents (0 included) had x as first term and $\frac{0}{3}x^3;\ \frac{1}{3}x^3;\ \frac{2}{3}x^3;\ \frac{3}{3}x^3$, etc. as second term; the progression 0, 1, 2, 3, etc., given by numerators of

$$\frac{0}{3}x^3;\ \frac{1}{3}x^3;\ \frac{2}{3}x^3;\ \frac{3}{3}x^3,\text{ etc.}$$

coincided with the n expressing the exponent of the binomial $\left(1-x^2\right)$. The idea was that the same correspondence could hold when the exponents were rational and not only integer, i.e. 1/2, 3/2, 5/2, etc.

Newton had to determine the string of terms, after the second ones, for each of the series originated by binomials with rational exponents. He made also this step through a *second analogy,* which is here explained.

The series originated by binomials with integer exponents (0 included, and then, 1, 2, 3, etc.) have the denominators of the coefficient of x (which are 1; 1, 3; 1, 3, 5; 1, 3, 5, 7; …) increasing by the arithmetic progression, while the numerators increase according to "Pascal triangle" (this was

---

first; and m/n is the numerical index of the power of $P + PQ$, whether that power is integral or (so to speak) fractional, whether positive or negative".

generally also called "Tartaglia's triangle" and "Oughtred's analytic table" by Newton) as follows:

```
1
1  1
1  2  1
1  3  3   1
1  4  6   4   1
1  5  10  10   5   1
1  6  15  20   15  6   1
...         ...           ...
```

The "Triangle" can be disposed according to another particular configuration (which is more useful for the next considerations):

1	1	1	1	1	1	1	1	...
0	1	2	3	4	5	6	7	...
0	0	1	3	6	10	15	21	...
0	0	0	1	4	10	20	35	...
0	0	0	0	1	5	15	35	...
0	0	0	0	0	1	6	21	...
0	0	0	0	0	0	1	7	...

The law of formation, as appears in this "triangle", is such that the element of place (i, j), where i is the index of line and j is the index of column, is obtained by adding the element of place $(i-1; j-1)$ with the element of place $(i; j-1)$.

The same law of formation is "algebraically" exposed in the table that follows, where *a, b, c, d* are generic numbers:

$\frac{0}{2}$	$\frac{2}{2}$	$\frac{4}{2}$	$\frac{6}{2}$	$\frac{8}{2}$	…
a	a	a	a	a	…
b	$b+a$	$b+2a$	$b+3a$	$b+4a$	…
c	$c+b$	$c+2b+a$	$c+3b+3a$	$c+4b+6a$	…
d	$d+c$	$d+2c+b$	$d+3c+3b+a$	$d+4c+6b+4a$	…

Newton extended the interpretation of this table from the traditional *Pascal triangle* to a triangle where there are also new *inserted columns,* corresponding to the mentioned fractional exponents 1/2, 3/2, 5/2, etc.[19] The table becomes:

$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	…
a	a	a	a	a	a	…
b	$b+a$	$b+2a$	$b+3a$	$b+4a$	$b+5a$	…
c	$c+b$	$c+2b+a$	$c+ 3b+3a$	$c+4b+6a$	$c+5b+10a$	…
d	$d+c$	$d+2c+b$	$d+3c+3b+a$	$d+4c+6b+4a$	$d+5c+10b+10a$	…

Therefore we have this latter tabulation ("algebraical") readapted, or dilated to make spaces in the "pores" to the new *insertions*. If the same dilatation is done on the "arithmetical" tabulation, corresponding to the last ones here considered, let us note that the first line of terms, starting from the "heading line" made of 0/2, 1/2, 2/2, 3/2, etc., has all the squares filled with the number 1 (see the following table), while the second one is formed by the string 0/2, 1/2, 2/2, 3/2, etc.

In this way, the squares under the columns with "fractional headings" 1/2, 3/2, 5/2, etc., remain "uncovered" like in the figure:

$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	…
1	1	1	1	1	1	…
0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	…
0		0		1		…
0		0		0		…

Consequently, let us consider, for example, the third line and let us try to fill the uncovered squares with appropriate numbers. We have to use the numbers made explicit in the other squares of the same line, by equalizing

---

19 This is an typical approach of the "heroic period" which saw the birth of the infinitesimal calculus: it is an approach done *by analogy*, which has played an important rule in the development of the mathematical sciences.

each of these numbers with the corresponding literal expression present in the "readapted general tabulation"; therefore we are able to write the equations:

$$\begin{cases} c = 0 \\ c + 2b + a = 0 \\ c + 4b + 6a = 1 \end{cases}$$

that are useful to "cover" the resting "blank" spaces of the third line.

