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Some fixed point results for multi-valued mappings in partial metric spaces

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Abstract
In this paper, we obtain some fixed point results for multi-valued mappings in partial metric spaces. Our results unify, generalize and complement various known comparable results from the current literature. An example is also included to illustrate the main result in the paper.

MSC: 46S40; 47H10; 54H25

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1 Introduction and preliminaries

The fixed point theory is one of the most powerful and fruitful tools in nonlinear analysis. Its core subject is concerned with the conditions for the existence of one or more fixed points of a mapping T from a topological space X into itself, that is, we can find $x \in X$ such that $Tx = x$. The Banach contraction principle [1] is the simplest and one of the most versatile elementary results in fixed point theory. Moreover, being based on an iteration process, it can be implemented on a computer to find the fixed point of a contractive mapping. It produces approximations of any required accuracy, and, moreover, even the number of iterations needed to get a specified accuracy can be determined. Recently, Samet *et al.* [2] introduced a new concept of α-contractive type mappings and established various fixed point theorems for such mappings in complete metric spaces. The presented theorems extend, generalize and improve several results on the existence of fixed points in the literature.

In 1994, Matthews [3] introduced the concept of a partial metric space and obtained a Banach-type fixed point theorem on complete partial metric spaces. Later on, several authors (see, for example, [4–31]) proved fixed point theorems in partial metric spaces. After the definition of the partial Hausdorff metric, Aydi *et al.* [28] proved the Banach-type fixed point result for set-valued mappings in complete partial metric spaces.

The aim of this paper is to generalize various known results proved by Nadler [32], Kikkawa and Suzuki [33], Mot and Petrusel [34], Dhompongsa and Yingtaweesittikul [35] to the case of partial metric spaces and give one example to illustrate our main results.

We start with recalling some basic definitions and lemmas on partial metric spaces. The definition of a partial metric space is given by Matthews [3] (see also [7, 29, 30]) as follows.

Definition 1 A partial metric on a nonempty set X is a function $p : X \times X \to [0, +\infty)$ such that the following conditions hold: for all $x, y, z \in X$,

(P_1) $p(x,x) = p(y,y) = p(x,y)$ if and only if $x = y$,
(P_2) $p(x,x) \le p(x,y)$,
(P_3) $p(x,y) = p(y,x)$,
(P_4) $p(x,z) \le p(x,y) + p(y,z) - p(y,y)$.

The pair (X,p) is then called a *partial metric space*.

If (X,p) is a partial metric space, then the function $p^s : X \times X \to [0,+\infty)$ given by $p^s(x,y) = 2p(x,y) - p(x,x) - p(y,y)$ for all $x, y \in X$ is a metric on X.

A basic example of a partial metric space is the pair $([0,+\infty), p)$, where $p(x,y) = \max\{x,y\}$ for all $x, y \in [0,+\infty)$.

Lemma 1 *Let (X,p) be a partial metric space, then we have the following*:

(1) *A sequence $\{x_n\}$ in a partial metric space (X,p) converges to a point $x \in X$ if and only if* $\lim_{n\to+\infty} p(x,x_n) = p(x,x)$.
(2) *A sequence $\{x_n\}$ in a partial metric space (X,p) is called a Cauchy sequence if the* $\lim_{n,m\to+\infty} p(x_n,x_m)$ *exists and is finite*.
(3) *A partial metric space (X,p) is said to be complete if every Cauchy sequence $\{x_n\}$ in X converges to a point $x \in X$, that is*, $p(x,x) = \lim_{n,m\to+\infty} p(x_n,x_m)$.
(4) *$\{x_n\}$ is a Cauchy sequence in (X,p) if and only if it is a Cauchy sequence in the metric space (X,p^s)*.
(5) *A partial metric space (X,p) is complete if and only if the metric space (X,p^s) is complete. Furthermore*, $\lim_{n\to+\infty} p^s(x_n,z) = 0$ *if and only if*

$$p(z,z) = \lim_{n\to+\infty} p(x_n,z) = \lim_{n,m\to+\infty} p(x_n,x_m).$$

Remark 1 ([7], Lemma 1) Let (X,p) be a partial metric space and let A be a nonempty set in (X,p), then $a \in \overline{A}$ if and only if

$$p(a,A) = p(a,a),$$

where $\overline{A}$ denotes the closure of A with respect to the partial metric p. Note A is closed in (X,p) if and only if $\overline{A} = A$.