By counting, we obtain:

$$c = 0;\ \ b = -\frac{1}{8};\ \ a = \frac{2}{8}$$

and therefore:

$$c + b = -\frac{1}{8};\ \ c + 3b + 3a = \frac{3}{8};\ \ \ c + 5b + 10a = \frac{15}{8};\ \ldots$$

The table becomes:

$\frac{0}{2}$	$\frac{1}{2}$	$\frac{2}{2}$	$\frac{3}{2}$	$\frac{4}{2}$	$\frac{5}{2}$	$\ldots$
1	1	1	1	1	1	$\ldots$
0	$\frac{1}{2}$	1	$\frac{3}{2}$	2	$\frac{5}{2}$	$\ldots$
0	$-\frac{1}{8}$	0	$\frac{3}{8}$	1	$\frac{15}{8}$	$\ldots$
0		0		0		$\ldots$

By progressing similarly, we can "cover", *ad libitum*, how many squares we want. In addition to this, Newton discovered, what he underlined in the *Epistola posterior*, that it is possible make these "coverings" without solving systems of equations, but proceeding – as Leibniz said – with *cogitatio caeca*, that is through the automaticity which gives the *infinite product*

$$\frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \times ...,$$

where m has the value of the heading number of the column.

If m=1/2, limiting the product at the first two factors, we obtain -1/8, while limiting it at the first three factors, we obtain 1/16, etc.

It is possible to make these products explicit by the form

$$\binom{m}{k} = \frac{m(m-1)(m-2)...(m-k+1)}{k!},$$

with m rational number and k integer, and it is known as “binomial coefficient”. It gives all the numbers constituting either the columns with “entire heading” and those with “fractional heading”.

Therefore, the *generalized triangle* (complete with the *inserted columns*) is as the following table shows:

$$\begin{array}{cccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & ... \\
0 & \frac{1}{2} & 1 & \frac{3}{2} & 2 & \frac{5}{2} & 3 & ... \\
0 & -\frac{1}{8} & 0 & \frac{3}{8} & 1 & \frac{15}{8} & 3 & ... \\
0 & \frac{1}{16} & 0 & -\frac{1}{16} & 0 & \frac{5}{16} & 1 & ... \\
... & & & ... & & & ... &
\end{array}$$

For example, it is possible to use the coefficients of the second column of the table in order to obtain the series:

$$x - \frac{\frac{1}{2}x^3}{3} - \frac{\frac{1}{8}x^5}{5} - \frac{\frac{1}{16}x^7}{7} - \frac{\frac{5}{128}x^9}{9} + ...$$

which expresses the surface of the circular segment.

Newton considered again the developments of binomials under square roots (which had been the starting point of his reflection, studying Wallis’ works) and noted that they can be obtained simply by omitting the denominators 1, 3, 5, 7, …, from the terms of the “series expressing the surfaces” (*Epistola posterior* [Gerhardt 1971, 1, p. 124]):

> “ […] et ad hoc nihil aliud requiri quam omissionem denominatorum 1, 3, 5, 7, etc. in terminis exprimentibus areas”.

Consequently:

$$\left(1-x^2\right)^{\frac{1}{2}} = 1 - \frac{1}{2}x^2 - \frac{1}{8}x^4 - \frac{1}{16}x^6 - \frac{5}{128}x^8 - ...$$

In this way Newton was able to calculate the power of a binomial for every exponent ("dimension"), integer, fractional positive or negative. Also the "formula of the binomial coefficients", ascribed to Newton, derives from this and will have later an autonomous life.

Newton proceeded to a check in order to "test" the lawfulness of the operation ("[…] ut probarem has operationes" – *Epistola posterior* [Gerhardt 1971, 1, p. 124]) which allowed him the passage from the "series of the surfaces" to the "series of the curve": he multiplied, amid the cases considered as paradigmatic examples, the development $1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6-\frac{5}{128}x^8-\ldots$ by itself (by means of the "long multiplication") and obtained just $1-x^2$.

After obtaining (Newton wrote "demonstrated") the certainty of his conclusion (that is the development in series of the square root of the binomial $1-x^2$) he added – *Epistola posterior* once more, p. 125– to begin to try if, viceversa, those series, which give the roots of the binomial $1-x^2$, could be extracted with the arithmetical method. The attempt was successful, and he gave, as an example, the scheme of the operation of the square root (here already explained by us, *sub species* of the square root of $\left(a^2+x^2\right)$, with more annotations):

$$
\begin{array}{l}
\sqrt{1-x^2} \quad \Big(1-\frac{1}{2}x^2-\frac{1}{8}x^4-\frac{1}{16}x^6 \text{ etc.} \\
\underline{1 \qquad} \\
0-x^2 \\
\underline{\quad -x^2+\frac{1}{4}x^4} \\
\qquad 0 \quad -\frac{1}{4}x^4 \\
\underline{\qquad -\ \frac{1}{4}x^4+\frac{1}{8}x^6+\frac{1}{64}x^8} \\
\qquad\qquad 0 \quad -\frac{1}{8}x^6-\frac{1}{64}x^8
\end{array}
$$

Newton gave this scheme without any explanations regarding the "hidden calculus" which he used in the execution of the algorithm.

Let us note that it is possible to get to a result of this form also by using the "binomial formula" applied to $\sqrt{a^2+x^2}$.

If we want to develop in series $\sqrt{a^2 + x^2}$, through the "binomial formula"

$$(a+b)^r = \sum_{i=0}^{n} \binom{r}{i} a^{r-i} b^i = \binom{r}{0} a^r + \binom{r}{1} a^{r-1} b + \binom{r}{2} a^{r-2} b^2 + \binom{r}{3} a^{r-3} b^3 + \ldots,,$$

with r fractional, $\binom{r}{i} = \dfrac{r(r-1)(r-1)\cdot \ldots \cdot (r-i+1)}{i!}$ and where $\binom{r}{i}$ is the *binomial coefficient*, we obtain:

$$\sqrt{a^2 + x^2} = \left(a^2 + x^2\right)^{\frac{1}{2}} =$$
$$= \binom{\frac{1}{2}}{0} \left(a^2\right)^{\frac{1}{2}} \left(x^2\right)^0 + \binom{\frac{1}{2}}{1} \left(a^2\right)^{\frac{1}{2}-1} \left(x^2\right)^1 + \binom{\frac{1}{2}}{2} \left(a^2\right)^{\frac{1}{2}-2} \left(x^2\right)^2 + \binom{\frac{1}{2}}{3} \left(a^2\right)^{\frac{1}{2}-3} \left(x^2\right)^3 + \ldots$$

Since

$$\binom{\frac{1}{2}}{0} = 1,\ \binom{\frac{1}{2}}{1} = \frac{1}{2},\ \binom{\frac{1}{2}}{2} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!} = -\frac{1}{8},\ \binom{\frac{1}{2}}{3} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)}{3!} = \frac{1}{16},$$
$$\binom{\frac{1}{2}}{4} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\left(\frac{1}{2}-3\right)}{4!} = -\frac{5}{128}, \ldots,$$

we obtain:

$$\sqrt{a^2 + x^2} = a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^5} - \frac{5x^5}{128a^7} + \ldots$$

4. IS THERE A RELATION OF PRIORITY BETWEEN THE TWO NEWTONIAN ALGORITHMS?

Newton added (*Epistola posterior* [Gerhardt 1971, 1, p. 125]) the following consideration, which aroused the surprise of some scholars:

> His perspectis neglexi penitus interpolationem serierum, et has operationes tanquam fundamenta magis genuina solummodo adhibui. Nec latuit reductio per divisionem, res utique facilior.

That is, the acquired knowledge of the development in series of powers of the roots of the binomial $1-x^2$ made him completely leave the "method of interpolation of the series" (and so the "binomial formula") and he put as "more genuine" basis only this operation of extraction of root (for the new calculus he just thought of). He added that he had not let out the development in series done through the division ("long"), a proceeding of easy execution.

Previously, in the same *Epistola posterior*, Newton specified to have got to the development of the root of a binomial (for example $\sqrt{1-x^2}$) before knowing (… *antequam scirem extractionem radicum*) the algorithm, *more arithmetico*, of extraction of the root.

Newton tried to complete Wallis' project; he wanted to "square" not only the figures under arcs of curves of the form $\left(1-x^2\right)^0$, $\left(1-x^2\right)^1$, $\left(1-x^2\right)^2$, …, but also those ones produced by the same binomials when the exponent was a fractional number, and in trying to give an answer to this problem he used the "binomial formula" which he found before having possessed the algorithm *more arithmetico* of the extraction of square root, as he said. He could stopped there but, even if he brought many innovations in the mathematic sciences (he was the author), he was however prisoner of the tradition: he needed an algorithm *more arithmetico* which could be put as basis of his theory of the series, and also other authors tried in the *more geometrico* the certainty of their argumentations, even Baruch Spinoza in his *Ethica ordine geometrico demonstrata*, published posthumous in 1677.