Now, we state the following definitions and propositions of a very recent paper of Aydi *et al.* [28].

Let $CB^p(X)$ be a collection of all nonempty closed and bounded subsets of X with respect to the partial metric p. For any $A \in CB^p(X)$, we define

$$p(a,A) = \inf\{p(a,x) : x \in A\}.$$

On the other hand, for any $A, B \in CB^p(X)$, we define

$$\delta_p(A,B) = \sup\{p(a,B) : a \in A\},$$
$$\delta_p(B,A) = \sup\{p(b,A) : b \in B\}$$

and

$$H_p(A,B) = \max\{\delta_p(A,B), \delta_p(B,A)\}.$$

Proposition 1 [28] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have*

(1) $\delta_p(A,A) = \sup\{p(a,a) : a \in A\}$.
(2) $\delta_p(A,A) \leq \delta_p(A,B)$.
(3) $\delta_p(A,B) = 0$ *implies that* $A \subseteq B$.
(4) $\delta_p(A,B) \leq \delta_p(A,C) + \delta_p(C,B) - \inf_{c\in C} p(c,c)$.

Proposition 2 [28] *Let (X,p) be a partial metric space. For any $A,B,C \in CB^p(X)$, we have*

(1) $H_p(A,A) \leq H_p(A,B)$.
(2) $H_p(A,B) = H_p(B,A)$.
(3) $H_p(A,B) \leq H_p(A,C) + H_p(C,B) - \inf_{c\in C} p(c,c)$.

Lemma 2 [28] *Let A and B be nonempty closed and bounded subsets of a partial metric space (X,p) and $h > 1$. Then, for all $a \in A$, there exists $b \in B$ such that $p(a,b) \leq hH_p(A,B)$.*

The following result was proved by Aydi *et al.* in [28].

Theorem 1 *Let (X,p) be a partial metric space. If $T : X \to CB^p(X)$ is a multi-valued mapping such that, for all $x,y \in X$,*

$$H_p(Tx,Ty) \leq kp(x,y),$$

where $k \in (0,1)$. Then T has a fixed point.

2 Main results

Now, we characterize the celebrated theorem of Kikkawa and Suzuki [33] in the framework of partial metric spaces.

Theorem 2 *Define a strictly decreasing function Θ from $[0,1)$ onto $(\frac{1}{2},1]$ by $\Theta(r) = \frac{1}{1+r}$. Let (X,p) be a complete partial metric space and $F : X \to CB^p(X)$ be a multi-valued mapping. Assume that there exists $r \in [0,1)$ such that*

$$\Theta(r)p(x,Fx) \leq p(x,y) \quad \Longrightarrow \quad H_p(Fx,Fy) \leq rp(x,y) \tag{2.1}$$

for all $x,y \in X$. Then there exists $u \in X$ such that $u \in Fu$.

Proof Let $x_0 \in X$ be arbitrarily chosen. For all $x_1 \in Fx_0$, we have

$$\Theta(r)p(x_0,Fx_0) \leq \Theta(r)p(x_0,x_1) \leq p(x_0,x_1)$$

and, by the condition (2.1), we get

$$H_p(Fx_0,Fx_1) \leq rp(x_0,x_1).$$

Let $h \in (1, \frac{1}{r})$, by Lemma 2, there exists $x_2 \in Fx_1$ such that $p(x_1, x_2) \leq hH_p(Fx_0, Fx_1)$. Using the previous inequality, we obtain

$$p(x_1, x_2) \leq hH_p(Fx_0, Fx_1) \leq hrp(x_0, x_1).$$

Now, we have

$$\Theta(r)p(x_1, Fx_1) \leq \Theta(r)p(x_1, x_2) \leq p(x_1, x_2)$$

and, by the condition (2.1), we get

$$H_p(Fx_1, Fx_2) \leq rp(x_1, x_2).$$

By Lemma 2, there exists $x_3 \in Fx_2$ such that

$$p(x_2, x_3) \leq hH_p(Fx_1, Fx_2) \leq hrp(x_1, x_2) \leq (hr)^2 p(x_0, x_1).$$

Continuing in this way, we can generate a sequence $\{x_n\}$ in X such that $x_{n+1} \in Fx_n$ and

$$p(x_n, x_{n+1}) \leq k^n p(x_0, x_1) \tag{2.2}$$

for all $n \in \mathbb{N}$, where $k = hr < 1$.

Now, we show that $\{x_n\}$ is a Cauchy sequence. Using (2.2) and the triangle inequality for partial metrics (p_4), for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+m}) - p(x_{n+1}, x_{n+1}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+m}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+m}) - p(x_{n+2}, x_{n+2}) \\ &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + p(x_{n+2}, x_{n+m}). \end{aligned}$$

Inductively, we have

$$\begin{aligned} p(x_n, x_{n+m}) &\leq p(x_n, x_{n+1}) + p(x_{n+1}, x_{n+2}) + \cdots + p(x_{n+m-1}, x_{n+m}) \\ &\leq k^n p(x_0, x_1) + k^{n+1} p(x_0, x_1) + \cdots + k^{n+m-1} p(x_0, x_1) \\ &\leq \left(k^n + k^{n+1} + \cdots + k^{n+m-1}\right) p(x_0, x_1) \\ &\leq \frac{k^n}{1-k} p(x_0, x_1) \to 0 \end{aligned}$$

as $n \to +\infty$ since $0 \leq k < 1$. By the definition of p^s, we get

$$p^s(x_n, x_{n+m}) \leq 2p(x_n, x_{n+m}) \to 0$$

as $n \to +\infty$, which implies that $\{x_n\}$ is a Cauchy sequence in (X, p^s). Since (X, p) is complete, by Lemma 1, the corresponding metric space (X, p^s) is also complete. Therefore, the sequence $\{x_n\}$ converges to some $u \in X$ with respect to the metric p^s, that is,

$\lim_{n\to+\infty} p^s(x_n,u) = 0$. Again, by Lemma 1, we have

$$p(u,u) = \lim_{n\to+\infty} p(x_n,u) = \lim_{n,m\to+\infty} p(x_n,x_m) = 0. \tag{2.3}$$

Next, we show that

$$p(u,Fx) \le rp(u,x)$$

for all $x \in X\backslash\{u\}$. Since $p(x_n,u) \to 0$ as $n \to +\infty$, there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n,u) \le \frac{1}{3}p(u,x)$$

for all $n \in \mathbb{N}$ with $n \ge n_0$. Then we have

$$\begin{aligned}\Theta(r)p(x_n,Fx_n) \le p(x_n,Fx_n) \le p(x_n,x_{n+1}) &\le p(x_n,u) + p(u,x_{n+1}) - p(u,u)\\ &= p(x_n,u) + p(u,x_{n+1})\\ &\le \frac{2}{3}p(u,x) \le p(u,x) - p(x_n,u)\\ &\le p(x_n,x)\end{aligned}$$

and hence $H_p(Fx_n,Fx) \le rp(x_n,x)$. Since

$$\begin{aligned}p(u,Fx) &\le p(u,x_{n+1}) + p(x_{n+1},Fx)\\ &\le p(u,x_{n+1}) + H_p(Fx_n,Fx)\\ &\le p(u,x_{n+1}) + rp(x_n,x),\end{aligned}$$

letting $n \to +\infty$, we obtain

$$p(u,Fx) \le rp(u,x) \tag{2.4}$$

for all $x \in X\backslash\{u\}$.