About the succession of his discoveries (at first the "binomial formula" and then the algorithm of extraction of the root and immediately afterwards that ones of the "long division") it is credible that it was just like Newton himself indicated. More precise informations are also necessary. The *more arithmetico* algorithm (that we can define "algebraical algorithm") of extraction of root, presented by Newton, implies many passages (even if the comprehension of them is favourished by the fact that the procedure is of recursive form), which lead to neglect the "powers of superior order or dimension. Without the support of the "binomial formula", the attempt to find the algorithm *more arithmetico* of the root perhaps would have been very hard.

At the conclusion of this interpretative line, we could wonder why Newton gave – *Epistola posterior* – more emphasis to the success obtained in connection to the algorithm *more arithmetico* of extraction of root than to the "binomial formula". A possible answer could be that Newton himself probably realized that the "binomial formula", admirable aesthetically, is the fruit of risky formal manipulations and it is not provided with the maximum of the "authenticity", or "rigour", which, in comparison with his time, could be asked to a scholar and accepted by the resting members of the community of the experts of the mathematic sciences.

REFERENCES

- [Bombelli 1966], R. Bombelli, *L'Algebra*, Milano, Feltrinelli, 1966 [in 5 "libri"].
- [Edwards, Jr 1979], C.H. Edwards Jr., *The Historical Development of the Calculus*, New York-Heidelberg-Berlin, Springer-Verlag, 1979.
- [Gerhardt 1971, 1], G.W. Leibniz, *Mathematische Schriften*, edited by C.I. Gerhardt; anast., Hildesheim-New York, G. Olms, 1971, 1.
- [Gerhardt 1971, 5], G.W. Leibniz, *Nova Methodus pro maximis et minimis itemque tangentibus, quae nec fractas, nec irrationales quantitates moratur, et singulare pro illis calculi genus*, «Acta Eruditorum», Octobris 1684, pp. 467-473, in *Mathem. Schr.*, see above.
- Niccolò Guicciardini, *Isaac Newton on Mathematical Certainty and Method*, MIT Press, 2009.
- [Neugebauer 1974], O. Neugebauer, *The Exact Sciences in Antiquity*, Providence, Rhode Island (U.S.A), copyright by Brown University Press, 1957; Italian translation *Le scienze esatte nell'antichità*, Milano, Fertrinelli, 1974.
- [Newton 1707], I. Newton, *Arithmetica Universalis; sive de Compositione et Resolutione Arithmetica Liber*, Cantabrigiae, Typis Academicis, 1707.
- [Newton 1723], I. Newton, *Analysis per Quantitatum Series, Fluxiones, ac Differentias: cum Enumeratione Linearum Tertii Ordinis*, Amstaelodami, Sumptibus Societatis, 1723; first edition Londini, ex officina Pearsoniana, 1711.
- [Newton 1740], I. Newton, *La Méthode des Fluxions, et des Suites infinies*, translation of G.L. Leclerc Comte de Buffon, Paris, Chez De Bure l'aîné, 1740; first edition edited by J. Colson, London H. Woodfall, 1736.

- [Olds 1968], C.D. Olds, *Continued Fractions*, in «New Mathematical Library» published by Random House Inc., New York (USA), copyright Yale University, 1963; Italian translation F*razioni continue*, Bologna, Zanichelli, 1968.
- [Palladino 1995], F. Palladino, *Le quantità utili al ragionamento e alla ricerca*, in *Geometria, Flussioni e Differenziali*, edited by M. Panza and C.S. Roero, Napoli, Istituto Italiano per gli Studi Filosofici, La Città del Sole, 1995, pp. 397-441.
- [Pensivy 1987-1988], M. Pensivy, *Jalons historiques pour une épistémologie de la série infinie du binôme*, «Sciences et techniques en perspective», Université de Nantes, volume 14, année 1987-1988.
- [Picutti 1984], E. Picutti, *Storia del triangolo numerico*, in *Uomini e Numeri*, «Le Scienze-Quaderno n. 18», Novembre 1984.
- [Tannery 1894], P. Tannery, *Un fragment des Métriques de Héron*, Zeitschr. f. Math. u. Phys., T. XXXIX, 1894, Hist.-lit. Abth., p. 13-16; Bull. des Sc. Math., II Série, T. VVIII, 1894, p. 18-22.
- D.T. Whiteside (edited by), *The mathematical papers of Isaac Newton*, Cambridge, University Press, 1967-68, volumes I and II.