Next, we prove that

$$H_p(Fx,Fu) \le rp(x,u)$$

for all $x \in X$ with $x \ne u$. For all $n \in \mathbb{N}$, we choose $v_n \in Fx$ such that

$$p(u,v_n) \le p(u,Fx) + \frac{1}{n}p(x,u).$$

Then, using (2.4) and the previous inequality, we get

$$\begin{aligned}p(x,Fx) &\le p(x,v_n) \le p(x,u) + p(u,v_n) - p(u,u)\\ &= p(x,u) + p(u,v_n)\\ &\le p(x,u) + p(u,Fx) + \frac{1}{n}p(x,u)\end{aligned}$$

$$\begin{aligned}&\leq p(x,u)+rp(u,x)+\frac{1}{n}p(x,u)\\&=\left(1+r+\frac{1}{n}\right)p(x,u)\end{aligned}$$

for all $n\in\mathbb{N}$. As $n\to+\infty$, we obtain $\frac{1}{1+r}p(x,Fx)\leq p(x,u)$. From the assumption, we have

$$H_p(Fx,Fu)\leq rp(x,u).$$

Finally, if, for some $n\in\mathbb{N}$, we have $x_n=x_{n+1}$, then x_n is a fixed point of F. Assume that $x_n\neq x_{n+1}$ for all $n\in\mathbb{N}$. This implies that there exists an infinite subset J of $\mathbb{N}$ such that $x_n\neq u$ for all $n\in J$. From

$$\begin{aligned}p(u,Fu)&\leq p(u,x_{n+1})+p(x_{n+1},Fu)\\&\leq p(u,x_{n+1})+H_p(Fx_n,Fu)\\&\leq p(u,x_{n+1})+rp(x_n,u),\end{aligned}$$

letting $n\to+\infty$ with $n\in J$, we get

$$p(u,Fu)=0=p(u,u).$$

By Remark 1, we deduce that $u\in Fu$ and hence u is a fixed point of F. This completes the proof. □

It is obvious that Theorem 1 of Aydi *et al.* follows directly from Theorem 2.

The following theorem is a result of Reich type [36] as well as a generalization of Kikkawa and Suzuki type in the framework of partial metric spaces.

Theorem 3 *Let (X,p) be a complete partial metric space and let $F:X\to CB^p(X)$ be a multi-valued mapping satisfying the following*:

$$\theta p(x,Fx)\leq p(x,y)\quad\Longrightarrow\quad H_p(Fx,Fy)\leq ap(x,y)+bp(x,Fx)+cp(y,Fy)\tag{2.5}$$

for all $x,y\in X$, nonnegative numbers a, b, c with $a+b+c\in[0,1)$ and $\theta=\frac{1-b-c}{1+a}$. Then F has a fixed point.

Proof Let $h\in(1,\frac{1}{a+b+c})$ and $x_0\in X$ be arbitrary. Let $x_1\in Tx_0$. By Lemma 2, there exists $x_2\in Fx_1$ such that

$$p(x_1,x_2)\leq hH_p(Fx_0,Fx_1).$$

Since $\theta p(x_0,Fx_0)\leq\theta p(x_0,x_1)\leq p(x_0,x_1)$, we have

$$\begin{aligned}p(x_1,x_2)&\leq hH_p(Fx_0,Fx_1)\leq h\big(ap(x_0,x_1)+bp(x_0,Fx_0)+cp(x_1,Fx_1)\big)\\&\leq h(a+b)p(x_0,x_1)+hcp(x_1,x_2)\\&\leq\frac{h(a+b)}{1-hc}p(x_0,x_1).\end{aligned}$$

Continuing in a similar way, we can obtain a sequence $\{x_n\}$ of successive approximations for F, starting from x_0, satisfying the following:

(a) $x_{n+1} \in Fx_n$ for all $n \in \mathbb{N}$;

(b) $p(x_n, x_{n+1}) \leq k^n p(x_0, x_1)$ for all $n \in \mathbb{N}$,

where $k = \frac{h(a+b)}{1-hc} < 1$. Now, proceeding as in the proof of Theorem 2, we deduce that the sequence $\{x_n\}$ converges to some $u \in X$ with respect to the metric p^s, that is, $\lim_{n\to+\infty} p^s(x_n, u) = 0$. Moreover, (2.3) holds by Lemma 2.

First, we show that

$$p(u, Fx) \leq \left(a + \frac{b}{\theta}\right)p(u, x) + cp(x, Fx)$$

for all $x \in X\backslash\{u\}$. Since $p(x_n, u) \to 0$ as $n \to +\infty$ under the metric p, there exists $n_0 \in \mathbb{N}$ such that

$$p(x_n, u) \leq \frac{1}{3}p(u, x)$$

for each $n \geq n_0$. Then we have

$$\begin{aligned}
\theta p(x_n, Fx_n) \leq p(x_n, Fx_n) &\leq p(x_n, x_{n+1}) \\
&\leq p(x_n, u) + p(u, x_{n+1}) - p(u, u) \\
&= p(x_n, u) + p(u, x_{n+1}) \\
&\leq \frac{2}{3}p(u, x) \leq p(u, x) - p(x_n, u) \\
&\leq p(x_n, x),
\end{aligned}$$

which implies that

$$\begin{aligned}
H_p(Fx_n, Fx) &\leq ap(x_n, x) + bp(x_n, Fx_n) + cp(x, Fx) \\
&\leq ap(x_n, x) + \frac{b}{\theta}p(x_n, x) + cp(x, Fx) \\
&= \left(a + \frac{b}{\theta}\right)p(x_n, x) + cp(x, Fx)
\end{aligned}$$

for all $n \geq n_0$. Thus we have

$$\begin{aligned}
p(u, Fx) &\leq p(u, x_{n+1}) + p(x_{n+1}, Fx) \\
&\leq p(u, x_{n+1}) + H_p(Fx_n, Fx) \\
&\leq p(u, x_{n+1}) + \left(a + \frac{b}{\theta}\right)p(x_n, x) + cp(x, Fx)
\end{aligned}$$

for all $n \geq n_0$. Letting $n \to +\infty$, we get

$$p(u, Fx) \leq \left(a + \frac{b}{\theta}\right)p(u, x) + cp(x, Fx)$$

for all $x \in X\backslash\{u\}$.

Next, we show that

$$H_p(Fx, Fu) \le \left(a + \frac{b}{\theta}\right) p(x,u) + cp(u, Fu)$$

for all $x \in X$ with $x \neq u$. Now, for all $n \in \mathbb{N}$, there exists $y_n \in Fx$ such that

$$p(u, y_n) \le p(u, Fx) + \frac{1}{n} p(x,u).$$

From

$$\begin{aligned} p(x, Fx) &\le p(x, y_n) \le p(x,u) + p(u, y_n) - p(u,u) \\ &= p(x,u) + p(u, y_n) \\ &\le p(x,u) + p(u, Fx) + \frac{1}{n} p(x,u) \\ &\le p(x,u) + \left(a + \frac{b}{\theta}\right) p(u,x) + cp(x, Fx) + \frac{1}{n} p(x,u) \\ &= \left(1 + a + \frac{b}{\theta} + \frac{1}{n}\right) p(x,u) + cp(x, Fx) \end{aligned}$$

for all $n \in \mathbb{N}$, it follows that, as $n \to +\infty$,

$$(1-c) p(x, Fx) \le \left(1 + a + \frac{b}{\theta}\right) p(x,u)$$

and so $\theta p(x, Fx) \le p(x,u)$. Thus we have

$$\begin{aligned} H_p(Fx, Fu) &\le ap(x,u) + bp(x, Fx) + cp(u, Fu) \\ &\le \left(a + \frac{b}{\theta}\right) p(x,u) + cp(u, Fu) \end{aligned}$$

for all $x \in X \setminus \{u\}$.

Finally, if, for some $n \in \mathbb{N}$, we have $x_n = x_{n+1}$, then x_n is a fixed point of F. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. This implies that there exists an infinite subset J of $\mathbb{N}$ such that $x_n \neq u$ for all $n \in J$. Now, for all $n \in J$, we have

$$\begin{aligned} p(u, Fu) &\le p(u, x_{n+1}) + p(x_{n+1}, Fu) \\ &\le p(u, x_{n+1}) + H_p(Fx_n, Fu) \\ &\le p(u, x_{n+1}) + \left(a + \frac{b}{\theta}\right) p(x_n, u) + cp(u, Fu). \end{aligned}$$

Letting $n \to +\infty$ with $n \in J$, we get

$$p(u, Fu) = 0 = p(u,u).$$

By Remark 1, we deduce that $u \in Fu$ and hence u is a fixed point of F. This completes the proof. □

The following theorem is a generalization of a result of Dhompongsa and Yingtaweesittikul [35] to the setting of partial metric space.

Theorem 4 *Let (X,p) be a complete partial metric space and let $F : X \to CB^p(X)$ be a multi-valued mapping such that*

$$\theta p(x,Fx) \leq p(x,y) \quad \Longrightarrow \quad H_p(Fx,Fy) \leq \lambda\bigl(p(x,Fx)+p(y,Fy)\bigr)+\mu p(x,y) \tag{2.6}$$

for all $x,y \in X$, where $\theta = \frac{1}{2\lambda+\mu+1}$ with λ, μ nonnegative real numbers and $0 \leq 2\lambda+\mu < 1$. Then F has a fixed point.

Proof Let $h \in (1, \frac{1}{2\lambda+\mu})$ and $x_0 \in X$ be arbitrary. Following the same proof of Theorem 3, by replacing $\theta = \frac{1-b-c}{1+a}$ in the proof by $\theta = \frac{1}{2\lambda+\mu+1}$, we can obtain a sequence $\{x_n\}$ such that

(a) $x_{n+1} \in Fx_n$ for all $n \in \mathbb{N}$;

(b) $p(x_n,x_{n+1}) \leq k^n p(x_0,x_1)$ for all $n \in \mathbb{N}$,

where $k = \frac{h(\lambda+\mu)}{1-h\lambda} < 1$.

Now, proceeding as in the proof of Theorem 2, we deduce that the sequence $\{x_n\}$ converges to some $u \in X$ with respect to the metric p^s, that is, $\lim_{n\to+\infty} p^s(x_n,u) = 0$. Again, from Lemma 2, we have

$$p(u,u) = \lim_{n\to+\infty} p(x_n,u) = \lim_{n\to+\infty} p(x_n,x_m) = 0. \tag{2.7}$$

Next, we show that

$$p(u,Fx) \leq \mu p(u,x) + \lambda p(x,Fx)$$

for all $x \in X\backslash\{u\}$. Since $p(x_n,u) \to 0$ as $n \to +\infty$, there exists $n_0 \in \mathbb{N}$ such that $p(x_n,u) \leq \frac{1}{3}p(u,x)$ for all $n \geq n_0$. We have

$$\begin{aligned}
\theta p(x_n,Fx_n) &\leq p(x_n,Fx_n) \leq p(x_n,x_{n+1}) \\
&\leq p(x_n,u) + p(u,x_{n+1}) - p(u,u) \\
&= p(x_n,u) + p(u,x_{n+1}) \\
&\leq \frac{2}{3}p(u,x) \leq p(u,x) - p(x_n,u) \leq p(x_n,x).
\end{aligned}$$

Now, using the conditions (2.6) and (2.7), we obtain

$$\begin{aligned}
p(u,Fx) &\leq p(u,x_{n+1}) + p(x_{n+1},Fx) \\
&\leq p(u,x_{n+1}) + H_p(Fx_n,Fx) \\
&\leq p(u,x_{n+1}) + \lambda p(x_n,Fx_n) + \lambda p(x,Fx) + \mu p(x_n,x) \\
&\leq p(u,x_{n+1}) + \lambda p(x_n,x_{n+1}) + \lambda p(x,Fx) + \mu p(x_n,x)
\end{aligned}$$

for all $n \geq n_0$. Letting $n \to +\infty$, we get

$$p(u,Fx) \leq \lambda p(x,Fx) + \mu p(u,x),$$

as desired.

Next, we show that

$$H_p(Fx,Fu) \le \lambda p(x,Fx) + \lambda p(u,Fu) + \mu p(x,u)$$

for all $x \in X \setminus \{u\}$. By Lemma 2, for all $n \in \mathbb{N}$, there exists $y_n \in Fx$ such that

$$p(u,y_n) \le p(u,Fx) + \frac{1}{n}p(u,x).$$

Clearly, we have

$$\begin{aligned} p(x,Fx) &\le p(x,y_n) \le p(x,u) + p(u,y_n) - p(u,u) \\ &= p(x,u) + p(u,y_n) \\ &\le p(x,u) + p(u,Fx) + \frac{1}{n}p(x,u) \\ &\le p(x,u) + \lambda p(x,Fx) + \mu p(u,x) + \frac{1}{n}p(x,u) \\ &\le \left(1 + \mu + \frac{1}{n}\right)p(x,u) + \lambda p(x,Fx) \end{aligned}$$

for all $n \in \mathbb{N}$. Hence, as $n \to +\infty$, we get

$$(1-\lambda)p(x,Fx) \le (1+\mu)p(x,u)$$

and so $\Theta p(x,Fx) \le p(x,u)$ since $\Theta \le \frac{1-\lambda}{1+\mu}$. Now, using the condition (2.6), we obtain

$$H_p(Fx,Fu) \le \lambda p(x,Fx) + \lambda p(u,Fu) + \mu p(x,u)$$

for all $x \in X \setminus \{u\}$.

Finally, if, for some $n \in \mathbb{N}$, we have $x_n = x_{n+1}$, then x_n is a fixed point of F. Assume that $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. This implies that there exists an infinite subset J of $\mathbb{N}$ such that $x_n \neq u$ for all $n \in J$. From

$$\begin{aligned} p(u,Fu) &\le p(u,x_{n+1}) + p(x_{n+1},Fu) \\ &\le p(u,x_{n+1}) + H_p(Fx_n,Fu) \\ &\le p(u,x_{n+1}) + \lambda p(x_n,Fx_n) + \lambda p(u,Fu) + \mu p(x_n,x) \\ &\le p(u,x_{n+1}) + \lambda \frac{1+\mu}{1-\lambda}p(x_n,x) + \lambda p(u,Fz) + \mu p(x_n,x), \end{aligned}$$

letting $n \to +\infty$ with $n \in J$, we get

$$p(u,Fu) = 0 = p(u,u).$$

By Remark 1, we deduce that $u \in Fu$ and hence u is a fixed point of F. This completes the proof. □

Now, we give one example to illustrate Theorem 3.

Example 1 Let $X = \{2,3,4\}$ and $p : X \times X \to [0,+\infty)$ be a partial metric on X defined by

$$p(2,2) = p(3,3) = 0, \qquad p(4,4) = \frac{9}{20}, \qquad p(2,3) = p(3,2) = \frac{2}{5},$$
$$p(2,4) = p(4,2) = \frac{1}{2}, \qquad p(3,4) = p(4,3) = \frac{9}{19}.$$

Let $F : X \to CB^p(X)$ be defined by

$$Fx = \begin{cases} \{2\} & \text{if } x \in \{2,3\}, \\ \{2,3\} & \text{otherwise.} \end{cases}$$

It is easy to see that $\{2\}$ and $\{2,3\}$ are closed in X with respect to the partial metric p. Now, we have

$$H_p(F2,F2) = H_p(F3,F3) = H_p(F2,F3) = H_p\bigl(\{2\},\{2\}\bigr) = 0;$$
$$H_p(F4,F4) = H_p\bigl(\{2,3\},\{2,3\}\bigr) = 0;$$
$$H_p(F2,F4) = H_p(F3,F4) = H_p\bigl(\{2\},\{2,3\}\bigr) = \frac{2}{5};$$
$$p(2,F2) = p\bigl(2,\{2\}\bigr) = 0; \qquad p(3,F3) = p\bigl(3,\{2\}\bigr) = \frac{2}{5};$$
$$p(4,F4) = p\bigl(4,\{2,3\}\bigr) = \frac{9}{19}.$$

If we choose $a = \frac{3}{4}$, $b = \frac{1}{8}$ and $c = \frac{1}{10}$, the multi-valued mapping F satisfies the hypotheses of Theorem 3 and so has a fixed point. To such end, it is enough to show that (2.5) is satisfied in the following cases.

Case 1. $x = 2$ and $y = 4$. Now, $\theta p(2,F_2) \le p(2,4)$, where $\theta = \frac{31}{70}$ and

$$H_p(F2,F4) = \frac{2}{5} \le \frac{21}{50} \le \frac{3}{4}p(2,4) + \frac{1}{8}p(2,F2) + \frac{1}{10}p(4,F4).$$

Case 2. $x = 3$ and $y = 4$. Now, $\theta p(3,F3) \le p(3,4)$ and

$$H_p(F3,F4) = \frac{2}{5} \le \frac{9}{20} \le \frac{3}{4}p(3,4) + \frac{1}{8}p(3,F3) + \frac{1}{10}p(4,F4).$$

Case 3. $x = 4$ and $y = 3$. Now, $\theta p(4,F4) \le p(4,3)$ and

$$H_p(F4,F3) = \frac{2}{5} \le \frac{9}{20} \le \frac{3}{4}p(4,3) + \frac{1}{8}p(4,F4) + \frac{1}{10}p(3,F3).$$

Case 4. $x = 4$ and $y = 2$. Now, $\theta p(4,F4) \le p(4,2)$ and

$$H_p(F4,F2) = \frac{2}{5} \le \frac{43}{100} \le \frac{3}{4}p(4,2) + \frac{1}{8}p(4,F4) + \frac{1}{10}p(2,F2).$$

Thus all the conditions of Theorem 3 are satisfied. Here $x = 2$ is a fixed point of F.

On the other hand, the metric p^s induced by the partial metric p is given by

$$p^s(1,1) = p^s(2,2) = p^s(3,3) = 0, \qquad p^s(2,3) = p^s(3,2) = \frac{4}{5},$$
$$p^s(4,3) = p^s(3,4) = \frac{189}{380}, \qquad p^s(4,2) = p^s(2,4) = \frac{11}{20}.$$

Note that, in the case of an ordinary Hausdorff metric, the given mapping does not satisfy the condition (2.5). Indeed, for $x = 2$ and $y = 4$, the condition $\theta p^s(2, F2) \leq p^s(2, 4)$ is satisfied. But the condition $H(F2, F4) \leq ap^s(2, 4) + bp^s(2, F2) + cp^s(4, F4)$ is not satisfied.

In fact, we have

$$H(F2, F4) = H\bigl(\{2\}, \{2, 3\}\bigr) = \frac{4}{5},$$

$$p^s(4, F4) = p^s\bigl(4, \{2, 3\}\bigr) = \frac{11}{20}$$

and

$$\frac{4}{5} \geq \frac{3}{4}\left(\frac{11}{20}\right) + \frac{1}{8}(0) + \frac{1}{10}\left(\frac{11}{20}\right) = \frac{187}{400}.$$

Competing interests
The authors declare that they have no competing interests.

Authors' contributions
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